Chapter 13
Perceiving and Using Structures When Determining the Cardinality of Sets: A Child's Learning Story

Priska Sprenger and Christiane Benz

13.1 Introduction

Early childhood education in mathematics is often limited to the most obvious mathematical activities children show at this age: counting and determining sets. Researchers (cf. Brownell et al., 2014; Gasteiger & Benz, 2018; van Oers, 2004) and official policy educational documents agree that early mathematics education should be based on central mathematical concepts, and enable continuous learning and a broad understanding of mathematics (Brownell et al., 2014; Gasteiger, 2015). Not only mathematical contents are relevant for a broad understanding of mathematics, but also mathematical processes or proficiency strands, for example, problem-solving or reasoning (cf. Australian Curriculum, Assessment and Reporting Authority, 2014; Department for Education, 2013; National Council of Teachers of Mathematics, 2000). Nevertheless, the arithmetical content is still one important part in early mathematical educational concepts and policy documents—also in relation to the background of early intervention or prevention for problems in learning mathematics in school. Therefore, we will look at and analyze some aspects of early numerical and arithmetical development.

13.2 Role of Structures for Numerical and Arithmetical Development

Structures play an important role in numerous models for number and arithmetic concept development (cf. Baroody, Lai, & Mix, 2006; Lüken, 2012; Mulligan & Mitchelmore, 2018; Mulligan, Mitchelmore, English, & Crevensten, 2013). In the

P. Sprenger (✉) · C. Benz
University of Education Karlsruhe, Karlsruhe, Germany
e-mail: priska.sprenger@ph-karlsruhe.de

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hypothetical development trajectory for key aspects of early number and arithmetic development, Baroody et al. (2006) emphasize the importance of structures both for the number concept and for the arithmetical development:

> Conceptually based VNR [verbal number cognition] enables a child to see (decompose) collections of two (a whole) as one and one (into its parts) […]. […] Experiences [of] decomposing and composing small, easily subitized collections may be the basis for constructing an informal concept of addition (and subtraction) (Baroody et al., 2006, p. 193).

Structure or structuring, which can be defined as the way in which various elements are organized and related (Mulligan & Mitchelmore, 2013), can be seen as decomposing and composing (visible) objects and therefore it is an underlying concept for the part–whole-relations because "this composing process fosters an understanding of part-whole-relations and vice versa" (Baroody et al., 2006, p. 193).

> A part–whole concept and experience with composition and decomposition may underlie an understanding of "number families" or the different-names-for-a-number concept (a number can be represented in various ways because a whole can be composed or decomposed in various ways) and is one key link between number and arithmetic (Baroody et al., 2006, p. 195).

Thus, it is not surprising that Resnick, already in Resnick, 1989, pointed out that "probably the major conceptual achievement of the early school years is the interpretation of numbers in terms of part and whole relationships" (p. 114). Referring to Baroody et al. (2006), (de)composing collections of objects can nurture the part–whole understanding. If children (de)compose collections of objects, they switch the focus from individual items to perceiving and identifying structures of parts. Hunting (2003) describes this ability as an important step for part–whole reasoning, which in turn contributes to numerical development. If the switch from focusing on individual items to perceiving and identifying structures of parts is so important, different aspects of this "switch" have to be examined, so that children can be supported to achieve this switch. Therefore, in this study we look at different possibilities to perceive items in collections and also how perception is used for the determination of cardinality.

The evaluations of the learning story presented in this study are based on a theoretical model that distinguishes between two processes: the process of perceiving sets and the process of determining cardinality. These processes can run one after the other, or coincide with each other, for example during subitizing (cf. Fig. 13.1 and Schöner & Benz, 2018). In Fig. 13.1, possible relationships between the two processes are illustrated.

Each of these two processes can be divided into three different subgroups. The different ways of perceiving a set allow different ways to determine the cardinality as, for example, the use of a counting strategy, a derived facts strategy, or the use of known facts. If the elements of a set are perceived as individual elements, the only possibility to determine the cardinality is to use the counting strategy "counting all." If a set is perceived in (sub-)structures, "counting all" would also be a possible strategy to determine the cardinality. Furthermore, in this case, in addition to "counting on" (four, five, six, seven, eight) or "counting in steps" (four, six, eight), noncounting-derived facts strategies (three and three equals six and two more is eight) can also be used to determine the cardinality.

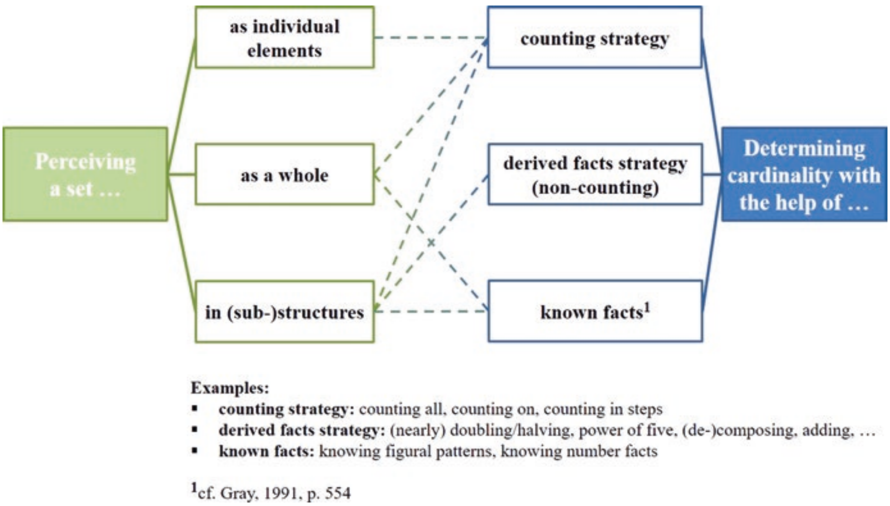

Fig. 13.1 Two processes: Perception of sets and determining cardinality (Schöner & Benz, 2018)

13.3 Research Question

In this chapter, we aim to answer the following research question based on a child's learning story:

How do the perception of structures and the use of structures determine the cardinality of a set change based on an implementation?

13.4 Design of the Study

Ninety-five children from nine different kindergartens aged from 5 to 6 years were interviewed three times. The study of Schöner and Benz (2018) describes that at the first interview (T1), 102 children were interviewed. Some children have left the study, for example due to a move to another city. The children were divided into a treatment group (n = 55) and a control group (n = 40). Only the treatment group took part in an implementation phase (cf. Fig. 13.2). Luca, the boy in the presented learning story, was a member of the treatment group. In the first interview (T1), at the beginning of the last year in kindergarten in September 2015, Luca was five years and two months old. Then, an implementation happened for four months. After the implementation period, the posttest interview (T2) proceeded in February 2016. The children were given the same tasks again to investigate the development in perceiving and using structures to determine the cardinality of sets. The third interview (T3), at the end of the last year in kindergarten, was conducted as a follow-up interview in July 2016 (cf. Fig. 13.2).

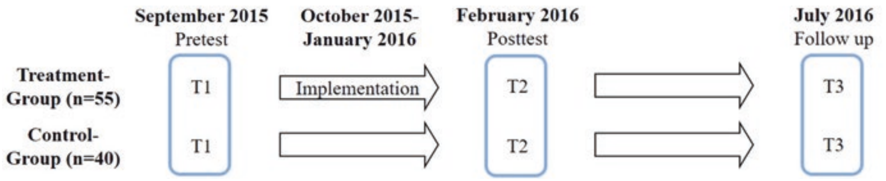

Fig. 13.2 Timeline of the study

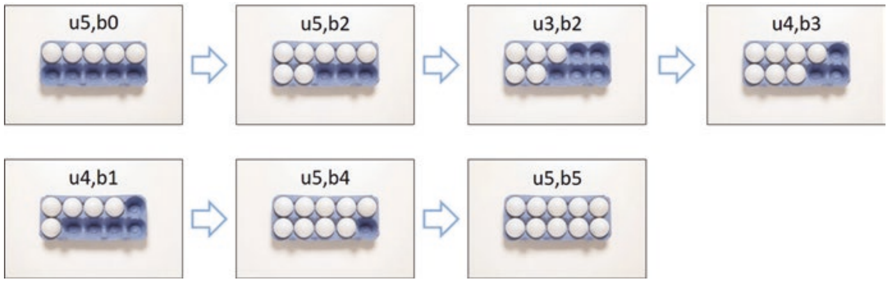

Fig. 13.3 Order of the presented items

The study is designed in a panel design, so the same children were interviewed three times (T1, T2, T3) to evaluate whether and how they perceive and use structures for determining the cardinality of the presented sets. To get some deeper insights into the perceiving process of the children, the research tool eye-tracking was used. With eye-tracking, it is possible to detect the eye movements of the children while they are perceiving and determining the cardinality of sets. The whole interview consists of four different parts (in three of them, the research method eye-tracking is used: unstructured pictures of dots, egg cartons, and daily life pictures). In this study, the focus is on the part with the egg cartons for ten eggs which has an equivalent structure as the tenth frame. It is the typical size of egg cartons in Germany.

13.4.1 Tasks

In this part of the study, pictures of egg cartons with the quantities two, three, four, five, seven, nine, and 10 were presented to the children on a monitor. In the present learning story of Luca, only the sets with cardinality ≥ 5 are considered. In Fig. 13.3, the order of these items is illustrated.

Because there are three pictures with the cardinality five and two with cardinality seven, it is necessary to name the egg cartons individually. “u3,b2” means, for example, “three eggs on the upper row” and “two eggs on the bottom row” (cf. Fig. 13.3). These abbreviations were not visible to the children. They are useful in this chapter to facilitate communication about individual egg cartons.

Before the pictures were presented, the children had been told that the interviewer would like to know how many eggs they saw. They were asked to say the number as soon as they knew it and they had as much time for determining the quantity as needed. As soon as they said a number, the interviewer asked how they came to the result (cf. Schöner & Benz, 2017; Schöner & Benz, 2018). First, a closed egg carton could be seen. Then, the carton was opened. After the child said a number (cf. Fig. 13.5, phase 1) and explained how it came to the result (cf. Fig. 13.5, phase 2), the carton was closed again.

13.4.2 Implementation

After the first interview (T1), the treatment group got a collection of different materials and games, like the game "I spy with my little eye" which will be explained below. These materials and games offered the opportunity to discover and facilitate the structured perception of sets in a playful way. The children in the control group did not get the materials. Additionally, in order to observe a development in perceiving and using structures through an everyday support of the children in kindergarten and at home, the kindergarten of the control group did not work with any special mathematical training program in perceiving and using structures. That means that, during the test period, the kindergarten teachers worked with mathematics in exactly the same way as before the research project. The normal routines of the kindergartens of the control group were therefore not altered.

During the four months of the implementation phase (cf. Fig. 13.2), the kindergarten teachers in the treatment group were instructed to use these materials with the children one to three times each week for 30 min (cf. Schöner & Benz, 2017). During the entire implementation phase and until the end of the kindergarten year in July 2016, the materials were kept in the kindergartens and were always freely accessible to the children. During the implementation phase, the educators were asked to keep a diary by writing down their activities with the children and their observations in order to get additional information. Luca was mentioned by name, so statements about his development can be given from the perspective of his educators.

The ten-egg-cartons were part of the provided materials. The kindergarten teachers were instructed on how to use the materials and how to ask questions in order to gain insights into the children's ideas and their ways of thinking. Additionally, they got a handbook with different ideas and examples. One of these ideas was a modification of the game "I spy with my little eye:" Sets of eggs with quantities ranging from 1 to 10 are sorted in the egg cartons in such a way that the upper row is always filled first. Thus, the numbers are represented up to 10 with a five-structure (cf. Fig. 13.4).

After the cartons are filled, a collective conversation about the "appearance" of the different number-pictures can take place. Afterwards, the egg cartons are closed and mixed. Now, a child (or a kindergarten teacher) takes an egg carton and looks inside it. He or she describes which number-picture he or she sees. There are different

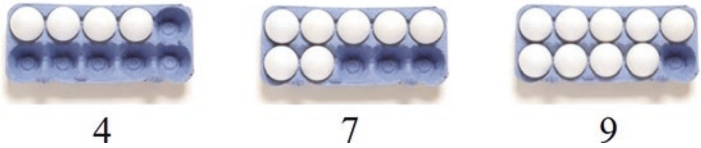

Fig. 13.4 "I spy with my little eye" – Three examples

requirements: Filled rows and places and/or empty rows and places can be described. Hence, a description of the number six can be different. For example: "The upper row is full. In the bottom row it is just one egg" or "The upper row is full. In the bottom row, four places are empty or in the bottom row four eggs can be placed" or "There are still four empty places or in the bottom row it is still place for four eggs." The game is finished when all egg cartons have been described. The child who knows how many eggs there are in the carton will receive the carton. Whoever guesses the most boxes wins. If the game is played in this way it is very challenging, because children have to rely on internal pictures. In order to help the children to build up internal, structured images of the individual numbers, a variation of the game may be played. If all cartons are open during the whole game, children can see the pictures when somebody else describes it and link picture and description. Some helpful questions for both ways of playing are "can you describe how the number-picture looks like?" or "can you say how many eggs there are in a full egg carton without counting every single egg?"

Another focus can be established if cartons can be filled without any restriction. So, one number can be displayed by many different ways in an egg carton. Here, the focus can be placed on different ways of decomposing a set of objects and therefore different ways of decomposing numbers. Here, one possible question could be: "How can you put *n* eggs in the carton (for ten eggs)?" (Benz, 2010, p. 28). There was, on the one hand, a description of the games and, on the other hand, a lot of possible questions and impulses in the handbook the kindergarten teachers got. These questions and impulses were helpful in supporting the use of the learning opportunities of the games regarding the perception of structures and the structural use to determine the cardinality.

13.4.3 Aspects of Data Analysis

In the evaluation, a distinction is made between three different types of data: the *observation aspects,* the *eye-tracking data*, and the *explanation* (cf. Fig. 13.5). Each piece of data leads to hypotheses about the perception process and the determination process. On the one hand, hypotheses about these processes are generated on the basis of the observations which are made during the interview, such as gestures, sounds, or promptness of the answer (cf. Fig. 13.5, observation aspects and explanation). On the other hand, additional information is gained during phase 1,

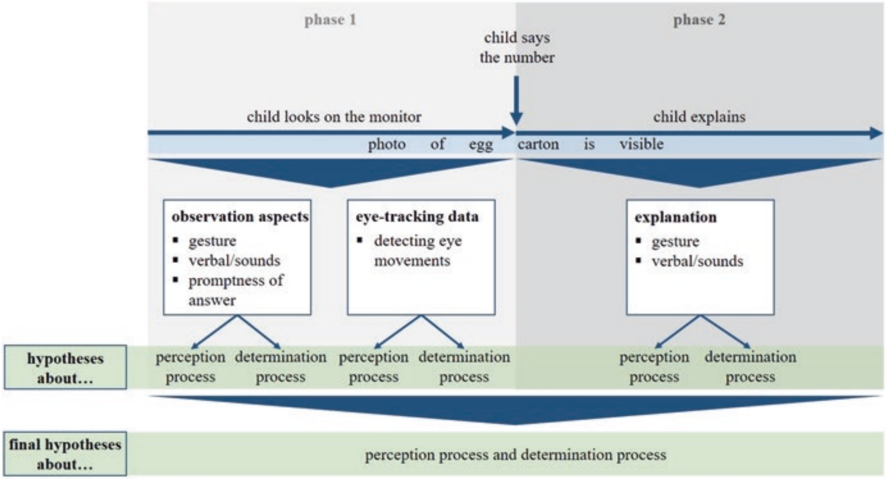

Fig. 13.5 Differentiation of aspects of analysis

by detecting the eye movements with the help of the eye-tracker (cf. Fig. 13.5, eye-tracking data).

The eye-tracking data can provide insights into children's processes of perception (see paragraph below "Data Analysis by the use of Eye-tracking technology"). These insights can be used to form hypotheses about perceiving structures (perception process) and about determining the cardinality (determination process). A typical observation with eye-tracking data is that the children's gaze often oscillates between two subsets when a structure is perceived. So, the eye-tracking data first lead to a hypothesis about the perception process. In most cases, it is possible to generate a hypothesis on the determination process from the hypothesis on the perception process. This is the case, for example, when the eye-tracking data show that each egg was fixed individually. In this case, the hypothesis for the determination process would be "counting all." If the eye-tracking data show a pendulum motion between two subsets, then a perception process can be concluded as a determination process. In the learning story described below, a special observation can be made during the analysis of the eye-tracking data. First, a pendulum movement between two subsets is visible and then the fixation of each individual egg. In this case, both the perception process (structural perception) and the determination process (counting all) become visible (cf. Fig. 13.9). Regarding the observation aspects, it became apparent in the course of the evaluations that, often, only a hypothesis on the determination process and none on the perception process can be made. An example of this is "counting all" as a strategy for determining cardinality (determination process). In this case, no interpretation of perception is possible, because it is not clear if the child perceived the set as individual elements or in (sub-)structures (cf. Fig. 13.6). Explanations, which can be assigned to a structural perception and use, are, for example, "there are four and three and that is seven. I know that" or "In the upper row there are three and below two, that is together five."

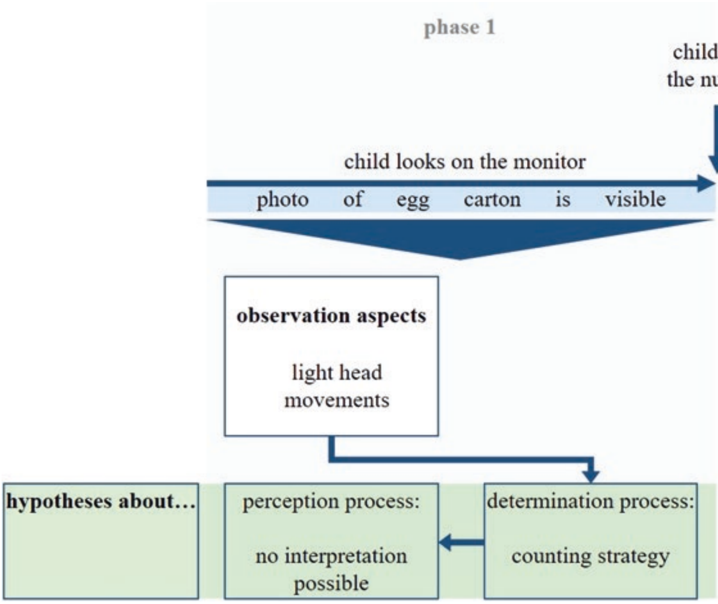

Fig. 13.6 Observation aspects

The final hypotheses about the perception and determination process are gained from all three different types of data (cf. Fig. 13.5). The evaluations are therefore based on a hypothesis-generating method (for more details, cf. Schöner & Benz, 2018). In the following example, it becomes clear that the three-level evaluation process is very complex. The three data types, observation aspects, explanation, and eye-tracking data, are interrelated and complement each other.

13.4.4 Data Analysis by the Use of Eye-Tracking Technology

The eye-tracking data is a collection of the eye movements that the children made during the interview. The analysis of eye-tracking is based on the hypothesis that "eye movements provide a dynamic trace of where a person's attention is being directed in relation to a given visual display" (Jang, Mallipeddi, & Lee, 2014, p. 318). To follow these dynamic traces of the children, there are different ways of visualizing the eye-tracking data: the *GazePlot*, the HeatMap, or the Cluster. Each of these visualization types are dynamic representations of the selected media. In the study presented here, mainly the GazePlot-data was evaluated. It is a helpful tool to evaluate the eye-tracking data, because the order in which the child fixed the single objects is shown by numbers written on the dots. The dots reflect an eye-fixation and the diameter of the dots indicates the duration of each fixation.

The longer the child looks at an object, the larger is the diameter of the dot in the GazePlot visualization. To evaluate the data, the GazePlot-video was interpreted by replaying it several times. In the video, the dots appear one after the other and the course of the child's gaze becomes visible. The data of the GazePlot-video can also be represented in another way, namely as an Accumulate-Graphic. Here, all dots are shown on one picture (cf. Fig. 13.8). In the present study, it is important, not only to look at the eye-tracking data, but also to connect it with the observation aspects and the explanation (cf. Fig. 13.5). With this three-level evaluation process, more reliable assumptions can be made about the perception and the determination of the children.

13.4.5 Example from the Data Analysis

In order to show how the collected data was evaluated, an example from the posttest (T2) with the item "u5,b0" is described in the following. It is an example from the interview with Luca. The egg carton with five eggs in the upper row was presented to Luca. Light head movements could be observed and then he said "five." The interviewer asked how he found out that there were five and he answered that he had counted quietly. Looking at the observation aspects of Luca, light head movements could be observed, which leads to the hypothesis that he probably used a counting strategy to determine the cardinality. No hypothesis about his way of perceiving is possible (cf. Fig. 13.6). At an early stage of the evaluations, the hypothesis "perception as individual elements" was established if a "counting strategy" was observable in the observation aspects (cf. Schöner & Benz, 2018). In the course of analyzing the data of all children who were interviewed, it was decided not to draw any more hypotheses about the perception process in this case, as it has been shown that children of this age very often have counted the number, but still perceived a structure (cf. Example "Liam" in Schöner & Benz, 2017). To sum up, it can be said that the data of the observation aspects of Luca does not automatically lead to the hypothesis that he perceived the set as individual elements.

In the data of explanation (cf. Fig. 13.7), Luca said that he counted the eggs quietly. This statement allows no hypothesis about the perception process, because one cannot conclude from this statement alone whether he perceived the set as individual elements or perhaps in a (sub-)structure. For the determination process, the hypothesis "counting strategy" can be generated because he says that he has counted (cf. Fig. 13.7).

In Fig. 13.9, the eye-tracking data indicate that Luca's gaze oscillates between the left and the right side of the eggs. Then, he fixed every single egg one after the other. These two types of perception (in substructures and as individual elements) become visible in the eye-tracking data if the GazePlot-Graphic is divided into two parts. In the first part of the GazePlot-Graphic (cf. Fig. 13.8, above), it becomes visible how his gaze oscillates back and forth.

In the first part of the graphic, it can be seen that his gaze oscillates between the left and the right side of the presented eggs (cf. Fig. 13.8, above). This is a typical

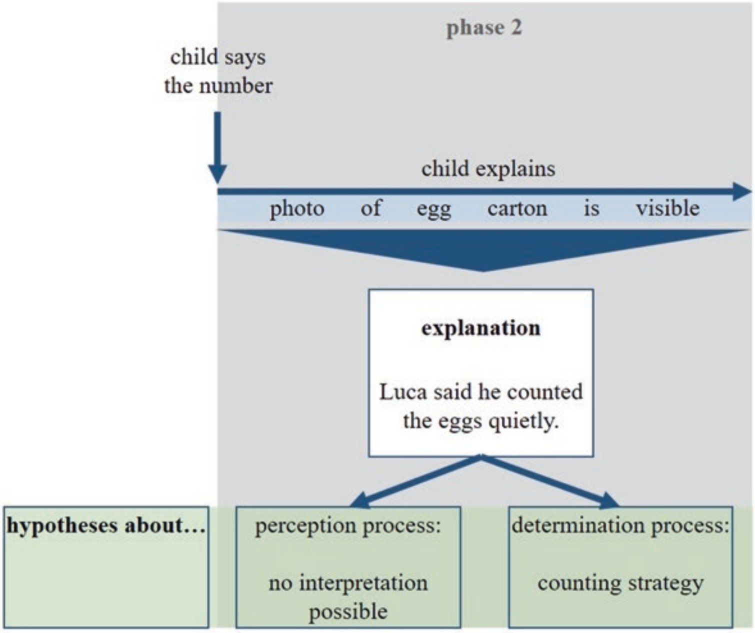

Fig. 13.7 Explanation

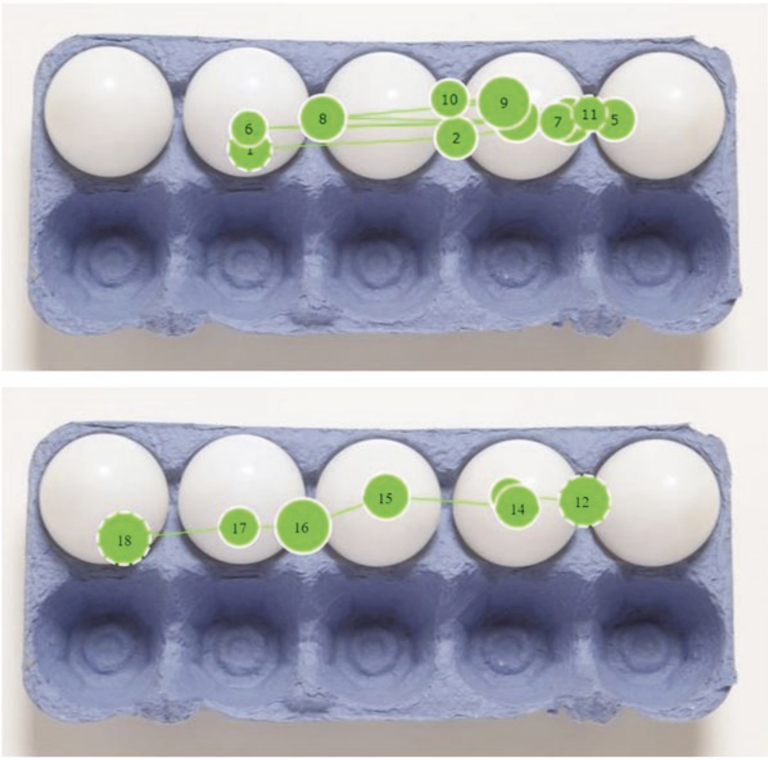

Fig. 13.8 Accumulate GazePlot-Graphic part one (above) and part two (below)

movement when two subsets are perceived (cf. Schöner & Benz, 2018). The data on the first part of the GazePlot-Graphic can lead to the assumption that Luca perceived two eggs on the left and three eggs on the right. This is an interpretation based on the evaluation of all eye-tracking data of the 95 children who participated in the study. These comparative data from all eye-tracking data could only be obtained by analyzing the observation aspects and the explanation in a three-level evaluation process. The second graphic (cf. Fig. 13.8, below) shows that Luca fixed every single egg. This leads to the hypothesis that he counted every single item. Luca started with the egg on the top right and then fixed each egg separately from right to left. The evaluation of all eye-tracking data shows that the first egg the children have counted is often not very clearly fixed. This can also be seen in the example of Luca, because the first fixation dot is not in the middle of the egg, but on the left side. Another phenomenon is that sometimes two gaze-dots are exactly on top of each other. In this case, it is dots two and three (numbered 13 and 14, respectively, in Fig. 13.8, below). This does not mean that Luca counted this egg twice, because not every fixation dot corresponds to a counting step (Fig. 13.9).

Presumably, Luca cannot yet use the perceived structure to determine the cardinality and uses his familiar counting strategy "counting all."

The final hypotheses on the two processes are that Luca perceived the set in structures and used the counting strategy "counting all" to determine the cardinality. Hypotheses on the perception process could not be made, either on the basis of the

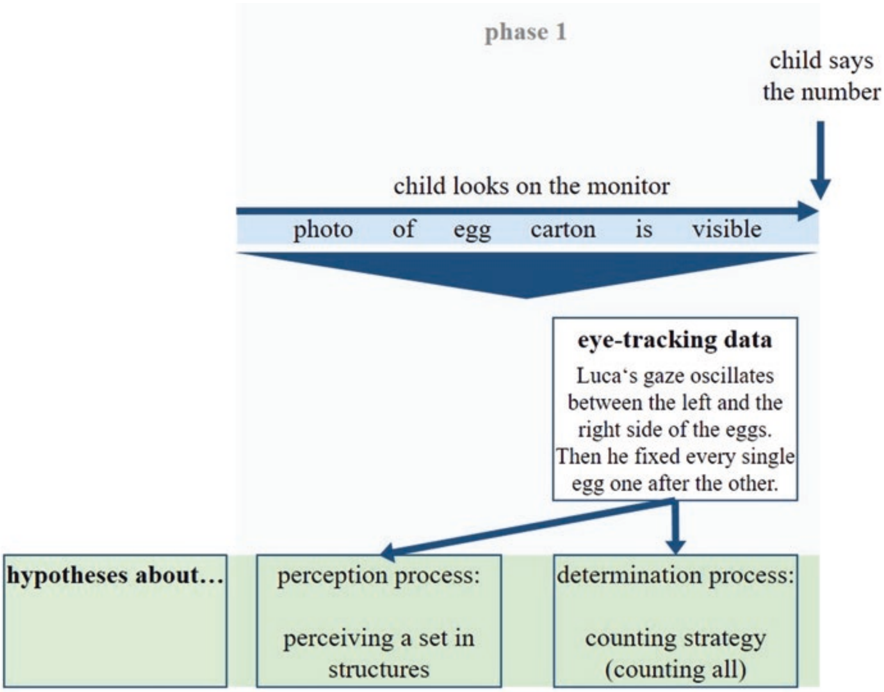

Fig. 13.9 Eye-tracking data

data from the observation aspects or on the basis of the data from the explanation. It can therefore be stated that without the eye-tracking data, no hypothesis on the perception process would have been possible.

13.5 Results of the Learning Story of "Luca"

The observation of the educators refers to the mathematical situations that were guided. Referring to the observations of the kindergarten teachers, Luca was difficult to motivate for the mathematical activities. He often quickly lost interest. The kindergarten teachers responded to this fact, for example, by forming smaller groups in order to support him and in order to take his interests into account. This worked successfully – according to the educator's statements – and he could later be reintegrated into the whole group. In November 2015, the kindergarten teacher described Luca's difficulties imagining the arrangement and finally the cardinality of the sets of eggs described by another child in the game "I spy with my little eye." Even an open, empty egg carton to see the ten-structure did not help him at this stage. Four weeks later, in December 2015, he still had difficulty (with the same activity) in giving a verbal explanation, but most of the time he could name the number immediately, if somebody else described it. By comparing the results of the three interviews (T1, T2, T3) with this information, it can be concluded that the analysis of the data supports the kindergarten teacher's observations regarding perceiving and using structures to determine the cardinality (cf. Fig. 13.10).

Figure 13.10 shows the final hypotheses of the three individual interviews of the learning story of Luca. Only the quantities with a cardinality ≥5 are considered. On the left part of the bars, Luca's way of perceiving a set is presented (gray: no interpretation possible; dark yellow: perceiving a set in (sub-)structures and, on the right part of the bars, his way of determining the cardinality is shown (violet: counting strategy: counting all; light yellow: structural use). The results are divided into the three parts: T1 (pretest), T2 (posttest), and T3 (follow-up).

In the pretest (T1), Luca counted every single egg aloud for each item to determine the cardinality and also pointed, with his finger, to the corresponding egg. No interpretation about his way of perceiving the set was possible because, on the one hand, his finger gesture interrupted the connection to the eye-tracking camera and, on the other hand, there was no additional observation, like an explanation, which could lead to a hypothesis about perception. The fact that he counted every single egg does not automatically lead to the hypothesis that he did not perceive a structure. The observations in the posttest (T2) lead to the hypothesis that Luca always perceived a set in structures but he could not use the structures to determine the cardinality. In the example "u5,b0" (T2), described in detail above, the observations obtained using the eye-tracker indicated that Luca had perceived a structure (cf. Fig. 13.8). For the remaining items, Luca verbally named the structure, and the existing eye-tracking data confirmed this observation. To answer the question on how many eggs there were, he again consequently used the counting strategy

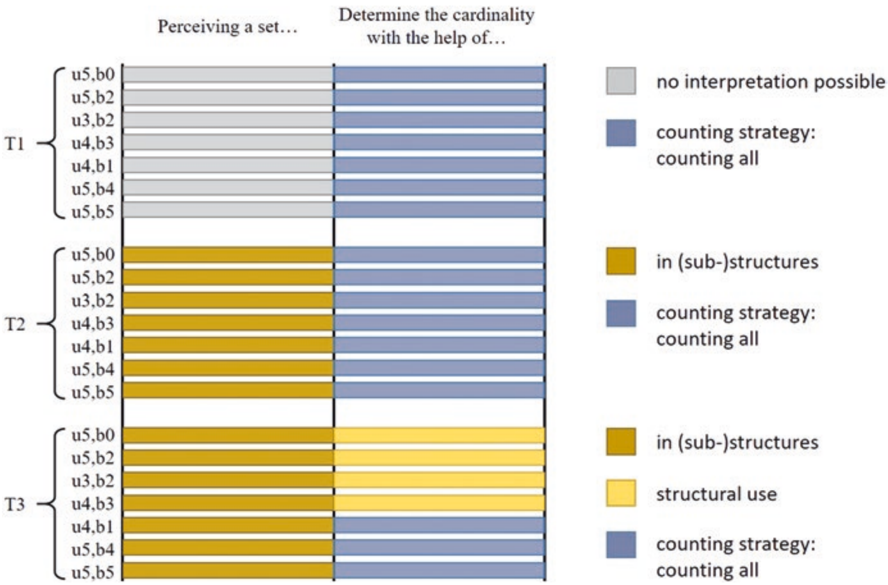

Fig. 13.10 Results of the learning story of Luca

"counting all." In the follow-up interviews (T3), the eye-tracking data for all seven items led to the hypothesis that Luca again perceived a structure. In four of the shown items (u5,b0; u5,b2; u3,b2; u4,b3), Luca was now able to use the perceived structure for determining the cardinality (cf. Fig. 13.10). Only in three remaining items (u4,b1; u5,b4; u5,b5), he continued to use the counting strategy "counting all" to determine the cardinality.

13.6 Summary and Discussion

The research question in this chapter was "how does the perception of structures and the use of structures to determine the cardinality of a set change based on an implementation?" Based on the learning story of Luca, a development can be seen. In the first interview (T1), he counts all the eggs individually and loudly, and points with his finger to each individual egg. No statement can be made about the perception process. After the implementation phase (T2), we can observe that he is able to perceive a structure in the presented sets, but still counts each egg individually to determine the cardinality. By structuring (decomposing) the presented set, the basis for a part–whole understanding is initiated (cf. Baroody et al., 2006, p. 193). At the end of the last kindergarten year (T3), Luca was able to use the perceived structure partially to determine the cardinality of the sets. It cannot be safely assumed from this single learning story that these results can also be transferred to other children and other learning environments. But still, the assumptions as stated above can be made.

The presented material (egg cartons) is used for the implementation phase as well as for the interviews with the children. For this reason, the study could be accused of "teaching to the test." It must be taken into account that also other materials were used for the implementation. There are activities with structured materials like "finding pairs" with the egg cartons where the existing structure is in focus, as well as unstructured materials such as glass nuggets, where the children can structure the set themselves. It is also very important for children, especially in regard to primary school, to learn reliable structures that they can use for calculating strategies (Lüken, 2012). This could be enhanced in primary school by using certain materials, for example, the ten-frame, a typical didactical presentation used in primary school, which has a ten-structure (like the egg cartons), and extends the number space to 100. Therefore, perceiving structures and using them for noncounting strategies are valuable skills which can serve as a basis for the development of a part–whole-understanding, and later for the development of calculating strategies. It is shown that children at this age are able to develop a perception of (sub-)structure and a structural use of determination strategies. The study is based on a hypothesis-generating procedure. In the future, some hypotheses will be tested in a statistical examination and significances will be calculated.

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