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Mathematics Education in the Early Years

Results from the POEM4 Conference, 2018

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Introduction

The fourth POEM Conference was held on 29–30 May 2018 in Kristiansand, Norway, at the University of Agder, following the conference held in Frankfurt am Main, Germany, in 2012 (Kortenkamp et al., 2014); Malmö, Sweden, in 2014 (Meaney et al., 2016); and Karlsruhe, Germany, in 2016 (Benz et al., 2018). POEM is a conference comprising invited participants, and it is a conference for researchers working in early childhood mathematics education. In 2018, 9 years after the launching of the early years mathematics thematic working group at CERME 6 (The Sixth Congress of the European Society for Research in Mathematics Education in Lyon, France), we are still faced with the critical question: In which way—and how much—should children be "educated" in mathematics before entering primary school (Levenson, Bartolini Bussi & Erfjord, 2018)? In a working atmosphere, focusing on interaction and exchange, participating researchers addressed this question in order to start and maintain a research network in early mathematics learning.

Over the 2 days of the conference, three plenary sessions were provided by Professor Maria Giuseppina Bartolini Bussi from the University of Modena and Reggio Emilia, Italy; Professor Lieven Verschaffel from the Catholic University of Leuven, Belgium; and Professor Luis Radford from Laurentian University, Canada. These plenaries engaged the listeners in three different directions.

Bartolini Bussi did her talk on the theme "Early Years Mathematics: Semiotic and Cultural Mediation", in which she reflected on in which ways the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008) has had an impact on national standards in Italian pre-school mathematics. She exploited the compatibility of semiotic mediation with Bishop's idea of mathematical enculturation (Bishop, 1988) through his six fundamental mathematical activities: counting, locating, measuring, designing, playing, and explaining. She also elaborated on how the theory of semiotic mediation has been drawn upon, over the years, to interpret and design mathematical activities also for pre-school children. Furthermore, she reported and analysed several original examples of mathematical enculturation involving children from 0 to 6 years old.

Verschaffel gave a talk on “Young Children’s Early Mathematical Competencies: The Role of Spontaneous Focusing Tendencies”, in which he reflected on spontaneous focusing tendencies amongst young children and their role in early mathematical development. He drew on neurocognitive research results on children’s early mathematical abilities, but he also launched a new line of research focusing more on dispositional aspects of children’s early mathematical competencies. The most studied dispositional aspect so far is children’s spontaneous focus on numerosity (SFON), but Verschaffel presented research results, including theoretical insights, diagnostic tools, and educational recommendations, with respect to other aspects as well: numerical symbols (SFONS), quantitative relations (SFOR), and mathematical patterns and structures (SFOPS). His main point was that these aspects are not addressing what children may think and do when taught mathematics but rather what young children spontaneously focus on without any teaching.

The theme of Radford’s talk was “Play and the Production of Subjectivities in Kindergarten”, in which he elaborated on his theory of knowledge objectification to also encompass a process of subjectification. He reflected on this theme by drawing on a situation in which children were playing mathematical games in a kindergarten context. Radford analysed these situations with a particular interest in mind. He took the audience through his ideas aiming for understanding in what ways mathematical ideas emerging through play allow children to inscribe themselves in the social world in which they participate and are members, i.e. a process of subjectification. Thus, in this way, he recast a more usual research approach within early years mathematics education research, namely, how play allows children to develop mathematical ideas.

All three plenary speakers have written a chapter based on their talks. These chapters are placed first in this book as Chaps. 1–3. These are followed by selected revised chapters submitted for the conference. The review process of the chapters comprised three rounds. Initially, each author submitted his/her chapter to the conference. These versions of the chapters were presented and discussed at the conference. Based on these discussions, questions, and comments from the audience, the authors submitted a full revised chapter to the editors. The editors then launched the first round of review. In the first round, each chapter was sent to two reviewers. The reviewers were both participants at the conference and international experts in the field of early years mathematics education research. The reviews were collected by the editors and submitted to the authors. The authors then revised and resubmitted their chapters accompanied by a letter in which all changes and modifications were described with respect to the reviewers’ comments. The second round of review comprised the editor’s careful and thorough reading of the revised chapters, making sure that the reviewers’ comments and prompts were dealt with as well as raising new concerns, questions, and ideas for revision. These reviews were sent to the authors, urging them to consider the criticisms made. The authors then submitted their third version of their chapters to language control before proofreading was done. The final versions of the chapters resulting from this profound review process are included in this book.

Over the 2 days of the conference, five parallel sessions also took place. There were three parallels in each session, and every session included the presentation and discussion of two chapters, making a total of 30 chapters being presented and discussed in total. A selection of these chapters is thematically divided and commented upon below. Based on the research areas in focus of these chapters, we have categorised them into three themes: (1) children's mathematical reasoning, (2) early years mathematics teaching, and (3) parents' role in children's mathematical development.

Children's Mathematical Reasoning

This section comprises 15 chapters, Chaps. 4–18, whose object of study is children's mathematical reasoning with respect to various mathematical concepts. The chapters encompass a vast variety of theoretical lenses used to study early childhood mathematics education.

Tsamir, Tirosh, Barkai, and Levenson study how pre-school teachers implement patterning tasks in their teaching and in what ways the children engage in copying repeating patterns and comparing the patterns. The teachers in their study introduced a strand of beads representing an AB-structured pattern, and they asked the children to make a similar strand of beads. However, the children had to use different colours than the original one. Then the children were asked to compare their own pattern with the model pattern. A similar task was carried out with an AAB-structured pattern. Tsamir et al. found that the children were using two different strategies when copying the AB-structured pattern and a third strategy when copying the AAB-structured pattern. The authors hypothesise that this third strategy might signal an intermediate position in children's development of structure recognition. These researchers also found, interestingly, that more children recognised the AAB-structured pattern than the AB-structured pattern.

Lüken also presents a study of patterning; in her case, the focus is on children's strategies when working with repeating patterns. Her concern is on children's processes rather than the product of such pattern activities. She conducted interviews with 159 children aged 3–5 years while they were working with patterning tasks, and she coded their strategies in five strategy categories. Her results show a wide variety of strategies being used and with a tendency that the older children more often used strategies considered as more advanced. A main insight is that most young children are not using the unit of repeat in patterning activities.

In their chapter, Björklund and Reis report from a study of how pre-schoolers use fingers in numerical reasoning when working on arithmetic problems. Variation theory of learning is used as a theoretical lens in analysis of 133 observations of 4–5-year-olds work with arithmetic problems and particularly how they use their fingers when working with the problems. Three main different kinds of finger use were found, where two of the three kinds of finger use were based on expressions of number knowledge that did not contribute to solving the problems. Debating on whether finger use should be advocated or not in early mathematics, Björklund and Reis take the stance that it is how fingers are used rather than its use per se that must be given concern.

Bjørnebye and Sigurjonsson take a study of 3–4-year-old children in outdoor navigation tasks in kindergarten as its empirical basis. According to the authors, these children initially did not master cardinality for exact enumeration but had previously engaged in an experimenter-led articulated “kangaroo-two” feet activity with physical tagging of 1–4 dotted arrays. The reported study in the chapter concerns a new experimenter-led intervention where the children individually navigated across a 50 dots circle. The conceptual metaphor theory is used as a theoretical lens to analyse the children and the experimenter actions and verbal and non-verbal utterances while engaging with the navigation task. Their findings indicate that the children in their verbal and physical communication were able to use number metaphors. Bjørnebye and Sigurjonsson mention “rooster-one”, “kangaroo-two”, “monkey-three”, and “frog-four” as examples of such metaphors. They argue that the spatial structured language design as evident in their tasks could develop children’s enumeration skills efficiently exceeding a basic cardinality insight.

Similarly, as Bjørnebye and Sigurjonsson, Lossius and Lundhaug also take outdoor activity in kindergarten as their departure. However, a clear difference in the studies is that Lossius and Lundhaug do not use adult- or experimenter-led activities as their focus but rather look at a children-initiated spontaneous outdoor activity in a kindergarten. By adopting theories of instrumental and pedagogical situations in kindergartens, they characterise the problem-solving situations in the activity and how mathematics is involved in the activity. Despite that the children only implicitly themselves used mathematics to solve a practical task, the kindergarten offered support of their mathematical ideas and is argued as a pedagogical purpose in the kindergarten teachers’ contributions. Based on this insight, Lossius and Lundhaug argue that awareness of the features of mathematical problem-solving could support kindergarten teachers to be able to support and develop mathematical problem-solving in the outdoor environment.

De Simone and Sabena discuss how strategy games may be exploited to develop mathematical reasoning and argumentation competencies in kindergarten. Five-year-old children were involved in a teaching experiment based on the game “13 buttons”. The game is played by two players, which alternate and play one against the other, starting from the initial situation of 13 buttons (or other tokens) displayed on a line. Each player, in his turn, can take one, two, or three buttons. It is not possible to skip the turn. The one who takes the last button loses the game. They analysed their data by adopting two theoretical frameworks, the game theory and the structure of attention frame. Their results indicate that children in game activity settings experience different aspects of mathematical thinking, such as making choices and checking their consequences, identifying regularities and relationships, producing conjectures, and explaining them. The results also point out the key role of the teacher in prompting children to develop these processes, and in particular, three different kinds of successful prompts are suggested.

In the study of Mellone, Baccaglini-Frank, and Martignone, insights are provided with respect to how 5–6-year-old children engage with quantities of rice. According to Mellone et al., rice is a mathematically interesting substance since it can be viewed both as a continuous and a discrete substance. The children were asked to compare quantities of rice and whether there was "as much [rice] as" in different piles of rice, having the opportunity to take advantage of various provided artefacts. From a semiotic mediation perspective, the authors analyse the children's strategies in solving the task as well as the developed situated signs of the children. From this, Mellone et al. identify those signs that teachers later may use as pivot signs in their teaching aiming for mathematical meanings of measurement.

The aim of the study of Severina and Meaney is to investigate the semiotic resources children utilise when asked to come up with mathematical explanations within hypothetical situations. Severina and Meaney analyse video data from a situation in which 4–5-year-olds and a kindergarten student teacher interact in discussions of both real and imaginary page layouts of a photo book. The authors analyse the children's reasoning as they argue about the number of photographs in the various layouts. The children's reasoning is characterised by the use of oral language, gestures, and physical objects. According to Severina and Meaney, these results indicate that younger children, than previously suggested, can use mathematical ideas in their provided explanations of hypothetical situations – explanations afforded by the use of semiotic resources.

As Mellone et al. and Severina et al., semiotic perspectives are also at stake in the chapter written by Di Paola, Montone, and Ricciardiello. They report from a case study involving kindergarten children aged 5–6, who were asked to build a Lego block and then to discuss drawings of such blocks observed from different points of view. These authors emphasise the didactic potentiality of the use of an artefact, useful to construct mathematical meanings concerning the coordination of different points of view. Di Paola et al. hypothesise that the alternation between different semiotic systems, i.e. graphical system, verbal system, and system of gestures, is important in the children's mathematical learning. The theory of semiotic mediation is suitable to design the teaching sequence and to analyse the collected data. This chapter reports some preliminary results concerning the validity of the hypothesis about the potentiality of using the combination of artefacts as tools.

Sprenger and Benz study children's development of number and arithmetic concepts by emphasising the important role of structures in these developments. In supporting a child's perception and use of structures, Sprenger and Benz analyse this child's ways of determining the cardinality of a set of objects. Drawing on a pre- and post-design and follow-up design study, in which interventions focused on playing games emphasising structures were carried out, these authors argue that the child changed the way he perceived structures in sets. Moreover, the child took advantage of the perceived structures to determine the cardinality of various sets.

Maier and Benz' chapter present a study of children's conception of geometrical shapes. In order to investigate the geometrical concept formation of 4–6-year-old children in two different learning environments, around 80 English and German children were given several tasks concerning the conception of geometrical shapes.

Different aspects of children's concepts are illustrated, as, for example, how children explain the shape of a triangle; how they perceive different kinds of triangles; what kinds of examples they choose as circles, triangles, and squares; and how their explanations and their choices of shapes, as well as their drawings, go in line with each other. Their results indicate that the children often cannot apply a comprehensive definition of a shape to different representations of that respective shape. Thus, it can be concluded that a definition that is learned by heart, without an understanding of what this definition means, does not contribute to a comprehensive concept formation.

Palmér and Björklund present a study where the emphasis is on mathematics teaching framed with narratives. The fairy tale "Goldilocks" is used in two examples, one with children aged 1–3 and one example with children aged 4–5. The empirical examples are used to illustrate the complexity and to reinforce the focus of the chapter, a content analysis of the Goldilocks story. These authors investigate what mathematical concepts can be explored and framed within the story and what challenges the story imposes for learning mathematical concepts. The result of the analysis shows that the mathematical content in the story entails a complexity that may hinder the emergence of mathematical learning objects if the story is played out true to its original form. Palmér and Björklund conclude that unless a pre-school teacher has an advanced understanding of the mathematical concepts in question, the story makes it difficult to frame children's concept exploration in profitable ways.

Policastro, Almeida, Ribeiro, and Jakobsen aim to achieve a more profound understanding of connections between 5-year-old kindergarten children's and kindergarten teachers' insights into measurement. Through a lens of one teacher's specialised knowledge in mathematics, these researchers investigate the subtleties of this teacher's ways of fostering mathematical discussions. These discussions are launched to empower the involved children's mathematical learning processes with respect to measurement. Policastro et al. analyse one taught video-recorded lesson on measurement tasks. Their results show, particularly in contingency moments, how the teacher's specialised mathematical knowledge informs decisions during the teaching to sustain relationships between the children's mathematical contributions and the teacher's mathematical insights.

Breive's concern is on the coordination of turn-taking in small groups of pre-school children aged 5–6 working with addition problems. Breive uses a multimodal interpretative perspective as the basis for her analysis and draws on the theory of knowledge objectification as outlined by Radford. Her findings pinpoint that the way children organised their turn-taking resulted in different kinds of materialisation of their mathematical thinking and how multiplicative structure emerged. Breive proposes that children's early multiplicative thinking could be fostered by small group settings and by using various equal group addition problems with hands and fingers.

Tzekaki's theoretical study aims to provide insights into characteristics of early childhood mathematical activity. Tzekaki takes as a point of departure that preschoolers, in order for them to develop mathematical ideas, generally are engaged in games, various tasks, and situations comprising mathematical objects and content such as measuring, counting, and recognising of different geometrical shapes and patterns. Nevertheless, Tzekaki argues that whether these children are considered to act and think mathematically and learn mathematical concepts depends on what we as researchers define mathematical thinking and acting to be. This researcher thus seeks to characterise what she calls genuine mathematical activity in early childhood, culminating in arguing generalisation as the essential component of mathematical activity. Examples from mathematics teaching practices in early childhood are provided which demonstrate the importance of generalisation.

Early Years Mathematics Teaching

This section comprises five chapters, Chaps. 19–23, whose object of study is mathematics teaching in early childhood education. The chapters emphasise both the teaching of teachers and early childhood education students' teaching of mathematics.

Bruns, Carlsen, Eichen, Erfjord, and Hundeland report from a cross-country study of early childhood education students' situational perception of the mathematics involved in learning situations with children in a kindergarten environment. The respondents were students in their educational training to become professional kindergarten teachers. Bruns et al. conducted a study with ten students, five from Austria and five from Norway. The students watched seven video vignettes prepared by the researchers and individually wrote down their observations and reflections after looking at each of the seven vignettes. Each vignette was a video recording of a situation in a kindergarten, and no sound was offered. Through the use of qualitative content analysis of the students' written responses, three categories of responses were found: process, learning, and mathematical concepts. Their findings showed clear differences between Austrian and Norwegian students' responses when it came to the category mathematical concepts where the density of these responses was much higher for the Norwegian students. The main difference between Austria and Norway is the students' background in early years mathematics, where the Norwegian students had met mathematics as part of their educational training to become kindergarten teachers. Based on their findings, Bruns et al. conclude that the kindergarten teachers' situational perception of mathematical concepts is less influenced of general aspects of education and stronger influenced by their knowledge background in mathematics.

Flottorp's chapter does, as Bruns et al., focus on early childhood education students. The attention in this study is on how the students reflect on ways of supporting children in spontaneous mathematical situations. The data is collected from students' reports from kindergarten and interviews with them afterwards. The framework of the knowledge quartet is used in the analysis. Flottorp investigates the challenges early childhood education students face in spontaneously supporting children in measuring activities and how these students reflect on being active versus passive. The study reveals that not only contingency is involved in spontaneous situations. Many aspects of the knowledge quartet are necessary for being able to act in the moment.

Similarly to Bruns et al., Fosse, Lange, and Meaney study kindergarten teachers' responses to photos of children working with mathematics. Fosse et al., in focus group interviews, were particularly interested in whether problem-posing and problem-solving were explicitly discussed since previous research has found this not to be evident. By utilising Bishop's universal mathematical activities of explaining and playing in their analysis of kindergarten teachers' responses, they found four components in problem-posing and problem-solving being paid attention to. These were the routine or nonroutine nature of the problems, known or unknown problem-solving strategies, body actions or verbal explanations, and playing by exploring different scenarios or following rules. Fosse et al. state that their identification of kindergarten teachers' responses, capturing their discussion of problem-posing and problem-solving, contribute as a way to characterise kindergarten teachers' professional knowledge.

Measurement is the mathematical scope in the chapter by Keuch and Brandt. Their concern is on how kindergarten teachers in small group oral interaction with children support learning of measuring length and mass (weight). They adopt interactional linguistics as a theoretical basis to analyse the use and meaning of adjectives in the interaction between the kindergarten teachers and the children. They found 14 different occasions where measuring was in focus in such small group interactions, nine on length and five on weight. Their findings are twofold. Keuch and Brandt argue that interaction focused on measuring gives rich possibilities to develop an understanding of adjectives. Despite this, they only found a few examples that the kindergarten teachers exploited the learning opportunities, and they argue that the kindergarten teachers do not seem to possess sufficient language awareness to exploit these opportunities for learning.

In the study of Sæbbe and Mosvold, the research focus is at the complexity of the work of teaching mathematics in kindergarten. These authors analyse instances of mathematics teaching in Norwegian kindergartens. As a result, Sæbbe and Mosvold not only propose mathematical tasks that characterise the work of teaching mathematics in kindergarten but also anticipate that the work of teaching mathematics in kindergarten is constituted by these and similar tasks. This testifies to the complex nature of mathematics teaching in kindergartens situated within a social pedagogy tradition, in which free play and learning from everyday situations are highly valued.

Parents' Role in Children's Mathematical Development

This section comprises the final two chapters of the book, Chaps. 24 and 25, whose object of study is children's mathematical development within the home environment.

In Lembrér's chapter, parents, as well as other family members, are recognised as young children's first educators who contribute to their learning of mathematics knowledge and skills. The focus of this study is on how parents describe their children's engagement with mathematics at home. She is interested in what parents value in the mathematics activities that their children engage in at home. Data were collected from nine Norwegian parents through photo-elicited focus group interviews, where the parents' own photos were used as stimuli for the interviews. The analysis was done using a narrative approach to identify how parents saw their children authoring, sense-making, collaborating, and using non-verbal communication which gave insights into the values they held. Results indicate that the parents valued their children learning numbers, counting skills, early measuring concepts, and use of money.

As Lembrér, Anderson and Anderson also focus on the home context. They report on two "pedagogical" at-home, play-based activities for evidence of two mothers' capacity to establish and sustain mathematics as a goal while addressing each child's role throughout the event. Their attention is on ways in which parent-child activity may unfold in families and underscore a need to learn more from parents about the pedagogical practices that make sense to them. Periodically over the course of 2.5 years, six pre-schoolers were videotaped as they participated with family members in at-home activities, such as baking cookies or reading a storybook. The chapter offers a detailed account of two cases where the mothers' and children's interaction during two pedagogical activities are analysed.

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About the Editors

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Chapter 1
Early Years Mathematics: Semiotic and Cultural Mediation

Maria G. Bartolini Bussi

1.1 Introduction

1.1.1 What Does Early Years Mathematics Mean?

According to the POEM literature (Benz et al., 2018; Kortenkamp et al., 2014; Meaney, Helenius, Johansson, Lange, & Wernberg, 2016), early years means children aged 2–8. Yet, in the international literature, mathematical concepts (whatever they mean) are addressed also for younger children. For instance, a famous paper by Karen Wynn (1992) addressed *addition and subtraction by human infants* with ingenious experimental settings:

> A group of 32 infants were divided randomly into two equal groups. Those in the "1+1" group were shown a single item in an empty display area. A small screen then rotated up hiding the item from view, and the experimenter brought a second identical item into the display area, in clear view of the infant. The experimenter placed the second item out of the infant's sight behind the screen. Thus, infants could clearly see the nature of the arithmetical operation being performed, but could not see the result of the operation. The "2-1" group was similarly shown a sequence of events depicting a subtraction of one item from two items. For both groups of infants, after the above sequence of events was concluded, the screen was rotated to reveal either one or two items in the display case. Infants' looking time at the display was then recorded […]

For "addition," the trials alternated between a two-item (possible) outcome and an one-item (impossible) outcome. The same was done for "subtraction," alternating between a two-item (impossible) outcome and an one-item (possible) outcome. In both cases infants looked longer to the impossible outcomes. Wynn's conclusion follows:

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> Here I show that 5-month-old infants can calculate the results of simple arithmetical operations on small numbers of items. This indicates that infants possess true numerical concepts, and suggests that humans are innately endowed with arithmetical abilities.

The words used by Wynn are taken from mathematics (e.g., calculate, operation, addition, subtraction, results, and numerical concepts). Doridot and Panza (2004), in contrast, claim that it is very important to distinguish between innate perceptual capacities (similar to the ones recorded by Wynn) and the constitution of arithmetic as a system. They put their comment in a paper addressing a much larger issue, i.e., the possible contribution of cognitive sciences to the philosophy of mathematics. For them, mathematics is supposed to be what Barton (2008) calls NUC Mathematics (i.e., Near-Universal Conventional Mathematics), that is,

> the smaller, formal, conventional world of academic mathematics as it is exemplified in schools and universities all over the world. (p. 10)

Barton contrasts NUC Mathematics with a QRS system, that is,

> any system that helps us deal with quantity or measurement, or the relationships between things, or space, shapes or patterns. (p. 10)

When we speak of early years mathematics, we hint, in most cases, at a QRS system rather than to NUC Mathematics. In this perspective, infants' perceptual sensitivity to possible and impossible outcomes recorded by Wynn (1992) may be considered the germs of a QRS system. This interpretation supported my personal trajectory in this research field.

1.1.2 My Personal Trajectory

In the 1980s, I started to collaborate with the municipal preschools in Modena, my home city. I worked with a famous Italian educator (Sergio Neri), who was later appointed in the National Committee for the first definition of national programs for preschools (Orientamenti, 1991[1]). The Orientamenti were divided into six fields of experience, one of which was mathematics (named *space, ordering, and measuring*). I was asked by Neri to outline this field. I did it following Bishop's (1991, 2016) idea of *mathematical enculturation*, with the six "universal" activities: *counting, locating, measuring, designing, playing, and explaining.* The Orientamenti are still the guidelines for school design, although summarized versions were later issued as standards (Indicazioni, 2012[2]). Standards did not repeal the Orientamenti but coordinated a short outline of them with the standards for grades 1–8 aiming at longitudinal continuity.

In 2012, I was appointed as a leader of the WG13 on Early Years Mathematics for CERME 8 (Antalya) and CERME 9 (Praha). In that context, I started a very

[1] http://www.edscuola.it/archivio/norme/decreti/dm3691.html

[2] http://www.indicazioninazionali.it/J/

fruitful collaboration with Ingvald Erfjord and Ester Levenson, who cochaired the group with me (Levenson, Bartolini Bussi, & Erfjord, 2018). In 2013, the pedagogical coordination of Modena launched a new action research program[3] with the extension of the original program to toddler centers too (the so-called ZEROSIX program hinting at kids' age). The program was organized in 3-year cycles, with about 30 + 30 participants (30 toddler center educators and 30 preprimary school teachers[4]).

Between the years 2012 and 2018, I cochaired with Xu Hua Sun, the first ICMI Study on early mathematics (Bartolini Bussi & Sun, 2018), who addressed, for the first time, the issue of arithmetic in primary school, with some possible extension to preprimary too.

1.2 Some Outcomes of the Preschool Program

The preschool research was realized by the collective work of an action research group comprising a mathematical advisor (the author of this chapter), a small number of pedagogical coordinators, and some dozens of preschool teachers. Some documentation is reported in a multimedia.[5]

1.2.1 The Giant Slavonic Abacus

The quoted multimedia has a whole section on the giant slavonic abacus (Bartolini Bussi, 2013). In 2008, all the municipal preschools were given a giant Slavonic abacus (Fig. 1.1). Teachers themselves had designed it with 40 beads because this number meets the most common needs of school activity (e.g., counting children in the roll and counting the days per month in the calendar). The large size fosters large body gestures (even steps for younger children) to move the beads. Intentionally, schools received a dismantled abacus, as most teachers agreed that the very assembling could have been an important part of the exploration of the artifact.

[3] The complete action-research program addresses several school subjects: number and space (*mathematics*); knowledge of the world (*science*); body and motion (*gym*); from gesture to sign (*drawing*); talking with children; oral narration; philosophy with children. The last one is for preprimary teachers only. Every educator/teacher is enrolled for 3 years in one of the above programs. The yearly structure of the program is roughly outlined by Bartolini Bussi (2013).

[4] In Italy the term *educatore/educatrice* (educator) is used in the toddler centers, while the term *insegnante* (teacher) is used in the preschools. There are different ways and rules for preservice education and for recruitment, although recently the coordination of the so-called zerosix services has been issued (http://www.gazzettaufficiale.it/eli/id/2017/05/16/17G00073/sg). In this chapter I shall use the two different terms accordingly.

[5] http://memoesperienze.comune.modena.it/bambini/index.htm

Fig. 1.1 The giant slavonic abacus: exploration and representation

Some mathematical meanings were analyzed by the research group:

- Partition, to separate counted beads from beads to be counted;
- One-to-one correspondence, between beads and numerals;
- Cardinality, given by the last pronounced numeral;
- Sequence of early numerals, to be practiced in counting.
- Place value (early approach), as beads are divided in tens.

A system of suitable tasks was collaboratively produced by the research group, with the aim of fostering children's productions of different voices: the presence of different voices allowed the teacher to orchestrate *a polyphony of voices,*[6] which, according to the theoretical framework of semiotic mediation, nurtures the individual construction of mathematical meanings during collective interaction. This way, the same artifact is looked at from different perspective. Some tasks tested with 4- and-5-year-old children follow.

[6] The term *voice* is used after Bakhtin (1981). See also Bartolini Bussi and Mariotti (2008).

1.2.1.1 Task 1: The First Impact (The Narrator's Voice)

Tasks are different if the abacus is assembled (A) or if it is dismantled (B): (A) What is it? Have you seen it before? What is its name? (B) What have I carried today? What do you see? Do you know objects with many beads? Such tasks, used in either small or large group discussions, aim at evoking earlier experiences and involving children. At the end, the name may be introduced (in Italian the word *pallottoliere*—has the same root as pallina—bead). This task fosters the emergence of the *narrator voice*.

1.2.1.2 Task 2: The Structure of the Artifact (The Constructor's Voice)

How is it made? What do we need to build another one? How to give instruction to build another one? Such tasks, used in either small or large group discussion, aims at identifying the components and naming them in a correct way, and describing the spatial relationships between them. They foster the emergence of the *constructor voice*. After a discussion, individual drawing tasks are given: draw our abacus. The previous verbal analysis of the structure of the artifact fosters the production of very detailed drawings, with, for instance, the right number of beads and the realistic representation of legs and other parts (Fig. 1.1, right).

1.2.1.3 Task 3: The Use of the Artifact (The User's Voice)

The task is functional to the context where the artifact is used. For instance, it may be used to keep the score in skittles or to count the present children during the call. It may be given in small or large groups. How do you use it to keep the score? How do you use it during the call? This task fosters the emergence of the *user voice*.

1.2.1.4 Task 4: The Justification for Use (The Theoretician's Voice)

In this case too, the task is connected with the context. Children are asked to explain "why does it work to keep the score?" and similar. This is a very difficult task that fosters the emergence of the *theoretician voice*, to explain what mathematical meaning or process makes us sure that the function is effective. This task may be given indirectly, showing a puppet that makes mistakes and encouraging children to comment and to correct it, if they do not agree, explain why.

1.2.1.5 Task 5: New Problems (The Problem Poser and Solver's Voice)

These last tasks were not designed in advance but emerged together with creative solutions in classroom activities. For instance, in a classroom, children proposed to use the Slavonic abacus to plan the preparation of tables for lunch. They suggested registering, on the abacus, the number of children for each table on a different line. When they realized that the tables to be set were 5, they explained that there were not enough lines and decided to create a new line on the floor, lining up 10 small cubes and moving them accordingly. Such self-posed tasks may foster the emergence of the *problem poser and solver voice*.

The system of tasks for the giant Slavonic abacus may be usefully referred to in literature on mathematics education. The emergence of the narrator voice in the first task is related to *devolution* (Brousseau, 1986), as it fosters the personal involvement of children in the tasks. The emergence of the constructor voice in task 2 is related to Rabardel's *instrumentation* (1995), as it concerns the component of the artifact as an object. The emergence of the user voice in task 3 is related to Rabardel *instrumentalization* (1995), as it fosters the emergence of individual utilization schemes. The emergence of the theoretician voice in task 4 is related to *mathematical meanings*, hence, it is consistent with the Vygotskian approach to school subjects through semiotic mediation (Bartolini Bussi & Mariotti, 2008). The emergence of problem poser and solver voice in task 5 shows that, in spite of the teacher's guidance, children's creative ideas have space to be developed. However, those tasks were not exercises of application of that literature, but rather experiments designed, realized, interpreted, analyzed, and generalized, in collaboration between teacher educators, teachers, and pedagogists in a dialogic way, exploiting the literature.

1.2.2 The Time Tube

Later, other activities were designed by the action research group to complement the activities of counting practiced in the schools. One of them aimed at introducing the activity of measuring and estimating, which concerns approximate processes. The activity spanned over a long period (several weeks or even months), hence contrasting the focus on only short-term processes. To construct a context suitable and motivating for young children too, a special artifact[7] was designed. The activity was carried out with 4-and-5-year-old children in more than 20 schools. The children were already familiar with the number line in the form of a monthly calendar. A day-by-day tear-off calendar was added. A cylinder tube of plexiglass with no graduation was gradually filled with small balls made by crumpling tightly each day sheets: every day, a child tears off the old sheet (yesterday), crumples it very strongly, and throws it into the time tube (Bartolini Bussi, 2015).

The past "goes" into the tube, the present is visible on the front of the pad, and the future is still hidden (in the calendar on the wall). If the teacher suggests to mark the level after 1 month, it may approximately define a unit. Guessing games can be played, like "What will the level be on Christmas day?" and the conjectures can be checked some weeks or months later. What is into play, from the arithmetic point of view, is approximation, as the level of the crumpled balls only gives an approximate idea of the time duration (Figs. 1.2 and 1.3).

[7] The notion of artifact (and, in particular, cultural artifact) is elaborated below.

Fig. 1.2 Some images of the time tube activity (our data 2014–16)

Fig. 1.3 Some images of the activity in the calendar corner (our data 2014–16)

However, actually deeper issues are discussed by children, concerning narrative and metaphorical thinking as the short excerpt from 5-year-old children's conversation with their teacher shows the following:

Teacher:	but according to you, where do the days go by?
S.:	in the tube!
Teacher:	I do not mean the cards we tear … the things we did yesterday, for example, where do the things we have done yesterday go? The days we have already passed?
E.:	the past days have gone behind us.
C.:	they end up in our mind, they are free to turn in head…
M.:	no, they do not end up in our mind because there are our thoughts: there is no space!
R.:	the days hang on thoughts and go in the head. The thoughts are free in the head.

The time tube was creatively invented by the research group: the members were likely to be inspired by traditional artifacts like sand clepsydras, but produced something new. First, the global time duration was much longer than usual, not minutes (as in the standard clepsydras), but weeks and even months with just an item (a crumpled ball) added every day; second, it was explicitly related to other activities concerning time, proposed to the same children, fostering narrative and metaphorical thinking. In this sense, it was a *cultural artifact* in a twofold interpretation, as related, on the one hand, to the traditional culture of sand clepsydras and, on the other hand, to the classroom culture of reflection on time.

1.2.3 Semiotic Mediation at the Preschool Level

The two abovementioned examples may be interpreted by means of the Theory of Semiotic Mediation (TSM) as developed by Bartolini Bussi and Mariotti (2008) after a Vygotskian approach. TSM was first developed by the two authors drawing on the studies by Bartolini Bussi on manipulative artifacts in both primary and secondary schools, and by Mariotti in secondary schools on Information and Communication Technologies (Bartolini Bussi & Mariotti, 2008). The following scheme (Fig. 1.4) was used at all school levels, but it is presented here with explicit reference to the example of the time tube. The example of the Slavonic abacus is discussed by Bartolini Bussi (2013).

Figure 1.4 shows the scheme of the process of semiotic mediation around the time tube artifact. The task is *to foresee the level of the crumpled ball for Christmas*; the mathematics knowledge at stake is *estimation*; the traces are found in *conversations, graphical representations, gestures*, and so on, produced during the activity; the mathematical "texts" (if any) are the *conclusions* constructed by the children under the teacher's guidance to answer the given task. The observations made immediately before Christmas (hence more than 2 months after the beginning of the activity with semiotic activity developed every day with increasing complexity) allow children to check whether the proposal was correct or which of the different proposals (if more appeared during the process) was correct and to argue about the rationale of either proposal.

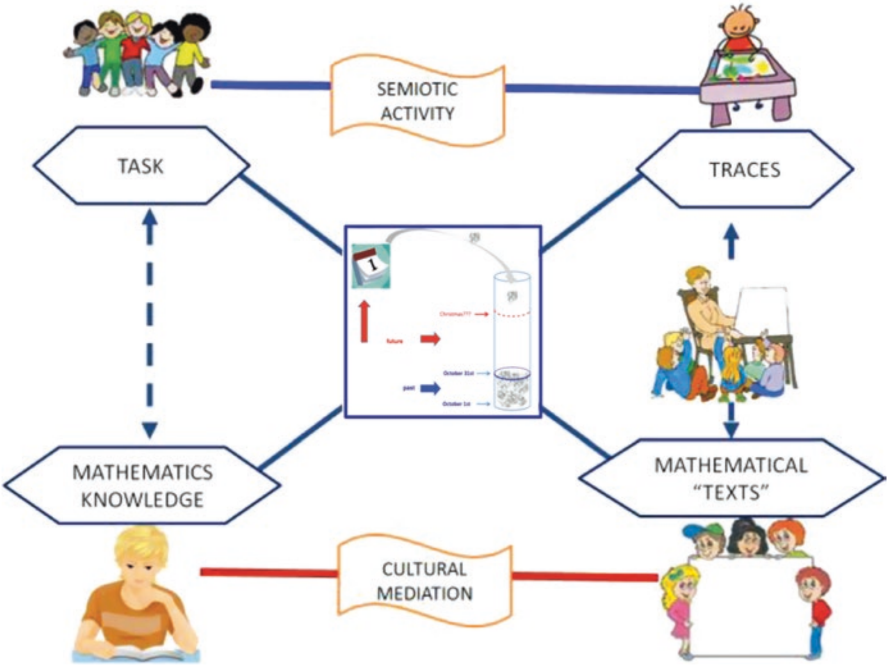

Fig. 1.4 The semiotic mediation scheme

1.3 The Shift to Toddler Centers (and to Prelinguistic Kids)

1.3.1 From Semiotic Mediation to Cultural Mediation

When the action research program was extended to toddler centers in 2013, the research group reflected on the possible interpretations of the semiotic mediation scheme. The research group was aware that some nodes of the scheme were tricky: what kind of *tasks* were suitable for very young children? Were the tasks to be explicitly posed by the educators or self-posed by the children under the stimulus of the context? In the first case, what might have been the function of *semiotic means* (language, gestures, gazes)? What kind of *mathematics knowledge* could have been focused on? It was clear enough that NUC mathematics was not an appropriate reference, but which parts of a QRS system might have been referred to? What kind of *traces* could have been observed? Very young children are, in many cases, prelinguistic, hence just gestures and gazes were likely to be considered as language precursors (see, for instance, Corballis, 2002). And which kind of mathematical "texts" (if any) might have been constructed and shared by adults and children?

To foster continuity between toddler centers and preprimary schools, a promising theoretical construct seemed to be the notion of *artifact*. Artifacts are used (either intentionally or sometimes even unintentionally) by educators to define the context

Fig. 1.5 Free play with everyday artifacts (*pasta*) and chairs (our data 2013)

of children's activity. The artifacts are culturally biased, although professional journals for educators tend to present a kind of "universality," especially when commercially structured material is concerned (Fig. 1.5).[8]

If one visits either toddler centers or preschools in different countries, s/he may discover different space organization, different furniture and artifacts, and different building locations, even before entering the details of classroom interaction. Classroom culture shows from the very beginning. The differences convey different values and traditions for early childhood education that are in the background of every institutional educational choices. When I was in Kristiansand for the POEM conference (May 2018), thanks to the kind and generous support of the local organizers, I had the possibility to visit primary and preprimary schools: everything looked different from the places I was familiar to in Italy and in many other countries. The preprimary school building was in a forest and on the shore of a lake; young children (1 year old) were free to explore the environment climbing in the rain, and to be exposed to situations that Italian parents (and maybe also teachers) might have considered not suitable (too dirty, too cold, too wet) and even "dangerous" for the children. It was, for me, a

[8] See, for instance, https://www.gov.mb.ca/fs/childcare/resources/pubs/equipment_toddler.pdf

short, but full lively immersion in outdoor education, which I only knew through literature. The language barrier fostered the attention on gesture and gaze observation. The strong and deep cultural differences in preschools were the object of famous ethnographic studies (Tobin, Hsueh, & Karasawa, 2009; Tobin, Wu, & Davidson, 1989). Yet, educators and teachers are likely to not always be aware of this cultural bias and seem convinced that just one model is the correct one (e.g., either a child-centered or a teacher-centered model and either an indoor or an outdoor model).

Since the beginning of the ZEROSIX action research program, it was clear that the notion of semiotic mediation, as focused mainly on language traces, was not the best way to cope with the extension to younger (prelinguistic) children. A crucial step in this process was represented by the careful reading of Arievitch and Stetsenko (2014), and by the continuous dialogue with them. The starting point of the two Vygotskian scholars was to consider semiotic mediation (or sign mediation) as just an aspect of cultural mediation.

> The transformational power of sign mediation was the centerpiece of Vygotsky's programmatic attempt to eliminate the gap between external activities and the human mind – the direction wherein many of his hallmark achievements lie. Yet we argue that Vygotsky did not provide a sufficiently coherent explanation of cultural mediation. Most significantly and quite paradoxically, Vygotsky did not consistently apply his own, quintessentially developmental approach to this key construct: he did not offer a developmental account of cultural mediation. (Arievitch & Stetsenko, 2014, p. 217)

In the following, I offer examples produced and observed by the members of the action research group intertwining our data with Arievitch and Stetsenko's arguments (see also Arievitch, 2017; Stetsenko, 2016).

1.3.2 The Newborn Experience in a Cultural Context

According to Arievitch and Stetsenko (2014),

> The infants' earliest encounters and experiences take the form of joint activities […] including feeding, bathing, dressing, going to sleep, and so on. […] [they are] communal, collaborative, social, and culturally mediated processes of being held and touched, fed and nursed by the other person, with actions of looking, touching, and hearing all being initially performed together with the other, as intricate parts of activities arranged and orchestrated by caregivers. (Arievitch & Stetsenko, 2014, p. 229)
>
> Adults engage in joint activities with infants according to social rules and conventions of their culture and their understanding of cultural practices, including those of infant care and human contact. (ibidem, p. 230)

Newborns are social beings and everything starts from social relationships, which are culturally biased. There are cultures where newborns are tightly bound and kept bound for either weeks or months; there are cultures where newborns are recommended to be placed naked on the mother's naked stomach to experience skin to skin contact for as long as possible. There are sound reasons often depending on the context, but there is not the best "universal" way to behave, although Western pediatricians are sometimes supposed to have discovered the ultimate right answer.

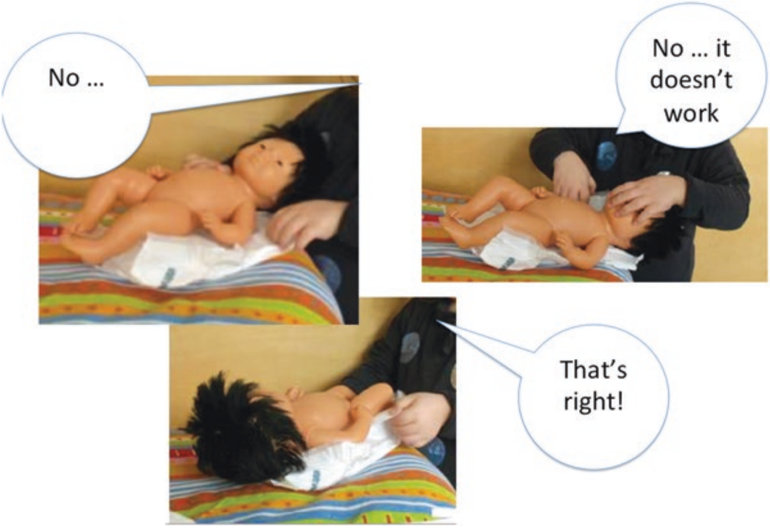

Fig. 1.6 A girl, a doll, and a diaper (our data: 2013)

Yet, whoever had the experience of more than one child, in different years (as the author of this chapter), has experienced the change of pediatricians' advices, every time with apparently good reasons for change.

A funny example may be observed by tourists who visit China: very young children wear open-crotch trousers or a split pair of pants,[9]; these trousers encourage children to urinate and defecate without lowering the pair of pants. The child simply squats, eliminating the need for diapers. The shift to potty training is much easier and faster than in the West. This early experience creates a context consistent with the QRS system, as far as relationships between one's own body and space context are concerned. The diaper experience of Western children is different. In the toddler centers, children experience routines, with adults taking care younger children. Figure 1.6 shows a 2-year-old girl who is taking care of a doll and tries to reproduce the observed gestures of caregivers. She is commenting on her efforts (speaking to herself) to orient the doll in the right position (a spatial experience).

Another episode was reported by an educator, who narratively described the strong efforts of a 2-year-old girl who tried (and succeeded) to tie a puppet on her shoulders with a scarf, imitating the gestures of her mother who came from a culture where this way of carrying babies was common.

In their cross-cultural study on motor development, Adolph, Karasik, and Tamis-LeMonda (2009) report different routines (suspension and shaking by ankles, arms, and head) which would be astonishing (and maybe considered "dangerous") for Westerner caregivers.

[9] https://en.wikipedia.org/wiki/Open-crotch_pants

1.3.3 Vygotsky's Hints to Non-Verbal Mediation: The Inner Visual Field

Vygotsky's reliance on narrow linguistic interpretations of verbal meanings as the only mediators of the mind is highlighted and criticized by Arievitch and Stetsenko (2014). There are, however, in Vygotsky's work, some hints to a more complex system of signs. For instance, Vygotsky reads as follows:

> Our experiments demonstrate two important facts: (1) A child's speech is as important as the role of action in attaining the goal. Children not only speak about what they are doing; their speech and action are part of one and the same complex psychological function, directed toward the solution of the problem at hand. (2) The more complex the action demanded by the situation and the less direct its solution, the greater the importance played by speech in the operation as a whole. Sometimes speech becomes of such vital importance that, if not permitted to use it, young children cannot accomplish the given task. These observations lead me to the conclusion that children solve practical tasks with the help of their speech, as well as their eyes and hands. This unity of perception, speech, and action, which ultimately produces internalization of the visual field, constitutes the central subject matter for any analysis of the origin of uniquely human forms of behavior. (Vygotsky, 1978, p. 25–6)

The complex system of visual perception, speech, and action fits well with the observation of children in our culture, which gives value to language and verbal communication. Yet, cultural anthropology (e.g., Morgan et al., 2015) documents other ways of transmission of tool-making teaching (e.g., imitation/emulation and gestural teaching), although the social transmission of technology in primitives seems to be enhanced by direct teaching and, in particular, by verbal teaching.

The internalized visual field is likely to become a zone of *mental experiments, where practice problem-solving is found.* An example of its functioning may clarify the power of this construct in problem-solving. The following game was tested in many preprimary classrooms (Briand, Loubet, & Salin, 2004). The artifact is given by a large collection of matches and a small collection (around 15) of empty match boxes where a small hole has been produced to allow a single match to be introduced inside (Fig. 1.7). The game is proposed by the teacher to a child (4–5 years of age), in a group of children with the following task: *You must put just one match in each of the boxes. You cannot open the box. Only when you say that you have finished, we shall open all the boxes. If there is exactly one match in each box, you win. But if there is no match or more than one match, you lose.*

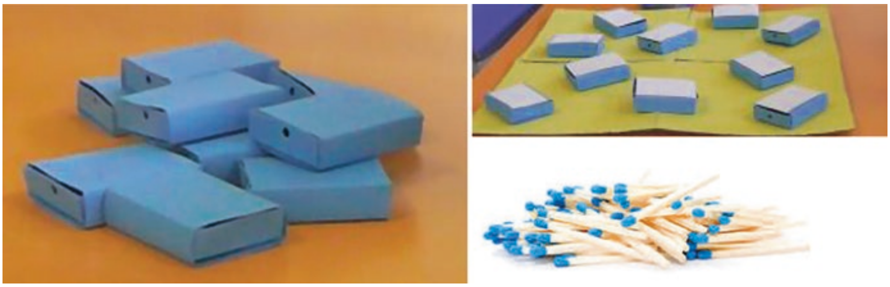

Fig. 1.7 The artifact: match boxes and matches (our data 2017)

In the first trial, the boxes are placed on the table and can be displaced. In a second trial, the boxes are stuck on a cardboard and cannot be displaced. What we observed in several experiments is that, in the first trial, children very quickly invent strategies (the same strategies in different preschools and even different countries): for instance they put aside the filled boxes (as if they were counting them), or, they shake a box to check whether something is already inside. Neither displacing nor shaking is possible any more when the boxes are stuck on the cardboard. Hence, children have to look at other strategies. The task is difficult. Adults are soon likely to split the process into steps: first they prepare just one match close to each box and then regularly put them in the boxes. This strategy may be interpreted as drawing on a mental experiment: the player knows the desired result and prepares one match for each box before acting, hence is able to change the natural time sequence, drawing on the verbal reconstruction of the process, while young children mechanically follow the order given in the task and fill a box before taking another match. Only later, after having played, lost, and discussed with the teacher and with their peers, they are likely to produce a more advanced strategy. Our interpretation is that through the unity of perception, speech, and action, they are eventually able to exploit the internalized visual field.

1.3.4 Vygotsky's Analysis of the Pointing Gesture

One of the most famous examples from Vygotsky (1978) concerns the genesis of the pointing gesture. An infant tries to grasp an object, but cannot because it is too far away. When the adult hands the object to the infant, s/he socially supplies the indicatory meaning to the infant's grasping. As a result, the infant realizes that s/he does not actually need to grasp an object in order to procure it. Her/his grasping movement then reduces in scope (becoming pointing), which can direct adults to fetch objects that are further away, such as a dropped toy (Vygotsky, 1978). Vygotsky's comment is crucial:

> An interpersonal process is transformed into an intra personal· one. Every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first, between people (interpsychological), and then inside the child (intrapsychological). (Vygotsky, 1978, p. 56)

This is the genesis of the so-called imperative or richiestive gesture to be contrasted with the *declarative gesture* (not considered by Vygotsky, as far as I know), where children direct the adult's attention to a referent in order to indicate its existence and share interest in it. As stated in the comprehensive volume by Kita (2003) about *pointing*, some authors do not agree in Vygotsky's reconstruction (e. g., Wilkins, 2003), claiming that finger pointing is not as universal as it would have been if related to the universal grasping movement. However, pointing gestures by means of other body parts (e.g., the head or the lips) seem to refer to some limited cultural groups living in remote regions.

The ZEROSIX group suggested to observe the emergence of pointing gestures, in both imperative and declarative functions in the toddler centers. Moreover, the intention was not only to observe but also to foster this emergence, creating specific educational activities which were likely to encourage children to produce these gestures with the educator. The following excerpt with photos is taken from a paradigmatic educator diary (*emphasis* in the original).

> Every morning, after breakfast time, we prompt reading image books with familiar objects. One of the books includes a kite also. Two kites are close to the ceiling of the room. I can show the kite in the book and the real kites. One morning I *asked* children "Where is the kite like that of the book?" *pointing* at the book and *looking* at them. And Caterina (11 months), *listening* to me, *pointed* to the kite at the ceiling and *looked* at me in response.
>
> I then *smiled* at her and *looked* at the real kite, *pointing* at it, while she *looked* at me with the intention of *sharing the interest*. I produced and she produced a *declarative* gesture with her finger and showed aware that we were only *talking* about the kite but could not *touch* it or *pull* it down. A routine has produced a piece of knowledge, a shared meaning, by imitation. Caterina has repeated my gesture and showed me to have understood through a *non-verbal communication* (gestures and gazes).
>
> In another situation, I *offered* an object from far. The child wished to touch and have it. The *richiestive gesture* is different from the declarative gesture. I repeated the routine with all the children, creating the possibility of observing each other and of enjoying the pleasure of waiting one's own turn.

The educator is not simply observing, but fostering the child's process and introducing a conscious educational perspective (e.g., *waiting one's own turn*) (Fig. 1.8).

This diary reports the early steps of what Arievitch and Stetsenko (2014) call *collaborative meaning making.*

Fig. 1.8 Declarative and imperative gestures (our data: 2017)

> At the earlier, pre-semiotic (or proto-semiotic) stages, the guiding activity of the adult is performed in its fully-fledged form, whereby the adult's actions intertwine (fuse) with those of the child within joint activities providing foundations for collaborative meaning making. Later in ontogenesis, the guiding role of others gradually takes on a more distanced and condensed (abbreviated) form represented by activities "crystallized" first in objects (according to their cultural use) and then in symbols and signs. At even later stages, children themselves begin to use this condensed and abbreviated guidance embodied in signs for orienting and self-regulating their own activity. (Arievitch & Stetsenko, 2014, p. 218).

1.3.5 A Replica of a Classical Experiment: Learning to Drink from a Cup

In the action research program, we have encouraged educators to replicate some classical experiments mentioned by Vygotsky and his collaborators. The aim was not to test the correctness of their analysis, but rather to understand better the reported process. Learning to drink from a cup is described as follows by Leont'ev (1981):

> When an adult first tries to give a baby a drink from a cup, the touch of the liquid evokes unconditioned reflex movements in the child that strictly correspond to the natural conditions of the act of drinking (cupping the hands as a natural water-holder). The baby's lips push out, forming a pipe, the tongue is advanced, the nostrils contract, and sucking motions are performed. The cup is not yet seen here as an object that determines the way of performing the act of drinking. The baby soon learns, however, to drink properly from the cup, i.e. its movements are reorganised so that the cup is now used appropriately to its purpose. Its rim is pressed down onto the lower lip, the baby's mouth is distended, the tongue takes up a position in which its tip just touches the inner surface of the lower jaw, the nose trils expand, and the liquid flows from the tipped cup into the mouth. A quite new functional motor system arises that performs the act of drinking and incorporates new elements. (Leont'ev, 1981, p. 305)

This careful description was reconsidered by an educator who reported the process of a 5-month-old girl who was learning to drink from a cup (*Denise and the wate*r). She videotaped the same girl every week for some months starting from the initial offer by the educator and ending at Denise's autonomous drinking taking the cup with both hands. The correspondence between the observed process and the process reported by Leont'ev was astonishing, and was debated in the collective meetings. This example suggested to analyze, at least theoretically, what could have been the process if not a standard cup but a sippy cup was offered. In this way the peculiar function of the cultural artifact was analyzed and the reasons for choosing either was debated (Fig. 1.9).

1.3.6 The Effects of Social Relationships

All the examples reported above highlight the adults' implicit or explicit functions as cultural mediators. This view has the potential to change the approach to early childhood education. The observation of the so-called spontaneous children behavior gets

Fig. 1.9 Denise and the water (2016: our data)/right: a cup and a sippy cup

Fig. 1.10 Traces in corn flour (our data, 2016)

a new sense and induces the search for which kind of adult–child interaction (if any) was in the prehistory of an episode. This is not an easy process as, in the West, observations are mainly focused on children (learner centered).

An activity in our toddler centers is fostering children to leave traces in corn flour or to saw corn flour on the floor. This use of flours (or sands) is a very popular activity in some cultures (e.g., *sona* in Africa, Gerdes, 2007, or *kolam* in the Tamil culture, Ascher, 2005) (Fig. 1.10).

We used corn flour to leave the possibility of exploring by the mouth, without incurring in celiac disease allergy as it may happen with wheat flour. Sometimes additional artifacts were introduced (e.g., spoons, sticks, combs, rakes, or straws). We have collected hundreds of photos of children blowing through a straw in the flour and observing the effects. The big number of photos of playing children might suggest that this action is a children "spontaneous" creation. This was the early interpretation during a meeting. But suddenly we discovered that, among dozens of photos, where children playing alone were photographed, there was just one where, at the same table, an educator was photographed. When we asked for details about the activity, she explained that, actually, she prompted the activity: the children had to learn how to blow in order to produce what they wished. Hence, the activity was not "spontaneous" at all, but suggested by the adult (Fig. 1.11).

Fig. 1.11 Blowing in corn flour with a straw. Left: children; right: the educator. (our data: 2017)

Another example was told by an educator:

> We collected a lot of objects from families and put them at the disposal of young kids (less than 12 months). They were quite interested by cell phones but not by an "old" phone, although it was coloured, movable and producing interesting noise.

The interpretation might be that cell phones evoked joint activity with parents, while the old telephone did not evoke anything.

The above examples clearly show the (sometimes hidden) intertwining of the adult's and children's action.

> infants at first dance the dance without knowing that they do so. (Arievitch & Stetsenko, 2014, p. 231)

Artifacts, the actions and signs (gestures, words, other kinds of production) of all the participants in the interaction, are the cultural means which structure the children's learning process.

> Similarly to Bakhtin (1981, p. 341) who has argued that "becoming a human being is the process of selectively assimilating the words of others" our account suggests that cultural means and signs represent the actions of others and thus carry in themselves the history of human activities. Before semiotic mediation, the child's actions and the involved objects assume meanings through the guiding role of the adult. Meanings are born in joint action and undergo several substantial transformations before they become the meanings of words. Therefore, when the developmental path "from social to individual" and "from external to internal" in Vygotsky's law of cultural development is expanded to include the earliest stages of child–adult joint activity, it becomes possible to chart a continuous progression in the mastery of ever-new activities that are engendered by ever-new forms of cultural mediation, without any ontological breaks among diverse forms that these activities take, including the most complex and sophisticated (i.e. mental) ones. (Arievitch & Stetsenko, 2014, p. 237)

1.4 Concluding Remarks

In this chapter, I have outlined some outcomes of the more than 5-year-long ZEROSIX action research program addressing educators of toddler centers and teachers of preprimary schools for mathematics (*numbers and space*). In the two

3-year cycles, more than 50 teachers and 50 educators have been involved. The mathematics in play is, in most cases, part of a QRS system, rather than of a NUC mathematics, because of the young children's age. All the reported examples concern either quantitative aspects (the slavonic abacus, the time tube), relational aspects (problem-solving), or spatial aspects (routines). The cases of pointing, learning to drink, and learning to blow in the flour concern the development of some specific motor systems that are fundamental for the further development. The continuity between toddler centers and preschools exploited the construct of cultural artifacts and the approach through semiotic and cultural mediation, consistent with the line traced by Arievitch and Stetsenko (2014), in order to critically overcome the breaks emerging from the activities with prelinguistic and linguistic children. This is just a part of the story, as with older children (in preprimary school), the connections with NUC mathematics (numbers and arithmetic) became more evident and paved the way toward the most complex and sophisticated cultural forms of the primary school.

The signs produced during the adult–child "dance," led by adults in the beginning, make the collective meaning evident. Our position fits well with Arievitch and Stetsenko's (2014) claim:

> There is a dynamic continuum of different forms of mediation. From this perspective, mediation does not begin with, but rather develops into mediation by external signs and later culminates in the internalized ability to guide and self-regulate one's own activity. (Arievitch & Stetsenko, 2014, p. 218)

In all cases the social relationships between children and adults are emphasized: they happen within and are biased by the already shared (or the to-be-shared) culture. The presence of examples from other cultures in our elaboration is a leitmotiv, although we are not engaged in comparative studies, but aim to exploit Bateson's statement, which says,

> *Information* consists of differences that make a difference. (Bateson, 1978, p. 99)

In other words, following Stetsenko, knowing other cultures gives one the possibility of

> seeing the world through a new lens, while learning not only to understand new culture(s) but to also see one's own culture and oneself from a newly acquired distance. These experiences highlighted, with striking clarity, the prescience of Bakhtin's words that "our real exterior can be seen and understood only by other people, because they are located outside us in space and because they are others," whereby "[a] meaning only reveals its depths once it has encountered and come into contact with another, foreign meaning" (1986 , p. 7). (Stetsenko, 2017, p. 17)

This position is consistent with the approach to a *cultural transposition*, as elaborated by Mellone, Ramploud, Di Paola, and Martignone (2019), and exploited in all our programs of teacher education and development in schools.

This chapter on early years mathematics presents neither the conclusion of the action research program nor a fully-fledged theory of ZEROSIX development for mathematics. The present members of the action research program (and the future members of similar programs as well) will continue to design, implement, interpret, and redesign a repertory of meaningful ZEROSIX experiences within our culture and to discuss and expand theoretical reflections.[10] In this way, the relationship between theory and practice appears reciprocal and dialogical. The detailed diaries of the educators and the teachers of the ZEROSIX group are paradigmatic examples, putting them all at the crossroad of theory and practice. We agree that

> Kurt Lewin's famous expression that there is nothing more practical than a good theory could thus be expanded, in the spirit of Vygotskian approach, by the mirror expression – that there is nothing more theoretically rich than a good practice. (Stetsenko & Arievitch, 2014, p. 235)

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[10] A rich collection of annotated meaningful experiences is in preparation (Bartolini Bussi, Botti, Riley, Matematica ZEROSEI: un approccio culturale, i.e., ZEROSIX mathematics: a cultural approach).

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Chapter 2
Young Children's Early Mathematical Competencies: The Role of Mathematical Focusing Tendencies

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2.1 Introduction

The past 10–15 years have witnessed the emergence of a remarkably productive and highly influential line of research on the development of young children's early mathematical knowledge and skills (such as counting, subitizing, comparing numerical magnitudes, number recognition, number line estimation, simple arithmetic, and basic mathematical patterns and structures), on their association with school mathematics, and on their stimulation in the home, preschool, and beginning elementary school environments (Andrews & Sayers, 2015; Aunio & Niemivirta, 2010; Baroody & Purpura, 2017; Torbeyns, Gilmore, & Verschaffel, 2015; Verschaffel, Torbeyns, & De Smedt, 2017). In this line of research, children are explicitly prompted to use their mathematical knowledge and skills, and they are assessed in terms of their ability to deal with the task at hand.

So, these studies take an "ability" perspective on children's early mathematical development and stimulation. In doing so, they concentrate on children's early mathematical knowledge and skills, thereby ignoring other possibly relevant aspects of young children's early mathematical competence. One such aspect, which will be the theme of the present chapter and relates to the dispositional side of children's early mathematical competence, is their inclination or tendency to attend to and focus on numerosities, Arabic numerals, quantitative relations, and patterns in their environment. We emphasize that these "mathematical focusing tendencies," as we will call them in this chapter, are not about what children think and do when they are instructed or guided to the mathematical entities, relations, or patterns in the situation, but about what they think and do when there is no explicit instruction or guidance to focus on them.

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The basic claim underlying this latter line of research is that, besides having different abilities with respect to the abovementioned distinct elements or aspects of early mathematical competence, young children also demonstrate different mathematical focusing tendencies when exploring, describing, and organizing their everyday world.

Furthermore, it is argued that this tendency to focus on the mathematical aspects of a situation will trigger self-initiated practice of the corresponding mathematical knowledge and skills. Thus, if some children are more prone to focus on these mathematical aspects of various situations, this will be more beneficial for their mathematical development compared to children who are not (Hannula-Sormunen, 2015).

During the past 10–15 years, researchers have started to empirically investigate these various mathematical focusing tendencies in young children, their development, their concurrent or predictive relation to children's mathematical achievement, and their stimulation by means of particular interventions (Hannula & Lehtinen, 2005; Verschaffel et al., 2017). So far, this research has largely focused on children's spontaneous focusing on numerosity (SFON), but to a much lesser extent, similar efforts have been made for Arabic number symbols (SFONS), quantitative relations (SFOR), and mathematical patterns (SFOP).

We start our overview of the available research on these mathematical focusing tendencies with the research on SFON. Afterwards, the emerging research on SFONS, SFOR, and SFOP is reviewed. In these overviews, we will give a broad picture of the international research scene, with special attention to the research done by our own research team.

2.2 SFON

As stated above, Hannula-Sormunen and colleagues were the first to hypothesize that within children's early mathematical competencies, there exists a separate and domain-specific attentional process of spontaneously focusing on numerosity (SFON). Hannula, Lepola, and Lehtinen (2010, p. 395) defined SFON as "a process of spontaneously (i.e., in a self-initiated way not prompted by others) focusing attention on the aspect of exact number of a set of items or incidents." According to these authors, this spontaneous focusing of attention on exact numerosity is needed for triggering exact number recognition processes (Hannula-Sormunen, 2015), affects the amount of children's self-initiated practice in recognizing and operating with exact numerosities in their everyday surroundings (Hannula, Mattinen, & Lehtinen, 2005), and, as such, will have a significant positive impact on their numerical skills and broader mathematical development (Hannula et al., 2010). Reversibly, more elaborated numerical and mathematical ability may further strengthen children's SFON tendency. So, the relationship between SFON and mathematical ability is assumed to be reciprocal.

Because the instruments aimed at assessing SFON must capture whether children *spontaneously* use their available exact numerical knowledge and skills in situations where they are not explicitly guided, stimulated, or instructed to do so, these instruments must meet several strict methodological criteria. A major criterion is that the experimenter is not allowed to give any instruction before or feedback during the testing of SFON that could help the child figure out which are the relevant aspects of the task (Hannula, 2005; Hannula & Lehtinen, 2005; Hannula-Sormunen, 2015).

The most frequently used type of SFON task is the *Elsi Bird Imitation task*, wherein the child is instructed to imitate the experimenter's playing behavior with toys, i.e., feeding berries into the beak of a toy parrot. For instance, the experimenter puts two berries, one at a time, into the parrot's beak. Immediately afterwards, the child is asked to imitate the experimenter's behavior (Hannula et al., 2010). Importantly, during the introduction and further administration of the task, no instructions are given to treat the task as numerical, implying that it is up to the child whether (s)he regards exact numerosity as a relevant aspect of the task. Whereas some children may put the exact number of berries into the parrot's beak or otherwise demonstrate spontaneous attention to the numerosity of the berries (e.g., by verbally counting the berries they put into the parrot's beak), others may put a random number of berries, or all the available berries into the toy parrot's beak, without paying any attention to numerosity. A SFON score is given on an item as soon as the child is observed using the correct number of objects or otherwise saying something that shows that he/she is spontaneously attending to the quantitative aspect of the situation (e.g., making a quantitative statement about the number of berries to be used) even without using the correct number of berries. (For a more detailed description of the various SFON imitation tasks and their scoring, see Hannula & Lehtinen, 2005; Hannula et al., 2005; Rathé, Torbeyns, Hannula-Sormunen, De Smedt, & Verschaffel, 2016). Meanwhile, several other action-based SFON tasks have been developed by Hannula and colleagues (see Hannula et al., 2005; Hannula & Lehtinen, 2005; Rathé, Torbeyns, Hannula-Sormunen, De Smedt et al., 2016).

Batchelor, Inglis, and Gilmore (2015) designed a less behaviorally and more verbally based type of task, the *Picture task*, wherein the child is shown three different cartoon pictures displaying both nonnumerical and numerical information (e.g., about a little girl making a walk in the forest) and is requested to verbally describe as precisely as possible what is in the picture. If the child spontaneously refers to exact numerosities—correct or not—in his or her verbal description of the picture, (s)he gets a SFON score of 1 per trial (For a more detailed description of the available SFON picture measures, see Batchelor et al., 2015; Rathé, Torbeyns, Hannula-Sormunen, De Smedt et al., 2016).

Since the early 2000s, Hannula-Sormunen and her colleagues set up a large set of studies on the variety in children's SFON and its association with other early numerical and mathematical competencies.

The first series of cross-sectional and longitudinal SFON studies revealed that typically developing children at the ages of 3–12 years largely differ in their tendency to spontaneously focus their attention on exact numerosity (Hannula-Sormunen, 2015). In these studies, SFON showed a reasonable within-subject stability across two or three different SFON tasks, or even across years

(Hannula-Sormunen, 2015). Furthermore, results showed a positive relationship between children's SFON and the development of early mathematical skills (i.e., subitizing-based enumeration, counting, number sequence, and basic arithmetical skills) (Hannula & Lehtinen, 2005; Hannula, Räsänen, & Lehtinen, 2007), later mathematical achievement in elementary school (Choudhury, McCandliss, & Hannula, 2007; Hannula et al., 2010; Hannula-Sormunen, Lehtinen, & Räsänen, 2015), and even rational number conceptual knowledge by the end of elementary school (McMullen, Hannula-Sormunen, & Lehtinen, 2015), suggesting that children with higher SFON tendencies before school age acquired a clear mathematical advantage in elementary school. Moreover, path models revealed evidence for the reciprocal influence of SFON and early mathematical ability (Hannula & Lehtinen, 2005). In addition, subitizing-based enumeration was found to be an important mediator of the association between SFON and object counting skills (Hannula et al., 2007). Finally, longitudinal investigations suggested the domain-specific role of SFON in children's mathematical development by showing that SFON acts as an important predictor of later arithmetical but not of reading achievement (Hannula et al., 2010; Hannula-Sormunen, Lehtinen et al. 2015).

Motivated by the outcomes of the first SFON studies, Hannula et al. (2005) and Mattinen (2006) also investigated whether it was possible to enhance children's SFON tendency in day care by providing 3-year-old typically developing children a 4-week SFON enhancement program. During the training period, the day care personnel observed and kept a record of incidents when children spontaneously focused their attention on numerosity while also purposefully guiding children's attention to exact numbers involved in everyday behavior, such as during eating, picking up toys, and outdoor activities. In addition, these preschool children played a set of structured numerical games with numerosities from 1 to 3, such as numerosity matching games. They also had a board of variable numbers of animal figures on the wall of the playroom at the day care center. The results of this quasi-experimental study showed that it was possible to enhance children's SFON tendency by means of social interaction in preschool settings, although effects were only obtained for children who already had demonstrated some initial SFON tendency.

During the past few years, a large number of other researchers have started to investigate SFON outside of Finland, including Australia (Gray & Reeve, 2016), Belgium (Rathé, Torbeyns, De Smedt, Hannula-Sormunen, & Verschaffel, 2018; Rathé, Torbeyns, Hannula-Sormunen, & Verschaffel, 2016), China (Tian & Siegler, 2015), Ecuador (Bojorque, Torbeyns, Hannula-Sormunen, Van Nijlen, & Verschaffel, 2016), Germany (Poltz, Wyschkon, Hannula-Sormunen, von Aster, & Esser, 2014), Israel (Sharir, Mashal, & Mevarech, 2015), Italy (Sella, Berteletti, Lucangeli, & Zorzi, 2016), Switzerland, Germany (Kucian et al., 2012), the UK (Batchelor, 2014; Batchelor et al., 2015), and USA (Edens & Potter, 2013; Tian & Siegler, 2015), and have thereby addressed both similar and new research themes. Some studies looked into additional theoretical questions on the development of SFON by examining spontaneous nonverbal counting among toddlers (Sella et al., 2016), the spontaneity and malleability of SFON (Chan & Mazzocco, 2017), the role of cognitive factors such as symbolic fluency (Batchelor et al., 2015). Others focused on the role of environmental factors, such as spontaneous activity choice during kindergarten play (Edens & Potter, 2013), the quality of early mathematics education (Bojorque et al., 2016), home numeracy experiences (Batchelor, 2014, Study 3), numerical picture book reading (Rathé, Torbeyns, Hannula-Sormunen, & Verschaffel, 2016), and broader cultural influences (Tian & Siegler, 2015) in the development of SFON. Other studies focused on methodological issues by investigating the psychometric properties of the SFON tasks (Batchelor, 2014, Study 1; Batchelor et al., 2015). A final series of studies focused on the relation between SFON and mathematical difficulties, such as Developmental Dyscalculia (Kucian et al., 2012) and (low) math profiles (Gray & Reeve, 2016). Generally speaking, these new studies replicated the main Finnish findings mentioned above (e.g., Batchelor et al., 2015; Bojorque et al., 2016; Gray & Reeve, 2016; Poltz et al., 2014; Tian & Siegler, 2015). However, they also pointed to important issues concerning the measurement, development, and enhancement of SFON (for a more detailed discussion of these issues, see Rathé, Torbeyns, Hannula-Sormunen, De Smedt, et al., 2016).

2.3 SFONS

One important characteristic of the SFON studies reviewed in the previous section is that they always used numerosities as stimuli and thus exclusively presented numbers in a nonsymbolic format. As a result, children's spontaneous attention for Arabic number symbols has not yet been addressed in this research. In the context of the PhD project of Sanne Rathé, we are addressing this gap by exploring the existence of a separate tendency of spontaneously focusing on Arabic number symbols (SFONS) within children's mathematical competencies.

Whereas SFON is defined and operationalized as children's tendency to spontaneously focus on the exact *numerical magnitude of sets* and use that information in their action (Hannula-Sormunen, 2015), SFONS refers to a similar but potentially separate attentional process whereby children spontaneously attend to and use *Arabic number symbols* in their everyday surroundings (Rathé, Torbeyns, De Smedt, & Verschaffel, 2019). As is the case for SFON, we assume that children's tendency to spontaneously attend to Arabic number symbols acts as an important contributor to children's early mathematical development. Analogous to the role of environmental print in the development of letter knowledge (Neumann, Hood, Ford, & Neumann, 2013) and the role of SFON in the development of counting skills (Hannula et al., 2007; Hannula & Lehtinen, 2005), SFONS tendency might provide children with more self-initiated learning opportunities to practice their knowledge of Arabic numerals, which in turn may enhance their further mathematical development.

So far, SFONS has been measured by means of a Picture Description task. In this task, which is presented and administered in a similar way as the SFON Picture Description task of Batchelor et al. (2015), the child is asked to describe as precisely as possible the content of three different cartoon pictures showing numerical as well as nonnumerical information. As in the SFON Picture Description task of Batchelor et al. (2015), some of this numerical information is nonsymbolic, but each picture also contains a small and familiar Arabic number symbol (1, 2, or 3) that is meaningfully integrated into the depicted situation. In line with previous SFON research, no mathematical hints or instructions are provided during task instruction or performance, so the child's attention is not directed or guided toward the Arabic number symbols in the pictures. Children score SFONS in a trial when they mention the correct, an incorrect, or even a nonspecific number symbol while describing the picture (see Rathé et al., 2018, for a detailed description of the task).

In a first cross-sectional study (Rathé et al., 2018), we investigated whether SFONS can be observed in children from the 3 years of Flemish kindergarten (i.e., K1, K2, and K3) and explored whether SFONS is related to early mathematical competencies, including SFON, numerical abilities, and teacher ratings of mathematical competence. The participants were 111 kindergartners aged 2 years and 4 months to 6 years and 2 months who completed measures of SFONS, SFON, and numerical abilities (i.e., Arabic numeral identification, verbal counting, and counting objects). Kindergarten teachers were asked to rate their children's mathematical competence on a 4-point Likert scale. Findings showed that children from the three kindergarten years differed in their tendency to spontaneously attend to the Arabic number symbols in the pictures, and SFONS correlated with age. Furthermore, children's SFONS and SFON scores were not significantly associated in the three kindergarten years, except for a marginally significant association in K3. Interestingly, children's SFONS significantly related to their numerical abilities in K1 and K2, and teacher ratings of mathematical competence in K3. In K2, these associations remained significant when age, average word count (i.e., the average total number of words formulated during the picture tasks), and SFON were taken into account, which suggests that SFONS is a unique contributor to individual differences in children's early mathematical competencies.

These findings provided the first evidence for the existence of SFONS and its role in early mathematical development, raising interesting questions about the psychological possibility and educational value of its early stimulation. However, there are several questions concerning the construct of SFONS which first need to be addressed by future (longitudinal) research.

First, is there some developmental priority of SFONS and SFON? Does one mathematical focusing tendency develop before the other? In our study, we observed more SFON than SFONS in the three kindergarten years, but given the cross-sectional nature of our data, we are limited in drawing conclusions regarding this question. Second, are SFONS and SFON separate constructs, and how does the association between SFONS and SFON develop with age? Does their association increase with age and do they merge into one construct? How does SFONS influence the development of SFON and vice versa? Our cross-sectional data revealed no significant association between children's SFONS and SFON in K1 and K2, and only a marginally significant association in K3. On the one hand, these findings suggest that SFONS is different from SFON, but the slight developmental trend in the data might also indicate that SFONS and SFON tend to merge with age. Third, are both SFONS and SFON involved in the development of early mathematical abilities? Do their contributory roles change with age? Our data suggest that both SFONS and SFON are important in the first years of kindergarten, but their role might diminish when children enter formal education. Finally, where do these individual differences in SFONS and SFON come from? What are the developmental roots of these mathematical focusing tendencies? One hypothesis is that these individual differences in SFONS stem from qualitative and quantitative differences in the home and classroom numeracy environment, but given the reciprocal nature of the association between SFON and early mathematical skills, it is also likely that individual differences in SFONS and SFON are partly due to individual differences in early mathematical abilities.

We are currently addressing the abovementioned issues by means of a 3-year longitudinal study, in which we are following the developmental trajectories of 181 kindergartners' SFONS and SFON in relation to their early numerical abilities and mathematics achievement while taking into account their language ability, nonverbal IQ, and home and classroom environment. Preliminary results from the first data wave of the longitudinal study show that a two-factor model with a separate SFONS and SFON factor provided the best fit to the data, indicating that SFONS constitutes a unique tendency within children's mathematical competence. Replicating the cross-sectional data, SFONS significantly related to early numerical abilities and mathematics achievement in 4- to 5-year-old children, also after taking into account age, parental education, language ability, nonverbal IQ, and SFON.

2.4 SFOR

Shortly after the first publications about SFON, the Turku team proposed another spontaneous mathematical focusing tendency, namely spontaneous focusing on quantitative relations (SFOR) (e.g., McMullen, Hannula-Sormunen, & Lehtinen, 2013). While SFON refers to noticing numerosity as an aspect of situations that involve one discrete quantity, SFOR refers to noticing quantifiable relations between (at least) two quantities. As for the two previous spontaneous focusing tendencies, the overall claim is (1) that children can also recognize and use quantitative relations without explicit guidance to do so, (2) that there are individual differences in how often children spontaneously focus on quantitative relations that cannot be entirely explained by their ability to use their relevant mathematical abilities, and (3) that these individual differences in SFOR are (predictively) associated with their (later) mathematical development. Taking the principles for designing proper measures of SFON in mind (Hannula-Sormunen, 2015, see Sect. 2), McMullen and colleagues developed several SFOR tasks (for an overview, see McMullen, 2014), including behavioral tasks such as the Bread task, and tasks that were rather verbally based such as the Teleportation task.

In the Bread task (see McMullen et al., 2013), two stuffed dogs are fed pieces of bread that are cut into different proportions from a whole (i.e., halves, thirds, quarters, or sixths). For instance, in the first item, the experimenter has two halves of which (s)he gives one to the first stuffed dog, and the child, who has a set of four fourths, is asked: "Watch what I do carefully, and do it in exactly the same way" for the second dog. Children's matching strategies are then coded with regard to the most mathematically advanced level utilized as involving quantitative relations, numerosity, or other.

The Teleportation task, also devised by McMullen and colleagues (McMullen, Hannula-Sormunen, Laakkonen, & Lehtinen, 2016), involves a cover story telling that a set of supplies containing three sets of objects was sent from earth through space with a teleportation machine. However, when doing so, the objects are transformed. Children are asked, first, to describe the transformation in their own words in as many ways as possible, and, second, to draw what they expect to happen with a different numerosity of the same objects. When doing so, learners can pay attention to the various nonmathematical changes (e.g., in terms of the colors or shapes of the objects) and also to the quantitative relation between the original and final numerosity of the three sets. Children's responses on the items are then coded with regard to the most mathematically advanced level utilized as involving the intended quantitative relations or nonquantitative aspects.

Those studies using SFOR tasks further the understanding of how children recognize and utilize mathematical aspects when not explicitly guided to do so, complementary to SFON. More specifically, in the first SFOR study, McMullen et al. (2013) investigated SFOR in children aged 5–8 using behavioral SFOR tasks such as the Bread task. There were substantial differences in children's use of quantitative relations and numerosity. The number of matching strategies based on quantitative relations tended to increase with age.

In follow-up studies, the tasks developed to measure SFOR were further validated. In the first part of the study of McMullen, Hannula-Sormunen, and Lehtinen (2014), kindergarteners to third graders completed the abovementioned SFOR tasks and then completed a variant of these tasks with explicit guidance to focus on quantitative relations. Again, there were substantial differences in children's SFOR tendencies, which could not be fully explained by their ability to solve the guided task using quantitative relations. Similar results were found in third to fifth graders, who were presented with verbally based SFOR tasks and a guided version of these tasks. Individual differences in SFOR remained after taking into account children's guided performance, both within and across grade levels (McMullen et al., 2016).

The SFOR tendency has been examined in relation to other aspects of mathematical development. In the second part of the aforementioned study of McMullen et al. (2014), which was longitudinal in nature, first graders completed behavioral measures of SFOR tendency and a measure of fraction knowledge 3 years later. SFOR tendency was found to predict later fraction knowledge, suggesting that it plays a role in the development of fraction knowledge. Later studies, using verbally based SFOR tasks that were presented to children in upper primary education, confirmed that SFOR is an important predictor of rational number knowledge (Van Hoof, Degrande, Verschaffel, & Van Dooren, 2016) and rational number knowledge development (McMullen et al., 2016), even after controlling for a range of known predictors of rational number knowledge (McMullen et al., 2016; Van Hoof et al., 2016). More recently, by following third to fifth graders longitudinally over a period of 4 years, McMullen, Hannula-Sormunen, and Lehtinen (2017) showed that SFOR has a reciprocal relation with rational number knowledge, each being uniquely predictive of the other. Moreover, SFOR was found to predict algebra knowledge 3 years later, even after taking into account nonverbal intelligence and rational number knowledge.

As for the other spontaneous focusing tendencies, further research on SFOR is needed. First, the conceptualization of SFOR evokes several questions. Several authors noticed that quantitative relations were so far unilaterally conceptualized as *multiplicative*, and therefore operationalized by means of tasks that were multiplicative in nature (Degrande, Verschaffel, & Van Dooren, 2017; Vamvakoussi, Vraka, Lioliousi, & McMullen, 2016). In this respect, Vamvakoussi et al. (2016) stated that many of the SFOR tasks require even more than merely noticing a simple multiplicative relation; they also require proportional reasoning, at a quite advanced level. Therefore, they plea for examining whether multiplicative tasks that minimize the requirement for proportional reasoning could be devised. At a more fundamental level, Degrande et al. (2017) confronted children with a more open version of the same Teleportation task, containing items that could be interpreted and solved in terms of *multiplicative relations* as well as an *additive relation*. Based on these data, they suggested that SFOR might be better conceived as a bundle of different and competing focusing tendencies on different types of quantitative relations that may evolve differently throughout the development and may affect the development of mathematical knowledge and skills differently. Hence, besides the question "to what extent do children focus on quantitative relations?", the question "If so, on what kind of quantitative relations do children spontaneously focus?" appears to be a relevant one too. The latter question is the focus of the PhD thesis of Tine Degrande.

Second, and relatedly, all authors point to the need for further improvement of the reliability and validity of the currently available SFOR measures. In this respect, Vamvakoussi et al. (2016) argued that previous SFOR tasks exclusively involved tasks containing quantitative relations between discrete quantities. However, the nature of the quantities in the task may impact the type of relations children focus on. In tasks containing discrete quantities, children might be more prone to attend to absolute than relative quantities and also to additive rather than multiplicative relations (e.g., Ni & Zhou, 2005). Therefore, they designed SFOR tasks containing continuous quantities that needed to be compared. For instance, in their Elf task, children had to prepare a magic potion by using a picture of the ingredient needed for the potion as well as an object of reference, with a ratio of 1:2 or 1:4 between both. After that, children had to select the ingredient needed among five three-dimensional objects: the ingredient in question, the object of reference, and three more objects similar to the SFOR object but varying in length (one obviously shorter and two obviously longer than the SFOR object) and color. An SFOR response was coded when either the ingredient in question was chosen or when the child's behavior indicated that (s)he had noticed the relation.

Third, while research has begun to document the origin and development of the SFOR tendency in children in relation to mathematical knowledge and skills, little is known yet about the origins of individual differences in SFOR, as well as about the relation between SFON and SFOR. The fact that many of the SFOR measures require that children first focus on the numerosities in order to quantify the relation (Vamvakoussi et al., 2016) limits our understanding of the relation between both tendencies, and the estimation of their separate contribution to the development of mathematical knowledge and skills (McMullen, 2014).

Finally, research on the origins of individual differences in SFOR is closely related to research on its enhancement. The first steps have been taken in this direction. McMullen, Hannula-Sormunen, Kainulainen, Kiili, and Lehtinen (2019) found that a 7-week-long intervention program in which sixth graders used mobile technology to explore quantitative relations in their everyday life was successful. However, it still needs to be further explored under which conditions educational interventions to enhance SFOR may be profitable (McMullen, 2014; Vamvakoussi et al., 2016).

2.5 SFOP

Inspired by the research on the three previous spontaneous mathematical focusing tendencies, several researchers have suggested that similar processes might exist for mathematical patterns (e.g., Seo & Ginsburg, 2004; Verschaffel et al., 2017). In their conceptualization of "Awareness of Mathematical Patterns and Structures" (AMPS), Mulligan and Mitchelmore (2009) also tend to go beyond the pure ability aspect of early mathematical competence, by stating that AMPS may consist of "two interdependent components: one cognitive (knowledge of structure) and one meta-cognitive (a tendency to seek and analyze patterns)" (p. 38). However, neither in their assessment nor in their intervention materials, they have tried to specifically and explicitly address this spontaneous focusing aspect, given that all their assessment and instructional tasks are explicitly aimed at, respectively, measuring and stimulating children's knowledge and skills with respect to patterning and structuring.

Recently, new tasks have been designed to explicitly assess young children's SFOP as part of two broader studies (Sharir et al., 2015; Wijns, De Smedt, Verschaffel, & Torbeyns, in press). In a first study, focusing on children's Recognition of Mathematical Structures (i.e., ROMS; an overarching term for all possible spontaneous mathematical focusing tendencies), Sharir et al. (2015) developed three tasks. The two verbal tasks, one with pictures and one with geometrical shapes, were similar to the picture tasks used in the research on SFON and SFONS, whereas the nonverbal task resembled the action-based tasks used in SFON research. Each of the three ROMS tasks included quantities (e.g., xx), mathematical patterns (e.g., xxx xxx), and arithmetic series (e.g., x xx xxx). Again, there was no use of any phrase which could have suggested that the tasks were somehow mathematical or quantitative.

In a second study, which is part of an ongoing large-scale longitudinal study about the development and stimulation of early core mathematical competencies, we are addressing, in the context of the PhD project of Nore Wijns, young children's patterning competencies. Besides tasks measuring children's patterning ability, we also included a measure that is aimed to catch their SFOP, namely the Tower task. This Tower task emerged from the finding that some children, even with disadvantaged backgrounds, spontaneously engage in patterning activities (Fox, 2005; Garrick, Threlfall, & Orton, 2005; Seo & Ginsburg, 2004) and the suggestion that children with such a spontaneous engagement in patterning activities might have more advanced patterning abilities than children with no such spontaneous interest in patterns (McKillip, 1970). The child is presented with a set of blocks of different colors (five yellow, five red, and five blue) and asked to make a tower with (all) these blocks. Importantly, we administered this task at the very beginning of the interview, before the administration of the complementary patterning ability tasks, and we also restrained from using the word "pattern" in the task instructions. When the child had finished the task, the experimenter took a photo of the child's construction, and then categorized the photo as either "pattern," "sorting," or "random." In order to get categorized as "pattern," there had to be at least two full units of the assumed repeating or growing pattern and even the start of a third unit present in the child's tower. A construction was categorized as "sorting" when all blocks were sorted by color. All other constructions were named "random" since there was no clear evidence of a pattern.

Sharir et al. (2015) presented the three ROMS measures as well as a curriculum-based mathematical reasoning task including number word sequence production, counting of objects, numerical sequence order, and basic arithmetic skills to 113 4–6-year-old children from four Israeli kindergartens. Results indicated that young children could spontaneously recognize not only quantities presented in random order but also multiplication patterns and arithmetic series. Quantities did appear to be the easiest to recognize spontaneously, followed by multiplication patterns which are in turn easier to recognize spontaneously than arithmetic series. Children's scores on the nonverbal ROMS were significantly higher than those on the verbal ROMS and older children significantly outperformed younger children on most ROMS types. Furthermore, a factor analysis on the whole set of ROMS tasks revealed three different factors, explaining 47% of the variance, namely ROMS verbal based on pictures, ROMS verbal based on geometrical shapes, and ROMS nonverbal, implying that the data were classified according to the three ROMS representations (pictures, geometrical shapes, and nonverbal) rather than according to the ROMS types (quantities, multiplicative patterns, and arithmetic series). Finally, the different types of ROMS explained 34% of the variance in mathematics reasoning, while children's age and mothers' education added an additional 14% of the explained variance.

In our own ongoing longitudinal study with 391 4-year-olds from a wide range of socioeconomic backgrounds, we administered the abovementioned Tower task together with a measure for early patterning ability and for early numerical ability (Wijns et al., in press). The early patterning ability measure consisted of three activities, namely extending (i.e., What comes next?), generalizing (i.e., Make the same pattern using different materials), and identifying the unit of repeat (i.e., Reconstruct the pattern when hidden). Each of the three activities was implemented with six repeating (e.g., AABAAB) and six growing (e.g., AB, AAB, and AAAB) patterns. The measure of early numerical ability consisted of 88 items that were selected based on recent research (see Andrews & Sayers, 2015; Jordan, Kaplan, Olah, & Locuniak, 2006; Purpura & Lonigan, 2013) and comprised subtests measuring verbal counting (i.e., “Count as high as you can”), dot enumeration (e.g., “Count the dots”), object counting (e.g., “Give me N stones”), symbolic and non-symbolic comparison (e.g., “Which number is the largest”), number order (e.g., “Which number comes before/after N?”), number recognition (e.g., “Which number is this?”), and verbal arithmetic (e.g., “I put N stones in a box and add/subtract M, how many stones are in my box?”). As expected, the photos showed a wide variety of constructions, ranging from towers that manifestly did not involve any pattern (49%) to towers revealing sorting behavior (14%) or very systematic patterning behavior (37%). We also related children’s behavior on the Tower task to their scores for patterning ability and to their numerical ability. Children who constructed a pattern had a patterning ability and numerical ability similar to children who sorted the blocks per color. However, children who made a random arrangement had a lower patterning ability and numerical ability than children who did make a pattern.

As for the other mathematical focusing tendencies, several issues related to SFOP require further research. Concerning the conceptualization of SFOP, it seems reasonable to theoretically differentiate SFOP both from other mathematical focusing tendencies and from the patterning ability. Particularly the theoretical distinction between SFOR and SFOP seems less clear. Empirical evidence for conceiving SFOP as a construct different from the other mathematical focusing tendencies is absent since the only available factor analysis by Sharir et al. (2015) did not yield empirical evidence in favor of that differentiation. Maybe different results might be found when using other available measures of these different mathematical focusing tendencies.

As far as the measurement of SFOP is concerned, the attempts that have been made by Sharir et al. (2015) resemble the action-based and the picture tasks of other mathematical focusing tendencies discussed in the previous sections. However, one might argue that the need to include the more complex elements of mathematical structures and series into these simple tasks runs the risk of confronting the child with quite artificial scenes or actions wherein the structural or serial element is very prominently present, making them less suitable as measures of children’s spontaneous focusing tendencies. In this respect, the open Tower task used in the study of Wijns et al. (in press) may therefore capture children’s spontaneous focusing tendency to mathematical patterns and structures more adequately, but our experiences with the coding of the children’s constructions force us to warn of the difficulty in making a clear and objective distinction between a tower with and without a pattern, when only relying on the child’s externally observable actions and final product. A solution might be to collect additional verbal description data, but given the children's young age it is questionable whether they will be able to provide sufficiently rich and valid explanations of their construction. More research on this measurement issue is absolutely necessary.

Furthermore, whereas the available studies yield already some interesting findings about the interindividual variation in SFOP and its relation with other spontaneous focusing tendencies and with patterning ability, as well as with mathematical ability in general, future—preferably longitudinal—research is needed to yield better insight into the development of SFOP and its relation with other components of children's early and later mathematical development.

Finally, in line with the intervention studies addressing other mathematical focusing tendencies, training of SFOP might stimulate the development of patterning ability and mathematical ability in general. Such intervention studies could provide evidence for possible causal associations between SFOP, patterning ability, and mathematical ability. Clearly, the number of intervention studies aimed at the design, implementation, and evaluation of innovating learning environments that pay intensive and systematic attention to the development of young children's patterning ability is increasing drastically (Pasnak, 2018), but, to the best of our knowledge, in these studies the role of the tendency to attend to mathematical patterns and structures is hardly addressed in the intervention and/or assessment part of these intervention studies.

2.6 Conclusion and Discussion

During the past decades, early mathematical competencies have been widely recognized as a significant predictor of later mathematical performance (Andrews & Sayers, 2015; Aunio & Niemivirta, 2010; Baroody & Purpura, 2017; Verschaffel et al., 2017). Most of this research has been conducted with a view to characterize and stimulate the kinds of mathematical knowledge and skills that underlie this predictive relationship, using tasks wherein the children are explicitly asked to activate and demonstrate these mathematical abilities. During the last 10–15 years, there has been a growing interest in another aspect of children's early mathematical competence, namely their tendency to pay attention to numerical and mathematical aspects of an everyday situation or event when their attention is not drawn explicitly to these aspects. A main idea underlying this latter research is that in everyday situations or actions when there is no specific externally imposed attentional focus or guidance, before children put their mathematical knowledge to use, they need to pay attention to aspects that are amenable to quantification or mathematization. It is argued that such focusing on numerical and other mathematical aspects of a situation may elicit self-initiated practice of the corresponding mathematical abilities, resulting in an advantageous mathematical development over children who do not (Hannula-Sormunen, 2015). So far, most of this research has been done on young children's spontaneous focusing on exact numerosity (SFON), but to a much lesser extent, similar efforts have been made for Arabic number symbols (SFONS), quantitative relations (SFOR), and mathematical patterns (SFOP). In this chapter, we have reviewed these different lines of research and already discussed, for each of them, issues concerning their definition, measurement, development, and stimulation. In this final section, we reflect upon these issues in a more general and integrated way and propose some lines for future research.

2.6.1 Conceptual Issues

Even though serious efforts have already been made to clearly define these various mathematical focusing tendencies, several conceptual queries remain.

First, since most empirical studies involved only one or at most two of these spontaneous focusing tendencies, the relation between SFON, SFONS, SFOR, and SFOP remains unclear. Further theoretical reflection and empirical research is needed to answer questions such as: Is it theoretically appropriate and empirically warranted to distinguish between these four spontaneous mathematical focusing tendencies? Are these four tendencies conceived and defined at the appropriate level of specificity? Do they jointly cover the complete set of spontaneous mathematical focusing tendencies? For instance, the explicit and deliberate presence of the term "exact" in the definition of SFON makes this spontaneous focusing tendency highly specific to exact numerosities, raising the question why children's spontaneous focusing tendency on *approximate* numerosities must be left out of the scope. Similarly, in its definition and operationalization, SFONS is restricted to spontaneous focusing on Arabic number symbols. But what about possible individual differences between young children in their spontaneous focusing on *verbal number words* in daily conversations, stories, etc.? With respect to SFOR, it is remarkable that all tasks are about multiplicative relations between discrete quantities. As suggested by Degrande et al. (2017), there are good reasons to differentiate between different SFOR tendencies so that other kinds of mathematical relations, especially additive relations, are also included. Finally, with respect to SFOP, it should be clear that only a very small portion of the enormous variety of mathematical structures and series is being addressed in the initial elaborations of this concept by Sharir et al. (2015) and Wijns et al. (in press).

2.6.2 Measurement Issues

As described above, Hannula et al. (2005) listed five criteria that needed to be met in order to validly measure children's SFON. Essentially, all these criteria have been taken over by other scholars who have designed measures not only of SFON but also of the other mathematical focusing tendencies reviewed above. Whereas it is clear from the descriptions of all these measures that all researchers did serious attempts to meet these criteria when designing the materials, the instructions, and the scoring criteria of their measurement tasks, for all four tendencies important queries can be raised with respect to one or more of these criteria that jeopardize the (content) validity of these measures. Also, partly as an inevitable result of these criteria and partly as a consequence of the young age of the children at stake, there are problems with the reliability of most measures. Continued and intensified methodologically oriented research is needed with a view to analyze and improve the reliability and validity of these measures. Meanwhile, given this state of affairs with respect to the psychometric qualities of the various measures, it may be recommendable to assess SFON, SFONS, SFOR, and SFOP by means of a varied set of tasks covering these constructs.

2.6.3 Developmental Issues

As amply documented above, research has shown that already remarkably early in their development, many children demonstrate a tendency to attend not only to simple numerical elements or aspects of their environmental world but also to mathematically more advanced mathematical elements such as mathematical notations, relations, and patterns. Moreover, significant interindividual and developmental differences in all these mathematical focusing tendencies have been reported. However, several questions remain unanswered.

The first set of important questions relates to the development of the mathematical focusing tendencies. It seems very reasonable to assume, on theoretical grounds, that SFON, which involves spontaneous attention to nonsymbolic numerosities, develops both before SFONS, which involves attending to symbolic number representations as well as before the mathematically more complex focusing tendencies of SFOR, and SFOP. Empirical evidence supporting these hypotheses is scarce, but the scarce available evidence tends to support them: Rathé et al. (2018) found that early in their development, children's SFON scores exceed their scores for SFONS, while Sharir et al. (2015) found that quantities are easier to recognize spontaneously than multiplication patterns and arithmetic series.

Second, the question raises what causes that some children focus their attention on the numerical and mathematical elements in their environmental world and others do not, and that these focusing tendencies grow over time. As documented above, most research aiming at answering this question has been done in the domain of SFON. This research has yielded some indications favoring the plausible hypothesis that—just as what happens with the ability aspects of early mathematical competence—these mathematical focusing tendencies are at least partly affected by the intensity and quality of the early mathematical experiences children experience at home and in preschool. Although some studies provided first evidence for this hypothesis (Batchelor, 2014, Study 3; Hannula et al., 2005), other studies showed mixed findings (Edens & Potter, 2013; Rathé, Torbeyns, Hannula-Sormunen, & Verschaffel, 2016), but this may be due to measurement problems involved in the assessment of (the quality of) these mathematical environments. Future longitudinal studies should continue to explore the contribution of environmental factors to the development of these mathematical focusing tendencies. This future research should include more specific and direct measures such as observations of parents' number talk (Gunderson & Levine, 2011; Ramani, Rowe, Eason, & Leech, 2015) and the quality of classroom activities (Bojorque et al., 2016). Besides these environmental factors, various domain-general factors, such as symbolic fluency (Batchelor et al., 2015), inhibition (Clark, Pritchard, & Woodward, 2010), working memory (De Smedt, Janssen, Bouwens, Verschaffel, & Ghesquière, 2009; Passolunghi & Lanfranchi, 2012), language (Moll, Snowling, Göbel, & Hulme, 2015; Purpura & Ganley, 2014), and general motivational orientations such as task orientation and social dependence orientation (Lepola, Hannula-Sormunen, & Lehtinen, 2019) as well as domain-specific such as early mathematical abilities (Hannula et al., 2005), may also influence a child's score on a mathematical focusing tendency test. Further research, preferably applying a multi-method approach combining qualitative and quantitative methods, is needed to further our insight into the factors that affect the development of these mathematical focusing tendencies.

The third set of questions relating to development involves how the development of the mathematical focusing tendencies is related to the development of their mathematical ability. Based on the available evidence, particularly in the domain of SFON and SFOR, it seems reasonable to claim (1) that the mathematical focusing tendencies of children measured early in their development are uniquely related to children's concurrent and later mathematical performance up to many years later, (2) that there is a reciprocal relationship between the dispositional and ability components of children's mathematical development during their early and elementary mathematical development, and (3) that the main reason for this reciprocal relation is that children who have a higher mathematical focusing tendency are also more inclined to engage in self-initiated reflective practice, playing with number and mathematical relations and patterns which in turn will enhance their mathematical reasoning (Hannula et al., 2005), while children with low mathematical focusing tendencies will be less inclined to engage in such mathematical activities. Even though theoretically very plausible (Hannula, 2005) and already supported by some empirical findings (e.g., McMullen et al., 2017), convincing empirical evidence for the latter explanatory mechanism is lacking.

2.6.4 Promotion of Mathematical Focusing Tendencies

As far as the promotion of mathematical focusing tendencies is concerned, little is known about the possibility and effectiveness of promoting mathematical focusing tendencies in young children through specific interventions. Research so far has provided some initial evidence that children's SFON tendency can be stimulated by means of targeted interventions, which is a promising approach for improving young children's mathematical development (Hannula et al., 2005; Hannula-Sormunen, Alanen, McMullen, Kyttälä, & Lehtinen, 2015). As mentioned above, Hannula et al. (2005) and Mattinen (2006) showed that SFON tendency can be promoted by means of guided focusing activities in preschool and kindergarten settings, but these intervention studies have raised critical questions, both about the spontaneous nature of children's focus on numerosities after an intervention that is explicitly aimed at increasing children's SFON, and about the broader pedagogical question whether education *should* stimulate young children's tendency to attend primarily at the quantitative and mathematical dimension of their environmental world (Verschaffel et al., 2017). While some first steps in the direction of stimulating other spontaneous tendencies such as SFOR (e.g., see McMullen et al., 2019) have been taken, the same questions can arguably be asked for these other mathematical focusing tendencies.

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Chapter 3
Play and the Production of Subjectivities in Preschool

Luis Radford

3.1 Introduction

Play has always been a popular topic in early childhood education. And, one way or another, it has been associated with the more general question of children's development. Indeed, despite the impressive variety of conceptions of play (see, e.g., Elkonin, 2005), play has usually been considered either as a *source* of development or as a *window* through which one can grasp the current state of the child's development.

In the latter view, play appears as a kind of methodological tool. This is the case of Piaget's conception of play. In observing children play, the children's understanding of rules can be made apparent. Reasoning along this line, Piaget (1948) suggested a series of successive stages which children undergo in play: children travel from a motor or individual understanding of rules where the driven force is the child's desires, to an egocentric stage where although playing together each child plays "on his own" (p. 16), to incipient and, later on, developed stages characterized by social forms of collaboration.

In the former view, by contrast, play appears as something that can potentially *influence* the child's development. For instance, Smirnova and Gudareva argue that "Play is of special importance for the formation of the child's motivational sphere and voluntariness" (Smirnova & Gudareva, 2017, p. 252).

This chapter is about children playing mathematical games in a preschool setting. However, it goes in a different direction. Indeed, in this chapter, I am not interested in exploring how play allows children to develop mathematical ideas (the *play-as-a-source* view mentioned above that confines play to a mere facilitator of knowledge construction and intellectual growth). Nor am I interested in what we can learn about development in observing children play (the *play-as a-window* view

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that confines development to a natural unfolding process). I am interested in something different: I seek to understand how, through mathematical ideas and play, children and their teachers coproduce themselves and, at the same time, are produced by their cultural–historical context.

To pose the problem of teachers and students as entities that coproduce themselves and, at the same time, are produced culturally and historically, is to adopt a theoretical position about humans that is at odds with the classical view articulated during the Enlightenment and that has come down to us through the work of Rousseau, Pestalozzi, Piaget, and the mathematics education movement of the twentieth century epitomized in constructivism. In the enlightened tradition—that is, the European intellectual movement of the late seventeenth and eighteenth centuries that broke with tradition and emphasized individualism and reason (Horkheimer & Adorno, 2002)—the individual is portrayed as a constructor of ideas and the origin of her feelings, meaning, and intentionality. Kant, perhaps, the most enlightened philosopher of the Enlightenment, illustrates this idea of the individual better than anyone else: the Kantian individual is a subject of reason, the crafter of her/his own destiny, and the origin and source of meaning and knowledge. The result is a self-sufficient and substantialist conception of the individual: the *self-made* subject. In this context, the child appears as a *given entity*; that is, someone who, in order to develop her own intellectual capacities, simply needs a stimulating social environment (Martin, 2004).

In this chapter, I take a different route: I draw on a dialectical materialist philosophy and its conception of the human. Instead of being the origin of knowledge, feelings, meaning, and intentionality, the individual is conceptualized as an entity in flux, in perpetual becoming—an entity who through practical activity (like play) is continuously inscribing herself in the social world and, in doing so, she is continuously produced and coproducing herself within the limits and possibilities of her culture. In the first part of the chapter, I consider some theoretical ideas—such as subjectivity, subjectification, being, and becoming. These ideas frame the dialectical understanding of the child and her production in play offered here. In the second part, I discuss some video data that come from my current research in preschool settings. The last part of the chapter is an attempt at showing that the question of the production of individuals is immersed in ethical issues that mathematics education can no longer avoid taking into account.

3.2 The Production of Individuals In and Through Play

At first sight, exploring the production of individuals in and through play may seem an esoteric endeavor. Why, indeed, could such a problem be interesting from the point of view of mathematics education? Two of the major theories in our field—constructivism and the theory of didactic situations (see, e.g., Radford, 2018a)—charted a research agenda for themselves and the theories that followed where our problem at hand hardly finds a niche. While constructivism is oriented toward the investigation of the child's "construction of increasingly powerful conceptual structures and the development of intellectual autonomy" (Cobb, 1988, p. 100), the theory of didactic situations is oriented toward the creation of the didactical conditions that are conducive to the diffusion of mathematical knowledge (Brousseau, 1997). As we can see, the problem of the individual is left unproblematized in both theories.

I draw here on the theory of objectification (TO)—a Vygotskian theory of teaching and learning (Radford, 2008, 2018b)—that inscribes itself in a different educational project: it posits the goal of mathematics education as a political, societal, historical, and cultural endeavor aimed at the dialectical creation of reflexive and ethical individuals who critically position themselves in historically and culturally constituted mathematical practices and who ponder new possibilities of action and thinking.

As a result, in the TO, the focus is not on the mathematical content alone; the focus is not only on *knowing* (the dimension of knowledge) but also on *becoming* (the dimension of the subject or the individual). As a result, a cogent understanding and explanation of how learning happens should include accounts of how students come to know (knowing) and to be (becoming). Therefore, instead of being something esoteric, the problem of the production of individuals in and through play (or other educational settings) appears as something of great importance.

To avoid misunderstandings, I hasten to say that I do not see the production of individuals as the deterministic result of social forces shaping an inert *tabula rasa* subject. However, I do not see the production of individuals as the mere auto-production of the self either. What I have in mind is a production of individuals whose most distinctive feature is to be *dialectical*: individuals are projects of life in the making; they produce reality as much as reality produces them.

To look at children and teachers in this dialectical manner is to depart from the view of the world as "some eternal and objective network of causal factors converging on [the individuals] to shape an unresisting, passive blob to their external pregiven [cognitive] structures" (Wartofsky, 1983, p. 188). To look at children and teachers in a dialectical manner is also to depart from the view that conceives of individuals in general and children in particular as "self-contained homunculus, radiating outward in development from some fixed configuration of traits, dispositions, or preformed potencies" (Wartofsky, 1983, p. 188). That is, a view where children and teachers appear as the origin of their own experience and the product of their own life. Unfortunately, we tend to believe that the experiences through which we allegedly auto-craft ourselves are something direct. We tend to forget that the way we experience ourselves and come to constitute ourselves as subjects is mediated by culture and history. As Michel Foucault notes

> The experience we make of ourselves seems to us to be the most immediate and the most original; but it has in fact its historically formed patterns and practices. And what we believe to see so clearly in us and with such transparency is given to us in fact through deciphering techniques painstakingly constructed throughout history. (Foucault, 2017, pp. 29–30)

I want to contend that it is only through a genuine dialectical understanding of individuals and their social, cultural, and historical contexts that we can unravel what Stetsenko and Ho (2015) call "one of the most complex paradoxes of human existence" (p. 224).

> This paradox is about being one among many, that is, about being a unique individual in an essentially communal world shared with others. The paradox involved is that human beings are singular and unique individuals, yet they are also profoundly relational and deeply social, sharing with other people no less than the existential grounding of life in all of its expressions and forms. (Stetsenko & Ho, 2015, p. 224)

It is against the background of this most complex paradox of human existence that, in this chapter, I want to continue exploring a line of inquiry that I outlined in previous papers (Radford, 2014, 2018c), where the central idea is that all educational settings—play included—ubiquitously produce not only knowledge but individuals too. Since we are entering almost uncharted territory, I need to introduce some theoretical constructs. I need to delve into more detail on the question of the individual and the role cultures play in the process of knowing and becoming. To do so, I need to start from the beginning. I need to start with a brief discussion of a symbolic structure that, in each culture, defines the space of agentic maneuvering of the individuals and provides them with a definite sense of personhood.

3.3 Semiotic Systems of Cultural Signification

The starting point of the theoretical position that I want to explore here is that human subjectivity is entangled with its social, cultural, and historical contexts. Cultures, indeed, provide their individuals with the raw material of what they are. For instance, the very fabric of human subjectivity in ancient Mesopotamia was intertwined with the individuals' participation and their positioning in social and cultural activities such as agriculture, animal husbandry, or participation in religious events or military campaigns. These social and cultural activities out of which a sense of self emerged were, in turn, shaped by the political, religious, and economic structures that provide the individuals with meaning to their life. It is in this context that individuals in ancient Mesopotamia learned to live and die (see, e.g., Crawford, 1991; Kramer, 1963; Reade, 1991). And so do we, in our own cultural–historical context. And because these contexts are different, we find ourselves confronted by a different range of possibilities concerning rights and obligations from those encountered by the Mesopotamians, the ancient Greeks, Chinese, etc. We find ourselves in front of a world with different political, economic, and legal apparatuses and, as a result, with a different space of *agentic maneuvering*. The scope of the space of agentic maneuvering is both facilitated and constrained by a *symbolic superstructure*. This symbolic superstructure encapsulates the distinctive features of a culture—for example, its thematization of meaning production, the relationship between mind and reality, and the understanding of reality itself.

Symbolic superstructures have always puzzled philosophers, sociologists, and anthropologists. For example, adopting a Kantian position, Ernst Cassirer speaks of *symbolic forms*. Symbolic forms operate ubiquitously. They structure experience. For Cassirer (1955), language is the symbolic form *par excellence*: it is through language that, according to Cassirer, all forms of thought find meaning and expression. Abandoning the Kantian perspective, Hegel proposes a more dynamic vision in which the mind is considered as advancing historically (Hegel, 2001). Writing from a sociological perspective, Castoriadis (1987) speaks of the collective creation of symbolic webs that provide the individuals with the means to overcome the real and imagine new things. From the social, historical, and cultural educational perspective in which the theory of objectification is inscribed, the question of the symbolic superstructure is articulated around the material production of life in all its spheres, and particularly around the production of knowledge, mainly around dominant forms of knowledge production and their political–economic character. In the theory of objectification, the symbolic superstructure is termed *Semiotic Systems of Cultural Signification* (SSCS). They are dynamic systems that originate in the practical and sensuous activity of the individuals. They comprise ideas about the nature of the world (e.g., the nature of mathematical objects and their way of existing), truth (e.g., how truth is and can be established), and the nature of the individuals.

SSCS are full of tensions, as are the activities from where they emanate. They have a (implicit, explicit, or both) normative function and necessarily convey political and ethical views; for example, how we show ourselves to others, how we are expected to behave socially and to be recognized by others.

To understand the operativity of SSCS and how the individuals' deeds are embedded in a web of historical, political, legal, and economic relations that circumscribe the concept of self, let me mention an example from premodern times. The example comes from a county court in medieval England, where a prestigious blacksmith individual, Richard Bourdeaux, was offended publicly by a lower-status butcher, William Webbe (for details, see Shaw, 2005). This act, which was socially sensed as a disruption of rules governing the honor ethic and hierarchically structured social order that defined premodern life, went to court. The insult was seen as an offense against God and the hierarchical status of the town. The sentence included a repentance about social behavior and a monetary penalty. While the repentance was issued as a means to protect and validate the structural relationships between the social categories of people involved, the monetary penalty was a way to repair the offense to society in the form of a charitable donation to help with the restoration of a church. An excerpt from the court record reads as follows:

> For this reason, the said William begged (*supplicavit*) the said Richard, out of respect for God and for charity's sake, in view of the entire meeting, that he earnestly hoped he would pardon him his abusive language (*maledictum*) and the slander (*verba de dicto Ricardo malelocuit*) he had spoken. Then the said Richard, at the request of the master and burgesses, remitted and relaxed to the said William all the said fine and evil deed (*malefactum*) on condition that he never in the future publicly or openly say or proclaim defamatory words about Richard, such as he previously spoke so violently and harmfully, on threat of 40s. sterling to be paid to the current or future Master within two weeks of the relapse. And the said money should be applied to the restoration of St. Cuthbert's Church. (Shaw, 2005, p. 127)

Commenting on this medieval example, Diehl and McFarland note that "being successful in disputes over honor was predicated, at least partly, on the ability of disputants to justify their [social] position by appealing to cultural beliefs about what persons like them (and their opponents) should or should not do" (Diehl & McFarland, 2010, p. 1735). Those cultural beliefs about persons and what they can do can only make sense through the effects of *Semiotic Systems of Cultural Signification*. They operate through a complex web of political, legal, and economic relations and come to shape the concept of self, offering a spectrum of socially recognized positions and providing them with an agentic space for human action. The agentic space for human action is organized through a conception of the nature of the individuals that demarcates, in particular, the ethics of situated actions. Such agentic space is enforced through a legal system that vigilantly seeks to keep society and its individuals in a certain harmony.

3.4 Being, Becoming, and Subjectivity

Now, the relationship between the cultural "raw material" conveyed by the SSCS and the concrete individuals should not be seen in a causal or mechanical sense. On the one hand, as humans, we are unavoidably affected by our cultural–historical concrete context. This is part of our ontological makeup. It is part of what it means to be human. This is the point that the seventeenth-century philosopher Baruch Spinoza (1989) made in his *Ethics*—a book that had a tremendous influence on Vygotsky and Marx (Fischbach, 2014). However, individuals are not simply affected. They are affected in a *reflexive* manner. What reflexivity means here is that, in addition to being affected by their cultural–historical concrete context, individuals *react agentically* to such context. Vygotsky used to say that what distinguishes us most from other species is not intelligence, but free will (del Rio & Alvarez, 1995; see also Tappan, 1998).

Thus, while what emerges from the effects of affection—that is, the subject—bears the imprint of its culture, it always emerges as something different—different to others and to itself: the resulting subject is an "I" whose formula is "I ≠ I."

This formula captures the conception of the individual as, on the one hand, a dynamic concrete living agentic entity always in flux, in transformation, and, on the other hand, an entity whose agentic dimension can only be understood against the backdrop of culture and history. To refer to the individual in the aforementioned sense, I shall use the term *subjectivity*. To specify its sense a bit further, I need to turn to two related terms first: *being* and *becoming*.

Being, as I understand it here, is a *generative capacity* constituted of *cultural conceptions* of living in the world: ways of conceiving of oneself and of being conceived; ways of positioning oneself and of being positioned. In the previous example from medieval England, *being* includes those ways in which blacksmiths, butchers, cathedral builders, priests, soldiers, etc. conceived of themselves and were conceived by others. Those ways of conceiving of oneself and of being conceived by others are continuously *materialized* in the deeds and activities of the individuals. What materializes, however, does not coincide with the capacity that engenders it, for this capacity is a cultural, general, latent capacity. *Being* can only show itself through its materializations in the concrete world, where it can be recognized as what it is. *Being* a butcher, for example, is materialized in the *deeds* of William Webbe, as *being* a blacksmith is materialized in the *deeds* of Richard Bourdeaux. The always unfolding materialization or instantiation of *being* is related to *being*, but it does not coincide with it. William Webbe's deeds do not coincide with "butcher" (in the same way as the idea of a triangle does not coincide with any of its materializations). The materialization of *being* has a technical name: its name is *becoming*.

Now we can come back to the concept of subjectivity. A *subjectivity* is a unique sentient cultural concrete subject (William Webbe, Richard Bourdeaux, or a student or a teacher in our case) whose specificity results from the fact that it is continuously *reflectively* affected by *being* through its concrete materializations—an entity always in a process of becoming: an unfinished and unending project of life. Moreover, because it is constantly reflectively affected by *being*, a subjectivity is an entity that "is inseparable from the space of moral issues [of its culture and] from how one ought to be" (Taylor, 1989, p. 112). To be a subjectivity is "being able to find one's standpoint in this space, being able to occupy, to *be* a perspective in it" (p.112).

Empirically speaking, subjectivities are investigated through what I have termed in previous papers as *processes of subjectification* (Radford, 2012, 2018b). That is, the activity-bound processes where, coproducing themselves against the backdrop of culture and history, teachers and students (and individuals in general) *come into presence.*

In the next part of the chapter, I seek to understand how, through play, children and their teacher coproduce themselves and, at the same time, are produced by their cultural–historical context. I draw on video data that come from my current research in preschool settings.

3.5 Playing a Mathematical Game

In general, two contemporary trends can be discerned about the role of preschool. One of them considers preschool as a space of socialization and play suitable for the intellectual and physical growth of the child. The other trend is not in opposition to the first one, but it considers preschool as a preparation for school. While the former is usually immersed in the romantic view of the child of the Enlightenment, the latter is more preoccupied with school readiness. While the former usually advocates free play, the latter usually advocates learning in settings that follow a similar—although simplified—structure to what children will find in Grade 1. Furthermore, the latter view gives special attention to literacy and numeracy. Without expecting that children acquire deep concepts of numbers and forms, preschools are considered as channels to ensure the children's first contact with mathematics. This is the case in the Canadian province of Ontario, where my example comes from. My example comes, indeed, from a preschool classroom of 4- to 6-year-old children and is about a mathematical game whose goal is to introduce children to counting.

The emphasized presence of mathematics at the preschool level is coherent with the purpose of Ontario's vision of the school: the preparation of the young for a highly technological society characterized by quick change and adaptability. Of course, this emphasis on mathematics (and language) is not something new. Since the dawn of the twentieth century, mathematics came to occupy a privileged position in the school curriculum of those countries that saw in industrialization the path toward modern society. Mathematics became the ally and support of the new capitalist forms of production. To a large extent, the main problem of twentieth-century educational reform was the problem of massive schooling to train the young in the participation and development of a technological society (Radford, 2004). One century later, things have not changed much. Capitalism has not vanished. It has become transnational, diversified, and globalized. It is, hence, not surprising that the preschoolers I see entering the school every morning start the day with activities around counting. They start by singing, that is true. However, the content of the singing is about counting (see Fig. 3.1, Picture 1):

One little lamb in my house that jumps and turns around.
One, two, three, four, five.
One, two, three, four, five.
One, two, three, four, five.
It helps me fall asleep.

If the school has to create producers, consumers, technologically oriented minds, and "entrepreneurs"—as an important official document in Ontario insists again and again (see Ontario Ministry of Education, 2014, pp. 1, 3, 4, and *passim*)—counting has to be the starting point.

The mathematical game that I discuss involved two players and concrete artefacts to play it: a plastic sheet that contained two rows made up of 10 squares with

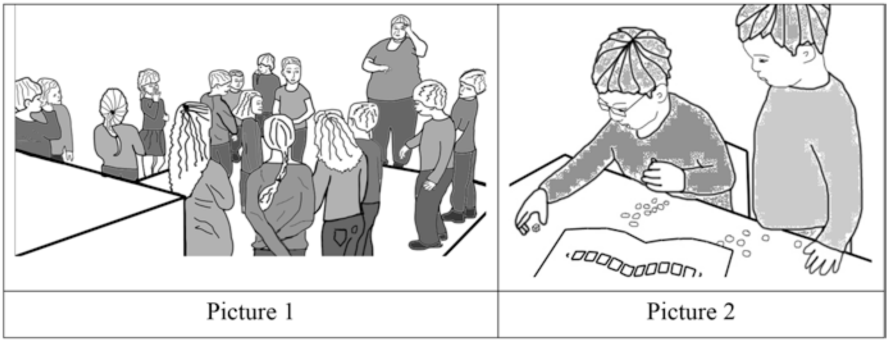

Fig. 3.1 Left, the children with their teacher singing an arithmetic song. Right, two players and the concrete material to play a mathematical game

space enough for the children to place a small plastic bear in each, 10 bears of one color for one child, and 10 bears of another color for the other child, and one dice (see Fig. 3.1, Picture 2).

In the second part of the game, which is the focus of my discussion, the children started with empty rows. The rules were as follows:

(a) Taking turns, each child had to place on her/his row the number of bears that corresponded to the number shown by the dice after the child rolled the dice.
(b) The winner is the child who filled her/his row first.
(c) To fill the row, the child had to roll the dice and obtain the exact number of points on the dice as the number of spaces left on her/his row.

To demonstrate the rules, the teacher played a game with a child in front of the class. Then, the class was divided into groups of two.

There were several mathematical notions involved in the game, such as producing a numerosity (the points shown by the dice); counting the numerosity (quantity) either perceptually or with their fingers and/or words; determining the number; choosing a quantity of bears that corresponds to the number; and placing the bears on the row and determining whether or not the game has been finished.

There were also some social dimensions involved in the game, such as subjecting oneself to the rules of the game; articulating one's actions with those of the other child; and paying attention to the various phases of the game.

Here is an account of the game played between Carl and Jack.

Jack rolls the dice and gets 6. With a tone of satisfaction, he says "6!" and proceeds to place six bears on his row while counting aloud. Carl follows Jack's actions. He waits for Jack to finish putting the bears on the corresponding row. When Carl is done, he says, "OK. My turn, my turn!" Carl takes the dice, rolls it, and says "Oh! 2!" He takes one bear at a time and places them on his row while counting aloud. Jack follows Carl's actions. Carl finishes placing his bears, moves the dice close to Jack's row and says, "OK, it's your go."

So far so good. The children have taken turns and moved the bears according to the game's rules. Unfortunately, things went badly right after. Here is the continuation of the game: Jack picks up the dice and rolls it. The upper face shows two points (see Fig. 3.2).

Jack is not happy with the result, picks up the dice again, puts it in his hands, shakes his hands vigorously, and lets the dice fall. He utters, "5!" Satisfied with the result, he starts adding bears while counting "1, 2, 3, 4." He is puzzled as he realizes that he does not have enough bears. Carl has been looking at what Jack does, apparently without fully understanding Jack's actions. Carl does not seem perturbed by the fact that Jack has ignored the first result (the dice showing 2 points) and has rolled the dice again.

At this moment, a child from another group calls the teacher and Carl's attention moves to that group. In the meantime, Jack is busy reordering his bears on his row. Thirteen seconds later, Carl's attention comes back to Jack. Jack is still reordering his bears on his row. Carl stretches his arm and tries to get the dice, which is in front

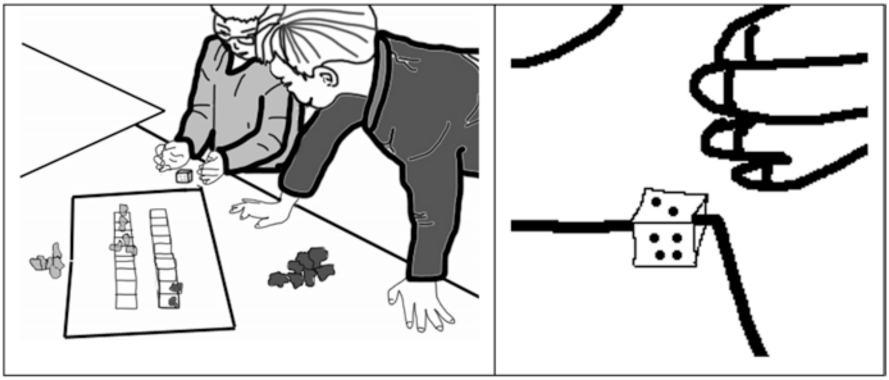

Fig. 3.2 Left, Jack rolls the dice and gets 2 points. Right, a close-up of the dice

Fig. 3.3 With his left arm, Jack (left) prevents Carl (right) from taking the dice

of Jack. Jack prevents Carl from taking the dice (see Fig. 3.3) and says, "So, it's … wait! OK, it's …."

Carl does not pay attention to Jack and says, "OK, my [turn], I …" Jack interrupts and says, "No, wait! Wait! Wait!" After some physical struggle, Carl succeeds in getting the dice. Jack continues, "So, it's 1, 2, 3, 4, 5, 6" and keeps on placing and counting bears: "1, 2, 3, 4." Carl is not paying attention to what Jack does. Carl rolls the dice twice. Jack finishes counting and puts his arms in a victory position. He utters, "I won! I won! I won! I won! I won! I won! Look!" Carl turns the dice in his hand, and when he finds the 6-point face, he stops and starts counting the points: "1, 2, 3, 4, 5, 6 … 6!" He tries to start putting six bears on his row. Jack puts his arms on the page covering all the bears to prevent Carl from placing his bears. Jack says, "I won! … Me, I won!" Carl moves his body toward the page and in a very frustrated tone says, "Ughhhhhh!" (see Fig. 3.4, Picture 1). Jack insists, "Me, I won!" Carl replies, "Me is getting mad at you!" Jack responds, "Me, I won! Won!" Jack takes the dice and shakes it vigorously as if to start a new game. Carl exclaims "No!

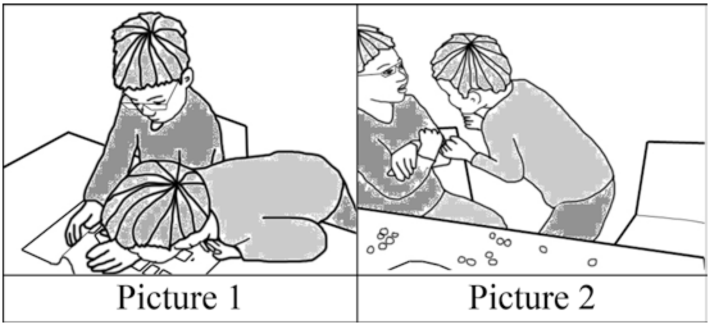

Fig. 3.4 Carl showing his frustration to Jack

JACK . . . Ughhhhhhh! No! This is enough!" He succeeds in getting the dice. "My was only when [I] have this" (he points to 6 on the dice) "So, my turn." Jack answers, "No, you didn't get that! . . . You did like (he pretends to hold a dice in his hand and to move it around) flip, flip, flip and then you found 6! Um, Carl cheated, he does like flip, flip, flip, flip! … (pointing at Carl) Cheater! Cheater! Cheater! Cheater!" Carl reacts with his body. He comes very close to Jack as if he is going to hit him (see Fig. 3.4, Picture 2).

3.6 The Inscription of the Children in the Social World

3.6.1 The Role of Rules

Generally speaking, following social rules is a crucial step toward inscribing oneself in the social world. A rule, indeed, provides a normative dimension and an agentic space of action that, as far as the rule is followed, keeps in principle the individuals' interaction within the scope of the socially expected. The medieval example discussed above provides an example of transgression. Not all rules are explicit. And even when they are—depending on the complexity of the behavior, duties, responsibilities, etc. that they target— rules may become objects of *interpretation*, for a rule is, by nature, *general*: it applies not only to a specific case, but a range of potential (i.e., not yet produced concrete) cases. The rules of the mathematical game played by Carl and Jack were explicit. The rules do not make a distinction between players. In this sense, the mathematical game's rules introduced above have a homogenizing effect on the children.

In the first part of the episode, we see how, drawing on the game's rules, the children come to position themselves in the game: they take turns, they wait for the other child to play, they even collaborate in sharing the dice; they seem to accept their responsibilities and the responsibilities toward the other player. Still, the inscription of the children in the social world is not an easy task. They have to pay attention to the evolution of the game; they have to wait for the other player to finish placing his bears. Moreover, to do so, they have to *control themselves*. As our data suggest, usually, in playing competitive games in preschool, it does not take long for the rules of the game to be broken. When Jack rolls the dice twice, he transgresses the social dimension of the rule. He seems to be aware of it. Figure 3.2, left, shows Jack rolling the dice and getting two points on the upper face of the dice. Disappointed, he picks up the dice again and shakes his hands vigorously with a sneaky smile on his face, which may mean something like: "I know that I should not be doing this, but …" Since Carl does not react, he continues playing seriously as if nothing had happened. We saw above that, right after, Carl got distracted and his attention moved to another group. The result is a *rupture* in the children's collaboration that was present in the early part of this game. The collaboration includes a *coordination* of actions (e.g., taking turns) but also *paying attention* to what each player does. Part of collaboration is indeed to pay attention to others, even if it is not one's turn. To maintain his attention on the game is a tremendous task for Carl, who is 1 year younger than Jack. In turn, although Jack's attention is on the dice and his bears, he does not realize that Carl is not paying attention. Jack is focused on his own actions. When Carl's attention comes back to the game, it is focused on taking his turn, regardless of the position of the game. *The rules that hold the children together and oriented the processes of subjectification in the first part of the game are no longer there.* The rules, which provided the children with rights and duties before, have evaporated. As a result, the social and theoretical common ground embodied in the rules of the game disappeared. The positioning of the children in the social world no longer has a shared reference. Without a shared reference, the connection and mutual recognizance that the children achieved before are lost. The relationship to the other takes a different turn. Impulse, desire, and imposition now drive the children's processes of subjectification. It is in this context that Jack draws on the stock of cultural categories at his disposal (the category of "cheater") to disqualify Carl. Carl, who exhibits a lesser mastery of the language than Jack, does not like to be called a cheater and responds with unarticulated phrases and with frustrating emotions expressed verbally ("Ughhhhhhh!") and with threatening body language (Fig. 3.4, Picture 2).

3.6.2 The Role of the Mathematical Content

In addition to the rules, the mathematical content required in the game also offers the children an important support to inscribe themselves in the social world. Indeed, the mathematical content offers the children entrance into a *shared space* of counting. For to play the game, the children have to count following the *same* culturally and historically constituted way of counting—they have to follow a same arithmetic and its counting principles. It would be a mistake to think that counting, as the children do in this game, is something natural. As shown by anthropological and ethnomathematical research, not all cultures count in the same way and not all count the same things (see, e.g., Lancy, 1983; Owens, 2001). Despite the presence of the bears, their colors, the plastic sheet with the rows, the dice, etc., the apparently concrete arithmetic these preschool children are playing targets an *abstract* form of arithmetic thinking that will be required in the abstract commercial exchange network that the children will find in society. The arithmetic that the children are encountering is, in fact, already economic and politically oriented toward a certain way of living and dealing with events in the world. The *Semiotic Systems of Cultural Signification* that ubiquitously operates in the school, the school system, and society as a whole *naturalize* this way of counting and its importance in children's education. It is only as a result of the effects of the *Semiotic Systems of Cultural Signification* that we end up assuming that counting things as the children do in this game is something obvious, necessary, and natural.

In short, the children's coproduction as subjectivities and their inscription in the social world takes place in processes (the process of subjectification) that occur as children engage in classroom activity—in this case, an activity around a mathematical game. Two important elements in these processes are (a) the manner in which children do (or fail to) subject themselves to the social rules and (b) the necessarily ideological stance of the content that they are learning. By ideological, I do not mean something that is purposely misleading (like a false consciousness). Following Voloshinov (1973), by ideological I mean that all theoretical content (like the arithmetical one conveyed by the game) is unavoidably the bearer of a *vision* or *idea* of the world—hence the term *ideo*logical. This is why the rules and the mathematical content are both also part of the very fabric of the children's subjectivity and their inscription in the social world.

There is still a third very important element in the children's inscription in the social world: the teacher.

3.6.3 The Teacher as the Embodiment of an Ideal Form

In a landmark paper, *The problem of the environment*, Vygotsky (1994) called attention to the fact that the settings in which children live are replete of "ideal forms" or "models" (p. 348) of behaving, thinking, speaking, doing, and so on, and argued that their greatest characteristic is not that these cultural and historically constituted ideal forms are already there in the environment or in society. Their greatest characteristic consists of how these ideal forms exert a real influence on the child. But how can this ideal form exert such an influence on the child? Vygotsky's (1994) answer is: under particular conditions of interaction between the ideal form and the child. Following Vygotsky's idea, I want to submit that the teacher is an *embodiment* of ideal forms—forms about knowledge, but also about *being*. In interacting with the children in classroom activity, teachers bring to the fore, and make available to the children, features of knowledge and *being* that are relevant in teaching and learning. To explain my point, let us come back to the classroom episode and continue with

Fig. 3.5 From left to right, Jack, Carl, and the teacher

what happened in the children's game right after Carl expressed his unhappiness and frustration to Jack (Fig. 3.4, Picture 2).

At this point, the teacher came to see Carl and Jack. She put herself close to Carl and, in a calm tone, asked him to sit down (see Fig. 3.5).

Jack was furious; pointing to Carl, he said loudly, "Cheater!" Carl defended himself responding, "Me no cheater." Carl turned to the teacher and, in a complaining tone, told her, "He does not want to listen to me!" In a patient, supportive, and comforting tone, the teacher responded to Carl with a question: "He doesn't listen to you?" In a discouraged tone, Carl responded with a brief "No!" Taking him seriously, the teacher asked, "What are you trying to tell him?" In the meantime, Jack pointed to Carl and shouted, "He, he cheats!" The teacher turned to Jack, and in the same calm tone she talked to Carl said, "OK. Stop saying that." Jack explained, "He was doing like (making some gestures with his hands) … and found 6." Coming back to Carl, the teacher asked him in a calm manner "What … what do you want to tell him?" Carl did not articulate a full answer and barely said, "Uh …" Then, the teacher invited the children to continue the game. Talking to both children, she said, "Whose turn is it?" Carl responded, "Me, me, me rolled like that but he didn't listen." In a comforting tone, the teacher said, "OK. Roll it [the dice] again. We'll restart [the game]."

At this point, the children started collaborating again. They started taking turns, paying attention to the other, putting the bears on their row and counting aloud. The teacher remained with them for 12 s and, having succeeded in calming both children, left to see another group.

What happened? In her interaction with the children, the teacher was able to calm them down. The teacher made available for the students forms of *being* (more specifically, forms of behaving and addressing the other) that were not within the children's reach. The teacher was able to show in a concrete way how to listen and how to care. She also showed empathy to the children. By showing empathy, she was able to connect with them and provide the reconstruction of a social, fluid, and dynamic structure where the children could reorganize their deeds around the rules of the game.

3.7 Synthesis and Concluding Remarks

In this chapter, I attempted to explore the question of the production of subjectivities in preschool. The question is based on a conception according to which individuals are *affected* by their cultural–historical context. However, as I pointed out, this affection should not be understood in a mechanical or causal sense: it should be understood in a *reflexive* manner. What this conception means is that while individuals are living agentic entities in a continuous process of transformation, the scope and parameters of their agentic dimension can only be understood against the backdrop of culture and history. It is in this sense that I talk about individuals coproducing themselves and, at the same time, being produced by their cultural–historical context.

One of the fundamental manners in which individuals are reflexively affected by their cultural settings is by the manner in which cultures offer their individuals a range of traits about how to show to, and position oneself in, the world. This is the idea of the concept of *being* that I introduced above. *Being*, I suggested, is as a *generative capacity* constituted of *cultural conceptions* of living in the world. *Being* is an ontological category, subsumed in symbolic superstructures that I termed *Semiotic Systems of Cultural Signification.* These systems operate ubiquitously through a complex web of historical, political, legal, and economic relations. It should not come as a surprise that schools, as places of preparation to life in society, draw from those *Semiotic Systems*, implicitly as well as explicitly. The Ontario system of education, for instance, seeks to produce "graduates who are personally successful, economically productive and actively engaged citizens" (Ontario Ministry of Education, 2014, p. 1). These three traits or forms of being obey a historical tradition anchored around the Enlightened concept of the child mentioned in the Introduction of this chapter, a specific Canadian conception of life around the individuals' community, and a contemporary economic urge to move successfully in the direction of the global economy, an urge that translates into the insistent inclination toward the formation of entrepreneurial minds.

Preschool is, in this context, the first step in the long journey on which children are about to embark. It is in preschool that they make the first contact with numbers, shapes, and social life. It is there that they start meeting, in a more or less structured way, the forms of *being* that society has to offer—for instance, to be an "actively engaged citizen," which includes knowing how to live by social rules. But *being*, in any of its cultural forms, is always something latent, a kind of archetype, something that in order to be perceived or noticed by the children, have to be *materialized* in the concrete world. Its materialization is what I called *becoming*.

In the first part of the game, the children followed the rules of the game. They enacted a way of *being*—*being* a good citizen. And because the game was about introducing children to abstract counting (as required in the counting of merchandise, the calculation of their prices, etc.), we could argue that the game is also about introducing the children to the sphere of (to use the Ministry's expression) the "economically productive," for how could you be economically productive if you do not know how to count? Then, we saw that the game was disrupted. Carl and Jack stopped following the rules. We also saw the crucial role that the teacher came to play. The teacher actively participated *along with the children* in recreating a social context where Jack and Carl could resurface and find manners of becoming a presence in the world, manners of agentically positioning themselves again in socially accepted ways (e.g., politely addressing the other and waiting for their turn).

The teacher's and the students' success in recreating a fruitful social context to continue the game calls attention to the fact that such an enterprise would not be possible without the reciprocated willingness to repair what was lost and mutual trust. As an embodiment of culturally and historically constituted ideal forms, the teacher was able to make available for the children traits of *being* of an ethical nature, such as genuine listening ("What are you trying to tell him?"), caring, answerability, and empathy. In practicing empathy, I would like to contend, the teacher is not just showing compassion. She is touching upon perhaps one of the most central features of what makes us human, namely the recognition of our fragility in the fragility of the other. I do not claim, though, that the children recognized those ethical traits of being as such. What I could claim is that, in playing the mathematical game, the children made the experience of those traits, that they sensed them, that those traits might have become objects of consciousness (not necessarily theoretical consciousness), and that, hopefully, those traits will become orienting parts of their subjectivity and their future deeds. Moreover, if this is so, I think that mathematics education can no longer ignore the centrality of the question of ethics in teaching and learning.

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Part I
Children's Mathematical Reasoning

Chapter 4
Copying and Comparing Repeating Patterns: Children's Strategies and Descriptions

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4.1 Introduction

The importance of engaging young children with pattern activities is supported by mathematicians, mathematics education researchers, and curriculum developers. Pattern exploration and recognition may support children as they develop mathematical skills, such as skip counting, and the invention and use of arithmetic strategies, such as adding "doubles" (Sarama & Clements, 2009). Exploring patterns may also enhance children's deductive reasoning skills as they generate equivalent patterns using different media and learn to predict what comes next in an existing pattern (Greenes, Ginsburg, & Balfanz, 2004). In addition, recognition and analysis of patterns may provide children with the opportunity to observe and verbalize generalizations as well as to record them symbolically (English & Warren, 1998). At the preschool level (ages 4 and 5 years), exploring repeating patterns has specifically been emphasized because of the relative ease with which children can recognize the basic unit which repeats and imagine its relationship with successive elements in the pattern (e.g., Papic, Mulligan, & Mitchelmore, 2011). Repeating patterns are patterns with a cyclical repetition of an identifiable "unit of repeat" (Zazkis & Liljedahl, 2002). For example, the pattern ABBABBABB… has a minimal unit of repeat of length three. According to the Israel National Preschool Curriculum (2008), "patterning activities provide the basis for high-order thinking, requiring the child to generalize, to proceed from a given" unit, "to a pattern in which the unit is repeated in a precise way" (p. 23).

Whether spontaneously or teacher led, children most often engage in duplicating, copying, and extending activities (Fox, 2005). Results of these activities are often described in general, stating success rates of children performing some tasks and typical errors (e.g., Rittle-Johnson, Fyfe, McLean, & McEldoon, 2013). Even

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when strategies are mentioned, they are often described in general terms, giving one or two examples (Papic et al., 2011). Yet, Klein and Starkey (2004) argued that "if mathematics standards and instruction aspire to be developmentally sensitive, they should reflect an awareness of the sometimes subtle changes that occur in children's early mathematical thinking as they progress toward full understanding of a concept" (p. 344). This study focuses on the subtle differences in children's strategies as they engage with a copying task, as well as children's verbal utterances when comparing the pattern they constructed with the pattern they copied. Children were shown a strand of beads with an AB pattern (and then with an AAB pattern) and requested to make a strand similar to the one shown but using different colored beads. They were then asked to compare the model strand with the one they constructed. In the next section, we discuss different types of pattern tasks, including copying, duplicating, and translating patterns.

4.2 Duplicating, Copying, and Comparing Patterns

Duplicating a pattern "involves making an exact replica of a model pattern" (Rittle-Johnson et al., 2013, p. 378), that is, constructing the same pattern using the same materials. The activity of copying a pattern is less clear. Papic et al. (2011) described several types of tasks, calling them all copying tasks. The first type was requesting children to create an exact duplication of block towers with identical blocks. The second type was to copy a block tower using tiles, and the third was to copy a block tower by drawing it with colored markers. Copying a pattern using different materials is sometimes called an abstraction task, or a translation task, as the child is requested to translate the structure found in one medium to another (Rittle-Johnson et al., 2013). This type of activity is considered more complex than duplicating a pattern using identical elements and may promote recognition of the unit of repeat (Sarama & Clements, 2009).

In relation to the complexity of duplicating and copying patterns, Papic et al. (2011) described different stages of structural awareness. Children who could duplicate a two-block tower (AB) with blocks, or copy it with tiles, or by drawing the tower, were considered to be at the prestructural stage. Children who could copy a four-block tower (ABAB) were considered to be at the emergent stage, those copying a six-block tower ABABAB were considered to be the structural stage, and those copying towers with a three-color repetition (ABC × 2 or ABC × 3) were considered to be at the advanced structural stage. Rittle-Johnson et al. (2013) based their construct map for repeating patterns on the level of abstraction needed to complete the task. They claimed that duplication tasks were at the lowest level, whereas translating a pattern from one material to another was more complex. Sarama and Clements (2009), in their description of the developmental progression for pattern and structure, noted that by the age of 4 years, most children can duplicate AB-structured and ABB-structured patterns, and that by the age of 6 years, children can translate patterns into new media.

When copying patterns, one strategy children use is the "direct comparison" strategy, where children copy a pattern by matching one item at a time. Papic et al. (2011) illustrated this strategy by showing a figure of a child lining up the tower he was building right next to the model tower. A second strategy, often used by children when copying an AB pattern, is the "alternation" strategy, because it is used most often when the elements alternate (Papic et al., 2011). Children using this strategy focus on the sequence, making comments such as "first red, then blue, then red, then blue." A third strategy is recognition of the unit of repeat. We will call this the unit strategy. Instead of a child saying out loud that after red comes green (alternation strategy), the child says red–green, pause, red–green, pause, and so on in a rhythmic pattern. The rhythm emphasizes that the two go together as a unit.

Studies of children's engagement with copying and duplication patterns vary with regard to the number of elements presented in the model pattern. For example, Klein and Starkey (2004) reported that when asked to copy the pattern ABABABAB, some children constructed ABABABABBBBBB. We can surmise from this that extra materials were given to the child and not just enough to duplicate the exact pattern. In a different study (Rittle-Johnson et al., 2013), two full units of repeat were presented to the children and children were offered exactly enough identical tiles to build two complete units and one partial unit of the model pattern. Another variable when copying a pattern might be its starting point. This issue arose when children were shown the pattern GGRGGRGGRGGR and asked to use the same tiles and make a new pattern that was the same as the presented pattern (Warren, 2005). One solution was RGGRGGRGGRGG. Note that this solution is actually the original pattern, just read from right to left. The researcher pointed out that the starting point of a pattern might be an important attribute of a pattern in the eyes of children. In other words, is it critical to "read" the pattern from left to right, or can we view it from right to left?

While most studies offer details of the unit of repeat, and the number of repetitions the unit appears in the model pattern, other details, such as the exact number and types of elements given to children with which to copy the pattern, are often left out. This study extends those studies by offering details of the elements (in this case beads) given to children in order to construct the pattern, as well as detailing how those loose elements were presented. Might these factors also affect children's ability to successfully copy a pattern and might they affect strategies used to copy the pattern? This leads us to our first two research questions: (1) Do children use different strategies when copying AB and AAB patterns, and if so, what are they? (2) What can we tell about the ways in which a copying task is presented, and the strategies children use to copy the pattern? Note that because the colors of the beads were different from the model pattern, we do not call this a duplication activity. On the other hand, because children used the same materials (beads) as the model strand, we do not call this a translation task. Instead, we use the term copying.

Often, the same strategies used when copying a pattern may be found when a child is requested to compare two patterns verbally. In our previous study (Tsamir, Tirosh, Levenson, Barkai, & Tabach, 2017), we investigated kindergarten children's ability to express similarities and differences between patterns with the same structure and patterns with different structures. In one instance, when describing the difference between two strands of beads, one with an ABB structure and one with an AB structure, a child said, “Here, there are two purples and one yellow and here everything is one.” We claimed that this description indicates the child’s recognition of the minimal unit of repeat. On the other hand, when asked to describe the similarity between two strands of beads, each with an ABB structure, one child said, “Here are two yellows (pointing to one strand) and here (pointing to the other strand) are two pinks. Here (in the first strand) is one green and here (in the second strand) is one purple.” In this case, the child was matching a section of one strand against a section of the second strand and did not refer to the unit of repeat.

Most studies have either investigated children’s ability to copy or duplicate a pattern, while other studies investigated children’s ability to verbally compare two patterns. However, taking into consideration that “action experience [is] a simple yet powerful tool for learning throughout development and into adulthood” (Kontra, Goldin-Meadow, & Beilock, 2012, p. 732), it might be that results would differ if children were first asked to copy a pattern (i.e., physically act), and then asked to compare (verbally) the one constructed with the model pattern. The next two research questions explore this possibility: (3) Are kindergarten children able to recognize the similarity between the structures of two repeating patterns, one presented to them, and one they constructed as a copy, and if so, how do they express this similarity? (4) Is there a relationship between being able to copy a pattern and being able to express the similarity of the structures of the patterns?

4.3 Methodology

This study took place within the context of a professional development course for preschool teachers, focusing on patterning for young children (for details regarding the professional development course, see Tirosh, Tsamir, Barkai, & Levenson, 2018). Twenty-three preschool teachers participated in the program. All had a first degree in education and between 1 and 38 years of teaching experience in preschools. During the program, teachers were introduced to different patterning tasks as a tool for promoting their mathematical and pedagogical knowledge for teaching patterns in preschool. For the final project of the program, teachers were instructed to choose two of the tasks that were presented and analyzed during the course and implement and video-record those chosen activities with one child (aged 4–6 years). Those videos were then analyzed and discussed together in terms of the children’s solutions.

In this study, we focus on the enactments of a copying activity by the teachers, as well as the children’s strategies when copying the pattern and when describing similarities and differences between the constructed pattern and the model pattern. The task, as presented to the teachers during the professional development course, was as follows:

Present to the child a strand of beads (we will call it a necklace) with an AB structure. Place on the table a string (or wire) and a number of different colored beads—use two different colors and make sure not to use the same colors as those presented on your necklace. Ask the child to create a necklace that has the same pattern as your necklace.

After the child has finished constructing his or her necklace, ask the child the following: "How are these two necklaces the same?" After the child answers, repeat this question, until the child seems to have nothing else to add. Then ask the child: "How are these two necklaces different?" Repeat this question again, as was done for the first question.

Repeat the whole procedure, but this time present a necklace with an AAB pattern.

Although the number of repetitions of the minimal unit of repeat in the necklace to be presented was not specified, during the course it was discussed that children should be shown at least three repetitions of a unit if we wish to encourage the recognition of a pattern.

Ten teachers (T1–T10) implemented this task, each with one child (C1–C10) in a quiet corner of the classroom. We note that the task was presented to the teachers by the teacher educator, who demonstrated how the task might be implemented, along with the instructions given above. Furthermore, this task was not meant to be an instructive task, but instead an evaluation task in the sense that it was meant to assess children's ability to copy and compare various repeating patterns. That being said, as with other tasks (Tirosh, Tsamir, Barkai, & Levenson, 2017), not all teachers implemented this task in the same way, and some teachers did intervene in the middle.

4.3.1 Data Analysis

When analyzing the teachers' implementations, we looked for the following possible variations in the way the task was presented and implemented: the number of repetitions shown in the teacher's necklace, the number of beads offered to the children to make their own necklace, the colors of the beads given to the children for stringing (were there only two different colors and were those colors different than those of the teacher's necklace?), and the teachers' instructions.

Children's strategies when copying the necklaces and the final necklaces were analyzed. Three copying strategies were mentioned above: direct comparison, alternation, and unit strategies. However, since one of the patterns to be copied was AAB, and the colors of the beads did not merely alternate, we use the term "succession strategy" to describe a child who strung one bead after another, based on what comes next. Regarding the final necklace, we noted if it had the same pattern structure as the teacher's necklace, how many repetitions of the unit of repeat were in the final necklace (especially in comparison to the teacher's necklace), if the child ended the necklace with a complete or incomplete unit of repeat, and if the child left over unnecessary beads.

When analyzing children's comparisons of the necklaces, we adopted the coding system of Tsamir et al. (2017), who assigned levels of structure recognition based on children's verbal statements and hand motions when engaging with a similar activity. Thus, children who made no utterances or gestures that referred to the unit of repeat were assigned Level 0. Level 1 was assigned to children who used either a "matching one at a time" strategy or a "succession" strategy. Finally, children who were able to abstract the unit of repeat were assigned Level 2. To summarize, levels of structure recognition were as follows: 0—no recognition of structure, 1—possible recognition of structure, and 2—recognition of structure. Two researchers independently coded the transcripts and were in agreement for all codes, except for one, which will be presented in the findings.

4.4 Findings

This section begins by describing engagement with the AB-structured necklace and then continues by describing engagement with the AAB-structured necklace. It then looks more closely at specific children, describing similarities and differences between the ways they engaged with each pattern. For both tasks, teachers used the same verbal instructions, telling the children to make a necklace like the one the teacher had and then to compare the necklaces.

4.4.1 Copying and Comparing AB-Structured Necklaces

Table 4.1 shows the variations in teachers' presentations and the final necklaces of the children for the AB-structured necklace. Only C3's necklace did not have an AB structure. T1 and T4 gave the children the exact amount of beads in order to copy the given pattern, while T9 gave one bead less. None of the children who were given extra beads strayed from the given pattern, and where there were extras of one color, those extras were left on the table unused.

Out of the nine children that created an AB-structured necklace, six used the succession strategy, taking each bead out of the basket of beads as it was needed, one C bead, stringing it, one D bead, stringing it, and so forth. A variation of this strategy was when one child took out a handful of beads from the basket, but still strung one bead at a time. Two children seemed to recognize the unit of repeat, first taking out both C and D, either taking them out together from the bowl of beads, or one from each bowl of beads (see Fig. 4.1), and then stringing them one at a time.

C7 used the unit strategy consistently. However, C6 started off by taking one bead at a time (see Fig. 4.2a). He did this twice, completing the first unit. He then took out together one unit (see Fig. 4.2b) and strung that unit. After that, he took out

Table 4.1 Copying an AB-structured strand of beads

	The teacher		The child	
	Pattern presented	Unstrung beads	Pattern built	Unstrung beads
1	ABABABAB (4 repetitions)	4 × C, 4 × D in one container	CDCDCDCD	None
2	ABABABAB (4 repetitions)	Many of both C and D in one container	CDCDCDCD (The teacher stopped her)	Many
3	ABABABABA (4 repetitions and 1 extra)	Many of both C and D in one container	CDDCDDCDD (different structure)	Many
4	ABABABABA (4 repetitions and 1 extra)	5 × C, 4 × D in one container	CDCDCDCDC	None
5	ABABABABABABABAB (8 repetitions)	Many of both C and D in one container	CDCDCDCDCDCDCDCDC	Many
6	ABABABABABABA (6 repetitions and 1 extra)	7 × C, 9 × D in one container	CDCDCDCDCDCDCD	2 × D
7	ABABABABA (4 repetitions and 1 extra)	Many of both A and B in separate containers	ABABAB	Many
8	ABABABABA (4 repetitions and 1 extra)	10 × C, 13 × D in separate containers	CDCDCDCDCDCDCDCDCDCDCD	3 × D
9	ABABABABA (4 repetitions and 1 extra)	4 × C, 4 × D in 2 separate piles on the table	CDCDCDCD	None
10	Showed a picture of a necklace 	Many beads of many colors in one container	CBCBCBCBC	Many

Fig. 4.1 C7 takes out C and D beads at the same time

Fig. 4.2 (**a**) C6 takes out one bead at a time, (**b**) then two at a time, and (**c**) places pairs on the table

three more pairs, laid them out on the table (see Fig. 4.2c), and strung the beads one at a time, after which he continued taking out one unit at a time and stringing that unit until he was done. C10 used what seemed like a mix of directly copying a pattern, along with a succession strategy. C10 counted the number of yellow beads in the teacher's picture of a pattern and then counted out the exact same number of orange beads from the bowl. He then counted the number of green beads in the teacher's pattern and counted out the exact same number of green beads as those in the picture. He then proceeded to string the beads one at a time. Note that T7 placed different colored beads in different containers (see Fig. 4.1), while T6 placed both colored beads in one container (Fig. 4.2). C10 placed many beads of many colors in one container. This way of presenting the unstrung beads might also have affected the children's strategies when copying the pattern.

After the children copied the necklace, they were asked to say how the necklace they constructed was the same and different from the teacher's necklace. Only children who successfully copied the pattern were analyzed further. In addition, T10 skipped this step of the task. Table 4.2 summarizes the children's strategies when copying the necklaces, as well as their level of structure recognition when comparing the patterns.

Regarding how children described differences between the necklaces, most children related to the different colors of the beads; others, where it applied, related to the length of the necklace. Regarding their descriptions of similarities, approximately half of the seven children who used a succession strategy exhibited Level 1 recognition of structure. For example, C6 stated, "This (pointing to the A bead in one necklace) is similar to this (pointing to the C bead in the second necklace). And this (pointing to the B bead) is similar to this (pointing to the D bead)." This description essentially mimics the succession strategy. Three of those who used a succession strategy did not exhibit any structure recognition. For example, when C1 asked to say what was the same about the necklace she built and the one the teacher had shown her, C1 replied, "they both have strings." When asked if there was anything else that was the same, she said, "they both have beads." When asked a third time, C1 remained quiet. Most surprising was C7, who used a unit strategy but did not exhibit in her verbal description any recognition of structure. Taking into

Table 4.2 Copying strategies and levels of structure recognition for AB patterns

Child	AB/Not AB	Strategy	Level of structure recognition
1	AB	Succession	0
2	AB	Succession	1
3	Not AB	–	–
4	AB	Succession	1
5	AB	Succession	0
6	AB	Succession and unit	1
7	AB	Unit	0
8	AB	Succession	0
9	AB	Succession	1
10	AB	Succession	(Was not asked to compare the patterns)

consideration that nearly all of the children used a succession strategy, it is not so surprising that none of the children exhibited Level 2 recognition of structure, that is, recognition of the unit of repeat. Regarding Level 2 of recognition, there was one child, C4, who, although we decided not to code her description at Level 2, might be considered on the verge of this level. When asked to say how the patterns were similar, the following exchange occurred:

C4: Here there is a color (pointing to the first bead on the first necklace) and here there is a different color (pointing to the second bead on the same necklace).
T4: And that is similar?
C4: No.
T4: So, how are the two [necklaces] similar?
C4: A color and a different color, a color and a different color, a color and a different color (as she talks, she uses her finger to move along one necklace).

On the one hand, C4 begins by stating the minimal unit of repeat, a color and another color. In addition, she uses general terms, "a color and another color." This generalization might also indicate that she recognizes this as the unit of repeat in both necklaces. Yet, when asked again, she feels each bead in turn on only one necklace, alternating the words, "a color" and "another color," and does not indicate that she recognizes this pattern in the second necklace.

4.4.2 Copying and Comparing AAB-Structured Necklaces

Table 4.3 shows the variations in teachers' presentations and the final necklaces of the children for the AAB-structured necklace.

Taking a look at the children's final necklaces, we note that three children (C5, C7, and C9) did not end up with AAB-structured patterns and that each of those three were different from each other. This might have been four if T6 had not requested C6 to check his strand of beads, causing C6 to correct himself.

Table 4.3 Copying an AAB-structured strand of beads

	The teacher		The child	
	Pattern presented	Unstrung beads	Pattern built (comments)	Unstrung beads
1	AABAABAAB (3 repetitions)	6 × C, 5 × D in one container	CCDCCDCCD	2 × D
2	AABAABAAB (3 repetitions)	Many of both C and D in one container	CCDCCDCCDCC (was stopped by the teacher)	Many
3	ABBABBABB (3 repetitions)	Many of both C and D in one container	CCDCCDCCD (note the reversed direction of the pattern)	Many
4	AABAABAAB (3 repetitions)	6 × C, 3 × D in one container	CCDCCDCCD	None
5	AABAABAABAABAABA (5 repetitions and 1 extra)	Many of both C and D in one container	CDCDCDCDCCDCD (different structure)	Many
6	AABAABAABAAB (4 repetitions)	15 × C, 13 × D in one container	Begins with CDCDCD. The teacher intervenes. Final product CCDCCDCCDCCDCCD	5 × C, 8 × D
7	AABAABA (2 repetitions and 1 extra)	Many of both A and B in two separate containers	AABBAABBAABBABB (different structure)	Many
8	AABAABAAB (3 repetitions)	8 × C, 7 × D in one container	CCDCCDCCDCCD	3 × D
9	AABAABAAB (3 repetitions)	6 × C, 3 × D in two separate piles on the table (In the middle, the teacher adds 5 × D.)	CCDCCDCCDDDDDD (5 extra Ds after 3 repetitions of CCD)	None
10	Showed a picture of a necklace 	Many beads of many colors in one container	CDDCDDCDD	Many

Although C9's final strand has five extra D beads at the end, it might be that C9 added those extra beads because the teacher added those extra beads while C9 was in the middle of constructing his necklace. Perhaps, C9 thought that if the teacher added beads, they must be used. One also might say that C3's strand was different from that of the teacher's necklace, because the teacher had shown ABB and not AAB. However, as this study took place in Israel, where Hebrew is read from right to left, it may be that the child "read" the pattern from right to left.

Table 4.4 summarizes the children's strategies when copying the necklaces, as well as their level of structure recognition when comparing the patterns. Out of the eight children (including C9) who ended with a correctly copied pattern, three

Table 4.4 Copying strategies and levels of structure recognition for AB patterns

Child	AAB/Not AAB	Strategy	Level of structure recognition
1	AAB	Succession	0
2	AAB	Succession and two-one	2
3	AAB	Succession and two-one	0
4	AAB	Succession	2
5	Not AAB (CDCDCDCDCCDCD)	–	–
6	First not AAB and then AAB	Two-one	1
7	Not AAB (AABB)	–	–
8	AAB	Succession	0
9	Not AAB CCDCCDCCDDDDDD	Two-one	1
10	AAB	Succession and two-one	(Was not asked to compare the patterns)

children consistently used the succession strategy, taking out one bead at a time, stringing each bead in its turn. C6 and C9 consistently took out two C beads at once, strung them, and then took out a D bead, strung that bead, and repeated. While this strategy does not necessarily imply recognition of the unit of repeat, it may be more advanced than the succession strategy. In fact, before beginning to string the beads on her string, C2 sang rhythmically to herself "pink, pink, green, (pause), pink, pink, green, (pause), pink, pink, green." We will call this strategy the "two-one" strategy, and consider it to be an in-between strategy, between the succession and the unit strategy. Three children used a mix of the succession and the two-one strategy. None of the children took out three beads together as a unit.

When asked to compare the patterns and say what was similar, (again) there were children who indicated that both necklaces had strings (C8), or they simply did not answer (C1) and were thus coded at Level 0 of structure recognition. C9, coded at Level 1, placed the two necklaces side by side and said, "They are similar because this has two reds (points to the first necklace) and this has two pinks (points to the second necklace). This is yellow (pointing at the first necklace) and this is blue (points to the second necklace)." C2 was a bit hesitant in the beginning. It was only after the teacher's third request that she said the following: "The same… for instance… uh… green green (pointing to one necklace) and this is orange orange (pointing to the second necklace). And this is yellow and this is pink." At this point, it seems that C2 is at Level 1 because she is matching the beads on one necklace to the other. But then C2 continues, "And each time it goes the same and each time it goes the same." We infer from this last part that C2 combines the two beads of one color with the one bead of a second color into one unit that she recognizes will now repeat over and over again. Thus, we labeled her at Level 2.

Above, we claimed that the "two-one" strategy might be considered as an in-between strategy, between the succession and the unit strategy. This seems to go hand-in-hand with the children's verbal descriptions. Of the four children (C2, C3, C6, C9 and not including C10) who used a "two-one" strategy at some point, one (C3) did not exhibit any structure recognition when comparing the patterns, two (C6, C9) exhibited limited structure recognition (Level 1), and one (C2) exhibited recognition of the unit of repeat (Level 2).

4.4.3 Comparing the AB and AAB Patterns

Looking first at the necklaces constructed by each child, nine children successfully copied the AB necklace, while six children successfully copied on their own the AAB pattern (with another two copying the pattern, C6 with help from the teacher, and C9 with extra beads on the tail). Table 4.5 summarizes each child's strategies and levels of verbal structure recognition for each pattern. Three children who used a succession strategy when copying the first pattern used this strategy when copying the second pattern. Three other children who only used a succession strategy for the AB pattern used a mix of succession and two-one strategies, hinting at a possible recognition of the unit of repeat. Interestingly, C7, who had used a unit strategy for the first pattern, hinting at her recognition of the unit of repeat, did not create an AAB pattern like the teacher did for the second necklace, seemingly making up her own pattern (AABB).

Regarding verbal descriptions when comparing two AB patterns, four children exhibited recognition of structure at Level 0, four children at Level 1, and no children exhibited recognition of structure at Level 2. When comparing two AAB patterns, three children exhibited recognition of structure at Level 0, two children at

Table 4.5 Strategies and levels of structure recognition per child per pattern

	AB		AAB	
Child	Strategy	Structure recognition	Strategy	Structure recognition
1	Succession	0	Succession	0
2	Succession	1	Succession and two-one	2
3	–	–	Succession and two-one	0
4	Succession	1	Succession	2
5	Succession	0	–	–
6	Succession and unit	1	Two-one	1
7	Unit	0	–	–
8	Succession	0	Succession	0
9	Succession	1	Two-one	1
10	Succession	–	Succession and two-one	–

Level 1, and two children exhibited recognition of structure at Level 2. In addition, among the six children who were analyzed for both tasks, four children exhibited consistent levels of structure recognition, two at Level 0, and two at Level 1. To sum up, although children had greater success copying the AB-structured necklace, their strategies and verbal descriptions indicated greater structure recognition for the AAB-structured necklaces.

4.5 Discussion

Before we answer and discuss the questions asked in the beginning of the chapter, we note that when comparing the final results of both tasks, findings indicate that copying an AB-structured pattern is easier for children than copying an AAB-structured pattern. This is in line with previous studies that noted that the AB structure is usually the simplest for children to duplicate, copy, and extend (Sarama & Clements, 2009). This study extends those studies by comparing the AB-structured to an AAB-structured pattern, a pattern not specifically investigated previously.

Do children use different strategies when copying AB and AAB patterns? Regarding strategies, for both structures, more children used a succession strategy, either alone or in combination with another strategy, than any of the other strategies. None of the children seemed to use a direct comparison strategy, as suggested by Papic et al. (2011). While C10 did count out in the beginning exactly how many beads of each color she needed, she then proceeded to string them, alternating colors, without glancing again at the teacher's strand. It could be that none of the children used this strategy because of the materials used in the task. When stringing beads, one has to hold one end of the string up and concentrate on getting the string through the bead. Then, one has to concentrate on not letting the strung beads slide off. In other words, unlike copying block patterns with blocks, or laying down tiles to copy a tile pattern, the materials used here might have challenged children to go beyond matching one item at a time. Interestingly, two children used the unit strategy when copying the AB pattern, but none used this strategy when copying the AAB pattern. Perhaps, abstracting the unit of repeat in an AAB is more difficult. Perhaps, taking out three beads at once, especially if all the beads are placed together in one container, is physically more difficult than taking out two beads at once, especially for small hands. An additional strategy came to the fore when copying the AAB pattern, the two-one strategy. This strategy hints that children are not merely looking at one element at a time, but are looking to see how the elements are combined, that two of the same go together and then one different. This strategy might be a prelude to abstracting the unit of repeat and may specifically be promoted by the AAB or the ABB structure.

This study also suggests an intermediate level task that promotes structure recognition. While previous studies have discussed the transition from duplicating, to extending, to translating patterns (e.g., Sarama & Clements, 2009), this study focused on copying, differentiating it from duplicating and translating. For teachers, the discussion of how materials can affect the level of challenge is important, as they scaffold children's development. This relates to our second research question: What can we tell about the ways in which a copying task is presented and the strategies children use to copy the pattern? Looking back at the differences in teachers' presentations, in all presented patterns except one (T7's AAB pattern), teachers showed at least three whole repeats of the minimal unit of repeat. In the one case when this was not done, the child did not copy the teacher's pattern structure. Yet, the amount of beads given to children with which to make their necklaces, as well as the placement of the beads in one or two containers, hardly seemed to affect children's ability to copy the pattern. Only one child added extra incorrect beads, and that was perhaps because his teacher surreptitiously added those beads as the boy was stringing his necklace. However, because children were not asked to specifically reflect on the pattern they had created, how they created it, and what influenced their decisions regarding how to create it, we cannot know for sure what did or did not influence their constructions.

The last two questions we address are related to each other: Are kindergarten children able to recognize the similarity between the structures of two repeating patterns and if so, how do they express this similarity? Is there a relationship between being able to copy a pattern and being able to express the similarity of the structures of the patterns? To begin with, there seemed to be more recognition of the structure for the AAB pattern than for the AB pattern. In fact, none of the children regressed, and two children went from Level 1 to Level 2 of structure recognition. It could be that after having some experience comparing AB patterns, children were more aware of the structure in the second set of patterns. It could also be that the children thought they were expected to verbally describe what they had just done, and thus, their verbal expressions did not necessarily reflect their perception of pattern structure. However, it may be that more complex structures encourage children to take a closer look at structure.

Regarding the relationship between children's ability to verbally describe pattern structure and to physically copy a pattern, we note that not all children who successfully copied the patterns were able to verbalize the similarities in the structures. This is in line with previous studies that found success in other patterning tasks, such as extending patterns or choosing appropriate continuations to a given pattern, does not always go hand in hand with the ability to verbalize pattern structure (e.g., Rittle-Johnson et al., 2013; Tsamir et al., 2017). In fact, in one of our previous studies, we found that some children who did recognize pattern structure when comparing two patterns were not always able to choose appropriate continuations to a given repeating pattern. Yet, in that study, children were asked to compare pictures of two ABB-structured patterns. In this study, children were first asked to copy a pattern and then compare the patterns. Interestingly, the results of both studies were about the same. In both studies, approximately half of the children did not verbalize any aspects of structure, a quarter of the children indicated possible recognition of structure, and a quarter of the children recognized structure. While the current study is limited by the small sample, taken all together, these studies hint at a possible disconnection between physically manipulating objects and verbalizing structure.

In fact, depending on how a child strings the beads when copying a pattern, abstracting the unit of repeat might actually be inhibited, as the child focuses on what bead to place next. A future study might first ask children to compare two patterns and then to copy them. It might be that by first focusing on the pattern, without manipulating items, more children would use the unit strategy when requested to copy a pattern.

As children are still language learners, their verbal comparisons may not always reveal what they are noticing. Teachers can help children verbalize the structure by talking about how they constructed the model necklace. Teachers can also suggest to the children that they discuss what they are doing, while they are in the moment of constructing their necklace, and then reflect on their strategies for copying a pattern. This might help children connect the physical with the verbal. Teachers can also invite children to reflect on how their strategies could be more efficient (e.g., by taking out of the container two or three beads at a time), thus emphasizing the unit of repeat. After this reflection, and after children have had more opportunities to perceive the unit of repeat, they can be asked again to compare two strands.

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Chapter 5
Patterning as a Mathematical Activity: An Analysis of Young Children's Strategies When Working with Repeating Patterns

Miriam M. Lüken

5.1 Introduction

It is at the heart of mathematics education that it is not all about the right solution but about the way a solution is found. We clearly differentiate strategies in arithmetic and look at HOW children solve a task (CCSSI, 2010; Houlihan & Ginsburg, 1981). It is part of a sound mathematical diagnostic to not only survey if a child can solve a task like 8 + 6 correctly, but also to analyze if the child solves it by counting or maybe by using the law of constancy as 7 + 7 (DEET, 2001). In general, we try to guide children's learning from their informal (counting-)strategies to more advanced, formal strategies (Sarama & Clements, 2009).

For repeating patterning activities, which are part of preschool and primary school curricula in many countries, strategies do not seem to be considered as relevant, yet (NCTM, 2013; Sekretariat der Ständigen Konferenz der Kultusminister der Länder in der Bundesrepublik Deutschland, 2005). Recent psychology-based research on young children's patterning competencies mainly focuses on correct solution frequencies or error types (Rittle-Johnson, Fyfe, McLean, & McEldoon, 2013). The relation between patterning skills (measured by correct performance) and early mathematic knowledge/arithmetic achievement is measured and quantified (Lee, Ng, Bull, Pe, & Ho, 2011; Ngyen et al., 2016; Warren & Miller, 2013), and patterning skills are related to both working memory and relational knowledge (Fyfe, Evans, Eisenband Matz, Hunt, & Alibali, 2017; Miller, Rittle-Johnson, Loehr, & Fyfe, 2016). Intervention studies show effects of teaching patterning on children's arithmetic skills (Kidd et al., 2013, 2014; Pasnak et al., 2015). In summary, pattern knowledge seems to be significant for children's mathematical development. But why? Specifically: What is the mathematics in a repeating pattern? How are

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patterning activities mathematically important? And why is looking at children's patterning strategies worthwhile?

To address these questions, the chapter first considers the mathematical foundation of repeating patterns and how it relates to other mathematical contents in primary and secondary school. Second, research studies on young children's patterning strategies are reviewed, and five strategy categories which are the basis of this chapter's data analysis are introduced. The empirical study that is the subject of this chapter, then, explores how young children address patterning activities in different mathematical and nonmathematical approaches and how these strategies develop with age. Finally, selected activities are evaluated with regard to their potential to elicit strategies that address the pattern's mathematical structure.

5.2 Repeating Patterns and Mathematics

A repeating pattern is a periodic sequence of elements that can be reduced to a smallest subset—the unit of repeat—which is repeated in the form of a geometric translation and, thus, creates the repeating pattern. Repeating patterns, therefore, have a cyclic structure (Liljedahl, 2004). Based on this definition, a mathematical approach to a patterning activity would be the recognition and use of the pattern's structure, i.e., the unit of repeat and its cyclic repetition.

A lot of mathematical topics in primary and secondary school are based on this same or a very similar structure. To show this, I will take three different perspectives on the repeating pattern's structure.

First, from a geometric mapping point of view, I interpret the unit of repeat as a basic figure. Every congruence mapping then shares the same idea: a basic figure is multiplied by a mapping with certain specifications. Frieze patterns and tessellations contain a basic figure that is repeated (like in the simpler repeating patterns) by a geometric translation. Furthermore, in axially symmetrical and rotationally symmetrical figures, a basic figure can be found; only the mapping specifications are different.

As a second approach, the unit of repeat is interpreted cardinally as a unit of the same size. This structure forms the basis of every base ten number representation, like the ten- or twenty-frame or the hundred-abacus. Other examples of this structure are multiplication as addition of equal parts, or analog, division as partitioning in equal units. Furthermore, understanding the repetition of a unit is the basic idea of measuring, and even the concept of fraction as part whole is based on the comprehension of partitioning a whole into *equal* parts.

My third perspective focuses on the periodicity of repeating patterns. The experience of a regularly recurring sequence can be made in the sequence of unit digits of any arithmetic sequence; for example, when counting in steps by fives (**5**, 1**0**, 1**5**, 2**0**, 2**5** …) the ones digit alternates. The simplest and in daily life most often used example might be the sequence of digits in our decimal system that recur from 0 to 9 through every place. The decimal expansion of rational numbers is nothing else than a repeating pattern, in which the unit of repeat is even explicitly marked (e.g., $11/37 = 0{,}297297297297\ldots = 0{,}\overline{297}$). Finally, one might consider the periodicity of trigonometric functions (sin(x), cos(x)).

Although the examples given above are only exemplary illustrations, they clearly show that the basic concept of repeating patterns underlies a lot of other mathematical contents up to secondary school. Therefore, activities that help children recognize a repeating pattern's structure may support laying a foundation for an understanding of other mathematical topics.

5.3 Patterning Strategies

There are few studies that specifically look at the process of solving patterning tasks, the way children think about repeating patterns, and the strategies they employ. Three of these studies are described in the following; all three interviewed children aged 3–5 on various patterning tasks.

Rustigian (1976) might have been the first who—under a problem-solving approach—described "response techniques" (p. 189) which children employed in the course of working on the three patterning tasks *reproduction* (copy with the model pattern in view while child responds), *identification* (select a structurally identical pattern), and *extension* (continue the pattern to one side). Her techniques are specific to the task and describe in detail the children's approach to finding a solution. Six different techniques are listed for both the activities *reproduce* and *extend*. Both category lists start with a *random* and end with a *correct placement technique*. In the categories in between, children's responses, on the one hand, focus only on relationships of similarity, for example, repeating a single element of the given sequence. On the other hand, children's responses focus on relationships of similarity and difference. Developmental hierarchies are suggested for the *reproduction* and the *extension* techniques.

Papic, Mulligan, and Mitchelmore (2011) in their study used similar and additional patterning tasks (*copy with and without the model pattern being in view, create, explain*, *extend*). They did not only describe task-specific strategies but formulated five main categories in which the children's solution strategies fell into. Papic et al. (2011) suggested that their strategy categories have an increasing order of sophistication, starting, similar to Rustigian (1976), with strategies where children choose and place elements randomly (*random arrangement*). Strategies that match items one-by-one were frequently observed and make up the category *direct comparison*. The most common strategy in this study was *alternation,* where children focus on the sequence of individual colors. For strategies in the fourth category, children are able to identify and use the unit of repeat (*basic unit of repeat*). Strategies where children demonstrate and express simple generalizations about the unit of repeat were sorted into the most sophisticated category *advanced unit of repeat*.

In the most recent study on patterning strategies, Collins and Laski (2015) suggested that patterning tasks can be solved using either a *one-to-one appearance matching strategy* or a *relational similarity strategy*. For a one-to-one appearance matching strategy, children match superficial features without considering the pattern's underlying structure, e.g., copy a pattern by matching the color or shape of each item in the pattern, one at a time. In order to mentally represent, abstract, and manipulate the unit of repeat, relational similarity strategies are required. The children in Collins and Laski's (2015) study also used strategies where elements are placed randomly, sorted by color or shape, or used for building. This third strategy category is called *off-task errors*.

In an effort to bring the findings of the different studies together and systematically investigate patterning strategies for a variety of patterning tasks, I conducted a longitudinal study describing the development of six children's repeating patterning strategies during their three years of Kindergarten (see Lüken, 2018). The observed patterning strategies could be assigned to five superordinate categories. As these strategy categories form the basis for this chapter's analyses, they are further explicated:

1. The first, most basic strategy category is called *no reference to pattern besides reproduction of the pattern's gestalt.* All strategies where children choose elements based on guessing, personal preference, or random selection belong to this category. A common example is using different colors or shapes than those represented in the pattern while copying or extending a pattern. General characteristics of this category's strategies are that they refer neither to the specific features of the elements nor to the regularity of the pattern. Still, most children will arrange the pattern's objects in a line, thus recognizing the linear arrangement. Their general perception seems to focus on the external shape. Put simply, two patterns are the same if they have the same shape or form (i.e., gestalt).
2. In the second strategy category, *attention to singular characteristics*, children's strategies show an understanding of singular aspects of the pattern. For example, they use either the same colors or the same shapes as in the pattern, or they purposely recreate the same length. However, they do not recreate the pattern's structure. Little regularity can be found in the children's patterns altogether. The general view on patterns seems to be: Two patterns are the same if they consist of the same elements (e.g., colors).
3. The idea of regularity initially becomes visible in the third strategy category *comparison & classification*. Children compare the pattern's elements and highlight sameness within and between patterns on a basic, very concrete level (e.g., "The yellows are the same."; "*Three* purples and *six* blacks here. *Three* yellows and *six* oranges here."). A common strategy for extending a pattern is to look at the pattern's beginning, and to compare and match the extension step by step with the beginning. This procedure shows an emerging sense for some kind of regularity within the pattern, although the specific regularity is not yet graspable for the child.

4. The growing awareness of regularity within the pattern can be observed in the strategies that belong to the fourth strategy category *focus on sequence*. The strategies focus on the relations between successive elements of the pattern, e.g., "Next to green is purple, next to purple is orange, next to orange is green, next to green is purple, next to purple …." Other typical strategies for this category are alternating colors or cycling through the elements of the pattern over and over again, even chanting them rhythmically. The children are aware that the elements are ordered in a regular way, without explicitly grasping the structure. The elements of the pattern are rather seen as strung together. Children are not yet able to break the pattern down into the units of repeat.
5. In the last, most advanced strategy category, *view of unit of repeat*, the strategies show the children's understanding of the pattern's structure. The children know that there is a smallest part that produces the sequence—they are able to identify this unit of repeat and use it during the tasks.

When we now look at these strategies which children employ when solving a patterning task, not all of them refer to the repeating pattern's structure. Only the strategies in the fifth category make use of the pattern's mathematical structure, the unit of repeat, and its repetition. Still, other mathematical approaches become visible in some of the other strategy categories. In category 3, children compare and classify, some even enumerate the number of different objects. These are basic mathematical activities. The strategies in the fourth category show the use and understanding of regularity and succession. Children use the relationship between consecutive elements when predicting an unknown element. In this way, it is only possible to predict the next element in the sequence, and then the next, one element after the other, starting from the last known element (i.e., $a_{n+1} = f(a_n)$). This type of thinking is called *recursive* thinking (McGarvey, 2012; Wijns, Torbeyns, De Smedt, & Verschaffel, 2019). In contrast, *functional* thinking in a repeating pattern context would be to identify the unit of repeat and to use the pattern's structure to predict any element of the sequence (i.e., $a_n = f(n)$; Wijns et al., 2019). This mathematical approach can be seen with the strategies of the fifth category.

Threlfall (1999) is an expert on the topic of repeating patterns in the early primary years. Among his reasons for working with these structures, is his belief that they develop a sense of sequencing and regularity. He has found that one way in which children can succeed in creating or extending a repeating pattern is through a rhythmic approach. As shown above, strategies based on the rhythmic approach would be categorized as a focus on the sequence or recursive thinking, and, therefore, belong to the fourth category. However, in line with my considerations on the mathematics in repeating patterns, Threlfall (1999) claimed that in order to generalize the pattern, a rhythmic approach is not sufficient. It is essential that the child develops a perception of the pattern's unit of repeat. This argument results in my question, how far do children develop this perception in early childhood (without instruction). As Threlfall (1999) already suggested, we cannot infer a perception of the repeating unit from a correct solution but need to consider the child's way to get to her correct (or wrong) solution—the child's strategy.

Therefore, this chapter addresses the following research questions: What strategies are employed by children aged 3, 4, and 5 when solving repeating patterning tasks? What differences in the distribution of frequencies for the various strategies are found between the age groups? And, in particular, to what extent do children use strategies from the fifth category (view of unit of repeat)?

For informing early childhood education settings, it might also be worthwhile to investigate if some tasks are more helpful than others for challenging children to use or refer to the unit of repeat. I am, therefore, going to exemplarily analyze the distribution of strategies for selected tasks and discuss some issues I found with these tasks.

5.4 Method

5.4.1 Setting and Participants

Consent was obtained for 159 children attending 14 kindergartens[1] in a metropolitan area in Germany. The sample consisted of 54 children of age 3 (30 girls, M_{age} = 3;6, *SD* = 2.7 months, range = 2;11–3;11, 76% speaking German as family language), 65 children of age 4 (33 girls, M_{age} = 4;5, *SD* = 3.2 months, range = 4;0–4;11, 82% speaking German as family language), and 40 children of age 5 (15 girls, M_{age} = 5;4, *SD* = 3.9 months, range = 5;0–5;11, 80% speaking German as family language).

None of the participating kindergartens were using a specialized curriculum focused on patterning, and teachers reported doing no repeating patterning activities at all (which I consider representative of German kindergartens). Therefore, it might be suggested that the findings of this study shed light on children's informal patterning knowledge and its organic development.

5.4.2 Tasks and Materials

Eight patterning tasks were designed to test children's strategy use in working with repeating patterns. They were based on items that are long known in mathematics education (Burton, 1982; Sarama & Clements, 2010) and are also published in research studies (e.g., Papic et al., 2011). In several prestudies, the tasks were adapted and tested. The eight patterning tasks are listed in Table 5.1, organized by the order in which they were administered.

[1] German kindergarten comprises the three years before school entry, i.e., children start kindergarten when they are 3 years old.

Table 5.1 Description of patterning tasks

	Task	Instruction
1	Explain	"Please, tell me about the pattern. What's the same? What's different?"
2	Copy (*the model pattern is in view while child responds*)	"Create the same pattern as mine. Use the same colors."
3	Copy (*the model pattern is hidden while child responds*)	"Create the same pattern as mine. Use the same colors."
4	Repair	"A cube is missing. What color is the missing cube?"
5	Extend	"What comes next?"
6	Name the last element	[*The pattern is extended by the interviewer with 3 (AB), 4 (ABC), 5 (ABCC) uncolored cubes*.] "Look, these cubes have lost their color. Imagine we recolor them according to the pattern. What color would be this last cube?"
7	Translate	"Use these counters [*different material and colors*] to create the same pattern."
8	Identify the unit of repeat	"Cut the pattern into parts that are the same."

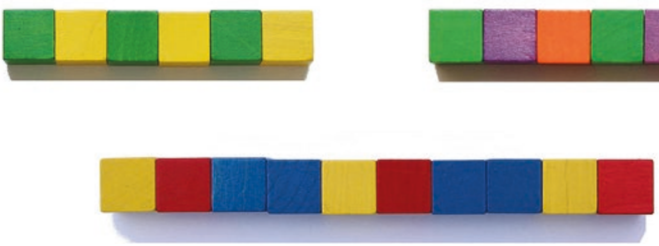

Fig. 5.1 AB pattern (green, yellow), ABC pattern (green, purple, orange) (top), and ABCC pattern (yellow, red, blue, blue) (bottom)

All tasks were conducted consecutively with three repeating patterns that differed according to the length of their unit of repeat. The pattern units contained two (AB), three (ABC), and four elements (ABCC), with only three elements being different in the last, four-element pattern. As shown in Fig. 5.1, the patterns were presented with three (AB) and two (ABC and ABCC) instances of the repeating unit. Whereas the ABCC pattern ended with an additional partial unit, the first two ended with a complete unit. All patterns were constructed with colored wooden cubes, choosing the dimension of color over the dimension of shape. Since 3-year-olds were interviewed, I assumed it easier to communicate about color than relying on children's knowledge of names for different shapes. A brief color-matching test was administered to each child in order to screen for color blindness. If the child did not specify the colors by herself/himself during the explanation task, the interviewer pointed to each colored cube, one at a time, and asked the child to name the color.

No children were excluded for color blindness. Numerous cubes in six different colors were available for the children to choose from during all tasks. For the translation task, counters were offered in four colors that differed from the cubes' colors. For the identification tasks, all three patterns were presented with three full units of repeat. In this way, it was not possible for the child to identify two equal units that were nonminimal.

5.4.3 Data Collection and Analysis

The children worked one on one with a researcher in a quiet room in their kindergarten. The session took 45 min on average, and it was split over two days if the child showed signs of fatigue. The interviews were video-recorded so that the interviewer was released from taking any notes, giving her the freedom to completely engage with the child. In addition, observations of all actions, gestures, and the exact wording were available for the analyses.

The answers to every task with all three patterns (i.e., 24 answers for each child) were coded by strategy and correctness. To establish interrater reliability, a second rater coded 20% of the answers; agreement was high (95%).

5.5 Results

As Table 5.2 displays, all age groups used patterning strategies from the complete range of strategy categories. However, the proportion of categories differed largely according to the age groups. The 3-year-olds mainly used strategies from category 1. Half of the strategies used by the 4-year-olds also belonged to category 1; the other half was distributed nearly evenly among categories 2–4. The majority (63%) of the strategies employed by the 5-year-old children were divided into the categories 3 and 4. Therefore, the older children used more sophisticated strategies than the younger children. Strategies from category 5 were almost never used by the 3- and 4-year-olds; the proportion of category 5 strategies for the 5-year-old children was under 10%. I conclude that with the vast majority of children who started formal schooling 4–10 months later, an understanding of the unit of repeat was not developed yet.

Table 5.2 Distribution of frequencies (%) of strategy categories (all tasks) for 3-/4-/5-year-olds

%	1. No reference to pattern besides reproduction of the pattern's gestalt	2. Attention to singular characteristics	3. Comparison & classification	4. Focus on sequence	5. View of unit of repeat
Overall	81/50/14	11/19/15	5/16/34	2/13/29	1/2/8
AB	57/25/7	26/25/13	10/20/27	6/27/44	1/3/9

Table 5.3 Distribution of frequencies (%) of strategy categories (selected tasks) for 5-year-olds

%	1. No reference to pattern besides reproduction of the pattern's gestalt	2. Attention to singular characteristics	3. Comparison & classification	4. Focus on sequence	5. View of unit of repeat
Copy with view_AB	2.5	2.5	70.0	17.5	7.5
Copy with view_ABC	5.0	0	90.0	2.5	2.5
Copy with view_ ABCC	15.0	0	80.0	2.5	2.5
Translate_ AB	22.5	12.5	17.5	42.5	5.0
Translate_ ABC	22.5	32.5	20.0	15.0	10.0
Translate_ ABCC	45.0	17.5	30.0	2.5	5.0
Identify_AB	5.0	40.0	42.5	5.0	7.5
Identify_ ABC	12.5	27.5	52.5	2.5	5.0
Identify_ ABCC	17.5	22.5	52.5	2.5	5.0

If we explicitly look at the distribution of frequencies for each of the patterns individually, it becomes apparent that for the easiest AB pattern all age groups used more advanced strategies more frequently (see Table 5.2). That is to say, the more complex the pattern was, the less sophisticated were the strategies.

The following results show the finely fanned out distribution of frequencies for three selected tasks (*copy with the model pattern in view, translate, identify*) for each pattern (see Table 5.3). The tasks are selected for their common inclusion in research studies (*copy*) or are suggested to help children focus on the unit of repeat (*translate, identify*). The findings are described and interpreted in turn using the example of the 5-year-olds.

Copy with the model pattern in view: The distribution of frequencies for this task is striking and exceptional compared to all other tasks. Seventy to ninety percent of all strategies employed while copying a visible pattern belonged to category 3 *comparison & classification*. Looking closer at the kind of strategy, it quickly became apparent during the analysis that the majority of 5-year-olds used a compare and match strategy, meaning they executed a one-to-one correspondence. They did this—contrary to the finding of the overall strategy use—although they showed more advanced strategies for other tasks.

Translate: During the analysis, I found it remarkable that 14% of the 5-year-olds were able to correctly translate an AB pattern, but also created an AB pattern when translating an ABC pattern. Looking at their strategy use, it became apparent that the successful children mainly used two strategies. Some of the children argued about similarities in color intensity and matched, in a one-to-one manner, a dark color with another dark and a light color with another light (category 3). The others very confidently created a repeating AB pattern step by step, some even arguing about alternation of color (category 4). A large proportion of the children who were using an alternation strategy with the AB pattern kept the strategy of alternating two colors for translating the ABC pattern (now coded category 2).

Identify: The task *identify* was the only task where it was difficult to fit the strategies into the five strategy categories. It rather became a classification of children's solutions than of strategies. Category 1 comprised solutions where children cut the pattern into parts which were of different length, and no regularity could be found whatsoever. In category 2, the patterns were cut into equal parts (mostly single cubes) without regularity regarding the color. The most common strategy made up category 3: cutting the patterns into single cubes and sorting them by color. There were no strategies or solutions that involved the succession of elements (category 4). In category 5, all children cut the patterns into units of repeat immediately after they had been asked to do so. As this task seemed difficult to explain verbally, all interviewers offered help to every child that produced a wrong solution during the first try. We contrasted two different parts that the child had produced previously and asked if the child thought the parts were really the same. If the child negated, the child was encouraged to try again. If the child cut the pattern correctly into units of repeats during the second try, the solution was then coded category 4. Hence, only the categories 4 and 5 contained correct solutions. This means that a child either had an understanding of the unit of repeat, i.e., the structure of the pattern and consequently was able to cut the pattern correctly or did not have an understanding of the unit of repeat and, therefore, was not able to solve the task *identify* correctly. This finding is different from the other tasks where children could produce very well a correct solution without having perceived the pattern's structure.

5.6 Discussion

This study considers strategies that children employ while working on repeating pattern tasks as an important diagnostic tool for assessing their understanding of the pattern and its structure. The results show that the 3-year-olds' strategies are mainly based on an understanding of the pattern's shape and, to a lesser extent, on the perception of singular, external characteristics of the pattern, like color. Regarding an AB pattern, half of the 4-year-olds show those same strategies. The other half of the 4-year-olds' strategies refer to basic regularities in the pattern (e.g., "The greens are the same, the yellows are the same.") or regard the succession of the specific colors (e.g., "Next to green is yellow, next to yellow is green, next to green is yellow, …"). It is interesting that the 4-year-olds' strategies are distributed nearly evenly among the first four strategy categories for the AB pattern, thus showing a broad developmental range. The 5-year-olds' main strategies belong to categories 3 (*comparison & classification*) and 4 (*focus on sequence)* and are thus displaying an emergent understanding of regularity, order, and succession. These findings go in line with Papic et al.'s (2011) categories *direct comparison* and *alternation,* which were observed most frequently for their overall sample, too.

Hence, the data show huge differences regarding the use of patterning strategies between the 3-, 4-, and 5-year-olds. The tendency is that older children use strategies focused on regularity and structure more often than younger children. It has to be said, though, that the strategy categories do not constitute a developmental stage model. One limitation of the study is that it is a cross-sectional study, and only trends between groups (not a development on an individual level) can be stated. Hence, it cannot be accounted for that every child's strategies will progress through all five categories. Furthermore, by sorting the participants into age groups for an easy comparison, some information gets lost. Children are possibly born only days apart but belong to different age groups. This is a limitation of working in a quantitative way.

What developmental mechanisms might underlie the improving understanding of repeating patterns? As the children did not receive any instruction on repeating patterning other than what they might have experienced at home or observed on television, the development is presumably not due to instruction. A possible explanation could be the general cognitive development in early childhood. Previous studies have shown that working memory is particularly important for helping preschoolers identify, re-create, and learn about patterns (Miller et al., 2016; Rittle-Johnson et al., 2013). Increases in working memory capacity are thought to allow young children to transition from focusing on singular aspects of a task to coordinating attention to two dimensions (Case & Okatomo, 1996). Furthermore, children between 3 and 5 years undergo significant development in their language ability. Children's explanations form the basis of categorization, and with more elaborate language, it is more likely that the interpretation of a child's strategy reflects her true thinking.

Another main finding is the variability of strategy regarding the difficulty of pattern: The more complex the pattern was, the less sophisticated were the strategies which the children employed. This goes in line with findings suggesting that children use more basic strategies in calculating when asked to solve unknown, complex tasks (Siegler, 1988).

Radford (2012) argued that the ability to discern and generalize patterns and mathematical structure in general does not develop spontaneously; rather it depends on cultural influence or some kind of education. Sarama and Clements (2009) noted that being able to recognize the unit of repeat may not develop until the age of 6 years. However, this study found that, in single cases, children as young as 3-year-olds are capable of recognizing structure. Still, only 8% of all strategies used by the 5-year-olds hint at a recognition of the unit of repeat. Since formal schooling in Germany starts when children are 6 years old, and patterns are part of the curriculum, it would be interesting to see the extent to which school education on patterning fosters the understanding of the pattern's structure. It would also be interesting to compare 3- to 5-year-old children's strategy use in countries where formal schooling starts earlier and/or patterning is part of the preschool curriculum.

This chapter also specifically reflects on some selected patterning tasks and how these tasks challenge children to use the pattern's underlying structure. A very common task with children of various ages is to *copy* a pattern while the model pattern is still displayed in front of the child. It is interesting to see that children who are otherwise showing advanced strategies, even referring to the repeating unit, regress to doing a one-to-one correspondence for this task. From this finding, I conclude that this kind of copying task is not challenging the children to use the repeating unit in finding a solution and, consequently, it might be an inappropriate task in an educational setting for older children, i.e., most 5-year-olds.

The task *translate,* where a model pattern is recreated with different materials, is often considered a helpful task for children to shift their attention from the superficially different characteristics of two patterns (e.g., color, shape) to the underlying identical structure (e.g., unit of repeat consists of two different elements) (Hoenisch & Niggemeyer, 2004; Warren & Cooper, 2006). Being able to translate patterns into new media is considered a more advanced stage by Sarama and Clements (2009, p. 331) than being able to copy, extend, or repair a pattern. Rittle-Johnson et al. (2013) even interpret a correct solution in a translation task as the child's ability to abstract the pattern's structure. Hence, this task is called an "abstraction task" (p. 381). Looking at the strategies, it becomes obvious that children are able to correctly translate a pattern into different material without having an understanding of the unit of repeat. This goes especially in line with the findings of Collins and Laski (2015), which highlight the one-to-one matching strategy also for the translation tasks. Furthermore, it seems that some 5-year-olds have developed an understanding of "pattern" as an alternation of two colors, a succession of colors with a certain regularity. Or, put differently, the AB pattern has become the prototype example for a repeating pattern. If asked to produce a pattern, this prototype is reproduced, regardless of its particular structure. In the work on patterning with children, it is, therefore, important to ask for the children's view of similarities and differences between the two patterns and what strategy they used to create the same pattern in the different medium.

The only task children gain a correct solution exclusively with an understanding of the unit of repeat is *identify*, i.e., breaking the pattern down into the repeating units. Although children show many different ideas for breaking a pattern down into parts, there are no differing strategies that lead to the correct solution. As it is the specific aim of the task to put the repeating part and with it the pattern's structure into focus, it might be a valuable task in teaching children the pattern's structure. Similar tasks would be, for example, to circle the unit of repeat, place a string around it (Papic et al., 2011), or build a tower with a repeating pattern and request the child to build the smallest tower that still keeps the same pattern as the one already built (Rittle-Johnson et al., 2013).

This argument leads to learning environments that teachers create around patterning activities in school. What kind of patterning tasks do teachers choose? What explanations do they give, and which strategies do they (sub-)consciously foster?

Do they link repeating pattern's structure to other mathematical content with a similar structure?

My hope is that the findings of this study convince teachers to ask more questions like "How do you do it? How do you know?" and to look closer at the process of patterning instead of the correct solution. The goal is to make patterning more of a mathematical activity.

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Chapter 6
Preschoolers' Ways of Using Fingers in Numerical Reasoning

Camilla Björklund and Maria Reis

6.1 Background and Aim

Young children encounter numerical questions on a daily basis; they are for instance asked how old they will be on their next birthday and most answer without hesitation with a number of raised fingers; they are asked how many days there are until their birthday, which is answered by raising one finger at a time while saying the weekdays in succession until the day of the birthday and a number of fingers are raised. The fingers are used in numerical situations in both cases, but how the fingers are used to end up with the answer differs. In the first case by showing a pattern that likely represents a number, and in the second case by keeping track of the number of days counted. Furthermore, fingers could be used to structure numbers in that the abstract relation between and within numbers becomes visible, as a whole hand can be seen as three and two fingers but also as one and four fingers together. Different ways of using fingers may be based on the situation, as the first examples. However, recent studies bring fore that the children's way of using their fingers may also be an expression of how numbers are experienced with implications for their numerical reasoning proficiency (Björklund, Kullberg, & Runesson Kempe, 2019). This is the issue we aim to discuss here—in what ways do preschoolers, without formal education in arithmetic problem-solving strategies,[1] use their fingers in numerical reasoning, and what do the differences tell us about their arithmetic skills? The research questions we ask are specifically: (1) What meaning of numbers are expressed by the children in their ways of using fingers? And (2) How are children's ways of using fingers related to their arithmetic skills?

[1] Compulsory education with formal mathematics learning goals begins in Sweden the year children turn 7, at the time of the study.

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Arithmetic skills is a broad and complex field of knowledge, which according to the large body of research commonly include knowledge of (symbolic) representations of numbers and basis in principles such as one-to-one correspondence, cardinality, and stable order (Gelman & Gallistel, 1978). Fingers are considered to be an aid in comprehending such knowledge and also in a more instrumental way to keep track when reciting the number sequence (Fuson, 1988). Due to the fact that most people have 10 fingers, these may also support the apprehension of the base-10 system and necessary conceptual knowledge of numbers' part-part-whole relations. However, empirical research is needed to find out if and how children's use of fingers are related to their proficiency in solving basic arithmetic tasks.

Fingers are known to be used globally as a kind of natural tool to keep track of counted items (Ifrah, 1985) and as representational systems that different finger patterns carry (Bender & Beller, 2012). Finger use is however not an innate ability since there is a broad variety in how fingers are used in different cultures and also to the extent fingers are used in the first place. The more advanced ways of using fingers for counting purposes, such as expressing tens with different finger patterns, have in common that they bridge the concrete countable items and the cognitive act of making use of fingers as representations for the quantity of items. There is in other words both a cognitive and a cultural dimension of finger use.

Our intention in this chapter is to contribute to a recent debate about finger use for mathematical problem-solving (Moeller, Martignon, Wessolowski, Engel, & Nuerk, 2011). Particularly neuro-scientific findings advocate the use of fingers, whereas researchers in mathematics education show a prolonged use of fingers among students with mathematical difficulties in later school years (Geary, Hoard, Byrd-Craven, & DeSoto, 2004; Ostad, 1998), which could indicate a relation to children not developing necessary knowledge about the abstract feature of numbers. However, the literature, both earlier work in the field of mathematics education (Fuson, 1988) and recent work in neuro-psychology (Berteletti & Booth, 2015), are restricted to advocating for or against (any kind of) use of fingers (see also Boaler, Chen, Williams, & Cordero, 2016). The findings supporting the benefits of finger use for arithmetic problem-solving are interesting for the field of mathematics education, but we wish to discuss distinctly different ways of using fingers in numerical reasoning, to clarify the benefits and hindrance of finger use in early childhood. To study this, we used observations of Swedish 4- and 5-year-olds, who have not before attended formal education in arithmetic strategies or counting. Our approach is experiential, based on Variation Theory of Learning (Marton, 2015; Marton & Booth, 1997), meaning that we do not only observe the ways in which children act but also interpret their actions as expressions of *ways of experiencing* the numbers they encounter in arithmetic tasks.

6.2 Research Review

The literature on children's arithmetic skills development is vast (see Baroody, Lai, & Mix, 2006; Baroody & Purpura, 2017; Carpenter, Moser, & Romberg, 1982) and many have described children's use of fingers in this process (e.g., Baroody, 1987;

Fuson, 1982). First, there has to be made a distinction between using fingers to represent (cardinal) numbers and as (iconic) symbols for numbers. The former allows the child to add units in a cumulative manner, producing a set, while the latter is a static image connected to a number word, which may or may not have an ordinal meaning. Research generally treats finger use as if fingers carry a numerical meaning and thus represent cardinal numbers, but exceptions of different finger systems and even absence of finger use are found in different cultures. Finger use for calculating purposes is thereby not evidently naturally occurring, it is rather a culturally induced tool that has to be learnt (Bender & Beller, 2012).

The use of fingers is evidently prosperous when keeping track of a smaller amount of items, but there is a natural constraint in the limited number of fingers on our hands. There has to be an extension of fingers' meaning and instrumental use that extends the perceived fingers at hand and include principles of the numerical system (Steffe, Thompson, & Richards, 1982). Children's use of fingers in counting activities starts from counting with perceptual unit items. This is the most primitive way of counting because the counted items have to be present. All numerical reasoning is then perception-bound until the child develops a more advanced way of counting with figural unit items, such as fingers that may represent items that are not present. Nevertheless, Steffe et al. conclude that even though children can make use of figural units (fingers) to represent quantities, they often fail in solving arithmetic tasks because of their inability to coordinate the problem-solving. Research of early numerical encounters where fingers are used in different ways gives strong support to this by concluding that some ways of expressing numbers with fingers are more prosperous than others. Eventually, most children stop using their fingers once they learn number facts and advanced arithmetic strategies. Children's finger use in arithmetic problem-solving and how children experience their fingers is nevertheless found to be related to arithmetic proficiency (Reeve & Humberstone, 2011). There is also evidence of a relationship between number knowledge and body knowledge long after a person has stopped using his/her fingers in arithmetic problem-solving. Rusconi, Walsh, and Butterworth (2005) conclude, based on neuro-cognitive studies, that finger calculation is "an almost universal stage in the learning of exact arithmetic" (pp. 1610).

Even though using fingers is commonly observed, it is nevertheless a rather complex ability since using fingers in arithmetic problem-solving requires that the child perceives fingers as representations of numbers, giving a cardinal and ordinal value to quantities that may not be present. Steffe et al. (1982) show how the perceptual attachment (what you count have to be counted *on,* literally) hinders children, particularly when a task is presented verbally only or the number range extends ten. Knowing that it is possible to use fingers as representations for numbers is a prerequisite but not enough to aid the problem-solving. The child also needs to coordinate the problem in that finger patterns or extension of fingers while counting is a way to construct *sets* to which other sets can be added or removed. This is significant knowledge for arithmetic skills development since constructing sets and making use of the part-whole structure created as finger patterns are found to be decisive for many preschool children's arithmetic proficiency (Björklund et al., 2019).

Fuson (1982, 1988) shows how children use fingers as an aid for counting when numbers exceed the subitizing range (more than the child can perceive in a non-counting process, approximately three units). Baroody (1987) also makes substantial efforts to systematically describe how young children use their fingers and points out the benefit of structuring numbers with finger patterns as “shortcuts” in counting (see also Björklund et al., 2019). Typical ways of using fingers are as follows: *keeping track* such as counting (up) the first addend on one hand, then the second addend on the other hand, and finally counting all fingers raised together, thus keeping track of the numbers added. According to Baroody (1987), this is a way of producing a pattern by adding single units. It also works when subtracting but is difficult to carry out, as the child has to count backwards and simultaneously keep track of counted units (thus keeping track with fingers unit by unit). Baroody also describes what he calls “pattern recognition,” where the child does not need to count to know the number of units. The child then recognizes the first addend as a pattern on the fingers, then the other addend and furthermore recognizes the total number of both parts when seeing both patterns together on their fingers. This latter strategy is also described by Brissiaud (1992), but he argues that this way of recognizing finger pattern sets (as called by Brissiaud) should precede any counting strategies and be presented as the primary way of determining numbers by young children.

There are also other scholars advocating a structural approach, such as Davydov (1982) and Schmittau (2004), in that constructing units (primary to counting singles) facilitates the advanced way of perceiving numbers as part-part-whole relations, which, for example, by Baroody (2016), is a breakthrough in the development of arithmetic skills. Gattegno (1974) presented an overview of the number structures to be found in finger patterns and argued that systematic training enhances children’s flexibility in recognizing finger patterns and their counterparts, for example, in a ten structure (6-4 and 4-6 by folding six fingers, leaving four unfolded and vice versa, promoting commutativity).

As shown above, there are observations of children’s different ways of using fingers and that there may be a relation to arithmetic skills, either as an instrumental tool to keep track of counted items or as a bridge toward more advanced ways of handling numbers in arithmetic problem-solving. In educational studies, it has been found that finger use is more common among children with mathematics difficulties in primary school (Geary et al., 1998), but Neuman (1987; 2013) makes a clear distinction that it is not finger use per se that induces mathematics difficulties. Her studies are in agreement that finger use may indeed become a cumbersome way to solve arithmetic problems, but it is rather due to different ways of experiencing numbers that hinder children in their arithmetic development. She, in line with Gattegno (1974), supports the idea that children benefit from using finger patterns, but in ways that promote seeing number structure and relations between and within numbers—in other words, experiencing fingers as representing a part-part-whole relation as opposite to experiencing fingers as separate single units. In a recent study (Björklund et al., 2019) with 5-year-olds, it is shown that children who use finger patterns to structure arithmetic tasks and thereby use fingers as a strategy to make parts and whole "visible" are most often successful in their arithmetic problem-solving. Those who cannot use fingers in this way never solved the problems in that study.

6.3 The Study

This study is part of a larger project (FASETT),[2] in which we investigate a systematic pedagogy that promotes a structural approach to numbers through the use of finger patterns and its effect for developing arithmetical skills in preschool years. This leads us to the present study, where we direct attention to the relation between finger use and numerical reasoning. Our study is based on Variation Theory of Learning (Marton, 2015; Marton & Booth, 1997) and thus based on an experiential approach to how children learn the meaning of numbers. According to Variation Theory, learning is the change in ways of seeing (or experiencing) phenomena in the surrounding world. A certain way of seeing, for example, numbers constitute those aspects of numbers that a child is able to discern in a particular situation. Some aspects of numbers are prominent when counting single units (for example, the ordinal feature of numbers) and others come to the foreground when showing a finger pattern in an instant gesture (for example, cardinality and part-part-whole relations). If the child is only able to see some, but not other aspects, this may limit what the child is able to do with numbers, that is his/her arithmetic skills. In our study, this theoretical framing induces that when children use their fingers in a problem-solving situation, it is interpreted as an expression of the way they experience the task and the numbers presented in the task.

An interview with children was conducted to screen their knowledge of numbers and arithmetic skills and in particular to find out what meaning of numbers they expressed when using fingers to solve arithmetic tasks. These interviews were task-based, covering basic number knowledge, finger patterns as representations of numbers and arithmetic problems within the number range 1–10. No manipulative material was present, or pen and paper, the children were though encouraged to use their fingers if they wanted to since this was one of the questions of interest. The children were 4–5 years old. The interviews were conducted in the children's own preschool by trained interviewers. Each interview lasted for approximately 20 min. Of the participating children in the FASETT project ($n = 103$), we had 99 children with their legal representatives' written consent to document the interview with video selected for this study. Since the purpose was to investigate finger use in detail, those children who we could not collect video data from were excluded. The interviews were thus video-recorded to ensure reliable data for analyzing finger use, acts that can sometimes be very subtle or quick.

[2] FASETT is funded by the Swedish Research Council grant no. 721-2014-1791.

The study design consists of two parts. To find the different ways of using fingers and how this relates to the children's ability to solve arithmetic tasks, we first made an overview of the participating children's basic finger-use abilities related to how they experience numbers in two "show a number" tasks:

(a) Show me how old you are on your fingers.
(b) Show me six fingers.

When children learn that specific number words are used to describe quantities, they begin to understand that numbers are unique for a certain amount of items (Sarnecka & Gelman, 2004). The most common way of testing children's understanding of numbers' cardinal feature is by asking them to "give-a-number," that is to create a set of objects, usually by counting one item at a time, adding them to complete the requested number of items (Wynn, 1990, 1992). Give-a-number tasks do however direct attention to counting and producing a set in additive manners, while in our study we ask the children to "show" a number, which is more likely to induce an instant finger pattern rather than adding single units. For the purpose of our study, this distinction was important since it allowed the children to express their way of seeing numbers without being biased into counting procedures if another way of creating numbers was a possible alternative.

Secondly, for this particular study, we chose four arithmetic tasks from the interview for a thorough analysis of the children's ways of using fingers in arithmetic problem-solving. The same tasks were given to all children with an exception if a child had given incorrect or no answers to the first two tasks (the interview was then ended):

(c) You have two sea shells and receive five more, how many do you have then? (2 + 5=).
(d) If you have ten candies and eat six of them, how many are left? (10–6=).
(e) You have three glasses, but are going to set the table for eight people, how many more glasses do you need? (3 + _ = 8).
(f) On the morning of your birthday party, you blew up balloons. At the party, three balloons broke, and there were only six balloons left. How many balloons did you blow that morning? (_–3 = 6).

The tasks were developed to cover different part-part-whole relations: (c) straight forward addition of two parts, (d) removal of one known part from the whole, (e) missing addend, and (f) missing whole. These tasks thus cover both addition and subtraction, of which (e) and (f) may be solved by the inversion principle. As Vergnaud (1979) states, the construct of a task is important to consider when studying children's concept knowledge. Our tasks thereby cover numbers 1–10, which were all given verbally and with no other manipulatives than children's own fingers since we aimed to study the way of experiencing the tasks' part-whole structure, to which finger patterns or counting strategies were possible to enact.

6.4 Analysis

In accordance with the Variation theory of learning (Marton, 2015), we conjectured that children act in a problem-solving situation in ways that reflect their way of experiencing, in this case, the numbers in the arithmetic task. Answers to the "show a number" tasks were thus coded in accordance with the child's preferred way of creating the number on their fingers: Counting single units to six, showing a finger pattern instantly and no/wrong answer. We then selected observations from those children who spontaneously, or after being encouraged by the interviewer, used their fingers when encountering the four arithmetic tasks. Of the 99 children, 59 were finger users and 40 children chose not to use their fingers when trying to solve the tasks. The 59 finger users were thus selected for further analysis. The arithmetic tasks performed by the finger users were then coded for correct/incorrect answers, followed by a thorough qualitative analysis of different ways of experiencing numbers represented by fingers.

In total, our data consist of 133 unique observations of finger use in arithmetic tasks that is 59 children × the number of tasks they tried to solve by using fingers. The children did not necessarily use fingers in all four tasks they were given. We then did a microanalysis of each observation to differentiate the ways in which children were using their fingers. The observations were coded based on the purpose of using fingers that the children expressed. Based on Variation theory, we then interpret these ways of using fingers as qualitatively different ways of experiencing numbers. Three categories emerged in the analysis:

1. Fingers as an image of numbers.
2. Fingers to create numbers of single units.
3. Fingers to visualize the structure of numbers.

Fingers as an image of numbers can be related to what Bender and Beller (2012) describe as iconic symbols, which are culturally informed ways of representing a number word, however, does not necessarily have a cardinal or ordinal meaning. *Fingers to create numbers of single units* is an expression of added ones constituting a number. Fingers are then figural units that can be counted or used to keep track of counted single units (c.f. Steffe et al., 1982 "figural units"). Last, *fingers to visualize the structure of numbers*, which also has been observed by Baroody (1987) as pattern recognition in arithmetic problem-solving. The three categories of ways of using fingers were then related to the success rate in the arithmetic tasks, which strengthened the interpretations of the children's ways of experiencing the numbers in the tasks. We will in the following section present these categories with examples from the data and in relation to correct/incorrect answers given, to discuss what the ways of using fingers entail for arithmetic problem-solving skills in the early years.

6.5 Results

All children were able to show with their fingers how old they were (either 4 or 5) and furthermore, all of them showed this with an instant finger pattern (see Table 6.1). The second question concerned the ability to create a finger pattern: 8 of the children did not attempt to show any fingers, or, created a finger pattern that did not represent six as in six raised fingers. In 15 cases the children counted and raised one finger at a time ending when six was uttered and six fingers were raised. Finally, 36 children showed a finger pattern of six instantly, without counting, either as a whole hand and a thumb on the other hand or any other finger combination of six. This overview was important as a basis for the thorough analysis of how different ways of using fingers influence on arithmetic problem-solving skills. Now, we knew that all the children were able to show a set of fingers, but there were differences in abilities and ways to create a finger pattern as a representation for a specific number.

The findings from the "show a number" tasks directed our attention to the differences found in how children created numbers with their fingers. To create a set (number) represented with fingers may indicate an understanding of the cardinal meaning of numbers. If so, this would have an impact on the children's ways to try to solve arithmetic tasks. As Steffe et al. (1982) have shown, there seems to be a critical change when children not only create quantities and represent numbers with fingers but also are able to coordinate patterns as addends in arithmetic tasks. However, this initial task could not illuminate if the children experienced numbers expressed as finger patterns in a cardinal sense.

There are three main categories of ways to use fingers found in our analysis. The differences are seen as expressions of different ways to experience numbers. And accordingly, there are differences in the success rate found within the categories (see Table 6.2): Fingers as an image of numbers (category 1) and Fingers to create numbers of single units (category 2) entail mostly incorrect answers. These categories are distinctly different from using fingers to visualize the structure of numbers

Table 6.1 Ability and way to create a number among the 59 children who used fingers in their arithmetic problem-solving

Question	No answer or wrong answer	Counting single units to six	Showing a finger pattern instantly
Show me how old you are on your fingers	0	0	59
Show me six fingers	8	15	36

Table 6.2 Frequency of correct and incorrect answers sorted by way of using fingers

Category	Correct answers	Incorrect answers	Total
Category 1	1	16	17
Category 2	2	20	22
Category 3	61	33	94

(category 3), which most children express themselves doing and consequently solve many of the given arithmetic tasks. These results show that there are significant differences in how children are using their fingers and that finger use per se cannot determine whether a child is able to solve an arithmetic task or not.

In the following text, we will describe the characteristics of these categories and also proceed to a more detailed analysis of differences in meaning within and between the categories.

6.5.1 Fingers as an Image of Numbers

A total of 17 observations (13%) show that children are using fingers to present an answer to the arithmetic task that is characterized as an image of one of the numbers heard in the task. No arithmetic structure or operation that would be necessary for solving the task is addressed by the children. Nevertheless, numbers (in words or in fingers) are related to the question "how many."

We suggest that fingers are, in Excerpt 1, used as an image of a number, which is expressed in the example and utterance "is this six?". The only case where a correct answer is given in this category we interpret as a "lucky guess" since the child is holding his hands with some fingers folded unchanged from the previous task, looking at his hands and says "look, seven!" which happened to be the correct answer (2 + 5 = _).

These observations and similar cases are interesting because the child is constructing a finger pattern which most often is assumed to have a cardinal meaning. The majority of children in this category instantly show a finger pattern for the

Excerpt 1

Interviewer	At the party, three balloons broke, and there were only six balloons left. How many balloons did you blow up that morning?
Charlie	*Mmm. What*?
Interviewer	Well, three were broken and six were whole.
Charlie	(folds up thumb, index and middle fingers on the right hand, then all five fingers on the right hand) *Is this six? No, it's five*. (folds up thumb on the left hand and then all ten fingers) *Is this six?*
Interviewer	How did you show six before?
Charlie	*One, two* (starts unfolding thumb and index finger, then showing all fingers but left thumb) *like this.* (lays down her right hand on the table but holding up the other hand with four raised fingers) *four.*
Interviewer	Yes, four. Mmm, how many balloons did we have from the beginning then?
Charlie	*Three*.

number “six” in the interview. However, when finger patterns are used as an image of a number, it does not aid their numerical reasoning since the image has no cardinal meaning and thus cannot be considered a part included in a larger quantity. The images/numbers do not seem to be related to other images/numbers that would be expected in a true cardinal sense. The children rather pick up a number heard in the task and represent it with their fingers, as an image, not as a *composite* set.

6.5.2 Fingers to Create Numbers of Single Units

Fingers are by some children used to help create a number, by counting each finger as a single unit (17% of the observations). The order of the counting sequence seems to be experienced as closely attached to each finger, as the children start counting from one and either touch each counted finger or extend one finger for each said counting word. It differs thereby from the previous category in which children instantly showed or tried to show a finger pattern. This category is building on the counting procedure where an answer is produced by counting one finger at a time. However, the children using fingers in this way do not necessarily relate the created number or finger pattern to other numbers or try to operate with the created pattern such as to add another addend by “counting on” (see Steffe et al., 1982).

This child acts in a typical way for this category. A number (either some number heard in the verbal task or a seemingly random number) is created by counting fingers as single units. This way of experiencing numbers does not emphasize any arithmetic operation or structure to operate on. It is rather the question “how many” that prompts a specific procedure of counting on the counting sequence in correspondence with single fingers.

The child, in Excerpt 2, created a number by adding single units. However, the result of the addition cannot be seen as related to the task at hand. In Excerpt 3, on the other hand, the child is struggling to create the number six, but seems to

Excerpt 2

Interviewer	You have two sea shells and receive five more, how many do you have together then?
Sam	(looks at his hands with all fingers unfolded) *A lot.*
Interviewer	Is it possible to solve it?
Sam	*Yes.* (points with his right index finger on each finger on the left hand while counting) *One, two, three, four, five.* (changes to pointing with his left index finger, continuing counting and pointing at each finger on the right hand) *Six, seven, eight, nine, ten.*
Interviewer	Is it ten?
Sam	*Yes*

Excerpt 3

Interviewer	When your birthday party is over, three balloons broke, and there were only six balloons left. How many balloons did you blow up that morning?
Robin	(starts with all fingers unfolded, then folds all fingers but the thumb on the left hand and folds the right thumb, leaving four fingers unfolded on the right hand, then looks at the interviewer)
Interviewer	You had six and three were broken. How many were there from the beginning?
Robin	(unfolds index and middle finger on the left hand, now showing three fingers on the left and four fingers on the right hand, then folding back and remaining the left thumb and four fingers on the right hand again) *This is six. With four.* (moves his hands closer together) *Or should the thumb be there as well?* (unfolds the right thumb and counts each finger) *One, two, three, four, five, six. Then I need it.* (starts counting all fingers again) *One, two, three, four* (points at his left thumb) *What is this then?*

experience that there is a relation between numbers and the finger patterns he creates. He seems to know the pattern for "four" but is unsure whether the thumb should be included in creating "six" or not. Numbers, as they are presented on the fingers, seem to lack cardinal meaning as he asks "what is this called then" pointing at his unfolded thumb on his left hand (the "sixth" finger). Similar observations have been made also by Brissiaud (1992) and Neuman (1987), by them called "word tagging" or "numbers as names," meaning that each finger is given a number name, that *is* the number. Such meaning in the use of fingers will not aid the child in experiencing the arithmetic structure or that numbers can be added or subtracted. Images or names cannot be added to another image or name in the same way as numbers with a cardinal meaning can. The difficulty the children experience when trying to create a number with their fingers by adding single units is to know when the added "ones" are enough and if the child recognizes a finger pattern ("is this six") to know which fingers (ones) to include in the pattern. Consequently, the children do not experience numbers in ways that would help their solving arithmetic tasks.

6.5.3 Fingers to Visualize the Structure of Numbers

The two categories presented above are similar in the sense that the children act to create some representation of numbers with their fingers (however differently). These stand in bright contrast to the last category (70% of the observations), in which children are using their fingers to *visualize the structure of the numbers* in a task and thus experience a part-part-whole relation in the arithmetic task. By structuring the parts and the whole on their fingers the children act in somewhat different ways to find missing addends or adding units to create a sum. It is necessary to create a set on the fingers among these children too (as the previous categories were examples of), but the difference is shown in that these children seem to experience numbers not as isolated images or as single units (on the counting sequence), but as *composite sets* that are possible to extend or divide to make new (larger) sets or to find a part that is missing.

The child, in Excerpt 4, creates the numbers on his fingers as patterns, without counting. When folding the set representing the eaten candy, the missing part remains visible to him and he seems to recognize the relation between the whole (ten), the first part (six), and the missing part (four).

Our observations show that even though some children "see" the structure of numbers as a pattern of composite sets, some children need to create the sets, for example, by counting single units. Still, when they experience that the created number can be seen as part of a larger whole, they use their fingers to structure numbers and the relations within the given task.

Children in this category create finger patterns that are resembling the patterns shown in categories 1 and 2. However, the patterns are experienced in a distinct way, as the children are, in the former categories, not able to experience their pattern as a

Excerpt 4

Interviewer	If you have ten candies and eat six of them, how many are left?
Kim	(puts both hands on the table) *I eat six* (folding all five fingers on the right hand and left thumb, looks at the remaining four unfolded fingers) *Four!*

Excerpt 5

Interviewer	You have three glasses, but are going to set the table for eight people, how many more glasses do you need?
Tintin	(shows index, middle and ring finger on the left hand). *I think I have to count* (pointing and counting now unfolded little, ring and middle finger on the left hand) *One, two, three* (then pointing at index finger and thumb) *one, two* (switches to the right hand's thumb, starting to count) *three, four. Is it this many. Eight?*
Interviewer	Eight children.
Tintin	*Okay, I have to count them all then.* (pointing and counting all fingers on his left hand, continuing on the right hand's thumb, index and middle finger) *One, two, three, four, five, six, seven, eight.* (folds the last three counted fingers, unfolds the fingers again) *I have already three glasses on the table.* (folds the five fingers on his left hand, shows the fist to the interviewer) *Five more!*

composite set or as a part-part-whole relation. When children are able to experience the finger pattern as composite sets, they are able to discern the relation between and within patterns, which seems to aid them in solving the task, giving correct answers in 65% of the observations within this category.

In summary, children who create finger patterns as representations for numbers (either as images or as single units) have to extend their experiences to include the numbers/patterns in a structure in which the relationship to other numbers is the basis for their operation. Such an extension allows for more prosperous ways of solving arithmetic tasks, in which fingers are powerful tools for visualizing this structure.

6.6 Conclusions

The ways children use their fingers are in the vast majority of cases consistent over the different tasks. In those rare cases (9 of 59 children) where the same child uses fingers in different ways across tasks, it can be noticed that the child starts seemingly with a wild guess but changes her strategy in the middle of the task, or the child starts creating or illustrating a number but does not continue using the created numbers to model the task in similar ways as in the other tasks. Reliability in this sense is very difficult to establish since the children's acts are sometimes very subtle and sometimes interrupted which may give a false outcome of the child's expression. However, our aim is not to categorize children but to analyze unique observations of different ways of using fingers, to which our 133 observations provide a sufficient basis of data.

The observations show that merely creating a finger pattern is not sufficient to solve arithmetic tasks. Critical for solving the tasks is an awareness of numbers as a composite set, which can be created either by adding or removing single units (counting) or by experiencing and structuring the part-part-whole relation as finger patterns. However, by merely looking at children's ways of creating a finger pattern, it cannot be interpreted how the child experiences the meaning of numbers and thus is able to solve an arithmetic task. It is in the problem-solving act that the meaning appears in the way the child makes use (or does not make use) of the created set. Similarly, a child may very well create a number on her fingers by counting single units/fingers, but unless the cardinal meaning is experienced in the number/created finger pattern, it does not help the child experiencing the arithmetic relations in the task. These findings support the common knowledge of part-part-whole as an important aspect in arithmetic problem-solving (Baroody, 1987, 2016; Björklund et al., 2019). Our study though contributes to the field of knowledge with empirical evidence of how this is manifested by preschool children who have not yet taken part in formal mathematics education or been taught counting strategies of any formal kind.

By tradition, it is accepted for younger children to use fingers as an aid in calculations in their early years. Our interest in this field is thus pedagogical—the important distinction found in our study is that seemingly similar ways to use fingers, sometimes are used to represent isolated numbers and sometimes to operate on the part-part-whole relation within an arithmetic task. Fingers are excellent aid to create (keep track of) units that compose a set or to represent numbers as a pattern, but these sets/patterns need to be extended toward carrying a relational meaning (as parts and whole), as the observations in category 3 show that quite many preschool children are doing. These findings have significant pedagogical implications to educational practice with young children because counting on your fingers or illustrating numbers as images does not per se facilitate numerical reasoning that is necessary for arithmetic problem-solving.

Steffe et al. (1982) conclude that even though children can make use of fingers to represent quantities, they often fail in solving arithmetic tasks because of the inability to coordinate problem-solving. Recent research also shows that some ways of expressing numbers with fingers are more prosperous than others—critical seems to be children's ability to recognize numbers' part-part-whole relations (Björklund et al., 2019). We can relate our findings in this study to these observations in that using fingers as an image of numbers (category 1) or as single units (category 2) will not support children in finding answers because of their lack in experiencing these sets as related to other sets. Our contribution to these known observations is the way we explain *how* seemingly similar ways of using fingers differ in numerical meaning.

In our experience, counting single units or presenting a finger pattern as a representation for numbers are common acts in preschool. Furthermore, it is observed in most cultures that different counting sequences on fingers are developed to keep track of counting (Bender & Beller, 2012). Nevertheless, finger counting is not an innate ability but a culturally implemented tool for enumeration, which can be used for structuring numbers in prosperous ways. It would thus be beneficial, based on the results of this study, to promote the use of fingers in arithmetic problem-solving, but in a goal-oriented way where arithmetic structure (part-part-whole relation) is emphasized.

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Chapter 7
Young Children's Cross-Domain Mapping of Numerosity in Path Navigation

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7.1 Introduction and Background

Within a continuous view of cognition that sees learning as a synthesis of inborn and constructed representational systems, there is consensus that cognition is grounded in a limited set of innate domain-specific core knowledge systems for representing objects, actions, numbers and space (Spelke & Kinzler, 2007). One of these cognitive capacities supports navigation in the three-dimensional room, while the Approximate Number System and the Object Tracking System support spatially organised non-verbal representations of the cardinality of estimated values of sets and the exact quantification of small sets, respectively (Feigenson, Libertus, & Halberda, 2013; Piazza, 2010). These reciprocal functioning neurocognitive mechanisms account for basic number sense in space and time, and they guide and constrain the learning of the symbolic and cultural aspects of the number concept (Feigenson, Dehaene, & Spelke, 2004; Piazza, 2010). A model that reflects appropriation of cultural tools for representing numbers is the knower-level theory (Lee & Sarnecka, 2010; Wynn, 1992), which is based on the Give-N-task (Schaeffer, Eggleston, & Scott, 1974; Wynn, 1990). Typically, the Give-N-task uses a puppy to familiarise the test situation, and the experimenter asks: "Can you give the puppy one/two/three/four/five item(s)?" Based on the child's production of the requested set, the experimenter asks: "Is that one/two/three/four/five item(s)?" A child who shows consistency in producing a maximum of two items on the Give-N-task is labelled a C2-knower. The knower levels one to four (i.e. C1–C4) are categorised as subset-knowers (Le Corre & Carey, 2007), which reflects that the mapping is constrained by a limited-capacity system associated with the Object Tracking System and hence also subitising (Kaufman, Lord, Reese, & Volkmann, 1949). In order to illuminate that the cardinal meaning of higher number words is learned in a

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qualitatively different way, the group that master the use of the cardinal principle (Gelman & Gallistel, 1978) for exact enumeration are named CP-knowers. Wynn (1990, 1992) reported that most children do not understand the cardinal word principle until the age of 3.5–4 years, and the study of Levine, Suriyakham, Rowe, Huttenlocher, and Gunderson (2010) predicts that the conceptual breakthrough of understanding that a number word reflects the set as a whole occurs at 46 months of age. Hence, an educational challenge is to create models that enhance transformation between inborn aptitudes and culturally achieved representations of numbers (Gallistel, 2011), and in particular for subset-knowers who struggle to master exact enumeration. Informed by discoveries in neurocognitive science, which argue for the advantages of guided approaches that build on knowledge of the brain's cognitive architecture (Kirschner, Sweller, & Clark, 2006), the participants (mean age 51 months) were, prior to this study, engaged in an intervention using a spatially structured language to support body-spatial-verbal coupling. For example, they articulated "frog-four" as they physically mapped the elements in a four-dotted array using four limbs (i.e. the feet and hands). Likewise, they said "monkey-three" as they imitated an itching monkey by placing one hand on the head, while the other hand and the feet tagged the elements in a three-dotted array. Based on these shared experiences, we ask the following: What inhibits and scaffolds C2- or C3-knowers' mapping of spatial structured knowledge of numerosity across conceptual domains in a navigation task? The results will be analysed according to the theoretical framework outlined in the next section.

7.2 Theoretical Framework

Conceptual Metaphor Theory (CMT) as introduced by Lakoff and Johnson (1980) in "Metaphors We Live By" is regarded as a prominent theory in interdisciplinary metaphor studies (Gibbs, 2009). CMT posits that metaphors are primarily a conceptual phenomenon characterised by complex mental ensembles of schemas. Of particular significance is the class of image schemas, which refers to spatially structured mental pictures or representations of assemblies of objects, bodily orientation, movement and physical interaction that structure and guide abstract thoughts and reasoning (Lakoff & Johnson, 1980). Image schemas with their perceptual and conceptual nature provide "a bridge between language and reasoning on the one hand and vision on the other" (Lakoff & Núñez, 2000, p. 31), and therefore they possess a mediating role between the embodiment and formation of complex concepts such as the idea of numbers. Based on this, CMT holds that metaphors are cross-domain mapping in the conceptual system (Lakoff & Johnson, 1980). Consequently, the tension created by metaphors possesses an epistemic function as it addresses instability and transfer of meaning between two cognitive domains. Central to a conceptual mapping is thus what the source and targeting domain have in common, and this set of shared features or similarities is termed the "ground". For example, a metaphorical mapping between the spatial domain in the form of the visual perception of the legs of a kangaroo onto a semantic targeting domain (e.g. "kangaroo-two") has the imaginative rationality reflected in the cardinal value "two" as it is a shared ontological feature (i.e. a part of the ground). The dissimilarities between the two compared domains create a metaphorical "tension". It is important to note that the instability in any conceptual metaphorical mapping might hinder understanding, but the dissimilarities might also point to the "grounds" zone of proximal development (cf. Vygotsky, 1978). To further develop our understanding of the epistemological potential of conceptual metaphorical mappings, we will apply the term cognitive conflict, which refers to a psychological state that involves a discrepancy between mental representations (including image schemas) and experience, or between different cognitive structures (Waxer & Morton, 2012). For example, a cognitive conflict might emerge when a quadruped body posture does not allow a one-to-one correspondence to an assembly of three items on the ground, and the cognitive tension in terms of the mismatch in the transfer of quantity might be resolved by raising one hand or one foot. Moreover, a principal tenet of CMT is that abstract concepts, for example numbers, are structured by several layers of mutually supporting and overlapping conceptual metaphors (classified as structural, ontological and orientational metaphors), which in turn are based on some concrete representations or complex bodily gestalts (e.g. a one-legged pose imitating a howling rooster), which are referred to as grounding metaphors (Lakoff & Johnson, 1980).

The notions subitising, Approximate Number System, Object Tracking System and pattern recognition might further develop our understanding of how vision, bodily gestalts and everyday experiences such as navigation, hop, gait, crawling and the balancing and physical tagging of objects might conflate with innate numerosity in "metaphorical mappings so that the inferences of the source domains will map correctly onto arithmetic" (Lakoff & Núñez, 2000, p. 102). The Object Tracking System (OTS), also termed the "parallel individuation system", is a cognitive system for tracking from 1 to 3 or 4 objects in parallel (Piazza, 2010). The term "individuation" emphasises that through this mechanism the objects are perceived as specific entities at a given spatial location. Perceptual subitising refers to an intuitive and direct perception of the numerosity of a small set of objects (Clements, 1999), and some neurocognitive scientists claim that perceptual subitising emerges from the OTS system (Piazza, Fumarola, Chinello, & Melcher, 2011). Another core knowledge system, the Approximate Number System (ANS, also termed "analogue magnitude"), sub-serves rapid non-verbal estimated representations of quantities in an analogue fashion, and its precision relates to the size of the set (Feigenson et al., 2013; Piazza, 2010). The ANS might capture more-and-less-relations by comparing approximated values of arrays of dots or sequences of actions and sounds, thereby supporting non-symbolic ordering of sets, addition and subtraction. Moreover, the notion pattern recognition describes a process where visual perception of a learnt configuration (i.e. a blend of ontological and structural metaphors), possibly supported by OTS and ANS, is mapped onto conceptual metaphors. For example, visual perception of a patterned group of objects might be recognised to share spatial and quantifiable features with a bodily gestalt, and a goal-directed (i.e. orientational metaphors) articulated body-spatial coupling might transfer additional layers of meaning in the process of abstraction.

In this study, we see the core knowledge systems as mental representations (or rather image schemas) that might support navigation, subitising, pattern recognition and spatial structured transitive relations of quantities and the synthesis of these processes. This stance is further based on the dialectically functioning, rapid and spontaneous nature of these inborn capacities for supporting perception in space and time (Spelke & Kinzler, 2007). This concurs with the basic assumptions of CMT that metaphors are usually used instinctively and naturally as an integrated part of ideas, thoughts and reasoning in real life (Lakoff & Johnson, 1980). Moreover, the notion of metaphorical mapping (i.e. conceptual mapping) provides a tool for analysing the direction and coherence in the projection of numerosity across conceptual domains, and we will use this conceptual framework to guide and structure our methodological choices as outlined in the next section.

7.3 Methodology

7.3.1 The Intervention and Case Selection

Upon the written consent of their parents, 15 children of 3–4 years old in a Norwegian kindergarten participated in guided outdoor sessions over a 2-month period. The intervention focused on articulated body-spatial mapping of numbers in 1- to 4-dotted arrays (regular shape for sets with three and four dots) using corresponding number metaphors (i.e. "cock-a-doodle-doo-one", "kangaroo-two", "monkey-three" and "frog-four"), see Fig. 7.1. The selection of these four number metaphors was based on the aim of building on children's prior knowledge of animal behaviour, and to include established bodily expressions in the representation of the numbers 1–4. Moreover, the articulated body-spatial coupling could also represent novel aspects concerning numerosity to specific animal behaviour, for example a head-scratching monkey imitation might represent the set of three elements (see Fig. 7.1). The navigation task was administered as a post-test of the intervention to examine the participant's ability to use, adapt and transform their spatial structured knowledge in a novel context involving navigation and free use of number metaphors in a multi-dotted unstructured array.

Fig. 7.1 Articulated body-spatial mapping of "monkey-three" and "cock-a-doodle-doo-one"

The selection aimed to include extreme cases from the intervention group (Flyvbjerg, 2001) in terms of choosing participants who struggled to master the cardinal principle for exact enumeration. Thus, using this biased criterion on the scores on a Give-*N*-test (Wynn, 1990) that was taken 2 weeks after the navigation task, two C2- and six C3-knowers (four girls, four boys; mean age 4:3, range 3:11–4:9; average participation 9 sessions and 7 h) were included in the present study.

7.3.2 Procedure of the Navigation Task

The navigation task (see Fig. 7.2) was contextualised in a circle ($d = 3$ m) with 50 arbitrary distributed dots ($d = 0.1$ m). In order to familiarise the participants with the context of the navigation task, the 50-dotted array was used in a modified Give-*N*-test (cf. Bjørnebye, Sigurjonsson, & Solbakken, 2017) as preparation. Using the criterion of at least two successes out of three trials (Wynn, 1992), the scores on this warm-up task showed that six of the participants mastered articulated bodily production of requested number metaphors (e.g. "Can you jump a frog-four?") linked to cardinal values from 1 to 4. The C3-knowers Al and Val (all names are anonymised) did not master production of "cock-a-doodle-doo-one" and "monkey-three", respectively.

Coloured lines outside the circle marked the start (A) or the end (B) of the trial (see Fig. 7.2). The experimenter presented the task: "You are to jump from the red line to the white line (pointing), and you must tell what you jump". After the completion of a trial, feedback was provided if the child skipped articulation ("remember to say what you jump") or double-tagged ("do not use the same dot twice"). During action, the experimenter could give hints to remind the participants of the aim of the task (e.g. "You are to jump to"), provide encouragement (e.g. "and then") or address unclear body-spatial-verbal coupling (e.g. "You said monkey-three?").

Fig. 7.2 Array with 50-dots used in the navigation task

7.3.3 Operationalisation and Data Analysis

The present investigation is a case study which combines pattern matching and cross-case synthesis to develop general descriptions that fits the task behaviour of the eight C2- and C3-knowers (Yin, 2009). Pattern-matching logic involves comparing empirical patterns with theoretical predicted ones (Yin, 2009). Thus, informed by our theoretical framework, we assumed that articulation (e.g. "frog-four"), experimental gestalt (e.g. a four-limbed grounding pose) and body-spatial coupling (i.e. physical tagging of spatial structures) mediated signatures of cross-domain mapping of numbers. As subset-knowers seldom use counting procedures for exact enumeration (Le Corre, Van de Walle, Brannon, & Carey, 2006), we presupposed that a rapid series of simultaneously expressed embodied parts were supported by core knowledge structures for representing numbers. Moreover, we assumed that their knower level as assessed in the Give-N-task provided an indication of their capacities to produce exact numbered sets across modalities (e.g. kinaesthetic, semantic and body-spatial coupling). In this way, we also examined how the classification of knower-level behaviour (Lee & Sarnecka, 2010; Wynn, 1992) conflated with Conceptual Metaphor Theory. Concerning navigation, we looked at precise and flexible actualisation of the spatial layout and the measurable (i.e. distance and angle) and topological properties of the affordances (Gibson, 1979), in a manner that preserved orientation to the targeting line B. Based on assumed diversities in the gross motor proficiency of 47- to 57-month-old children (i.e. gait, jump and body coordination), we hypothesised that the participants would show varied competence in actualisation of the affordances (Gallahue, Ozmun, & Goodway, 2012).

The data analysis consisted of three main stages: First, based on codes (e.g. type of number metaphor expressed verbally and physically, coherence in body-spatial-semantic mapping) and thick descriptions from the transcript of the video-recorded material, shared patterns and diversities for the eight participants were analysed and compared to the predicted patterns as outlined in the operationalisation. Second, in order to generate conjectures of a classificatory framework that captured similarities and diversities across and within the subset-knower level (Gerring, 2004), different categories of cases were compared and discussed. Finally, we synthesised the findings. To support the qualitative analysis, selected numerical data were extracted from the empirical material. The credibility and reliability in the data were supported by crosschecking the coding of the data, which has been conducted by two researchers.

7.4 Results

This section presents the general results of the task solution of the eight participants. Then, in order to provide an empirical basis for comparing patterns across and within cardinal knower level and to perform a cross-case synthesis (Yin, 2009), the next part presents rich descriptions which highlight similarities and differences in main categories and levels of performance.

7.4.1 General Results

Each of the participants produced two to four movement trajectories across the 50-dotted circle in 22 tasks. Along these trails, using the hands and legs for tagging, there were 106 physical expressions of numbers distributed across three series of three parts (e.g. notated as 4+2+3), six series of four parts, seven series of five parts, four series of six parts and two series of seven parts. Three C3-knowers and one C2-knower embodied all four number metaphors, while the rest did not include the "cock-a-doodle-doo-one" gestalt in their repertoire. Regarding the frequency of different embodied containers, a four-limbed gestalt was applied 36% (36/106) of the time, a two-legged-tagging was applied 26% (28/106) of the time, a three-limbed gestalt was applied 25% (26/106) of the time and a one-legged tagging was applied 15% (16/106) of the time. Most verbalisations (i.e. 102/116 = 88%) were considered to precede or overlap with a part of the body-spatial coupling process, while two remained unarticulated and the rest pointed to a completed tagging. Double tagging of dots occurred in 5 out of 22 series and most frequently after a monkey-three or frog-four body pose. In some series, extensions of an initial tagging occurred. For example, one C3-knower said "kangaroo, frog-four" as she extended a two-dotted-feet-tagging to include the hands in a four-limbed gestalt. In 91% (96/106) of the embodied containers, there was conformability in the cross-modal mapping of numerosity between the experimental gestalt and the spatial and linguistic domain. With the exception of one deviant mapping, the rest of the verbal-body-spatial couplings showed coherence between two out of the three observable modalities of quantities. However, some children corrected their misuse of semantic expression in situ, while others "stepped out" for visual examination to adjust their articulated body-spatial coupling.

Concerning navigation and accounting for guiding hints, all but one series ended in the predetermined coloured line. Most of these movement trajectories reflected a goal-directed path from A to B. However, five out of 22 series had intermediate embodied containers adjacent to the arc of the circle, suggesting off course. Four of the participants walked, two combined jumping and walking, and one C2 and one C3-knower jumped between each embodiment. The group of children that walked seemed to search for a tagging opportunity until an approximated regular-shaped configuration appeared in front of them. For example, two-dotted arrays had to be in parallel with their visual field, and the configuration and orientation of a set of three items had to appear as regular in a "2-1 structure". In contrast, the children who jumped showed proficiency in adapting their body posture to sets with irregular layout.

Based on pattern matching and analyses of the eight children's behaviour, three categories of task performance emerged:

1. Children who failed in some of the cross-modal mapping of numerosity.
2. Children who had problems in navigation and pattern recognition.
3. Children who showed coherence in multimodal use of numerosity and goal-directed appropriation of the affordances.

The next section provides rich descriptions and excerpts highlighting unique and shared aspects of these categories.

7.4.2 Rich Description of Three Groups of Task Solutions

The in-depth presentation of three groups of task solutions presented below aims to show differences in performance and shared patterns based on prerequisites of using a spatial language, as well as on different abilities of goal-directedness, fluency and precision in the cross-modal mapping in the navigation task. In particular, task behaviour in the first group with Al (C3) and Val (C3) aims to highlight inconsistency in cross-modal production of numerosity. They also demonstrate task performance for children with gaps in prior knowledge as they failed to produce one of the number metaphors in the warm-up task. The second group with Ed (C2) and Max (C3) represents children who opted to walk and showed problems with pattern recognition and navigation. Although Elias (C3) and Rae (C3) also belong to this group, their task performances will not be presented as their behaviour does not add any significant information to this category. Finally, Liv (C2) and Amy (C3) represent the group of participants that opted to jump and showed fluency in navigation and precision in body-spatial-semantic coupling.

Al (C3) produced 4+4+3+3+2 (43 s), 4+1+3+2+3 (32 s, see Fig. 7.3), 4+2+3+4+2+3 (30 s, see Fig. 7.4) and 4+2+1+2+2+4+3 (53 s). Al opted to walk and took a long time to initiate a coupling. Two trails had intermediate parts adjacent to

Fig. 7.3 Al embodies 4+1+3+2+3 and articulates "frog-four, cock-a-doodle-doo-one, monkey-three, kangaroo-two, cock-a-doodle-doo-one" (Closed curves illustrate tagged sets)

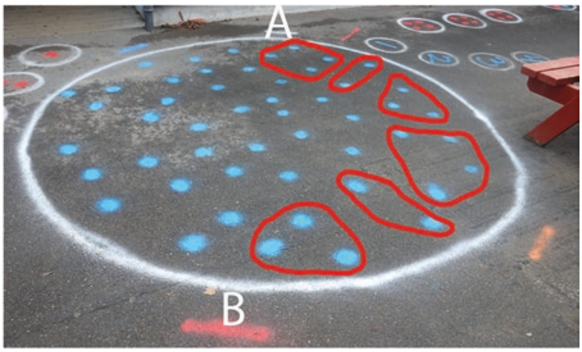

Fig. 7.4 Al embodies 4+2+3+4+2+3 and says "frog-four, kangaroo-two, cock-a-doodle-doo-one, frog-four, kangaroo-two, cock-a-doodle-doo-one"

the arc of the circle. In the third series, Al extended a two-footed tagging to a four-limbed coupling while verbalising "kangaroo, frog-four". Al performed six errors in the one-to-one correspondence, all related to three-limbed gestalts. On five occasions, his body-spatial coupling was correct, but he said "cock-a-doodle-doo-one". For example, in the embodied series 4+4+3+3+2, Al articulated "frog-four, frog-four, cock-a-doodle-doo-one, cock-a-doodle-doo-one, kangaroo-two". Notably, in the fourth mapping, Al made a three-limbed coupling onto two dots and said "cock-a-doodle-doo-one". Al overlapped once, between the two "frog-four"-couplings in this series. The following excerpt starts from the second tagging in the series 4+1+3+2+3 (see Fig. 7.3) and shows the only correct mapping of numerosity three:

Al: [Al uses one foot and tags a dot] Cock-a-doodle-doo-one [Al walks and uses one hand and both feet and tags three dots and immediately utters] Heah? [Al looks up at the experimenter with a questioning expression on his face, holds his position and waits for 3 seconds] Heah?

Experimenter: [Pause for 2 seconds] What do you call this?

Al: [Pause for 2 seconds] Monkey-three [Al stands up, walks, using both feet he tags two dots] kangaroo-two [Al walks and uses one hand and both feet and tags a set of three dots] cock-a-doodle-doo-one [Al walks to B]

Val (C3) produced 4+4+2+4+2 (68 s, see Fig. 7.5) and 3+4+4 (14 s). Val walked and articulated all parts in parallel with the coupling process. The following excerpt is from the first trial:

Val: [Val walks from A and tags two dots with her feet] Kangaroo [Val extends the coupling to include her hands to tag four dots] frog-four [Val stands up and turns slightly to the right, and faces a three-dotted set in a 1–2 structure (i.e. "arrowhead" pointing towards her, see Fig. 7.6). She uses both hands and tags the paired dots with her feet placed on either side of the odd dot (see Fig. 7.7). Her four-limbed gestalt partly overlaps the previous tagging] monkey-three [Val stands up, walks and completes the series]

Fig. 7.5 Val embodies 4+4+2+4+2 (Dotted curve illustrates gait pattern)

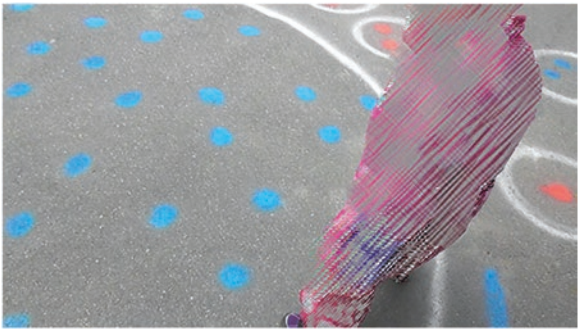

Fig. 7.6 Val faces three dots shaped as an "arrow-head" pointing towards her

Fig. 7.7 Val aims to tag three dots, uses four limbs and says "monkey-three"

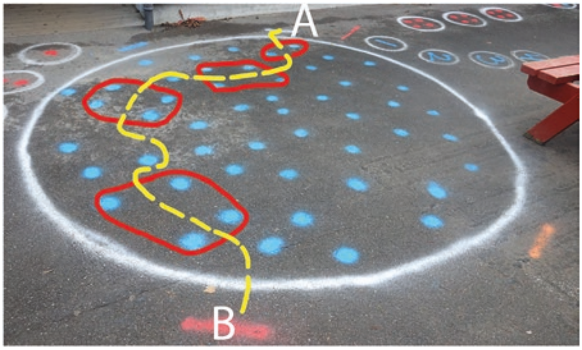

Fig. 7.8 Ed produces 2+4+4+4

In the first embodiment of the second series, the three-dotted configuration held a regular "2-1-structure" and Val tagged it correctly.

Ed (C2) produced 3+3+2+3+4+4 (56 s) and 2+4+4+4 (45 s, see Fig. 7.8). Five articulations were in parallel with a coupling process, while three verbalisations pointed back on a completed embodiment and two double-footed spatial couplings remained unarticulated. Ed walked and his bodily direction seemed to be determined by the orientation and shape of the identified pattern. Ed overlapped twice in

Fig. 7.9 Ed says "that is for the return"

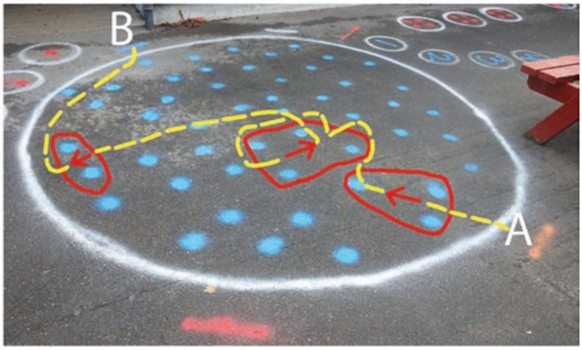

Fig. 7.10 Max produces 3+4+2 (Arrow illustrates bodily direction in tagging)

the first series. After the third coupling in the second trial, he turned around and initiated the tagging of a three-dotted set, but he reconsidered, pointed and said "that is for the return" (cf. Figs. 7.8 and 7.9).

Max (C3) produced 3+4+2 (35 s, see Fig. 7.10), 2+3+4 (23 s), 4+4+2+2 (32 s) and 4+3+4+3 (22 s). Two movement trajectories had embodied containers adjacent to the arc of the circle. The following excerpt is from the series 3+4+2:

Max: [Max walks from A and uses both feet and one hand and tags three dots] Monkey-three

[Max stands up, walks around and initiates a tagging, but he does not complete it, then turns, stares at and partly circles the configuration] Here [in an exaggerated tone and pointing, he tags four dots] monkey, frog-four [Max stands up, walks and halts] Here [Max tags two dots while his bodily direction is about 180 degrees from the previous tagging and 90 degrees from the A-B line (see Figs. 7.7 and 7.11)]

Experimenter: What is that?
Max: Kangaroo-two [Max turns to the right and walks to B]

Fig. 7.11 Max steps out of course

Fig. 7.12 Max tags an irregular assembly

In the series 4+4+2+2, Max nearly inaudibly said "frog" as he tagged an irregular configuration of four dots (see Fig. 7.12), and he stepped out and looked at the container with a body language that projected doubt before he completed the series.

Liv (C2) produced 1+2+3+4+2+1 (28 s), 1+2+1+3+4+2 (18 s) and 1+1+3+4+2 (26 s). Liv articulated most parts in parallel with the body-spatial coupling (15/17), and she jumped and adapted her embodiment in a manner that maintained direction towards the targeting line B. On one occasion, she extended a one-legged tagging to a two-feet coupling while verbalising "cock-a-doodle-doo, kangaroo-two". In the third and fourth tagging in the second task, she said "frog-four" as she tagged first one and then three dots (coded as failures). The following excerpt starts from the third tagging in the first trial (see Fig. 7.13):

Liv:	[Liv tags a three-dotted assembly and articulates in parallel] Frog-four [She moves "out" to the right of her trajectory and looks at the tagged array (cf. dotted curve in Fig. 7.13)]
Experimenter:	Was it a frog-four?
Liv:	No, a monkey-three. [She articulates and re-tags the assembly and completes the sequence in a jumping fashion]

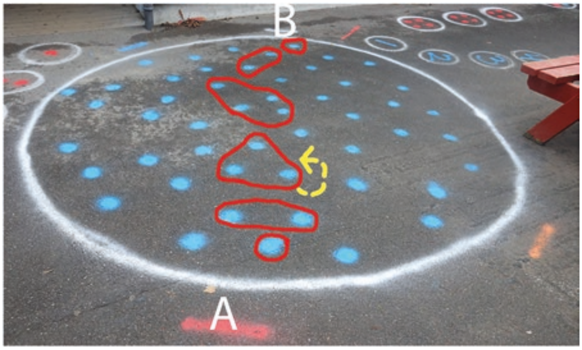

Fig. 7.13 Liv experiences a cognitive conflict in 1+2+3+4+2+1

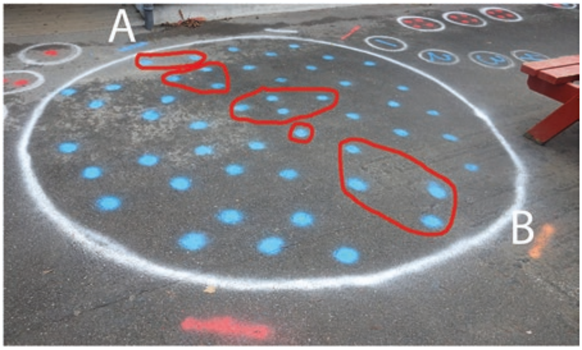

Fig. 7.14 Amy jumps and embodies 2+3+4+1+4

Fig. 7.15 Amy in goal-directed, rapid and precise body-spatial couplings

Amy (C3) produced 2+1+3+4 (25 s), 1+2+3+4+4 (12 s) and 2+3+4+1+4 (17 s, see Figs. 7.14 and 7.15). Recurring patterns were jump and rapid and coherent body-spatial-semantic coupling and adaption to the configuration of the set in a goal-directed manner.

7.5 Discussion

This study investigates the use of a spatially structured language to cultivate inborn aptitudes in navigation and non-verbal representations of numerosity to enhance 3- and 4-year olds' skills in exact quantification. Using CMT to guide our analysis, we assume that body-visual coupled patterns and the spatial structure reflected in articulated number metaphors mediate salient signatures of cross-domain mapping of numerosity. In the task, the ANS supports initial comparison, discrimination and identification of estimated magnitudes and configurations of relevant sets; for example the spatial structure of an assembly of less dots (e.g. 3) is more jumpable than the larger group (e.g. 5). Based on increased visual attention, the mapping between the OTS (perceptual subitising) onto the semantic domain (e.g. "monkey-three") or the body-based conceptual metaphor (e.g. a three-limbed gestalt) might support transfer of exact numerosity. Moreover, the pauses in between each embodiment create a timeframe for orientation and perception of the geometry in the array, in relation for instance to the distance, angle and topological properties of boundaries of potential sets, and thereby establish coherence in cultivation of inborn and learnt cognitive capacities for navigation and for representing numbers.

In over 90% of the body-spatial couplings, the tension created by the cross-domain conceptual mapping preserves the numerosity as a shared ontological feature (i.e. the "ground"). Moreover, 40% of the participants' self-governed embodied mappings reflect cardinal values that exceed their assessed capacity of abstract representations of numerosity (c.f. the Give-N-task). For C2-knower Ed, his mappings of numerosity three and four (i.e. 8 out of 10) are coherently mediated across semantic and visual-motoric modalities. However, when operating within his cardinal knower level, he has problems mapping the body-spatial representation to the semantic domain as none of the two "kangaroo-two" gestalts are verbalised. Concerning contrasting performance levels in using a spatial structured language, Liv (C2) and Amy (C3) integrate all four number metaphors in each task, while Ed (C2) and Max (C3) show rigidity as they apply only two metaphors in three out of the six series and do not use "cock-a-doodle-doo-one" at all.

The overall results also show a mismatch in the intermodal mediation of numerosity (i.e. 10/106), and we illustrate this with the task solution of Al (C3) and Val (C3), who fail to produce one of the number metaphors in the preparation task. In four navigation trails, Al maps five three-limbed spatial couplings onto the semantic expression "cock-a-doodle-doo-one". Compared to the warm-up modified Give-N-task where he maps the verbal instruction "cock-a-doodle-doo-one" onto a monkey-three gestalt, this suggests that the mismatch is bidirectional and grounded in semantics. Hence, for Al, this underlines that metaphorical expressions are only surfacing manifestations of more deeply grounded conceptual metaphors (Lakoff, 1993). Moreover, in one coupling, Al produces a similar articulated gestalt onto a two-dotted configuration, and thereby he shows discrepancy in the mapping of numerosity from the visual source domain (i.e. numerosity two) onto the linguistic (i.e. numerosity one) and motoric (i.e. numerosity three) target domains. Another example of failure in the one-to-one correspondence concerns the issue of pattern recognition. Consistent with the warm-up task, Val (C3) identifies an approximately regular three-dotted assembly with a "1-2-structure" (i.e. "arrow-head" pointing towards her) and produces a four-limbed gestalt and articulates "monkey-three" (see Fig. 7.7). Thus for Val, in contrast to Al, this suggests that the discrepancy in the intermodal transfer of numerosity is primarily a body-spatial issue. Ed (C2) also addresses this potential visual-motoric tension, as he points to a similar oriented configuration and remarks "that is for the return" (see Fig. 7.9). With reference to the intervention, this suggests that guidance embracing conceptual metaphors (e.g. "monkey-three" gestalts), which hold the idea that body-spatial coupling only applies to oriented regularly structured configurations, might inhibit the use of the core knowledge structures in cross-domain mappings of quantities (cf. Fig. 7.1).

However, for most participants, this type of tension creates a cognitive conflict (Waxer & Morton, 2012). For example, Liv (C2) steps out of the "container" to visually examine the discrepancy provided by the cross-modal mapping of numerosity (see dotted curve in Fig. 7.13). In addition, Amy (C3), Max (C3) and Elias (C3) adjust their initial articulations during an embodiment to match the body-spatial coupling, as they said "monkey, kangaroo-two", "monkey, frog-four" and "frog, cock-a-doodle-doo, monkey-three", respectively. We offer two possible explanations for the emergence of and solution to these cognitive conflicting tensions, which are backed up by claims saying that ANS supports numerical processes of small numbers and interacts with the OTS (Feigenson et al., 2013; Piazza, 2010), and that subitising and ANS is activated in respectively tactile and haptic modalities (Gimbert, Gentaz, Camos, & Mazens, 2016; Riggs et al., 2006). The first hypothesis is based on an initial use of ANS for estimation of the numerosity of a set of dots, or for perception of a more-and-less relation based on discrimination of distinct or overlapping groups. Either way, the imprecise representation of a jumpable quantity is thus mapped onto the semantic domain (e.g. Elias: "frog, cock-a-doodle-doo"). In the next phase, based on increased visual attention and possibly supported by tactile and haptic sensory information, the OTS supports the conceptual mapping onto the embodied and linguistic domain to a correctly verbalised representation (e.g. Elias: "monkey-three"). The second hypothesis is based on the assumption that an initial visual pattern recognition was simultaneously mapped onto the bodily and semantic domain (e.g. Max: a four-limbed coupling and says "monkey"). Following this line of reasoning, the spatial structured discrepancy in the visual-linguistic transfer of numerosity is, via the body-semantic mapping, corrected to hold the cardinal value as a shared feature (e.g. Max: "frog-four"). Regardless, and epistemically speaking, this underlines that the "ground" in different mappings interacts according to a proximal zone that might include coherence in transfer of numerosity across conceptual domains in general and in spatial layout of the items in particular. This claim is further supported in Al's (C3) behaviour, as his only correct cross-modal mapping of numerosity three (i.e. 1 out of 7) occurred right after a "rooster-one" tagging (see Fig. 7.3). On the basis that the other three-limbed couplings were mapped onto the same semantic unit (i.e. "rooster-one"), this suggests that the conflict, linguistically expressed as "Heah?" and physically as halting with a questioning facial expression, emerges due to a close temporal and spatial contiguity. Moreover, the task's incorporated potential to establish balance between perception and conflicting ideas that include the invariant property of numerosity is exemplified by Max's (C3) first path (see Fig. 7.10), where he circles around and examines a potential set from different angles and eventually uses the Eureka-word "Here" when he recognises a jumpable pattern. In addition, when Max hesitates and partially articulates the metaphor (i.e. "frog") while embodying an irregular configuration (see Fig. 7.12), his body language says "Was that really a frog-four?" as he subsequently examines the tagged part. Liv (C2), on the other hand, shows flexibility as she maps a similar irregular configuration of four items onto the linguistic expression "frog-four" and afterwards projects the perceived whole onto an embodied part-part structure (i.e. 1 + 3).

Concerning actualisation of the measurable and topological properties of the 50 dotted array, some children jump from one body-spatial mapping to the next in a goal-directed manner, which suggests a coherent visual-motoric appropriation of distance, angle and spatial layout of the affordances. This is exemplified in the six series of Liv (C2) and Amy (C3), where they adapt their body position to the affordances and maintain the orientation towards the targeting coloured line in producing 31 parts in 126 s (i.e. average frequency of 4.1 s/part). In contrast, others such as Ed (C2) and Max (C3) walk and use substantially more effort to decode and interpret the visual information (on average 8.9 s/part), and their bodily angle seems to be determined by the regularity and orientation of the perceived set. Max illustrates this in the series 3 + 4 + 2 (see Fig. 7.10), where his corporal direction changes 180° between consecutive body-spatial couplings. Moreover, his "astray" movement trajectory suggests that his aim was to identify patterns that fit rigid ideas of how the spatial structure of the number metaphors ought to be mapped onto the embodied domain.

7.6 Summary and Concluding Remarks

Based on a design targeting the integration of innate capacities and learnt representations of spatial structured magnitudes, this qualitative study aimed to analyse subset-knowers' cross-domain mapping of numerosity in a bodily path navigation task. The main findings are that the eight C2- or C3-knowers mastered articulated bodily production of numerosity that exceeded their cardinal knower level. The examination also points to the task's inherent epistemic potential in creating and solving cognitive conflicts in situ, and thus it underlines that, "Bodily projections are especially clear instances of the way our bodies shape conceptual structure" (Lakoff & Johnson, 1999, p. 34). The analysis also shows quality differences in subset-knowers' ability to perform visual pattern recognition and in use of a spatial structured language to enhance body-spatial coupling. We suggest that the most prominent feature of the participants who walked is that task solution for them becomes an issue of pattern matching and that a rigid conceptualisation of the spatial structure of the number metaphors hinders appropriation of the affordances across different arrangements of the dots. In contrast, participants with a high level of proficiency in the navigation task are characterised by flexibility in metaphor usage, coherence in the body-spatial semantic mapping of numerosity and rapid, adaptive and goal-directed visual-motoric coupling across spatial layout and the shape of the boundaries. Hence, the group of jumping children actualises fundamental properties of the concept of cardinality (Fuson, 1988), and their fluent appropriation of the affordances suggests support for and synchronisation of core knowledge structures and culturally achieved concepts. However, prior to the navigation task, the participants did receive guidance on how to use selected number metaphors for regular-shaped body-spatial coupling of small numbers. In light of CMT, which holds that abstract concepts as numbers consist of a cluster of partly overlapping conceptual metaphors (Lakoff & Johnson, 1980), such focus might indirectly hide other salient aspects of the concept. Hence, and particularly in regard to the participants who did not master production of all the number metaphors prior to the task, we acknowledge that other conceptual metaphors might support the synthesis of navigation and multimodal production of numbers in qualitatively different ways. However, compared to the Give-N-task, which basically concerns children's ability to map a linguistic representation of a memorised number word onto a sequentially produced exact numbered set, we argue that by reversing the mapping order and building on autonomy the navigation activity possesses the potential to utilise authentic movement patterns and inborn capacities in cross-domain mapping of numerosity in an outdoor scene. Even though a small number of cases and tasks limit the study, the in-depth inquiry might contribute to knower-level behaviour as it reveals shared proficiencies and qualitative differences in task performance across and within knower level. In light of a consensus that exact enumeration of large sets build on children's ability to label small sets (Mix, Sandhofer, Moore, & Russell, 2012), further research should address how subset-knowers might map the cardinal value linked to number metaphors onto number words.

To wrap up, from a CMT stance, we have examined the qualities, constraints and entailments of a limited number of conceptual mappings of numerosity in a navigation task, and the idiosyncratic and the cross-case analyses show that spatial structured metaphors relate to the source and target domain differently as they coexist in a complex network. Hence, our findings show that the C2- and C3-knowers are more diverse than the cardinal knower level indicates.

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Chapter 8
Mathematical Problem-Solving Visualised in Outdoor Activities

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8.1 Introduction

A problem-solving approach to learning of mathematics has for decades been an important goal for school mathematics (Pólya, 2004; Schoenfeld, 2016; Stanic & Kilpatrick, 1989). In the early years mathematics, research has tended to focus on children's mathematical problem-solving primarily in relationship to number sense (Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Charlesworth & Leali, 2012; Rogers, 2004; Tarim, 2009). In contrast, problem-solving has not been discussed in the same way in relation to other mathematical topics, particularly when situated in outdoor activities in kindergarten.[1] Yet problem-solving is an instinctive part of children's daily life as they make sense of the world. Children face many new situations with curiosity and intelligence because the world is new to them.

Problem-solving "has been used with multiple meanings" (Schoenfeld, 2016, p. 1). Lesh and Zawojewski (2007) phrase it in the following way: "No strategy, process, behavior, or characteristic should be expected to always be productive for every problem, nor for every stage in learning or problem-solving." (p. 778). We, therefore, define problem solving as the process that occurs when children meet challenges they do not immediately know the answer to or how to reach an answer. Consequently, there are many opportunities to stimulate problem-solving during outdoor activities in kindergarten.

In Norway, nearly every child between the ages of 1 and 6 attends kindergarten (Stabekk, 2017). Norwegian kindergarten is based on play-oriented guidelines, following a sociocultural tradition (Engel, Barnett, Anders, & Taguma, 2015). The

[1] When we use the term Kindergarten, it relates to children between 1 and 6 years old that is the year span in Norwegian kindergartens.

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guidelines in the national Framework Plan for the Content and Tasks of Kindergartens (Norwegian Directorate for Education and Training, 2017) regulate the rules, content and tasks that should be undertaken in Norwegian kindergartens. The framework plan includes different learning areas. "Quantities, space and shapes" is the learning area related to mathematics. Yet explicit discussion of problem-solving is relatively new in the kindergarten curriculum. The previous Framework Plan (Norwegian Ministry of Education and Research, 2006) made no reference to problem-solving. In the 2017 Framework Plan, the requirement for kindergartens to engage children in problem-solving has been made prominent:

> The learning area shall stimulate the children's sense of wonder, curiosity and motivation for problem-solving …
>
> By engaging with quantities, space and shapes, kindergartens shall enable the children to … investigate and gain experience of solving mathematical problems and find pleasure in mathematics …
>
> Staff shall… stimulate and support the children's capacity for and perseverance in problem-solving. (Norwegian Directorate for Education and Training, 2017, p. 53–54)

With the new focus on problem-solving, we decided to investigate and characterise mathematical problem-solving situations in kindergartens and to look specifically at outdoor situations. Outdoor activities are considered a central part of the work of kindergartens, with every child expected to spend time outside during their days in kindergarten, regardless of the weather. For example, Moser and Martinsen (2010) found that children in Norwegian kindergartens spend 70% of the time outside during the summer and 30% during the winter. In kindergartens which specifically focus on nature and outdoor play, the children spend even more time outdoors, 87% of the time in the summer and 79% in the winter (Lysklett, 2005). Moser and Martinsen are critical of the outdoor activities happening in the kindergarten. They questioned whether the outdoor environment acts as a pedagogical space for play, learning and development and raised the question about the extent that staff critically reflect upon their outdoor practice (Moser & Martinsen, 2010). Similarly, Kaarby and Tandberg (2017) questioned if the absence of an explicit understanding of the value of being outdoor implied a form of hidden curriculum. In light of these concerns, we are particularly interested in mathematical problem-solving in an outdoor environment as a way of understanding the possibilities of combining play, learning and development.

Although little research has been done on this area, Lee (2012) conducted a case study in New Zealand over a 4-month period in which she videotaped 32 children aged between 13 months and 3 years. In her case study, she described how the children explored mathematics in their outdoor play. She focused on natural play among the toddlers and excluded play situations that included interactions between children and adults. Her findings showed that many children engaged in problem-solving in their play and used a variety of strategies to solve these problems.

We wanted to follow up on Lee's (2012) research and investigate what constitutes mathematics problem-solving in outdoor situations in Norwegian kindergartens. In Sweden, Delacour (2016) investigated how two preschool teachers implemented an outdoor realistic problem situation for children aged 4–5 years. She discussed the communication between the children and the adult in the outdoor setting, and how the teacher followed up the children's interests in a planned activity. However, the majority of activities in kindergartens are not pre-planned, but rather evolves based on the children's own interests. Therefore, this study wanted to investigate the problem-solving opportunities that occur during daily activities with a focus on the interaction between adults and children. Consequently, our research question is: What kind of mathematical problem-solving can be identified in the communication between children and the teacher in an outdoor context?

8.2 Theoretical Frameworks About Mathematical Problem-Solving

In regard to mathematical problem-solving, researchers have proposed the use of a range of different theoretical frameworks for investigating problem-solving interactions, depending upon whether it is the teacher, the child or the problem-solving process which is the focus. In order to answer our research question, it has been important to consider problem-solving interactions from all the three perspectives.

The teachers' role in supporting children to become "good" problem solvers has been discussed for many years in relationship to school mathematics. Sometimes the teacher presents problems so that the children have an opportunity to learn some mathematics. For example, in Carpenter et al.'s (1993) study, the kindergarten teachers generally presented problems, with the purpose of having children learn mathematics. In this situation, the focus is on how the teacher considered that mathematics will be learnt through problem-solving.

At other times, mathematics is used as a tool to solve a problem so that learning mathematics is not the goal; rather it is the solving of the problem. In this case, the teacher wants the child to learn to solve problems with the mathematics having only a use value within that problem-solving context. A child could engage in problem-solving with the mathematical aspects being more or less visible, depending on how they perceive the goal of the activity. Nevertheless, children's engagement with the problems can provide information to the teacher. Charlesworth and Leali (2012) stated that problem-solving "provides a window into children's mathematical thinking" (p. 373). In Charlesworth and Leali's description of children's problem-solving, mathematics is more or less a tool for problem-solving for the children.

In considering the teaching of problem-solving, English and Sriraman (2010) summarised the findings from a range of studies to suggest that it is not possible to identify just one or a few strategies to teach every child to do problem-solving. This is because the problem needs to be something new for the child to explore. If a teacher gives children detailed instructions on how to solve the problem, then the child's own exploration and learning will be reduced.

When the focus shifts to the child and their engagement with problem-solving, the mathematics can be backgrounded. For example, in research on higher-order thinking, Lesh and Zawojewski (2007) discussed instructional strategies that encouraged problem-solving. They described an effective problem solver as being among other things a person that could break up a complex problem into subtasks, do each subtask and exhibit self-regulation. Further, they suggested that brainstorming and trying alternatives was not good or bad but a way to revise the problem solver's way of thinking. According to Lesh and Zawojewski, the teacher should be aware of the children's beliefs and dispositions (approaches to learning). Burkhardt cited in Schoenfeld (2016, p. 22) discussed the difficulties for the teacher when they teach problem-solving at school both mathematically, pedagogically and personally. The teacher would not necessarily know the students' strategies. This means that the teachers need to be professionally confident in the different areas of mathematics to know if the students' strategies are productive or not, both for solving the problem and for learning more about mathematics or about problem-solving. The teacher needs to know how and when the students need support without giving them the answers. The teacher does not know what will happen if they give problem-solving task to the students, so they have to deal with the uncertainty meaning that they need confidence in handling the situation. These difficulties are maybe even stronger in kindergarten because kindergarten teachers have less education in mathematics.

Based on Lesh and Zawojewski (2007) research, Copley (2010) described an effective problem solver in early childhood as a person who perseveres, takes risks, tries alternative strategies, remains flexible and focus attention on the problem. Copley indicated that an effective problem solver also tested hypotheses and exhibited self-regulation. Although problem-solving is a complex area and children are likely to learn to be effective problem solvers in different ways, teachers do have an impact on supporting children to be effective problem solvers. The teachers' role includes helping children focus their attention on the problem, give them sufficient time to solve the problem and to remain flexible when the children solve the problem. Further they have a role in letting children try alternatives, take risks and exhibit self-regulation. As part of our investigation, we wanted to determine if Copley's characteristics also occurred in mathematical problem-solving in outdoor activities.

When problem-solving is the focus of studies, Polya's description of mathematics as problem-solving is acknowledged as foundational in considerations of the role of problem-solving in mathematics (Schoenfeld, 2016). Polya described four stages in problem-solving: "(1) Understand the problem, (2) Device a plan, (3) Carry out the plan and (4) Look back" (Pólya, 2004). He situated problem-solving as the focus for mathematics instruction in schools, where the teacher provides a problem-solving activity which the pupils then solve. His model describes a process that supported students' problem-solving processes as it gave them a metalanguage for discussing how they could improve their strategies. Nevertheless, it is not easy to see how Pólya's (2004) stages could be used by kindergarten children, who tend to engage spontaneously in problem-solving in kindergartens with the process happening dynamically. In all of the frameworks described so far, there is an expectation

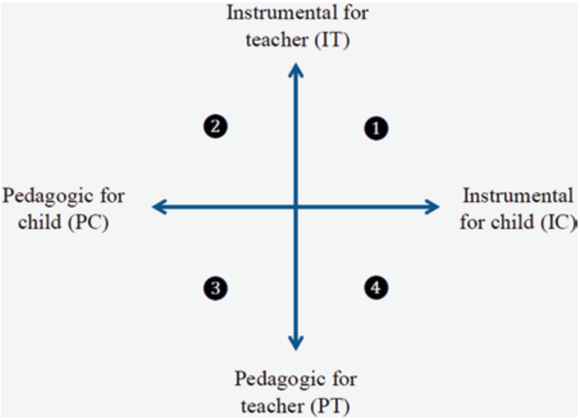

Fig. 8.1 The Didaktic Space Framework from Helenius et al. (2016, p. 161)

that the teacher has planned a problem-solving situation, with the purpose for that situation identified beforehand.

Given the tensions around the role of mathematics in problem-solving, we also considered the didaktic space (Helenius et al., 2015; Helenius, Johansson, Lange, Meaney, & Wernberg, 2016) to discuss the role of mathematics in outdoor problem-solving activities. "The 'didaktic space' utilises a 'k' in order to highlight the Nordic-German rather than the Anglo-Saxon notion of 'didactic'." (Helenius et al., 2016, p. 160). In their framework, the purpose of the problem-solving is considered from the child and the teacher perspectives, thus allowing the mathematics to be seen within spontaneous situations. The didaktic space is a theoretical framework described by Helenius et al. (2015, 2016) and is based on Walkerdine's (1988) classification of instrumental and pedagogic tasks. The primary purpose of an instrumental task is to solve a practical task, and using/learning mathematics is needed to solve the problem. In a pedagogic task, the focus is on the teaching/learning of mathematics and problem-solving is the vehicle for achieving this. Helenius et al.'s (2016) framework expands that of Walkerdine (1988) by distinguishing between whether it is the teacher or the child who focuses on solving the problem or on learning the mathematics, as is shown in Fig. 8.1.

If both the teacher and the child are focusing on solving the task so that mathematics is the tool to solve the problem, the situation is considered to be in quadrant 1. In quadrant 2, the child would consider the activity as being about learning mathematics while the kindergarten teacher's focus would be on the problem itself. In quadrant 3, both the child and the teacher are focused on teaching/learning mathematics. In quadrant 4, the teacher is focused on the teaching/learning of mathematics, while the child or children have their focus on solving the problem.

In our analysis, we identified spontaneous problem-solving situations that included elements of numbers, measurements, shapes, spatial thinking or reasoning, which are highlighted in the mathematics component of the framework plan (Directorate for Education and Training, 2017).

8.3 Methodology

The data for this study came from observations in a Norwegian kindergarten that took place during the autumn of 2017. The observations included video and audio recordings and field notes. In this kindergarten, the children were outdoors from 10 am to 3 pm each day. The first author collected the data over 7 non-consecutive days, by following a group of 22 children, aged 5–6 years old, 2 kindergarten teachers and 2 assistants. These days normally began with planning and packing for the day's excursion, then travelling to an outdoor environment, such as a forest, the seaside, a park, a valley, or climbing up a mountain, before returning to the kindergarten. The aim was to give the children new environments to explore.

8.4 Data

From the data set, one problem-solving situation was chosen for analysis using the different theoretical frameworks. To answer the research question, we wanted to analyse how the kindergarten teacher interacted with the children in a spontaneous situation initiated by a child in an outdoor setting. In this situation, the children and the kindergarten teacher were communicating about a problem-solving event.

As most previous research has focused on children's problem-solving with number ideas (Carpenter et al., 1993; Charlesworth & Leali, 2012; Rogers, 2004; Tarim, 2009), we selected a situation that involved measurement. According to Sarama and Clements (2009) and Zöllner and Benz (2016), certain ideas are common across different measurement situations. At an early age, children learn to use language to represent quantities or magnitudes. They also engage in comparing objects directly and in this way recognise equality or inequality. Later on, children come to understand how to use units to measure, usually non-standard and then standard units, and connect numbers to quantities.

The outdoor environment had the advantage of plenty of space, and it was easy to involve a lot of children. The outdoor setting also provided access to natural materials such as sticks, trees, and stones, which allowed the children to use their own creativity and imagination about how this material could be used. The nature had rich access to tools that might be used for measurement activities. The advantage was that tools as for example sticks did not have predefined properties in the same way as measuring tools for example a ruler or a tape measure. In nature the children were allowed to explore the material and find a suitable tool to solve the problem, adding an extra dimension to the problem-solving activity.

8.4.1 A Child-Initiated, Problem-Solving Activity

The episode happened outside the kindergarten building while a group of nearly 20 children (aged 5–6 years old) and 2 kindergarten staff members were waiting to begin their daily excursion. A boy (4 years old) from another part of the kindergarten arrived requesting help from a kindergarten teacher as a spade had fallen in-between some planks on a wooden boat and the gap between the planks was too narrow to retrieve it easily. The spade had a handle with a hole at the top.

In this episode, we have named the participants as: kindergarten teacher “Kt”; Child 1, the boy that initiated the problem, “C1”; Child 2, “C2”; Child 3 “C3” and Child 4, “C4”.

The transcript begins with the kindergarten teacher and the boy, C1, standing and looking at the lost spade through the slats of the boat.

1 Kt: How should we solve this? Maybe we could fish it up in some way?
2 C1: Could we use a stick?
The boy ran and fetched a stick, put the stick in the hole, but it could not reach the spade.
3 Kt: It is too short.
The boy ran and fetched a small spade and wanted to use it to grasp the lost spade. He returned and looked down on the lost spade with his small spade in his hands, then moved his eyes and looked at the small spade and after that returned his eyes to the lost spade. Finally, he took a few steps back.
Other children came along, and one child carried a spade of the same size as the one that had fallen down. This spade was longer than the first one. C2 looked at the kindergarten teacher and said:
4 C2: Is it possible to use this?
Then the children and the kindergarten teacher took a glance at the spade and down the hole.
5 Kt: It is not a good idea to push the new spade in the hole. This spade is so wide so you have to push it down to get it into the hole in the boat and that could easily result in losing two spades. What else do you think we can use?
One of the children discovered a plank that is slightly loose on one side. He lifted the plank and rotated it slightly to make the hole a bit bigger.
6 C3: Could we use a rope?
The kindergarten teacher walked inside and got some string. C3 tried to fish up the spade with the string, and finally the string reached the spade. However, the string just hung down in the hole and the spade stayed where it was. The children were quiet, looking down in the hole.
7 Kt: Do we have a long stick to use?
The children started searching in the nearby area.
8 C2: We could get a stick in “Trollskogen” (“The troll forest”, an area close to the kindergarten, approximated 250 metres from the boat).
9 Kt: Yes, that was a good idea.
The kindergarten teacher and the children moved towards “Trollskogen”. On the way, they passed the shed and on the top shelf in the shed, they found a long pole. It was difficult for the children to reach up to the pole, so the kindergarten teacher picked up the pole and they brought it back to the boat. More children were coming along and the dialogue around the boat continued.

10 C4: I want to see, I want to see.
About 10 children gathered around the boat, looking for the missing spade. They all wanted to try to fish up the spade by using the pole. Suddenly the bell rang and the kindergarten teacher had to leave with her own group for their day trip. She asked one of the other employees if they could help. The kindergarten teacher only told the other adult about the spade in the hole, not about the children's engagement in the problem. The other employee removed the loose wooden plank, put his hand down the hole and picked up the spade. Then he reinserted the wooden plank.

After this episode, the first author talked to the kindergarten teacher who said:

> Just before the bell rang, I was thinking that this activity could take up the children's mind for quite a long time, because everyone wanted to fish up the spade. My thought was that this was such a wonderful opportunity to explore measurement and their logical thinking, both the distance down to the spade and their suggestions of how one actually should grab the spade at the bottom of the boat. I was the one that got the children into this activity, but I was also the one that stopped their activity. Could we have engaged with it for longer and let the children solve it themselves? Would they have managed to fish the spade up on their own if I had told my colleague about the children's activity instead of just telling him about the spade in the hole?

8.5 Analysis and Discussion

In order to answer the research question, we analyse the interaction from different perspectives: the mathematics; the child and the teacher, using different theoretical frameworks. Using these frameworks provides an understanding of what comes into focus and how this contributes to understanding the research question.

8.5.1 An Analysis and Discussion of the Measurement Understanding

As noted earlier, the problem-solving involved the children engaging with different measurement ideas. In this situation, the problem seemed to have two parts: one was to reach down to the lost spade, and the other one was how to grab the spade and get it up. The first one involved a measurement understanding, related to the distance down to the lost spade. According to Sarama and Clements (2009), an important understanding about measurement is identifying the attribute to be measured, in this case the length. In solving the problem, the children seemed to understand that they had to focus on the distance between two points, the boards on the boat and the spade under the boards. This indicates that they could identify the attribute, length, as being important in solving the problem.

When they tried a stick which was too short, they did not try the same stick twice as they understood that the length of the stick did not change. They also did not suggest items smaller than this stick indicating that they understood transitivity in that if the length to the lost spade is longer than the length of the stick and the stick is longer than other sticks, then the length to the lost spade must be longer than the lengths of the other sticks (Bush, 2009). In the episode, the children showed competence in comparing different objects to determine if they were long enough to reach down to the lost spade. After trying a stick, child 1 ran and fetched a small spade and wanted to use it to grasp the lost spade. He used his eyes to make a comparison because he moved his eyes from the lost spade to the small spade in his hands and finally looked back to the lost spade. Then, he took some few steps back, perhaps because he concluded that his small spade was too short without having to physically make a comparison. It did not seem that the children chose items to reach down to the spade randomly. Instead they adjusted their choice of objects from what they learnt from using the previous one.

According to Lesh and Zawojewski (2007), problem-solving can involve finding an appropriate tool to solve the problem. In this episode, the children explore sticks and other spades as tools to solve the problem. One of the benefits of an outdoor environment is that there is a variety of potential tools available. In this case, where a direct comparison is needed between the tool and the length between the boat slats and the lost spade, sticks of different lengths seem more valuable for this problem than rulers or measurement tape with a fixed length that indicates standard units of measurement. In this environment, the children had the possibility to choose different objects based on their own creativity and fantasy as the most appropriate tool to reach down to the lost spade. This problem-solving situation is, therefore, in alignment with the Framework Plan's emphasis on following children's own interests.

In this interaction, the first two objects are too short, while the third tool is longer but too wide. Finally, the string had an appropriate length, but they were not able to get the string around the spade so they could not solve their actual problem of retrieving it. However, to get as far as they did, the children engaged in a lot of comparisons, which is one of the initial forms of measurement understandings (Sarama & Clements, 2009). The kindergarten teacher described what the children were doing with certain adjectives of length, such as short and long, and it seemed that the children understood the meaning of them in this context, which is noted as another competence of measurement (Zöllner & Benz, 2016).

8.5.2 An Analysis and Discussion Using Copley's Characteristics of an Effective Problem Solver

To analyse the situation by highlighting how the teacher supported the child's role as a problem solver, Copley's (2010) description of an effective mathematical problem solver was used. Copley (2010) described an effective problem solver in early childhood as a child who perseveres, takes risks, focuses attention on the problem, remains flexible and tries alternative strategies, by testing hypotheses and exhibiting self-regulation.

In the episode, the kindergarten teacher helped the children to focus on the problem and encourage the children to try alternative strategies. For example, when Child 2 suggested using a spade the kindergarten teacher told the children why it was probably not a good idea, but then she followed up with the question, "What else do you think we can use?" In this way, she did not take the onus for the problem-solving off the children but encouraged them to preserve with trying out alternative ideas even after she had rejected the initial suggestion as not being appropriate.

When a child suggested using a rope, the kindergarten teacher did not say that using a rope would not work but allowed the children to explore the suggestion. The children got the opportunity to try and fail, but their action and utterances showed that they were adjusting their understanding of the different qualities that an object would need if it was to be used for rescuing the lost spade. When child 1 suggested using a stick and it was too short, no other child suggested using a stick of the same or smaller size. Instead, the children came up with alternative strategies, based on their reflections on their previous attempts. The kindergarten teacher gave the children some time to solve the problem before she had to leave. After the episode, the kindergarten teacher reflected on how she could have maintained the problem-solving activity among the children. In this reflection, the kindergarten teacher showed that she had thought about how it would have been possible to give the children more time to solve the problem, thus supporting the children in persisting with the problem. Thus, Copley's characteristics of an effective problem solver support our understanding of how the teacher can support children to learn how to engage in mathematical problem-solving in outdoor situations.

8.5.3 An Analysis and Discussion Using Polya's Problem-Solving Stages

Polya's four stages of problem-solving provide the possibility of analysing the episode by highlighting the children's engagement with problem-solving. It is possible to see that the children engaged trying to understand the problem, the first of Pólya's (2004) stages. The children recognised that they needed enough information about the problem as well as solution strategies to identify the conditions for solving the problem. It seemed that they did this as they were also devising plans.

The children's plan seemed to be to identify items, initially by guessing and later by refining those guesses based on previous information from trying them out. However, there was no explicit discussion about alternative strategies until each idea was tested.

The third stage, "Carry out the plan", involves actively solving the problem and often this seems to be easier than stopping to plan how to solve the problem. Certainly, the guessing and trying in this episode were actions for carrying out the plan and also provided a basis for redefining their understanding of the problem and the sorts of strategies that would be successful.

According to Jacobbe (2007), the last stage "look back" or "reviewing the solution" is often the most neglected of all of Polya's stages in school. In this episode, the last stage, "look back", was not explicit in the children's problem-solving processes, rather it seemed to occur when it was found that the string could reach the spade but not bring it up. However, the children did not use metalanguage for discussing what they did that might be useful to solve similar problems in the future. Instead, the refining of the choice of tools could be seen as a result of reflection on what they had learnt from trying out items that were not appropriate. Polya's stages provide some understanding of the problem-solving situation, but aspects related to how the children interacted with the teacher and with each other were not highlighted as it had been, for example from the analysis using Copley's ideas about problem solvers.

8.5.4 An Analysis and Discussion Using the Didaktic Space

To get an understanding of the relation between mathematics and problem-solving in relationship to the interaction between the teacher and the children, we analysed the situation using the framework of the didaktic space. In the example, the kindergarten teacher did not impose their own view on the problem-solving process but tried to prompt the children to engage with the problem. By allowing the children to try out different ideas, she seemed to be respecting their suggestions in order for them to learn from their own experiences.

We classify the start of utterance 1 "How should we solve this?" as pedagogical (Helenius et al., 2016) as it seemed that the kindergarten teacher had a pedagogic purpose to encourage the child's reasoning abilities in relation to solving the problem. The kindergarten teacher asks the child for a solution instead of just telling the child how to find the solution.

In contrast, when the kindergarten teacher then suggested fishing the spade up in some way, it could be that she was now more interested in solving the problem and not in teaching the children about mathematical ideas, so we classify the end of utterance 1: "maybe we could fish it up in some way" as belonging to quadrant 1 as both the teacher and the children had an instrumental purpose. However, it may also be that in order to take the child's perspective the teacher asked a question that she herself already knew the answer to, which would then mean that this part of the utterance is also pedagogical in that it was designed to support the children learning about problem-solving strategies. The children's interest was not on learning or teaching others about mathematical ideas. Their interest remained purely on solving the problem. The mathematics and the problem-solving were pathways for solving the problem, so their purpose in engaging in it was instrumental.

In the utterances 2 and 3 there is a switch from an instrumental to a pedagogic situation. Child 1's utterance, "Could we use a stick?", is classified as instrumental because it focused on the actual problem and not on the mathematical ideas that could be learnt. The boy compared the length of his spade with the length from the top of the wooden plank down to the lost spade, showing that mathematical ideas about comparing lengths were essential for solving the problem. Still, his main purpose is to solve his problem and not on learning mathematical knowledge. This makes his utterance instrumental.

The kindergarten teachers described his physical exploration in words by saying, “It is too short”. In this way, she switches attention away from the solving of the problem, to ensuring that the child has an opportunity to learn the appropriate mathematical language. Thereby we classify utterance 2 and 3 as belonging to quadrant 4, where the purpose is instrumental for the child and pedagogic for the kindergarten teacher. Utterances 4 and 5 suggest that the situation remained instrumental for the children as they were focused on solving the problem. Engaging in direct comparisons, the children used their mathematical knowledge about measurement to try to solve the problem, rather than learning about how to make a direct comparison. Therefore, these utterances belong in quadrant 1, as both the child and the kindergarten teacher are focused on the problem.

It is not until the kindergarten teacher’s final question in utterance 5, where she seemed to take on a more pedagogical focus by asking about the children’s other ideas, “What else do you think we can use?”. The teacher’s discussion about the risk of putting another spade down the hole could also be pedagogical, as it both used measurement terms to do with distance and also provided a model of logical reasoning to describe her wish for them not to try with the other spade.

Utterance 6: “Could we use a rope?” is an example of a situation that was instrumental for the child because the child was focused on the problem. An adaptation of the child’s suggestion was tried and failed, not because it had inappropriate measurement attributes but because it was an inappropriate way to fish up the spade.

Then, the kindergarten teacher returned the focus to measurement, “Do we have a long stick to use?”. This suggests that the teacher saw this as a pedagogical situation but allowed the children to stay with their instrumental focus. Thus, we classify this interaction as belonging to quadrant 4, instrumental for the child and pedagogic for the teacher.

8.6 Implication for the Kindergarten Teacher’s Role

In the kindergarten teacher’s reflection after the episode, she reflected on how she could have prolonged the activity. She questioned whether she could have done something else to support the children’s own problem-solving activity even more. As discussed in the theoretical framework section, Burkhardt cited in Schoenfeld (2016, p. 22) discussed the difficulties for the teacher when they teach problem-solving in school both mathematically, pedagogically and personally. In this episode, the kindergarten teacher needed to know what kind of mathematics she could highlight for the children in solving this problem. This would include knowledge about how to support children's measurement understanding, as well as children's knowledge about arguing and logical thinking. The kindergarten teacher would also need knowledge about how to support the children pedagogically, by deciding what to ask questions about and how to enquire about children's own ideas about their exploration. The kindergarten teacher did not know what would happen next when the children tried to solve the problem. That is an uncertainty, challenging the kindergarten teacher personally.

A discussion related to theory might be a way of learning how to overcome such challenges. Copley's (2010) characteristics of problem solvers, Pólya's (2004) stages and the theory of the didaktic space (Helenius et al., 2015) could be theoretical tools to help the kindergarten teacher to identify problem-solving situations in outdoor education and to reflect on how to support children's own understanding in similar situations. Kaarby and Tandberg (2017) suggest that if the kindergarten teacher education to a greater extent acknowledged subject-specific play and learning experiences and discussions, it may help prospective kindergarten teachers to see the variety of possibilities that exist within outdoor activities, in this case related to mathematical problem-solving.

Another element in this situation is that it is not given that the mathematics is clear for the children while they are trying to solve the problem. In fact, it might be detrimental to the problem-solving if the kindergarten teacher had stopped to highlight the mathematics. Maybe it had been a better idea to have a discussion about the episode another day so that the children had an opportunity to reason about the experience? Lossius (2012) wrote about how pedagogical documentation might be the starting point for professional discussions with the children. In this situation, the kindergarten teacher could have taken a picture of the boat itself and let it be the starting point for a common later reflection. It also could give the kindergarten teacher an opportunity to reflect on how to ask pedagogical questions allowing children to use mathematical terms in their discussion about their previous problem-solving activity. This is in line with Pólya's (2004) stage "look back".

8.7 Conclusion

This episode shows an example of how mathematical problem-solving occurs in an outdoor child-initiated activity. In the situation, mathematics was a tool that the children used implicitly to solve a practical task in the outdoor environment. However, the kindergarten teacher was able to use the child's instrumental focus to raise measurement awareness, suggesting that she had a pedagogical purpose. The didaktic space theory provided a nuanced tool to discuss how the kindergarten teacher switched from instrumental to pedagogic perspectives and vice versa. Both Copley's (2010) ideas about problem-solving and Polya's four stages of problem-solving also provided some insight into what the children were doing. However, they did not show how the kindergarten teacher used the problem-solving situation to highlight measurement ideas about direct comparison, to support the problem being solved and also to raise these points as being valuable for the children to reflect on.

The kindergarten teacher used the children's interest and ideas to drive the problem-solving, while also providing opportunities for them to engage with direct comparisons, an initial measurement understanding. The kindergarten teacher used mathematical terms to describe the children's experiments, such as "short", "wide", "long". In this way, the teacher had a clear pedagogical stance in relationship to the children engaging in the problem-solving situation, "My thought was that this was such a wonderful opportunity to explore measurement and their logical thinking, both the distance down to the spade and their suggestions of how one actually should grab the spade at the bottom of the boat".

According to Moser and Martinsen (2010), Norwegian kindergartens spend a lot of time in the outdoor environment, but kindergarten teachers do not take advantage of this environment in supporting learning. In this chapter, we have focused on a situation where the kindergarten teacher listened to children's interests in the outdoor problem-solving activity and supported the children to solve the problem on their own, while also raising mathematical ideas. Such awareness of the potential of similar problem-solving situations might be fruitful to stimulate mathematical learning in the outdoor environment.

Mathematical problem-solving is a complex area, and the kindergarten teacher needs knowledge both in mathematics and in problem-solving. In a planned activity, the kindergarten teacher has the time to prepare and think of how to support and engage children in problem-solving, but the knowledge requirements are higher in spontaneous situations that happen in the outdoor environment. According to Kaarby and Tandberg (2017), learning opportunities may not be recognised by kindergarten teachers in the outdoor context. The discussion in this chapter around one episode may help kindergarten teachers to develop a language to talk about mathematical problem-solving in the outdoor environment and an understanding of how to support mathematical problem-solving in similar situations.

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Chapter 9
Making Choices and Explaining Them: An Experiment with Strategy Games in Kindergarten

Marina De Simone and Cristina Sabena

9.1 Introduction

The development of mathematical thinking from the time of childhood is recommended by the National Standards in most countries and is getting increasing attention in mathematics education research (see for instance the recent ICMI Study 23, Bartolini Bussi & Sun, 2018). Many studies focus on number development and, more generally, on numeracy, whereas less attention is dedicated to other topics (e.g. spatial thinking or probability), or to less content-specific aspects such as problem-solving and argumentation. Furthermore, as Levenson, Bartolini Bussi, and Erfjord (2018) point out as a future direction for research in early years mathematics, "there is little research into the nature and task design of mathematical activities and teacher's orchestration that might foster children questioning and children's own investigation" (p. 112).

In our research, we explore how strategy games may be designed and orchestrated at kindergarten level in order to promote children's mathematical processes such as reasoning, making choices, identifying regularities and relationships, producing conjectures and explaining them. We consider these processes as key tools to give young children "access to powerful mathematical ideas" (Perry & Dockett, 2008). However, their development is hardly to be achieved without a careful task design and teachers' guidance.

Research addressing primary and secondary school levels has shown that mathematical games may constitute potential learning tools for mathematics teaching and learning, in particular with respect to spatial reasoning, mathematical

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abstraction, higher level thinking, decision making and problem-solving (e.g. Ernest, 1986). By referring to Harvey and Bright (1985) and Oldfield (1991), Mousolides and Sriraman define a task or activity as a “pedagogical appropriate mathematical game” when the following criteria are met:

> has specific mathematical cognitive objectives; students use mathematical knowledge to achieve content-specific goals and outcomes in order to win the game; is enjoyable and with potential to engage students; is governed by a definite set of rules and has a clear underlying structure; involves a challenge against either a task or an opponent(s) and interactivity between opponents; includes elements of knowledge, skills, strategy, and luck; and has a specific objective and a distinct finishing point (Mousolides & Sriraman, 2014, pp. 383–384).

Taking a design-based perspective (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), we designed and carried out a study in an Italian kindergarten, based on strategy interactive games in which children were asked—in addition to playing in a supportive atmosphere—to make choices in order to win a game, and to explain them. The design is in line with the Italian National Guidelines, which underline the importance of developing, starting in kindergarten, children’s “rational thinking”, in order to allow them to “tackle problems and situations based on certain elements and being aware of the limitations of statements regarding complex issues that do not lend themselves to univocal explanations” (MIUR, 2012, p. 10, translation by the authors). Children are introduced to strategy games within a *guided play* setting (Levenson et al., 2018), in which the teacher not only offers activities and material to children but also interacts with them, in ways that support the development of their mathematical ideas and reasoning processes. We agree with those approaches in mathematics education, inspired by phenomenological perspectives on cognition, that stress how mathematical learning requires a development of scientific ways of seeing and focusing things. As underlined by Radford (2010, p. 4), students must be taught to “see and recognize things according to ‘efficient’ cultural means” and to convert their “eye (and other human senses) into a sophisticated intellectual organ”. Namely, it is necessary to promote a “lengthy process of domestication” (*ibid.*) of the way they are looking at things while learning mathematics.

In the next section we introduce the two main theoretical pillars of our study, namely the game theory for framing the design and the analysis of the activities from a global point of view, and a phenomenological frame that will allow us in detail to analyse children’s foci of attention and their evolution during the teaching experiment.

9.2 Theoretical Framework

Strategy games are strategic interaction problems in which two or more decision makers can control one or more variables that affect the problem’s results. Each individual’s situation is fully dependent on the move of the opponents, and the players know this. The decisions of each player influence the final result of the game. Therefore, every player should think, not only about his/her possible moves but also about what other players should do if they want to construct a successful strategy. According to Game Theory, usually the best strategy to win the game is discovered through a backward induction process that starts from the last move and goes back to the first one (von Neumann & Morgenstern, 1947).

Strategy games' features are in line with the previous definition of a pedagogical appropriate mathematical game. More precisely, our research is based on the assumption that these games constitute suitable contexts to develop key competences related to problem-solving and argumentation, from the time of childhood. As a matter of fact, strategy games are based on *making and comparing choices*, hence they may be related both to *planning* processes (e.g. to figure out winning strategies for the game) and to *control* processes (e.g. choosing suitable semiotic resources to represent the possible outcomes of a certain move), which are important features of genuine *problem-solving* (Martignone & Sabena, 2014). If inserted within a suitable educational design in which the students are asked *to give reasons* for their choices and to explain what they observe in order to win the game, strategy games may be also be suitable activities to develop processes of argumentation. Previous results in primary school seem to confirm such a hypothesis (Sabena, 2018); to our knowledge, research on this topic in kindergarten level is underdeveloped.

In order to find a winning strategy of a game, students need to identify regularities and to relate these regularities. This requires focusing *attention* on specific aspects of the game, ranging from more global ones to more detailed ones. Mason (2008) defines attention as "the manifestation of will, of intention. It is not a thing, but its influence can be inferred, though certainly not observed, in others. It is observation: it is the medium through which observation takes place" (*ibid*., p. 4).

When facing a mathematical problem, students focus their attention on it in different manners. In this perspective, attention has "micro qualities" that characterize *how* students are attending, rather than to *what* they are attending. In particular, Mason (*ibid*.) outlines five forms of attending: *gazing at the whole, discerning details, recognising relationships, perceiving properties, and reasoning on the basis of specific properties*. These forms of attention are often intertwined in students' mathematical activity and they are not organized in a hierarchical structure.

Gazing at the whole encompasses "gazing not really focused on anything in particular, yet taking in the whole" (*ibid*, p. 6). Rather, *discerning details* is referring to the process of making distinctions. For example, in Fig. 9.1, we can focus our attention by discerning some details of this strip: it is made up of aligned squares, some squares are greyed out and others are white, the 5 squares seem identical, etc.

Then, "*recognising of relationships* between discerned elements is often an entirely automatic development from discerning details" (*ibid*, p. 7), but it could be difficult to be aware of a relationship between two or more elements embedded in the activity. In the previous example (Fig. 9.1), it is the relationship between 2, 3

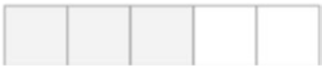

Fig. 9.1 The example of the strip (from Mason, 2008)

and 5 that is essential. For example, we could formulate the following relationships: 2/5 squares are white or 3/5 squares are grey. More formally, taking the entire strip as a unit, 3/5 of the strip is grey or 2/5 is white.

Perceiving properties is activated when a subject is aware of a possible relationship and looks for elements to reinforce/confirm it. Particular relationships are considered, therefore, as examples of general properties, i.e. particular relationships are instantiations of properties. In the example of the strip (Fig. 9.1), a student perceives a *property related to fractions* between 2, 3 and 5 if he looks for and finds other particular relationships that confirm this property. For instance, in order to distinguish which of the two fractions 3/5 and 5/3 are represented, one should know some information on what represents the unit in the strip, that is a specific property related to fraction models. Or, more simply, a child could perceive a *property related to addition*, and understand that 3 grey squares and 2 white squares equals 5 squares in all.

Finally, *reasoning on the basis of specific properties* refers to the utilization of axioms, theorems and definitions for constructing mathematical thinking. For example, reasoning about equivalent fractions on the basis of their definition.

The different forms of attending are strongly influenced and shaped by knowledge: as all phenomenological perspectives underline, attention and knowledge are deeply interrelated. In educational settings, teachers therefore have an important role in guiding learners' attention: "indeed perhaps the only thing they [teachers] can actually do for learners, is to direct learners' attention" (Mason, 2008, p. 1). This theoretical premise constitutes the background on which the teacher's role has been foreseen in our study.

9.3 The Teaching Experiment

Following a design-based methodology (Cobb et al., 2003), we designed and carried out a teaching experiment in a kindergarten school in Italy, involving twenty 5-year-old children and one teacher. The kindergarten involved in the research was a voluntary school. The teaching experiment was conducted once a week over a period of 3 months. Each intervention lasted about 2 h. Children were involved in playing some strategic interaction or probability games and in reflecting on how they could make the best choices in order to win the game. During this teaching experiment, we proposed to children three different games: a strategic game called the "Thirteen buttons game", a game about the spatial orientation inspired by the problem of the seven bridges of Königsberg and a probability game. In this chapter, we focus on the teaching experiment based on the "Thirteen buttons game", an adaptation from the "Race to twenty" (Brousseau, 1997).[1] "Thirteen buttons game" was the first game introduced to children in our study, for a period of about 1 month (10 h).

[1] The game has been adapted from the "Race to 20" to the "Thirteen token game" by Valeria Perotti for the "Gruppo di ricerca disciplinare" I.C. in Pianello Val Tidone, with the support of Donatella Merlo, teacher-researcher from the University of Torino.

The first author was present in the classroom as a participant observer and collaborated with the teacher in all phases of the design experiment. A master' student was also present and helped with videotaping the activities and with transcribing the dialogues.

9.3.1 *The* Thirteen Buttons *Game*

The game is played by two players, which alternate and play one against the other, starting from the initial situation of 13 buttons (or other tokens) displayed on a line. Each player, in his turn, can take 1, 2 or 3 buttons. It is not possible to skip the turn. The one who takes the last button loses the game.

As the reader may check himself/herself, the winning strategy involves starting as the second player and in taking the buttons according to the following number sequence: 4-8-12 (multiples of 4). In this way, the other player is obliged to take the last button in the final move and loses the game. As "the run to twenty", the game may be connected to the Euclidian division (13: 4 = 3, rest 1). More generally, in these kinds of games, the player needs to find out, for any move by the opponent player, the right move in order to win the game: hence such processes may be related to the logical scheme of coordinating a universal qualifier (for any move of my opponent player…) with an existential one (there exists a winning move), as is the case in many mathematics theorems. At kindergarten level, this is not, of course, the mathematical refinement that is meant to be considered. The *Thirteen buttons* game was chosen in order to propose an inquiry situation in which children could be engaged in making choices and checking their consequences, identifying regularities and relationships, and producing conjectures and explaining them. Even if, in playing the game, children can rely on immediate recognition of quantities up to 3–4 (subitizing), the identification of the complete winning strategy is far from being immediate and constitute a suitable inquiry situation for children, also in upper school level.

9.3.2 The Didactical Choices

Children were organized in two groups of 10, and each group was involved for 10 h in the experimentation, which has been organized in three phases. In the first phase (the first lesson, 2 h), the game is introduced as a mysterious game that the teacher and the researcher (introduced to children as a new teacher) are playing. After the initial hypothesis expressed by the children on the goal of the game, the rules are made explicit. Children are then organized in pairs and start playing, one pair after the other, in front of the other children, the teacher and the researcher.

During the second phase (three lessons, 6 h), in order to help children understand some regularities underpinning to the winning strategy, a new element has been

Fig. 9.2 The dinosaur with the analogical representation for numbers from 1 to 13

introduced: a dinosaur with 13 crests, each crest providing an analogical representation of a natural number through dots. In this way, a sort of analogical number line from 1 to 13 (Fig. 9.2) is provided. The dinosaur was introduced 1 week from the first meeting, when the rules of the game were discovered. This was pre-planned by the researcher in the organization of the experiment. We used dots for representing numbers because not all pupils engaged in the activity knew numbers written in digits. Putting buttons on the top of the crests aimed to make visible to pupils the position of each button (in light yellow in Fig. 9.2).

In the new game, an ordinal sense for natural number is introduced and associated with the buttons. On the other hand, the analogical representation is meant to scaffold children to establish which number of the sequence is considered, grounding on subitizing processes and on counting.

In the third phase (lasting 2 h), children played again without the dinosaur, simultaneously and autonomously. This phase allows the researcher and the teacher to observe if children have understood how to win the game and, more generally, which strategies they follow without the scaffolding of the analogical dinosaur line.

Since the first phases, the teacher and the researcher's roles consist of playing with children, and of prompting them to make explicit their ideas and to refine their observations; also, teacher and researcher may ask directly about the choices made by the children, especially in case of failure, so to enhance their reflection. The focus on the verbalization of incorrect procedures is planned ahead with the aim of getting children to verbalize their choices as much as possible. This way, also children who observed the matches could intervene and help their mates through creating a positive atmosphere, in which losing the game was not considered as a failure but as an occasion to reflect. Furthermore, in each phase, after a playing time, some time is dedicated to reflect on what happened during the game. In these "reflective moments", the teacher and the researcher interacted with the children in order to prompt their attention towards important aspects of the game and to foster the production of explanations on what they observed.

9.4 Data Analysis

We will briefly report on each phase and detail selected episodes, in order to report how children's focus of attention evolved and how they were engaged in meaningful mathematical processes, such as reasoning, making choices, identifying regularities and relationships, producing conjectures and explaining them.

The episodes are selected from the video recording of the entire cycle of activities, obtained through a mobile video camera. We video-recorded the mathematical activity of all the children but, for our analysis, we have selected the most meaningful excerpts. In the transcriptions, all children's names have been changed.

9.4.1 Phase 1: Introduction to the "Thirteen Buttons Game"

To introduce the game, we brought a metal box containing many buttons of different colours and shapes. We asked two randomly selected pupils to collect 13 buttons from this box, and put them in a line (Fig. 9.3). We agreed with the students that the buttons were all equivalent, although some of them had different colours and shapes.

The activity is introduced through some matches being played between the teacher Maria and the researcher Marina, without any explanation. The children observe and are then involved in a discussion on how the game works. The researcher opens the discussion with some questions: "what do you think this game is about?", "who has won the match?", "why has the winner won?". The last move—taking the last button— is soon identified as a *losing move*:

1 Silvia: She took the last button and she lost the game.
2 Bruno: And then if a button remains there, it means that Maria takes it and so Marina wins.

After the initial hypothesis expressed by the children on the goal of the game ("it's the 'take-the-buttons-game'", "who takes more buttons wins the game"), the buttons are all counted and the rules are made explicit by the teacher and the

Fig. 9.3 The buttons' box and the buttons displayed in a line

researcher. Children are then organized in pairs and start playing, one pair after the other, in front of the other children, the teacher and the researcher. Sometimes they play against Maria or Marina. In this phase, the children insist on being the first player; they almost always take the maximum number of buttons that they can, that is 3, saying that "taking more" increases their chances of winning. Also, they show a tendency to take the buttons without following a precise order, so, for instance, they take them from both sides of the line. We remark that this behaviour is allowed by the rules of the game, which do not establish anything about which buttons are to be taken, or any order to be followed.

These first naive considerations will change for most children during the development of the teaching experiment. During the first phase, almost all children understood that taking the second last button allowed them to win. They did not spontaneously verbalize this choice, but just enacted it. This is an example of a match between Marina (the researcher) and a pupil, Pia: Pia begins the match and she takes three buttons, then Marina also takes three buttons, Pia takes three buttons, Marina takes two buttons and, in this moment, Pia is very careful to take the second last button and to stop.

We will now analyse, more in detail, one example from this first phase, in which the researcher (Marina) interacts with Stefano, after playing with him. Marina has just won a match against Stefano and asks the child to reflect on why he has lost. In order to narrow down Stefano's attention to a specific aspect of the game, Marina produces the same configuration that Stefano was facing in his second last move: four buttons are left on the floor. During the match, in front of this configuration Stefano had taken two buttons; hence Marina could take one button and win (because Stefano had to take the last one). The reader is reminded that when four buttons are left on the floor, a player has the possibility to win the game, taking three buttons and leaving the last one to the opponent.

After setting the configuration with four buttons, Marina asks Stefano *what he could have done to win*:

3 Marina (researcher): How many buttons should you take to win?
4 Stefano: One, two, and three (*pointing with his finger to the buttons thathe intends to take,* Fig. 9.4*, and stressing the number "three"*)!

Focusing on the specific configuration with four buttons, Stefano is now able to identify the winning move. Referring to the structure of attention frame (Mason, 2008), we can say that, thanks to the researcher's intervention, Stefano is *discerning the details* in this specific configuration, and in this way he is discovering a *local strategy*, i.e. a strategy that allows him to win the game in a specific configuration that can be reached during a match. However, at this moment, Stefano is not aware of the conditions that allow him to reach such a configuration with four buttons; nor that if

Fig. 9.4 Stefano's pointing gestures while counting to find the right move

he repeats this strategy backwards, he will obtain other winning positions. In other words, he, as well as the other children, has not yet *recognized the relationships* between the positions of the buttons, which will allow them to seize the *global strategy* to win the game (consisting in the 4th, the 8th and the 12th positions). This will be the goal of the second phase, in which the representation with the dinosaur is introduced.

9.4.2 Phase 2. The Discovery of the "Magic Buttons"

The second phase lasts for about three lessons (about 6 h) and it is (again) a pair of children playing in front of the class. In this phase, the dinosaur with the analogical number line is introduced. Children discover that there are special buttons that allow a player to win (the 4th, 8th and 12th position): they call them "magic buttons".

The discovery starts from the 12th position: if a player manages to take it, he/she wins the game because the opponent player is obliged to take the last button.

5 Luca: This is important (*he indicates the 12th crest*), because if we take this button (*Luca imagines that the button is placed on the crest*), the other [*player*] loses the game.
6 Researcher: Which is this button?
7 Luca: (*he counts the dots in the crest,* Fig. 9.5) 12, the button in the crest with 12 dots.

As we see in the excerpt, the penultimate button is recognized as the number 12 after the researcher has asked pupils to name the button that Luca was pointing at. The analogical representation on the dinosaur's crest is used by the child to identify this number (in Fig. 9.5, he is counting the dots in the crest).

The discoveries about the 8th position, and later the 4th one (going backwards as usual in these kinds of games) occur later, during the reflective moments that the researcher sets between series of matches. We can see an example in the following excerpt, in which a child, Giuseppe, focuses his attention on the 8th button:

8 Researcher: What have you discovered thanks to the matches played on the dinosaur?
9 Giuseppe: The 8th button is the most important, it is magical

Fig. 9.5 Luca counts the dots in the 12th crest of the dinosaur

10 Researcher: Can you show me which one is it?
11 Giuseppe: 1, 2, 3, 4, 5, 6, 7, 8 (*he is counting the crests until 8, then he indicates the 8th position,* Fig. 9.6a)
12 Researcher: Why?
13 Giuseppe: It is important because…and then it is also important this one here (*indicating the 12th button,* Fig. 9.6b), and then this one is left (*indicating the 13th button*) and the other loses.
14 Researcher: So, we discovered that there exist at least two magic buttons, one on the crest with 8 dots and the other on the crest with 12 dots.

Answering to the request of the researcher, Giuseppe is verbalizing the strategy that he followed during his matches, and which allowed him to win: taking the 8th and then the 12th button, he calls the ones belonging to such a strategy "magic buttons" (and the other children will soon share this name also). Asked to explain why the 8th button is a "magical one", he explains that it is "important" and immediately links it to another important button, the 12th (line 13, Fig. 9.6a, b), specifying that taking it allows a player to win (because the other player loses). In this way, Giuseppe is making an important step in order to build a global strategy to win the game, by *coordinating the discerning of the details and the gazing at the whole* strategy to win the game. This coordination requires discovering which buttons are subsequent winning moves of the game, and are therefore related to each other.

The discussion continues and the researcher prompts children to clarify why the button in the 8th position is a winning one. To reach this goal, she involves them in imagining a hypothetical situation during a match:

15 Researcher: Why, if I take the 8th button, do I surely win? Let's pretend that Felice and I have played and that I have taken the 8th button (*she organizes the buttons in order to reproduce this configuration,* Fig. 9.7). What happens if I take the 8th button? That is to say, if I arrived at the 8th button, I stopped and I took it?
16 Luca: You win
17 Researcher: Why, Gaia? Explain what Felice can do now.
18 Luca: If he takes three buttons (*he takes 3 buttons from the configuration*), you take 1 and you win, you.

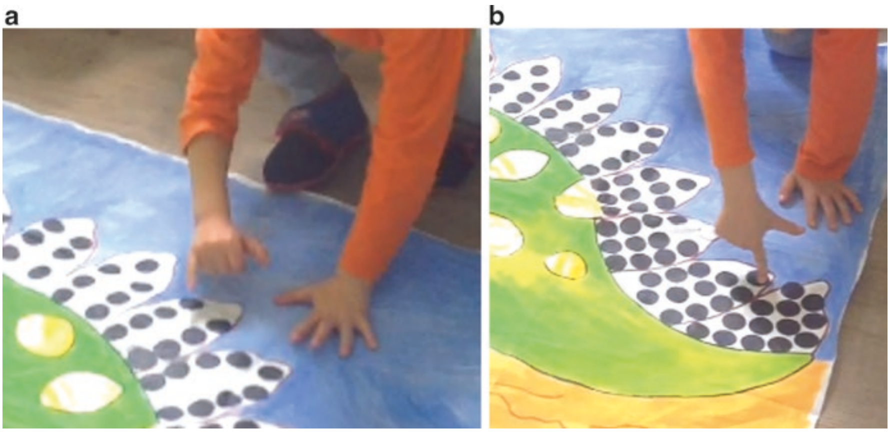

Fig. 9.6 (**a**, **b**) Giuseppe indicates the 8th and then the 12th button on the dinosaur line

Fig. 9.7 The researcher sets the imagined game configuration to discuss, with five buttons

19 Researcher: Ok, but can he take 2, for instance?
20 Luca: Yes (*placing back the buttons and taking 2 of them*), but you take two and you win.
21 Pietro: You take two and you win.
22 Gaia: If he takes two (*indicating Felice*), but you (*indicating the researcher*) you take one, he wins.
23 Researcher: This idea that Gaia said is very interesting, because we are imagining to know how to play well, but if I did not know what to do to win, then what Gaia said could happen: that is if Felice takes two buttons, it could be the case that I don't know that in order to win I have to take two new buttons, because I don't know that the 12th one is magical, and so what Felice does?
24 Gaia: He takes 1
25 Researcher: And who is going to win?
26 Gaia: He wins (*referring to Felice*)
27 Researcher: So, summing up, if we have four buttons, if he takes three buttons, I take
28 Many children: One button and you win
29 Researcher: If he takes two buttons, I take
30 Many children: Two buttons and you win
31 Researcher: If he takes one button, what do I have to do to win?
32 Many children: Three
33 Researcher: Ok? Do you all agree?
34 Many children: Yes

Explaining why the 8th button is a winning one requires explaining that, for any opponent's choice (1, 2 or 3 buttons), the player can make the winning choice (3, 2 and 1 buttons, respectively). And that such a choice leads to a win because it allows taking the 12th button, so leaving the last one to the opponent. The children have well established that the 12th button must be taken in order to win, so this last part of the argument is not focused here anymore. The discussion on the first part of the argument is introduced through an *imagined match* between the researcher and a child, Felice (lines 15–21). Luca immediately imagines one of the possible moves that Felice can do, that is taking three buttons (line 18). The researcher then guides an *argument by cases*, corresponding to all the possible moves that Felice could do (lines 19–21). Imagining all the possible moves requires Luca (and all the children who are silently attentive) to foresee a sequence of two moves, *discerning the details* of the situation, and also *discerning the relationships between these details.*

Even if the researcher is scaffolding the reasoning by cases—which is indeed a complex reasoning because it requires to consider simultaneously different choices and consequences—the children appear to follow the argumentation process, thanks to the imagined moves in the fictitious match. This leads, for instance, Gaia to propose an alternative move for the researcher so that Felice could win the game (lines 22–26).

From line 27 onwards, the researcher sums up all the different cases, and now, many children participate in answering her questions. This shows that they were also actively participating in the previous part, in which they were active listeners to the exchange between the researcher and Luca.

In the following excerpts, we see how the children's focus of attention shifts to *recognizing relationships* because they begin to understand a global strategy that allows them to win the game from the beginning, generalizing the relations among different buttons and moves. This is realized through the identification of the 4th button as a winning one, together with the 8th, and hence the 12th. The match between Greta and Giovanni is a catalyst for this discovery:

35 Greta takes three buttons
36 Giovanni takes one button (the 4th one), stops and comments: I took 1

37 Researcher:	Why did you take 1?
38 Giovanni:	Because I win anyway
39 Researcher:	Why? Can you repeat because it was not clear. You took the button in which position?
40 Many children:	The 4th
41 Giovanni:	I did it just because I wanted to win and I knew how to. If she now takes 2, then I take 2 and I win.
42 Researcher:	And if she takes 3?
43 Giovanni:	I win immediately (because it would take just one button to win)
44 Researcher:	Ok, so that button has been a smart move. Which is?
45 Alex:	The fourth

After taking the button in the 4th position, we see that Giovanni stops and realizes that he will win the move, no matter the move of his opponent (line 38: "I win anyway"). He is *gazing* the mathematics *as a whole* and he concludes that he will reach the 8th position and hence win the game (line 41). In reaching this conclusion, he does not need to go through all the details about what happens after taking the 8th button, which has already been discussed and is not in the forefront in the child's attention. This shows that this child has *recognized the relationship* that exists between taking the 4th and the 8th button and winning the game. The other children are closely considering what Giovanni is saying and at the end of the match, when Marina asks which is the button that was crucial to win the game (line 44), it is another child, Alex, who answers (line 45).

9.4.3 Phase 3: The Dinosaur Goes on Holiday

In the third phase, the game has been played again without the dinosaur. In general, in this phase children selected buttons following their sequence in the line (1–13), rather than randomly as in the first phase. This helps them to recognize the "magic buttons" that were identified with the dinosaur.

We report a short excerpt in which Giorgio shows to the researcher the "magic buttons" before the beginning of a match:

46 Researcher: Giorgio, could you show me the "magic buttons"?
47 *Giorgio points to the fourth button (*Fig. 9.8*)*
48 Researcher: Then?
49 Giorgio: This one (*Giorgio points to the 12th button,* Fig. 9.9)
50 Researcher: Then?
51 Giorgio: This one (*Giorgio points to the 9th button,* Fig. 9.10)
52 Researcher: Are you sure?
53 Elia: You have to count to know which is the eighth button
54 Researcher: It's right! You have to count from where you have started. Where did you start?
55 Giorgio: There, 1, 2, 3, […], 8 (*he counts until the eighth button*)
56 Researcher: Ok! Who begins?
57 Giorgio: You!

We see that Giorgio recognizes the 4th and the 12th winning buttons even without the support of the dinosaur. He makes a mistake in indicating the 8th one, and points to the 9th one (line 51). Elia suggests to count in order to find out which is the 8th button (line 52). Following his suggestion, Giorgio counts up to eight, starting from the first button in the line. Finally, when the researcher asks who is going to begin the game, Giorgio reacts immediately and proposes that his mate can start:

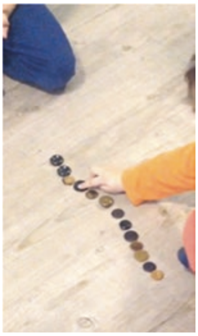

Fig. 9.8 Giorgio points to the 4th button

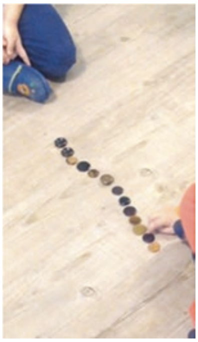

Fig. 9.9 Giorgio points to the 12th button

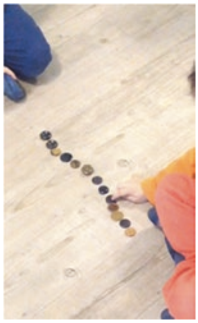

Fig. 9.10 Giorgio points to the 9th button

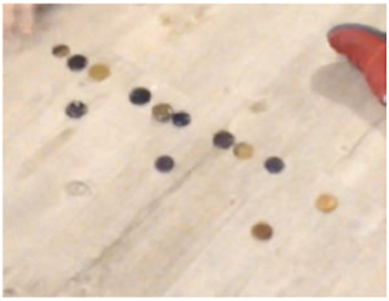

Fig. 9.11 The magic buttons are made evident by shifting them from the line

this indicates that he knows how to apply the winning strategy from the beginning of the game.

At the end of the third phase, almost all children manage to identify the winning buttons without the support of the dinosaur, but sometimes they need to mark them before beginning the match, moving them to the side of the line (Fig. 9.11).

It is through this semiotic trick that most children manage to *understand* the structure of the winning strategy *as a whole,* and make clear that, in order to win, it is not convenient to play as a first player.

9.5 Conclusion

The didactic activity based on the "Thirteen buttons" game has been designed to foster the development of basic competences related to mathematical problem-solving and rational thinking from an early age. In particular, in our study we investigated how the game can be exploited by the teacher to introduce children to making choices and checking their consequences, identifying regularities and relationships, producing conjectures and explaining them.

From the first encounter with the game, to the final identification of the "magic buttons" that allow a player to win, we could actually identify an *evolution in the children's focus of attention* (Mason, 2008). At the beginning, thanks to the ploy of making the teacher and the researcher play without any explanation, the children's attention is focused on the rules of the game, and in understanding who is winning. Then, during the first game moment they get immersed in the game, but soon they start to discern details about specific choices that can be made in order to win: this leads them to identify the 12th button as a magic button. The discerned details are then organized to foresee a sequence of two consecutive moves through an anticipatory thinking, which leads to the discovery of the 8th magic button.

The number representation with the dinosaur played a role in such an organization because it allowed the children to consider the buttons in an ordered way from 1 to 13. Discerning details is not enough to understand all the sequences of the winning buttons, which requires focusing the attention to the game in a global way, and to identify or recognize the relationships between the various positions of the buttons. In upper school levels (e.g. primary school), such relationships may be related to multiplication or division schemes between natural numbers.

In the study, the didactic design has been based on the alternation of playing moments and reflective moments guided by the researcher (who took the role of the teacher), and on the mediation of an analogical representation of the ordered sequence of numbers from 1 to 13 (the dinosaur).

Playing moments are those moments in which children are familiarized with the game, between themselves or with the researcher. The focus of attention is on participating in the game (possibly on winning), and feeling pleasure and enjoyment are essential parts of the game. *Reflective moments* are organized by the researcher so that children may stop playing and dedicate time to reflect on the game situations, without being immersed in the playing action. Throughout our analysis, we could observe how intertwining reflective moments and playing moments are crucial in engaging children in fundamental mathematical processes such as making hypotheses and checking them, identifying relationships, producing arguments to support a claim. To this purpose, the role of the researcher/teacher appears as crucial and from our data we found *three different kinds of prompts* that are particularly successful:

Invite children to *reflect back* on the matches they have just played, on what they have *discovered* when playing, and on *why* their discoveries are true (see phase 2, lines 5–11) or *why* they made the winning move (lines 35–39);

Ask children *to explore other possibilities* that they did not consider when playing. Indeed, in this way children may be introduced to the actual *existence* of other possibilities—which is crucial to strategy games, as in genuine problem-solving, as well as to argumentation processes. This has been accomplished in particular by asking children to *imagine a hypothetical situation* that could occur during a match: in this way, the researcher prompted the children to clarify why the button in the 8th position was a winning one (see phase 2, lines 12–31 and 39–42).

Ask children to *make explicit their strategy before playing* (see phase 3, lines 46–57).

From our observations it is particularly important to get children involved in *imagined matches* in which they can *foresee possible moves*. Asking the children to imagine a match without playing it has allowed children to make explicit all the possible moves that a player could do and to analyse their consequences in the game. The ideas emerging during such reflective moments have then been exploited in the subsequent actual matches. Previous studies carried out with similar games in primary school suggest that such imagined fictitious matches may be enacted by older students, also without the teacher's mediation, and may be exploited in the argumentation processes that involve reasoning through using cases (Sabena, 2018). In this study, we provide evidence of their educational role, also at kindergarten level, at least when the teacher suitably organizes the reasoning through using cases, coordinating it with the imagined matches, which become present during the discussion.

The mediation of the number sequence representation with the dinosaur has not been deeply analysed, and this constitutes a limit of this study. We have evidence that, thanks to the representations, children could easily consider the buttons in an ordered way from 1 to 13, and that this order is also kept during the third phase, when the "dinosaur goes on holiday". This appears to confirm that the pedagogical choice to build on the ordinal sense of numbers, rather than a cardinal one, did support children to identify the structure 12th-8th-4th buttons, then reversed to 4th-8th-12th. Another option that could be explored is to substitute the analogical representations with the black dots in the crests with the numbers written in Arabic digits. We hypothesize that this change would not make a fundamental difference in the design; of course, it requires that children are at ease with the Arabic digit representations of numbers, up to 13. Another option could be to start playing from the beginning on a number line, whereas a rather different choice would be allowing children to continue playing with much less scaffolding. Also, many different didactic choices could be explored through considering how children are organized during the play, which could be different from children playing in pairs in front of the class (e.g. allowing multiple free plays among peers and then an open discussion). All these choices are deeply affected by pedagogical principles, and only some of them are coherent with our phenomenological perspective on mathematics learning, which also shaped the researcher's/teacher's role in the design. We would be pleased if other researchers may explore other didactic designs, based on different theoretical premises. We will continue our research in the direction of establishing whether this kind of didactic design and specific prompts made by the teacher may constitute good scaffolding for students in activating mathematical thinking processes (also with respect to metacognitive skills) in similar or different situations.

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Chapter 10
Measuring Rice in Early-Childhood Education Activities: A Bridge Across Discrete and Continuous Magnitudes

Maria Mellone, Anna Baccaglini-Frank, and Francesca Martignone

10.1 Introduction

Measurement of different kinds of magnitudes is one of the goals presented in the section "*learning about the world*" in the Italian National Guidelines for the primary school curriculum (MIUR, 2012). Moreover, within the international panorama, measurement is considered to be a necessary and fundamental concept to master when learning to reason mathematically (NCTM, 2000; OECD, 2010). Measurement is considered a grounding core of mathematics school curricula because of its special role and pervasiveness in so many aspects of the practical and social life. Indeed, measuring quantities is a common action of our daily life, although it requires culturally sophisticated operations in terms of both action and abstraction. For example, in order to measure quantities, we choose a convenient sample to be compared to the quantity to be measured. Then, one has to somehow count how many times the unit of measure fits in the quantity to be measured and deal with a reasonable approximation based on the goal of the measurement. This planned action and its connections with numbers have been one of the crucial triggers of the human social and cultural evolution (Aleksandrov, Kolmogorov, & Lavrent'ev, 1965).

Since measuring is a complex cultural practice that involves links to numbers, in order to introduce it in school, some appropriate pedagogical conditions need to be created. We believe that such pedagogical conditions must stem from and be

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coordinated with children's naïve skills in counting and quantifying. With this aim, we carried out an explorative study, to analyze children's skills, upon their entrance in first grade, when working with a particular substance, rice, which can be treated as an intermediate between discrete and continuous quantities. The idea to use rice is related to the evidence coming from the fields of both neuroscience and psychology that at the origin of human numerical insights there is a strong link between the management of continuous, uncountable quantities and large amounts of discrete, countable objects (Gallistel, Gelman, & Cordes, 2006; Piazza, 2010).

In this study, we present some qualitative results, focusing on how young children manage the task of comparing quantities, judging whether there is "as much [rice] as" in different piles. We report on different strategies activated by the children to accomplish the tasks they were given. We analyze these strategies reflecting on which aspects can be potentially used by a teacher in order to build mathematical meanings associated with the measuring process. Moreover, acknowledging the crucial role of artifacts both at a psychological and at a social level (Vygotsky, 1978), we offered the children different *artifacts* to choose from and work with if they thought it might help. The idea of an artifact is very general and encompasses several kinds of productions of "human beings through the ages: sounds and gestures; utensils and implements; oral and written forms of natural language; texts and books; musical instruments; scientific instruments; tools of the information and communication technologies" (Bartolini Bussi & Mariotti, 2008, p. 746).

10.2 Theoretical Background of the Study

Historically and pedagogically (natural) numbers are firstly seen as tools for counting. Most approaches in mathematics education prevalently assign, with minor differences, a primitive and dominant role to natural numbers and to the action of counting discrete, countable magnitudes. The management of countable magnitudes usually consists of counting the discrete entities of which it is composed. In general, approaches from the Western tradition tend to introduce natural numbers before introducing measurement. For example, Sfard (1991) proposes a reconstruction of the number concept within the process/object dialectics, according to which the development of a mathematical object always starts as a process. For example, the process "subtracting" as "taking away" is eventually reified into an object, in this case "subtraction of integers." In Sfard's perspective, the counting process constitutes the starting point, whereas the measuring process appears only at a later stage, when rational numbers are generated. A different approach was proposed by the Russian psychologist Davydov, who placed the experience of measuring continuous quantities as preliminary to the introduction of numbers (Davydov, 1982). According to Davydov, counting itself may be conceived as the particular measuring process of sets of discrete objects in which the unit of measure is the discrete object itself. Therefore, Davydov suggests that in early education, managing continuous quantities should precede the introduction of natural numbers. This crucial idea has found several followers and initiatives all around the world; one among others is the project Measure Up (see, e.g., Dougherty & Slovin, 2004).

In our research, we are not arguing in favor of assigning priority neither to the discrete nor to the continuous approach. Instead, we wish to reflect on possible effective strategies of early cultural mediation in order to create solid links between the discrete and the continuous aspects of numbers (in the direction of Iannece, Mellone, & Tortora, 2009). This approach seems to be in line with neuroscientific findings about the preverbal mathematical systems. Two systems for numerical quantification, which children are equipped with before symbolic learning (spoken or written language, number systems, symbol systems, and so on), have been identified: the object tracking system (OTS or "parallel individuation") and the approximate number system (ANS or "analogue magnitude") (Piazza, 2010). The first system is specialized in recognizing the numerosity of small groups of objects (usually up to four) by subitizing, while the second one provides "*an analogical representation of quantities, in which numbers are represented as distributions of activation on the mental number line*" (Dehaene, 2001, pp. 10–11). What is especially interesting is that the second system is activated not only for comparing and manipulating continuous quantities but also for perceiving and processing discrete quantities in an approximate way. Moreover, there is evidence of existing links between these abilities regarding approximate estimation of nonsymbolic magnitudes (e.g., sets of dots or grains of rice), approximate evaluation of spatial extension, and the numerical symbols (Piazza, 2010). Moreover, recent research (e.g., McMullen, Hannula-Sormunen, & Lehtinen, 2014) showed how children's tendency to spontaneously focus on quantitative relations can be used as a predictor of their following knowledge of rational numbers. In particular, during McMullen et al.'s longitudinal research project (McMullen et al., 2014), a group of children was followed for 4 years. In particular, these children's ability to evaluate quantities of cake or of rice in first grade positively correlated with their abilities to manage fractions 3 years later (their conceptual knowledge of fractions).

We took inspiration from these studies to design a particular task with the aim of exploring processes related to these existing links between the approximate estimation of large amounts of discrete quantities and symbolic systems.

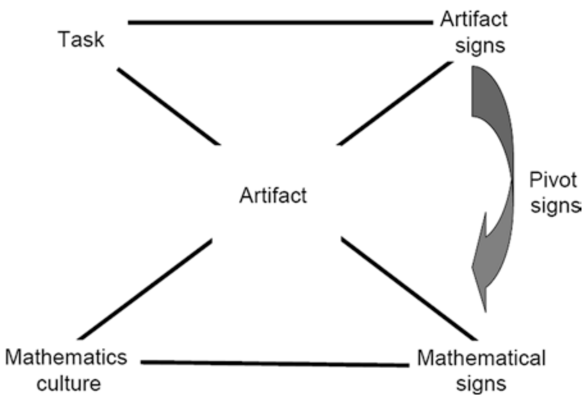
This study was set up with the goal to gather information on strategies that children, raised in the Italian culture, may produce, based on possible cognitive links between the system of approximate estimation of large amounts of discrete quantities and the symbolic system. Moreover, we focus this study on a specific age range: that of children at the beginning of primary school, when the exposition to symbolic systems has already started, but is not very advanced. We deal with *spontaneous daily concepts* (Vygotsky, 1987) that could be activated by children managing a particular kind of substance (rice) that is *in between* discrete and continuous magnitudes. We also offered the children a variety of artifacts to choose from if they thought any of these would be helpful, so that we could analyze the use of the particular artifact and, then, the potential semiotic activity produced in accomplishing the task. We take the perspective offered by the Theory of Semiotic Mediation inspired by Vygotsky (1978, 1987) and developed by Bartolini Bussi and Mariotti (2008) specifically for mathematics education. The Theory of Semiotic Mediation states that the progressive and intentional introduction of artifacts in educational activities can play a crucial role if teachers use them as "instruments for *semiotic mediation.*"

> […] the teacher may guide the evolution towards what is recognizable as mathematics. In our view, that corresponds to the process of relating personal senses (Leont'ev, 1964/1976, p. 244 ff.) and mathematical meanings, or of relating spontaneous concepts and scientific concepts (Vygotsky, 1934/1990, p. 286 ff.) (Bartolini Bussi & Mariotti, 2008, p. 754).

In particular, to analyze the role of artifacts in processes of semiotic mediation, Bartolini Bussi and Mariotti (2008) use a set of interpretative tools from Rabardel's instrumentation approach (Bèguin & Rabardel, 2000; Rabardel, 1995), according to which a subject, engaged in a goal-directed activity, can build schemes of instrumented action for an artifact. The artifact, together with the *utilization scheme* a subject has developed for using it to accomplish a task, becomes an *instrument*. In Bartolini Bussi and Mariotti's framework (see the schema in Fig. 10.1), the instrument is a tool with a double potential: it is a tool for the pupil to accomplish a given task and also a tool for the teacher to use in the task of helping pupils construct mathematical meanings stemming from the meanings that emerged in the situated context.

According to the Theory of Semiotic Mediation (see Fig. 10.1), classroom activities can be arranged around the use of an artifact and specific appropriately designed tasks presented to the students. In the design of the activities, the artifact is closely related to particular mathematical content and should be used by the students to solve the assigned tasks. During their solving processes, students produce signs called *artifact (or situated) signs* (they could be words, phrases, drawings, gestures, etc.), which may present a certain autonomy from the artifact and task context. In this case, the situated signs can also be used as *pivot signs* by the teachers in order to create explicit connections to mathematical content. Eventually, pivot signs can be transformed into more formal *mathematical signs* (Bartolini Bussi & Mariotti, 2008). Pivot signs are particularly situated signs produced in a specific context (including a task and an artifact), but that also can be put in relation with more formal mathematics. In this sense, they present a shared polysemy: they may refer to specific instrumented actions, to oral or written language, and, at the same time, they are used to link to mathematical signs. The polysemy of the pivot signs can be used by teachers as hinge for fostering the transition from the context of the specific task and artifact to the context of mathematics. Indeed, children, as in the case of our interviews, use a variety of terms and gestures with situated meanings; among these

Fig. 10.1 A diagram summarizing the Theory of Semiotic Mediation and the role of pivot signs (Bartolini Bussi & Mariotti, 2008, p. 757)



signs, we can identify those with a potential of leading to mathematical signs. Situated signs with this potential are good candidates for the teacher to use later to foster the development of mathematical meanings.

Since the interviews analyzed in our study proposed practical activities involving interaction between a researcher and a child, we carried them out outside the classroom context. Thus, they cannot be considered classroom activities. This is an important difference with respect to the activities traditionally studied and described using the Semiotic Mediation framework introduced above. Indeed, the idea of pivot signs was introduced to study the teacher's use of particular situated signs when guiding the students in the construction of mathematical meanings. Pivot signs arise as situated signs, with meanings relative to a particular artifact, and they are used by the teacher in relation to selected mathematical signs with culturally accepted and shared mathematical meanings. Pivot signs act as hinges between situated meanings and more general mathematical meanings. In our study, we label the signs identified in the interviews as *potential pivot signs*, underlining that they have the potential to later become pivot signs in classroom activities. The specific research questions we address in this study are as follows: What strategies (in relation to selected artifacts) do children at the beginning of primary school use to evaluate small quantities of rice? Which potential pivot signs can be identified in young students' attempts to evaluate quantities of rice?

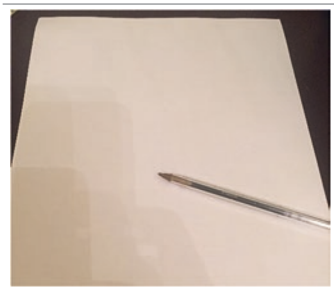
Indeed, knowing which signs that later could be used by the teacher as pivot signs can be quite useful for a teacher when designing or carrying out educational activities related to the same mathematical content, in our case measurement of length, surface, and volume. We believe that such knowledge should become part of teachers' Pedagogical Content Knowledge (Shulman, 1986). Therefore, it is important to conduct analyses of students' responses, even in a setting like that of our interview, in order to identify such signs.

10.3 Methodology

In order to gain insight into the strategies mobilized by young children to compare and to measure, we chose to work with a substance typically treated as continuous but that can also be treated as many discrete entities, i.e., uncooked rice; and we asked children to judge whether there is "as much as" of a certain quantity of rice. Indeed, we wanted children to be able to propose strategies through which they could treat the substance as continuous, but we were just as interested in observing whether some children (and which ones) would opt for strategies handling the rice discretely, as many grains.

We interviewed 5- to 6-year-old children (14) from an Italian public school at the beginning of first grade (in the third month from the beginning of the school year). The children did not all have the same preschool experience, and two had not attended preschool at all. During the first months of first grade, the teacher had introduced a variety of counting activities with artifacts, such as a horizontal abacus with which children would count and straws for counting and representing small numbers. The class had also been taught to write the symbols of natural numbers up to 9. The children were interviewed one at a time through clinical interviews (Hunting, 1997), outside the classroom setting, in social interaction with the interviewing researcher (one of the authors). The experimental set up consisted in the following material and script: the interviewer picked up a bag containing about ½ kg of rice, and then poured about 200 grains onto the table lifting the bag and slowly letting the rice pour out. Then, she asked the child (phase 1): “Now can you please give me *as much* as you have in front of you?” When the child had finished making another pile of rice, the interviewer asked: “How are you sure they are the same [pointing to one pile of rice and then to the other]?” and she offered the children a variety of artifacts (Table 10.1) to choose from if they thought one would be helpful to answer the question (phase 2).

Table 10.1 Artifacts offered to students

Picture of the artifact	Description of the artifact and possible expected utilization scheme
	Spoon—measure the rice in spoonfuls, treating it as a continuous substance, possibly recalling utilization schemes developed outside of school
	Transparent plastic cups—compare the quantities of rice giving them the same practically two-dimensional shape
	Unbendable straws—“line up” the grains of rice of the pile to measure, and measure the rice in “full-straws”
	20 cm ruler with millimeter markings—use strategies to measure a dimension of two-dimensional or three-dimensional arrangements of the rice
	A pen and paper—to write something down, possibly to help in the counting process or to draw something

The approximate number of grains was chosen to allow the counting process for the comparison, but at the same time to make it a difficult and maybe even a discouraging task, in order to allow different strategies to emerge.

As introduced in the previous paragraph, we attribute a crucial role to the use of artifacts: after an answer to the first request, the interviewer offered the children the artifacts in Table 10.1. By introducing these artifacts, we wanted to gain insight into the children's utilization schemes related to the chosen artifacts and to the management of continuous substances, or whether they preferred to count. The artifacts chosen were a spoon, some transparent plastic cups, some straws, a ruler, and a pen and paper. The rationale of the choice of these artifacts is the following: (1) the spoon could be used in the attempt to measure the rice in spoonfuls, treating it as a continuous substance, possibly recalling utilization schemes developed outside of school; (2) the plastic cups could be useful for comparing the quantities of rice giving them the same practically two-dimensional shape[1]; (3) the straws were chosen due to a previous study (Mellone, 2008) in which a child had used a straw to "line up" the grains of rice of the pile he had to measure, and then proceeded to measure the rice in "full-straws"; (4) the ruler was offered in case some children felt inclined to use strategies in which they wanted to try to measure a dimension of two-dimensional or three-dimensional arrangements of the rice; and (5) a pen and paper were offered if children wanted to write anything down, for example, for helping themselves in the counting process or for drawing something. The interviewer would set all these items in front of the children during each interview, but did not insist on having them use anything if they did not immediately want to. We chose not to propose other artifacts, like scales, because we preferred not to involve the concepts of mass or gravity.

10.4 Analyses

In the analysis carried out in our study, in addition to the children's strategies and utilization schemes with the artifacts, we intended to identify some *potential pivot signs* produced by these children during the accomplishment of the task. We refer to these signs as potential pivot signs because they have the potential of evolving toward mathematical signs linked to the measurement process; indeed, these signs can be built on in the design of later teaching interventions grounded in the semiotic mediation framework. We will point out all these aspects in the excerpts we chose to present in the next section.

[1] The about 200 grains of rice form a thin layer in the plastic cup, if the quantities to compare were slightly larger, we expected that the cups might be used to compare the heights of the rice in each cup.

10.4.1 Strategies Before Introduction of the Artifacts

After a first round of analysis of the strategies adopted in the first phase of each interview, we found that the main distinction can be made on whether the children's strategies are oriented toward an evaluation of the amount of rice in terms of numerosity of the grains, surface, or volume occupied by the quantities of rice considered.

10.4.1.1 Evaluation of Numerosity

Four of the fourteen children focused on numerosity activating, at least initially, counting strategies. The children who chose to do this seemed quite confident in their counting skills, but only two seemed to really be acting in accordance with Gelman and Gallistel's (1978) counting principles, and to be using the correct strategies and number words even when dealing with large numbers that had not yet been introduced at school.

> Child 1: These are more [pointing to the larger pile] because I counted them. And these here were one hundred eighty and these here fewer…one hundred…one hundred twenty-three.

This child chooses to measure a set of discrete objects (the pile of rice) using as a unit of measure the single discrete object (a grain of rice), and then to count how many units there are in the group. The child uses the comparison between the numbers he found by counting as a means to compare the cardinality of the two piles. Here, the number words and the comparison made "these are more [...] one hundred eighty [...] these here fewer [...] one hundred twenty-three," are "situated signs" because they are related to the particular task, but they also refer to a standard measurement strategy potentially extendable to other situations. These are, thus, argued to be potential pivot signs for a subsequent teaching intervention about counting discrete quantities.

10.4.1.2 Evaluation of Surface

Three children considered the properties of the piles, seeing them as two-dimensional objects. One child formed small flattened piles and rounded them into similar circular shapes. When the interviewer asked how this is helpful, he answered:

> Child 2: This is a lot big and this is big

The "a lot big" [it: "tanto grande"] seems to be deictic, as if it indicates a specific quality of that pile, almost saying that the pile is "very large," and the other "big" is used to express that the second pile has a similar property as the first, but it is a bit smaller. In this sense, the child expresses a comparison which is not quite yet a comparison between *measurement*s, but can become the starting point to introduce the relationship between two quantities.

Another child tried to compare the amounts by flattening the piles: he seemed to focus on the two dimensionality of the shapes obtained. In this case, the identification of the same "shape" as the same surface occupied can ground the processes about the area comparison. When asked to explain what he was seeing and doing, he said:

> Child 3: I have done like this [he scatters the two piles so that they roughly occupy the same surface, (Fig. 10.2)] […] I look at the rice, and that this and this are the same.

In another case, a child had trouble explaining what she was evaluating, but her manipulation of the rice and her manual control suggest that she was considering the two-dimensional form. In response to the interviewer's request to explain how she was sure, she said:

> Child 4: They are the same because … uhm …because this one has the amount of this [she highlights the limits of the two piles with her hands] because they are equal, because this is equal to this.

In this case, the potential pivot sign are not only the words, but also the child's gesture of limiting the two piles in order to identify congruent shapes.

Indeed, the gesture led to the generation of similar shapes for the piles. This could later, through discussion and comparison with other proposed strategies (e.g., the strategy of child 6 showed below), be described as volumes with the same base (2D surface) of which the heights can be compared to establish which is greater.

10.4.1.3 Evaluation of Volume

Five children more or less explicitly referred to the volumes of the piles. After making the two handfuls, a little girl said:

> Child 5: This here is fatter [she then rearranges the pile making it more compact].

The control on the rice is essentially visual and tactile. It seems that in making the pile more compact with her hands she *feels* something, but the term "fatter [it: 'ha più ciccezza']" suggests that she is substantially considering both the surface and the height, thus the volume occupied.

Fig. 10.2 Child 3 flattening piles of rice

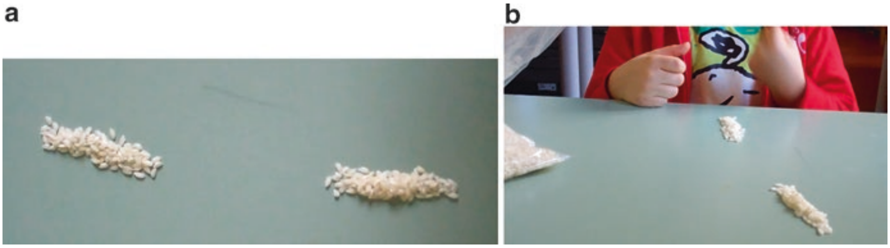


Fig. 10.3 (**a**, **b**) Child 6 comparing piles of rice using volume

Fig. 10.4 Child 7 comparing piles of rice using volume

Then, there were children who showed more organized strategies, such as a child who, after forming two elongated shapes (Fig. 10.3), said:

> Child 6: Mine is lower and yours is higher and it means that I have less and there, you there have more.

It would have been sufficient for the child to add something like: "if/since they have the same base area," and her method would have been mathematically exact. We interpret the expression as a potential pivot sign, since it can go beyond the particular task we are considering. Here, the unit of measurement is precisely the selected surface given by the shape of the base: the volume is measured by layers of equal surfaces. Similar strategies that depend on the measurement of volume were used by other children: in particular one who, after generating the second pile, checked the amount looking at the height:

> Child 7: It went a bit up [pointing to the first pile] and also here it went a bit up [...] Wait [he makes the gesture of moving his index finger horizontally from a pile at the other Fig. 10.4] [...] No, here there is still some missing [he adds 3 or 4 grains and makes the pile compact again] Yes, now I'm sure.

Here, concerning the previous strategy, the idea seems quite similar except for an initial focus of giving piles the same base area. Moreover, we consider the gesture of the finger that moves from one level to another as a potential pivot sign, which is essentially isomorphic to the recognition of the difference in the volumes of water contained in identical cylinders through the level (Fig. 10.5) proposed by Davydov (1982). Therefore, this sign can be used in the design of future measurement activities with continuous quantities.

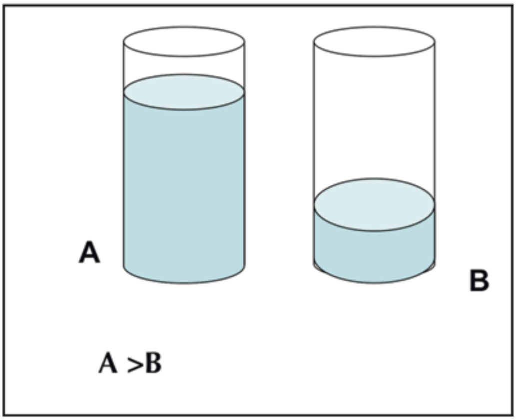


Fig. 10.5 Difference in volumes of water in identical cylinders

Fig. 10.6 Two-dimensional strategy comparing surface rectangles with one dimension evaluated by the ruler

10.4.2 Strategies with Artifacts

Some children exhibited sophisticated counting abilities during the interview and successfully compared the piles based on the numerosity of the rice in them. We note that for most of these children, the counting strategy seems to be satisfying enough for them to choose not to use the artifacts in the second phase of the interview. On the other hand, the children who chose to count, but had trouble due to difficulties in remembering the number words for the numbers after ten, tended to use new strategies for evaluating the quantities of rice in the second phase of the interviews.

When the interviewer proposed to use artifacts, two children chose the ruler to evaluate **one-dimensional** measures of the rice piles. One of these children had initially decided to count the rice grains in each pile, but gave up. He then decided to use the ruler. To do this, he lined up the two piles of rice on the sides of the ruler (Fig. 10.6) and said that to make the piles the same he had to add some rice to the second pile. He did this, lining up the piles again and again after each addition of rice. He then said:

Child 8: I measure it like this. […] I measure, like, I measure if they are the same or not.

A careful analysis of the video indicates that the child is not reading the numbers on the ruler, so the instrumented action scheme is not the conventional one the artifact was constructed for. Another child also used the ruler to compare the rice quantities without reading the numbers on the artifact. She arranged the rice along the entire length of both sides of the ruler, as if she was trying to confirm the fact that the two piles contained the same amount of rice. In this case, the ruler seemed to be used as an axis of symmetry (Fig. 10.7).

The word "measure" in these examples refers to comparing quantities, although no reference to number is made; we see the ruler (or a part of it) as being used as a unit for a one-to-one comparison between lengths. However, this is a potential pivot sign, because in a later whole-class activity its situated meaning could evolve toward its cultural meaning. For example, the teacher could make explicit the differences between the child's process and the culturally approved process of *measuring*.

A **two-dimensional** evaluation of the size of the rice piles was accomplished by some of the children through two main different processes: an evaluation of the length of one of the curved sides of the surfaces occupied by the piles or an evaluation of the surface by using a selected unit of area. We will focus on this second process. One child chose the straws and picked up a handful (six of them), which suggested she was not trying to establish a one-to-one correspondence between straws and grains. Then, she held the straws flat over each flattened rice pile, barely touching it, (Fig. 10.8) and said:

Child 9: They look equally big [it: "grosse," it could also mean "fat"] if I put them like this. […] Like to count we would put thousands, thousands, thousands [it: "migliaia"]…

Although the child could use the straws to flatten the piles and change their shape, this does not seem to be the case, as she barely touches the rice with the straws, as if not to change them after her earlier flattening. In this case, the straws seem to be used to measure the area. Also, in this case the utilization schemes were not those for which the artifact was constructed: the student showed the interviewer

Fig. 10.7 One-dimensional strategy comparing lengths of "symmetric" piles

Fig. 10.8 Two-dimensional strategy comparing surfaces with straws lined up

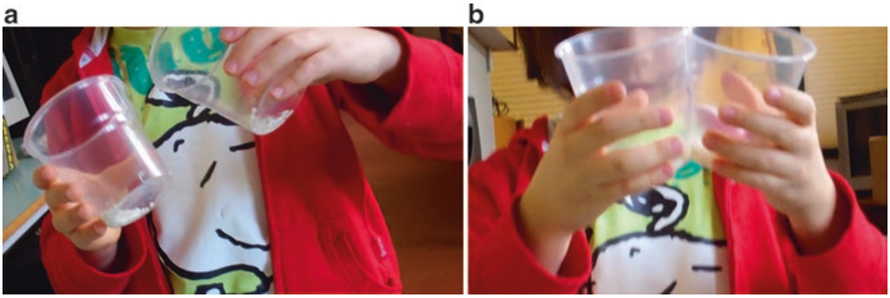


Fig. 10.9 (**a**, **b**) Three-dimensional strategy comparing heights of the volumes of rice in cups

that the same number of straws covers both piles, and she seemed to be referring to a straw as "a thousand." It is interesting that the word *a thousand* is already present in some children's vocabulary (another girl writes "1000" on paper). Although it is not appropriate for this context because the grains of rice are in the order of hundreds, the word seems to be somewhat connected to an idea of orders of magnitude (here it probably is used as a mathematical synonym of "a lot") and to an idea of possibility of using a counting strategy for rice, perceived as a discrete substance, and it can be considered as a potential pivot sign.

To evaluate volumes using **three-dimensional** strategies, one child decided to use the ruler to measure the height of the piles of rice, while four children chose the plastic cups. One of these four children is Child 6, who before being introduced to the artifacts referred to the heights of the piles (Fig. 10.3b). She chose the plastic cups and placed the two piles of rice one in each cup. The cups turned out to be rather large compared to the quantity of rice considered (the rice barely covered the bottom of the cup). So the child inclined the cups trying to obtain the same degree of inclination (Fig. 10.9a, b) and said:

> Child 6: These are more because they do not stay [it: "stare," it could also mean "fit"] and they go farther forward than the others, and the others are fewer because they reach to here.

The child seems to be trying to adapt the artifact to her initial idea by obtaining the same surfaces and then comparing levels in the cups: the same heights of the piles would indicate equivalent volumes. This idea seems to be expressed in the potential pivot signs “they go farther forward” or “they reach to here”: These signs have specific contextual connotations, but they are potentially exploitable in a process of semiotic mediation leading to culturally accepted forms of volume evaluation, such as using a graduated measuring cup and using expressions referred to precise levels (frequently identified through numerical values) of the measuring cup.

Since the student says that the piles do not contain the same amount of rice, the interviewer asked her to modify the piles so that they contained the same quantity. Now she added enough rice to each pile so that the cups properly set down on the table contained rice up to the same height. The interviewer asked her if she was sure that now each pile contained as much rice as the other, and the child answered:

> Child 6: I am not sure, counting is better, but [I would have to] count up to a thousand and a hundred.

Here, we can notice that because the degree of precision required in the comparison is not clarified in the task, the children can choose the approximation that they consider satisfying or easier to obtain.

Then, the interviewer asked the child to show what she was looking at; the child put the two cups close to each other (Fig. 10.9b) and said: “the difference.” This could also become a pivot sign, in that here it indicates a specific contextual aspect, the difference in height of the rice in the cups, but it can also refer to a general strategy linked to the result of the operation of subtraction. The situation is analogous to the one in Davydov (1982) (Fig. 10.5, in which “difference” has a double meaning. It refers to the result of a comparison and to the result of the operation of subtraction, related not only to the classical meaning of “taking away” but also to the identification of the quantity to add or subtract in order to obtain equality).

10.5 Conclusions and Future Perspectives

The analyses presented in this study show different strategies carried out by children at the beginning of first grade when asked to evaluate and manage quantities of rice, a substance that can be treated as continuous or discrete. This led to the identification of a set of situated signs that could later be used by teachers as pivot signs when constructing the meanings of discrete and continuous measurement of area and volume. As discussed earlier, we label these signs as *potential pivot signs* because they potentially can become pivot signs in semiotic mediation activities (Bartolini Bussi & Mariotti, 2008).

The design of educational interventions during classroom activities is not the focus of the current study, so it is not included. Indeed, the interviews we conducted serve as a preliminary basis assessing schoolchildren’s incoming informal knowledge. Building on the results of this research, we plan to design an intervention framed within the theory of semiotic mediation, in which the teacher builds on the children's initial signs identified here, to then lead them to refine their own strategies and develop cultural and more formal approaches and meanings related to *measurement*.

Moreover, classroom activities using "as much as" can be framed as a problem in a narrative context that could further motivate children to compare the rice quantities according to a certain goal insert in the narration (see also the activities with the rice proposed in McMullen et al., 2014). This type of activity can pave the way for gradually more formal reflections on measuring instruments or even, more generally, on the theme of approximation.

Obviously, we could have chosen other artifacts, different from the ones we worked with here, and these may be included in later activities with rice, always with the awareness that the children may use these artifacts with nonconventional and personal utilization schemes. These sorts of activities should be intertwined with other ones involving numbers, shapes, areas, and volumes; together they can be used to broaden the mathematical meanings gradually constructed. With this aim, it could be insightful to conduct deeper analyses of the children's strategies and to identify relationships between the children's strategies before they use the artifacts and the utilization schemes for their chosen artifacts. This direction of research promises to be quite fruitful.

Another development of this study could be to relate the strategies used by the children to some of the neuroscientific models described in recent studies. In this respect, we note that many of the strategies we identified, especially the ones based on comparison of spatial properties of the piles of rice, may be grounded in cognitive abilities that deal with manipulation of nonsymbolic magnitudes (e.g., sets of dots) and an approximate sense of spatial extension. This direction of research could be quite useful for educational purposes, because, as suggested by Piazza (2010), there is evidence of existing links between these abilities and those involved in managing numerical symbols. Indeed, Piazza argues that cognitive abilities that deal with manipulation of nonsymbolic magnitudes and those involved in an approximate sense of spatial extension can lead to the development of neural mappings that strengthen the meaning of the numerical symbols and that consequently strengthen many mathematical skills based on them.

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Chapter 11
The Semiotic Resources Children Use in Their Explanations of Hypothetical Situations

Elena Severina and Tamsin Meaney

11.1 Introduction

In this study, we analyse young children's explanations about the amount of photos in imaginary layouts in a photo book to document the semiotic resources the children use to explain and justify their mathematical thinking. To do this, we use Donaldson's (1986) definitions of empirical and deductive explanations as a basis for our analysis. In the past, very young children were not considered capable of providing explanations about hypothetical situations. In a seminal study, Donaldson (1986) suggested that the ability to explain requires linguistic skills (such as the use of causal connectives), cognitive abilities (such as making distinctions between cause and effect, a reason and a result, a piece of evidence and a conclusion) and a certain level of understanding of the phenomenon being explained. She described earlier studies that indicated that children did not have the necessary understanding to describe a causal relationship until the age of 7–8 years. However, she also highlighted significant problems in the research methodologies used in these studies, indicating that more in-depth research on this is needed. In mathematics education research in the 30 years since her book was published, there has been no research which has looked at the semiotic resources that young children use in causal relationships about hypothetical events.

Understanding how children provide explanations is important, as they are essential in both using and learning mathematics. According to Yackel (2001), mathematical explanations and justifications serve particular communication functions. The main function of explanations is to clarify aspects of mathematical thinking that may not be completely clear to others, while the function of mathematical justifications is to respond 'to challenges to apparent violations of normative mathematical activity' (Yackel, 2001, p. 6).

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Mathematical explanations and justifications can be related to Donaldson's (1986) four modes of explanation: empirical (what has happened to cause…?), intentional (for what purpose…?), deductive (how do you know that …?) and procedural (how do you DO…?). Each kind of explanation has distinctive features and also fulfils specific functions. An empirical explanation includes a directional indicator, through the inclusion of words, such as 'because' or 'so', which shows that something is caused by something else. This would be the equivalent of Yackel's (2001) explanations. An intentional explanation provides insights into the actor's reason for doing something. A procedural explanation describes the steps leading to a particular goal so that someone else could achieve the same goal. In these explanations, the temporal order is more important than information about causality, and so although they would appear to be similar to mathematical explanations, they do not necessarily provide information about mathematical thinking. In contrast, but similar to Yackel's justification, a deductive explanation uses evidence through logical reasoning to support why something is the case:

> The evidence may be observable, or it may take the form of a rule or of a 'given' fact. The role of causal connectives in deductive sentences is to make explicit the links in the deductive process, rather than causal relations between events (Donaldson, 1986, p. 104).

There is limited previous work on young children using mathematical thinking to discuss imaginary happenings. Furthermore, it is known that describing non-physical things or events requires more explicit information to be provided than when discussing physical things or events. For example, Meaney's (2011) research on a six-year-old child showed that discussions about time needed to be more explicit than measurement of length, which could be judged by the eye and thus did not need to be described verbally. In Saar's (2013) research in which 6- and 7-year-old children in a Swedish preschool class discussed the amount of imaginary cats that could be present in a picture in their mathematics textbook, the children gave different explanations for their thinking. These explanations included that there might be cats in the town that were not seen in the picture because they were hiding, or they had gone on holiday to the country or were ghost cats that were hard to see. The fact that they were discussing cats that could not be seen required the children to orally express their previous experiences with cats and on their awareness of happenings, such as going on holidays and ghosts. Their descriptions of what might have resulted in the cats not being visible can be considered as deductive explanations, from Donaldson's (1986) perspective. The discussions provided them with opportunities to discuss the idea that a number represented a specific amount of something, regardless of whether it was real or imaginary.

However, explanations of young children do not rely solely on speech. Other research has shown that younger children support their oral explanations with semiotic resources, such as gestures and physical objects. For example, Johansson, Lange, Meaney, Riesbeck, and Wernberg (2014) showed how three children, aged 4–5 years, who were playing with glass jars, predominantly used gestures to convey meaning to each other, but used verbal language in mathematical explanations to the teacher. They concluded, 'explanation is not a reflection of the thinking but actually constitutes the thinking' (Johansson et al., 2014, p. 898). Consequently, our research question is: what are the semiotic resources young children use in naturally occurring situations to explain the presence of hypothetical objects?

11.2 Semiotic Resources and Explanations

In this study, we consider children's production of explanations as a *semiotic process* – a process that actualises knowledge through signification (Radford & Sabena, 2015) of words, gestures and artefacts, etc. Artefacts and gestures have no specific meaning before being used in interactions. In a specific context, speech, gestures and the manipulation of artefacts are combined to convey meaning to others (Radford & Sabena, 2015). The different explanations described by Donaldson (1986) are likely to include combinations of oral language, gestures and manipulations of artefacts. However, how the combinations are enacted will differ depending on the explanation's function. Thus, it is important to identify the semiotic resources being used and their functions.

As Johansson et al. (2014) noted, gestures play an important role in young children's explanations. Sabena (2008) defined gestures in mathematical classrooms as 'those movements of hands and arms that subjects (students and teachers) perform during their mathematical activities and which are not a significative part of any other action (i.e. writing, using a tool, …)' (p. 21). Gestures are usually performed in a limited area in front of the body, between the shoulders, eyes and waist.

In studies on embodied knowledge in mathematics (see, for example, Alibali & Nathan, 2012; Roth, 2001; Sabena, 2008), McNeill's (1992) taxonomy of four basic types of the gestures is often used. The four types are: beat, deictic, iconic and metaphoric gestures. These types are connected to other contributors to the interaction, like speech, or artefacts and are context-dependent (Sabena, 2008). According to McNeill (2005), the same gesture can belong to several types, depending upon the meaning it is conveying and the function it fulfils in a particular context.

Beat gestures are simple, non-pictorial rhythmic gestures that indicate temporal or emphatic structure. An up-and-down movement, a flick of a hand and tapping motions used to emphasise certain utterances are examples of beat gestures (Roth, 2001).

Deictic gestures are gestures used to indicate objects, events or locations, often with an extended index finger, but sometimes with other fingers or the entire hand. Abstract pointing, according to McNeill (1992), identifies an abstract or non-present object, 'the speaker appears to be pointing at empty space, but in fact the space is not empty; it is full of conceptual significance' (p. 173). When providing an abstract deictic gesture, the movement of the pointing finger or hand can follow different paths, depending on the meaning being conveyed (Sabena, 2008). Goldin-Meadow (1998) suggested that abstract pointing does not occur until children are 10 years old.

According to Alibali and Nathan (2012), *iconic gestures* depict literal aspects of meaning, through the shape or motion trajectory of the hand(s), such as when cupped hands are used to indicate a cup. Iconic gestures are complementary to the speech, as they 'refer to the same event and are partially overlapping, […] but each presents a somewhat different aspect of it' (McNeill, 1992, p. 13). They provide information not only about the object being represented, but also the speaker's particular point of view about that object (Alibali & Nathan, 2012).

Metaphoric gestures are 'images of the abstract' (McNeill, 2005, p. 39). They depict 'the concrete metaphor for a concept, a visual and kinesic image that we feel is, in some fashion, similar to the concept' (McNeill, 1992, p. 14). The concept is given form in the imagery of objects, space, movement, etc. For example, when a speaker cups his/her hands while saying 'I have an idea', the metaphorical meaning is associated with the abstract concept of an 'idea' being held in the empty space in the hands, which makes use of the iconic nature of the holding-the-object gesture (McNeill, 2005). In mathematics, many metaphors are spatial and, thus, tend to rely on metaphorical gestures to carry the meaning, 'metaphors that involve space and action are readily expressed in metaphoric gestures that reflect the spatial structure of the underlying images' (Alibali & Nathan, 2012, p. 255).

As both iconic and metaphoric gestures are pictorial, they are considered by many researchers to belong to the broader category of *representational* gestures (Alibali & Nathan, 2012). Representational gestures are used to model an object or to simulate a process in order to support the thinking or its expression (Roth, 2001). They, therefore, provide a way of conveying important meaning when children are explaining their mathematical thinking.

11.3 Methodology

The data is a videoed discussion of the number of photos in a photo book. They came from a wider data set, collected in a Norwegian kindergarten. The question for the wider study was about the kind of mathematics young children engaged with when involved in photography activities. This particular video was interesting because of the imaginary nature of the objects being discussed and the diverse resources used in the explanations. In the 2:40-min video, a kindergarten student teacher began by showing the photo book to four 5-year-old children and asking them how many pictures were on the page. After a response, one child described the amount of photos in an alternative layout. The alternative layout seemed to include both existing photos and photos that the child imagined and which could not be seen by others. It was the child's explanation which allowed others to visualise them; therefore, we have labelled these photos *imaginary* and the layouts *alternative*.

The video camera was placed behind the children so that the photo book and the kindergarten student teacher were in focus. A second camera did not record as it should have, resulting in some of the children's gestures not being visible. After watching the video numerous times, episodes were identified in which alternative layouts were discussed. All gestures were described and together with screenshots, overlaid with drawings of the hand movements, were added to the transcript of the children's verbal utterances. We use square brackets in the transcript to identify when speech was accompanied by a gesture. The gesture is described in italics in round brackets. Our analysis first identified the semiotic resources the children used, such as types of gestures, oral language and artefacts, and then we connected them with Donaldson's (1986) different kinds of explanations.

11.4 Results

The children's explanations described how many photos could be placed on the actual or alternative, single- or double-page layout. As was the case with Saar's (2013) research, the children used the counting of real and imaginary objects, to justify their claim for a specific amount of pictures being placed on a page, based on their spatial positioning. These explanations combined oral language with gestures that illustrated how they visualised the alternative layouts as well as pointing to other artefacts, such as number cards which showed patterns of objects, linked to specific numerals.

11.4.1 Introducing an Imaginary Layout

The dialogue started with the kindergarten student teacher, T, asking the four children about how many photographs were on the page, to which they immediately answered, 'three'. As the children said 'three', Child 2 (C2) pointed at the book, made one vertical beat movement, as if counting, and then moved her index finger from left to right and then down. This gesture may have been used to support her silent counting.

After T emphasised that there were *three* photos on *one* page with words and gestures, C2 then offered a suggestion for an alternative layout.

C2: If we [divide it] (*C2 bends forward to touch the book*), [then it will be five] (*C2 uses the side of her palm to draw a line in the middle of the bottom photo: from the top (*Fig. 11.1a*) to the bottom (*Fig. 11.1b*))*

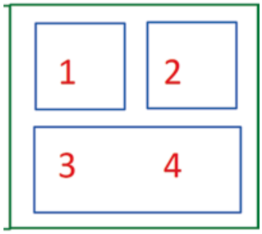
T: Then it will be [five. If you divide it here]… (*T draws a line with her index finger from the top to the bottom in the middle of the bottom photograph.*)

C2: Yes, I did not count. (*C2 shaking her head.*) [I just said it.] (*C2 takes her hands up from her knees and moves them to the sides and back to the knees.*)

Using a combination of words, gestures and interactions with an artefact, C2 gave a deductive explanation about why she considered that there would be five photos in the alternative layout. She used the words 'if we divide it', and a cutting gesture that continued the vertical line between the two pictures all the way to the bottom of the right page, by touching it with the side of her finger (see Fig. 11.1). Such division introduces the idea of separation of one real photo in two imaginary objects. Although she seemed to have miscounted the amount of photos, her explanation showed that she understood the need to indicate *how* she was dividing the page in order to justify the amount of photos.

Fig. 11.1 Beginning (**a**) and end (**b**) of C2's gesture

Fig. 11.2 C1's counting path



C2's cutting gesture has some deictic qualities. However, the dynamic motion of it highlighted the trajectory of the cut, suggesting that the gesture was iconic, as it resembles a typical cutting gesture made with a knife (see e.g., Edwards, 2009). The cutting gesture could be seen as having a spatial metaphoric component, as drawing the line made the two places for the new photos visible. Synchronising the gesture with speech extended the meaning of the verbal utterance, 'divide it', with an indication of *how it should be divided*.

After the teacher's comment, C2 seemed to realise that 'five' was not the correct amount of photos. The beat gesture in which she raised her hands from her knees that accompanied the speech could have indicated her uncertainty about the amount of photos in the alternative layout.

Building on the alternative layout introduced by C2, T held her hand on the imaginary division line and asked the children about the amount of photos. The children suggested 'four' and 'five'. The difference in the answers seemed to come from different counting strategies and provided information about the children's understanding of quantifying (what should be counted) and their ability to distinguish between real and imagined pictures.

T continued to hold her finger in the middle of the bottom photo, when C1 responded to T's invitation to count by saying 'one, two, three, four' while pointing at each place on the page (Fig. 11.2). C1 counted from left to right: first, two real photos at the top, and then – two imaginary parts of the photo that T 'split' with her hand (Fig. 11.3). T's hand was an important semiotic resource for C1, as it highlighted the

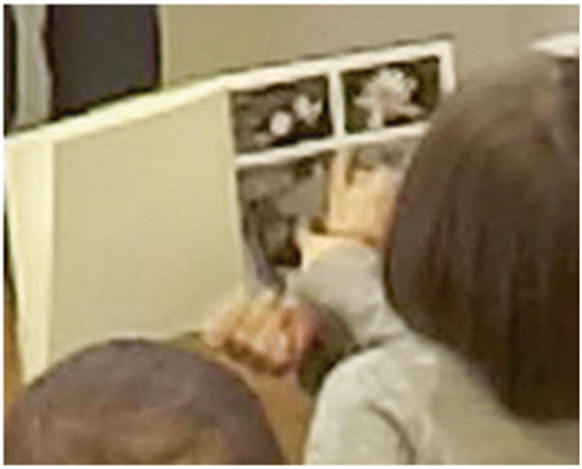


Fig. 11.3 C1 saying 'Three'

Fig. 11.4 C2 points and looks at the number card of ten

imaginary division line. What began as a beat gesture connected with the counting of real photos seamlessly became an abstract deictic gesture synchronised with the child's oral utterances, connected with the counting of imaginary photos.

C1's actions provided information about how counting could be done to justify a particular amount; thus, it can be considered a deductive explanation, in which what was to be quantified was identified. The evidence for the amount was provided through the counting, supported by pointing with the index finger to where the imaginary photos would be. Nevertheless, it should be noted that in this deductive explanation, the logical connectives, which Donaldson (1986) suggested showed the relationship between cause and effect, were missing.

All children agreed with four, including C2. Nevertheless, as T turned the page, C2 looked at the card for number ten on the wall. She placed her hand on the closest drawing of five (see Fig. 11.4) and said, 'Yes, but we have to see…', indicating some remaining uncertainty. However, T and the other children did not respond to her utterance. It may be that by placing her hand on the number card, C2 is using a deictic gesture to highlight the pattern, as part of a procedural explanation about there being five pictures in her alternative layout.

In this episode, the children used speech, without logical connectives, gestures, physical objects – the photo book and number cards – as semiotic resources, when giving deductive and procedural explanations. In providing a deductive explanation, C2 used a cutting gesture towards the photo book that could be iconic (describing how to make a cut) or metaphoric (the way to visualise the imaginary photos). C1 gave a further deductive explanation by locating both real and imaginary photos with a beat and abstract deictic gesture, respectively. C1 seemed to use T's hand as a semiotic resource to visualise the division line. C2 also gave a procedural explanation for determining that there were five images on her imaginary layout using a deictic gesture to indicate the pattern of five in the number card for 10. The children's gestures were often in connection with specific objects and were synchronised with the speech in all three cases. The explanations gave insights into how the children viewed the spatial structure of the alternative layout, which combined real and imaginary photos on the page. Both dividing the page and counting the photos indicated positions for the imaginary photos. However, while the cutting gesture created the position for the imaginary photos, the counting grouped both the real and imaginary objects together on the page.

11.4.2 Adapting Gestures

In a later episode, C4 repeated C1's beat gesture, while touching the page and counting out aloud, suggesting that the children could interpret each other's use of gestures in explanations and then use or adapt them in their own. This episode also showed how C2 adapted her earlier cutting gesture to show where an imaginary photo might be placed, indicating a different kind of adaptation.

T opened a new double-page layout and asked the children how many photos there were. C4 moved forward and touched one photo at a time, saying 'One! Two! Three!', using a beat gesture to count the two photos on one page and the single photo on the opposite page. Simultaneously with C4's 'one!' and 'two!', C2 pointed twice towards the book with a straight hand and the index finger (Fig. 11.5), but held the hand in the air when C4 said 'Three!'. C2 could have been using a deictic gesture to follow C4's counting, as C2's gestures were initially synchronised with C4's speech. As C4 leant back, C2 turned towards the photo book, while pointing with her index finger at four different positions on the page, as if she was counting silently. This suggests that she was using an abstract deictic gesture to count photos in an alternative layout, adapting the ideas and actions of C1.

About 1 s after finishing her counting, C2 tried to get the attention of the others by moving closer to the book and starting to explain her ideas, by saying 'if we do it' and using a cutting gesture similar to the one used in the previous example, but by only using the index finger.

Fig. 11.5 C2 counts in the air

Fig. 11.6 C2 splits with the finger

T: But what is this? (*T lifts the book over her head, C3 sits down, C2 stands up, moving closer to the photo book*)

C2: Hey, [if we divide it in two]… (C2 *draws a line from top to bottom in the middle of the left page with the left index finger (see* Fig. 11.6*), but fails to get attention of the others)*

After this failed attempt, C2 tried again to get the attention of the others and succeeded:

C2: Hey! (*All become silent for one second*) If we [divide it] in two (*C2 draws a line from the top of the left page to the bottom with her left palm formed as a cup, see* Fig. 11.7. *Hand relaxes.)*, we'll get [three] (C2 *scratches her ankle with the left hand)*, I think. [No] (*slips the grip of the right hand to keep balance*), four. (*Pause of about half a second*) [One, two, three, four.] (*Right-hand goes up and C2 touches each space of an imagined picture on the left page with her index finger as she counts, see* Figs. 11.8 and 11.9. *Sits down after she finishes the counting.*).

T: Yes, completely right, C2.

This time, her repeated metaphoric gesture (Figs. 11.6 and 11.7) is synchronised with 'if we divide it in two' and indicates where the real photos should be split according to the alternative layout. On Fig. 11.7, C2's hand forms a cup with the palm partly turned upwards, seemingly indicating how the imaginary photo should

Fig. 11.7 C2 splits the page with the cupped palm

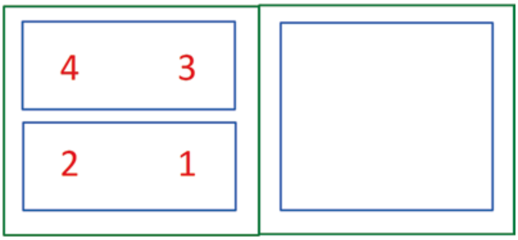


Fig. 11.8 C2's counting path

Fig. 11.9 C2 points at the second imaginary photo

be positioned in space. The metaphorical meaning was transformed from the process of cutting to being a container for the new imaginary photo. In this way, the gesture lost its iconic component and gained an abstract deictic form.

C2's oral language suggested that she was uncertain about the amount of imaginary objects, 'We'll get three, I think. No, four'. C2 seemed to count two parts to the left of the imaginary division line and only one to the right, which the cupped palm enclosed.

C2 corrected herself almost immediately, perhaps because of the symmetry of the arrangement. C2 seemed to have a good understanding of the quantifying (what should be counted) aspect of counting, while still struggling with distinguishing the different imagined objects (parts of the page as places for photos). It may be that the cupping gesture helped her to visualise the placement of the imagined photos, but at the same time distracted her from seeing all the photos in the alternative layout.

The uncertainty about the total amount of pictures and the pause immediately before C2 started to count aloud suggests that C2 was trying to find a way to resolve her uncertainty. After the pause, C2 connected physical spaces to her counting through the abstract deictic gesture. She counted slowly, pointing at one imaginary photo at a time, first right and left parts of the bottom photo and then right and left parts of the top photo (Fig. 11.8). 'One', 'two' and 'four' were synchronised with the pointing, but her finger touched the book a little bit before C2 said 'three'. This may have been because C2 recalled where the imaginary division line was. The beat nature of the gesture was not clear as the pointing was performed slowly.

C2 started her counting path from the imaginary photo closest to her hand, bottom right, and proceeded to the left and up, while C1 (Fig. 11.2) started at the top left and proceeded to the right and down. The counting paths provide insights into important differences between these explanations. When C1 counted, she gave a new explanation and used T's hand to visualise the division line, while C2 continued her interrupted explanation (due to her scratching) and had to remember where her imaginary division line had been. However, both C1 and C2 seemed to structure their counting path in a similar way, by using the real photo split into two by the imaginary division line as a semiotic resource to keep track of which imaginary photos had already been counted. The counting path can be seen as providing extra meaning to C2's deductive explanation and insights into her understanding of the layout, as well as revealing where the counting might have been challenging for her.

The purpose of C2's deductive explanation was to convince herself and others of the total amount as was the case with her previous explanation. However, this one had a different structure as, in addition to making the imaginary division line visible, she indicated where an imaginary photo would be placed. This gesture provided an elaboration of the deductive explanation, by using the metaphorical division line to illustrate an alternative layout and an abstract deictic gesture to indicate the position of the photos. It seems that counting by pointing provided her with another way to convey meaning, as the objects to be counted was placed in the space created by the first gesture.

In this interaction, C2 and C4 used a range of semiotic resources to convey meaning, but in two different situations. T's question resulted in C4 providing a deductive explanation in which he used his counting to confirm that the total amount was three. C4 counted real objects with words and by touching the photos, using a beat gesture. For her alternative layout, C2 adapted the beat gesture to an abstract deictic gesture to verify the layout's capacity. C2 used two metaphoric gestures to illustrate the alternative page layout. The first one was to repeat the cutting gesture, while the second one was new and performed with a cupped hand, indicating where a photo could be placed. Therefore, the second gesture could be considered to have abstract deictic qualities. The combination of the two gestures that supported the visualisation of the placement of the imaginary photos could indicate that C2 understood she needed to give precise arguments so that others could follow her reasoning. Nevertheless, logical connectives remain absent in her speech with the gestures being the main semiotic resources for conveying meaning about the relationship between the layout and the photos.

11.4.3 Clarifying What Is Discussed

In the next part of the transcript, C1 combined C4's and C2's previous explanations by indicating that she was visualising an alternative layout which included C2's imagined photos on the left page and the actual photo on the right page.

C1: [One, two, three, four, **five**.] (*C1 stands up and touches one place at a time with her index finger as she counts, see* Figs. 11.10 and 11.11)

Although C1 used an abstract deictic gesture when counting imaginary photos and a beat gesture when counting the real one, the structure of this deductive explanation is different from those given earlier. C1 seemed to have accepted C2's imaginary division line, by counting by touching four places for the imaginary photos

Fig. 11.10 C1 counts

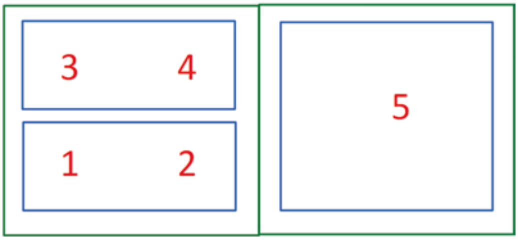


Fig. 11.11 C1's counting path

(Figs. 11.10 and 11.11) without having to make a cutting gesture herself. Comparison of the counting paths used by C1 (Fig. 11.11) and C2 (Fig. 11.8) suggests that C1 may have been creating her own alternative layout by extending C2's imaginary layout to include a real photo on the right page, so being the first to count across the double-page spread. It is interesting to note that C1's counting path is a reflection of C2's: C1 counted from left to right, while C2 counted from right to left. This might be explained by C2 sitting to the right of C1, and therefore C2 had to stand up, take a step forward and stretch out her hand in order to reach the left page of the photo book, while C1 just had to stand up. Also, in case of C2s' counting path (Fig. 11.8), her view of the right page seemed to be blocked by her right hand, which was not the case for C1 (Fig. 11.11), making it more natural to proceed to the right page.

In this episode, C1 used the same semiotic resources as the others, namely counting aloud, abstract deictic and beat gestures. However, the imaginary division line identified previously by C2 seemed to have been accepted by the children. It may be that the children's positioning to the photo book influenced their choice of counting path, but there were too few examples to be certain of this. It may be that where the child's hand was positioned for counting in the photo book may have influenced the possibilities for creating the alternative layouts and reasoning about them. Therefore, we suggest that the use of semiotic resources could have affected how explanations were developed and were not just vehicles for conveying already-established meaning.

11.4.4 Alternative Support for Deductive Explanations

The children continued to identify alternative layouts of the same page. In this episode, C3 responded to C1's explanation by indicating a non-symmetrical division of both pages (Fig. 11.12).

C3: But what, but what if we [divide it] (*C3 puts his right arm with the palm straightened, making a vertical line across the right page and then drags the arm down* (Fig. 11.12*, line 1 and* Fig. 11.13), and then we [divide it] (*C3 puts his right arm with the palm straightened, making a vertical line across the left page and drags the arm down* (Fig. 11.12*, line 2 and* Fig. 11.14), and then we divide (*inhales*) [it] (*C3 puts his right arm with the palm straightened so that it continues the horizontal line between upper and lower photos on the left page to the right page and drags the hand to the right* (Fig. 11.12*, line 3 and* Fig. 11.15), then it will be six. (C3 *sits down*).

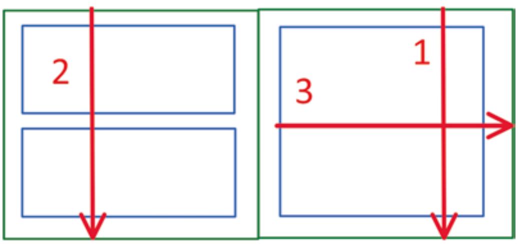


Fig. 11.12 C3 cutting

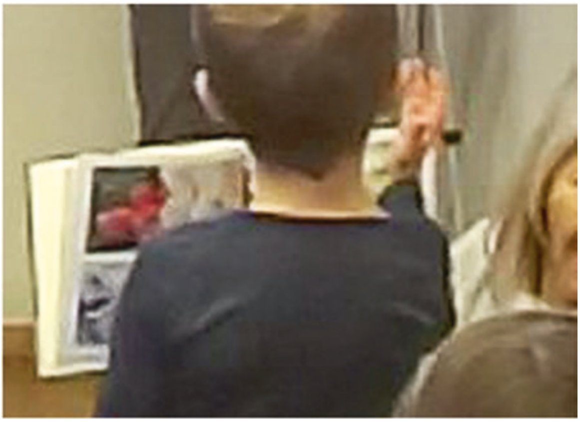


Fig. 11.13 C3's arm at line 1

Fig. 11.14 C3's arm at line 2

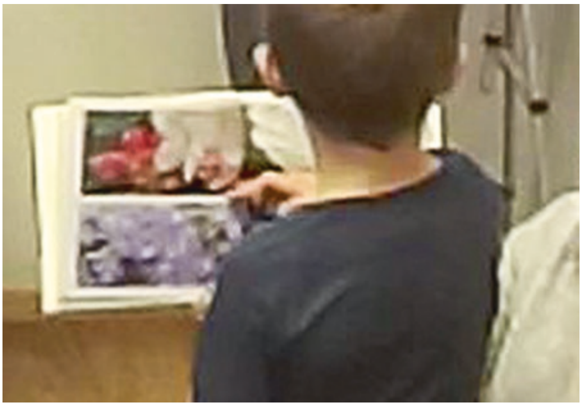


Fig. 11.15 C3's arm at line 3

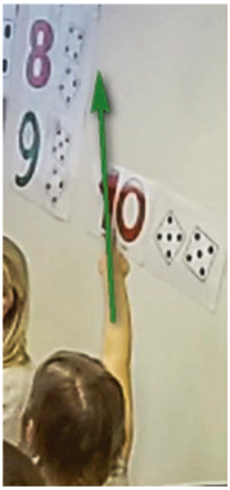


Fig. 11.16 C2 counts cards as support

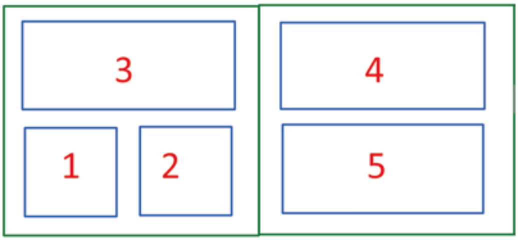


Fig. 11.17 C4's counting path

Whereas C2 used a finger or a palm to make her cutting gestures (Figs. 11.6 and 11.7), C3 used his entire arm to illustrate the imaginary division lines. Also, while C2 appeared to divide *photos*, C3 seemed to divide *pages*, as he used the masculine form of 'it' in Norwegian, which is relevant for pages, but not for pictures. The structure of C3's vertical and horizontal cutting gestures was slightly different, but the orientation of the imaginary division lines seemed to always remain in focus. The vertical cutting gestures did not follow the separation between real photos, which had been the case for C2 when producing her alternative layouts. It did have two strokes: first C3 placed his arm on the page along an imaginary division line while saying 'divide', and then moved the arm down along the line while saying 'it'. The first stroke had iconic characteristics, as his arm split the page into two pieces, while the second stroke had abstract deictic characteristics, as C3 was drawing an imaginary division line in the alternative layout. C3 performed the cutting movement for the horizontal division with his hand. Before dividing the right page, he carefully aligned his fingertips with the separation space between the two photos on the left page. C3 seemed to have used the position of the real photos as a semiotic resource to divide the right page into two parts horizontally. As C3 was drawing an imaginary division line, the gesture can be considered an abstract deictic one.

It is not entirely clear how C3 arrived at his answer ‘six’. It is possible that he was referring to the amount of page parts after the division, in that the vertical cuts on the left page produce two parts and the horizontal and vertical cuts on the right page produce four parts. This is different from what the other children had done where they kept the real photos as a basis for seeing the parts after the cutting. Nevertheless, C3’s explanation can be classified as deductive, because its function seemed to be to justify his answer. Unlike other explanations, it included the logical connective, ‘if … then’, indicating a causal relationship between the splitting actions and the result.

T seemed to be uncertain about the division of the page and the amount of photos and suggested another counting round. However, the children were not keen, except for C2 who stated, ‘I can count them!’ while pointing with her index finger at the wall with the number cards. T was not looking at C2 and turned the page. C2 then responded with, ‘Hey!... I wasn’t looking at [this]’, while pointing to the photo book and then ‘I was looking at [this]’ while pointing at the number card for eight on the wall (Fig. 11.16).

Although somewhat limited, this could be a deductive explanation, as C2 seemed to be indicating that with the help of the number charts she had determined the number of photos in C3’s layout. The pointing gesture had characteristics of two types: deictic (use of the index finger to indicate the direction of the number cards) and beat (gesture used at a high speed to emphasise the meaning). The counting card, showing a representation of ‘eight’, seemed to be C2’s evidence for verifying that C3’s layout had eight photos.

In this episode, C3 and C2 used similar semiotic resources to what had been used previously, that is representational, deictic and beat gestures, the photo book and the number cards. However, C3’s adaption of the cutting movement seemed to result in it fulfilling iconic and abstract deictic functions. C3 did not use counting, neither verbally nor with gestures, to support his claim that the correct result was six, but instead he used a logical connective to support his deductive explanation. C2 used the number cards in her deductive explanation (*what* the right amount is), while previously she had used the number cards in a procedural explanation (*how* one can count the imaginary objects).

11.4.5 Explicit Explanation

In the final episode in the transcript, the children gave more explicit explanations that drew on a broader range of semiotic resources. It may be that the teacher’s intervention supported them to realise that others might not always be able to follow their reasoning.

After the last episode, T lifted the photo book and said in a low voice, looking at C4: ‘How many do you see here?’. C4 immediately ran forward and counted, ‘One, two, three, four, **five**!’, while touching one real photo with his index finger per count (Fig. 11.17). Starting simultaneously with C4, C2 made seven pointing gestures towards the book in the air with a straight hand and the index finger. As C2 finished, T said, ‘Oh, five’. As in ‘Adapting gestures’, C4 counted the amount of real objects, with his pointing gesture having beat characteristics, while C2 counted by pointing at non-present objects using an abstract deictic gesture. In both cases, the children tried to figure out the amount of photos in different layouts. In C4’s case, the counting formed the evidence for the deductive explanation to the group, while C2’s use of gestures only suggests her thinking process.

C2 then stated ‘I see seven!’ and provided a justification for her solution by counting aloud up to seven and touching the location of the imaginary photos as she verbalised her count (see Fig. 11.18). Her use of the abstract deictic gesture of touching the photo book to indicate the location of the non-existing objects made this alternative layout visible to others. In this way, the gesture gained metaphorical characteristics.

C2’s counting path (Fig. 11.18) has several features that distinguish it from the one she used earlier (see Fig. 11.8). C2 started at the top of the page as was the case with C1 (Fig. 11.2) and C4 (Fig. 11.17). As had been done previously, C2 counted the imaginary photos before the real photos, moving from right to left (1–2) and then anticlockwise from left to right (3–4). The change in the counting direction of imaginary and real photos indicates that C2 understood that she could start and finish her systematic counting at any location and the amount would be the same, one of Gelman and Gallistel’s (1978) principles of counting, order-irrelevance. C2 extended her counting across the double-page spread, moving from an imaginary photo at the right bottom corner of the right page (4) to the real image at the right bottom corner of the left page (5), then up (6) and down to the left corner (7). Although counting across the double-page spread had previously been done by C1 and C4, C2 moved from right page to the left. The direction of the counting is in alignment with our view that positioning with regard to the photo book might have influenced the choice of the counting path.

T then asked C2 ‘What are you doing now in your head? [One…] (*T points with her index finger at the top picture on the right page.*)’, explicitly inviting C2 with the gaze, gesture and speech to explain her thinking.

C2: I [divide it in two!] (*C2 stood up and used the side of her palm to draw a line in the middle of the right page from top (*Fig. 11.19a*) to bottom (*Fig. 11.19b*). C2 sits down.*)

T: You divide it in two.

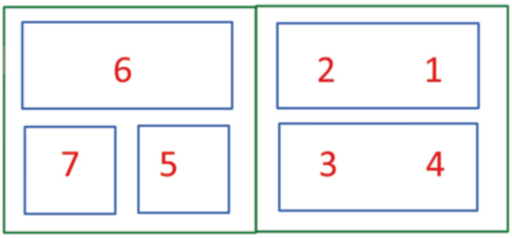


Fig. 11.18 C2’s counting path

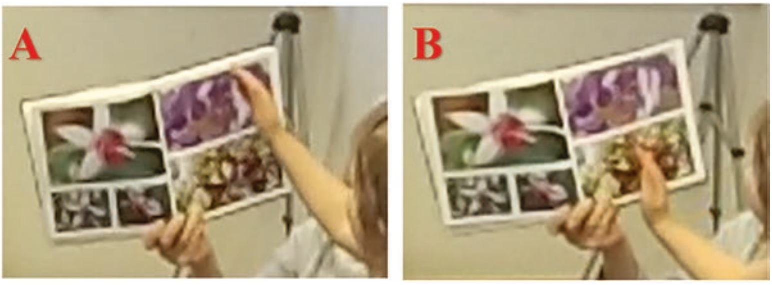
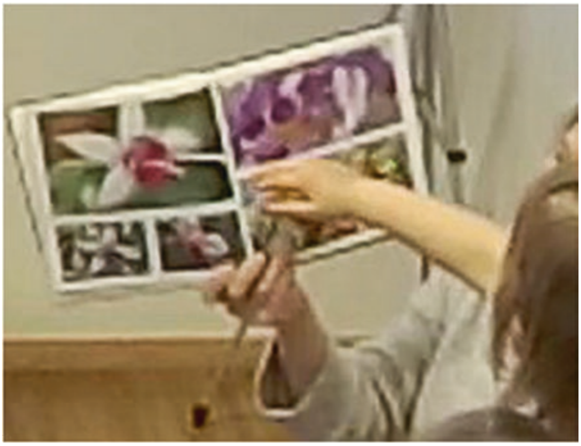


Fig. 11.19 C2 draws the first division line (**a**) Gesture start. (**b**) Gesture end

Fig. 11.20 C2 draws a horizontal line



C2: And this one [is already divided in two.] (*C2 stood up and drew the line with the side of her palm along the white space between the photos on the right page, see* Fig. 11.20. *C2 sat down.*).

T: This one was divided in two.

C2: And if we [have these] (*C2 stood up and rotated her hand an anticlockwise direction in the central part on the left page starting from the bottom right position (*Fig. 11.21*). C2 sits down.*), then it will be correct!

C2's explanation consisted of three steps, each communicated by speech and synchronised with a gesture. T repeated C2's words or described her actions after each step and each gesture was marked by C2 standing up and sitting down. The alternative layout was defined by the counting path. The gesture used to visualise the imaginary division created two pairs of imaginary photos in the alternative layout. Therefore, the gesture was both deictic because it indicated where she would cut, and metaphoric, as the drawing of the division line revealed the two places for imaginary photos on each real photo. The drawing of a horizontal line on the left-hand page in the alternative layout relied on the position of the actual photos. Therefore, this gesture seemed to be more an abstract deictic one than a representational one. C2 did not count as part of her explanation, and she seemed to assume

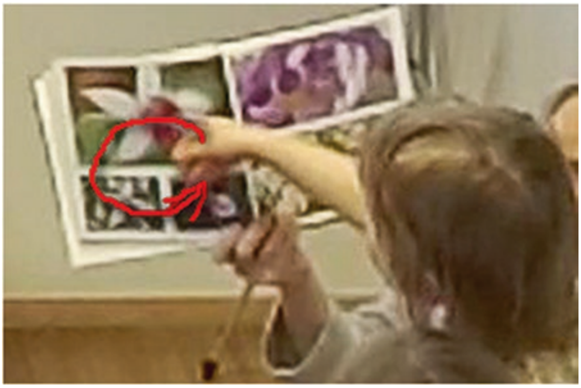


Fig. 11.21 C2 makes a circular movement

that her audience would remember the total amount of photos that she had given previously. Instead, she used gestures to illustrate how she built on the existing layout to create a new arrangement of real and imaginary photos. When she included three real pictures, she used a metaphorical gesture of drawing a circle to communicate 'those three photos on the left page', and completed her explanation with 'then it will be correct!'. When C2 performed the circular movement, it followed the counting path in Fig. 11.18 for 5–7, which could indicate a strong connection between counting and cutting as parts of the same explanation.

This explanation is more explicit than the ones the children used previously. The teacher's question prompted C2 to provide more details about her reasoning. She built on her claim that her imaginary layout had seven photos, by describing how she divided the photos on the right page to produce four imaginary photo places. She then connected the virtual and real photos with 'if we have these' real photos and concluded with 'then it will be correct'. Her explanation is deductive and contained an 'if … then' statement showing a causal relationship and also gestures.

In this episode, the children used similar semiotic resources to the ones they had used in previous episodes: oral utterances, gestures and the photo book. However, in this episode, C2 moved away from just using counting in her oral utterances to including the same logical connective as C3 had used. Whereas C4 only counted actual photos, C2 described an imaginary layout for the right-hand page and combined this with the actual photos on the left-hand page. This suggests that C2 was able to adapt her visualisation of different layouts and also respond to the needs of her audience by combining a range of semiotic resources to convey her meaning.

11.5 Discussion

In our analysis, we have described the semiotic resources used by five-year-old children to explain hypothetical situations, involving alternative layouts of photos. Like Saar's (2013) study of a preschool class in Sweden, the children, particularly

C2, seemed to enjoy the challenge of discussing these alternative layouts. The mathematical idea in both Saar's (2013) and our study was that a number represented a specific amount of something, regardless of whether it was real or imaginary, Gelman and Gallistel's (1978) principle of abstraction. In our data, the children drew on a range of semiotic resources, such as spoken language, gestures, and artefacts, to give predominantly deductive explanations about the amount of real and imaginary photos. Some of these semiotic resources were those which other researchers had previously considered too advanced for this age of children, such as the deictic gesture of abstract pointing. We conjecture that it may be the context of explaining the amount of photos in imaginary layouts, which prompted the children to use this range of semiotic resources. If this is the case, then the context of the situation can be considered as affecting the functions that the semiotic resources need to fulfil.

In our study, in all but one case, the 5-year-old children gave deductive explanations (Donaldson, 1986). Donaldson (1986) had suggested that children may not be able to understand the causal relationships, needed for a deductive explanation, until they were 7–8 years old. However, the children in this interaction showed an understanding of two causal relationships in their deductive reasoning. The first was that dividing the existing photos on a page would produce a new amount of photos. The second was that by counting real, real and imaginary, or just imaginary photos, a total amount of photos could be determined. As was the case also in Saar's (2013) study, the children in our study drew on their previous experience and observation of the real-world situations, such as of photos, number cards and page layouts, to support their explanations.

Donaldson (1986) suggested that linguistic skills like the use of logic connectives are needed for deductive reasoning. However, in the interaction, the children more often used other semiotic resources, such as iconic and metaphorical gestures, to indicate through a cutting motion how the page was to be divided. As such, they provided the logical links between the actual layout and the alternative layout that the child wanted to 'show' to the others.

As the layouts could consist of only real (C4), real and imaginary (C1, C2 and C3) or only imaginary photos (C2), both beats (C1, C2 and C4) and an abstract deictic (C1 and C2) gestures were used to point at specific areas on the page to convey evidence for the amount of photos being counted. C3's cutting gesture also had abstract deictic characteristics. These children's use of gestures in their explanations is interesting in that Goldin-Meadow (1998) suggested that children are not able to use abstract pointing until they are 10 years old. Nonetheless, in these deductive explanations, the children used abstract pointing to illustrate the logic behind the counting of imaginary photos.

As was the case with counting, the children often used artefacts, in connection with gestures, to convey meaning. The photo book and its existing page layouts clearly provided inspiration to the children for their alternative layouts. As well, the teacher's hand was used as a semiotic resource by C1 to convey extra meaning about her counting of imaginary photos for the first time. C2 used a deictic gesture to point at number cards on the wall when she provided her deductive explanation of amount of photos in C3's alternative layout, as well as in the limited procedural explanation about how she got a total amount of five photos in her first alternative layout. The cards provided extra meaning by illustrating how she imagined the alternative layout would look like.

Logical connectives in oral language were used only in a few cases. C3 used 'if…then' to provide some indication of the relationship between dividing the page and the number of photos in his imaginary layout. In C2's last explanation prompted by T's request for her to explain her thinking, she used 'and' to connect all her divisions, and 'if … then' to indicate how the actions would give the correct amount of photos, suggested previously. Except for these two cases, the logic linking the different parts of the explanations was mediated by following a fairly consistent explanation structure – description of the dividing of the actual photos and pages, followed by counting of the imaginary photos – through gestures and artefacts as well as speech.

It may be that the need to reason about the amount of imaginary photos created a context which stimulated the children to use these types of gestures at a much younger age than Goldin-Meadow (1998) suggested. Meaney's (2011) research has previously indicated that non-physical objects, such as time, require oral discussion and it may be that the hypothetical nature of the alternative layouts prompted the children to use a wider range of semiotic resources that would have been the case with just counting physical objects. These results indicate that further research is needed to better understand how context affects the range of semiotic resources that children draw upon.

11.6 Conclusion

In this chapter, we have described the semiotic resources used by 5-year-old children in naturally occurring explanations about hypothetical situations, in which they imagined alternative layouts. In the interaction, the children used a range of semiotic resources to give mostly deductive explanations. Similar to Johansson et al.'s (2014) suggestion from their research, we argue that the gestures, along with the spoken utterances, did not represent the children's internal thinking, but actually contributed to that thinking. It is in the deductive explanations about the division lines and the number of photos or the number of places on a page which supported the children to understand the relationship between them. The objects to be counted existed in a complex space where real photos and imaginary ones were combined in alternative layouts and it seemed that the counting paths were used by the children to visualise this complex space. The use of the semiotic resources in the different explanations provided some insights into the mathematical thinking of the children, as well as into what might have been challenging for them.

Given that our results are in contrast with those of other researchers (Donaldson, 1986; Goldin-Meadow, 1998), there is a need for further research to understand how children make use of non-verbal semiotic resources to justify their understanding about hypothetical situations. In particular, we consider that understanding the impact of different contexts on the semiotic resources that children chose to use in their explanations would be extremely valuable, as it seems that the hypothetical nature of what they were discussing affected their use of resources. This is because the explanations required the semiotic resources to take on specific functions in order to ensure the appropriate meaning to be provided.

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Chapter 12
Drawings, Gestures and Discourses: A Case Study with Kindergarten Students Discovering Lego Bricks

Benedetto Di Paola, Antonella Montone, and Giuditta Ricciardiello

12.1 Introduction

The Italian Ministerial Guidelines in their latest formulation (MIUR, 2012) propose for Kindergarten a mathematical area, named 'knowledge of the world'. In this area, they propose the following: 'Children continually explore reality, but need to learn to reflect on their experiences by describing them, representing them, and reorganizing them, according to specific educational goals. In this way it is possible to lay the foundations for the subsequent elaboration of scientific and mathematical concepts that will be introduced in primary school' (p. 21). Moreover, in the transition from kindergarten to primary school, the Italian Ministerial Guidelines emphasize the power of Geometry in involving students in activities such as the use of visualization, spatial reasoning, analysis of the characteristics of the figures, production of arguments, mathematics related to geometric relationships and modelling to solve problems (MIUR, 2012).

Moreover, we believe that the learners' capacity to visualize geometric relationships can develop, starting from kindergarten (Anning & Ring, 2004; Di Paola, Battaglia, & Fazio, 2016), as children sort, build, draw, model, trace and measure. Active children's involvement in the use of manipulatives is, indeed, fundamental in geometry. Such activities develop their skills in visualizing and reasoning about spatial relationships (Di Paola & Montone, 2018; Faggiano, Montone, & Mariotti, 2018). It has long been known that use of real objects and manipulative tools can be useful to support mathematics learning (Sowell, 1989; Montone, Faggiano, & Mariotti, 2017).

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The Theory of Semiotic Mediation (TSM) (Bartolini Bussi & Mariotti, 2008) offers an effective reference framework in order to study the relationships between artefacts, the actions they allow one to accomplish and how pupils use them to construct mathematical concepts. In the present work that fits into this research field, a sequence of activities has been created, to be carried out using a manipulative artefact, a Lego block and its drawing, and aimed at promoting the construction/conceptualization of the coordination of different points of view. The focus of the study is on investigating the alternation between different semiotic systems, graphical, verbal and gestures (Arzarello, Paola, Robutti, & Sabena, 2009). Our didactic assumption claims that the process of construction of the correct coordination of different points of view can be achieved through the mediation of specific artefacts.

In this chapter, we analyse a mathematical discussion concerning the drawings of Lego blocks realized by 15 kindergarten students (8 of them are 5 years old and the remaining 7 are 6 years old) connected with their use of gestures as a semiotic resource into the description of their drawings. This analysis is aimed to study the thought construction made by the children in the coordination of different points of view (Duval, 1998) observing a 3D object as a Lego block. This topic is not deeply studied in research. Therefore, we think that it should be interesting for the study of the geometrical and spatial thinking (Battista, 2007; Clements, 2004), starting from Kindergarten level (Radford, Edwards, & Arzarello, 2009).

12.2 Theoretical Framework

As stated above, in this study we refer to TSM. The main aspect of the TSM that we focus on in the design aspect of the teaching process is the semiotic potential. The semiotic potential of an artefact consists of the double relationship that occurs between an artefact, the personal meanings emerging from its use to accomplish a task (instrumented activity) and the mathematical meanings evoked by its use and that are recognizable as mathematics by an expert (Bartolini Bussi & Mariotti, 2008, p. 754).

In semiotic activities, various signs are produced: the ‘artefact signs’, the ‘mathematics signs’ and the ‘pivot signs’. The ‘artefact signs’ express personal meanings that often have a highly subjective nature and are linked to the learner’s specific experience and/or culture (Di Paola, 2016; Mellone, Ramploud, Di Paola, & Martignone, 2019) with the artefact and the task to be carried out. The ‘mathematics signs’ express the knowledge of mathematics which may evolve ‘artefact signs’. Finally, the ‘pivot signs’, with their hybrid nature, show the evolution between artefact signs and mathematics signs, through the linked meanings.

Such an evolution can occur together with specific semiotic activities, in particular, in the peer interaction during the task and in the collective discussions, accompanied by the expert guidance of the teacher. The collective construction of shared mathematical meanings is a complex process, where it is possible to distinguish between evolution paths (semiotic chains) described by the appearance and chains of different types of signs: artefact signs, mathematical signs and pivot signs (Bartolini Bussi & Mariotti, 2008).

> Through a complex process of texture, the teacher constructs a semiotic chain relating artifact signs to mathematics signs, expressed in a form that is within the reach of students. In this long and complex process, a crucial role is played by other types of signs, which have been named pivot signs. […] they may refer to specific instrumented actions, but also to natural language, and to the mathematical domain. Their polysemy makes them usable as a pivot/hinge fostering the passage from the context of the artifact to the mathematics context (Bartolini Bussi & Mariotti, 2008, p. 757).

Finally, we must underline the importance of organizing the teaching in such a way that it, during this evolution, may foster the collective production and development of signs through Mathematical Discussion (Bartolini Bussi, 2008).

In recent years, many researchers have studied and highlighted the role and coexistence of various semiotic resources that come into play in the processes of learning and teaching mathematics (Fandiño Pinilla, 2008). The words (written or spoken), the specific symbols of the discipline, gestures, body position and all other aspects related to the embodied nature of knowledge (Edwards & Robutti, 2014) are considered as fundamental mediators of the mathematical thinking of students and teachers alike, and not as mere accidental elements (Nemirovsky & Ferrara, 2009; Radford et al., 2009).

Radford et al., (2009) describe gestures as important sources of abstract thinking and as the very texture of thinking. Therefore, the activation of different cognitive and semiotic components together with the perceptive-motor and 'embodied' activities, such as the manipulation of materials or artefacts, drawing, gestures, body movements, and rhythms, support the student's thinking process (Arzarello et al., 2006).

12.3 Research Methodology

According to the main assumption concerning the TMS and the role of gestures as a semiotic resource used by young students (5–6 years old) in the learning process, a teaching sequence has been designed. However, the main hypothesis consists of alternating activities involving the manipulation of a Lego block, the use of drawing and a description of representations that could make the evolution of the significance concerning the coordination of different perspectives from different points of view. The teaching sequence was carried out in a pilot study with the participation of 10 children attending the last year of kindergarten (5/6 years old). In order to analyse the students' drawings, gestures and the related discourse, the teaching experiment was videotaped. Transcriptions were used in the analysis of the data to highlight the evolution of the signs from artefact signs to mathematics signs.

In this chapter, we will refer to data coming from the mathematical discussion. We will not only show the unfolding of the semiotic potential related to the artefacts, but also how the transition from the manipulation of the real object to its representation can foster the construction of mathematical meanings.

The teacher conducted the teaching experiment. She asked students to manipulate a Lego block observed from different points of view and to draw it, identifying the main characteristics related to all its parts (e.g. faces, angles and parallelism). The interpretations of the drawing are analysed by the teacher and researchers who observed the videotape of students' interactions and his/her gestures used to present his/her drawings to the teacher and to the other students.

The following questions guided the research conducted:

How does the use of different semiotic systems allow students to construct and conceptualize the correct coordination of different points of view?

12.4 Overview of the Teaching Sequence

The first teaching phase involves the Lego blocks, like in the following figure (Fig. 12.1).

Throughout the exploration, manipulation and observation of the blocks, each child describes his/her blue block to their fellow students. The child becomes familiar with the three-dimensional object, and through the use of personal symbols describes the shape, the presence of edges or 'points' and how many there are.

The second phase involves the drawing of the block. The request to draw the blue block freely on an A4 sheet has the aim of making the child express through the drawing one of the many points of view and the relations between the parts of the blue block compared to the chosen point of view.

As in the first phase, each child initially observes (through manipulation) the blue block, trying to grasp its shape, the presence of edges, 'circles', their relative number and so on. Later, the block is placed on a bench and the children line up in front of it. Children are then asked to draw the blue block again, now being able to see it. This setting is chosen to support more observation points of view of the children, and to link it to the position taken by the children with respect to the position of the blue block. We try to encourage the production of drawings of the same object, but with different points of view.

The third phase, executed in the days following those dedicated to the previous phase, was carried out in the kindergarten through a mathematical discussion aimed to analyse what students put in evidence producing the figurative representations of

Fig. 12.1 Lego block

the Lego block observed from different points of view. Children were asked to describe their drawing to the rest of their classmates. The teacher asked them to 'compare' the various drawings with one another. The selection of the drawing most similar to the blue block was made by the class, thanks to a discussion orchestrated by the teacher (Bartolini Bussi & Boni, 1995), and was designed to allow all students to independently assess the designs of their classmates and its accuracy in relation to the point of view used.

12.5 Analysis and Preliminary Results

In this section, we present the analysis of the drawings and the students' discourses through the emergence and evolution of specific signs. Data analysis is based on the transcriptions of students' interactions during their accomplishment of the task using their drawings, and finally on the transcripts of the collective discussions. A specific lens of analysis will regard the identification of key elements used in the drawing of real objects in order to begin to coordinate the different points of view, strengthening the ability to understand someone else's point of view. The listed protocols are only some of the most significant ones, in relation to the variety of strategies, identified in the drawings produced by children.

12.5.1 Different Points of View of Matilde, Laura and Elio

With regard to the research hypothesis, we will in this section present the analysis of some interesting episodes involving all children. In particular, three of them, Matilde, Laura and Elio, compare their different drawings and show not only the evolution of signs towards the mathematical meanings, but also the expected ability to use different points of view and recognize the necessity to coordinate them.

12.5.1.1 Episode 1

The children carry out this task: '*Take the blue block in your hands, touch it and look at all its parts, then draw it*'. During the following mathematical discussion, the teacher asks them to describe their own drawing and compare it with others' drawings. In the first part, the discussion focuses on the description of their drawings. In the following, the discussion focuses on the shape of the Lego block. Not all the children immediately realize that it would have been enough to draw one of the parts of the Lego block, but at the end of the discussion none of them seem to have any doubts about that. Then, Matilde intervenes and demonstrates an important but obvious point. She draws the circles on the figure with her index finger and then goes through the two rows of dots in parallel. Afterwards, she points with her index

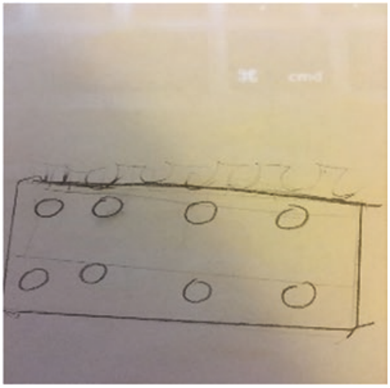


Fig. 12.2 The drawing of Matilde

and middle finger on the couple of dots, counts and says 'we must draw the Lego block, as we saw it'. At this stage, the discussion[1] is concentrated on the shape of the block and Matilde, seeing on her sheet (Fig. 12.2), tries to explain what she has done:

1. Matilde: In a first moment I have done… I did the shape of the square, then I made first one very small then as Elio, then I didn't like it, I erased it and before I did all these *r*, I didn't like it and I did it again.

   afterwards pointing with her index and middle finger on the couple of dots she counts and says:
2. … two, two, two, two… eight… because on that rectangle there were eight.
3. Teacher.: the rectangle?
4. Matilde: there were eight circles.

   Matilde tapping with her finger on each circle, touching them all
5. Teacher.: there were eight circles… you said you did the shape.
6. Matilde: yes, the shape …I did the same edge of the block put in this way.

   with her index finger running along the rectangle she had drawn, putting her hands one in front of the other, making a rectangle matching thumbs and index fingers together, the other fingers closed. She went down with her hands on the drawing, as holding the block in her hands, showing the view from above.

In the description (Point 1), concerning what she has done, Matilde tries to remember the block used in the previous phases and thinks about its top part. Then, she thinks back to the choices made and expresses a judgement on her drawing by declaring an indecision with respect to the choice of the point of view to be considered. Indeed, the task asks to draw the block, but it is not specified from what point of view. Matilde probably interprets this request first by answering with a side representation of the block. Subsequently, however, she considers her first representation not matching to the Lego block. Indeed, she often repeats 'I didn't like it'. Finally, she chooses the representation from above. The need to choose a point of view from which to observe is already beginning to emerge in Matilde's indecision.

[1] Transcripts and gesture (in *Italic*) descriptions extracted from the video-recording of the third phase, concerning the collective discussion about drawings and their comparing.

In the representation that Matilde does in the end (Point 2), she turns her attention to the circles, which strongly characterize the Lego block in the drawing. Indeed, finally she counts the circles as a fundamental element of the drawing and says 'because on that rectangle there were eight'. Consequently, her drawing is not just any rectangle, but the eight circles determine the characterization of the object drawn.

Then, the teacher (Point 3) mirrors the words used by Matilde, 'rectangle' to focus on the shape. But Matilde (Point 4) counts the circles two at a time, tapping the index finger and the middle finger on the sheet, giving importance to how the circles are arranged, parallel to two by two, probably because her attention is caught by the characterizing element of the drawing.

The teacher (Point 5) repeats the words used by Matilde, 'shape', and mirrors it. Finally, Matilde (Point 6), helping herself with her hands, seems to make a projection of the three-dimensional shape in the plan, like the shape left by the block on the sand. Here, the point of view from above began to emerge as a privileged point of view.

This episode shows the first evolution towards the mathematical meanings that are the aims of the teaching intervention. The intervention of the teacher is fundamental in inducing the pupils to express their personal meanings, and the different reformulations show how such meanings evolve from the description of the drawing to the idea of a shape (rectangle). According to Bartolini Bussi and Mariotti (2008), the passage from the context of the artefact to the mathematics context begins to appear.

12.5.1.2 Episode 2

The discussion continues and Laura intervenes, saying that she also did it the same way. Therefore, the teacher invites her to express her ideas. So she says:

7. Laura: The block… I did the same thing as Matilde, I took the block….
   she moved her hands as if there was a block laying on the sheet of paper (Fig. 12.3)
8. …that was in this position and I drew some lines, as if it was a ruler
   held a fictive block with one hand and drew the edge with the other one (Fig. 12.3)
9. … the lines with the block, the shape… we did it in the same position, identical

Fig. 12.3 The hands of Laura to simulate the block while she compares different drawings

Fig. 12.4 Laura sliding with her index finger on the upper side of the rectangle

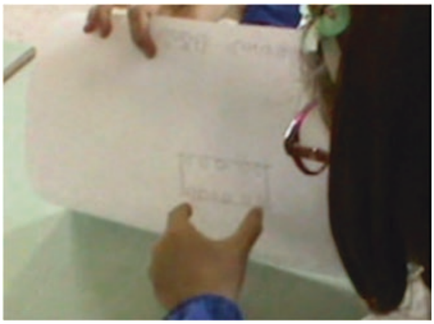


Fig. 12.5 Laura interprets Elio's drawing, of a block placing on a side face

10. Teacher: but why did you draw it in this way?... as if it was on the desk. What did you choose? Did you make a choice to draw it, did you?
11. Laura: In a first moment I thought to do it above… like Elio and Matilde, then I changed my mind and I wanted to…I choose to draw the circles here, in the middle…above here (Fig. 12.4).

    sliding with her index finger on the upper side of the rectangle
12. Teacher: and why? What is it "above here"?
13. Laura: yes, Elio, for example, drew it in this way.

    she lays the hand on one side of the rectangle, with the palm of the hand in front of herself
14. …from the side

    Laura raising the sheet of paper and pointing to the rectangle with thumb and index finger, as if she wanted to lean the block on a side face and says
15. ...Elio put the Lego Block in this way, so… (Fig. 12.5)

Afterwards mshe put the sheet on the desk and rotate both her hands, as is if she held the brick tight, in the movement of rotating the brick itself, changing her point of view from above to aside. In a first moment she used the whole hand to contain the brick; in a second moment she laid on the sheet a thumb and an index finger only, representing the edge of the base. With the rotation of the fingers, only, she emphasizes the thickness of the block which comes out from the sheet (Fig. 12.6).

In this episode, Laura (Points 7–9) compares her point of view with the one expressed by Matilde, deducing that it is the same. According to TMS and our assumption, Laura uses different semiotic systems: graphical, verbal and gestures. She, while speaking, uses drawing as a starting point to compare her point of view

Fig. 12.6 Laura rotaing the finger, only, to emphasize the thickness of the block

with that of others, and uses gestures to simulate the position of the real object on the desk relative to the point of view of the observer. She takes up the sign 'shape' used first by Matilde and then repeated by the teacher. The passage from the three-dimensional object that Laura simulates with her hands (Fig. 12.3), to her two-dimensional representation, is evident. Laura uses an iconic gesture and the word 'shape'. According to Arzarello et al., (2009), her gestures seem to be important sources supporting her thinking process.

At this point, the teacher (Point 10) asks her to clarify and to explain the reasons that guided her choices.

In Laura's next expression (Point 11), a rethinking can be noted. Laura first draws the block looking at it from one side, and then she crosses it out and draws the block again when she looks at it from above. While talking, Laura refers to the circles on the top of the block.

The characteristic of the block represented by the circles emerges again to confirm that children recognize it as an element that cannot be left out in the representation of the Lego block, because it is what distinguishes the Lego block from other rectangular blocks. It had also already emerged in the intervention of Matilde.

The teacher mirrors the words 'above here' and the intervention of Laura (Points 13–15) seems to us very interesting. The previous idea expressed by Matilde began to evolve and is shared. The shape of the block leads to recognizing a rectangle. Moreover, Laura tries to explain her drawing through the comparison with the drawings of other pupils. She interprets Elio's drawing, considers Elio's point of view and highlights the differences between the two drawings.

According to TMS and Arzarello's point of view, the alternation among different semiotic systems, graphical, verbal and gestures, promotes the evolution of the meanings.

Furthermore, the characteristic of the circles is an element that cannot be left out in the representation of the Lego block because it is what distinguishes the Lego block from other rectangular blocks. And it had already emerged in the intervention of Matilde.

12.5.1.3 Episode 3

At this point, Matilde intervenes again:

16. Matilde: Elio has done in this way.
    stretching her arms in front of herself, she placed them one in front of the other, as if she held the brick in her hands, she leaned on the chair back and stirred between her hands as if she could see a brick, seen from aside (Fig. 12.7)

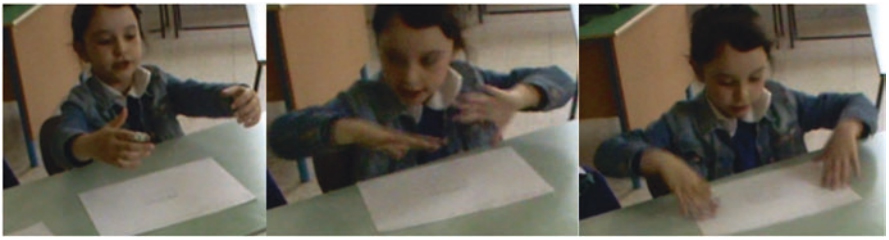
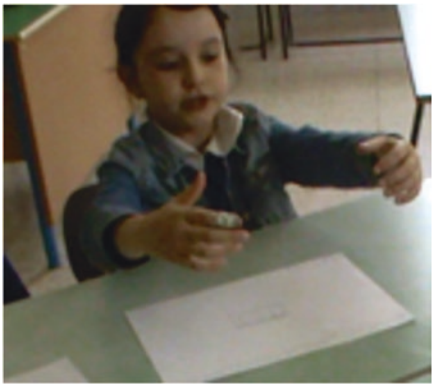


Fig. 12.7 Matilde's gesture to represent different points of view

Fig. 12.8 Matilde's gesture to represent the point of view from the side



17. …and Laura and me, <u>from the top</u>

    she puts her open hands with her palms down, as if she were touching the upper part of the brick, getting them closer to the desk and with the whole body she leans on the desk, looking at it from above
18. Teacher: What do you think, Elio?
19. Elio: Yes, it's true! I drew the block looking at the side, while Matilde and Laura looking it from the top.
20. Matilde: Elio in this way (Fig. 12.8) … instead Laura and me from the top (Fig. 12.9).

    she stretches her arms, puts once more her hands parallel to each other in front of herself, with the open palms (Fig. 12.8).

    she makes a quick gesture, putting her hands on the sheet and overlooks it with her whole body (Fig. 12.9).

    At this point, Laura intervenes, gesticulating in the space in front of her, opening her hands and moving them as if she touched the brick on its side, and says:
21. Laura: Elio looked at the block from the side. Matilde and me looked at it from the upper side, instead.

    Laura stood up and overlooked the desk, bended her head forward
22. Laura: Not in this way.

Fig. 12.9 Matilde's gesture to represent the point of view from the top

Fig. 12.10 Laura lifts the sheet and looks at its profile as if she saw the thickness of the block

Laura first lifted the sheet from the desk and she put it in front of herself, as if it was the side surface of the brick (Fig. 12.10).

Then she leaned the sheet on the desk and put her open hand on the drawing, flattening/ pressing it from above. She repeats the same gestures done by Matilde.

In this episode, the reference to the idea of different points of view clearly emerges, coming out of the comparison of three different drawings.

Matilde, confirming what Laura says and acting with gestures, interprets the drawings made by Laura, compares Laura and Elio's drawings and underlines the difference in the points of view from which they are drawn (Fig. 12.7).

The sign emerges in relation to the use of the manipulative artefact, the Lego Block and the drawing evolves, thanks to the shared discussion. In fact, 'from the top' can be considered a pivot sign, because on the one hand it may express the action of seeing the block from above, and on the other hand, it refers to the mathematics sign (view from above).

At this point, the teacher invites Elio to explain the point of view from which he observed the block when he drew it (Point 18.). Elio confirms and summarizes (Point 19).

Subsequently, Matilde recognizes two different points of view in the drawings. The sign 'this way' is a pivot sign, because it on the one hand indicates the drawings that Matilde is directly observing, on the other hand it refers to the side of real object that is represented (Point 20). The sign is matching with gestures (Figs. 12.8 and 12.9).

In the end (Point 21), it seems that Laura wants to describe her representation as a representation that takes into account multiple points of view at the same time, but she clearly has in mind that a different point of view has been used in her representation than in the one of Elio.

The analysis of episodes 2 and 3 shows the development of the complex system of meanings and the texture of their relationship, highlighting the expected alternation between different semiotic systems, graphical, verbal and gestures. Moreover, these episodes are showing how the meanings of the different points of view emerge once more. In addition, the final argumentation by Matilde and Laura shows how the use of the artefacts, the Lego block and the drawing, led to a consolidation of the mathematics meaning among the students. The process of the interpretation of the drawings played a fundamental role that conducts the pupils to compare different representations, to choose the adequate geometrical shape, the rectangle, to recognize in the drawing the characteristic element of the object, the circles. It is evident that the alternation among different semiotic systems, graphical, verbal and gestures, when pupils try to explain their argumentation, supports, as a fundamental mediator, the student's thinking process. Also, the process of construction of the correct coordination of different points of view has been achieved through the mediation of specific artefacts, the Lego block and the drawing.

12.6 Conclusions

This chapter reports some preliminary results concerning the validity of the hypothesis about the potentiality of using the combination of artefacts as tools of semiotic mediation. The analysis of the data coming from a teaching experiment clearly shows how the potentiality of each single artefact can be utilized and combined. In this way, it is possible to construct and develop mathematical meanings concerning the coordination of different perspectives, drawing three-dimensional figures from different points of view. Additionally, the analysis of the collected data shows the potentiality of the alternation among different semiotic systems, graphical, verbal and gestures, and how the different meanings could be related and integrated to develop expected meanings related to the mathematical notion of point of view.

The discussed results, considering the small number of children, cannot be said to have any general character. According to our hypothesis, we try to interpret the data collected in reference not only to the request to represent the Lego brick with a drawing, but also to a subsequent description made by the same children with gestures and words. The drawing is linked both to the representation of a three-dimensional object on a two-dimensional surface in relation to different observational points of view, and to the argumentative skills related to the natural language and gestures recorded during the discussion.

The analysis of the drawings produced by the children and the relative gestures used during the discussion allow us to highlight the evolution of signs and corresponding meanings. Here, we notice the presence of several visible parts drawn at the same time or the overlapping of several elements linked to different points of view.

Looking at the Mathematics teaching/learning phases in a vertical perspective that embraces all school grades, we believe that the discussed results can be a good starting point for reflection for researchers in mathematics education for future theoretical and experimental investigations on the development of the geometric thoughts in all its forms from the Kindergarten to University.

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Chapter 13
Perceiving and Using Structures When Determining the Cardinality of Sets: A Child's Learning Story

Priska Sprenger and Christiane Benz

13.1 Introduction

Early childhood education in mathematics is often limited to the most obvious mathematical activities children show at this age: counting and determining sets. Researchers (cf. Brownell et al., 2014; Gasteiger & Benz, 2018; van Oers, 2004) and official policy educational documents agree that early mathematics education should be based on central mathematical concepts, and enable continuous learning and a broad understanding of mathematics (Brownell et al., 2014; Gasteiger, 2015). Not only mathematical contents are relevant for a broad understanding of mathematics, but also mathematical processes or proficiency strands, for example, problem-solving or reasoning (cf. Australian Curriculum, Assessment and Reporting Authority, 2014; Department for Education, 2013; National Council of Teachers of Mathematics, 2000). Nevertheless, the arithmetical content is still one important part in early mathematical educational concepts and policy documents—also in relation to the background of early intervention or prevention for problems in learning mathematics in school. Therefore, we will look at and analyze some aspects of early numerical and arithmetical development.

13.2 Role of Structures for Numerical and Arithmetical Development

Structures play an important role in numerous models for number and arithmetic concept development (cf. Baroody, Lai, & Mix, 2006; Lüken, 2012; Mulligan & Mitchelmore, 2018; Mulligan, Mitchelmore, English, & Crevensten, 2013). In the

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hypothetical development trajectory for key aspects of early number and arithmetic development, Baroody et al. (2006) emphasize the importance of structures both for the number concept and for the arithmetical development:

> Conceptually based VNR [verbal number cognition] enables a child to see (decompose) collections of two (a whole) as one and one (into its parts) […]. […] Experiences [of] decomposing and composing small, easily subitized collections may be the basis for constructing an informal concept of addition (and subtraction) (Baroody et al., 2006, p. 193).

Structure or structuring, which can be defined as the way in which various elements are organized and related (Mulligan & Mitchelmore, 2013), can be seen as decomposing and composing (visible) objects and therefore it is an underlying concept for the part–whole-relations because "this composing process fosters an understanding of part-whole-relations and vice versa" (Baroody et al., 2006, p. 193).

> A part–whole concept and experience with composition and decomposition may underlie an understanding of "number families" or the different-names-for-a-number concept (a number can be represented in various ways because a whole can be composed or decomposed in various ways) and is one key link between number and arithmetic (Baroody et al., 2006, p. 195).

Thus, it is not surprising that Resnick, already in Resnick, 1989, pointed out that "probably the major conceptual achievement of the early school years is the interpretation of numbers in terms of part and whole relationships" (p. 114). Referring to Baroody et al. (2006), (de)composing collections of objects can nurture the part–whole understanding. If children (de)compose collections of objects, they switch the focus from individual items to perceiving and identifying structures of parts. Hunting (2003) describes this ability as an important step for part–whole reasoning, which in turn contributes to numerical development. If the switch from focusing on individual items to perceiving and identifying structures of parts is so important, different aspects of this "switch" have to be examined, so that children can be supported to achieve this switch. Therefore, in this study we look at different possibilities to perceive items in collections and also how perception is used for the determination of cardinality.

The evaluations of the learning story presented in this study are based on a theoretical model that distinguishes between two processes: the process of perceiving sets and the process of determining cardinality. These processes can run one after the other, or coincide with each other, for example during subitizing (cf. Fig. 13.1 and Schöner & Benz, 2018). In Fig. 13.1, possible relationships between the two processes are illustrated.

Each of these two processes can be divided into three different subgroups. The different ways of perceiving a set allow different ways to determine the cardinality as, for example, the use of a counting strategy, a derived facts strategy, or the use of known facts. If the elements of a set are perceived as individual elements, the only possibility to determine the cardinality is to use the counting strategy "counting all." If a set is perceived in (sub-)structures, "counting all" would also be a possible strategy to determine the cardinality. Furthermore, in this case, in addition to "counting on" (four, five, six, seven, eight) or "counting in steps" (four, six, eight), noncounting-derived facts strategies (three and three equals six and two more is eight) can also be used to determine the cardinality.

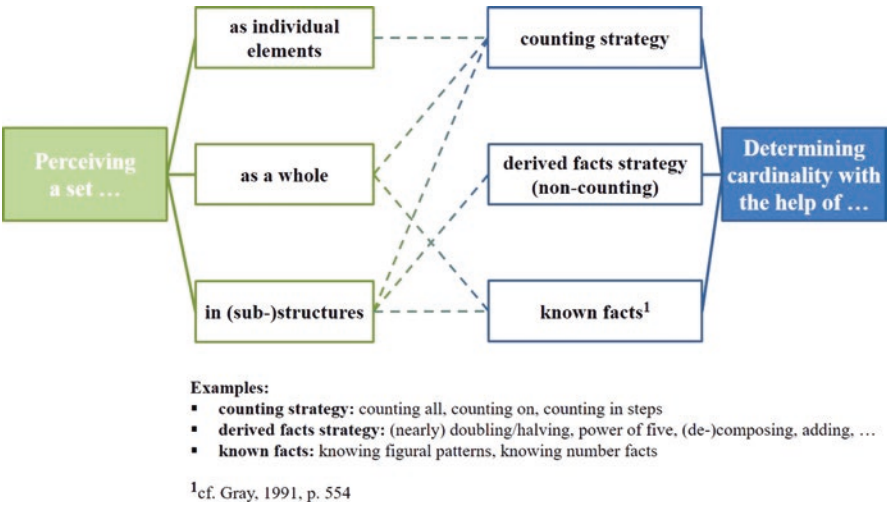

Fig. 13.1 Two processes: Perception of sets and determining cardinality (Schöner & Benz, 2018)

13.3 Research Question

In this chapter, we aim to answer the following research question based on a child's learning story:

How do the perception of structures and the use of structures determine the cardinality of a set change based on an implementation?

13.4 Design of the Study

Ninety-five children from nine different kindergartens aged from 5 to 6 years were interviewed three times. The study of Schöner and Benz (2018) describes that at the first interview (T1), 102 children were interviewed. Some children have left the study, for example due to a move to another city. The children were divided into a treatment group (n = 55) and a control group (n = 40). Only the treatment group took part in an implementation phase (cf. Fig. 13.2). Luca, the boy in the presented learning story, was a member of the treatment group. In the first interview (T1), at the beginning of the last year in kindergarten in September 2015, Luca was five years and two months old. Then, an implementation happened for four months. After the implementation period, the posttest interview (T2) proceeded in February 2016. The children were given the same tasks again to investigate the development in perceiving and using structures to determine the cardinality of sets. The third interview (T3), at the end of the last year in kindergarten, was conducted as a follow-up interview in July 2016 (cf. Fig. 13.2).

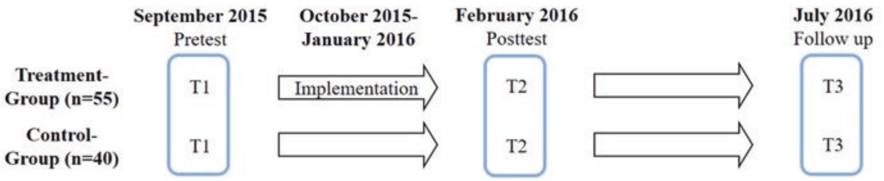

Fig. 13.2 Timeline of the study

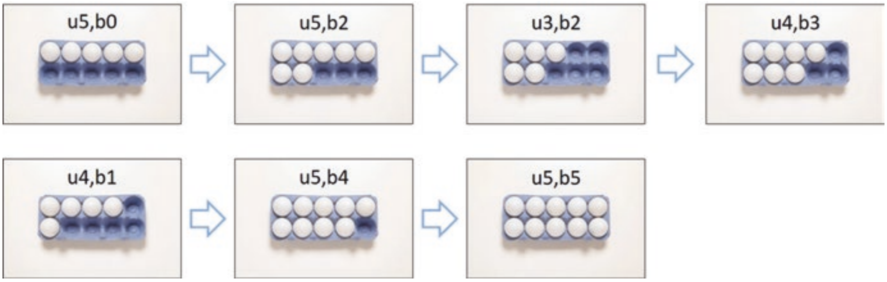

Fig. 13.3 Order of the presented items

The study is designed in a panel design, so the same children were interviewed three times (T1, T2, T3) to evaluate whether and how they perceive and use structures for determining the cardinality of the presented sets. To get some deeper insights into the perceiving process of the children, the research tool eye-tracking was used. With eye-tracking, it is possible to detect the eye movements of the children while they are perceiving and determining the cardinality of sets. The whole interview consists of four different parts (in three of them, the research method eye-tracking is used: unstructured pictures of dots, egg cartons, and daily life pictures). In this study, the focus is on the part with the egg cartons for ten eggs which has an equivalent structure as the tenth frame. It is the typical size of egg cartons in Germany.

13.4.1 Tasks

In this part of the study, pictures of egg cartons with the quantities two, three, four, five, seven, nine, and 10 were presented to the children on a monitor. In the present learning story of Luca, only the sets with cardinality ≥ 5 are considered. In Fig. 13.3, the order of these items is illustrated.

Because there are three pictures with the cardinality five and two with cardinality seven, it is necessary to name the egg cartons individually. "u3,b2" means, for example, "three eggs on the upper row" and "two eggs on the bottom row" (cf. Fig. 13.3). These abbreviations were not visible to the children. They are useful in this chapter to facilitate communication about individual egg cartons.

Before the pictures were presented, the children had been told that the interviewer would like to know how many eggs they saw. They were asked to say the number as soon as they knew it and they had as much time for determining the quantity as needed. As soon as they said a number, the interviewer asked how they came to the result (cf. Schöner & Benz, 2017; Schöner & Benz, 2018). First, a closed egg carton could be seen. Then, the carton was opened. After the child said a number (cf. Fig. 13.5, phase 1) and explained how it came to the result (cf. Fig. 13.5, phase 2), the carton was closed again.

13.4.2 Implementation

After the first interview (T1), the treatment group got a collection of different materials and games, like the game "I spy with my little eye" which will be explained below. These materials and games offered the opportunity to discover and facilitate the structured perception of sets in a playful way. The children in the control group did not get the materials. Additionally, in order to observe a development in perceiving and using structures through an everyday support of the children in kindergarten and at home, the kindergarten of the control group did not work with any special mathematical training program in perceiving and using structures. That means that, during the test period, the kindergarten teachers worked with mathematics in exactly the same way as before the research project. The normal routines of the kindergartens of the control group were therefore not altered.

During the four months of the implementation phase (cf. Fig. 13.2), the kindergarten teachers in the treatment group were instructed to use these materials with the children one to three times each week for 30 min (cf. Schöner & Benz, 2017). During the entire implementation phase and until the end of the kindergarten year in July 2016, the materials were kept in the kindergartens and were always freely accessible to the children. During the implementation phase, the educators were asked to keep a diary by writing down their activities with the children and their observations in order to get additional information. Luca was mentioned by name, so statements about his development can be given from the perspective of his educators.

The ten-egg-cartons were part of the provided materials. The kindergarten teachers were instructed on how to use the materials and how to ask questions in order to gain insights into the children's ideas and their ways of thinking. Additionally, they got a handbook with different ideas and examples. One of these ideas was a modification of the game "I spy with my little eye:" Sets of eggs with quantities ranging from 1 to 10 are sorted in the egg cartons in such a way that the upper row is always filled first. Thus, the numbers are represented up to 10 with a five-structure (cf. Fig. 13.4).

After the cartons are filled, a collective conversation about the "appearance" of the different number-pictures can take place. Afterwards, the egg cartons are closed and mixed. Now, a child (or a kindergarten teacher) takes an egg carton and looks inside it. He or she describes which number-picture he or she sees. There are different

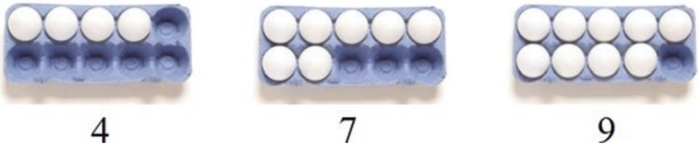

Fig. 13.4 "I spy with my little eye" – Three examples

requirements: Filled rows and places and/or empty rows and places can be described. Hence, a description of the number six can be different. For example: "The upper row is full. In the bottom row it is just one egg" or "The upper row is full. In the bottom row, four places are empty or in the bottom row four eggs can be placed" or "There are still four empty places or in the bottom row it is still place for four eggs." The game is finished when all egg cartons have been described. The child who knows how many eggs there are in the carton will receive the carton. Whoever guesses the most boxes wins. If the game is played in this way it is very challenging, because children have to rely on internal pictures. In order to help the children to build up internal, structured images of the individual numbers, a variation of the game may be played. If all cartons are open during the whole game, children can see the pictures when somebody else describes it and link picture and description. Some helpful questions for both ways of playing are "can you describe how the number-picture looks like?" or "can you say how many eggs there are in a full egg carton without counting every single egg?"

Another focus can be established if cartons can be filled without any restriction. So, one number can be displayed by many different ways in an egg carton. Here, the focus can be placed on different ways of decomposing a set of objects and therefore different ways of decomposing numbers. Here, one possible question could be: "How can you put *n* eggs in the carton (for ten eggs)?" (Benz, 2010, p. 28). There was, on the one hand, a description of the games and, on the other hand, a lot of possible questions and impulses in the handbook the kindergarten teachers got. These questions and impulses were helpful in supporting the use of the learning opportunities of the games regarding the perception of structures and the structural use to determine the cardinality.

13.4.3 Aspects of Data Analysis

In the evaluation, a distinction is made between three different types of data: the *observation aspects,* the *eye-tracking data*, and the *explanation* (cf. Fig. 13.5). Each piece of data leads to hypotheses about the perception process and the determination process. On the one hand, hypotheses about these processes are generated on the basis of the observations which are made during the interview, such as gestures, sounds, or promptness of the answer (cf. Fig. 13.5, observation aspects and explanation). On the other hand, additional information is gained during phase 1,

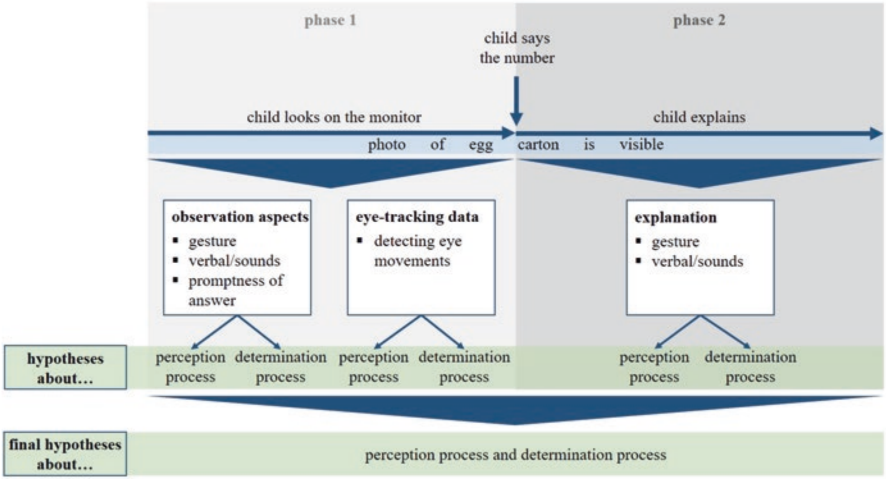

Fig. 13.5 Differentiation of aspects of analysis

by detecting the eye movements with the help of the eye-tracker (cf. Fig. 13.5, eye-tracking data).

The eye-tracking data can provide insights into children's processes of perception (see paragraph below "Data Analysis by the use of Eye-tracking technology"). These insights can be used to form hypotheses about perceiving structures (perception process) and about determining the cardinality (determination process). A typical observation with eye-tracking data is that the children's gaze often oscillates between two subsets when a structure is perceived. So, the eye-tracking data first lead to a hypothesis about the perception process. In most cases, it is possible to generate a hypothesis on the determination process from the hypothesis on the perception process. This is the case, for example, when the eye-tracking data show that each egg was fixed individually. In this case, the hypothesis for the determination process would be "counting all." If the eye-tracking data show a pendulum motion between two subsets, then a perception process can be concluded as a determination process. In the learning story described below, a special observation can be made during the analysis of the eye-tracking data. First, a pendulum movement between two subsets is visible and then the fixation of each individual egg. In this case, both the perception process (structural perception) and the determination process (counting all) become visible (cf. Fig. 13.9). Regarding the observation aspects, it became apparent in the course of the evaluations that, often, only a hypothesis on the determination process and none on the perception process can be made. An example of this is "counting all" as a strategy for determining cardinality (determination process). In this case, no interpretation of perception is possible, because it is not clear if the child perceived the set as individual elements or in (sub-)structures (cf. Fig. 13.6). Explanations, which can be assigned to a structural perception and use, are, for example, "there are four and three and that is seven. I know that" or "In the upper row there are three and below two, that is together five."

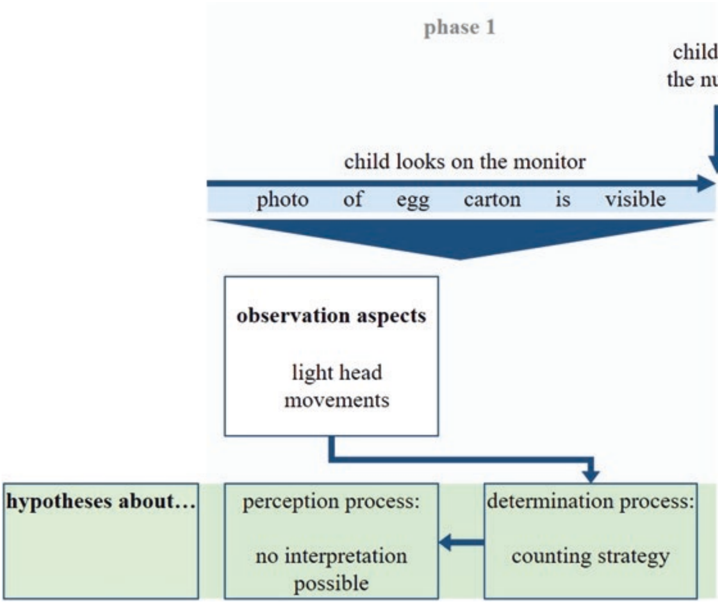

Fig. 13.6 Observation aspects

The final hypotheses about the perception and determination process are gained from all three different types of data (cf. Fig. 13.5). The evaluations are therefore based on a hypothesis-generating method (for more details, cf. Schöner & Benz, 2018). In the following example, it becomes clear that the three-level evaluation process is very complex. The three data types, observation aspects, explanation, and eye-tracking data, are interrelated and complement each other.

13.4.4 Data Analysis by the Use of Eye-Tracking Technology

The eye-tracking data is a collection of the eye movements that the children made during the interview. The analysis of eye-tracking is based on the hypothesis that "eye movements provide a dynamic trace of where a person's attention is being directed in relation to a given visual display" (Jang, Mallipeddi, & Lee, 2014, p. 318). To follow these dynamic traces of the children, there are different ways of visualizing the eye-tracking data: the *GazePlot*, the HeatMap, or the Cluster. Each of these visualization types are dynamic representations of the selected media. In the study presented here, mainly the GazePlot-data was evaluated. It is a helpful tool to evaluate the eye-tracking data, because the order in which the child fixed the single objects is shown by numbers written on the dots. The dots reflect an eye-fixation and the diameter of the dots indicates the duration of each fixation.

The longer the child looks at an object, the larger is the diameter of the dot in the GazePlot visualization. To evaluate the data, the GazePlot-video was interpreted by replaying it several times. In the video, the dots appear one after the other and the course of the child's gaze becomes visible. The data of the GazePlot-video can also be represented in another way, namely as an Accumulate-Graphic. Here, all dots are shown on one picture (cf. Fig. 13.8). In the present study, it is important, not only to look at the eye-tracking data, but also to connect it with the observation aspects and the explanation (cf. Fig. 13.5). With this three-level evaluation process, more reliable assumptions can be made about the perception and the determination of the children.

13.4.5 Example from the Data Analysis

In order to show how the collected data was evaluated, an example from the posttest (T2) with the item "u5,b0" is described in the following. It is an example from the interview with Luca. The egg carton with five eggs in the upper row was presented to Luca. Light head movements could be observed and then he said "five." The interviewer asked how he found out that there were five and he answered that he had counted quietly. Looking at the observation aspects of Luca, light head movements could be observed, which leads to the hypothesis that he probably used a counting strategy to determine the cardinality. No hypothesis about his way of perceiving is possible (cf. Fig. 13.6). At an early stage of the evaluations, the hypothesis "perception as individual elements" was established if a "counting strategy" was observable in the observation aspects (cf. Schöner & Benz, 2018). In the course of analyzing the data of all children who were interviewed, it was decided not to draw any more hypotheses about the perception process in this case, as it has been shown that children of this age very often have counted the number, but still perceived a structure (cf. Example "Liam" in Schöner & Benz, 2017). To sum up, it can be said that the data of the observation aspects of Luca does not automatically lead to the hypothesis that he perceived the set as individual elements.

In the data of explanation (cf. Fig. 13.7), Luca said that he counted the eggs quietly. This statement allows no hypothesis about the perception process, because one cannot conclude from this statement alone whether he perceived the set as individual elements or perhaps in a (sub-)structure. For the determination process, the hypothesis "counting strategy" can be generated because he says that he has counted (cf. Fig. 13.7).

In Fig. 13.9, the eye-tracking data indicate that Luca's gaze oscillates between the left and the right side of the eggs. Then, he fixed every single egg one after the other. These two types of perception (in substructures and as individual elements) become visible in the eye-tracking data if the GazePlot-Graphic is divided into two parts. In the first part of the GazePlot-Graphic (cf. Fig. 13.8, above), it becomes visible how his gaze oscillates back and forth.

In the first part of the graphic, it can be seen that his gaze oscillates between the left and the right side of the presented eggs (cf. Fig. 13.8, above). This is a typical

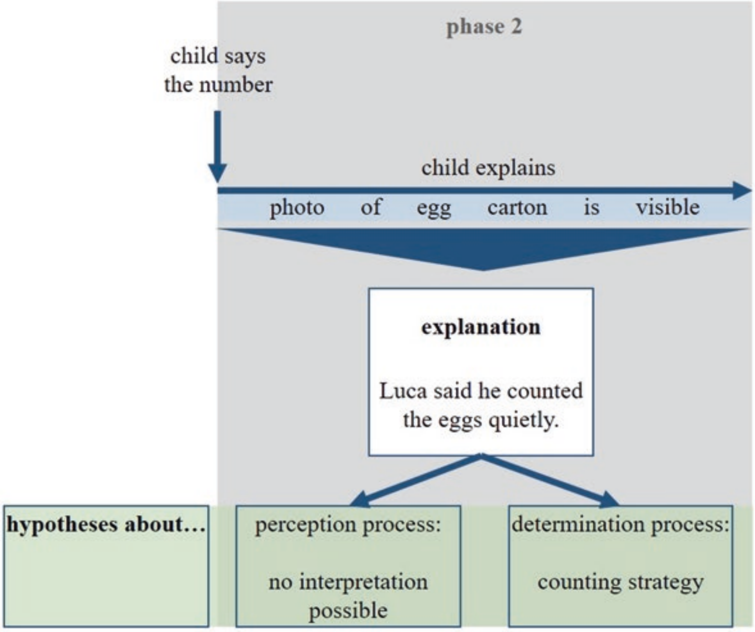

Fig. 13.7 Explanation

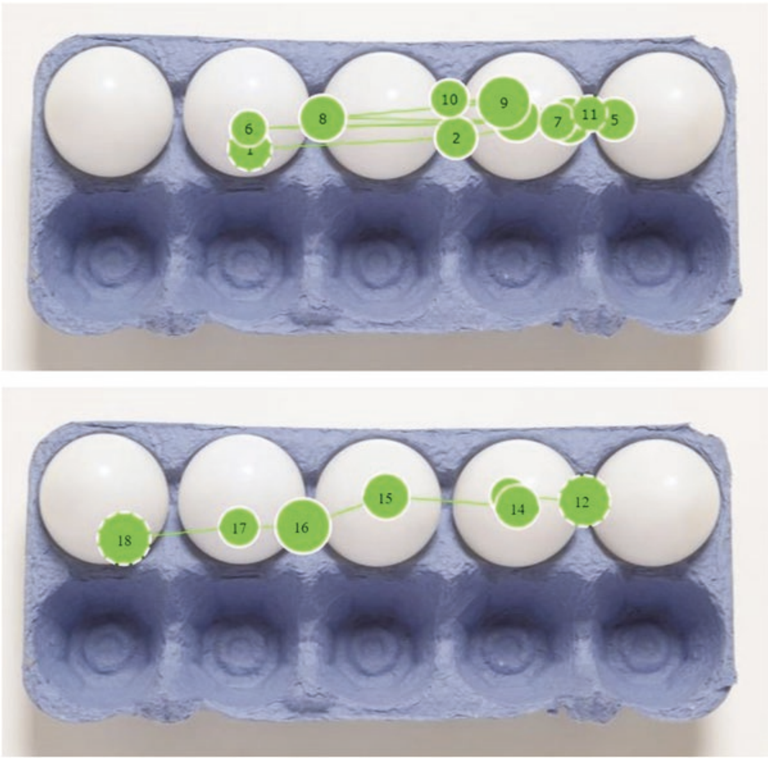

Fig. 13.8 Accumulate GazePlot-Graphic part one (above) and part two (below)

movement when two subsets are perceived (cf. Schöner & Benz, 2018). The data on the first part of the GazePlot-Graphic can lead to the assumption that Luca perceived two eggs on the left and three eggs on the right. This is an interpretation based on the evaluation of all eye-tracking data of the 95 children who participated in the study. These comparative data from all eye-tracking data could only be obtained by analyzing the observation aspects and the explanation in a three-level evaluation process. The second graphic (cf. Fig. 13.8, below) shows that Luca fixed every single egg. This leads to the hypothesis that he counted every single item. Luca started with the egg on the top right and then fixed each egg separately from right to left. The evaluation of all eye-tracking data shows that the first egg the children have counted is often not very clearly fixed. This can also be seen in the example of Luca, because the first fixation dot is not in the middle of the egg, but on the left side. Another phenomenon is that sometimes two gaze-dots are exactly on top of each other. In this case, it is dots two and three (numbered 13 and 14, respectively, in Fig. 13.8, below). This does not mean that Luca counted this egg twice, because not every fixation dot corresponds to a counting step (Fig. 13.9).

Presumably, Luca cannot yet use the perceived structure to determine the cardinality and uses his familiar counting strategy "counting all."

The final hypotheses on the two processes are that Luca perceived the set in structures and used the counting strategy "counting all" to determine the cardinality. Hypotheses on the perception process could not be made, either on the basis of the

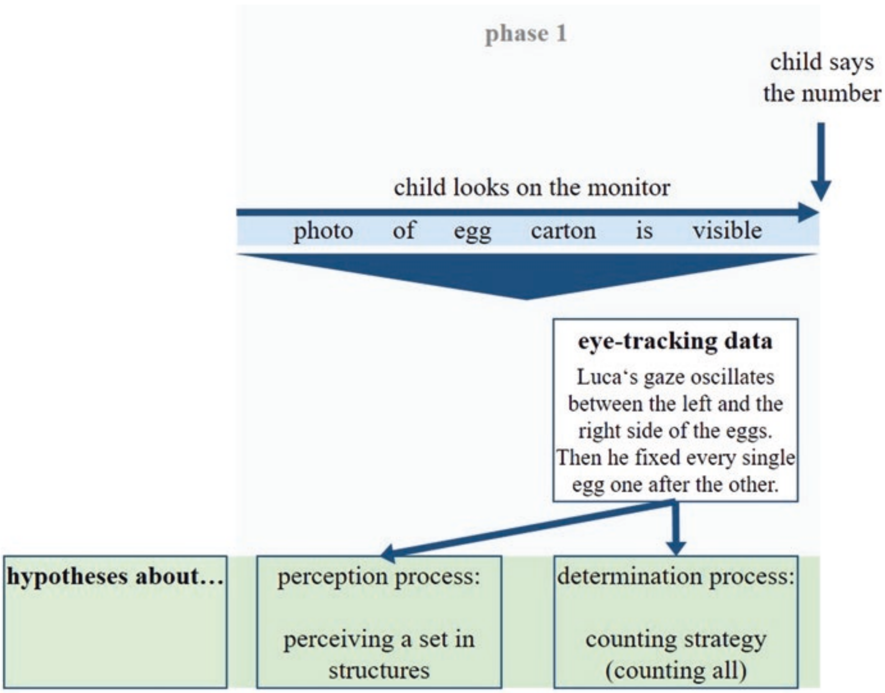

Fig. 13.9 Eye-tracking data

data from the observation aspects or on the basis of the data from the explanation. It can therefore be stated that without the eye-tracking data, no hypothesis on the perception process would have been possible.

13.5 Results of the Learning Story of "Luca"

The observation of the educators refers to the mathematical situations that were guided. Referring to the observations of the kindergarten teachers, Luca was difficult to motivate for the mathematical activities. He often quickly lost interest. The kindergarten teachers responded to this fact, for example, by forming smaller groups in order to support him and in order to take his interests into account. This worked successfully – according to the educator's statements – and he could later be reintegrated into the whole group. In November 2015, the kindergarten teacher described Luca's difficulties imagining the arrangement and finally the cardinality of the sets of eggs described by another child in the game "I spy with my little eye." Even an open, empty egg carton to see the ten-structure did not help him at this stage. Four weeks later, in December 2015, he still had difficulty (with the same activity) in giving a verbal explanation, but most of the time he could name the number immediately, if somebody else described it. By comparing the results of the three interviews (T1, T2, T3) with this information, it can be concluded that the analysis of the data supports the kindergarten teacher's observations regarding perceiving and using structures to determine the cardinality (cf. Fig. 13.10).

Figure 13.10 shows the final hypotheses of the three individual interviews of the learning story of Luca. Only the quantities with a cardinality ≥5 are considered. On the left part of the bars, Luca's way of perceiving a set is presented (gray: no interpretation possible; dark yellow: perceiving a set in (sub-)structures and, on the right part of the bars, his way of determining the cardinality is shown (violet: counting strategy: counting all; light yellow: structural use). The results are divided into the three parts: T1 (pretest), T2 (posttest), and T3 (follow-up).

In the pretest (T1), Luca counted every single egg aloud for each item to determine the cardinality and also pointed, with his finger, to the corresponding egg. No interpretation about his way of perceiving the set was possible because, on the one hand, his finger gesture interrupted the connection to the eye-tracking camera and, on the other hand, there was no additional observation, like an explanation, which could lead to a hypothesis about perception. The fact that he counted every single egg does not automatically lead to the hypothesis that he did not perceive a structure. The observations in the posttest (T2) lead to the hypothesis that Luca always perceived a set in structures but he could not use the structures to determine the cardinality. In the example "u5,b0" (T2), described in detail above, the observations obtained using the eye-tracker indicated that Luca had perceived a structure (cf. Fig. 13.8). For the remaining items, Luca verbally named the structure, and the existing eye-tracking data confirmed this observation. To answer the question on how many eggs there were, he again consequently used the counting strategy

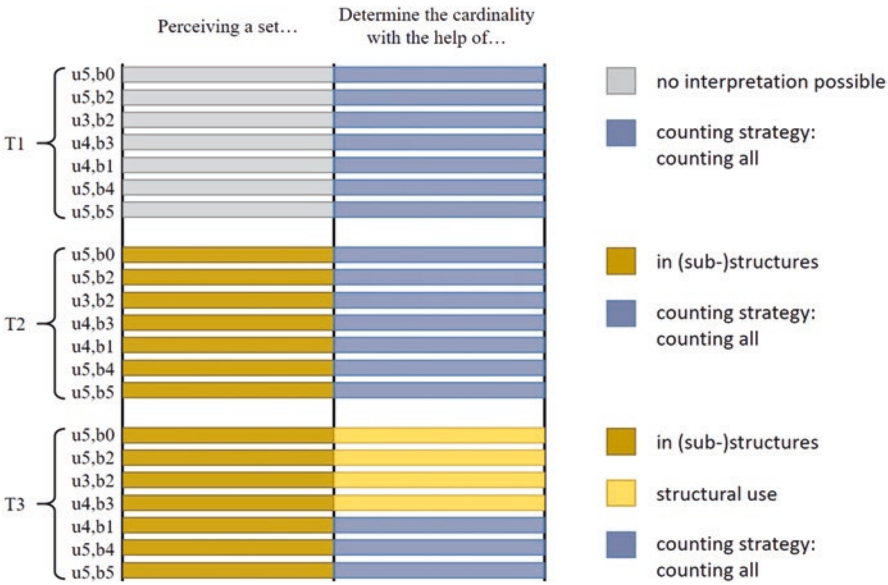

Fig. 13.10 Results of the learning story of Luca

"counting all." In the follow-up interviews (T3), the eye-tracking data for all seven items led to the hypothesis that Luca again perceived a structure. In four of the shown items (u5,b0; u5,b2; u3,b2; u4,b3), Luca was now able to use the perceived structure for determining the cardinality (cf. Fig. 13.10). Only in three remaining items (u4,b1; u5,b4; u5,b5), he continued to use the counting strategy "counting all" to determine the cardinality.

13.6 Summary and Discussion

The research question in this chapter was "how does the perception of structures and the use of structures to determine the cardinality of a set change based on an implementation?" Based on the learning story of Luca, a development can be seen. In the first interview (T1), he counts all the eggs individually and loudly, and points with his finger to each individual egg. No statement can be made about the perception process. After the implementation phase (T2), we can observe that he is able to perceive a structure in the presented sets, but still counts each egg individually to determine the cardinality. By structuring (decomposing) the presented set, the basis for a part–whole understanding is initiated (cf. Baroody et al., 2006, p. 193). At the end of the last kindergarten year (T3), Luca was able to use the perceived structure partially to determine the cardinality of the sets. It cannot be safely assumed from this single learning story that these results can also be transferred to other children and other learning environments. But still, the assumptions as stated above can be made.

The presented material (egg cartons) is used for the implementation phase as well as for the interviews with the children. For this reason, the study could be accused of "teaching to the test." It must be taken into account that also other materials were used for the implementation. There are activities with structured materials like "finding pairs" with the egg cartons where the existing structure is in focus, as well as unstructured materials such as glass nuggets, where the children can structure the set themselves. It is also very important for children, especially in regard to primary school, to learn reliable structures that they can use for calculating strategies (Lüken, 2012). This could be enhanced in primary school by using certain materials, for example, the ten-frame, a typical didactical presentation used in primary school, which has a ten-structure (like the egg cartons), and extends the number space to 100. Therefore, perceiving structures and using them for noncounting strategies are valuable skills which can serve as a basis for the development of a part–whole-understanding, and later for the development of calculating strategies. It is shown that children at this age are able to develop a perception of (sub-)structure and a structural use of determination strategies. The study is based on a hypothesis-generating procedure. In the future, some hypotheses will be tested in a statistical examination and significances will be calculated.

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Chapter 14
"A Triangle Is Like a Tent": Children's Conception of Geometric Shapes

Andrea Maier and Christiane Benz

14.1 Introduction

Because "children's minds are moldable" (Hüther, 2007, p. 70) and to a higher degree than previously supposed, the focus on early childhood education has increased over the last decade. Studies reveal that there is a coherence between a high number of mathematical activities in kindergarten and higher mathematical achievements in school (Krajewski, 2003) and moreover, there is no other time in life where one is capable of learning as much as in early childhood (e.g., Caspary, 2006). Thus, early knowledge of mathematics is often seen as a predictor for later school success. With respect to geometry, there are less studies concerning early childhood education, although geometry is an important aspect of the mathematical development in children (Clements, 2004). Children begin to build geometric concepts even before they enter school. In order to answer the question of how to best support the development of geometrical concepts and reasoning, one must first look at how children develop geometrical concepts. This chapter gives some insights into how 4- to 6-year-old children explain and draw triangles and identify different shapes (circles, squares, and triangles). Emphasis will be laid on the children's ways of explaining a triangle in comparison to the images of triangles they have in mind.

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14.2 Theoretical Background

Firstly, this chapter will illustrate how children develop geometrical concepts and geometrical thought, before showing how children aged between 4 and 6 can develop a comprehensive concept knowledge with reference to the single aspect of concept formation presented here in this chapter.

There are different ways how a concept can be built (Franke & Reinhold, 2016, p. 116ff.). Children mainly build concepts "by actively dealing with objects in connection with language" (ibid., p. 118). Thus, concepts are built through everyday experiences and through language. Furthermore, concept formation is often described as being related to categorization or classification, which will be described in the following chapter. Within cognitive psychology, there are two major theories trying to describe the processes of categorization and concept formation: the *classical view* and the *probabilistic (prototype) view*, differing mainly in answering the following questions: Is there a general description for all members of a class? And do all attributes specified in one consistent description apply to all examples of the class (Smith & Medin, 1981)? The classical view affirms both questions and assumes that all examples of a concept share common properties, which are both necessary and sufficient in order to describe the concept (Klausmeier & Sipple, 1980; Smith & Medin, 1981). The classical view assumes clear-cut boundaries which is defined by what belongs to a certain concept and what does not belong. For this reason, the classical view applies to mathematical concepts because mathematical examples always have precise and clear definitions. Hence, a triangle is defined as a (1) plane, closed figure, (2) with three corners and (3) three straight sides. On the other hand, the probabilistic or prototype view (Clarke, 2004; Reed, 1972; Szagun, 2008) takes into account characteristic features and not just defining features. Some members of a concept can be considered as "better examples" than others. The probabilistic or prototype view proposes the existence of "ideal examples," often described as "prototypes." In the case of triangles such "better examples" from a child's view can be equilateral or isosceles triangles, considering for example that the sides must be of the same length in order to be called a triangle. When regarding mathematical concepts, both views are often employed. The classical view because it resembles mathematical definitions (the definition of a concept) and the probabilistic view because it illustrates the individual picture, the image, one acquires about a shape.

> A concept name when seen or when heard is a stimulus to our memory. Something is evoked by the concept name in our memory. Usually it is not the concept definition, even in the case the concept does have a definition. It is what we call 'concept image'. (Vinner, 1991, p. 68)

The image of a concept is something nonlinguistic that is connected with the name of the concept. Still, the concept image often resembles only a few typical examples (often prototypes) and not the variety of examples that can be connected with the concept name (Franke & Reinhold, 2016, p. 123). At other times, the concept image may include examples that contradict the definition, being rather non-examples (Levenson, Tirosh, & Tsamir, 2011). The concept definition in combination with a variety of the respective images plays a crucial part in the formation of concepts.

There are two main types of models describing the development of geometric thought and reasoning. Some researchers emphasize a hierarchical structure, where one level develops after the other (Battista, 2007; Piaget, Inhelder, et al., 1975; Van Hiele, 1985) and in order to advance into a higher level, one must first acquire all the competencies at the present level and the single levels are connected to certain age groups. Because students often used the competencies of different levels for one task or switched between the levels depending on the task, other researchers suggested a more dynamic or "wave-like" development (Clements & Sarama, 2007; Lehrer, Jenkins, & Osana, 1998; Siegler, 1996), where the competencies develop at the same time with one aspect always being predominant. However, in both models, the development reaches from visual reasoning (as "most basic level" or "first dominant wave") to descriptive reasoning and finally to analytical reasoning.

However, there is no description yet what constitutes concept formation in kindergarten and primary school. From a further developmental view, originally describing competencies of secondary school children, aspects that constitute a comprehensive conception of geometric shapes (Vollrath, 1984) or respectively the aims of concept formation (Franke & Reinhold, 2016) are formulated. In the following summary Vollrath's as well as Franke and Reinhold's descriptions of a comprehensive concept formation are illustrated: (1) knowledge of the content of the concept, i.e., knowing definitions and properties, (2) capture of the range of the concept, e.g., distinguishing examples from non-examples, drawing own examples, building or designing own examples of shapes, (3) knowledge of the generic as well as the minor terms, and (4) being familiar with the applications of the concept. These aspects may also apply for early childhood but must be specified when applying them to a younger age group. For example, a 4- to 6 -ear-old child is not yet expected to give a perfect definition, as will be illustrated more precisely later. In this chapter, only some aspects of a comprehensive concept formation are illustrated: (a) how children explain the shape of a triangle (correlating to (1) "content of the concept"), (b) how they perceive different kinds of triangles, i.e., what kind of shapes they consider as being triangles, manifested in their drawings and identifications (see (2) "range of the concept") as well as (c) what kind of examples they choose as circles, triangles, and squares (correlating to (2)), and (d) how their explanation and their choice of figures as well as their drawings go in line with each other (consistency between (1) and (2)). For this reason, a brief background to these four aspects will be given.

14.2.1 Explaining Shapes

The knowledge children have about the content of a category of shapes is shown in their explanations, i.e., in the way they describe a geometric shape. Here, it is not yet important that they know the perfect definition but that they are aware of the single critical attributes, whether these are described formally, informally, or via gestures. Often, the children are able to describe (give a definition of) a shape correctly but are not able to link this explanation to respective examples (Levenson et al., 2011). Consequently, a definition (that is learnt by heart) does not indicate a concept knowledge if it is not connected to the respective examples of shapes.

14.2.2 Drawing Shapes

The drawings of children reveal what kind of image children have in mind and shows how far they "captured the range of the concept." Still, in order to draw an object correctly, it demands the *knowledge* as well as the *ability* to put this knowledge into praxis, the so-called drawing skills. If these are still undeveloped, a child is not able to draw a geometric shape even when he or she knows what it looks like. Piaget interpreted such a situation to be due to a lack of knowledge. He was not considering a lack of drawing skills if a child was not able to copy or draw a certain shape, which is one reason why his results were criticized (Battista, 2007; Freudenthal, 1983). Because drawing a shape correctly demands knowledge and drawing skills, Kläger (1990) highlights the importance to never regard drawings of children in isolation but to always complement these drawings with interviews. This was considered and conducted in a task by Burger and Shaughnessy (1986), who asked children to draw many different triangles. They found that younger children often vary their drawings by ending up with "new inventions," as for example a triangle with "zic-zac-sides"; older children vary their drawings more according to the nature of triangles (equilateral, isosceles, rectangular, or general triangles).

To summarize the findings concerning children's drawings, it can be stated that children cannot be "generalized," they draw what they see and know but also more and less than they see and know (Kläger, 1990, p. 15f.).

14.2.3 Identifying Shapes

The ability to distinguish between examples and non-examples of a certain category of shapes also illustrates whether children "captured the range of a concept," i.e., which representatives belong to a certain category and which do not, and also which attributes of a shape children consider as being crucial in order to be described as a certain shape. The *identifying shape task* was originally conducted by Razel and Eylon (1990), applied by Clements, Swaminathan, Hannibal, and Sarama (1999) and with minor changes adapted for this study. Here, the children were shown different arrangements of shapes (see Fig. 14.1), where they were asked to mark all the circles (left), all the squares (middle), all the triangles (right), and all the rectangles (not presented here). In both studies, circles and squares were more often identified correctly in comparison to triangles and rectangles (Clements et al., 1999; Razel & Eylon, 1990).

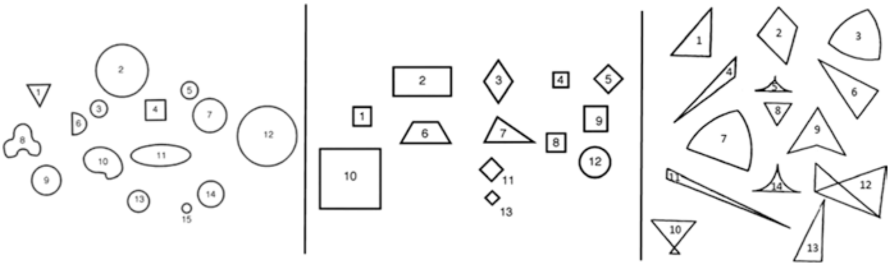


Fig. 14.1 Tasks of "Identifying shapes" (Razel & Eylon, 1990, in Clements et al., 1999, p. 211)

14.2.4 Coherence

Coherence between single aspects of a comprehensive concept formation is important to mention because one can only talk about a comprehensive concept knowledge if all aspects are fulfilled. This means that a child who is able to explain a shape correctly but is not able to show all kinds of examples does not have a comprehensive understanding of the concept yet. In this case the concept knowledge is inconsistent (Wilson, 1990, p. 31 f.). Wilson assumed in her research (ibid.) that children use definitions without any connection to examples. Other researches explain this as a natural developmental process that children may have high competencies at a certain task and lower achievements in another task (Bauersfeld, 1983; Senk, 1989; Siegler, 1996).

In order to investigate the role of instruction in geometric concept formation, this study investigates children of the same age but from two different learning environments. One from an English primary school, where the education is rather systematic and curriculum-based, and the expected competencies are described as "stepping stones." The other from a German kindergarten, where learning through play and an approach using "everyday mathematics" is at present the main concept. Thus, two different ways of fostering children's concept formation are at the basis for the interviews. The study complements previous studies that were focusing mainly on one aspect of concept formation.

14.3 Research Question

In this chapter the following research questions will be answered:

- Which tendencies in explaining, drawing, and identifying shapes are visible regarding the results of the German and the English children? What kind of images do they have in mind concerning certain geometric shapes (in this chapter focusing on triangles and squares)?

- In how far do these images (illustrated through their drawings as well as their identifications of triangles) match the children's definitions (shown in their explanations) of these shapes?
- Can these results or tendencies be explained by regarding the different ways of instruction (very instructive in the England preschool vs. rather constructive in the German kindergarten) in the two countries?

14.4 Design of the Study

The study comprises 77 children, 34 from England, and 43 from Germany aged between 4 and 6, who were interviewed at the beginning and the end of one school year. The children from England go to a local primary school (for children from 4 to 11 years old), and German children of the same age attend a kindergarten for children from the age of 3 to 6 (up to primary school). The elementary education in England is rather systematic and curriculum-based and the expected competencies are described as "stepping stones," whereas in Germany, learning through play and an approach using "natural learning situations" (Gasteiger, 2015, p. 258) is at present the main concept in kindergarten education.

The study was conducted in the form of qualitative interviews, taking about 30 min for each child. The order of the tasks—there were five tasks altogether, of which three are presented in this chapter—as well as the material was predetermined, but in accordance with the nature of qualitative interviews this order could be altered or complemented. There were two points of data collection (S1 and S2), one at the beginning of the school year or kindergarten year and the other at the end, without a planned intervention. Still, the different learning environments add their own natural interventions; the English children, in contrast to the German children, were instructed in geometry during the year.

14.4.1 Tasks

In this chapter, the results of some aspects of the *explaining triangles task*, the *drawing triangles task*, and the *identifying shapes task* will be illustrated, thus giving insights in the concept formation of the children. First, the children were asked to "explain a triangle to someone who has never seen a triangle before." In another task, they were asked to draw a triangle, then a triangle that looks a bit different from the first one and again another one looking different than the first two and so on. The children were asked to draw at least three triangles but were often asked to draw more (up to seven at the most) depending on whether their way of variation could be seen or not.

This study complements the original study by Burger and Shaughnessy (1986) by examining whether children also prefer *drawing* prototypical triangles and whether there is a correlation between the children's explanations and their drawings of a triangle. In the third task presented here, the children were asked to identify, in a collection of different shapes, the ones belonging to a certain category of shapes (cf. Fig. 14.1). Here, mainly the results of the “identifying triangles” task will be presented and also some tendencies concerning circles and squares.

14.4.2 Analysis

For the analysis of the children’s explanations, drawings, and identifications, different categories were generated and discussed. These categories were first formulated after a pretest ($n = 10$) and a thorough examination of the theory and then complemented and altered after the two points of investigation (S1 and S2) were conducted. Besides the interpretation of the qualitative data as small case studies, quantified details will also be given to show tendencies and to suggest hypotheses because quantitative details can be one aspect of qualitative reality (Oswald, 2010, p. 186). Moreover, significances (Wilcoxon test, Mann-Whitney U test as well as chi-square test) were calculated in order to highlight different tendencies between the English and the German children, as well as between the two points of investigation.

14.5 Results

14.5.1 Explaining Shapes

To answer the research questions, there will be an illustration of how the children explained a triangle when being asked to describe the shape of a triangle to someone who has never seen a triangle before. Here, five categories could be generated (cf. Table 14.1). As illustrated in the table, the majority of the English children used a formal definition (“a triangle has three straight sides and three corners”) at both points of investigation in order to explain the shape of a triangle, significantly more than the German children.

Table 14.1 Explaining the shape of a triangle

	(1) no explanation		(2) gestures		(3) comparisons		(4) informal		(5) formal	
	G	E	G	E	G	E	G	E	G	E
S1	30% (13)	15% (5)	21% (9)	6% (2)	9% (4)	6% (2)	30% (13)	9% (3)	16% (7)	59% (20)
S2	23% (10)	-	7% (3)	3% (1)	21% (9)	15% (5)	49% (21)	21% (7)	14% (6)	62% (21)

Shaded fields: significant differences between the two countries
Crosshatched fields: significant differences between the two points of investigation

The majority of the German children used “informal ways” (4) of explaining at both points of investigation (cf. Table 14.1). With the term “informal,” the explanations of the children were described who on the one hand knew the attributes of a shape but on the other hand lacked the formal descriptions and used their own words to explain a shape, as it is seen in the example of Lucca (cf. transcript Lucca below). He used “comparisons” (to everyday objects) (3) to explain what a triangle looks like, as for example: “a triangle looks like a roof,” “like a tent,” or “like the hat of a witch.”

Lucca (6;1 years, S2).

Lucca:	It looks like a roof. [comparison]
	It is straight at the bottom and at the top it is not straight, it’s sloped, so that it becomes a spike. [informal]
Interviewer:	What would you say is different to a circle?
Lucca:	The circle, it is, well, it has not corners. It is like a star without spikes [comparison].

The example of Lucca also shows that the answers of the children could often be grouped into different categories at the same time, thus multiple answers were possible. Therefore, the overall percentages could be more than 100% (cf. Table 14.1). At this task, it was very interesting that the majority of the English children were all using the same explanation: “a triangle has three straight sides and three corners.”

14.5.2 Drawing Triangles

In another task, the children were asked to draw a triangle, then to draw another one that looks different and another one being different to the first two and so on. Looking at the children’s drawings of triangles, it is evident that the majority of the children varied the triangles according to their size, i.e., “area dimension” (cf. Table 14.2). Because the children’s variations could be grouped into several categories, multiple answers were possible. Altogether, variations according to size were often combined with angle variations (“angular dimension”), but rather rare in combination to variations according to positions (triangles pointing in different directions). This indicates that the prototypical representation of an upright triangle is more familiar to the children than triangles pointing in different directions as for example an “upside-down-triangle” from the child’s perspective.

The category “shape,” as well as “missing attributes,” was considered as “not correct variations” because when varying a triangle according to its shape (e.g., changing the nature of the sides or drawing additional corners), it is not a triangle anymore (cf. Figs. 14.2 and 14.4). Additionally, when critical attributes as *three* sides, *straight* sides or *three* corners (cf. Fig. 14.3 (Lilly)) are missing, the shape cannot be described as triangle anymore. Jannis (Fig. 14.2) draws a triangle in “steps” in order to make it look different, whereas the triangles of Johannes (Fig. 14.4) “get feet” throughout the drawing task.

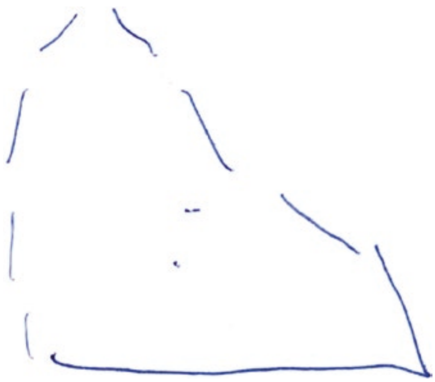
Table 14.2 Drawing triangles in different variations

	area dimension		angular dimension		position		everyday objects	
	G	E	G	E	G	E	G	E
S1	65% (28)	71% (24)	35% (15)	35% (12)	12% (5)	24% (8)	-	-
S2	84% (36)	97% (33)	58% (25)	65% (22)	28% (12)	26% (9)	2% (1)	3% (1)

	identity		shape		missing attributes	
	G	E	G	E	G	E
S1	5% (2)	9% (3)	21% (9)	32% (11)	9% (4)	15% (5)
S2	5% (2)	12% (4)	26% (11)	18% (6)	9% (4)	-

Crosshatched fields: significant differences between the two points of investigation

Fig. 14.2 Jannis (S2)



The illustrated tendencies and images give a brief impression of the results at this task. Still, a closer look should be given to the coherence of the children's explanations as well as their drawings of a triangle. Although the children could already explain a triangle quite well, there were quite a few children drawing non-triangles as variations of triangles and describing them as proper triangles. Especially as second or third triangle, the children often drew triangle-like shapes but for example with "rocky," "wavy," or "step-like" sides or additional "feet" (cf. Fig. 14.4).

Fig. 14.3 Lilly (S1)

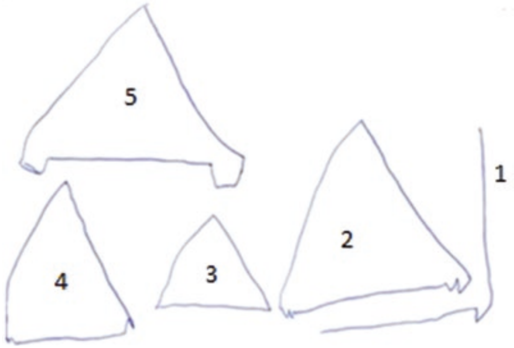


Fig. 14.4 Johannes (S2)

14.5.3 Coherence Between the Children's Explanations and Drawings

The following tables illustrate to what degree the drawings and the explanations of the children correlate (cf. Tables 14.3 and 14.4). Combined variations were summarized in the category "mixed" variations.

Children, who explained the shape of a triangle formally (mainly the English children) as well as the children who explained a triangle in informal ways but correctly (mainly the German children), varied the triangles at both points of investigation in most cases according to size, i.e., area dimension. One could also infer that the children, especially the English children, could explain a triangle with nearly perfect definitions, but still chose mainly triangles varying only in size and triangles that were pointing upwards. Thus, they did not apply the definition to all kinds of triangles. There were less children varying the triangles according to the position (different directions), but these children mainly explained in a formal way. The children who drew wrong variations (either according to shape or missing attributes) also explained mainly in a formal way. Some explained informally (and correctly) but still varied the shape of a triangle or did not draw some critical attributes (such as *straight* sides or *closed* figure or only *three* sides).

At the second point of investigation an overall increase in correct variations is visible and twice as many German children than before drew triangles in different variations ("mixed"). The number of wrong variations, especially in the case of the English children, decreased, so at the second point of investigation there is a higher correlation of correct explanations and correct variations than before.

Table 14.3 Coherence drawings and explanations (S1)

S1		explaining									
		no explanation		gestures		comparison		informal		formal	
		G	E	G	E	G	E	G	E	G	E
drawing	area	12% (5)	-	7% (3)	-	7% (3)	-	23% (10)	6% (2)	12% (5)	50% (17)
	angular	9% (4)	-	2% (1)	-	5% (2)	-	16% (7)	-	5% (2)	29% (10)
	position	5% (2)	-	-	-	-	-	2% (1)	-	-	15% (5)
	mixed	12% (5)	-	2% (1)	-	5% (2)	-	14% (6)	-	2% (1)	29% (10)
	shape	5% (2)	-	2% (1)	-	2% (1)	3% (1)	7% (3)	6% (2)	5% (2)	15% (5)
	missing attribute	-	-	-	-	2% (1)	3% (1)	5% (2)	3% (1)	-	12% (4)

Table 14.4 Coherence drawings and explanations (S2)

S2		explaining									
		no explanation		gestures		comparison		informal		formal	
		G	E	G	E	G	E	G	E	G	E
drawing	area	14% (6)	-	7% (3)	3% (1)	21% (9)	15% (5)	47% (20)	21% (7)	14% (6)	59% (20)
	angular	14% (6)	-	2% (1)	-	16% (7)	12% (4)	28% (12)	15% (5)	12% (5)	38% (13)
	position	7% (3)	-	-	-	5% (2)	6% (2)	16% (7)	-	5% (2)	21% (7)
	mixed	9% (4)	-	2% (1)	-	16% (7)	9% (3)	28% (12)	12% (4)	12% (5)	35% (12)
	shape	9% (4)	-	-	-	2% (1)	3% (1)	9% (4)	3% (1)	-	9% (3)
	missing attribute	2% (1)	-	-	-	2% (1)	-	2% (1)	-	-	-

14.5.4 Identifying Circles, Squares, and Triangles

In order to show what kind of images the children have in mind, when thinking about certain shapes, one task needs to be illustrated here as well: the "identifying shapes task." Here, the children had to distinguish between examples and non-examples of each class of shapes (in this case circles, squares, and triangles, cf. Fig. 14.1). Firstly, certain aspects of this task will be illustrated, before showing to what extent the children's explanations of a triangle matched their actual choices of different representations of a shape. Starting with the *identifying circles task*, the majority of the children in both countries marked all the circles and no other shapes. Only a few (about 10%) additionally chose the ellipse as circle. The choice of circles did not seem to be difficult for the children because for them a circle is the most familiar shape and can deviate only in its size.

Regarding the children's choices of squares, it was first examined whether the children chose (1) only all the squares, (2) all the squares and other figures, (3) not all the squares but no additional other figures, or (4) not all the squares and other figures. At the beginning of the school year (S1), about the same proportion of children (G: 47%; E: 44%) only marked all the squares as examples. Still, more than half of the English children (53%) marked "only squares but not all the squares." At the end of the school year (S2), there were significantly less English children who chose only all the squares (21%). Instead, the majority of the English children again marked "only squares but not all the squares" (68%). So, at the end of the school year, there were more children who only chose "squares but not all of them." Taking a closer look at which of the squares that were considered as "not being squares," it becomes obvious that the squares not lying in horizontal position were often not considered as being squares, but "diamonds" (rhombuses) instead (cf. Fig. 14.5):

"Turned squares" (nos 5, 11, and 13) were often described as "diamonds" by the children, so the choice of a square was depending on the position of the square.

Lucy (5;3 years, S2):

Lucy:	Now I've got all the squares.
Interviewer:	Are there any more squares?
	(Lucy points to square no. 5, cf. Fig. 14.5)
Lucy:	That is a likely square but it's a diamond.
Interviewer:	Why is it a diamond and not a square?
Lucy:	Because it goes that way (draws the shape with her fingers on the table) and not that way (draws a horizontal lying square).

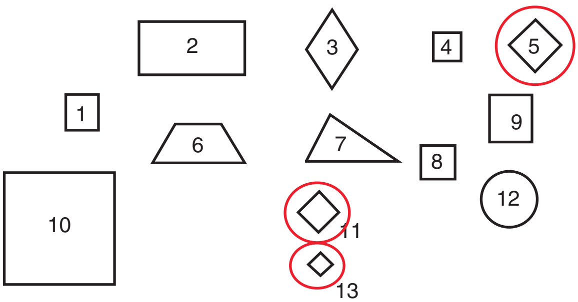


Fig. 14.5 Identifying squares—non-chosen squares are highlighted

Most of the English children who did not choose all the squares explained that "a turned square is a diamond" or "if you turn a square it turns into a diamond" (Georgia, 6;1,[1] S2). In a few cases, as in the example of James, the rhombus (cf. no. 3, Fig. 14.5) was marked as a square too, although it was realized that the "diamond" (as he described the rhombus) looks a bit different than the squares, describing it as "a bit bent." Most of the few English children who chose "all the squares and other shapes" chose the rhombus as additional square. There were some children, mainly German children, who marked all the squares and other shapes. Here, mainly the rectangle (no. 2, cf. Fig. 14.5) was chosen as additional square, but the rhombus (no. 3, ibid.) was also chosen quite frequently.

Even more difficult than *identifying squares* was the task to identify all the triangles correctly. This could also be due to the illustration (cf. Fig. 14.1, right) because now non-examples that are very similar to equilateral triangles are included, just with convex (nos 3 and 7, ibid.) and concave sides (nos 5 and 14, ibid.) or an additional concave corner (no. 9, ibid.). Here, hardly any of the children chose just all the triangles: only one German child at the beginning of the school year (S1) and two English children at the end of the school year (S2). The choices of the children are illustrated in Table 14.5, starting with the most frequent choices and ending with the least chosen triangle (no. 4, ibid.).

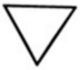
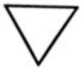
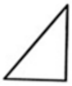
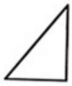
The majority of the children chose the equilateral triangle pointing downwards (no. 8, cf. Table 14.5), some even exclusively, as Tizian. Here, his explanations for not choosing some of the triangles ranged from "too thin" (nos 4 and 11, ibid.), "has a peak" (nos 6 and 11, ibid.), "one side is too long" (no. 1, ibid.). The children who did not choose this triangle (no. 8, ibid.) argued that "it is upside down" or turned the paper, arguing that in this position it would be a triangle. In the case of the triangles (nos 4 and 11, ibid.) which were chosen the least often, the children often reasoned that these are pointing in a "wrong direction," "pointing downwards" or were "too thin" and thus could not be triangles. As in the *identifying squares* task, some children here also considered the position of a shape when judging whether the respective shape was an example or a non-example. Children of both countries who did not choose the right-angled triangles as triangles often argued that these were "too straight," "too long at one side," or "are lacking one corner."

The non-examples are also presented according to their frequency of choice (cf. Table 14.6).

Here, the most frequently chosen non-examples can be compared to the least often chosen examples (nos 11 and 4, cf. Table 14.5). The most chosen non-examples resemble equilateral or at least isosceles triangles in an upward position, which is illustrated through the red triangles in the table. Many of the non-examples (cf. Table 14.6) were chosen with the same frequency as, or in some cases even more than, the two scalene triangles. Although the children often emphasized that a triangle must have "three straight sides," non-examples with curved sides or an additional corner were chosen as well. Regarding these tendencies, it becomes obvious

[1] 6;1 represents the age of 6 years and 1 month

Table 14.5 Identifying triangles—examples

	S1				S2		
No.	Examples	G	E	No.	Examples	G	E
8		81% (35)	82% (28)	8		91% (39)	85% (29)
1		74% (32)	88% (30)	1		79% (34)	88% (30)
6		63% (27)	97% (33)	6		67% (29)	88% (30)
13		67% (29)	85% (29)	13		65% (28)	88% (30)
11		42% (18)	50% (17)	11		40% (17)	53% (18)
4		44% (19)	47% (16)	4		33% (14)	44% (15)

that the chosen triangles resemble a certain image the children had in mind. This could either be "triangles in an upward *position*," "triangles where the *sides* are about the same length," or triangles that are not "too pointy" or "too scalene" (considering *angles*).

When the image of "triangles in an upward position" is dominant, triangle no. 8 (cf. Fig. 14.1, right) is not chosen. If the arrangement of the sides is regarded, only the equilateral (no. 8, ibid.) or the isosceles triangle (no. 6, ibid.) is chosen. Furthermore, if the angular dimension is considered, the scalene triangles (nos 4 and 11, ibid.) are not described as triangles. Depending on the predominant image of a triangle (e.g., equilateral triangles), non-examples that are similar to examples of triangles (cf. marked figures, Table 14.6) are chosen. These images that influenced the choices of the children are "ideal examples" or "prototypical examples" that could be triangles pointing upwards, equilateral, or isosceles triangles. Additionally, some attributes are not connected with a triangle as for example very "scalene" or "too skinny" triangles or triangles that are "too straight" (often right-angled triangles). For the children, such images are more influential when it comes to choosing examples than definitions, even when the child knows the formal definition (Vinner, 1991).

Table 14.6 Identifying triangles—non-examples

	S1				S2		
No.		G	E	No.		G	E
	Non-examples						
9		65% (28)	47% (16)	14		44% (19)	50% (17)
14		53% (23)	50% (17)	5		44% (19)	50% (17)
5		47% (20)	35% (12)	9		37% (16)	38% (13)
3		35% (15)	15% (5)	7		35% (15)	21% (7)
7		37% (16)	12% (4)	3		35% (15)	21% (7)
2		5% (2)	-	2		5% (2)	-

14.5.5 *Coherence Between Children's Explanations and Identifications*

The following cross tabulation (Table 14.7) illustrates the coherence of the children's explanations (either no explanation or two ways of correct explanations) and their identifications. A high number of the English children chose non-triangles as triangles although they were able to explain a triangle correctly in a formal way. Although the German children did not often explain in a formal way, there is a coherence visible between other correct explanations (i.e., through gestures, comparisons, or informal explanations) and triangle-like shapes in the results of the German children. The quadrangle no. 9 was chosen by a quarter of the English children who also explained a triangle formally. Here, only the three acute angles were considered, not the obtuse angle. Correct examples were recognized by most of the formally explaining children. Still, there were more English children who explained in a formal way (59% at S1 or 62% at S2 cf. Table 14.1) but did not identify upside-down triangles, rectangular triangles, or the two general but very scalene

Table 14.7 Coherence explaining and identifying triangles

S1	explaining: no explanation G	no explanation E	not formal (gestures, comparisons, informal) G	not formal E	formal G	formal E
identifying: non-examples without straight sides (no. 14; 5; 3; 7)	19% (8)	6% (2)	44% (19)	12% (4)	7% (3)	38% (13)
triangle-like quadrangle (no. 9)	23% (10)	9% (3)	40% (17)	12% (4)	9% (4)	26% (9)
triangle no. 8	26% (11)	12% (4)	49% (21)	18% (6)	14% (6)	50% (17)
triangle no. 1 and/ or no. 13	23% (10)	15% (5)	47% (20)	21% (7)	14% (6)	56% (19)
triangle no. 11 and/ or no. 4	14% (6)	3% (1)	26% (11)	15% (5)	7% (3)	32% (11)

S2	explaining: no explanation G	no explanation E	not formal (gestures, comparisons, informal) G	not formal E	formal G	formal E
identifying: non-examples without straight sides (no. 14; 5; 3; 7)	12% (5)	-	44% (19)	27% (9)	7% (3)	29% (10)
triangle-like quadrangle (no. 9)	7% (3)	-	26% (11)	18% (6)	7% (3)	18% (6)
triangle no. 8	19% (8)	-	72% (31)	32% (11)	14% (6)	50% (17)
triangle no. 1 and/ or no. 13	16% (7)	-	60% (26)	35% (12)	12% (5)	59% (20)
triangle no. 11 and/ or no. 4	9% (4)	-	30% (13)	15% (5)	12% (5)	35% (12)

triangles. The same applies for the German children who mostly did not explain formally but in other ways correctly. Still, their correct explanations (through gestures, comparisons, and informal explanations) could not always be applied to all triangles or respectively was applied for non-triangles as is shown in the high number of children choosing all kinds of non-triangles although explaining correctly. Thus, it is evident that the high percentage of formal definitions of the children was not in line with their selection of shapes. About two-thirds of the English children at both points of investigation gave a formal definition, whereas no English child at the beginning and only 6% at the end of the school year identified all the triangles and only triangles. Thus, in summary, the children could not link the definition they had in mind to certain examples of triangles.

14.6 Summary and Discussion

To summarize the results, the explanations of the children (especially in the case of the English children, among whom a large proportion already gave formal explanations) did not always match their drawings and identifications. The children's explanations partly matched their drawings for they drew mainly correct triangles or correct variations of triangles, respectively, but these drawings were rather one-sided: variations according to position (orientation) were drawn rather seldom and the number of wrong variations (i.e., variations concerning the shape of the figure or drawing figures missing critical attributes) was quite high.

Regarding the explanations of the children, although the majority of the children explained a triangle correctly, this “definition” could in most cases not been applied to different representations of a certain shape category. Especially in the case of the English children, the good explanation of a triangle (“three corners and three straight sides”) was in most cases not considered when drawing different kinds of triangles as well as when identifying different kinds of triangles. The children’s images or perceptions were more influential when they were judging a shape as belonging to a certain category than the definitions they had in mind. These images were often “ideal examples” or “prototypical examples” such as “triangles pointing upwards” and equilateral or isosceles triangles. Consequently, the children regarded noncritical attributes such as position, sides, or angular dimension as being decisive attributes. Moreover, the children often neglected critical attributes of a shape such as *straight* sides or *three* corners when the non-example resembled the prototype of a triangle. It was also evident in the drawings of the children that they preferred drawing triangles in an upright position, just varying in size.

Concerning squares, the children’s predominant images were “horizontal lying squares,” especially in the case of the English children, who chose significantly more often only squares in horizontal position. Children, who tended to also choose 45° turned squares, neglected right angles, if a non-example was very similar to a prototypical representation. The children of both countries had no difficulties to name, choose, and distinguish circles from non-examples. This was also the case in previous studies, where shapes were much more likely to be chosen the more they resembled a certain prototype (Clements et al., 1999; Tsamir, Tirosh, & Levenson, 2008). Furthermore, untypical examples were significantly less often correctly chosen as prototypes (Unterhauser & Gasteiger, 2018). With this, the main tendencies at these tasks as well as the coherence of the children’s explanations and the images they had in mind are illustrated and the first two research questions are answered.

The remaining question is whether this tendency to choose prototypical examples (and also prototypical-like non-examples) is a natural step in development (cf. Franke & Reinhold, 2016) or is due to a one-sided or “limited” teaching in school. One possible way to explain these tendencies and thus answering the research question is by regarding the different learning environments of the children. The English children chose less squares (mainly horizontal lying squares) at the end of the school year compared to the beginning. This leads to the assumption that this tendency is not due to a natural developmental process but due to the instruction in school. Regarding the material and the examples that are used in the primary school where the study took place, only equilateral triangles and only squares in horizontal position were shown and the illustrated 45° turned shapes were only rhombuses. Clements (2004) also speaks of a too limited use of prototypical examples of shapes in school. In this case it could be carefully stated that educational instruction that is too limited regarding examples, and that only aims on developing definitions that are learned by heart, could lead to a limited concept formation among children. With respect to the German children, the methods and materials of instruction cannot be taken into account because the German children of the study were not yet attending primary school. To summarize, it is evident that the learning environment is influencing the children's geometric concept formation.

14.7 Conclusion

The results of the study have shown that the children often could not apply a comprehensive definition of a shape to many different representations of that respective shape. Thus, it can be concluded that a definition that is learned by heart without an understanding of what this definition means does not contribute to a comprehensive concept formation. Therefore, one can suggest that instead of an isolated memorizing of definitions and the limited use of only prototypical representations, which can dominate children's thinking throughout their lives (Sarama & Clements, 2009, p. 216), the focus should be more on the ability to connect a concept with many different representations of that concept, even as early as preschool level.

So as to enhance a comprehensive formation of concepts, the children should have the opportunity to encounter many different examples of a shape category to explore the differences and similarities of the examples. Moreover, non-examples (e.g., lacking one decisive attribute such as straight sides or having "wavy" or "rocky" sides) could be included and accompanied by explanations differentiated from examples of a shape category. Thus, critical attributes could be mentioned and explored without learning isolated definitions. Moreover, children's own productions such as drawings or constructions of shapes could also foster the formation of concepts, and together with examples and non-examples, be a step toward building a valid concept formation in early childhood.

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Chapter 15
Framing Mathematics Teaching with Narratives: The Ambiguity of Goldilocks

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15.1 Introduction

Young children's interest in and ability to learn mathematics is rarely questioned today. Nevertheless, in both research and early childhood education, there is a diversity regarding what constitutes appropriate content and how early mathematics education should be designed (Palmér & Björklund, 2016). The debate often comes down to whether a specific method or material is suitable or not, rather than scrutinizing the learning potential or challenges that different methods and materials offer. The lacuna in the current debate is, therefore, how teaching may be designed to offer children profitable learning opportunities. In this chapter, we take our point of departure from a traditionally well-used method for teaching mathematics to young children—narratives. In the Nordic countries, preschool education is heavily influenced by creativity, play, fantasy and—not least—stories, where our aim with this chapter is to contribute with scientifically grounded knowledge to the research and teaching of early childhood mathematics.

In the Nordic countries, play-oriented activities withholding mathematical content are often set as a counterpole to systematic teaching of mathematical learning objects (Palmér & Björklund, 2016). This debate is similar to the dichotomy that Bennett (2005) describes between academic and social-pedagogical teaching approaches. This dichotomy becomes critical in relation to preschool mathematics since mathematics has been shown to be hard to implement as a natural component in children's play and, at the same time, hard to teach as meaningful content for young children (Björklund, 2014a; Björklund, Magnusson, & Palmér, 2018). To

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overcome these difficulties, material, drama and pictures are often mentioned as possible starting points for the teaching of early mathematics. Many preschool teachers use narratives, such as children's stories and fairy tales, in an attempt to make mathematics interesting and appealing to the children (Carlsen, 2013), and to frame the content in familiar contexts to support children's interpretation of meaning of what is intended for them to learn (Shiyan, Björklund, & Pramling Samuelsson, 2018).

The starting point for this chapter is a research project exploring play-responsive teaching in preschool (funded by the Swedish Institute for Educational Research, project no. 2016/112). One of the objectives in the project was operationalized as the preschool teachers enacted narrative play, and several of them chose to play the story of Goldilocks in their preschool groups. However, these explorations made visible several difficulties where the mathematics content in the story seemed to be hard to discern for the children. This experience, together with research showing contradictory results regarding whether narratives benefit or hinder mathematics learning, tells us that the question of using narratives in early childhood mathematics education includes a complexity that is not thoroughly investigated or known in preschools. In this chapter, we will therefore focus on both the benefits and challenges found in an analysis of the mathematical content in the story of Goldilocks. The specific research questions are:

- What mathematical concepts can be explored and framed within the story?
- What challenges for learning mathematical concepts does the story impose?

In the chapter, the use of the Goldilocks story is first illustrated by two empirical examples from the above-mentioned research project. After that follows a content analysis of the story itself, with the aim of answering the research questions.

15.2 Narratives as Pedagogical Tools

A narrative used as a pedagogical tool may be based on a known story, fairy tale, personal experiences or stories made up in the moment, where "narrative" implies that there is a logical frame that intertwines and relates phenomena and features. The narrative creates stringent episodes of meaning, which may, as suggested by Burton (2002), be important for learning abstract concepts and discerning internal relationships as a coherent whole. In preschool, mathematics education is preferably focused on mathematical content situated in familiar and meaningful settings for children. Deliberate use of communicative strategies in teaching mathematics to young children has shown positive long-term effects (Doverborg & Pramling Samuelsson, 2011; Pruden, Levine, & Huttenlocher, 2011). In such ways of teaching, the teacher makes use of and, together with the children, explores the meaning of mathematical concepts. The use of stories is thus one example of how mathematical content can be communicated, framed within a familiar setting, where children's interest can be directed towards mathematical concepts that are naturally occurring in the narrative framework (van Oers, 1996). Narratives are in this sense a natural, culturally and contextually situated, suitable way to introduce mathematical concepts to children. The beneficial aspects of narratives are usually centred around the opportunity they offer to intertwine features of both play and teaching, since the content is framed in familiar contexts and thus has relevance for the children and the problem at hand (Flevares & Schiff, 2014).

Studies exploring the use of picture books when teaching mathematics (van den Heuvel-Panhuizen & van den Boogaard, 2008; van den Heuvel-Panhuizen, van den Boogaard, & Doig, 2009) and studies comparing the use of picture books with other ways of teaching mathematics in preschool have yielded positive results (Hassinger-Das, Jordan, & Dyson, 2015; Jennings, Jennings, Richey, & Dixon-Krauss, 1992). Studies by Hassinger-Das et al. (2015) have highlighted that children who are taught mathematical concepts through reading picture books may develop conceptual knowledge and a flexible use of mathematical concepts. When investigating conceptual understanding, the children in the studies who were taught mathematics by narratives outperformed the children who were taught mathematics by task-based books. However, narratives may include many different features, which can draw attention away from the mathematical content towards other issues, explaining why the mathematical content may not be experienced as the primary object of learning by children (Pramling & Pramling Samuelsson, 2008). Furthermore, decoding the meaning of mathematical concepts involved in a narrative is a complex undertaking and relates to the child's ability to make use of cultural tools to structure their thoughts (Shiyan et al., 2018).

In preschool, teachers often tell narratives they know in a physical way, using facial expressions, voice, gestures and props to engage the children and to illustrate and emphasize the mathematical content. In a study by Carlsen (2013), it was made clear how a preschool teacher, while telling the story of Goldilocks, used her face, body and voice as well as questions and paraphrasing to emphasize the comparison words in the narrative (quantities, size, temperature and softness). Also, narratives are commonly used together with props in order to illustrate or reify components of the story or the mathematical content.

Thus, even though narratives are not exclusively used for teaching mathematics in preschool, there nevertheless seems to be many benefits. However, although the use of material and pictures can visualize mathematical content in narratives, they can also make it difficult for children to discern the intended mathematical objects and processes if they draw attention away from the mathematics towards other features (Björklund, 2014b; Dowker, 2005; Rathé, Torbeyns, Hannula-Sormunen, & Verschaffel, 2016). Furthermore, visualizing mathematical concepts by the use of manipulatives only facilitates understanding to a limited extent because of the abstract nature of mathematical objects (Lakoff & Núñez, 2000). Thus, there seems to be reason to more thoroughly investigate the pedagogical content and form in narratives for teaching mathematics in early childhood.

15.3 Methodology

As mentioned, the empirical examples in the chapter are from a research project on play-responsive teaching in preschool. In this study, preschool teachers made video documentations of play activities in their preschools, where one theme was narrative play. Several of the teachers chose to play the story of Goldilocks, of which two examples are used here as illustrations. "Goldilocks and the three bears" is a traditional story for children, originating from England and folklore stories (Tatar, 2002; see Appendix). The story is about a girl and a family of bears: Papa bear, Mama bear and Baby bear. The story is commonly used with props where teachers often choose bears in three different sizes to represent Papa bear, Mama bear and Baby bear, and the utensils and furniture that Goldilocks finds in the story usually correspond with the size of the bears.

The focus of this chapter is on the mathematical content in the story and in particular how the content may be experienced and thus facilitate learning in the specific setting of the narrative. To gain better knowledge of what aspects of mathematical concepts may be discerned in the story and thus what pedagogical potential it may have, we do a content analysis. We use Variation theory of learning (Marton, 2015) to analyse the mathematical content of the story. This theoretical framework makes visible what mathematical learning objects (concepts appearing in the narrative) are brought to the fore, and more specifically what challenges for learning the narrative entails. In this analysis, we focus on how different aspects of a mathematical concept are expressed and reified in the story, and thus direct our attention to the dimensions of variation that are possibly opened up for exploration. *Dimensions of variation* means the aspects of a mathematical concept that, through patterns of contrasting features, are made possible for the children to experience. Broadening the understanding of a mathematical idea is, according to Variation theory of learning, possible if a mathematical concept is contrasted; for example, the meaning of three will only be discernible (as an earlier unknown concept) if three is contrasted with a set of four or other sets while the objects constituting the sets are kept invariant. To generalize the idea of numbers is thereafter possible if the quantity of a set is kept invariant while other irrelevant aspects, such as colour, shape or other non-numerical features, are allowed to vary (Marton, 2015). Thus, the analysis is focused on potentials for learning; the mathematical concepts that may become possible to learn through contrasting and generalizing necessary aspects. This analytical approach reveals why some expressions in the story may entail challenges for young children to whom the concepts are novel.

Before presenting a content analysis of the story, two empirical examples of children playing Goldilocks with their teacher are presented. The aim of this is to illustrate the complexity that framing mathematics teaching within the story may entail. According to the theoretical framework of Variation theory of learning, children's concept development can be understood in terms of what aspects of a concept they are able to discern in a specific situation (Marton, 2015). Learning means a more differentiated way of seeing a phenomenon, and teaching means to offer the learner experiences that will enable them to differentiate necessary aspects of the phenomenon that they were not able to see before (Pramling Samuelsson & Pramling, 2013).

15.4 Two Empirical Examples

Two empirical examples are used to illustrate the complexity that framing the mathematics teaching within the story of Goldilocks may entail. The aim is not to analyse children's learning but to illustrate how the use of the narrative makes it possible or not for the children to differentiate necessary aspects of mathematical concepts that they were not able to see before.

15.4.1 Example 1

One teacher and four children aged 1–3 years are seated on the floor around a green piece of clothing illustrating the home of the Bear family. First the teacher picks up the props, three bears in three different sizes representing Papa bear, Mama bear and Baby bear. There are also bowls, chairs and beds in three different sizes.

Before the teacher starts to tell the story, they talk about which chair and which bowl belongs to which bear. It is obvious that the children are familiar with the story. The teacher starts to tell the story and at the same time she moves the bears. The episode below starts when Mama bear has made porridge.

Teacher	And in the little bowl she put Baby bear's porridge.
Child	Yes
Teacher	And in the middle bowl she puts her porridge. And in the big bowl [still acting as narrative-teller but starts talking with a deep voice] she puts Papa bear's porridge. Quite soon the bears are out on their walk and Goldilocks enters their house and starts to taste the porridge.
Teacher	Oh, here is porridge. Oh! That porridge was too hot. She then tastes the middle one, Mama bear's porridge. Oh! It was too cold. And then she tasted Baby bear's porridge. And she ate it all. Then she sat down on Baby bear's footstool. And it broke. She sat down on Mama bear's. It was so uncomfortable. And then she sat on Papa bear's. … What did she do then?
Child	Lay down in that bed [pointing at the middle-sized bed].
Teacher	Should she lie in that bed? [pointing at the middle-sized bed] Should we try? [She is laying down the doll in the bed that suits perfectly in size] No, it was not comfortable.
Child	Yes it was.

In this example we can follow (parts of) the traditional story both through the verbal narrative and the props that are used to illustrate the story. However, there seems to be some different interpretations of Goldilocks' conclusions, based on the

Picture 15.1 Goldilocks is trying Mama bear's bed

children's responses when listening to and watching the narrative played out. One notion that is critical in the story is when Goldilocks finds the perfect chair or the perfect porridge temperature, that is, something that is "just right" ("lagom" in Swedish). The children direct their attention primarily to the visual props, which are contradictive in meaning in relation to the verbal story (see Picture 15.1); for instance, when a perfect-sized bed is said to be "not comfortable" but visually fits the Goldilocks figure just right.

15.4.2 Example 2

The second example is a drama play where three children aged 4–5 and one teacher are acting out the different characters in the story. In this example it is again obvious that the children are familiar with the story. The teacher is the storyteller and Papa bear and when the episode below starts, the child acting as Goldilocks sits in a dress at a table with three bowls in different sizes.

Teacher	At the kitchen table Goldilocks found three bowls of newly made porridge. She had walked far and was hungry so now she wanted to taste the porridge.
Child	[leans over the bowl, takes a spoon in her hand and pretends to eat]
Teacher	The largest bowl, but it was too hot.
Child	Yuck [walks towards the middle-sized bowl]
Teacher	Then she tasted the porridge in the middle-sized bowl.
Child	[pretends to taste the porridge in the middle-sized bowl] Yuck!
Teacher	But it was too cold.
Child	[moves towards the smallest bowl]
Teacher	Then she tasted the porridge in the smallest bowl and it was just right, so she ate it all.

In this episode we find that the child is receptive to the narrative as it is told verbally. She acts in accordance with the story and moves from the largest via the middle-sized to the small-sized bowl, which gets approved.

Teacher	Now she was tired so she decided to rest for a while.
Child	[lays on top of a large couch] Yuck [kicking around].
Teacher	First she lay in the really large bed but it was too hard.
Child	[gets up and walks towards two mattresses, throws herself down on them, gets up and puts them in order].
Teacher	And then she tried the middle-sized bed.
Child	[lies down and gets up again].
Teacher	But it was way too soft.
Child	[walks to a doll's cradle].
Teacher	Then she tried the smallest bed.
Child	[sets the blanket on the bed]
Teacher	And it was just right, it was so comfortable.

The child continues acting according to the verbal story, accepting the smallest bed as "just right" even though it clearly does not fit her in real life as she has to put her legs and feet over the foot of the bed (see Picture 15.2).

In both examples, the teachers use the concepts little, middle and large as they talk about the bowls and they sometimes emphasize a concept with their voices, bringing attention to it as a demarcated phenomenon. However, there is an ambiguity visible in the illogical relationship between the size of the bowls and the temperature of the porridge in both examples. In the first example, the teacher never says "just right" about the porridge in the small bowl so the series is not completed. Instead she directly continues with Baby bear's footstool, which gets broken. In this example it seems like the big chair, Papa bear's chair, is the suitable one—since Baby bear's footstool gets broken and Mama bear's was too uncomfortable, while nothing is said about Papa bear's chair. Similarly, one child says that Mama bear's bed is comfortable, which can be understood as the "just right" one. An ambiguity in the relationship between the sizes of the beds in relation to "just right" also becomes visible in the second example, where the reasonably soft bed is actually too small for the child acting as Goldilocks.

As the three beds are different in size it makes sense for the children to discern size instead of softness, since the latter is not visible to the eye. Furthermore, regardless of the chosen props, size was the concept to be focused on in relation to the three bowls and the three chairs just before, which is why it may have made sense

Picture 15.2 Goldilocks tries the smallest bed, which was "just right"

for the children to continue to focus on size as the narrative implicates a direction towards generalizing concepts. Thus, based on both examples, one can question what mathematical content is brought to the fore and thus may be picked up as potential learning objects. Together, the two examples illustrate some of the complexity in framing mathematics teaching within this story. This was the starting point for the content analysis of Goldilocks presented below.

15.5 Content Analysis of the Goldilocks Story

The first piece of mathematical content found in the story is the number three. The number range is already comprehensible to most children in their early years and possible to subitize as a set of items (Wynn, 1998). Sarnecka, Negen, and Goldman (2017) also show that children from 2 to 3 years of age develop their sense of number words' cardinality, which means they become aware of the precise numerical meaning of numbers. Three objects should thereby be a comprehensible quantity to explore relations within by preschoolers.

A second piece of content found to be central in the story is seriation. Series or sequences re-occur throughout the story in several ways: size, warmth and softness. The bears and items related to each bear are relational in size. The other concepts are also relational in that there are different values of warmth and softness expressed. Piaget (1952) defines the act of making series as a relational one, where every item has to be related and compared to every other. In his studies, 4-year-olds were shown to struggle with making series of more than three elements. However, in a more recent study of toddlers who were given abundant time to explore item relations, the children made series of up to six elements (Reis, 2011).

A third piece of content of a mathematical kind is the relation to Goldilocks herself, as a reference point of what "just right" may be. The value of the units in the series is thus related to a specific reference. This content relates to the series in the story, as a tool for comparing the units. In mathematical activities where measurement is used as point of departure, this is known to be an essential and necessary aspect for young learners (Schmittau, 2004). Children, 2–3-year-olds, also use reference points as intuitive strategies to compare and sort items (Öhberg, 2004).

Thus, all of these pieces of content in Goldilocks are known to be within the range that preschool children should be able to make sense of as learning objects. The following analysis will direct attention towards how it is made possible to experience this content and make meaning from it, based on how the pieces of content are presented in the story.

15.5.1 What Mathematical Challenges Does the Story Impose?

The story is divided into three sections. Each section is illustrated by Goldilocks moving from one scene to the other, referring to Papa bear, Mama bear or Baby bear accordingly, in different setups. Series or sequences of three units are the dominant learning object occurring in the story. Thus, reference points become central in that every object or value (unit) in a sequence has to be compared to some unit kept invariant, to make a series. In this story, the visual props entail simple and easily comprehensible series (such as series of size), but the idea of series is challenged through the reference point being Goldilocks' preferences of "just right". What is considered "just right" is not consistent throughout the story as a successive increase or decrease of some value. The idea of series constitutes that the relational aspect is discerned. This dimension of variation is opened up for exploration as the size of the utensils and furniture occurring in the story are visual props and ordered in accordance with size. However, the story offers an additional, different approach:

> She tasted the porridge from the first bowl [Papa bear's]. "This porridge is too hot!" she exclaimed. She tasted the porridge from the second bowl [Mama bear's]. "This porridge is too cold," she said. She tasted the last bowl of porridge [Baby bear's]. "Ahhh, this porridge is just right."

The porridge is in accordance with the re-occurring series Papa bear–Mama bear–Baby bear, tasted first from the largest bowl, then the middle-sized bowl and last from the smallest bowl. But the temperature of the porridge is first too hot, then too cold and finally just right. One would assume that "just right" would be a temperature between hot and cold, but in the story, this sequence of warmness does not follow the increasing or decreasing structure that the sizes of the utensils entail. Although the size varies and thus makes it possible to explore the relative nature of size and the effect this has on the occurring series, the values of the warmth would need to be explored more explicitly according to the relative meaning. The same ambiguous relation between size and other elements of order also occurs in the second section.

> "This chair is too big!" she exclaimed, sitting on the first chair [Papa bear's]. So she sat in the second chair [Mama bear's]. "This chair is too big, too!" she whined. So she tried the last chair [Baby bear's]. "Ahhh, this chair is just right," she sighed. But just as she settled down into the chair to rest, it broke into pieces!

Obviously, the last chair, which was the smallest one, was *not* right for Goldilocks, since she broke the chair while sitting in it. And once more, a visible series is offered for the children to perceive (size), but it is not related to the size of Goldilocks, but to her own judgement.

> She lay down in the first bed [Papa bear's], but it was too hard. Then she lay in the second bed [Mama bear's], but it was too soft. Then she lay down in the third bed [Baby bear's] and it was just right.

The last section is similar to the first one in that the order of the occurrences goes from one extreme (too hard) to the other extreme (too soft) and finding the perfect softness in the last attempt. This is also inconsistent with the props used, starting with the largest bed, then the middle-sized bed and finishing with the smallest one. As the three beds are different in size, it makes sense for the children to discern different values of size instead of softness, which is not visible to the eye.

15.6 Discussion

In this chapter we presented two empirical examples from authentic preschool practice where the children's story Goldilocks was used as a narrative for teaching. These examples show difficulties occurring when the story is played either with small figures or as a drama where children themselves act out the story. These two examples are not chosen to illustrate how Goldilocks should or should not be used in preschool, but to illustrate the complexity of using the narrative for teaching purposes. What is discerned by the child while testing three beds that differ in size, colour, material and softness? What impression of "just right" is formed when you are lying in a bed in which your feet do not fit? Even if the teacher emphasizes softness by altering her voice, the child may connect her wording to other possible mathematical (or other) content. In our study, the story Goldilocks was told in accordance with the original traditional setting, which turned out to limit the opportunities to develop concept knowledge. Stories that are created with a pedagogical purpose as their primary goal may of course provide better opportunities to select and frame content that stand out and allow exploration to facilitate conceptual development (Björklund & Pramling Samuelsson, 2013).

Another empirical example of the difficulty in using the story of Goldilocks is an intervention focused on area using circles with older children (Ameis, 2001). In the intervention the focus was on "just right" as a middle value, with the aim of finding the "just right" area in between a too-large area and a too-small area. However, this study found, in accordance with our analysis, that "just right" is not the middle value, which is why the use of the story in Ameis' intervention may have confused children's understanding. Thus, our intuition from our and others' empirical investigations was that the mathematical content in the story entails a complexity that hinders the emergence of mathematical learning objects if the narrative is played out true to its original form, or if the mathematics is not made a piece of content for exploration.

The learning potential in the story could theoretically be seen as the idea of series. Other potential learning objects are size, softness and warmth, all of which are necessary to experience as relations of different values. The relative nature of the notion of size can in the examples be explored by comparing the visible props used to illustrate the story, which is not the case for softness and warmth. The series from one extreme, via an in-between value, to another extreme could be one way of opening up necessary dimensions of variations for learning the meaning of size, softness or warmth. Thus, the story has the potential to either provide a structure for exploring values in sequences, which is necessary for ordering them consecutively (a series), or for learning about the relative nature of the different concepts. If a child has discerned the necessary aspects of series (relative and relational), this could be beneficial for generalizing the idea to other features, such as the concepts of size, warmth and softness. According to Variation theory of learning, the generalization of mathematical ideas, for example series, is possible if the idea is kept invariant while other irrelevant aspects of the series are varying. However, in the story of Goldilocks, the two possible learning objects—series and the concepts of size, warmth and softness, with their respective aspects that children would need to discern—are fused together. This makes the story a mathematical challenge for young children who may not have an understanding of either values in sequences or the different concepts. Thus, it is questionable to what degree the narrative enables children to differentiate the necessary aspects of the phenomenon of series as well as size, warmth and softness, if the children had not been able to discern them before.

Based on the content analysis, using the narrative of Goldilocks for teaching mathematical concepts to young children can be seen as a challenge per se, which explains some of the challenges found in the empirical examples. An important issue to keep in mind is that the content analysis of the story was made *after* we experienced the widespread use of the narrative by the preschool teachers, so they did not have access to the content analysis before they conducted the activities. Not being aware of the illogical structure of the narrative makes it hard to elaborate its content with the children, resulting in the mathematical concepts becoming difficult for the children to discern.

In this chapter, we have only focused on one story, and despite the difficulties highlighted, we strongly encourage preschool teachers to use narratives for pedagogical purposes, not least because of the coherence and interrelationship between concepts that are enabled (Burton, 2002). Narratives have many features that may emphasize specific learning objects in ways that embrace young children's need for concrete experiences of mathematical relations. Furthermore, concepts may be communicated in ways that direct children's attention to the specific features, extending both their verbal resources and experiences (Carlsen, 2013; Pramling Samuelsson & Pramling, 2013). Nevertheless, as our analysis shows, there are challenges lying within this approach that need to be highlighted. It is not clear cut that children will experience a story in ways that enlighten the meaning of mathematical concepts if the narrative does not support such relations in comprehensible ways. If teachers make a thorough analysis of the mathematics content in stories, as exemplified in this chapter, the possibility of mathematics becoming an object of learning is likely to be increased. However, for this to happen, the teacher has to have an advanced understanding of the specific and general aspects of the concepts in question. Thus, although using narratives may be a strategy to overcome the dichotomy between academic and social-pedagogical approaches, our analysis indicates that preschool teachers need mathematical knowledge that enables them to orchestrate the activity in a way that makes it possible for the children to discern the intended aspects of the involved concepts in a profitable way.

Appendix: The Story of Goldilocks and the Three Bears

Once upon a time, there was a little girl named Goldilocks. She went for a walk in the forest. Pretty soon, she came upon a house. She knocked and, when no one answered, she walked right in. At the table in the kitchen, there were three bowls of porridge. Goldilocks tasted the porridge from the first bowl. "This porridge is too hot!" she exclaimed. So, she tasted the porridge from the second bowl. "This porridge is too cold," she said. So, she tasted the last bowl of porridge. "Ahhh, this porridge is just right," she said happily and she ate it all up. After she had eaten she was feeling a little tired. So, she walked into the living room where she saw three chairs. Goldilocks sat in the first chair to rest her feet. "This chair is too big!" she exclaimed. So she sat in the second chair. "This chair is too big, too!" she whined. So she tried the last and smallest chair. "Ahhh, this chair is just right," she sighed. But just as she settled down into the chair to rest, it broke into pieces! Goldilocks was very tired by this time, so she went upstairs to the bedroom. She lay down in the first bed, but it was too hard. Then she lay in the second bed, but it was too soft. Then she lay down in the third bed and it was just right. Goldilocks fell asleep. As she was sleeping, the three bears came home. "Someone's been eating my porridge," growled the Papa bear. "Someone's been eating my porridge," said the Mama bear. "Someone's been eating my porridge and they ate it all up!" cried the Baby bear. "Someone's been sitting in my chair," growled the Papa bear. "Someone's been sitting in my chair," said the Mama bear. "Someone's been sitting in my chair and they've broken it all to pieces," cried the Baby bear. They decided to look around some more and when they got upstairs to the bedroom, Papa bear growled, "Someone's been sleeping in my bed." "Someone's been sleeping in my bed, too," said the Mama bear. "Someone's been sleeping in my bed and she's still there!" exclaimed Baby bear. Just then, Goldilocks woke up and saw the three bears. She screamed, "Help!" And she jumped up and ran out of the room. Goldilocks ran down the stairs, opened the door, and ran away into the forest. And she never returned to the home of the three bears.

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Chapter 16
Kindergarten Teacher's Knowledge to Support a Mathematical Discussion with Pupils on Measurement Strategies and Procedures

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16.1 Introduction

Measurement is a core construct of mathematical learning, since it serves as a "bridge between the two critical domains of geometry and numbers" (Clements & Sarama, 2007, p. 517) and research has revealed that early cognitive foundations are not limited to number concepts (e.g. Mix, Huttenlocher, & Levine, 2002). Research focusing on early childhood education, especially in the context of measurement, is still limited, and even more so when considering the role of the teachers' knowledge in supporting (or inhibiting) the development of pupils' mathematical knowledge, and consequently students' performance (e.g. Boyd, Grossman, Lankford, Loeb, & Wyckoff, 2009; Hill, Rowan & Ball, 2005; Nye, Konstantopoulos, & Hedges, 2004). In that sense, it is essential to broaden the understanding of the impacts of teachers' knowledge on measurement in the development of children's learning in this domain. Amongst the diversity of teachers' knowledge conceptualisations, a common tendency is to consider such knowledge as specialised for the work of teaching mathematics and, in that sense, one of such conceptualisations assuming both mathematical and pedagogical knowledge as specialised concerns the Mathematics Teachers' Specialised Knowledge—MTSK (Carrillo et al., 2018).

More recent research indicates that young children are capable of engaging in situations and discussions involving substantive mathematical ideas (e.g. Baroody, Lai, & Mix, 2006; Clements, Sarama, & DiBiase, 2004). Children often engage in mathematical thinking and reasoning when exploring patterns, shapes, and spatial

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relations, as well as when comparing magnitudes and counting objects (Clements & Sarama, 2007). Furthermore, it has been reported that when young children have access to a guide asking key questions, they are more likely to go further with their mathematical thinking (Björklund, 2008), as well as advance to the higher stages of developmental progression in terms of measurement thinking (e.g. Sarama, Clements, Barret, Van Dine, & MacDonel, 2011; Szilagyi, Clements, & Sarama, 2013).

In this study, we aim to contribute to a deeper and broader understanding of kindergarten teachers' knowledge—considering the MTSK perspective—implied in supporting a mathematical discussion to foster children's knowledge, promoting their progression through the measurement thinking levels. Pursuing such an aim, we focus on one kindergarten teacher's practice and on the revealed knowledge when implementing a measurement task with 5-year-old children. Our research question is: *How does the teacher's (specialised) knowledge support and facilitate mathematical discussions to foster and develop the children's mathematical knowledge on concepts, procedures, and measurement strategies?*

16.2 Theoretical Framework

In order to acquire the notion of magnitudes, children need to go through different stages involving, among others, correct and adequate wording when expressing the magnitudes and the use of different resources and strategies for measuring. In that sense, understanding the measurement strategies and related processes is much more complex than putting them into practice.

Based on a previous work on *Learning Trajectories* (Clements & Sarama, 2004), Sarama et al. (2011) proposed a description of children's thinking and learning in the scope of length measurement, through a set of instructional tasks designed to generate mental processes or actions conceptualised to help children move through a developmental progression of levels of thinking. Later, Szilagyi et al. (2013) presented a set of descriptors with nine levels for children's mathematical thinking and learning when working in a length measurement context: (a) Length Quantity Recognizer; (b) Length Direct Comparer; (c) Indirect Length Comparer; (d) Serial Orderer to 6+; (e) End-to-End Length Measurer; (f) Length Unit Relater and Repeater; (g) Length Unit Relater; (h) Length Measurer; and (i) Conceptual Ruler Measurer.

The work discussed here is related to the ways children represent and communicate their reasoning, including linguistic and gestural communication, when comparing a certain measurement of 3D objects (toys, for example) and when measuring some of the magnitudes of such 3D objects (e.g. length using pieces of a string). In that sense, we focus on only four of the aforementioned levels for children's mathematical thinking and learning (b) to (e). (b) *Length Direct Comparer* includes the children's capacity for physically aligning two objects to compare length, understanding the end-to-end comparison, placing the objects side by side. Furthermore, it includes understanding that objects can be repositioned and rotated (mentally, then physically) in order to align and compare them; (c) *Indirect Length Comparer*, which concerns the children's capacity for comparing the length of two objects by representing it with a third object. This level includes the ability to build a mental image of a particular length, maintaining and, to a simple degree, manipulating such a mental image, to compare it with other objects, applying to the images an explicitly transitive order relationship: a > b and b > c makes a > c; (d) *Serial Orderer to 6+*, which concerns the children's ability to order lengths, organising a scheme in a hierarchy, creating a mental image of a scale in which the higher-order concept prevails. This level also includes the ability to estimate relative lengths, by a trial-and-error approach and, eventually, complementing it with a scheme that considers a series of objects organised in such way that each one must be longer than the one before it and shorter than the one after it (Szilagyi et al., 2013). It is important to highlight that when a person builds a mental image of a particular object's length, different meanings of the concept "length" are evoked and linked to the concept image—in Vinner's (2002) sense. Thus, several aspects related to the same concept, which might be expressed in different representation systems (graphic, linguistic or semiotic), or various properties of such a mathematical concept, are addressed; and (e) *End-to-End Length Measurer*, which concerns the children's capacity to lay units end to end, although (s)he may not recognise the need for units of equal length. As this level includes an "implicit concept that lengths can be composed as repetitions of shorter lengths" (Szilagyi et al., 2013, p. 586), it is expected that children intuitively use equal-sized units or avoid gaps between units.

Considering teachers' knowledge as a factor that impacts students' understandings and results (Boyd et al., 2009; Hill et al., 2005; Nye et al., 2004), when aiming at grounding children's understanding for improving their results, it is essential to devise ways for improving teachers' knowledge. From the diversity of teachers' knowledge conceptualisations, we assume the specialised nature of such knowledge in and for teaching mathematics. Thus, we consider such specialisation in terms of both the Mathematical Knowledge (MK) and the Pedagogical Content Knowledge (PCK), which are the two main domains included in the MTSK model (Carrillo et al., 2018). Considering such conceptualisations, three subdomains are considered in each domain.

When thinking about the MTSK related to measurement, such specialised knowledge is included in MK and PCK. In terms of the MK, it includes, for instance, knowledge associated with the foundations of the measurement activity (what is a measure), the principles and procedures of the measurement activity (what, how, and why to measure), and the reasoning associated with it (Clements & Stephan, 2004). Here, the first level associated with measuring is included, corresponding to a visual comparison in terms of the considered magnitude (e.g. length, weight, height). It requires that the teacher possesses knowledge of the different representations that can be employed in the kindergarten context concerning how gestural and linguistic representations need to be considered in an adequate manner. Such knowledge, associated with MK, is included in the Knowledge of Topics (KoT), a subdomain of the MTSK.

In terms of the PCK on measurement, ideally, teachers should be knowledgeable on, for example, the children's abilities and difficulties when working on a measurement task. For instance, how to measure, what to measure, and how (and why) to address the measurement topic, in terms of the nature of the task, the strategy of implementation, examples, and the resources used. Such knowledge is a part of the subdomain of Knowledge of Features of Learning Mathematics (KFLM). It is also a part of the PCK, the knowledge associated with the "awareness of the potential of activities, strategies and techniques for teaching specific mathematical content, along with any potential limitations and obstacles which might arise" (Carrillo et al., 2018, p. 12), termed as Knowledge of Mathematics Teaching (KMT).

Considering the teachers' role in a mathematical discussion, Stein, Engle, Smith, and Hughes (2008) proposed a set of five facilitating practice foci of "de-emphasizing the improvisational aspects of discussion facilitation in favour of a focus on those aspects of mathematical discussions that can be planned for in advance" (p. 231). Our focus will be on the aspects involving anticipations of children's thinking and responses, to plan some decisions to be made "about how to structure students' presentations to further their mathematical agenda for the lesson" (p. 231), which corresponds to (a) anticipating children's likely responses to mathematical tasks and (b) monitoring children's responses to the tasks during the implementation of the facilitating practice.

When the objective is promoting the development of a mathematical discussion, the conceptualisation and implementation of a task require the teacher to mobilise the content of his/her specialised knowledge. Such knowledge also grounds the anticipation of students' productions, in a direct relationship with the teacher's own space of solutions in the sense put forth by Jakobsen, Ribeiro, and Mellone (2014) and Mellone, Tortora, Jakobsen, and Ribeiro (2017). Such ability to anticipate students' productions and give meaning to such productions and comments grounds the decision-making process mainly in the so-called contingency moments—corresponding to "knowledge-in-interaction as revealed by the ability of the teacher to 'think on her feet' and respond appropriately to the contributions made by her students" (Rowland, Huckstep, & Thwaites, 2005, p. 266). Such knowledge mobilisation can occur, for example, when children face difficulties in using words correctly in the context of a mathematical discussion on a measurement activity when differentiating the "smallest" from the "biggest".[1] In such cases, the teacher needs to possess a specialised knowledge that would sustain her/his decision to extend the mathematical discussion to the development of an adequate mathematical vocabulary. For instance, when the pupil uses the term *tiny* to refer to a very small toy, the teacher decides to emphasise the need to reword the term to *small*, highlighting the direct relationship between the wording, the particular dimension(s) of the considered elements, and the classification assumed.

[1] In Portuguese, for example, because of the words smallest (*menor*) and biggest (*maior*) sound very similar, it is very common for children to experience difficulties using these terms appropriately.

16.3 Context and Method

The work reported in this study is a part of a broader research study focusing on kindergarten and primary teachers' knowledge and practices in geometry.[2] Here, we focus particularly on one kindergarten teacher's (Karina) knowledge on the topic of measurement when implementing a task and developing a mathematical discussion with her 5-year-old children from a Brazilian public school.

Data has been gathered through audio and video recordings of the meetings (where the task has been designed by the teacher and the first author); the implementation of the task with children and two interviews with the teacher (before and after the class). The video recordings focused on the children's and the teacher's actions (strategies and procedures) in and for measuring. The video recordings of the implementation were captured using three devices: one focusing on the teacher's actions, another focused on the children's actions, and the third one was positioned in order to capture the global scenario. During the task design, the discussion focused on aspects related to the teacher's MTSK on measurement (e.g. related to the mathematical goals and topics involved in the task and the type of resources chosen to be used during the task implementation—pieces of sticks or strings). The interviews immediately before the task implementation focused on the teacher's "lesson image", which correspond to Karina's vision of what will occur in the class and how it will occur, including children's interactions and responses (Schoenfeld, 2000). The interviews after the class focused on discussing Karina's perceptions of her implemented practice.

When preparing the task, the teacher had the explicit aim of promoting a mathematical discussion on measurement with pupils and, in order to do so, the task implementation comprised three stages. First, children were presented with a set of five aquatic animal toys (which, during the interaction, the teacher named as "whale, shark, orca, dolphin, and goldfish") and afterwards they were asked to identify the smallest and the biggest among the five animals. In the second stage, children had to organise the toys by "size order". In the third stage, focusing on measurement strategies and procedures, two strings of different lengths were given to the children and they were required to: (1) "measure the whale" using a string much longer than the toy and (2) "measure the whale" using short pieces of string (of the same size). During the implementation of the three stages of the task, the children were working in groups of four. Our focus here is on two specific episodes. The first episode is related to the second stage of the task, when the teacher instructed pupils to present the five toys in a queue considering "size order", and then started a discussion about the reasoning the pupils used to organise the toys in a certain way. The second episode pertains to the third stage of the task, when the teacher invited the pupils to measure one of the five toys (specifically, the whale), using short pieces of string (each of 5 cm length), invited them to explain their procedures, and (eventually) their "results".

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[2] Research Project "*Kindergarten and Early Years' mathematics teachers' specialized knowledge on geometry.*"

The video analysis—identification of mathematically significant episodes in terms of teacher's knowledge of measurement—was grounded on Sherin, Linsenmeier, and van Es' (2009) criteria for characterizing video clips of students' mathematical thinking: (a) windows into students' thinking, related to the sources given by the pupils, as responses or forms of communication (gestural, for example), which thus become evidence of students' thinking; (b) the depth of students' mathematical thinking, which is related to the pupils' exploration of substantive mathematical ideas; and (c) the clarity of students' thinking, which is associated with the ease of understanding the pupils' thinking shown in the video. These three categories of the aforementioned characterisation of students' mathematical ideas are graduated into three levels, denoted as "low", "medium", and "high". Particularly, here we focus the discussion on the moments involving two medium-level episodes concerning student thinking when understanding the task and working on the task. The episodes concerning student thinking when understanding the task refers to episodes when the student, even without providing detailed information about his/her thinking, revealed a sufficient understanding of the teacher's request. Complementarily, the analysis focused on the episodes identified when children were working specifically on the task, particularly referring to the mathematical thinking and learning revealed (Szilagyi et al., 2013). The selected episodes have been chosen due to their suitability for discussing evidence of children applying different levels of thinking during the ongoing mathematical discussion and that pertaining to Karina's knowledge sustaining such a discussion.

The episode selection implied activating our own "ability to read, hear, and understand the interactions and knowledge in action" (Ribeiro, Badillo, Sanchez-Matamoros, Montes, & de Gamboa, 2017, p. 3379) in order to effectively select potential episodes. Thus, such a selection was made considering three foci: (a) one of the student's positioning of the toys when organising the queue by "size order"; (b) the strategy one of the students used when measuring the toy (a whale); and (c) the teacher's revealed knowledge. Concerning the teacher's capacity to support a mathematical discussion, we focus on anticipating children's likely responses to mathematical tasks, and monitoring children's responses to the tasks during the implementation of the facilitating practice (Stein et al., 2008).[3] For the teacher's MTSK, different subdomains were considered as means of analysing the teacher's knowledge and practices both during the mathematical discussion and in the interviews.

16.4 Analysis and Discussion

During the meeting when the task was designed, Karina stated that she was aware of the importance of teaching measurement, although she admitted needing to broaden her own knowledge on such a topic ("I need to incorporate and have more

[3] We have to note that even if such facilitating practices where developed with older students, in the work we are developing with kindergarten and primary teachers, its core essence remains.

elements in order to be able to work with children more often."). Such verbalisation is perceived as a recognition of the teacher's need for developing and grounding her MTSK on measurement in order to sustain a mathematical discussion with her pupils.

16.4.1 Episode 1: Ordering by Length Vs. Height

The first selected episode is deemed particularly suitable for analysis due to the distance considered by the pupil between the goldfish (1) and the shark (2), as shown in Fig. 16.1, and the discussions that this prompted. Moreover, Karina explicitly mentioned that this kind of situation is a challenge for her, as she does not know very well how to proceed in such cases. She shared, "She shared, "I was not expecting a correct answer. In fact, I do not know if there is a correct answer in those situations … Is there one?". This scenario, even if anticipated, corresponds to a contingency moment (Rowland et al., 2005). This revelation points to the need for Karina to improve her MTSK on measurement and on the principles related to what is measurable when considering 3D objects (toys in this case).

The previous student's ordering of the toys and the distance between Toy 1 and Toy 2 may be related to the discrepant goldfish size (in the variety of dimensions) when compared with the other toys, corresponding to a medium evidence of student thinking (Sherin et al., 2009).

Karina [Putting the whale in the set of animals]: How would you guys organise the animals? Queuing, for example? Let Isadora have her queue first, then each of you will have yours.
Karina: But you must organise the toys in order of size. You must look at the size.
[Isadora organises her queue—Fig. 16.1]
Karina: Wait … Let's see Isadora's queue. You started from the biggest or the smallest, Isadora?
Isadora [pointing to the shark (2)]: With *this* one.
Karina: Which one is the first in your queue?
Isadora [pointing to the shark (2)]: This one.
Karina [referring to the goldfish (1)]: What about that one?
Isadora: Also.
Karina: Ah, also? And did you arrange by the size order?
[Isadora nods affirmatively]

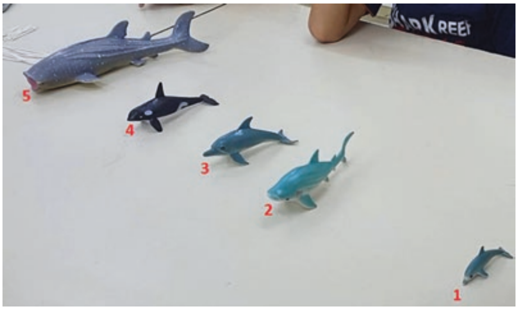

Fig. 16.1 Student's organisation by "size order"

Isadora made sense of the teacher's request (Fig. 16.1), although she did not provide detailed information about her thinking ("*With this one*") when Karina posed the question on what the starting element was—the biggest or the smallest (associated with KoT). The teacher's questioning led Isadora to a short list of answers (accompanied by pointing) and revealed aspects of the teacher's knowledge related to anticipating the students' answers/productions. This is grounded in the content of the different subdomains of the MK and her previous experience with measurement, which she recognises as being an important topic, although she admits not working with students and measurement often.

The dialogue continues: Karina [pointing to the whale (5)]: This one is big, isn't it?
Isadora [picking up the goldfish (1)]: And this one is *tiny*.

Although the pupil provides a brief verbal response, her gestural communication (pointing and picking up) is significant for the discussion and externalisation of her knowledge (understanding), allowing the teacher to make informed decisions grounded in the meaning given to the student's answers. Isadora seems to exclude the goldfish from the queue arrangement, putting her arms around the toy (Image 1 in Fig. 16.2), although she nods affirmatively to the teacher when prompted for a response. The pupil's response reveals that she knows how to compare the whale and the goldfish by size, even when these toys are apart from each other. Such a situation positions the child at a "Length Direct Comparer" level of the developmental progression in the *Learning Trajectory* of length measurement (Szilagyi et al., 2013).

Regarding the mathematical depth of the episode, it corresponds to what Sherin et al. (2009) label "low level", since Isadora seems to organise the toys in a way that is usual for her.

Karina [pointing to the orca, the dolphin, and the shark]: And among these three here, which one is the biggest among these three?
[Yuri points to the dolphin]
Felipe [pointing to the shark]: No, it is that one!
[Isadora points to the shark]
Karina [pointing to the shark]: That one? (Image 2, Fig. 16.2)
Karina: Let's have Felipe's queue? Would you order the animals now, Felipe? Look for the size. How would you organise them by their size? All of you will make a queue.

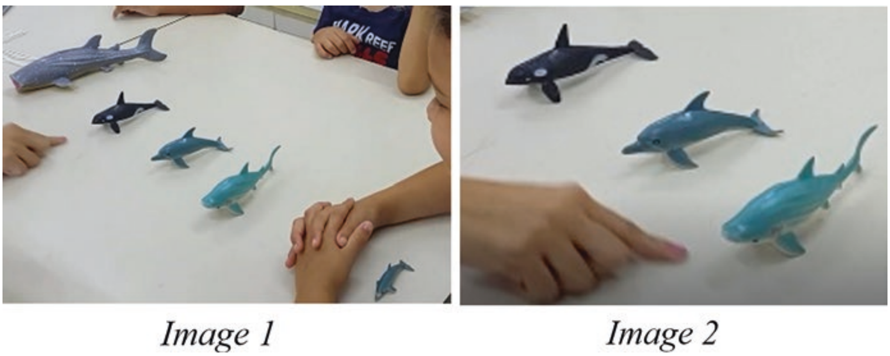

Fig. 16.2 Student's answer followed by the teacher's questioning

Even if the children provide information from different sources (e.g. responses, forms of communication), they do not provide clear information (low clarity, in terms of Sherin et al., 2009), because Isadora does not provide clues to identify her reasoning when arranging the toys. In the size-ordered queue that Isadora proposed, it is evident that the fish is located much further from the shark (Toy 1 and Toy 2 in Fig. 16.1). In addition, the other toys—dolphin, orca, and whale (labelled 3, 4, and 5 in Fig. 16.1)—are positioned closer to each other.

Such an unanticipated way of organising the animals (by height measurement instead of length) left Karina in a contingency moment (Rowland et al., 2005) as the students' and teacher's focus was not aligned (height and length), which also can be related to the teacher's knowledge of what can be measured. When ordering (one step in the measuring process) the animals, one can have a diversity of foci. Being aware of such diversity is part of teachers' specialised knowledge, which is associated with the set of (im)possible solutions for the same problem/situation—the elements of the space of solutions each one has/develops for each situation/problem (Jakobsen et al., 2014). This awareness allows the teacher to give meaning to children's productions, contributing consequently for promoting deep mathematical discussions which are intentionally grounded on such diversity and the implications of the focus of attention when ordering. In that sense, this lack of correspondence between what is asked for and what is answered—in terms of focus of attention—is linked both to the teacher's knowledge of what can be measured and the kind and nature of the given instructions (order by size), leading to the teacher's apparent space of solutions with one single element.

When focusing on Isadora's options and arguments—reflecting her knowledge—placing the shark first in the queue, corresponds to the fact that she considers this toy the tallest animal among the others (considering the height from the table to the belly of the animal). The goldfish is not included in Isadora's queue, probably due to her perception of similarity between the height of the goldfish and that of the other four animals, revealing her understanding of transitivity. If the instruction was to organise by "size order", two toys of the same size cannot be placed in the same position. Alternatively, because its height is so discrepant to the other animals, she does not know where to include it within the queue.

Isadora is able to physically align two objects to perform a comparison of a measurement (seen by the teacher as length, but assumed by the student as height), understanding the end-to-end comparison, placing the objects side by side, revealing knowledge associated with the "Length Direct Comparer" level (Szilagyi et al., 2013). She recognises that, among the three objects in the subset of the previous arrangement (toys labelled 2, 3, and 4 in Fig. 16.1), the biggest animal in terms of both length and height (as well as volume or weight) is the shark. The key question in the mathematical discussion that emerged from this situation is grounded in the duality of understanding of "comparing". For the student, comparison is being made between the heights of the objects, whereas, for the teacher, "organise the queue by size order" is assumed to refer to the length of the toys, as confirmed by the teacher in the post-class interview.

Furthermore, the teacher's request to "queue in size order" is related to a common daily activity performed by the children when entering the classroom, as they queue from the shortest to the tallest. The mental image (Vinner, 2002) evoked by the pupil is aligned with the mathematical way pupils consider length (referring to height, in their daily activity when perpendicular to the plane we walk on) which then, when associated with the word "queue"—"Indirect Length Comparer" (Szilagyi et al., 2013)—was the natural correspondence for the pupil. Furthermore, Isadora's arrangement of the toys considering their height corresponds on a hierarchical scheme with the higher-order concept of an element in an ordered series, creating a mental image of a scale—corresponding to what Szilagyi et al. (2013) termed as *Serial Orderer to 6+*.

The teacher's interventions were also related to monitoring students' responses to the tasks during implementation (Stein et al., 2008), but such interventions followed a path that did not necessarily consider the students' answers and reasoning—linked to the teacher's "space of solutions" (Jakobsen et al., 2014; Mellone et al., 2017) in the scope of measurement. Such lack of awareness of the multiplicity of meanings of the word/notion *size* (organising by "size order") related to the teacher's knowledge (Carrillo et al., 2018) of the concept definition and the concept image—in Vinner's (2002) sense. Such knowledge refers, from one perspective, to the knowledge associated with the principles and procedures related to the measurement activity (Clements & Stephan, 2004)—which has been established as the focus of the measurement activity. Thus, it is not something considered uniquely as an element of the teachers' knowledge, even if it is obviously an element of MTSK (in this case, it corresponds to the knowledge that pupils are required to possess/develop). From another perspective, a part of the teachers' knowledge refers to the different types of representations including, but not limited to, a gestural communication associated with the natural language in order to externalise the knowledge grounding the performed reasoning.

It is also noteworthy that, when the students compare the size of three of the five given toys (shark, dolphin, and orca), the shark is indeed the "biggest", whether comparing lengths or heights (as well as if considering the other "typical" measures—volume, weight/mass, perimeter, the "space" occupied on the table). In fact, the question the teacher posed in connection with the given example does not help her (the teacher) to identify the reasons (students' thinking) that led to specific object arrangements, whereby the goldfish is placed before the shark and not included in the queue. This corresponds to the KMT dimension on MTSK. Such difficulties in posing mathematically powerful questions, linked with the missed opportunities for exploring in depth some of the students' responses, are, according to the teacher, related to the fact that she does "not have much confidence, sometimes because, for example, there are situations I do not know very well how to find a way out". In that sense, such lack of confidence in and for exploring the mathematical topics—here related to some dimensions of measurement—is grounded in her awareness of "having something missing", and thus points to the need for improving her own knowledge (KoT and KFLM) related specifically to the measurement criteria and with anticipating the students reasoning and possible difficulties in the topic.

16.4.2 Episode 2: The Magnitude of Length

The second episode focuses on the magnitude of length (in particular related to perimeter) corresponding to a student production and measurement procedure anticipated by Karina during the interview. Before inviting pupils to measure the whale, Karina engaged with them in a discussion about the size of the pieces of string. Prompted by one of the pupils' statements (*These strings are very short*), she stimulated them to think about the number of pieces of string they would need to use to measure the whale.

Karina [showing the pieces of string]: Look guys! Now, this is the [piece of] string we will use to measure the whale.
Isadora: These strings are very short.
Karina [showing some strings]: Are these very short?
Felipe: Yeah!
Karina: How many pieces [of string] you think will be needed to measure the whale?
Felipe: A lot of them!
Karina: A lot of them? Hum…
Karina: [Looking to Felipe]: How many pieces of this very short string do you think you will need to measure the whale?
Felipe [Looking to the pieces of string]: Ahm … Thirty!

The pupils' statements about the length of the strings (Isadora and Felipe) reveal that the children are on the "Serial Orderer to 6+" level of the developmental progression in the Learning Trajectory of length measurement (Szilagyi et al., 2013), since they seem to organise a hierarchy scheme of a mental image of scale in which the higher-order concept prevails. Besides that, regarding the window of students' thinking revealed in the episode, it can be characterised as a medium level, since the teacher had to explore Isadora's statement (*These strings are very short*) more deeply in order to understand the evidence of the student's thinking.

This episode corresponds to a crucial situation for deciding to continue exploring the mathematical ideas on the use of a "short string" to measure the whale. Contrary to what occurred in the first episode (organising the toys by size-order), this situation had been anticipated by Karina when elaborating on the task (Ribeiro et al., 2017) and thus was not a contingency moment (Rowland et al., 2005) as she was prepared to deal with it. Consequently, she was able to support and develop a more fruitful mathematical discussion, engaging children in giving sense (mathematically) to what they say and allowing them the opportunity to justify their arguments (Stein et al., 2008).

Karina: Ok guys, do you agree that Felipe should start measuring the whale? Because he said he would need thirty [pieces of string] to measure the whale, wasn't that the case Felipe? You said thirty, right?
Felipe: Yes.
Karina: So, do you agree that Felipe should start?

When asking the pupil to confirm his estimate of the number of pieces of string he will use to measure the whale, Karina is investing into the development of the pupil's knowledge within the "Serial Orderer to 6+" level. It is related to Karina's knowledge of the principles and procedures related to the measurement activity, as well as the reasoning associated with it (Clements & Stephan, 2004) which is the content of Karina's knowledge related to the topic (KoT). During the interview, Karina emphasised the importance of providing several pieces of string for the pupils to develop their knowledge and understanding of the correct procedure for measuring the length of an object. She mentioned:

> Karina: I think that with a single piece it is difficult [for the children], because, then, you need to mark, go marking with your finger, to turn [indicating that the end of the string should be kept at the same point to iterate the unit] the string, right? I do not know … So, I think you will need … to provide to all of them, regardless of age, several pieces of string to measure.

Karina reveals her knowledge associated with the principles and procedures of measurement— in the same space of knowledge that her pupils are supposed to develop. This is associated with parts of KoT, related to knowing how to perform. However, her comment does not reveal the need for using a single unit of measurement to iterate, but instead, by "simplifying" the process, she provides several strings.When asked about the need for using several pieces of string for the measurement process, Karina mentioned:

> Karina: The first time I did something like this with them [pupils], I used several pieces of string. Maybe in a teaching process that lasts a long time, for instance, a year, you can work with this [possibility] … it takes time … they must play, think, explore … so then, I think they can, after a process, they can try … try to do without several strings.

When intentionally exploring the task allowing pupils to use several equal units (pieces of string with same length) Karina reveals her knowledge associated with pupil's features of learning the topic of measurement (KFLM), being this procedure of using multiple equal units an intermediate step for the development of the notion of measure. Her statement reveals, thus, simultanously, her knowledge associated with one of the main principles of the measurement activity—using one unit to iterate (KoT).This is an evidence that the teacher's MK influences her PCK.

When Felipe performed the measurement of the whale's "perimeter", he opted for positioning the pieces of string in a standard way (end to end, seemingly trying to avoid gaps and overlaps) and counting the number of strings used (Fig. 16.3).

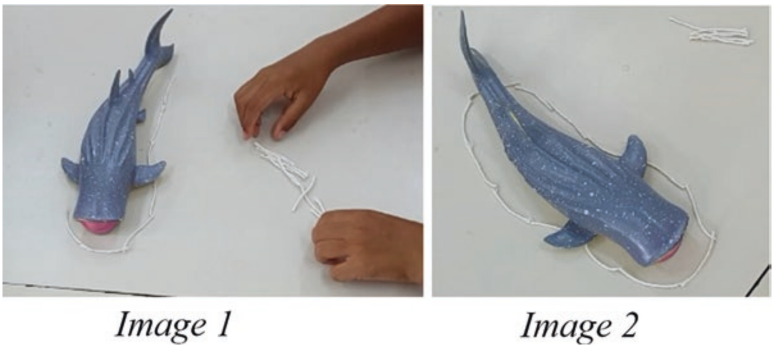

Fig. 16.3 Felipe's strategy for measuring the whale's perimeter

The pupil's arrangement of the pieces of string puts him at the "End-to-End Length Measurer" level, since it is evident that the child understands the necessity to queue the units and that the length can be expressed as a repetition of shorter lengths. Moreover, he uses "intuitive constraints to use equal-size units or to avoid gaps between units" (Szilagyi et al., 2013, p. 586), even if some of the pieces of string are not properly arranged, which may be due to the physical nature of the string or could be related to the children's difficulty in manipulating the string—not related to the mathematical notion of measurement.

After positioning the pieces of string, Karina questioned the pupil about the number of pieces of string he used to measure the whale.

Karina: Ok guys, come and take a look at how Felipe proceeded with measuring the whale.
Karina: Felipe, how many pieces of string did you use to measure the whale?
Felipe [starts counting one by one]: Thirteen.
Karina: Thirteen? And did you use more or less [pieces] than you thought [estimated] before?
[Felipe does not respond]
Karina: You said thirty before, and now you used thirteen.
[Felipe remains silent, looking at his arrangement of strings]
Yuri: It is a smaller amount.
Karina: Is it a smaller amount, Yuri?
[Yuri nods]

Even after considering the issue of the number of pieces of string to be used to measure the whale as a potential aspect to be explored with children, the teacher did not engage with pupils in this mathematical discussion (Stein et al., 2008). During the interview after the class, the teacher recognised that it would be a good opportunity to discuss quantities with the children, comparing thirty to thirteen (KoT—types of representation and number sense) by using the strings (KMT—the potentiality of the resource). However, she admitted to having consciously chosen not to use this opportunity for discussing quantities once she realised that she would need to spend (extra) time discussing the procedure employed by the pupil in order to help him develop his knowledge on this particular aspect of measurement (mobilising elements of both KoT and KFLM) but would prefer to do that in another moment with another task.

Continuing the discussion:

Karina: Look guys, Felipe used thirteen pieces of string to measure the whale. What do you think about his way of measuring?
Yuri: Imagination.
Karina: Imagination? What do you mean by "imagination"?
Yuri: Because he did it like this [gesturing as if he was doing a circle on the table, pretending he was skirting the toy] and then he put all the strings the way he wanted to do.

Yuri's statement reveals that he is interpreting his classmate's (Felipe) procedure as having created a mental image (Vinner, 2002) of the length to be measured. This possible interpretation puts Yuri both in the "Serial Orderer to 6+" and the "End-to-End Length Measurer" levels on the *Learning Trajectory*, which are expected to be developed in parallel (Szilagyi et al., 2013). Moreover, it can be said that Yuri is trying to give sense to Felipe's procedure by building a concept image

(Vinner, 2002) of what must be done when measuring a perimeter, linking it to the concept that lengths can be expressed as repetitions of shorter lengths.

Regarding the content of the teacher's knowledge to support the mathematical discussion grounded in Yuri's comment, within this situation, Karina encounters another contingency moment (Rowland et al., 2005). Karina has not anticipated that the notion of "imagination" would appear in the context of an experimental activity, in which children are required to effectively employ physical movements for measuring. It was not even related to the process of estimation. Such a dimension of the content of teachers' knowledge is related, on the one hand, to the notion of building a mental representation of a concept, in this case the measurement of a length (KoT), before proceeding to the activity of actual measuring. On the other hand, this knowledge is associated with the relationships between such a concept image and the resources applied to design the tasks aimed at generating learning opportunities, in this case the pieces of string, which are, mathematically linked to the notion of length (KMT—resources used).

Building a mental representation of a certain length can, amongst other possibilities, support children in developing knowledge related to estimating lengths (Policastro, Almeida, & Ribeiro, 2017) as well as other magnitudes. Thus, the notion of "imagination" Yuri evoked during the discussion is one of the core ideas teachers should explore more frequently in the context of kindergarten schooling, in order to develop children's mathematical thinking (Björklund, 2008). This would allow and contribute to the development of the foundations for more formalised learning in the years to come.

Since Karina was focusing on the procedure related to measurement, she questioned Felipe about one of the parts of the whale (the tail) that he did not consider when measuring its "perimeter" (see also Image 2 in Fig. 16.3) (Fig. 16.4):

> Karina: [pointing to the tail of the whale] Hey Felipe, tell me something … what about this part of the whale? Don't you need to measure it?
> Felipe: I forgot.
> Karina: Did you forget? So, what would you do now to fix it?

Immediately, the child moved one of the pieces of string he was using in the measurement (the one which was under the tail of the whale) and rearranged it on the line that was defining the "perimeter" of the toy (see Fig. 16.5). Then, he continued the process of including the units of measurement to complete the skirting, using three more pieces of string he had available.

Fig. 16.4 Teacher questioning the pupil on the "unmeasured" part of the perimeter of the whale

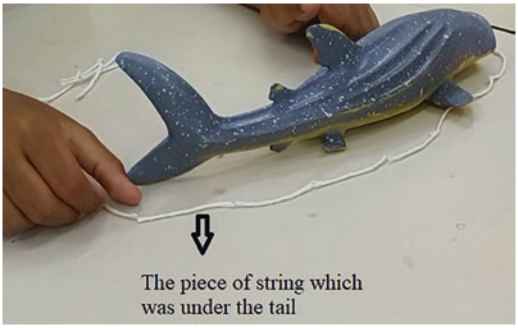

Fig. 16.5 Felipe's rearranging the pieces of string

It can be said that the pupil reveals a level of knowledge that can be considered at the "End-to-End Length Measurer" level, although he does not properly recognise the relationship between the number (verbalised) and the quantity of objects associated with such a number.

Karina: And now, how many pieces of string did you use, Felipe?
Felipe: [starts counting pointing with his finger to the pieces of string]: Seventeen.
Karina: Now you used seventeen? Ok.
Karina: Isadora, now it is your time to measure …

Felipe did not count the quantity of strings properly, because he pointed to both ends of one of the pieces of string on the line, effectively counting the same piece twice. This is a problematic aspect of children's knowledge related to what the focus of counting is: the extremes (points) or the units connecting those extremes. Such a situation is configured as an obstacle to ground students' learning since kindergarten, and it was a good opportunity for the teacher to develop a discussion with the pupils about the counting process, in order to also work jointly on the conservation of a quantity, starting from the 13 strings (the number Felipe already used) and adding three more.

Such an opportunity might not have been considered by Karina, since her focus during the task was on the procedure involved in the measurement activity. Alternatively, it might not be an element of Karina's space of solutions (Mellone et al., 2017) concerning the connections between the pupils' difficulties when establishing the correspondence between measuring (the procedure involved) and giving a final number to the measurement (counting the number of times they need to repeat the unit of measurement).

16.4.3 Final Comments

Considering the specialised nature of teachers' knowledge, to better understand the content of such knowledge and its impact on practice and students' understanding is not only about addressing "what" teachers know (in terms of the mathematical and pedagogical content), but also addressing "how" one needs to know what (s)he knows. In that sense, understanding teachers' knowledge requires taking into account that it "is not a kind of knowledge but a style of knowing that accounts for specialisation in mathematics teacher knowledge" (Scheiner, Montes, Godino, Carrillo, & Pino-Fan, 2019). In this perspective, analysing the role of MTSK in teachers' practice is a complex task that involves, amongst others, moving the focus from assessing the content of teachers' knowledge (from what they seem not to know) towards emphasising what they know and how they know it. Such analysis is needed in order to be able to contribute effectively to improving the quality of teacher education.

Here, the results enhance the role of teachers' specialised knowledge as one of the pillars that sustain the interrelationship between anticipating students' answers and the set of answers teachers would provide. This allows them to make informed decisions and implement significant mathematical practice even in contingency moments (Rowland et al., 2005) during ongoing mathematical discussions.

Here, we refer explicitly to the knowledge dimensions involving principles (what is measured) and measurement procedures (how we measure), as well as the relationships between natural and mathematical language in measurement contexts. However, the dimensions that involve knowledge about the children's difficulties and understandings, the kind and focus of the questions to pose, and examples to provide are also some of the required elements of teachers' knowledge in order to promote a mathematical discussion focusing on developing students' knowledge and understanding of the topic and the connection within and between topics. In that sense, and in order for teacher education to place its focus where it is most needed, and grounded in research results, some possible topics of attention that will guide our future investigations are as follows: (1) the content of teachers' knowledge in and for anticipating students' productions and comments related to a specific mathematical task; (2) the content of teachers' knowledge associated with posing questions following students' hypothetical reasoning; and (3) what the core elements of teachers' knowledge on measurement are, how they differentiate or complement the general dimensions of measurement, and how they can be promoted.

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Chapter 17
The Materialisation of Children's Mathematical Thinking Through Organisation of Turn-Taking in Small Group Interactions in Kindergarten

Svanhild Breive

17.1 Introduction

Children stand in constant relation to an ever-changing environment. To notice differences and similarities in the ever-changing environmental context and to recognise structures (generalities) from these differences and similarities are argued by many to be the essence of mathematical thinking (e.g. Mulligan & Mitchelmore, 2013; Radford, 2010). In kindergarten, children may experience mathematical structures in both free-play situations and in organised activities. For example, when children work in small groups to solve a mathematical problem they must coordinate and organise their actions in order to productively solve the problem. It is through this coordination of actions that mathematical structures emerge in the activity (Radford, 2010, 2013, 2015).

To understand more about how mathematical structures may appear in young children's activities, this study examines the characteristics of children's turn-taking while they work in small groups to solve addition problems. The aim is to reveal how children coordinate and organise their actions to move the activity forward and solve the problems. The analysis focuses on children's use of various semiotic means like gaze, word emphasis and gestures to organise their turn-taking and what mathematical structures are revealed through their joint activity.

This study addresses the following research questions:

- What characterises children's organisation of turn-taking while they work in small groups to solve addition problems?
- What role does children's organisation of turn-taking play in the materialisation of children's mathematical thinking in the joint activity?

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17.2 Theoretical Framework

In the research study reported here, I draw on Radford's (2013) theory of knowledge objectification, a cultural-historical theory of mathematics teaching and learning where learning is conceived as "social processes of progressively becoming critically aware of an encoded form of thinking and doing" (Radford, 2013, p. 26). It is through a complex coordination of semiotic means (language, artefacts, mathematical signs, gestures and other bodily actions) that mathematical ideas are mediated into our consciousness. Put another way, learning mathematics is to become critically aware of mathematical structures in the environmental context. However, this process does not happen all of a sudden, rather, there are layers of generality (Radford, 2010) which the subject gradually becomes more aware of. And it is through human activity and through a coordinated use of semiotic means that these generalities are materialised, which is brought into life and into our consciousness.

Roth and Radford (2011) use the term 'joint practical activity' to describe how humans jointly work together towards a mathematical object in the process of objectification. In their study, they show how a teacher and a student work together towards a mathematical object (a specific algebraic pattern) and how the algebraic pattern is materialised (brought into life) through the two participants' actions. Through complex coordination and tuning of different semiotic means, a space of joint action and intersubjectivity is created where thinking appears as a collective phenomenon (Radford & Roth, 2011).

This study investigates young children's joint practical activity working in small groups to solve addition problems. To understand more about the nature of the coordinated interaction, the movement of the activity, and the materialisation of children's mathematical thinking, the study focuses on children's turn-taking and especially how children organise their turn-taking by coordinating various semiotic means. In their description of a 'simplest systematics for organisation of turn-taking', Sacks, Schlegoff, and Jefferson (1974) characterise organisational features for turn-taking in conversation and describe how turn-taking is organised by two main types of 'turn-allocation techniques': a current speaker may select the next speaker, or a 'non-speaker' may self-select in starting to talk. In self-selected turn-taking, the potential next speaker must find a 'transitional-relevance place', which is a place where it is relevant for a transition in the conversation. Such transitional-relevance places are determined by clausal, phrasal and/or lexical principles which create conversational units, and by which the speaker may construct a turn.

In his investigation on how the next speaker in turn-taking is addressed by the current speaker, Lerner (2003) discusses a range of explicit and tacit 'techniques' for addressing the next speaker. The current speaker may select the next speaker using address terms (like 'you' or the next speakers name), or through gaze-directional addressing where the current speaker is directing his/her gaze to another participant while speaking. Although describing these ways of addressing the next speaker independently, Lerner (2003) emphasises that these methods are often used in concert with each other (like the use of an address term in concert with gaze or gestures). Tacit addressing is another method for addressing the next speaker, and this makes evident who is being addressed without using explicit address terms or other explicit means. Tacit addressing draws upon specific features of the current circumstances and through a specific composition of content and initiating actions, the next speaker is being selected. Lerner (2003) emphasises that both explicit and tacit address 'techniques' are context sensitive; however, tacit addressing cannot be considered without it. Similarly, Mondada (2007) emphasises the situatedness or context sensitivity of turn-taking. From a multimodal perspective, she investigates how participants, in a conversation, gradually establish him/herself as the next speaker through specific use of gestures. By using pointing gestures, while the current speaker is still talking, the participants establish him/herself as the next speaker. In her case study, the participants are sitting around a table with diverse artefacts (maps, documents, etc.) in the middle, and where everyone is engaged in reading, writing and considering these artefacts. In this context, the interaction is not primarily organised as a face-to-face exchange of talk but as a side-by-side exchange where the participants are not looking at each other (having eye contact), rather looking at the artefacts and their joint actions.

In the two segments that are examined in this study, the children were given addition problems (considering the semantic structure[1] of the problems). However, as will be shown in the results, these problems and the children's organisation of turn-taking while solving these problems promt rhythmic counting of groups and repeated addition which may be considered as key steps towards multiplicative reasoning. Multiplicative reasoning is distinguished from additive reasoning and traditionally considered as more complex (Anghileri, 1989;Greer, 1992 ; Mulligan & Mitchelmore, 1997). In additive reasoning, quantities of the same type are added, for example 5 apples plus 3 apples equals 8 apples. In multiplicative reasoning, quantities of different types are involved, for example 4 baskets with 3 apples in each basket equal 12 apples altogether. The example also illustrates the group structure, which is characterised by multiplication.

From research on the semantic structure of multiplicative situations (Greer, 1992; Mulligan & Mitchelmore, 1997) there have been found at least four different types of problems relevant for kindergarten and early school-years, where 'equal groups problems' (e.g. 4 baskets with 3 apples in each) is considered as one of the basic sematic groups. To solve equal groups problems, rhythmic counting and repeated addition, with diverse use of tools, are found as two key strategies that children use, and are key steps towards multiplicative reasoning with number facts (Anghileri, 1989; Mulligan & Mitchelmore, 1997).

What can be found from the substantial number of empirical research on the relationship between the sematic structure of addition and subtraction problems and children's strategies for solving these problems is that most of the research has focused on children's individual skills and individual problem-solving strategies (see Baroody & Purpura, 2017 for an overview). Similarly, research on children's understanding of multiplication and their multiplicative problem-solving strategies

[1] 'Semantic structure' refers to the way in which the problem is formulated, either in writing text or verbally, before the children start to solve it.

has also focused on children's individual skills (e.g. Anghileri, 1989; Greer, 1992; Lu & Richardson, 2018; Mulligan & Mitchelmore, 1997) where clinical interviews are often used as a method for collecting data. These studies fail, I hold, to see the contextual features of children's thinking.

In the literature above, rhythmic counting is seen as a means to reach multiplicative thinking. In Radford's (2013, 2015) theory, on the other hand, rhythm must be seen as an integral part of mathematical thinking. Thinking, in Radford's (2015) conception, is thought put into motion and it is through joint practical activity that mathematical thinking is brought to life (i.e. being materialised or actualised). Radford (2015) argues that "mathematical thinking happens in time. (…) Mathematical thinking not only happens in time but its most striking feature is *movement*" (p. 68). Rhythm is one structuring feature through which children's mathematical thinking may be materialised, and Radford defines rhythm like this: "In its general sense, the concept of rhythm tries to characterise the appearance of something at regular intervals and attempts to capture the idea of regularity, alternation, or something oscillating between symmetry and asymmetry" (p. 68). An important feature of rhythm is thus movement, and in accordance with Radford's conception of mathematical thinking, rhythmic counting (as referred to in the literature above) is not merely a means for multiplicative thinking, but rather a part of the multiplicative thinking itself.

Rhythm mediates several things, and one of the most important elements of rhythm is what Radford calls 'theme'. "Theme is the very important component of rhythm that moves us from memory to imagination and that provides us with the feeling of continuity of the phenomenon under scrutiny—the sense that something will happen next, or the expectation of a forthcoming event" (Radford, 2015, p. 81). Rhythm mediates that there is a regularity or a continuation of something and it gives the children possibilities for imagining what comes next. Another important element of rhythm is 'prolongation'. According to Radford (2015), "Prolongation is the component of rhythm where a phenomenon is *expressed*" (p. 81). Through rhythm, a mathematical phenomenon may be expressed or materialised. In this case the rhythmic counting that emerges from children's turn-taking materialises a structure fundamental for multiplication. The different elements of rhythm help to organise thinking and are essential components for the flow of thinking.

17.3 Methodology

The case study (s) reported in this chapter is part of a larger Study (S) on mathematical teaching and learning in kindergarten and is situated within a research and development project called the Agder Project.[2] Five kindergarten teachers (KTs) from the focus group of the AP and their groups of children participated in the Study. Data

[2] The Agder Project is funded by the Research Council of Norway (NFR no. 237973), The Sørlandet Knowledge Foundation, The Development and Competence Fund of Aust Agder, Vest Agder County, Aust Agder County, University of Agder and University of Stavanger.

was collected from 37 sessions during the academic year 2016/2017 (the intervention year of the project) where the five KTs implemented mathematical activities pre-designed in the project. The 37 sessions included activities with numbers, geometry, measurement and statistics and were organised as both whole group sessions and as small group collaborations. All observed sessions were video-recorded using two video cameras and focused on the participants' facial expressions as well as bodily actions.

This case study focuses on the joint activity and the coordination of turn-taking within two small groups of kindergarten children (aged 5–6) working on addition problems. The two segments[3] examined in this study were selected from the data set of 37 sessions, focusing on activities where children were challenged to solve addition problems in small groups, without extensive interference from the KT, and where the children showed willingness to solve the problems, that is they persevered in their effort to solve the problems. These criteria for selecting segments limited the data reported here to two segments from two different kindergartens (K1 and K2). In K2, the KT interfered in children's group work at the end of the segment, and therefore segment 2 is divided into two sub-segments (segments 2.1 and 2.2).

The KT in K1 implemented an activity called 'Treasure Hunt' where children searched for a treasure, and to get to the treasure the children had to solve mathematical problems en route. Each problem needed to be solved before the children could move on to the next problem. One of the problems in the activity, and which is the focus in segment 1, was formulated as follows: "Run around the nearest located tree three times each. How many times have you run around the tree altogether?"

The KT in K2 implemented an activity called 'Balloon Play'. In 'Balloon Play', the KT placed several balloons on a wall, each containing a mathematical task or problem. The children chose which balloon to burst with a drawing pin, and they worked in groups to solve the problem. One of the problems, which is the focus in segment 2, was: "Look at your hands, how many fingers have you got altogether in the group?"

The two segments were transcribed[4] and then analysed from a multimodal, interpretative perspective. The analysis was conducted (and refined) through iterative examination of the video recording and of the corresponding transcripts focusing on verbal and non-verbal actions which the participants used and made available to others for the purpose of moving the activity forward. The analysis focused on identifying verbal and non-verbal actions which seemed important for understanding the ongoing interaction in light of the formulated research questions. This fine-grained iterative analysis served as a means for interpreting the segment as a whole.

[3] A segment is here considered a self-contained part of a lesson with a distinct beginning and end.

[4] Transcription codes: (()) denotes non-verbal actions or contains explanations and interpretations necessary to understand the dialogue; _ denotes that the underlined word is emphasised; … denotes a pause in the verbal utterance; [] denotes that the utterance is cut off by another participant.

17.4 Results

In this section, three examples are presented to illustrate diverse ways in which turn-taking may be organised in small groups in kindergarten. The first example (segment 1) is taken from K1, where four children (Pia, May, Amy and Adam) collaborate to solve an addition problem. The example illustrates, primarily, how gaze in concert with word emphasis is used to address the next speaker and move the activity forward. From this way of organising turn-taking, a rhythmic counting is released and a multiplicative pattern emerges. The second example (segment 2.1) illustrates, primarily, how children self-select turns in the organisation of turn-taking. In this example, the children have different ideas for how to solve the problem, but during the activity they tune to one another and reach a compromise for how to solve the problem which they all support. In the third example (segment 2.2), the KT interferes in children's group work, and the turn-taking takes yet another form. The KT now strongly structures the turn-taking and helps the children re-establish joint attention and focus on a common strategy to solve the problem. The second and third examples (segments 2.1 and 2.2) are taken from K2, where three children (Lily, Eva and Mia) collaborate to solve an addition problem.

17.4.1 Segment 1, from Kindergarten 1 (K1)

Before the segment presented below, the KT and the children have reached post 2 in the 'Treasure Hunt' activity. The KT reads the problem for the children ("Run around the nearest located tree three times each. How many times have you run around the tree altogether?"), and the children immediately start to run around the tree. The children run around the tree three times each and then represent their runs with their fingers (each child shows three fingers to the KT and the other children). Then the KT initiates the second part of the task which is to figure out how many runs they have run around the three altogether. Figure 17.1 illustrates how the children are positioned when they try to solve the problem.

115	KT	But, how many times have you run altogether?
116	Pia	((Pia shows three fingers))
117	KT	All of you ((swipes her hand over the children, while keeping her gaze on Pia))
118	Pia	Aaah, we have to count! … One, two, three. ((Turns her gaze to May))

Fig. 17.1 Illustrates how the children in K1 are positioned when they try to solve the problem

119 May Four, five, <u>six</u>. ((Turns her gaze back at Pia))
120 Pia ((Turns her gaze towards Amy)) Amy, it's your turn to count! ((Points towards Amy))
121 Amy One, two, three, fo[ ]

In line 115, the children keep their gaze at the KT while she asks how many times they have run around the tree altogether. Pia shows three fingers to the KT and the KT looks back at Pia (line 116). The KT then, in line 117, emphasises "all of you" and swipes her hand over the children, but is still looking at Pia. In line 118 it seems that Pia gets an idea of how to solve the problem, because she immediately turns her gaze to May and says "Aaah, we have to count". To use Radford's (2015) terminology, Pia gets an idea (a thought) which is still pure possibility (the 'feeling' of a possible counting strategy), and which she has to put into motion. To put the thought into motion, that is to transform the idea into materialised thinking, she must interact with the other children. The idea includes the other children, because each child represents their runs around the tree and the mathematical thinking can (only) be materialised through their joint activity. Since the idea is pure possibility there is a risk to fail, and to succeed Pia is dependent on the other children's loyalty and their persistence to 'work out' the idea. Pia then, still in line 118, turns her gaze to her own fingers and starts to count "one, two, <u>three</u>". As she counts her third finger, she holds on to it and simultaneously moves her gaze to May. Both the gaze and the word emphasis address May as the next speaker. In line 119, May continues the initiated pattern, which indicates that she has got the 'feeling' of the idea, but which is still pure possibility that is about to be materialised. May looks down at her fingers while she counts "four, five, <u>six</u>", and as she counts her third finger she holds on to it and simultaneously moves her gaze back to Pia. May's gaze and word emphasis address Pia as the next speaker. Pia has already counted and therefore she turns her gaze further to Amy (line 120), but Amy has not paid attention to the ongoing interaction. She has been examining something on the ground and does not recognise that Pia has turned her gaze to her. Pia then approaches Amy verbally and says, "Amy, it's your turn to count!" and simultaneously points eagerly towards Amy. Amy then recognises that it is her turn to count and responds, "One, two, three, fou[ ]" in line 121. Since Amy has not paid attention to Pia and May's previous interaction, she has not recognised the pattern of the counting. She has recognised neither that she has to continue on seven, nor the rhythm of the counting. Amy starts to count from one and is about to continue further from three.

Pia recognises that Amy does not continue the same counting pattern as she and May initiated, and she interrupts Amy.

122 Pia No, not like that! … One, two, <u>three</u>! ((Pia counts slowly and keeps her gaze at Amy while she counts))
123 Amy One, two, <u>three</u>. ((Amy keeps her gaze at Pia while she counts))
124 Pia Ahrr. … ((Pia sounds a bit irritated, and then she turns her gaze to May's fingers and points at May's index finger))
125 May But I <u>have</u> already counted! … ((May sounds a bit irritated, she looks back at Pia with a resigned facial expression)) …
126 Pia But wait. Then we have to count one more time, since Amy counted one, two, three.
127 May One, two, <u>three</u>. ((then she turns her gaze at Pia and pokes Pia's hands)) Your turn!

128	Pia	((Pia looks down at her fingers, and holds on to her index finger for a while, before she starts counting)) … Four, five, six. ((Pia turns her gaze further to Amy))
129	Amy	Seven, eight, nine. ((Amy turns her gaze back to Pia))
130	Pia	((Pia turns her gaze further to Adam))
131	Adam	((Adam blushes, and looks down at his thumb)) …
132	Pia	Your turn! ((Pia points at Adam))
133	Adam	OK … Eleven, twelve, thirteen.
134	May	Thirteen altogether! ((May turns her gaze to the KT and smiles))
135	Pia	Yes. Thirteen!

In line 122 Pia interrupts Amy and says “No, not like that!”, and expresses both verbally and non-verbally (with a resigned facial expression), that Amy did not count as anticipated (in accordance with the initial idea). Then, still in line 122, Pia makes an attempt to correct Amy. She counts her fingers slowly and distinctly, “one, two, three”, with marked stress on “three”, while she keeps her gaze at Amy. By counting slowly and distinctly, Pia emphasises the rhythm in her counting—Amy is not supposed to count or say more than three counting words. It seems that Pia also tries to prompt Amy to count further, by keeping her gaze at Amy while she counts. In line 123, Amy imitates Pia’s actions. It seems that Amy understands the importance of the rhythm; however, she does not recognise that she has to count further. She counts slowly “one, two, three” while she keeps her gaze at Pia as if she needs Pia to confirm, accept or correct her.

Amy’s counting is still not in line with Pia’s initial idea, and in line 124 Pia expresses her frustration both verbally and non-verbally. In frustration she points at May’s index finger (which is difficult to understand why she did; perhaps it was just an attempt to keep the activity moving somehow). In line 125, May expresses, also a bit frustrated, that she has already counted and turns her gaze back to Pia with a resigned facial expression. This action does not really move the activity further. There is a pause in the interaction, where none of the children do anything, and the activity could have stopped at this point. However, it seems that Pia understands that something needs to be done, and she suggests, in line 126, that they start over. May accepts the idea and immediately starts to count from one, in line 127, and in the same manner as earlier she turns her gaze to Pia while she holds on to her third finger and says “three”. Again, the word emphasis and gaze address Pia as the next speaker. In line 128, Pia looks down at her fingers and holds on to her index finger for a while before she starts counting further from three, “four, five, six”. She seems concentrated, as if she wants to do it right and ensure that the activity moves forward in the desired direction, hence in accordance with the initial idea. As Pia counts her third finger, she holds on to it and moves her gaze further to Amy while she says “six”. This time Amy has payed attention to May and Pia’s counting strategy; she recognises the counting pattern (counting further, but not more than three numbers) and, in line 129, Amy counts further without hesitation. When she counts her third finger, she turns her gaze back to Pia and addresses Pia as the next speaker again. In line 130 Pia recognises that it is her turn, but without speaking she just turns her gaze further to Adam and addresses Adam as the next speaker. Adam has paid attention to the ongoing activity, but he has not yet contributed. Adam blushes as if he feels pressured. All the others have counted, and it is only him left. In addition, he knows that this is the second attempt to solve the problem, and the others would probably be disappointed if he failed. In line 132 Pia says "Your turn!" and expresses that she is impatient for him to count. In line 133 Adam says "OK" and after a little pause he counts "eleven, twelve, <u>thirteen</u>". Although he skipped counting 'ten', he still counted further, and the other children seem satisfied and accept thirteen as the final answer. May and Pia state, in line 134 and 135, that thirteen is the correct answer to the problem.

In most of the segment, gaze is used to organise turn-taking by addressing the next speaker (in 15 out of 21 turns). In many of the turns, gaze is used in concert with word emphasis (line 118, 119, 122, 123, 127, 128 and 129), typically in the turns where the current speaker counts and then prompts another person to count further. In some cases, gaze is also used in concert with a direct verbal prompt (line 120, 127 and 132), for example when the person being addressed does not pay attention or when the current speaker is impatient. Although turn-taking in the example above is mainly organised by addressing the next speaker, turn-taking is also organised by self-selecting in some cases (in lines 116, 122, 127, 132, 134 and 135). Although these turns have different reasons for being self-selected, they are still used to move the activity forward in some way. Or, as in line 122, the self-selected turn is used to move the activity in a different direction. In line 122, Pia interrupts Amy because Amy is not following the anticipated direction of the activity, as she does not act in accordance with the initial idea. Pia must re-direct Amy, and try to make her realise the initial idea.

After this segment, the KT makes another attempt to solve the problem together with the children and ensures that they get the correct answer. The KT asks the children to find as many pinecones as they have run around the tree and then to put them in a pile. To solve the task, they count all the pinecones that lie in the pile together.

17.4.2 Segment 2.1, from Kindergarten 2 (K2)

In K2 the children play 'The Balloon Play". The children are working in small groups to solve the problems. In one of the problems, the children are supposed to count how many fingers they have altogether on their hands. A girl named Lily immediately starts to solve the problem. Figure 17.2 illustrates how the children are positioned when they try to solve the problem.

6	Lily	Ten, twenty, thirty, forty, fifty … We have to count as well. … One, two [ ]
7	Eva	That is a lot slower
8	Lily	[ ] three, four, five then you have to count mine. ((Touches Mia's hand))
9	Eva	But we know that this is five ((points at Mia's right hand))
10	Mia	This is ten altogether ((Mia puts both her hands out to each side))
11	Eva	Yes
12	Lily	Ehm … this is ten … ((Lily counts Mia's right hand together with her own left hand as ten, and then continues on Mia's left hand and further to Eva's hands)), eleven, twelve, thirteen, ((she continues counting from thirteen to twenty-nine)), twenty-nine

Fig. 17.2 Illustrates how the children in K2 are positioned when they try to solve the problem

13	Mia	Twenty-nine! … It's twenty-nine! ((She turns her gaze to the KT))
14	Lily	Or, maybe not… ((Starts to count her own fingers mentally))
15	KT	Is it twenty-nine? ((The KT, who stands a little aside, recognises that the children do not find the correct answer))
16	Eva	Five, ten, twenty, thirty, forty, fifty ((Eva counts "five, ten" on Mia's hands and then "twenty, thirty" on Lily's hands and then "forty, fifty" on her own hands)).
17	Lily	((Lily continues to count by ones, and she counts her own ten fingers three times)) It's twenty-eight! We got twenty-eight!! ((She turns her gaze to the KT))

In line 6, Lily starts quite 'spontaneously', however a bit careless, to count by tens. She uses the correct counting words, but she does not really point at any fingers or hands when she counts. But then she changes her mind, and from the utterance "we have to count as well", it seems that she doesn't really think of counting by tens as a satisfying strategy to solve the problem. Perhaps she just 'plays' with the counting words (ten, twenty, thirty, forty, fifty) without really trying to solve the problem. But when she considers how to solve the problem, she chooses to count by ones. In line 7, Eva self-selects her turn by interrupting Lily's counting, and comments that Lily's strategy is a lot slower. Lily ignores Eva's comment and continues her counting by one strategy in line 8. Lily counts five fingers on Mia's right hand, and then she prompts Mia to count hers. She addresses Mia by the address term 'you' and a corresponding touch on Mia's hand. In line 9, Eva again self-selects her turn, this time in a suitable transitional-relevance space, and comments that they know that there are five fingers on Mia's right hand. Although Mia was addressed by Lily in line 8 to continue her strategy, she does not follow Lily's suggestion. In line 10, Mia states that it is ten fingers altogether on her two hands. She puts both her hands out to the side, which indicates that it is 'obvious' for her. Eva agrees with Mia in line 11 and confirms that it is ten fingers on two hands. In line 12, it seems that Lily accepts Eva and Mia's statements because she confirms that there are ten fingers on two hands (her left hand and Mia's right hand). And then she uses that derived fact to count further by ones. Lily continues with eleven on Mia's left hand, then she continues from sixteen on her own right hand, and then she continues from twenty on Eva's fingers. Eva and Mia are watching Lily's hands while she counts, and thus maintain joint attention. This indicates that everyone is now satisfied and support the strategy, and thus it is *their* strategy not only Lily's strategy, although it is Lily who counts.

None of them recognises that Lily makes a mistake, as she skips a finger when she counts eighteen. This results in an incorrect answer, as they end up with twenty-nine but should have had thirty. Mia accepts twenty-nine as the solution in line 13; however, Lily seems to doubt that the solution is correct. It is difficult to say whether she doubts the solution because she has an idea of what the answer should be or because she doubts the strategy that they used. Anyhow, in line 14 she starts to mentally count by ones as if she wants to check the answer.

The KT, who has helped another group of children and therefore stands a little aside, recognises that the children do not get the correct answer. In line 15 the KT asks "is it twenty-nine?" Lily continues to count by ones using her own fingers, and Eva in line 16 tries to use counting by fives or tens, but she mixes the two counting sequences. Eva counts a bit 'sloppy' without actually pointing at any hands or fingers. Again, it seems that she 'plays' with the words, but she is not really able to use it as a strategy to solve the problem. Mia partly focuses on what Eva does and partly focuses on what Lily does.

In the segment above, all turns are self-selected turns. In line 8, Lily is addressing Mia as the next speaker by using "you" and by touching Mia's hand; however, Mia does not respond to Lily's request. Instead, Eva takes the turn in line 9. Except from Eva's interference in line 7, the other self-selected turns are taken in transitional-relevance spaces. The joint activity, at least in the beginning, is characterised by disagreement which is identified by the way that the children interrupt each other. But the disagreement is not necessarily unproductive disagreement. The turn-taking is nevertheless moving the activity forward and gives possibilities to recognise diverse ways to solve the problem.

17.4.3 Segment 2.2, from Kindergarten 2 (K2)

In line 14 in the segment above, Lily starts to mentally count by ones using her own fingers. Simultaneously as Lily counts her own fingers, Eva tries to count by fives or tens, and Mia is partly focusing on what Eva does and partly focusing on what Lily does. The KT recognises that the group has problems to keep joint attention and to collaborate to solve the problem, and thus she interferes:

18	KT	Hmm … if you Lily, put your hands out. And you Mia. And then I. Maybe you can count how many fingers we have altogether Eva?
19	Eva	One … Emm … Five, ten, fifteen, twenty … No …
20	Lily	Yes. ((Lifts her left hand a bit up in the air)) Twenty
21	KT	Twenty, and then … ((Turns her gaze to Eva))
22	Eva	Thirty, forty ((points at Mia's right and left hand respectively))
23	KT	Is it thirty after twenty? … Twenty-one … ((she points at Mia's little finger when she says "twenty-one" and then moves her pointing finger to Mia's ring finger))
24	All	Twenty-two, twenty-three, twenty-four, twenty-five, twenty-six, twenty-seven, twenty-eight, twenty-nine, thirty ((the KT points at each of Mia's fingers respectively))
25	KT	Thirty ((whispers)) … If we take away the thumbs? If you take away your thumbs, how many fingers have you got then?

In line 18, the KT physically (but gently) takes Lily and Mia's hands and organises them so they are easy to operate on. Then she asks Eva to count. The KT organises whose hands should be counted, how the hands should be placed and who is going to count. The KT addresses Eva as 'the counter' and in line 19 Eva starts to count from one, but then she changes her mind and starts over counting by fives. First, she counts the KT's hands (five, ten) then Lily's hands (fifteen, twenty), and she is about to continue on Mia's hands, but then she stops and says "no". Probably she stops because she can't remember what comes after twenty in the counting sequence. Lily has paid attention to Eva's actions, and in line 20 she interferes and says "yes" and then lifts her hand and says "twenty". Lily confirms that she agrees with Eva's way of counting until that point and prompts Eva to continue to count from twenty. Lily does not offer any suggestion for how to continue, so Eva does not respond to Lily's actions. In line 21 the KT also repeats "twenty", while she holds on to Lily's left hand, and then she moves her hand to Eva's right hand and says, "and then …", which prompts Eva to continue counting. The KT prompts Eva to continue, but Eva still needs to figure out what comes after twenty. In line 22, Eva continues to count, but she is not consistent with her previous counting strategy, which was counting by fives. Instead, she continues counting by tens. The KT interferes in line 23 and asks, "is it thirty after twenty?" This might be a confusing question because thirty comes after twenty if you think of the number line, and it comes after twenty if you count by tens. However, thirty does not come directly after twenty if you count by fives, and this is, I think, what the KT means. The KT has a little pause, which might indicate that she considers how to continue, and then she initiates counting by ones by saying "twenty-one" and then points to the next finger which is about to be counted. The way that the KT initiates counting by ones is illustrated in Fig. 17.3. Then, in line 24, they all count the rest of the fingers by ones and solve the task together. In line 25, the KT confirms the answer as the correct answer, and then she initiates another problem for the children to work on, which is of the same type.

The nature of the turn-taking changes completely in line 18 when the KT interferes in the children's group work. In segment 2.1 (lines 6–17), the children self-select turns. From line 18 it is the KT that organises the turn-taking (except from line 20, where Lily self-selects her turn). The KT both self-select her turns and she addresses the next speaker, and the activity becomes quite structured. In line 18, the KT organises whose hands should be counted, how the hands should be placed and she addresses Eva as the next speaker. The KT gives Eva the role as 'the counter'. Eva accepts being addressed as the counter, and starts counting by fives in line 19. Lily self-selects her turn in line 20 and then she invites Eva to continue, but Eva does not accept Lily's invitation. Then the KT self-selects her turn in line 21 in order to move the activity further and again she addresses Eva as the next speaker by turning her gaze to Eva. Eva accepts being addressed and tries to continue counting. In line 23 the KT again self-selects her turn, but this time she does not turn to any particular child. Instead, she prompts everyone to count together.

Fig. 17.3 Illustrates how the KT in line 23 initiates counting by ones

17.5 Discussion

As mentioned above, turn-taking is used by the participants to adjust and move the activity further. Since thinking, in Radford's (2015) conception, is thought put into motion, turn-taking is one way that children put ideas into motion. To investigate the organisation of children's turn-taking is therefore a way to understand how children's (and the KT's) mathematical thinking is materialised through their joint activities. The discussion is organised around two issues: (1) The characterisation of children's turn-taking in the three segments reported in the result section, and possible reasons for the various ways in which children (and the KT) organise their turn-taking, and (2) The role of children's organisation of turn-taking in the materialisation of children's mathematical thinking in the joint activity.

17.5.1 The Characterisation of Children's Turn-Taking: Similarities and Differences in the Three Segments

The examples provided in the result section illustrate diverse ways that children (and a KT) organise turn-taking in small groups to solve addition problems. The turn-taking in the three segments are quite different. In the example from K1 there is mainly turn-taking by addressing the next speaker, and the children seem to agree on a common strategy. In segment 2.1 there is mainly self-selected turn-taking, and the children do not immediately agree on which strategy to use. In segment 2.2 it is mainly the KT that organises the turn-taking by taking and addressing turns. The structure that the KT brings into the activity by organising the turn-taking helps the children to focus on a common strategy and to re-establish joint attention.

There are probably several reasons for the differences in the turn-taking. However, one reason may be the way that the children are positioned and how the problems are formulated, which indicates that turn-taking is strongly dependent on context (cf. Lerner, 2003; Mondada, 2007). In the first example from K1, the children are working on a problem that invites all children to participate. All children are asked to run around the tree. The problem does not explicitly ask the children to represent their runs by their fingers, but this is, I hold, a likely strategy for modelling the problem. When the children are adding up their runs, they stand a bit apart from each other, and this may be a reason why they address the next speaker by gaze and word emphasis. The children use the fingers on one hand to represent their runs and the fingers one the other hand to count. And because they stand a bit apart from each other, they lift their gaze and emphasise the last counting word to address the next speaker. This way of taking turns may be regarded as face-to-face interaction (cf. Mondada, 2007), because the current speaker and the next speaker keep eye contact in the transition of turns.

In segments 2.1 and 2.2, the children are also working on a problem that invites all children to participate. The problem asks the children to count the fingers on all the children's hands. In segment 2.1 the children are standing quite close to each other while they solve the problem, which makes it possible to touch one another's hands for addressing the next speaker. Since they are standing quite close, they also have the possibility to count each other's hands and/or fingers (not only one's own fingers as in K1). This way of taking turns may be considered as side-by-side interaction (cf. Mondada, 2007), since the children do not (or very seldom) keep eye contact in the transition of turns (they usually kept their gaze on their hands/fingers). In segment 2.2, the KT interferes in the children's group work. Just before the KT interferes, in line 16 and 17, the children are not working together in a joint activity; rather they participate in separate activities. The KT recognises that the children have problems in collaborating, and she interferes to re-establish the joint activity. To achieve this re-establishment, there needs to be some structure to build the joint activity around, and the KT brings the necessary structure into the activity so that the children are able to focus on a common strategy again and act in a joint activity.

Another reason for the different ways in which children (and the KT) organise their turn-taking may be children's understanding of the problem and the degree of agreement of how to solve the problem. In segment 1 it seems that Pia gets a special organising role. Turns are often coming back to Pia, even when it is not her turn to count. Perhaps this is because Pia was the one who had the original idea for how to solve the problem. The idea was, in the beginning, pure possibility and Pia needed the other children to participate in a joint activity to put the idea into motion (cf. Radford, 2015). When the idea is put into motion, it seems that all children take up Pia's initial idea, however with various awareness of it, and through the joint activity the children's thinking becomes materialised. Because Pia is most likely the one who is the most aware of the idea, the other children trust Pia to organise the turn-taking to increase the possibility for the idea to be actualised.

In segment 2.1 there is disagreement, at least in the beginning, on how to solve the problem. Lily seems to focus on the answer and the 'safest' strategy to solve the problem. In the beginning of the segment Lily 'plays' with the words in the counting sequence by counting by tens. However, it seems that she realises that she is not able to use that strategy to solve the problem and she changes her strategy. Lily wants to count by ones, which is probably the strategy that they have used the most and which is then the 'safest' strategy to solve the problem. Eva and Mia's perspectives are perhaps a bit different. It seems that they are concerned about counting by fives or tens or that they at least use derived facts to solve the problem. During the activity the children struggle to tune in to one another to establish a common strategy. In the end of segment 2.1, the children compromise and combine the two strategies. They use a derived fact (there are ten fingers on two hands) and then they count further by ones. The way that the children compromise illustrates the flexibility of their thinking. Instead of accepting one of the suggested strategies (and discard the other), they compromise and combine the strategies into one common strategy.

In segment 2.2 the children have lost joint attention and work on separate activities. The KT takes a leading role in re-establishing the joint activity. The activity becomes quite structured, where the KT organises most of the turn-taking and where there is little room for disagreement. The KT does not decide which idea should be materialised (at least not initially), rather, she recognises the idea that the children want to put into motion and helps them materialise their thinking. The KT gives Eva the role to initiate an idea, and Eva initiates to count by fives. The structure that the KT brings in by taking and addressing turns helps the children to re-establish joint attention and to work in a joint activity again. Although it is Eva who is given the role as 'the counter', both Lily and Mia pay attention to Eva's actions, and the activity must therefore be recognised as a joint activity and materialised thinking as their joint thinking.

In all segments, whenever there is disagreement, we find that children interrupt each other, and it seems that interruption is not only moving the activity forward, but also adjusting the direction of the activity so that the children may focus on a common strategy.

17.5.2 The Role of Children's Organisation of Turn-Taking for the Materialisation of Children's Mathematical Thinking in the Joint Activity

The previous paragraph pointed to ways in which children organise turn-taking and some possible reasons for the various ways in which the children (and the KT) organised their turn-taking. This paragraph is devoted to a discussion about what role children's organisation of turn-taking plays in the materialisation of children's mathematical thinking. Since movement is the most striking feature of the (mathematical) thinking (Radford, 2015), the way that the children organise their turn-taking (through a complex coordination of various semiotic means) in order to move the activity forward reveals children's joint mathematical thinking and the way mathematical ideas are put into motion (Radford, 2013, 2015).

The semantic structure of the two problems may be considered as additive because they consider only one quantity (the number of runs around the tree or the number of fingers). However, the problems give possibilities for multiplicative thinking because the children are asked to add equal groups (cf. Anghileri, 1989; Mulligan & Mitchelmore, 1997), and as argued in the result section, children's joint activity (their complex coordination of various semiotic means) in the two examples do bring to life layers of multiplicative thinking (cf. Radford, 2013, 2015).

Segment 1 is particularly interesting for understanding how multiplicative thinking may be materialised through a joint activity. In segment 1 the children use gaze and word emphasis to address the next speaker (on every third number), and their coordinated turn-taking is especially important for materialising a specific feature of the thinking, namely rhythm. First of all, rhythm indicates that there is a regularity and a continuation. This component of rhythm is what Radford calls 'theme' (Radford, 2015). Already in line 118, Pia sets out with a rhythmic counting of three ("one, two, three"), which is a regular sequence of three counting words. However, the stress on "three" indicates that "three" is the end of the sequence and which may be followed by another regular sequence of three counting words. The rhythm gives possibilities for the other children to imagine what comes next and mediates that the sequence could be continued. The rhythmic counting indicates that *something* should re-appear, which in this case is three counting words. Through the following turn-taking a rhythmic counting sequence is then released: 1, 2, 3—4, 5, 6—7, 8, 9—11, 12, 13 (which should have been 10, 11, 12).

Rhythmic counting is emphasised as important in the transition from additive reasoning to multiplicative reasoning because it reveals the fundamental group structure of multiplication (Anghileri, 1989; Mulligan & Mitchelmore, 1997). In Radford's (2013, 2015) conception, however, rhythm (rhythmic counting) is embedded in the mathematical thinking itself, and is not merely a means for reaching 'another form' of thinking (in this case from additive reasoning into multiplicative reasoning). An important element of rhythm is 'prolongation', through which the phenomenon itself is expressed (Radford, 2015). As Radford (2015) argues, "They [elements of rhythm] are central features of the mediation of thought and the manner in which it becomes actualised in the students' reflections and actions. They are part of the materiality of thinking" (p. 78). Through the ongoing activity in segment 1 the multiplicative structure (at least some layers of the multiplicative structure) is mediated by children's joint rhythmic counting. The phenomenon that is materialised through this rhythmic counting sequence is repeated addition of three (3 + 3 + 3 + 3) and the number sequence 3, 6, 9, 13 (which should have been 12), both of which are elements of multiplication. Turn-taking is a way to structure the activity and to move the joint activity forward. It is turn-taking (children's coordination of various semiotic means) which releases the rhythmic counting that materialises the joint multiplicative thinking. To what extent children are aware of the multiplicative structure in their joint activity is of course an important consideration to make. The children are not yet able to multiply with number facts, that is to see that there are four groups of three, and then calculate $4 \times 3 = 12$. However, the way that the children are able to follow the same pattern (at least in the second attempt), indicates that they are aware of some layers of generality (Radford, 2010), that is some layers of the multiplicative structure in their interaction.

Above, I argued that the initial problem gives rise to possibilities for multiplicative thinking, because the children are asked to add equal groups. But the problem itself is not enough to materialise multiplicative thinking. After segment 1, the KT makes another attempt to solve the problem together with the children and ensures that they get the correct answer. The KT asks the children to find as many pinecones as they run around the three. Each child finds three pinecones, and on the request of the KT they gather them in a pile. To solve the task, they count all the pinecones which lie in the pile together. This solution strategy does not materialise the group structure of multiplication. The rhythmic counting disappears, so the problem itself is not enough to bring forth rhythm and a multiplicative pattern. The way that the children are placed (their positional location in space), the available artefacts and the way that the children organise the turn-taking is important for the materialisation of their multiplicative thinking. The mathematics is embodied in the children's use of word emphasis, gaze, gestures and their positional location in space, and these components are essential for the flow of thinking.

In segments 2.1 and 2.2 the turn-taking does not materialise mathematics in the same manner as in segment 1, where the turn-taking itself gives rise to rhythmic counting and thus materialises a multiplicative structure. However, the children seem concerned about the group structure of their fingers. They know that there are five fingers on one hand and ten fingers on two hands. In the beginning of segment 2.1 there is disagreement on what strategy to use for solving the problem. The disagreement in not necessarily unproductive disagreement. The way that Eva, May and Lily compromise at the end of segment 2.1 illustrates the flexibility in these children's mathematical thinking. The children in a flexible manner combine two strategies into one joint strategy. In segment 2.2, the KT helps the children to re-establish their joint activity (which in line 16 and 17 was split), and to organise their thinking. Eva, who was pointed out as 'the counter', initiates to count by fives. Again, rhythm seems to be an important part of the flow of thinking. Rhythm helps Eva to count by fives (in line 19) and later to count by tens (in line 22). In line 23 the KT problematises Eva's counting after twenty, and she invites all the children to count together, but she also changes her counting strategy. Eva initiated counting by fives, but in line 23 the KT initiates counting by ones. Again, the joint activity (strongly organised by the KT) materialises a flexibility. Together they flexibly combine two strategies and end up with a satisfactory solution.

Although the turn-taking itself does not materialise rhythmic counting in segments 2.1 and 2.2, rhythm is still an important feature in children's mathematical thinking. Already in line 6 Lily 'plays' with the counting sequence "Ten, twenty, thirty, forty, fifty". Lily does not point at any specific hands or fingers when she counts, however she still, somehow, rhythmically points to imaginary objects while she counts. After this, both Mia and Eva seem concerned about using the group structure of their fingers and their hands to solve the problem. In line 16 Eva makes another attempt to count by fives and tens. She 'plays' with the counting words "five, ten, twenty, thirty, forty, fifty" while she rhythmically points to imaginary objects. Again it seems that she is 'playing' with the counting words and not emphasising the correct use of words. The way that Eva changes her mind in line 19 (she starts counting by ones and then she changes her mind and starts counting by fives) indicates the 'fascination' she has for this type of counting. Although the children do not know exactly how to solve the problem with counting by fives or tens, the way that they ‘play’ with the rhythmic counting illustrates the importance of rhythm in their counting and also their ‘fascination’ for this type of counting.

As emphasised above, rhythm is an essential part of children’s mathematical thinking in both segment 1 and segments 2.1 and 2.2, and perhaps valued or prioritised higher than a correct sequence of number words or a correct solution. For example, in segment 1 Adam is not able to use the correct counting words (he counts 11, 12, 13 instead of 10, 11, 12), but he is true to the rhythm. It seems that because Adam is true to the rhythm, the other children accept the last counting word as the answer. This indicates the strong position that rhythm has in children’s thinking. Rhythm seems more valued by the other children than the answer itself. It is as if the children trust the rhythm and the regularity that it creates, and therefore they trust the answer that it gives (although it is incorrect). An important aspect of this interpretation is that all the children in this group are able to count ‘nine, ten, eleven, twelve’ in other settings, so to skip ‘ten’ is not a common problem for the children when they count.

To summarise, the findings from this study suggest that children’s various ways of organising turn-taking give rise to different ways in which their mathematical thinking is materialised. Segment 1 illustrates, in particular, how multiplicative structures emerge from the way that the children organise their turn-taking. In segment 1 the children stand a bit apart from each other which gives rise to a face-to-face exchange. The children address the next speaker by gaze and word emphasis, and the turn-taking reveals a rhythmic counting of groups and materialises layers of multiplicative thinking. In segments 2.1 and 2.2 the children stand closer, and the turn-taking may be identified as a side-by-side exchange. Segment 2.1 is characterised by disagreement, however not necessarily unproductive disagreement. The children’s turn-taking and disagreement result in a compromise, which illustrates how the children flexibly combine two strategies to solve the problem. In segment 2.2 the KT organises most of the turn-taking, which helps the children to restore the joint activity. Again the children (and the KT) flexibly combine two strategies to solve the problem. Rhythm is important in all segments, although it is not the turn-taking itself that releases this rhythm in segments 2.1 and 2.2. In segments 2.1 and 2.2 rhythm helps the children count by fives or tens, and the way that the children ‘play’ with the rhythmic counting of fives and tens indicates their ‘fascination’ for this type of counting. The study also illustrates how children’s turn-taking, and thus children’s mathematical thinking, seems to be dependent on contextual features like the formulation of the problem, available artefacts and the children’s positional location in space. The implications that can be drawn from this study is that KTs can promt children’s early multiplicative thinking by organising them in small groups and asking them to solve various equal groups addition problems with their hands and fingers.

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Chapter 18
Mathematical Activity in Early Childhood and the Role of Generalization

Marianna Tzekaki

18.1 Introduction

The importance of early childhood mathematics education is indisputable. Knowing that children come to school with relevant and interesting mathematical ideas arising from the use of mathematical elements or mathematical processes in their everyday life, education enhances rich programs with games, problems, playing activities, or constructions, etc. related to a wide range of mathematical concepts (spatial approaches, shapes, patterns, measures, numbers, probabilities, etc., see Aubrey & Godfrey, 2003; Baroody, 2004; Battista, 2006; Clements, 2004; Levenson, Tirosh, & Tsamir, 2013; Levine, Ratliff, Huttenlocher, & Cannon, 2012; Papic, Mulligan, & Mitchelmore, 2013; Sarama & Clements, 2009; van den Heuvel-Panhuizen & Elia, 2011). Every year, a huge amount of material is gathered by surveys, analyses, projects, suggested teaching approaches, applications, teaching materials, and related technology. Therefore, children, working with all these activities, artifacts, and technology, would be expected to develop mathematical ideas relevant and appropriate for their age and their way of thinking.

Nevertheless, there is one important question: do all these applications really make children act and think mathematically, and thereby, depending on their age, start developing mathematical meanings? Recently researchers have raised some concerns about young children's mathematical thinking in games or other tasks, and also about teachers' focus on children's mathematical development (cf., van Oers, 2013). They express reservations as to whether youngsters' ideas derived from these tasks or from their everyday life with mathematical objects (like patterns, numbers or shapes) are related to mathematical concepts (cf., Lüken, 2018). They argue that these acts or ideas appear to be "mathematics" from outside (that is adults' understanding), but in fact are not directly connected to mathematical knowledge. For

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example, there were questions on whether children act mathematically while reproducing, comparing or following the sequence in a patterning task (Lüken, 2018) and discussions about the criteria that make playing a mathematical activity (Dockett & Perry, 2010; Helenius et al., 2016; Holton, Ahmed, Williams, & Hill, 2001; van Oers, 2010). van Oers (2013) contends that everyday situations in which children find a result by counting or solve a simple problem, thus showing "mathematical behavior," do not ensure that they are able to see this knowledge in a more abstract or general way and use it in other problems. Earlier research findings (Carraher, Carraher, & Schliemann, 1985) indicated a significant divergence between the everyday use of numbers (street mathematics) and their conceptualization in mathematics education. Carraher and Schliemann (2002), expanding on the relationship between everyday and academic mathematics, examined the connections between concrete and abstract, whether local or general. They suggested activities in the classroom organized to shift students' thinking from a local to a more general context with a wide variety of (concrete) situations that would allow them to abstract relations and, thus, mathematical concepts.

In general, it would appear necessary to take a deeper look at the characteristics of "*genuine mathematical activity*" in young children's playing, problem solving or working with realistic or constructed situation and, therefore, to examine important elements that ensure that this activity is supportive to children's mathematical development. Although research on early mathematics has significantly advanced (e.g., English & Mulligan, 2013; Sarama & Clements, 2009), there are still few references regarding clarifications or statements on the meaning of mathematical development expected at this age. In this chapter, we will not present a research study, but based on research findings we will first attempt to specify the characteristics of mathematical activity in early years, and then to emphasize the importance of generalization as an essential component of this activity.

18.2 Analysis of Mathematical Activity in Early Years

In a previous study (Tzekaki, 2014), addressing the issue of early mathematical activity and attempting to clarify it, we started synthesizing a number of similar or complementary approaches that describe mathematical activity in general. According to these approaches, a mathematical activity is a special kind of endeavor that follows a specific way of processing problems or situations, with creative and flexible reasoning, documentation, symbolization, reflection, and generalization. Thus, mathematical activity is considered a situation in which a person or a group work to achieve something, keeping certain characteristics, properties, or behavior, while a task is a piece of this work to be undertaken or done. For Freudenthal (1983), this activity is *modeling of real situations,* for Brousseau (1997) solving a *situation-problem*, in the *epistemological triangle* of Steinbring (2005) it is linking situations and signs with concepts, for Noss, Healy, and Hoyles (1997) it is also making *connections*, and for Ernest (2006) it is a process of *symbolization.* Moreover, doing mathematics is problem-solving or modeling, connecting and representing, but also a way of thinking that transfers solutions, models or connections from specific situations to a more general content (Radford, 2006). These approaches that effectively show the complexity of seeking answers about doing mathematics make the task of combining them in order to find the specific characteristics of mathematical activity in early years even more complicated.

Young children come to school with everyday or intuitive ideas about numbers or quantities, shapes, spontaneous patterning, or measurement (Sarama & Clements, 2009). Undoubtedly, this initial knowledge could be considered as a basis for the development of further mathematical ideas, but there are many questions related to its nature. Could this knowledge be considered as mathematical, or is it just common, social, perceptual, and kinesthetic, related to specific experiences or needs but with no apparent correspondence to mathematical concepts? Many everyday situations involving playing or problem-solving or constructing experiences are meaningful for preschoolers and add to their previous knowledge, but usually young children have no motivation to think about them, to express their ideas, or to transfer them to new situations and view them in more abstract and general ways (Greenes, Ginsburg, & Balfanz, 2004).

Therefore, whether young children think or act mathematically or develop mathematical concepts depends on what could be considered as genuine mathematical activity for early years. Thus, in alignment with the previously presented approaches, children's activity in early years could be characterized as mathematical if it, first, concerns modeling of real situations, or solving different problems, but also linking or connecting or symbolizing elements and being transferrable to a more general content. Some typical examples of usual preschool activities with no wider application could perhaps clarify this aspect.

Composing shapes: In this task, children combine shapes of different sizes and orientation to produce a composite configuration given by its outline. Composing shapes to form a picture (as shown in Fig. 18.1) is a challenging task, encouraging preschoolers to reflect and anticipate the component pieces, to recall different shapes, to analyze and combine parts, to perceive and compare the attributes of shapes, and to make rotations and other transformations.

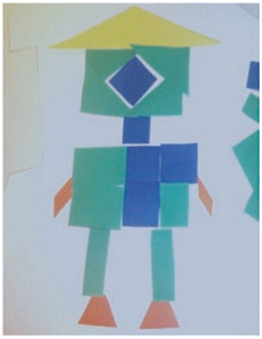

Fig. 18.1 Composing shapes

In short, this task is supporting a rich and rather general approach to geometrical shapes (Sarama & Clements, 2009).

However, research related to these kinds of tasks produced different levels of achievement (see Clements, Battista, Sarama, Swaminathan, & McMillen, 1997). Some children may choose shapes and put them together to fit the outline by trial and error, some are able to see smaller parts of simple shapes to be put together but they relate them mainly with general attributes, while others anticipate different parts of smaller shapes and compose them based on their relationships. These findings confirm that preschoolers very often complete this work without reflecting (e.g., when working by trial and error) or discussing what they did to achieve this result (shape selection, attributes, transformation, etc.). Thus, they do not have the opportunity to take advantage of the ideas involved and, thus, they do not gain relevant and aimed conceptual benefits.

Patterning: Children copy or continue repeating patterns like ABC using blocks (as shown in Fig. 18.2). Dealing with repetitive patterns and identifying common elements and structures in different situations is an important activity that lies at the core of mathematical development since it supports recognition of properties and relationships in different situations (Papic et al., 2013).

However, not every patterning activity is mathematical, and, consequently, in these kinds of tasks children often just reproduce the pattern matching items one by one, without "seeing" or focusing on the pattern's design, while others observe the sequence of different colors rather than the unit of repetition (Lüken, 2018). Without systematic reflection on what they do and how, as well as identification of the unit of repetition and its generalization (Threlfall, 1999), children are limited to imitating rather than showing a higher level of processing as implied by patterning.

Numbering: In various arithmetic tasks, children usually recite the arithmetic sequence (verbal counting) or count quantities of objects, fingers, knocks, etc. (object counting, cf. Baroody, 2004). Dealing with numbers is the most recognizable "mathematical task" of this age, although just working with numbers (even as part of their daily life) does not introduce children to arithmetic understanding or developing number sense. An "easy task" of change, in Nicol, Kelleher, and Saundry (2004), showed that children who generally know how to enumerate a quantity of objects or to give an answer to the question "how many..." had considerable

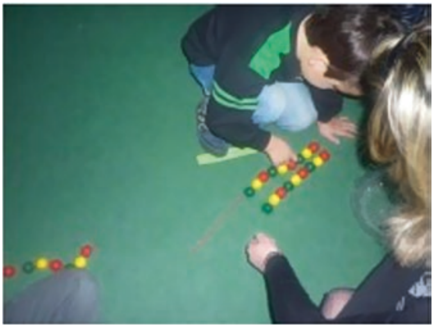

Fig. 18.2 Continuing a pattern

difficulties when asked to "change 8 to 4." They evidently used a counting procedure using blocks or their fingers and recited the number sequence, while many of them were not even able to find the answer. These results also confirm the argument that the simple use of counting in everyday situations is not spontaneously transferred to other situations. Thus, it does not automatically ensure the conceptualization of number sense, relationships, and structure.

Summarizing, the previous examples suggest that children working on a task, solving a problem, constructing or generally dealing with a mathematics-related situation or mathematical objects (such as numbers, shapes, or patterns) do not obviously and automatically develop some mathematical idea. Vygotsky (1934/1962) had, since 1934, been distinguishing between spontaneous and scientific concepts. He argued that while higher levels of understanding have their roots in everyday personal experiences, spontaneous and scientific concepts are dialectically related. Artifacts, social interaction and adults' scaffolding are needed to support the so-called "bottom to top" development of spontaneous to scientific concepts. Otherwise, youngsters remain linked to their specific actions or outcomes, focused on the empirical situation or the concrete material, and may not attempt to generalize or search for a wider explanation and deeper understanding (if they do not need to).

It is common that when children, in order to be driven to a wider understanding, are asked after accomplishing a task to connect their current with other practices, they often recall previous but rather personal experiences.

For example, after completing a spatial pattern such as  children, who were asked "*have you done something like this before?*" responded by saying "*yes, it is a castle, I have done it many times…*" or "*it is a train, I have to do the same…*," showing a connection to some previous experience but not to some other mathematical idea such as finding the pattern's repeating unit. Similarly, in another task preschoolers were provided with four paper squares and were asked to combine them in as many different ways as they could (see Fig. 18.3, e.g., "shape composer" in Sarama & Clements, 2009).

Completing this work with different results, only a few children were able to describe it by saying (with their own way of expressing their ideas) "*we made squares with other shapes, trying to make them differently…*," indicating an initial detachment from the specific situation and understanding at a more general

Fig. 18.3 Combining squares

level. Most of the pupils explained what they did by saying "*we made many houses*" or "*we made houses with many rooms*," without "seeing" a geometrical idea in this construction.

Concluding, in suggestions for early mathematics education, in which tasks, materials, situations, and actions are not obviously "mathematical," as is an arithmetic operation, an equation, or an algebraic expression, teachers are usually contented when children manage to measure, calculate, or find a pattern. However, if we are pursuing an authentic mathematical activity and, thus, the development of initial mathematical ideas, we need to thoroughly analyze children's endeavor in terms of actions and outcomes. For this purpose, combining different views, we can contend that an activity during early age could be considered mathematical if it displays characteristic actions and outcomes corresponding to those of the mathematical activity in general, that is *looking for common characteristics and relationships, recognizing repeating units in patterns and common structures in situations, analyzing and combining parts and unit parts, and encouraging children* to *make connections, express their ideas in words, to represent with signs or other symbols, to explain, justify, intentionally reflect, and generalize* (Helenius et al., 2016; Tzekaki, 2014).

18.3 Development of Mathematical Ideas and Generalization

As mentioned previously, literature related to early years' mathematics education provides many suggestions regarding tasks and material with important pedagogical and educational value (cf., Bryant, 1997; Greenes et al., 2004; Levenson et al., 2013; Papic et al., 2013; Sarama & Clements, 2009, etc.). Despite the volume of research related to early mathematics education, less focus is placed on the examination of the mathematical meaning developed by children or, in this sense, on their abilities to generalize from their experiences as an indispensable part of their mathematical development. Freudenthal (1983) argued that children are expected to manage realistic situations close to their interests and needs but their activity, though resulting from practical manipulation, had to direct them to transform the real objects to mental ones and thus to understand them at a higher level. Without this understanding, no matter how rich and challenging a task might be, it is doubtful whether it supports mathematical development. Many recent suggestions for early mathematics education (cf., Sarama & Clements, 2009) study the gradual development of children's thinking and propose developmental trajectories related to the progression of this thinking at different levels with activities and tasks. For this approach (or other similar approaches), we believe that a deeper insight into whether youngsters, following this succession of tasks, make the necessary connections and form the more generalized idea we are seeking is essential. A deeper awareness is also needed in the teaching practices while implementing these tasks in order to support a development of more general ideas.

A clarification regarding this consideration could be provided by a challenging task (related to 3D shapes) called "the dressmaker." For the needs of this work, children are given a cube and six squares of the same size as the faces of the cube. They are asked to cover the cube with the squares (sticking them together) and thus to "dress" it (Hejny & Jirotkova, 2006). The task is meaningful for them because of its playful constructional character, but also important from a mathematical point of view, as preschoolers, after "dressing" the cube, have in their hands several cube nets, to be examined and compared, without any teachers' intervention. However, what do young children "see" in these constructions more than different "dresses" for their cubes? We venture to suggest that, without other similar tasks that can give some meaning to these "dresses" (e.g., other solids, other nets, comparisons, etc.) but also appropriate discussions that generalize all these results, this interesting task remains without significant conceptual benefits.

In conclusion, reflection and generalization is an integral part of mathematical activity, even in preschool age, since mathematics is the product *of reflection over actions* (Duval, 2000). Thus, educational settings aiming at mathematical development should give to learners' opportunities to generalize (Mason, Drury, & Liz Bills, 2007). According to research findings, young children have the potential to reflect on their actions, to move from a local to a more general level and even to express more generalized statements (Tzekaki & Papadopoulou, 2017). There is a volume of research related to patterns concerning preschoolers' abilities to generalize or express abstract ideas and relations (e.g., Garrick, Threlfall, & Orton, 1999; Papic et al., 2013; Rivera, 2013). Certainly, developing these abilities requires long-term practice and appropriate management by the teachers to encourage children constantly, to "see" beyond the specific results of a task, a game, or a construction, seeking conclusions related to more general ideas.

18.4 A Teaching Approach Encouraging Generalization

As explained earlier, it is important for preschoolers to deal with situations and material related to their needs, interests, and their way of thinking, but in order to develop mathematical ideas they need teaching approaches that support them to extract more generalized ideas from their activity. Duval (2000) emphasizes the fact that mathematical development cannot be derived from actions and practice on particular objects or situations, but demands thinking over these actions, "… to take over the thought processes which enable a student to understand concepts…" (p. 56). It is, therefore, teachers' responsibility to develop a learning community in their classroom that discusses, explains, and justifies actions and outcomes related to their activity, reflects on them, and makes conclusions attempting to form more generalized ideas. More specifically, during or after finishing their tasks, teachers should orient and support children to systematically reflect on their actions and discuss their way of doing them or their methods, explain their decisions and reach some kind of closing remarks (Dougherty, Bryant, Bryant, Darrough, & Hughes Phannestiel, 2015).

These remarks aim at arriving primarily at *a first level of generality* (1) with conclusions drawn by children *out of particular tasks or activities* (e.g., ways of doing things) and related to recognition of characteristics or relationships that concern the specific content or fact. For example, statements such as "*this triangle has three sides*," or "*this pattern is green, red, yellow and again green, red, yellow*" refer to recognition of characteristics of specific shapes or patterns. At this level, the teacher's questions focus children's attention on the outcome or the method used in a(??) (still specific) activity, for example,

– How did you find out that this figure is a rectangle?
– How did you combine these shapes inside this outline?
– What is the design of this pattern?
– How can we change 6 to 3?
– What should you remember if we want to do this measurement again?

Later, at *a second level of generality* (2), children make announcements related to characteristics or relationships but regarding ideas with a wider application. For example, statements saying, "*a square has four sides*" or "*patterns have a design*" or "*3 dots+3 dots make 6 dots*" refer to more generalized approaches. At this level, the teacher's questions shift children's attention from specific to wider characteristics, for example,

– How do we recognize that a shape is a square?
– What do we observe to find the design of a pattern?
– How do we examine if two distances are equal?
– Is 5 and 3 always 8?

Finally, teachers could also target a *higher level of generality* (3) that concerns verbal expressions of rules connected to relevant concepts for this age. This level requires systematic and long-term engagement of teachers and students, but research shows that young children are eventually able to "formulate" even more generalized statements such as (the original verbal expressions of 6-year-old kids were kept) "…(a rectangle) *has upright angles, two same but smaller sides and the others bigger, also same…*," which, in the particular language of 6-year-old children, presents a statement with properties and relationships for rectangles. Similarly, in patterns or measurement children manage to express statements, such as "*…. we see all the design, where it starts, where it ends and what pieces it has...*" (Papadopoulou, 2017, p. 207) or "*…we start from the beginning, we don't put the sticks as they are, but in a row, one after the other… without overlapping…*" (Papadopoulou, 2017, p. 238). For these results, a teacher needs to discuss with children their general conclusions that summarize a series of activities and outcomes, for example,

– How do we identify a shape?
– How do we find the design of a pattern?
– How do we examine the equality of two distances?
– How do we know that 5 and 3 makes 8?

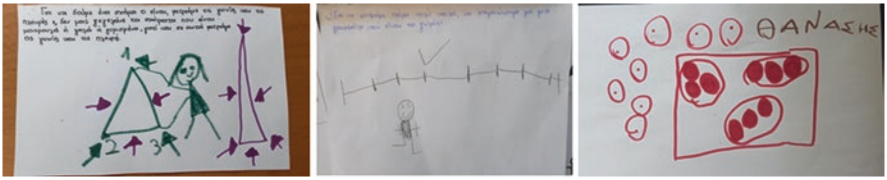

Fig. 18.4 Children's drawings related to "conclusions" about shapes, measurement, and numbers

It is important to underline that the levels of generalization that a preschool child can achieve, by improving thus her/his ability to draw general conclusions from specific tasks or to find common elements in different situations, follow a developmental trajectory similar to those of concepts: from *the specific to the more generalized and finally to the general*. Often, a semiotic activity in which children capture, post, and then use their "conclusions" in the classroom, but also correct or broaden them, can prove particularly effective (Fig. 18.4).

The above drawings, as well the examples that follow, come from an 8-month teaching intervention, the results of which were synthesized to provide an overall image of generalization in early childhood (Papadopoulou, 2017). The next examples are similar to those presented earlier and illustrate the different ways in which children face analogous situations. These differences could be explained by the fact that in the following excerpts the preschoolers are intentionally and systematically seeking more generalized conclusions, as they were working for months in a community with discussions, explanations, and justifications and attempts for generalization.

Composing shapes: Children combine shapes of different sizes and orientation (triangles, rectangles, trapezes) to produce a composite configuration given by its outline (as shown in Fig. 18.5).

After working with shapes in different tasks, the children of this class were gradually able to identify more general elements about shapes, e.g., "…*we count sides and angles to find what shape it is*…." Thus, after completing this composing task, in the discussion about the ways of doing and justifying it, these children, using properties and relationships previously identified, presented the following statements (the original words of the 6-year-olds have been kept):

Child A "…I put them together because their sides are equal…" (identifying equality of shapes' sides).
Child B "… I noticed that the sides are the same and not bigger and smaller…" (similarly) (Papadopoulou, 2017, p. 132).
Child C "…I knew that this was the right shape because, no matter how I turn a shape, it does not change at all, it's always the same shape…." (identifying shape's orientation) (Papadopoulou, 2017, p. 138).

Patterning: The children had to find a missing element in a geometric pattern AABC, as shown in the figure (Fig. 18.6).

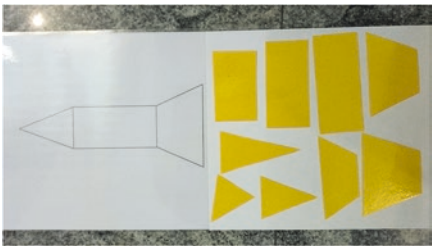

Fig. 18.5 Composing shapes

Fig. 18.6 A geometric pattern

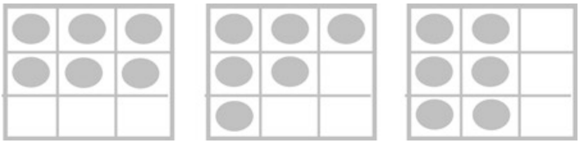

Fig. 18.7 Nine cells cards

Unlike the previous example with patterns, in this case preschoolers, having already worked with similar tasks, did not just match items one by one, but focused on the pattern's unit of repetition. In the discussion on the design of the pattern, they replied:

Child A "…circle, circle, triangle, square and then it starts again…"
Child B "… it starts circle, circle, triangle, and ends square, it is cut here…"
Child C "... I found it because it is circle, circle, triangle (*first repetition, the child presents the pattern rule*) and then here... (*for the second repetition*) I need to put the missing circle..." (Papadopoulou, 2017, p. 163).

Later, the children in this class presented a more general idea about patterns: "*…we look at the order of the first part* (meaning the pattern's unit of repetition*) and we put the other in the same order* ... (transferring this unit*)* ..." (Papadopoulou, 2017, p. 172).

Number activities: The children in this task had counters and nine cells cards as shown in the figure (Fig. 18.7).

They were asked to create all possible arrangements of six counters on their cards. At the end of this task, the whole class discussed the possible configurations and decided if there were more or not. The children gave a considerable variety of combinations, by moving the counters and explaining "*..., we make groups, we keep one and change the other…*" or a girl explained showing the cards "*here! 5+1, 4+2, 3+3, 3+2+1… it is 3+3, who doesn't know it…?*" (Tzekaki & Papadopoulou, 2019).

The process of generalizing and expressing conclusions is a complicated and demanding activity at such an early age as 6. It requires considerable practice so that young children become gradually able to combine their actions and thoughts and perceive them at a higher level. Therefore, systematic involvement and undoubtedly motivation are required, but the implementation of teaching practices aiming at developing generalizing abilities in preschoolers suggests impressive results (Tzekaki & Papadopoulou, 2017).

18.5 Closing

A number of studies related to preschool mathematics education indicate that systematic and long-term programs encouraging young children to infer from their actions and search for more general elements in a set of situations advance their potential to generalize. All studies underline the role of teachers in supporting them to focus on relational and structural elements of objects or situations and widen these elements to a broader and thereafter a more generalized level. Although generalization lies at the core of mathematical development, teaching practices in early childhood underestimate it, remaining often restricted in the successful completion of a task, game or construction. As it is an important component of mathematical activity in early age, research needs to thoroughly explore young children's generalization levels, examining the ways they reflect on their actions and methods and express their conclusions about more general ideas in the situations they work with. Thus, we hope to understand more clearly how and if preschoolers act and think mathematically and how we can advance their relevant knowledge.

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Part II
Early Years Mathematics Teaching

Chapter 19
Situational Perception in Mathematics (SPiM)—Results of a Cross-Country Study in Austria and Norway

Julia Bruns, Martin Carlsen, Lars Eichen, Ingvald Erfjord, and Per Sigurd Hundeland

19.1 Introduction

Lately, several authors have highlighted kindergarten teachers' (KTs') mathematics-related competence and its relevance to children's mathematical learning (e.g. Bruns, 2014; Dunekacke, Jenßen, & Blömeke, 2015; Gasteiger, 2014; Tsamir, Tirosh, Levenson, Tabach, & Barkai, 2014). Competence models describe structural elements of KTs' professional competence (Gasteiger & Benz, 2018a; Jenßen, Dunekacke, Eid, & Blömeke, 2015). Not only in the context of early childhood, authors currently promote approaches that conceptualise competence as a continuum integrating (math-related) knowledge and beliefs on the one side, skills in actual teaching (performance) on the other side and situation-specific skills as a structural relation in between (Blömeke, Gustafsson, & Shavelson, 2015; Gasteiger & Benz, 2018a).

In the context of early mathematics education, KTs' situation-specific skills are assumed to play a fundamental role as early mathematics education is mostly spontaneous and based on natural learning situations (Gasteiger, 2014; van Oers, 2010). A decision to intervene in a play situation in order to create a mathematical learning situation depends strongly on the mathematics a KT perceives and interprets in this situation. Still unclear is, however, how KTs' situation-specific skills and especially their situational perception is shaped. Goodwin (1994) assumed in his seminal work

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in the field of situation-specific skills (he called it professional vision) that situation-specific skills are developed in a professional community. It "consists of socially organized ways of seeing and understanding events that are answerable to the distinctive interests of a particular social group" (Goodwin, 1994, p. 606). From this perspective, two aspects seem especially interesting to us:

1. How situation-specific skills of beginning KTs are shaped
2. To what extent situation-specific skills of KTs sharing a professional community but with different backgrounds differ.

As research results reveal close relations between educators' situational perception and their professional knowledge (e.g. Dunekacke et al., 2015; Wittmann, Levin, & Bönig, 2016), one can assume that early childhood students show sound skills in situational perception. Moreover, from a mathematics education perspective, it is not only interesting how situational perception is related to other competence facets but also which concepts KTs focus on in typical kindergarten situations. There is, however, no study yet that examines KT students' situational perception in mathematics from this point of view.

Altogether, these open questions lead to the SPIM project on KTs' situational perception. The main interest of our research is to study (1) which aspects KTs perceive in typical kindergarten situations and (2) similarities and differences between the situational perception of KTs with different educational backgrounds. Therefore, we first elaborate on the competence facet situation-specific skills in general and secondly on KTs situational perception. In the following, we present the design of our study and the results. In the last section of this chapter, we discuss the results in relation to our research question.

19.2 Situation-Specific Skills: An Essential Part of Kindergarten Teachers' Competence

Teachers' situation-specific skills is the generic term for three aspects of teacher competence: situational perception, interpretation and decision-making. In the context of early mathematics education, situational perception is described as identifying mathematics in children's play and recognising everyday situations with mathematical potential (Björklund & Barendregt, 2016; Gasteiger & Benz, 2018a). Interpretation is related to the mathematical development of the child; decision-making focuses on the planning of mathematical activities for children as well as the spontaneous act of offering support in a natural learning situation (Gasteiger & Benz, 2018a). Theoretically, it is assumed that these situation-specific skills function as a bridge between disposition and performance (see Fig. 19.1). Concerning early mathematics education there is some research supporting this idea, as studies find relations between cognition (e.g. professional knowledge) and mathematics-related perception and decision-making (Dunekacke et al., 2015) as well as between beliefs and decision-making (Wittmann et al., 2016).

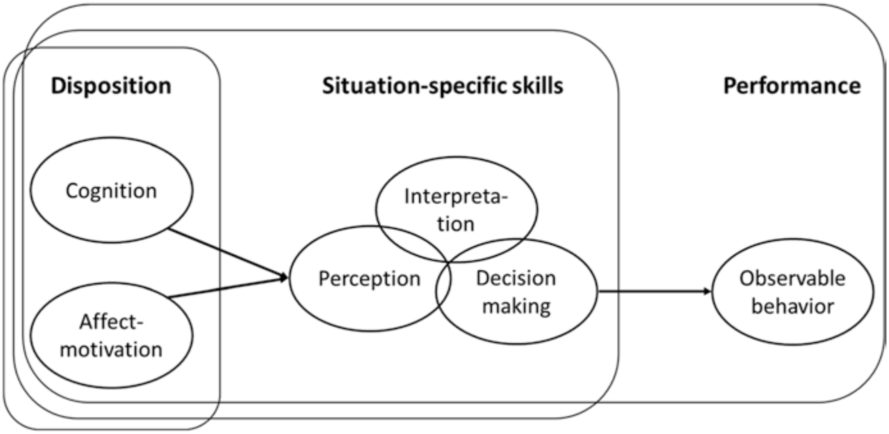


Fig. 19.1 Competence as a continuum (Blömeke et al., 2015, p. 7)

In addition to the relations between different aspects of competence, it seems reasonable to assume close relations between situational perception, interpretation and decision-making in KTs' everyday work. A typical kindergarten situation with mathematical learning potential is, for example, setting a table for lunch. However, the KT has to interpret this situation as one relevant for mathematical learning (van Oers, 2010) to use it as a learning situation. This interpretation in turn depends on the mathematical aspects a KT perceives in the given situation. If she/he cannot see the mathematical potential of the situation or only parts of it, she/he will most likely not use the full potential of the situation to support the mathematical learning of a child (Gasteiger & Benz, 2018b). In the context of setting a table one could see the potential of this situation to talk, for example, about counting strategies, analyse and interpret the child's counting strategy and ask the child, if she/he can think of another way to count the number of plates. However, if the KT does not recognise these mathematical concepts she/he probably will not address them (see also Björklund & Barendregt, 2016; Gasteiger & Benz, 2018b).

This example shows that sound skills in perceiving mathematical concepts in everyday situations in kindergarten can be seen as a precondition not only for interpretation and decision-making but also for good mathematics teaching in early childhood. Furthermore, while Lee (2017) found no statistical relation between noticing and interpreting, Gasteiger and Benz (2018b) show qualitative results indicating the impact of situational perception on KTs' pedagogical and didactical actions. These contrary results show that research on situational perception of KTs is still at the beginning. Therefore, it is worthwhile researching KTs' situational perception more closely.

19.3 Kindergarten Teachers' Situational Perception of Mathematics

Situational perception was first picked out as a central theme in the middle of the 1990s by Goodwin (1994) under the keyword professional vision. In his seminal work, Goodwin focused among others on a trial in the US against several police officers who were accused of unreasonable violence against a black man (the 'Rodney King trial'). Notably to Goodwin is that both sides in the trial—the prosecution and the defence—used the same videotape as evidence for their line of argument. Goodwin reasons from this that "the ability to see relevant entities is lodged not in the individual mind but instead within a community of competent practitioners" (Goodwin, 1994, p. 626). This leads him to the conclusion that professional vision can and has to be learned to perform successfully in a given job setting. At the beginning of the twentieth century van Es and Sherin (2002) adapted this concept of professional vision to the mathematics teaching context. They called this facet of teacher competence 'noticing' and defined it as "learning to identify what is noteworthy about particular situations" (van Es & Sherin, 2002, p. 573) which "involves using what one knows about the context to reason about a situation" (op. cit., p. 574). Therefore, the concept of noticing integrates the process of perception and interpretation.

Fitting to the described line of research, in the more general research field of early childhood education, observation and pedagogical documentation has emerged as an important research topic (e.g. Heiskanen, Alasuutari, & Vehkakoski, 2018; Schulz, 2015). Researchers from this perspective mainly focus on KTs' skills in and methods of observation and pedagogical documentation of children's development and describe observation and documentation as pedagogical activities indispensable for understanding and supporting children's learning (e.g. Knauf, 2015). The results indicate that KTs value observation and documentation as an important, but to some extent stressful, part of their work (op. cit.) and document some aspects of children's development intensively (Rintakorpi & Reunamo, 2017). Research, however, also shows that some KTs spend little time on observation and documentation (e.g. Fröhlich-Gildhoff & Strohmer, 2011), they have difficulties interpreting children's skills correctly (e.g. Bruns, 2014; Eichen, 2016) and lose sight of the documentation of, for example, mathematical development (Bruns, 2014).

The concept of situational perception in early childhood mathematics education can be traced back to some aspects of the described research fields. Definitions of situational perception, however, stress that situational perception is more than observing children's mathematical development: Gasteiger and Benz (2018a) define situational perception as recognising the mathematical relevance of play situations and everyday situations, Dunekacke (2016) as the identification of surface characteristics (as mathematical themes or materials) and more sophisticated characteristics (as the level of development of children or relationships between contributions of children). Both definitions have in common that they do not only focus on the learning process of the child but also include the perception of mathematics potential in different situations in a kindergarten setting. Based on this, it can be reasoned that the concept of situational perception overlaps more strongly with the concepts of noticing and professional vision than the concept of observation.

Empirical results on situational perception are still rare. Björklund and Barendregt (2016) assume based on survey data of Swedish KTs that teachers "seem to be quite perceptive of their environment and the mathematics that may be recognized within it" (p. 370)—except if it comes to situated mathematics learning, which is especially important to early mathematics education. Additionally, Lee (2017) found differences in KTs' situational perception related to years of teaching experiences as well as education background. Our research study that looks more closely into KTs' situational perception aims to add to this sparsely illuminated area of research.

19.4 Research Questions

Following the presented line of argument, it can be stated that situation-specific skills are a central aspect of KTs' competence. However, research on situational perception in mathematics is rare—especially regarding KTs. Existing research focuses on situation-specific skills of KTs in a more general way (e.g. Eichen, 2016; Knauf, 2015; Lee, 2017) or the relation between situation-specific skills in mathematics and other mathematics-related competence facets (e.g. Dunekacke et al., 2015; Wittmann et al., 2016). From a mathematics education perspective, however, it is not only interesting how situational perception is related to other competence facets but also which concepts KTs focus on in typical kindergarten situations. In addition, it seems to be an open question how KTs' situational perception skills develop. While Goodwin (1994) assumed that it is shaped by a professional community and should, therefore, be comparatively stable in different groups of KTs, Lee (2017) found differences in KTs situational perception with regards to their backgrounds. To address this question, a study comparing the situational perception of KTs in the same situation with different backgrounds—as, for example, KT students in Austria and Norway are—seems suitable. Altogether, this led to the main research questions of our project: What aspects do Norwegian and Austrian early childhood students perceive in video-situations from a typical kindergarten setting? What are the similarities and differences between Norwegian and Austrian students' situational perceptions of these video-situations?

19.5 Methods

To get some first answers to these research questions, we conducted a pilot study with students from Austria and Norway. As the goal of the study is to reconstruct the individual perception of the same situation, we chose a qualitative approach.

19.5.1 Sample

The Norwegian sample comprised of $N = 5$ students. The Norwegian students were on average about 23.5 years old; two of them were males. They were in the middle of their 3 years bachelor program in early childhood education. At the point of data collection, these students had studied courses in pedagogy, science, social science, health science, religion and ethics, and they were in the middle of taking a course called language, text and mathematics, from which mathematics comprises ten ECTS. Thus, the Norwegian students, at the point of data collection, had an educational university background in studying early years' mathematics. However, only from a theoretical point of view. The field practice of the course did not take part until after the data collection period. Nevertheless, the students had experience from field practice from the previous courses mentioned (8 weeks of practice in kindergarten all together). None of the Norwegian students had extensive work experience in early childhood settings.

The Austrian sample comprised of $N = 5$ students. The Austrian students were on average about 24.4 years old; all of them were females. The students had a bachelor's degree in pedagogy, social pedagogy or social work. All students were in the first master's semester (of four) for social education at the time of the study and visited a university course on early childhood education with four ECTS. Thus, the Austrian students had a theoretical background in early childhood education. They had, however, no special course or qualification in early mathematics education. In addition, none of the Austrian students had extensive work experience in early childhood settings.

19.5.2 Data Collection

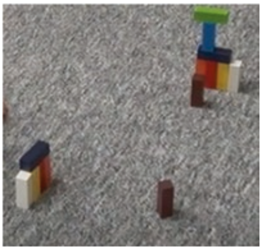
Following the approach of Gasteiger and Benz (2018a) as well as Dunekacke (2016), we define situational perception as recognising mathematical concepts in different materials and in children's activities in concrete situations. As the situations should be comparable in both groups, we did not use real-world observations but videotaped situations as a stimulus to collect data (Blömeke, 2013). Two of the authors created different mathematical learning opportunities in a kindergarten in Germany and videotaped the situations. In this study, we focus on vignettes capturing situations where children work with geometry in two and three dimensions. At the beginning of the study, the Norwegian researchers checked if the situations were comparable to situations in Norwegian kindergartens. Only videos that matched this criterion were included in the study.

Due to the different languages in the two countries, we decided to focus on mathematical concepts early childhood students may identify in activities and not in children's language. As young children's mathematical concepts are mostly tied to real objects, we focused on video-situations where children were actively engaged with varied materials and especially on activities that can be described as geometrical learning opportunities. To make sure that the students focused on the activities, we selected only videos that showed children in intensive engagement with these materials. Overall, we used seven video-situations that met the described criteria (fits the German and Norwegian kindergarten setting, active engagement in a geometrical activity, understandable without sound).

In the following, we will make a brief description of the content in the seven vignettes watched by the students. Half of the activities the children engaged in, shown in the vignettes, were initiated by the KT (vignette 1, 2 and 6), the other half was merely child initiated. If the teacher plays a role in the videotaped situation it is mentioned in the description (Table 19.1).

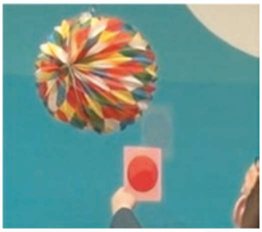
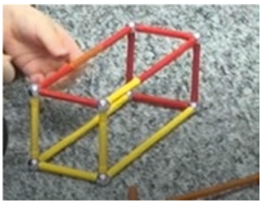
Using an open questionnaire, the early childhood students were asked to describe the mathematical aspects they saw in the situation. The students watched the vignettes only once and wrote down their observations and reflections immediately upon watching. The vignettes' duration was 70–130 s. Before watching the vignettes, the students were informed that they were to write down their answer to the following question: What mathematical aspects do you see in this situation? Together with the information: Please indicate where you have seen these aspects. The video was presented to the whole group and each student wrote his or her answer individually in the open questionnaire. For each of the seven situations, we analysed data from five students in each country.

Table 19.1 Description of the vignettes 1–7

Vignette number and illustration		Brief description of the content of each vignette
1		Vignette 1 shows two girls sitting back-to-back on the floor and later face-to-face oriented 90°. They are working with congruent coloured rectangular prisms and one of the girls tries to copy the other girl's construction.
2		Vignette 2 also concerns two girls making constructions based on prisms. This time different kinds of wooden bricks formed as straight square prisms, including cubes and oval shapes, are used.

(continued)

Table 19.1 continued

Vignette number and illustration		Brief description of the content of each vignette
3		Vignette 3 shows one girl working with red- and blue-coloured congruent wooden equilateral triangles. Next to her is an instruction sheet apparently with three sequences of three coloured triangles she is to make (only one of these sequences is visible on the video).
4		Vignette 4 shows one boy working with coloured congruent plastic cars of two sizes, a tray with squares with holes in the same distances where the cars can be put in place based on instruction cards.
5		Vignette 5 shows one girl and two boys working with magnetic spheres and sticks, and the girl builds a pentagonal prism with five congruent squared sides.
6		Vignette 6 shows several children holding paper sheets with different coloured circles in their hands, one girl has a camera, and they are in a room with coloured circles on the wall and a spherical-shaped paper creation hanging from the roof.
7		Vignette 7 shows a boy working with magnetic spheres and sticks. He has a working sheet in front of him and gets some advice from an adult in making a rectangular prism.

19.5.3 Data Analysis and Analytical Process

To analyse the collected data, we used qualitative content analysis (Mayring, 2015). Qualitative content analysis allows a systematic analysis of the content of a text within its context. We therefore defined the videotaped situations as the context of the answers and each of the answers as a recording unit. These units are the focus of

Table 19.2 Identified categories and their definitions

Category	Definition
Process	When associating the term *Process* with students' answers with respect to the vignettes, we mean answers focusing on what children do, i.e. their actions. We distinguish between a manifold focus (code P, dark blue) and limited focus on process (code p, light blue)
Learning	When associating the term *Learning* with students' answers, we mean answers focusing on children's opportunities to learn and whether the children have learned something. We distinguish between explicit focus (code L, dark green) and implicit focus (code l, light green) on children's learning processes
Mathematical concepts	When associating the term *Mathematical concepts* with students' answers with respect to the vignettes, we mean answers where students refer to mathematical concepts argued to be perceived in the vignettes. We distinguish between explicit and manifold focus on mathematical concepts (code M, dark red) and limited and occasionally imprecise focus on mathematical concepts (code m, light red)

the analysis. From these recording units we developed categories of perceived aspects inductively, meaning the categories were developed step by step guided by the answers of the students. Each of the categories represents a different aspect the students identified in the video. As the categories are developed inductively, these do not necessarily comprise all the aspects of the situations, and it is also possible that the students describe mathematical aspects that an expert would not ascribe to the presented situation. However, this analytical technique allows us to achieve insight into students' individual situational perceptions.

Firstly, the students' written responses were translated into English as none of the authors speaks both Norwegian and German. Secondly, we identified concepts implicitly and explicitly mentioned in the students' responses. Thirdly, we elaborated the categories below through the iterative process of the content analysis of the students' responses to the vignettes. The description of the categories thus was done in parallel with the analysis. In this way we were able to come up with category descriptions that comprised all the written responses from the Austrian and Norwegian students. We categorised the students' responses according to process, learning and mathematical concepts (see Table 19.2), first individually and then collectively. In most cases we immediately agreed on the category(ies) comprising one student's response to a vignette. In the cases where we initially disagreed, we discussed the written response thoroughly, watched the vignette in question one or two times more and arrived at a shared agreement about the category(ies).

Table 19.3 illustrates our coding within these categories:

19.6 Identified Categories from the Analysed Data

We have analysed five Norwegian students' and five Austrian students' answers to seven vignettes videotaped in a German kindergarten setting as outlined above. As mentioned, we identified three categories relative to the students' situational per-

Table 19.3 Examples of coding made based on students' responses to the vignettes

Code	Example	Student	Nationality	Vignette
p	"Maybe beginning of a pattern"	Student 2	Norwegian	Vignette 3
P	"The children played with shapes. Two children sit back to back. One child makes a building with shapes. The other child is then to make a copy of the first child's building"	Student 1	Norwegian	Vignette 1
l	"Spatial thinking. How do the figures fit with each other? Three-dimensional thinking"	Student 1	Austrian	Vignette 2
P/L	"Children have to build blocks according to a construction plan, had to distinguish between colours and shapes from building blocks"	Student 3	Austrian	Vignette 1
m	"Distances, large/small" "We see children who play with one-dimensional and two-dimensional shapes"	Student 2 Student 1	Austrian Norwegian	Vignette 6 Vignette 2
M/P	"Looks at a picture of two-dimensional shapes and tries to reproduce the shapes with help of physical bricks. The girl mirrors the pictures with three-dimensional shapes"	Student 3	Norwegian	Vignette 3

Table 19.4 Austrian and Norwegian students' foci relative to perception of process in the vignettes

Vignettes/	**Austria**					**Norway**				
Students	1	2	3	4	5	1	2	3	4	5
1	P	P	P/L	P	L	P	M/p	P/L	M/l	L/p
2	l	p	P/l	P	P	m		P/m	M/L	m/p
3	p/m	l	P	P	p/m	M/p	p	M/P	M/l	P
4	l	l	p	p/l	P	P/l	m	P/m	p/M	P
5	M/P	l	p	p	P	M/P	M	m	p/l	p/m
6	l/m	m	p	m	P/m	P/m	P/m	P/m	P/M	P/L/m
7	P/m	P/l	p/m	P	p/l	P/l/m	P/M	L/M	P/M	L/M

ception, process, learning and mathematical concepts. Below these categories are presented in this very order, one at a time. Table 19.4 presents our categorisation regarding the students' perception of process, whether they focused heavily or limitedly on process in their responses to the various vignettes.

In Table 19.4 we observe that the distribution is fairly equal between the Austrian and Norwegian students' perception of process in the vignettes, both in total and between students. However, we also observe that there are some differences between the students. With respect to vignette 2, 4 Austrian students' perception comprised a focus on process compared with two Norwegian students. With respect to vignette 6, two Austrian students' perception comprised a focus on process compared with all the five Norwegian students. At 16 occasions, we have coded the Austrian students' perception as heavily focusing on processes (code P) the children in the vignettes are participating in (coloured dark blue). The number for the Norwegian students is 17 in this matter. Additionally, at ten occasions we coded the Austrian students' perception as adopting a limited focus on processes (code p, coloured light blue). The number for the Norwegian students is 8. Amongst the Austrian students, there were nine occasions which did not focus at process at all (coloured white). The corresponding number for the Norwegian students is 10. Based on these numbers we argue that it seems as if both the Austrian students and the Norwegian students are occupied with the process aspect of the children's participation, and their perception seems pretty similar in this regard.

Table 19.5 below presents our categorisation regarding the students' perception of learning, whether they focused heavily or limited on learning in their responses to the various vignettes.

In Table 19.5 we observe that the distribution is fairly equal between Austrian and Norwegian students' perception of learning in the vignettes. Based on our analysis of the written responses to the vignettes, we also observed that the students were fairly equal with respect to how they described the children's opportunities to learn. At two occasions we have coded the Austrian students' perception as heavily focusing on learning (code L), which the children in the vignettes are participating in (coloured dark green). The number for the Norwegian students is 6 in this matter. Additionally, at ten occasions we coded the Austrian students' perception as adopting a limited focus on learning (code l, coloured light green). The number for the Norwegian students is 5. If we combine these numbers, we observe that the Austrian students' perception was coded as focusing on learning at 12 occasions. The corresponding number for the Norwegian students is 11. Since these numbers are almost

Table 19.5 Austrian and Norwegian students' foci relative to perception of learning in the vignettes

Vignettes/	**Austria**					**Norway**				
Students	1	2	3	4	5	1	2	3	4	5
1	P	P	P/L	P	L	P	M/p	P/L	M/l	L/p
2	l	p	P/l	P	P	m		P/m	M/L	m/p
3	p/m	l	P	P	p/m	M/p	p	M/P	M/l	P
4	l	l	p	p/l	P	P/l	m	P/m	p/M	P
5	M/P	l	p	p	P	M/P	M	m	p/l	p/m
6	l/m	m	p	m	P/m	P/m	P/m	P/m	P/M	P/L/m
7	P/m	P/l	p/m	P	p/l	P/l/m	P/M	L/M	P/M	L/M

Table 19.6 Austrian and Norwegian students' foci relative to perception of mathematics in the vignettes

Vignettes	Austrian students					Norwegian students				
	1	2	3	4	5	1	2	3	4	5
1	P	P	P/L	P	L	P	M/p	P/L	M/l	L/p
2	l	p	P/l	P	P	m		P/m	M/L	m/p
3	p/m	l	P	P	p/m	M/p	p	M/P	M/l	P
4	l	l	p	p/l	P	P/l	m	P/m	p/M	P
5	M/P	l	p	p	P	M/P	M	m	p/l	p/m
6	l/m	m	p	m	P/m	P/m	P/m	P/m	P/M	P/L/m
7	P/m	P/l	p/m	P	p/l	P/l/m	P/M	L/M	P/M	L/M

identical, we observe an equal amount of the Austrian students' perception and the Norwegian students' perception did not focus on learning at all from the vignettes (coloured white). Based on these observations we argue that it seems as if the Austrian students and the Norwegian students are averagely occupied with the learning aspect of the children's participation in the vignettes, and their perception is argued to be similar in this regard.

Table 19.6 presents our categorisation regarding the students' perception of mathematical concepts, whether they focused heavily or limited on mathematical concepts in their responses to the various vignettes.

Table 19.6 illustrates the difference between the Austrian and Norwegian students for each vignette with respect to the category mathematical concepts. We observe that the distribution between Austrian students' and Norwegian students' perception of learning in the vignettes is different. At one occasion we have coded the Austrian students' perception as heavily focusing on mathematical concepts (code M, coloured dark red). The number for the Norwegian students is 14 in this matter. Additionally, at eight occasions we coded the Austrian students' perception as adopting a limited focus on mathematical concepts (code m, coloured light red). The number for the Norwegian students is 12. In particular, we observe huge differences between the students with respect to vignettes 1, 2 and 4, where none of the Austrian students focused on mathematical concepts compared with nine of the Norwegian students. Moreover, in vignette 5 only one of the Austrian students focused on mathematical concepts compared with four of the Norwegian students. Despite these apparent differences, when focusing on the codes M and m, we see that in vignette 6 there is practically no difference between the two groups of students. In vignette 3 there is a small difference, as two of the Austrian students' responses are coded m while three of the Norwegian students' responses are coded M. In the five other vignettes we observe that quite many of the Norwegian students' responses are categorised as comprising mathematical concepts while very

Table 19.7 Summarising students' responses

	P	p	L	l	M	m	Total
Austrian	16	10	2	10	1	8	47
Norwegian	17	8	6	5	14	12	62
Total	33	18	8	15	15	20	109

few of the Austrian students' responses are categorised as comprising mathematical concepts.

Based on the analysis above, we summarise our categorisation of the students' perception of process, learning and mathematical concepts in Table 19.7.

From Table 19.7, when comparing the number of responses from the Austrian and Norwegian students in the two blue columns separately and the two green columns separately, we see that there are minor differences between the students concerning the answers coded as process or learning (16 + 10 (Austria) versus 17 + 8 (Norway) process responses; 2 + 10 (Austria) versus 6 + 5 (Norway) learning responses). The major difference between the Austrian and Norwegian students' responses is identified in the category mathematical concepts. The Norwegian students made 14 + 12 responses emphasising mathematical concepts in the children's activities, in contrast to 1 + 8 responses emphasising mathematical concepts among the Austrian students. The difference between the total amount of responses between Austrian and Norwegian students (47 versus 62) is thus explainable by the number of responses categorised as mathematical concepts.

19.7 Discussion

We set out in this study to come up with answers to the research questions: What aspects do Norwegian and Austrian early childhood students perceive from video-situations from a typical kindergarten setting? What are the similarities and differences between Norwegian and Austrian students' situational perceptions from these video-situations? We found that the students perceived what we have categorised as process aspects, learning aspects and mathematical concepts aspects. Both Austrian and Norwegian students' situational perceptions were characterised by all three categories. As regards the process and learning aspects perceived by the students, we found that there are negligible differences between the Austrian and the Norwegian students. Based on the data collected and analysed, the two groups of students placed explicit, quite similar emphasis with respect to process and learning aspects. We interpret this result as an argument supporting Goodwin's (1994) assumption that situational perception is related to the professional community. Although our sample consisted of early childhood students in different countries, with different learning experiences, they focused on similar aspects of process and learning.

However, we also found that there is a major difference between Austrian and Norwegian students' responses with respect to the category mathematical concepts. The Norwegian students made 14 responses that heavily emphasised mathematical concepts and additional 12 responses that comprised a limited emphasis on mathematical concepts in the children's activities. In contrast, we found only one response amongst the Austrian students that heavily emphasised mathematical concepts and additional eight responses with a limited emphasis on mathematical concepts in the children's activities. This difference seems remarkable, based on the fact that all students watched the same video vignettes. This result contradicts the assumption of Goodwin (1994) that situational perception is related to a professional community. The Norwegian students were in the middle of their education to become kindergarten teachers. Moreover, at the moment of data collection, they were studying a course with particular focus on early childhood education in mathematics. The Austrian students were enrolled in a university course in early childhood education, but no particular emphasis was put on mathematical aspects of early childhood education in that course. As the two groups of students had different learning opportunities concerning early mathematics education, we believe that the difference between the two groups of students is due to their educational background.

Summarising our results, we conclude that there is not one situational perception, but different facets of situational perception, at least the three facets: process, learning, and mathematical concepts. Further studies are needed to identify if there are even more facets. As our study used short video vignettes as a data collection tool, it could be possible that these facets are somehow influenced by our vignette selection. Additionally, it seems reasonable to assume that situational perception is tied to the whole context (e.g. knowing children, knowing play materials), not only to one situation, and situational perception might thus involve even more facets of situational perception. Our study therefore might underestimate students' situational perception.

While perceiving aspects of process and learning seems to be more closely connected to being part of the professional community of kindergarten teachers, perceiving mathematical aspects seems to need special attention in teacher training. Our study, however, only examines students. It is also possible that situational perception of mathematical aspects is further developed by the professional community in practice. Therefore, a study examining the situational perception of in-service kindergarten teachers with different backgrounds might be interesting. Nevertheless, the result of our study confirms the claim by Goodwin (1994) that professional vision, in order to be used successfully, has to be learned. The Norwegian students seem to have learned the mathematical concepts and mathematical aspects of early childhood education, as they were indeed able to utilise their insights successfully. They seem to have adopted what van Es and Sherin (2002) call 'noticing'. This facet of (kindergarten) teacher competence, we believe, is fundamental for orchestrating mathematical activities where children may engage themselves as well as experience and learn mathematics. This relationship is made explicit in the theoretical model of Blömeke et al. (2015) and Gasteiger and Benz (2018a), where perception is closely interrelated with interpretation and decision-making.

Nevertheless, even though all Norwegian students had at least one response that we coded as heavily emphasising mathematical concepts, we observe in Table 19.6 that there are severe differences among the Norwegian students, from one response coded M up to six responses coded M out of the seven vignettes. Varieties in the students' situation-specific skills thus reflect what we know from research (see Lee, 2017) as well as experience teaching and assessing this type of students for more than 10 years. Kindergarten teacher students demonstrate varying competences in mathematics relevant for early childhood education, due to varying background in mathematics as well as varying experience from early childhood education. We thus interpret our results as documenting the importance of addressing situation-specific skills in mathematics among kindergarten teacher students, as this is argued to be resulting from the students' own learning of mathematics relevant for early childhood education in mathematics in a kindergarten setting.

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Chapter 20
How to Support Kindergarten Children in Spontaneous Measuring Activities

Vigdis Flottorp

20.1 Background

In this section, I will first outline some significant features of the Norwegian kindergartens, as portrayed in official documents. I will focus on how learning is described. Secondly, I will refer to some studies concerning the role of the teacher in mathematics activities. I will outline some aspects of *measuring and comparing*, which is the mathematical subject in all the cases in the study. The section will conclude with the research questions.

In Norway, most children attend kindergarten, and the students are meant to work both with toddlers and 5- to 6-year-olds, without any difference concerning education. As a consequence, both the *Framework Plan for Kindergarten* (Ministry of Education and Research, 2017) and *The National Guidelinev for* (UHR—Teacher education group for revising national guidelines for Early Childhood Education, 2017) are the same regardless of the age of the children.

The Framework Plan for kindergartens emphasises that learning should occur in *everyday activities and children's free play*. The chapter on mathematics expresses the following:

> Kindergartens shall highlight relationships and enable the children to explore and discover mathematics in everyday life (…). (2017, p. 34)

Several studies show that kindergarten teachers to some extent are capable of recognising mathematics in daily life situations (Björklund & Barendregt, 2016; Ginsburg, Inoue, & Seo, 1999; Helenius et al., 2015; Østrem et al., 2009). However, this is valid for counting and numbers, but less applicable to spatial phenomena. This is consistent with an assessment of 82 kindergarten teachers, which examines the competence of the kindergarten teacher's mathematical knowledge. The study

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ranks the kindergarten teacher's ability to recognise children's knowledge in the area of number sense as the highest, second lowest in the area of measuring and lowest score for the area of spatial sense (Lee, 2010).

Teachers who recognise mathematical situations might easily be able to follow up children in these situations. The question is still what happens after they have recognised a mathematical situation. Do they just observe passively? Or do they act in the moment, by saying or doing something which stimulates and supports the children mathematically?

A *spontaneous situation* typically occurs in daily life and play situations. Since children initiate these situations, neither mathematical subjects nor approaches, examples or materials can be pre-fixed. Therefore, spontaneous situations may be harder to act upon than planned activities. Our knowledge about what occurs in spontaneous situations is scarce. This is a contradiction since the Framework Plan for kindergartens stresses the learning potential in daily life situations.

Several studies address the role and tasks of the kindergarten teacher. Van Oers (1996) investigates how children's mathematical thinking can be stimulated whilst playing shoe shop and finds that the question "are you sure?" might be productive for mathematical thinking. Erfjord, Hundesland, and Carlsen (2012) explore how kindergarten teachers develop their mathematical practice in a developmental research projects in kindergarten. Based on observations and interviews, they examine how the kindergarten teachers mature their mathematical practice. The teachers express being empowered in an inquiry approach, which enable them to be more explicit when communicating mathematical ideas to children. Benz (2016) examines the importance of reflection, based on in-service and pre-service course for early mathematics education. It seems that content knowledge and action-related competencies are interweaved by reflection. Lange, Meaney, Riesbeck, and Wernberg (2014) investigate the actions of a kindergarten teacher while children play with jars. The study shows that the teacher, through respectful listening and careful observation, succeeds in asking questions that sparkle the children's mathematical curiosity. Furthermore, the questions develop the activity.

The Framework Plan for Kindergartens in Norway underlines that it is the responsibility of the staff in kindergartens to promote learning:

> Kindergartens shall create a stimulating environment that supports the children's desire to play, explore, learn and achieve. Kindergartens shall introduce new situations, topics, phenomena, materials and tools that promote meaningful interaction. The children's curiosity, creativity and thirst for knowledge shall be acknowledged, stimulated and form the basis for their learning processes. (2017, p. 22).

The Framework Plan for kindergartens does not mention *teaching*. The emphasis is on process, not on learning outcomes. The tasks of the staff are demanding:

> The staff shall (…) build on creativity and play and be open to improvisation and the children's own contributions, alternate between spontaneous and planned activities, support and enrich the children's initiative (…), support the children's reflections on situations, topics and phenomena and create understanding and meaning together with the children (2017, p. 43)

20.1.1 Measuring

As mentioned, the mathematical subject of this study is measuring. According to Bishop (1988), measuring is an activity rooted in daily life. Small children also encounter situations in play and daily life which are related to measuring.

The *National Guidelines for Early Childhood Education* does not mention any specific mathematical subjects such as numbers, measuring or geometry. Instead, the guidelines indicate how each institution shall describe content, interdisciplinary subjects, organisation, working methods and evaluation forms (The Norwegian Association of Higher Education Institutions—UHR, 2017, p. 4).

The mathematics in kindergarten covers comparison, sorting, placement, orientation, visualisation, shapes, patterns, numbers, counting and measuring, according to The Ministry of Education and Research, (2017). At Oslo Metropolitan University (formerly Oslo University College), mathematics constitutes a limited part of the curriculum. Therefore, we concentrate on two main subjects rather than covering many superficially. The subjects are number and measuring.

The measuring subject was chosen since it is relevant also for toddlers who use their body to explore their surroundings (Palmer, 2012). Small children might, for example, experience the height up to the doorknob, as they use their bodies, and they will obtain this experience without uttering any words. Five- to six-year-old kids are involved in measuring activities, often in a different way, for example by using transitivity, measuring units or measuring scales.

Measuring in kindergarten is usually about comparing. In all forms of measuring, comparing an object with another is involved. Therefore, the term *measuring* applies to both comparison and measuring in this study.

The Freudenthal tradition introduces learning trajectories to describe three learning steps in developing the understanding of measuring (Buys & de Moor, 2005, s. 17). The easiest form of measuring is direct comparison, by placing two objects next to each other. The second method is to use a measuring unit, and the third is adopting a measuring scale.

The difference between the second and third levels is that the third level requires the ability to *read* written numbers. The second level demands counting units *orally*. There is a major cognitive difference between understanding a scale, for example with three-digit numbers, and grasping the meaning of counting a unit, for example a stick (Flottorp, 2018).

The research on the trajectories is limited. Some findings indicate that children might benefit from using a measuring scale before they fully understand what a measuring unit is (Sarama, Clements, Barrett, Van Dine, & McDonel, 2011). Others stress the importance of understanding the logic of a measuring unit. When children have grasped the measuring principles, they understand both arbitrary and standard units (Kamii, 2006, s. 158).

Transitivity is not mentioned in the overview above. However, it occurs in several of the recorded situations in this study. Measuring with transitivity means to compare two objects using a third one, without using units (Sarama & Clements, 2009, p. 275).

The research questions are: What are the challenges of Early Childhood Education students on supporting children spontaneously in measuring activities? How do the students reflect on being active versus passive?

20.1.2 The Knowledge Quartet

The Knowledge Quartet (KQ) was chosen as a theoretical approach because it appeared to be useful for my aim, namely to investigate what it takes to follow up kindergarten children mathematically in spontaneous daily life situations. The theoretical basis for mathematical didactics in kindergarten is scarce. *Mathematical Knowledge in Teaching* (MTK) (Even & Ball, 2008; Rowland & Ruthven, 2011) is based on a framework developed within school context, while *The Knowledge Quartet (KQ)* is adopted to a kindergarten context.

Knowledge covers a wide range of aspects. Shulman identified several categories: general pedagogical knowledge, knowledge of learners, knowledge of context, knowledge of purpose of teaching and learning. In addition, he formulated three content-specific knowledges: subject-matter knowledge, pedagogical content knowledge and curriculum (Shulman, 1986).

To find out what constitutes The Knowledge Quartet for smaller children, Rowland and Ruthven (2011) conducted a longitudinal study, observing and interviewing teacher students. Four major categories, called The Knowledge Quartet, were distinguished: *foundation, transformation, connection and contingency* (2011, pp. 200).

Foundation concerns subject knowledge, beliefs, awareness of purpose and identifying errors. The three other categories are based on foundation. *Transformation* refers to teacher demonstration, use of materials, examples and representations. *Connection* denotes connecting procedures and concepts, anticipating complexity, sequencing and recognising relevant concepts. *Contingency* addresses the ability to respond to children's ideas, to use opportunities and deviate from agenda. The "responding moves" seems to be some of the most difficult interventions for novice teachers to master (Brown & Wragg, 1993, in Turner & Rowland, 2011, p. 202).

The elements in the four major categories, *foundation, transformation, connection and contingency*, reflect different types of knowledge. An example is *beliefs* listed as a type of knowledge aligned with *anticipating* complexity.

Some studies focus on the difference between kindergarten and school concerning learning mathematics. Mosvold, Bjuland, Fauskanger, and Jakobsen (2011) adapted MKT to a kindergarten context. They found that an important difference between kindergarten and school is that the kindergarten teachers had to facilitate and use activities and play situations that enable children to experience mathematical ideas. This is very much in line with the Framework Plan for Norwegian kindergartens. In schools, on the other hand, the mathematics lessons are fixed to a certain schedule and textbook.

The focus in the chapter is spontaneous situations. *Contingency* occurs in both planned and spontaneous situations, as children's responses cannot be planned by the teacher. The concept of contingency is closely related to "competence of improvisation" (Erickson, 1982, in Krummheuer, 2012). This notion refers both to the content-related aspect of interaction as well as to the aspect of managing the ongoing course of interaction.

Improvisation is an important issue in early childhood education. Many situations in kindergarten require improvisation, and good improvisation depends on knowledge, preparation and training (Steinsholt & Sommerro, 2006). A jazz musician describes improvisation as an adaption of a pre-composed material (Berliner, 1994, in Jansen, 2014). A study on interactions between a kindergarten teacher and children revealed that what seemed to be improvisation was based on knowledge and observations. The actions of the kindergarten teacher were not coincidental (Odden, 2005, in Jansen, 2014, p. 57). In kindergarten it is crucial to develop pre-composed material, a repertoire that provides alternatives.

20.1.3 The Purpose of Education: Biesta

The theory of Biesta is introduced because it appeared to be relevant in the discussion of the goal of the children's activity and arguments about the students' choices. The discussions in the student groups exceed the purpose of the foundation domain. In KQ, the purpose of learning mathematics is taken for granted. Whether to teach mathematics or not is not disputed.

Biesta formulated a threefold question of the purpose of education: qualification, socialisation and subjectification (Biesta, 2015). Qualification denotes the purpose of being qualified for the future. Socialisation points to the purpose of the children being introduced to traditions where the aim is only partly educational. The third aspect, subjectification, represents how children "come to exist as a person, opposed to being an object" (2015, p. 77).

Biesta's three aspects relate to content, tradition and person. Biesta points out how the achievement in the domain of qualification dominates the education practice and discussion, leaving the other two aspects unfocused, especially in school. The Framework plan for kindergarten highlights children's need for care and play. Learning is mentioned as the last aspect. There are no learning goals in the Framework Plan for kindergartens in Norway. The subjectification domain relates to letting children investigate for themselves. It might be less important that the children fully understand some mathematical concepts in the process.

The Framework Plan for kindergarten emphasises that the learning area of mathematics "…shall stimulate the children's sense of wonder, curiosity and motivation for problem-solving. It covers play and investigation (Ministry of Education and Research, 2017), p. 53)". This way of formulating learning is connected to Biesta's subjectification, and is connected to the formative aspect mentioned in the Framework Plan for kindergartens. Biesta's concept of qualification can be related to school readiness. Although the Framework Plan for kindergarten emphasises care and play, the focus on learning and school readiness increases in Norway. Fosse et al. conducted a discourse analysis of central documents the last years. They found that mathematics has been more strongly linked to teaching and learning than to playing (Fosse, Lange, Lossius, & Meaney, 2018).

20.2 Methodology

The data consists of two parts; firstly, texts of students and secondly interviews. The student who records an activity in kindergarten is denoted the *recording student*. The student who comments the activity is called the *commenting student.* Each activity consists of four written parts, 1–2 by the recording student and 3–4 by the commenting student:

1. A record of an activity in kindergarten
2. A description of the mathematical concepts in the activity
3. A description of what contributed to the measuring activity, by the *commenting* student
4. A proposal for a follow-up activity, by the *commenting* student

The students were organised in groups and then interviewed. Three of the groups were taped. The students were given a choice whether to participate in the study or not, and all consented. The interviews resulted in 3 h with taped data. They were transcribed and the names were made unrecognisable. Then the recordings were deleted.

The groups were chosen based on which students I knew best, and how easy it was to find a suitable time for interviews. All the three groups turned out to be part-time students, which is a biased selection. Since the approach is qualitative, I will argue that the data still may shed light on the research questions.

The students' texts and the interviews are coded according to the categories of Quartet Knowledge—KQ (Turner & Rowland, 2011). Some aspects appear to be important, but hard to fit into the categories of KQ. These are coded "other".

The three groups that were taped had 14 members altogether. I chose one of the groups for closer analysis; this group had five members. The reason for choosing this group was that several of the KQ aspects appear in the coding process. Moreover, the discussion in the group was particularly elaborate. The purpose of mathematics in kindergarten was thematised in a way that exceeds the categories of The Knowledge Quartet.

The study contains some quantitative data, based on total amount of 214 students' texts. The purpose of collecting so many texts was to investigate two factors: the prevalence of *passive observers* and the frequency of *spontaneous situations* in the total material. A student is labelled *passive* when he or she does not say or do anything in the situation.

20.2.1 Limitations of the Study

As the observations are collected by students, I am not present in the situations. This challenges the validity of the findings.

The observations are not random notes but are recorded according to guidelines as a part of an assignment at the Early Childhood Education at Oslo Met. The students had to record the mathematical-relevant actions and words, including information on how the situations started and how it ended. Compulsory components of the assignment were to report a *measuring activity* which could be spontaneous or planned. Afterwards they were interviewed, based on their texts. The combination of texts and interviews illuminate the challenges of supporting children's daily life mathematical activities.

For a researcher, it is difficult to capture spontaneous situations by being present in a kindergarten. To use students as co-researchers might therefore offer a way to understand these situations better. An important part of the daily life in kindergarten is such situations.

The main purpose of the interviews is to give feedback to the students on their text, and secondly to supply answers to my research questions. Therefore, no interview guide was used. Thus, not all parts of the interviews were relevant for the research questions.

Furthermore, the interviews are conducted in the context of evaluation. Although the students who participated in the interviews knew they had passed, waiting for the right answer from the teacher may have affected their ability to reflect freely.

20.3 Analysis

20.3.1 The Quantitative Data

The study has a small amount of quantitative data which concerns the number of passive students and the amount of spontaneous situations. A student is labelled *passive* when he/she does not say or do anything in the situation, but just observes the children. If a student comes up with a contribution, he or she is labelled active.

I scrutinised the recorded situations and counted the episodes with passive students. Twenty per cent of the students were passive in the recorded situation. This is a high number, especially since they were asked to *participate* in the measuring activity.

Fifty-three per cent of the observations were spontaneous. There were considerable differences between full-time students, who had 48% spontaneous activities, and part-time students, who had 72% spontaneous activities. Part-time students worked 50–70% in kindergarten and easily captured spontaneous situations. This may also explain why all the situations in the taped group interviews were spontaneous, since all the students in the interviews were working part time.

The high number of passive students is related to the number of spontaneous activities. I will argue that when you plan an activity, you are usually active in the implementation, for example by explaining what the activity is about.

20.3.2 The Qualitative Data

In the following, I will revise five situations, based on the students' texts and the interview data. I use the term *situation* to denote an event in kindergarten recorded by the students. The situation could last for a short or longer period. Using the categories of KQ, I will investigate the students' reflections. The focus will be on the challenges of supporting children's mathematical activities.

The students' texts are excerpts.[1] I have focused on the parts which are the most relevant for the research questions and give enough information to create a picture of the events. Parts of the observation are reproduced verbatim, reflecting the details, which are regarded as relevant.

20.3.3 Relative Heights—Dorothea's Observation

The case was chosen because the student responded to the children in relevant ways but was not able to explain what was going on mathematically.

> A child stretches her arm up, asking if she reaches the head of the student. "Almost," the student says, and then takes a broom to measure her. Three more children want to do the same, and the student makes a mark on the broomstick for each of them. They use words like highest, lowest, the same and in the middle, and then line up according to their heights. A boy climbs on the table, measuring himself with the broomstick, concluding he is almost as long as the broomstick.
>
> They lose interest in the broomstick. Then one girl finds a ribbon and climbs on the table. While the student holds the ribbon, she investigates whether the ribbon is long enough to reach from the floor to herself. The other children gather and try to do the same.

The commenting student points out that Dorothea manages to use an informal learning situation, responding to the children's initiatives. She improvises by introducing the broomstick. These actions are aspects of *contingency*; she *uses opportunities* and responds to the children's ideas about height. The activity is prolonged because her actions keep them going on with measuring.

In her analyses of the situation, Dorothea claims the children use direct comparison, and that the broom is a measuring unit. She does not bring in the concept of transitivity in describing how the children use the broomstick to compare. Her

[1] The recording student has 300 words to record an event and 300 to analyse it, while the commenting student has 300 words to describe what contributed to the activity and propose a follow-up activity.

text reveals that she is confused about basic measuring concepts, lacking *subject knowledge.* Still, she is able to improvise in an adequate mathematical way.

20.3.4 Who Is Tallest?—Turid's Observation

The case was chosen because the student let the children find the solution themselves. She describes the mathematics in the situation precisely.

> The student and two five-year-olds lie on a mattress. One of the children, a girl, jokes about being the tallest one, while the boy is protesting. "How can we examine this?" the student asks. The girl suggests that they can lay down next to each other. They do, and the student asks if this is right. The girl gets on her feet and discovers that their feet are not aligned. "Our feet have to be at the same place," she says. "What do we do then?" the student asks. "Peter, you have to move your feet in line with Turid's feet. Isn't that correct?" the girl asks, she looks for confirmation, but does not receive any. The boy adjusts his feet and agrees the measuring to be correct. "We are like a staircase;" the girl says. "I am at the first step, you (the boy) are in the middle and Turid is at the highest step."

In her comment, the recording student describes the importance of the same *starting point*. She mentions how the children use actions and words, *ordering* the length of their bodies. The starting point and ordering are important aspects of the situation, concerning *subject knowledge.*

The commenting student points out how Turid is active by encouraging the children to *try out* their hypothesis, not giving answers, but motivating them to solve the problem themselves. Turid's *questions* are described as open ended—she urges the girl to check for herself and the girl consequently discovers the importance of having the same starting point.

As a follow-up activity, the responding student suggests measuring the mattress with the bodies of the children, arguing that introducing a measuring unit will expand the situation. Direct comparison is regarded as easier than measuring with a unit, so the follow-up activity might be interpreted as a sign of subject knowledge.

The recording student clarifies that the children are of equal length. During the discussion, the group suggests measuring bigger things like the circumference of the sandpit, underlining the importance of the children being of equal length. They do not mention the idea of *iterating* the unit, which might indicate that the relevant subject knowledge is not fully understood, or that the action-related competence is not fully developed.

20.3.5 Selling Water—Astrid's Observation

> After a meal, a girl and a boy stay at the table, where they are talking in a low and confidential tone. The girl suggests selling her glass of water to the boy, who is evaluating the offer. They put their glasses close to each other, nod in agreement and conclude that the boy has more water. The girl suggests she buys his glass of water since he had a lot. They are interrupted by a personal who asks them to hurry to the wardrobe.

The discussion about this his case made clear different views about the task of the pedagague. The students debated whether they should introduce the concept of *volume*, or whether the children should be allowed to investigate undisturbed, without focusing on volume as the learning outcome.

During the interview, the recording student Astrid clarifies that the glasses have equal form. She admits she does not recognise the relevance of this information to start with. To compare two glasses with equal form, you can just compare the height of the liquid in the two glasses. This is crucial for understanding the difference between the attributes of two-dimensional height and the three-dimensional volume.

Astrid admits that her subject knowledge is not thoroughly integrated: "My thoughts are spinning, but they are not coming out right in the moment. But after processing them for a while, they might come out another time", she says. The importance of reflection is consistent with the findings of Benz (2016), indicating that content knowledge and action-related competencies are interweaved by reflection.

The commenting student describes Astrid as a *passive observer*. After some discussion in the group, they argue that she is not exclusively passive. She does not stop the children from carrying out their investigations. She is not concerned that the children might spill the water. The children are permitted to carry on with their investigation when the meal is over and the other children have to go to the wardrobe, urged by one of the other staff. It is clear that Astrid recognises the children's activity as valuable. The student shows the ability to *deviate from the agenda*, a characteristic of *contingency*. Time and schedule of the day regulate the life in kindergartens, and interruptions are quite frequent in the recorded observations. "You have to explain mathematics to the staff in order to make them understand the potential learning in spontaneous daily life situations", one of the students stated.

The group discusses if Astrid's relatively passive role might be due to the lack of equipment. They conclude that she could have introduced glasses with different forms, different water bottles or leftovers like milk boxes, all things common in kindergartens. These could have led the children to investigate the difference between height and volume of the water.

As a follow-up activity, the commenting student suggests exposing the children to two glasses with different shapes. This activity might help to "recognise the difference between one-dimensional and three-dimensional properties". The suggestion reveals subject knowledge.

Astrid, the recording student, describes the situation as a role play about selling water. She emphasises the importance of letting the children investigate themselves.

> Astrid: …staying in that investigation is the most important thing in this situation. I am wondering about the role of the teacher. Should we point the children to certain facts, or is this not so important?

Astrid addresses the purpose of mathematics in kindergarten and the role of the kindergarten teacher. She stresses the value of children investigating by themselves, and the kindergarten teacher "not pointing to certain facts". This can be interpreted as a sound justification for a passive role, following Biesta's concept of subjectification.

Her utterance relates to the dichotomy between process and answer. Several of the students express that the best thing might be not to interrupt. On the other hand, one student in the group argues that one should not be so afraid of intervening, an utterance which can be connected to Biesta's qualification category:

> I think that most kids are excited about adults trying to expand on this. Of course, it depends on how they react. To tell them about *volume* cannot hurt anyone. And, this is, in a way, our responsibility when we have the possibility to bring it further.

Astrid reflects on the challenges of making relevant responding moves in the moment: "Afterwards it is possible to reflect on that there exists something else than height, and then introduce the word volume to the children", one of the students says.

20.3.6 An Empty Cardboard Box—Lina's Observation

The situation is chosen because it focuses on reasons for being passive.

> Two boys, three years old, are playing with a cardboard box, turning it around in different positions, making joyful noises. One boy is inside and the other outside. They close the flaps of the cardboard box, and both climb inside. When a third boy comes by, they tell him there is no room for him. This boy closes the flaps of the cardboard box, knocks on the "roof" and leaves the place.

Lina, the recording student, says the boys are experiencing volume, and that their bodies can be regarded as measuring units. As a follow-up activity, the commenting student suggests bringing in more cardboard boxes to examine how the size and form of different cardboard boxes influence how many children a cardboard box may contain. The students focused on the evident mathematical aspects of the situations, which might be regarded as subject knowledge.

The commenting student describes Lina as a *passive observer*. In the following discussion, the group elaborates positive aspects of the passive role, such as to observe carefully and evaluate whether children need help or not might express sensitivity to the children. They argue that the best choice might be to let the children investigate by themselves without interrupting, especially in short observations. A glance from a child can be a signal for wanting help. They argue that this requires a high degree of presence and is not a passive role.

The group discusses how the situation might have developed if Lina had been more involved. A classmate argues that she could have been more verbally active, naming the children's actions. Another student argues that this might distract the children, describing how 2-year-olds often communicate "stop, go away!", meaning that they want to be left alone.

20.3.7 Santa Hats—Lars's Observation

The situation is chosen because the children spontaneously use an arbitrary measuring unit and a rational number. The student seems to play a passive role, but his input is crucial, especially considering the age of the children.

> I am sitting at the breakfast table with Renate, four years old, when Johan, three years old, enters the room. He smiles knowingly and brings out two small Santas belonging to the kindergarten. They have been at his home for the weekend. Excited, he tells what he and the Santas have done during the weekend.
>
> "They are of equal length," he says. "How do you know that?" I ask and he responds that it is because their hats are the same length. He puts them on the table and compares them to each other. "They are the same height," he says. Renate claims that her hat is bigger than that of Santa. She runs to the wardrobe and grabs her hat. "Look, it is double the size!" she shouts. Johan starts to measure Santa's hat putting it upon her hat, while Lars puts his thumb as a mark. The children conclude that it is one and half Santa hat, and Johan is shouting: "Come, let us measure my hat as well!" and they run to the wardrobe.

No one in the student group recognises the usage of Santa's hat as a measuring unit at once. They do not mention rational numbers, or that Renate's hat is "one and a half" the size of a Santa's hat. This might be due to lack of subject knowledge. It can also illustrate the challenge of immediately discovering what the children are about to understand, which is related to recognising relevant concepts, and thus related to *connection*.

The commenting student describes how one single question posed by Lars starts the whole measuring activity. In addition, Lars helps them by using his thumb as a mark. These two contributions are the only ones the recording student makes. He is not described as passive. A classmate claims that he is "helping the children to investigate the mathematical ideas further". No one seems to recognise that the children manage to measure, almost by themselves, except Lars helping with his thumb.

The follow-up activity includes transferring the activities to the circle time. "The kindergarten teacher takes an initiative to a theme, which the children can develop according to their interests", the commenting student explains. The idea is to share the measuring of the hats with the other children, thereby letting them compare their own hats *directly*, focusing on the *verbal* side. "We can inspire the children to describe the differences between their hats by using words for comparison", says the commenting student. He describes the circle time as a form of guided play.

The follow-up activity does not take into account that the children already measure by a unit and use adequate words for comparison. It is unclear how the follow-up activity brings the children's mathematical ideas *further*. This might be due to the challenge of anticipating complexity. It can be a good idea to use direct comparison, but the proposal is not justified mathematically.

20.4 Discussion

The study reveals that it is easier to recognise mathematics in daily life than to act upon these situations. It might be surprising that the students seem to recognise *measuring* situations quite easily, an area where kindergarten teachers have the second lowest score in mathematical knowledge (Lee, 2010).

Some aspects of KQ seem to be more relevant than others. These are subject knowledge and awareness of purpose belonging to foundation, anticipating complexity, sequencing and recognising relevant concepts, and finally, the contingency aspects: the ability to respond to children's ideas, to use opportunities and to deviate from agenda.

In the presentation of The Knowledge Quartet, pedagogues and jazz musicians underline the importance of practice. The ability to improvise is connected to practice because it provides a repertoire of possibilities to act upon.

One might expect the contingency domain to be most frequent in spontaneous situations, but the analysis reveals that this is not the case. Contingency is regarded as the most difficult of the four categories of KQ, but it seems that other aspects of KQ are also challenging in a spontaneous situation. This applies especially to subject knowledge. Basic concepts of measuring are direct comparison, transitivity and measuring unit, labelled as subject knowledge. This turned out to be difficult to recognise and to be understood in depth.

The analysis reveals that the challenge of responding to children's ideas is closely related to recognising what kind of help the children need, and what they are about to understand, aspects concerning subject knowledge. The "responding moves" seem to be some of the most difficult interventions for novice teachers to master (Brown & Wragg, 1993, in Turner & Rowland, 2011, p. 202).

It seems difficult to use information from the spontaneous situation to plan a follow-up activity. The analysis reveals that the follow-up activities often are vague. Sometimes the activity is too easy, sometimes too difficult.

20.4.1 Reflecting on Practise with the Quartet Knowledge

This study is based on a task given in the beginning of the mathematics course. This is also mentioned by the students. "You should come back and interview us at the end of the course", one of them says. I will argue that even after the end of the mathematical course, they are still novices with a very limited repertoire. Other researchers have also questioned whether preservice teacher can be expected to provide suggestions on how to develop the children's mathematical understanding, also at the end of the course (Lembrér, Kacerja, & Meaney, 2016).

To what degree can the framework of KQ help novice teachers to develop themselves as teachers? I have described the aspects of KQ, which emerge from the data. My conclusion is that the focus has been more on the students' lack of knowledge than on what they master. This might be a consequence of the KQ being a *mapping* tool.

In The Knowledge Quartet it is not totally clear if knowledge is to be regarded as something static, or if it is contextual. Mason and Spence (1999) argue whether contingency has to do with *knowing how to act in the moment,* which does not coincide with *knowing about.* It is not a fixed knowledge that you possess or not. It is the ability to act, an ability that can be trained. They argue that *mental imagery for pre-paring* can be one way of training the ability (1999, p. 156).

20.4.2 The Active-Passive Dilemma

The students emphasize aspects of being passive, which in fact make them active observers. The study reveals that it can be challenging to decide when to be passive, when to intervene and what kind of responding moves might be relevant. The reasons for being passive turn out not to be exclusively lack of subject knowledge.

The passive role includes giving children time, space, equipment and possibility to play and to investigate. Time and space are scarce factors in kindergartens, so this might be a challenge.

The task of the kindergarten teacher is not always to find the right responding move mathematically, but to decide whether to intervene or not. Being passive is important and sometimes difficult. The study illustrates why it might be a good idea to withdraw and let the children play undisturbed. To investigate by oneself is regarded as the best way of learning (Polya, 1945/1971).

The framework of KQ addresses teaching situations with an active teacher. Thus, KQ seems to be less adequate for describing the passive role. In the study, the children initiate the situation and own the problem. This is different from a school context where the teacher poses the problem, often having to make some effort to get the children understand the problem.

This is consistent with the study of Mosvold et al. (2011), finding that the tasks of kindergarten teachers were different from the tasks of school teachers. In kindergarten, the task of the teacher is described as "facilitating and using activities and play situations that enable children to experience mathematical ideas" (2011, p. 1810).

Some of the utterances of the students reflect the qualification aspects of Biesta, for example the importance of introducing the children to the concept of volume and making them understand the difference between height and volume. One of the students in Astrid's observation, does not hesitate to introduce the concept of volume. He argues that "the kindergarten staff have a responsibility to do so". Other utterances reflect the subjectification aspect, stressing the value of the investigation itself, questioning whether it is right to "point them to certain facts".

20.5 Conclusion

In this study, I addressed two research questions: What are the challenges of Early Childhood Education students on supporting children's spontaneous in measuring activities? How do the students reflect on being active versus passive?

Since the study is qualitative, it cannot reveal the prevalence of different aspects of KQ, but it can indicate the nature of the challenges. The analyses reveal that the category of contingency is not as dominant as expected. The category of subject knowledge turns out to be frequent.

Further, mastering contingency does not always correspond with having subject knowledge. A student might be able to show contingency knowledge, but not subject knowledge. Mason and Spence (1999) argue that knowledge is not something fixed which you possess or not. It is *the ability to act*, an ability that can be *trained*. It is necessary to have time to reflect, to have integrated knowledge. The students have very little time to train. Therefore, it is hard to tell whether their main problem is lack of KQ-knowledge or lack of training.

Daily life situations are regarded as important potential learning situations. As all the situations in the study are spontaneous, it can be challenging to act in the moment, and it might be too hard for the students to start with. To design a follow-up activity based on observations of the children's activities might be easier when you have some time to think about it, and do not have to act on the spot. When the students have time to reflect in advance, they may have better chances to learn from their experience. This can give a slightly different meaning to "using daily life situations as learning activities".

The second research question has to do with the passive-active-dilemma. I started with the assumptions that many students are passive in mathematical situations because they lack subject knowledge. The reasons for being passive turn out to be more complex, and the students often show sound pedagogical judgement. Discussing this theme, they come up with many ways of being passive, stances which can be labelled as active.

This highlights the question about the purpose of mathematics in kindergarten. To what degree should we stress learning for the future, and to what degree should we focus on play and investigation for its own sake, not stressing the mathematical outcome? This is an important question concerning the purpose of mathematics in the early childhood education.

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Chapter 21
Kindergarten Teachers' Stories About Young Children's Problem Posing and Problem Solving

Trude Fosse, Troels Lange, and Tamsin Meaney

21.1 Introduction

In this chapter, we explore the stories told by 16 *barnehage*[1] teachers about the photos they and their peers had taken of children engaged in mathematics, as interpreted by the teachers. The photos, used as stimuli for focus group interviews, came from a larger research project. Here, we focus on how the teachers explicitly and implicitly described children's problem posing and problem solving in relationship to Bishop's (1988) mathematical activities of Explaining and Playing.[2] There are two reasons for undertaking such an investigation. First, the mathematical aspects of the Norwegian curriculum for *barnehage*, known as the Framework Plan (Kunnskapsdepartementet, 2017), has traditionally been based on Bishop's six activities (Reikerås, 2008), and, second, in the new curriculum for *barnehage* there is a stronger emphasis on problem solving than in the previous version. Our results form a basis for future developmental work with the *barnehage* teachers, but are also likely to be of value for other studies about problem posing and problem solving in early childhood mathematics education, which has received little focus in the past (Lowrie, 2002).

Given current global discussions about the need for instruction and/or construction in mathematics education in early childhood (see, e.g., Benz, Steinweg,

[1] We use the term "barnehage" for institutions providing early childhood education and care (ECEC) for 1–5-year-old children in Norway. Although this term is translated to "kindergarten", kindergarten carries a variety of meanings across countries, so we choose to use *barnehage* to recognise the specificity of ECEC in Norway.

[2] Bishop's 6 mathematical activities are capitalised to distinguish them from everyday meanings of the terms.

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Gasteiger, Schöner, & Vollmuth, 2018), we consider that a more in-depth understanding of the relationship between Playing and Explaining and problem posing and problem solving is important. Problem solving has been a component in mathematics school curricula for some time (Lester, 1994). However, in the last 20 years, there has been a shift in viewing it as something that students needed instruction in, to something that could support their construction of knowledge. As Palmér (2016) stated, the emphasis has "shifted slowly from a view where students first need to learn mathematics in order to become problem solvers to a view where problem solving is to be taught as content itself toward today's view that problem solving is a strategy for acquiring new mathematical knowledge" (p. 256). In discussing young children's learning of mathematics, Clements and Sarama (2007) stated that "problem posing on part of students appears to be an effective way for students to express their creativity and integrate their learning" (p. 143). However, they noted that very few empirical studies have been conducted on this, particularly involving young children.

Given that problem posing and problem solving have been linked to the construction of mathematical knowledge, we contend that they need to be seen as being related to two of Bishop's (1988) six universal mathematical activities, Playing and Explaining. Yet, Bishop (1988) did not explicitly mention problem posing or solving and so the connections are implicit. We provide extensive quotes from Bishop's (1988) writing about these two activities and then discuss the relationship:

> Clearly playing is a form of social activity which is different in character from any other kind of social intercourse which has been mentioned so far—playing takes place in the context of a game, and people become players. The real/not real boundary is well established and players can only play with other players if everyone agrees not to behave "normally".
>
> Could these characteristics be at the root of hypothetical thinking? Could playing represent the first stage of distancing oneself from reality in order to reflect on and perhaps to imagine modifying that reality? Certainly, Vygotsky (1978) argued that "the influence of play on a child's development is enormous" (p. 96) in that action and meaning can become separated and abstract thinking can thereby begin. (Bishop, 1988, p. 43)

Bishop (1988) described Explaining as:

> The sixth and final "universal" activity I call "explaining", and it is this activity which lifts human cognition above the level of that associated with merely experiencing the environment. It focusses attention on the actual abstractions and formalisations themselves which derive from the other activities, and where these are related to answering the relatively simple questions of "How many?", "Where?", "How much?", "What?" and "How to?", explaining is concerned with answering the complex question of "Why?". (1988, p. 48)

In these passages, Bishop (1988) proposed that Playing could be linked to answering how-to questions and the development of hypothetical thinking, which he related to the development of abstract thinking. Similarly, he linked Explaining to abstract thinking and connected it to answering why-questions.

Descriptions of problem posing and problem solving also make links to hypothetical and abstract thinking through enacting students' curiosity about the world. For example, Hiebert et al. (1996) advocated for students to make the subjects that they studied problematic, in that "students should be allowed and encouraged to problematize what they study, to define problems that elicit their curiosities and sense-making skills" (p. 12). Silver (1997) described the need for flexibility in problem posing in ways that are similar to the predicting, guessing and hypothetical reasoning that Bishop (1988) connected to Playing. In particular, based on the work of Brown and Walter (1983, cited in Silver, 1994), Silver emphasised the value of "what-if" and "what-if-not" questions when posing problems. By requiring people to pose "what-if" questions in order to explore something that is unknown, problem posing can be seen as being based on hypothetical thinking.

Similarly, problem solving can be linked to Explaining in that they both require abstract thinking to move beyond the experiential reality of a situation, abstracting out of it certain features while discarding others. In describing the problem solving of young children, Carpenter, Ansell, Franke, Fennema, and Weisbeck (1993) stated, "many problems can be solved by representing directly the critical features of the problem situation with an equation, a computer program, or a physical representation. Modeling also turns out to be a relatively natural solving process for young children" (p. 428). In this quote, identifying different representations of key features of a problem was connected to the problem-solving process, with hypothetical and abstract thinking considered as important for children constructing their own mathematical knowledge.

In this chapter, our focus is not on young children's actions, but on how teachers in a Norwegian *barnehage* considered young children to be engaging in problem posing and problem solving. With the extra emphasis on problem solving in the new curriculum (Kunnskapsdepartementet, 2017), it is valuable to document what teachers identified as young children's problem posing and problem solving. Yet, teachers may have difficulties identifying young children's problem posing and problem solving. In a study from New Zealand, Anthony, McLachlan, and Poh (2015) found that the early years teachers were more comfortable describing easily-identifiable mathematical situations than the mathematics occurring in free play. Identifying problem solving and problem posing may be difficult for the same reasons. So, to design the data gathering method, we needed to find ways to understand the *barnehage* teachers' implicit views about problem solving and problem posing.

21.2 Methodology

The data for this chapter comes from a much larger project. Dorota Lembrér, who also has a chapter in this book, designed the study in collaboration with the authors of this chapter. Data was collected using a photo-story methodology (see Black, Choudry, Pickard-Smith, Ryan, & Williams, 2016; Black, Williams, Choudry, Pickard-Smith, & Ryan, 2016; Clarke & Robbins, 2004; Hauge et al., 2018). As part of this larger study, fourteen teachers at one *barnehage* began by photographing children engaging in situations that the teachers considered involved mathematics.

A month later, they, and two others who had not taken photos, participated in focus group interviews (ten teachers in one interview and six in another).

A set of photos were chosen for each interview to ensure that all of Bishop's (1988) six activities were evident and that each teacher had at least one photo in the set. Lembrér, in consultation with Troels Lange and Tamsin Meaney, chose the photos as being representative of Bishop's (1988) six activities. The photos were first sorted into groups connected to the six activities (more details on this can be found in Lembrér's chapter in this book). Given that the photos could often be connected to more than one category, Lembrér, Lange and Meaney discussed the justifications for classifications that were not immediately clear. Although the researchers were clear on the classification of the photos, they did not assume that the teachers would see the connections to Bishop's six activities, even if they were familiar with them. Thus, the stories that they told about the photos had to be analysed without a preconception that they would be clearly about the activities we saw in the photos.

The use of the photos in the focus group interviews encouraged the teachers to tell stories. Initially, the teacher who had taken the photo was asked to describe what the child(ren) were doing. This was then followed by comments on this description and stories of similar events that occurred in the *barnehage*. More than one teacher contributed to each story. In one interview, there were explicit discussions of the problem solving the children were engaged in. This made us aware of the possibility of analysing the teachers' stories, in regard to the children's problem solving and problem posing.

Consequently, two of the authors, Fosse and Meaney, went through the transcripts identifying instances when Bishop's activities of Counting, Measuring, Locating or Designing were evident in the teachers' stories. In the categorisation, we looked for words that had to do with amounts (Counting), comparisons (Measuring), positions and movement (Locating) and shapes (Designing). We then related these four mathematical activities to Playing and Explaining. Table 21.1 provides an example of a story by T6 on how we colour-coded the original Norwegian transcript (here blue is for Counting, while yellow is for Measuring), before identifying links to Playing, Explaining and problem solving.

If a problem was not described explicitly by the teachers, we made a suggestion about what we considered the problem to be that the children seemed to be working on. We also noted if the teachers indicated whether the problem arose from the children's interest or if it was proposed by the teacher. Finally, we discussed our individual interpretations with Lange.

In the examples, the teachers were identified by the focus group interview they participated in (FG1 or FG2) and with an individual teacher identification number. Thus FG2:T1 refers to Teacher 1 in Focus Group Interview 2.

Table 21.1 Initial categorisation of one of a *barnehage* teacher's stories

Transcript	Explaining	Playing	Problem solving
Her leker vi Gullhår og de tre bjørnene Og her tenker jeg matematikken med liten, mellomstor, for det har de i skjeer også Og senger. Og det er en pappa-bjørn, en mamma-bjørn og den lille bjørnen. Så det er tre bjørner selv om det her bare vises to. Men de er veldig opptatt av. at det skal være tre. En liten og en imellom og en stor i stoler og i sengene	En liten og en imellom og en stor. I stoler og i sengene	Her leker vi Gullhår og de tre bjørnene	Men de er veldig opptatt av. at det skal være tre. En liten og en imellom og en stor i stoler og i sengene
Here we play Goldilocks and the three bears And here I think the mathematics with small, medium, because they have that in spoons too And beds. And it is a daddy bear, a mummy bear and the little bear. So there are three bears, although only two are shown here. But they are very preoccupied with that there must be three. One small and one in between and one big one in chairs and in the beds	One small and one in between and one big one, in chairs and in the beds	Here we play Goldilocks and the three bears	But they are very preoccupied with that there must be three. One small and one in between and one big one, in chairs and in the beds

21.3 Results

In the following sections, we present four stories as representative of the set of teacher stories. Each of these stories exemplified how Bishop's (1988) mathematical activities Explaining and Playing led us to identify four components that figured in stories about problem posing and problem solving. These components are brought to the fore in our description of our results.

21.3.1 Goldilocks and the Three Bears

In Scandinavia, the use of boxes filled with objects connected to specific fairy stories is very common in early childhood institutions. It is, therefore, unsurprising that a photograph was taken of children playing with one of these boxes, in this case, the one relating to Goldilocks and the three bears. Considered an everyday experience in *barnehage*, this photo was chosen for discussion in the focus group interviews.

Nevertheless, others may query this choice as researchers have previously raised concerns about the value of Goldilocks and the three bears with regard to young children's learning of mathematics. For example, Walkerdine (1988) criticised the assumption that young children's own families would match those of the bears in the story, with the father being the largest and the mother needing the middle-sized objects. Palmér and Björklund (Chap. 15) discussed how the comparison term "just right" changed its meaning during the telling of the story, making the mathematics complicated to convey to small children. They highlighted the importance of the teacher's mathematical knowledge in supporting the children to understand the mathematical ideas.

Figure 21.1 shows the photo that Lembrér (see chapter in this book) used in the focus group interviews with the teachers. FG2:T1, who had taken the photo, began by stating:

> *Her leker vi Gullhår og de tre bjørnene. Og her tenker jeg matematikken med liten, mellomstor, for det har de i skjeer også. Og senger. Og det er en pappa-bjørn, en mamma-bjørn og den lille bjørnen. Så det er tre bjørner selv om det her bare vises to. Men de er veldig opptatt av at det skal være tre. En liten og en imellom og en stor, i stoler og i sengene.*
>
> Here we play Goldilocks and the three bears. And here I think the mathematics with small, medium, because they have that in spoons too. And beds. And there is a daddy bear, a mummy bear and the little bear. So there are three bears, although only two are shown here. But they are very preoccupied with that there must be three. One small and one in between and one big one, in chairs and in the beds.

In this story, the problem the children worked on was not described explicitly. However, the teacher seemed to suggest that it was to do with clustering dissimilar items (bear, chair, bowl, spoon, bed) into three sets based on comparative size relationships (small, middle, big). To solve the problem, the children had to understand that it was the relationships in and between each set of similar objects that had to be maintained (e.g. small bear, small chair, small bowl, etc.). According to the teachers, the children achieved this. This can be seen in the following description given by FG2:T2:

> *Den lille har den lille skålen og den lille stolen og den lille skjeen, og lille sengen. Mammabjørnen er den mellomste og har det mellomste og pappa har det største, største, største.*

Fig. 21.1 Playing with the toys for Goldilocks and the three bears

Og så snakker de jo sammen sant, «Ja, den store skal ha den store, da må jo den lille ha den lille». Sant, at de snakker jo matematikk i tillegg til at de leker.

The little one has the little bowl and the little chair and the little spoon, and the little bed. The mummy bear is the middle and has the middle-sized and the daddy has the biggest, biggest, biggest.

And then they talk to each other, don't they, "Yes, the big one must have the big one, then the little one must have the little one". Really, they are talking mathematics in addition to playing.

Although the little bed might be bigger than the biggest spoon, it still belonged to the set of objects belonging to baby bear. Thus, the teachers seemed to suggest the children had posed a problem for themselves about making sets of dissimilar objects based on an abstract quality, relative size. The mathematics that the children engaged with when playing Goldilocks and the three bears, Palmér and Björklund (in press; Chap. 15) was seriation–ordering in a sequence according to a criterium. Using the work of Piaget (1952) and Reis (2011), they suggested that children as young as 4 years old might struggle to determine the relationship in this series. Although such problems might be difficult for young children to solve, Schoenfeld (1992) considered the solving of complex problems to be the heart of mathematics.

It also seemed that the children returned to this situation repeatedly, often varying the format. For example, FG2:T6 suggested that the children often role-played the fairy story:

Og ofte sier de sånn "ja, men du er minst, da må du være den lille bjørnen."

And often they say, "Yes, but you are smallest, then you must be the little bear."

The repetition of both the problem posing and problem solving suggests that, in Schoenfeld's (1992) terms, this kind of problem was routine. The aim of engaging in routine problems is to have children practice and acquire specific knowledge and skills, which in this case had to do with seriation. Yet, one of the differences between this situation, as identified by these teachers, and a school situation is that the children were in charge, thus making it play (Lange, Meaney, Riesbeck, & Wernberg, 2014). So, although it seemed a routine problem, it was chosen by the children, rather than being provided by adults.

The teachers raised aspects of the situation that could be linked to Explaining, because as Bishop (1988) stated, "Explaining is the activity of exposing relationships between phenomena" (p. 48). The teachers saw the children's representations and utterances as highlighting the relationships between objects. Also, FG2:T2's statement seemed to suggest that she saw the situation as being about classifying, something that Bishop (1988) considered to be a fundamental explanation in that it required an understanding of similarity. The objects could be seriated in ways that are homomorphic, that is they produced a mapping of little-medium-big to (or between) five materially different sets. According to the teachers, the children both talked about this in their allocation of roles and showed it through the clustering of artefacts.

As FG2:T2 stated, the teachers identified the situation as play. Yet, the teachers did not highlight aspects of Playing as a mathematical activity, except perhaps in regard to children adopting specific roles in the role-playing. As Walkerdine (1988) noted, young children can identify the mismatch between their own families and the bears in the story. Thus, it could be argued that by following the story and classifying the objects according to relative sizes, the children suspended reality to play within a set of rules that legitimised certain actions. Yet, "the 'as if' feature of imagined and hypothetical behaviour" that Bishop (1988, p. 23) identified as an essential component of Playing is not highlighted in the teachers' discussions. Instead it was the rule-bound nature of the game, which Bishop (1988) also considered to be part of Playing, that provided the children with opportunities to practice routine aspects of problem solving.

21.3.2 Train Crash

The second example is also a common experience in *barnehage*, that of children playing with toy vehicles, in this case a train. Trawick-Smith, Wolff, Koschel, and Vallarelli (2015) chose a wooden train as a focus toy because parents and teachers had nominated it as being used frequently in US preschools. In their research, the train set was found to support good-quality play, which included problem solving by children, regardless of socio-economic background or ethnicity.

As seen in Fig. 21.2, the child had placed a tunnel on the couch arm and a box on the floor so that a train could be pushed through the tunnel to land in the box. In describing the photo, FG1:T8 stated:

> *Her er en gutt som kjører tog, og så har han plassert en tunnel oppe på sofaen. Så kjører han toget gjennom tunnelen. Så har han flyttet kassen så toget treffer oppi. Så han har beregnet hvor kassen må stå, og hvor tunnelen må stå på sofaen for at han skal treffe.*
>
> Here is a boy who drives a train, and then he has placed a tunnel on the couch. Then he drives the train through the tunnel. Then he has moved the box so the train falls into it. So he has estimated where the box must be and where the tunnel must be on the couch for it [the train] to hit [the box].

In this description, FG1:T8 implicitly identified that the problem of having the train land appropriately as one the child had posed himself. However, FG1:T5 suggested that he may have gotten the idea from watching other children working on similar problems. Azmitia (1988) found that children learnt problem solving strategies from observing others, but in this case the teachers seemed to suggest that children could also learn to pose problems from watching others.

The teachers considered that posing and solving this kind of problem was not something the child could have done the previous year, when he was only a year old. Their comments suggest that, for this child, at this point in time this was a non-routine problem of the perplexing and difficult type described by Schoenfeld (1992). Yet, the teachers' stories suggest that it belonged to a set of problems that were

Fig. 21.2 The blurry train leaves the tunnel heading for the box

known to others in the *barnehage* and which the child would have previously observed being solved.

FG1:T9, described the situation explicitly as problem solving:

> *Han har i hvert fall skjønt dette med problemløsning at du må prøve deg frem. "Hva skjer hvis jeg gjør sånn? Nei, da skjer det. Hva kan jeg gjøre da?" Så det er klart at han har gjort seg noen erfaringer.*

> He has at least understood this about problem solving that you must try out things. "What happens if I do like this? No, then that will happen. What can I do then?" So it is clear that he has gained some experience.

The teachers indicated that by moving the tunnel and the box the child was using the problem-solving strategy of trial and error. Some time ago, Carpenter et al. (1993) found that many 5-year-old children used trial-and-error strategies to model division problems, suggesting that this problem-solving strategy is common in young children.

The teachers made implicit links between problem posing and problem solving and the mathematical activities of Explaining and Playing (Bishop, 1988). Their stories suggested that the child's actions could be considered as explanations, both of setting up a problem and of illustrating a method for solving it. In noting how the child may have learnt about this kind of problem from others, the teachers also considered that other children could learn from watching him:

> *Kanskje det er noen som ser på dette, og som vil prøve dette senere. Å se på andre, er jo å strekke seg litt lenger.*

> Maybe someone is watching this and will try this later. To look at others is to extend yourself a little further. (FG1:T5)

This child's actions were seen both as copying the explanations provided by other children and also as providing explanations for other children so they also could pose and solve similar problems. The explanations were physical, rather than verbal, perhaps reflecting the child's age. As Schunk (1987) stated, "young children may encode modeled events in terms of physical properties, whereas older children often represent information symbolically (e.g., language)" (p. 151).

Unlike the Goldilocks example, Playing appeared as trying out different "as-if" scenarios as part of the trial-and-error problem-solving strategy. As the teachers pointed out, this Playing had a clear goal. Once the goal was achieved, the problem was solved and the possibilities for Playing removed until another goal, such as changing the speed of the train, led to a change in the problem.

21.3.3 Counting Orange Boats

The children's problem posing and problem solving often had another purpose for the teacher than providing a mathematical learning opportunity. FG2:T1 explained how she had sat next to an unhappy child while he was eating pieces of orange and suggested he count the number of orange peel pieces, described as orange boats (see Fig. 21.3). Although spontaneous counting has been noted in early childhood situations (Ginsburg, Lee, & Boyd, 2008), it was the teacher who set up the problem in order to distract him from his unhappiness.

> *Og så satt jeg nå her med disse appelsinbåtene og plutselig fant ut at "Oi! det var mange. Skal vi telle dem?". Og da fikk vi en veldig grei samtale om å telle og sortere dem. Og den store appelsinen var i mange båter, sant. Og for hver båt han tygget og spiste så la han ned skall og så ville han at vi skulle telle dem på nytt. Peketelle. Telle på fingrene sine. Sånn telte han. Og så koblet han liksom båter til fingrene sine.*

Fig. 21.3 One to one matching of orange peel pieces and fingers

And then I sat here with these orange boats and suddenly found out that "Oi! there were many. Shall we count them?" And then we had a very good conversation about counting and sorting them. And the big orange was in many boats, right. And for each boat he chewed and ate, he put down the peel, and then he wanted us to count again. Pointing and counting. Counting on his fingers. That's how he counted. And then he kind of connected the boats to his fingers.

The teacher, FG2:T1, described this as an everyday situation which had some mathematics. She explained how the child found the problem challenging because he could not consistently count yet.

Det handlet veldig mye om å telle en av gangen. Så er det én, så blir det to, så blir det tre så blir det fire, og så gå tilbake og begynne på nytt igjen, sant. Ja, Nå er det sånn og sånn. Var det riktig? Han måtte vise meg fingrene sine, sant., Det som han så på hendene sine, han er fire.

It was very much about counting one at a time. It's one, then it becomes two, then it becomes three and then it becomes four and then go back and start over again, right. Yes. Now it's like this and this. Was that right? He had to show me his fingers, right.

In this situation, the child seemed happy to work on a problem posed by the teacher and to practice using counting as a problem-solving strategy. Other teachers considered this a typical kind of mathematical situation where something specific was counted.

Du har konkretene, og så skal du da telle fingrene, sant? Hvor mange ser du her, sant. Enten kan du gjøre sånn [peketelle?], sant, men du kan også telle på fingrene, sant?

You have the concrete, and then you count your fingers, right? How many do you see here? Either you can do it like this [pointing and counting?], right, but you can also count on your fingers, right? (FG1:T3).

There is no apparent purpose to the counting except to practice it, but the child actively participated and found it challenging. The teacher's story also suggests that the child took over the control of what should happen by insisting that he re-count the orange boats after he had eaten another one. The problem, of how many, was routine in that it was likely that the teachers offered similar opportunities where the solution strategy of counting was expected by both the teacher and the child. Nevertheless, as in the Goldilocks and the three bears situation, the problem seemed challenging and engaging for this child.

From the teachers' stories, it seemed that the child's actions, along with orally counting the boats, could be considered examples of Explaining in that they illustrated how the solution, the total amount of orange boats, was determined. Like the Train Crash stories, the physical actions seemed to provide the teachers with insights into the child's problem solving. For example, the teachers seemed to consider that by matching his fingers to the orange boats, the child provided extra information. In alignment with research (see, e.g., Moeller, Martignon, Wessolowski, Engel, & Nuerk, 2011), counting with fingers was mentioned by several teachers as an expected way to determine the total amount.

The teachers extended the discussion of the orange boats to discussions of other meal time situations in which mathematics could appear. Some of these, like laying the table, had some connection to different "what if" scenarios, but on the whole connections to Playing were not obvious. As was the case with Goldilocks, the teachers seemed to suggest that the main aim of the child's participation was to practice the expected problem-solving strategy, although in this case both the teacher and the child were aware that the child was still in the process of learning it.

21.3.4 Packing the Police Car

The final set of stories were to do with the photo in Fig. 21.4, in which a girl put different objects into a toy police car. Although she did not seem to be constructing something, the stories that the teachers told indicated that her experimenting resembled construction play, in which:

> Children experiment with building objects in order to learn more about the physical world and the laws that operate in the world. Higher-level thinking occurs when they attempt to solve problems that the construction materials (wood, clay, metal, and paper, for example) pose because the solution requires divergent rather than convergent thinking. (Bergen, 2009, p. 418)

FG1:T8 described how a child filled the police car with different combinations of objects, suggesting involvement in divergent thinking that explored a range of different problems and solutions.

> *Her har jeg en jente som har funnet en politibil. Så har hun satt en dame inni den og en mann baki. Og hun holdt på å sette alt oppi, prøvde å få plass til alt. Hun prøvde å se om det var plass til alle tingene i bilen, eller "må jeg ta noe ut for at menneskene skal kunne sitte inni den". Så hun satt og beregnet på å ta inn og ut og hva det var plass til. Så jeg tenkte at det handlet om romforståelse, hva det er plass til inni bilen. Hva er for stort til å være inni bilen?*

> Here, I have a girl who has found a police car. Then she has put a woman inside it and a man in the back. And she kept putting everything in it, trying to accommodate everything. She tried to see if there was space in the car for all the things, or "do I have to take something out for the people to be able to sit inside it?" So she estimated, taking in and out and what there was space for. So I thought it was about spatial understanding, what will fit in the car. What's too big to be in the car?

Fig. 21.4 Packing and unpacking the toy police car

In their comments, the teachers described the problem as being about capacity in regard to how much the child could fit into the car (FG1:T7 *Det var det hun prøvde på, hvor mange eller mye hun kunne ha oppi der.* That was what she tried, how many or how much she could have in there). Although the problem seemed to be a non-routine problem that the child had posed herself, the teachers identified it as being about an everyday situation that she might have been aware of from home:

> *Så er det sikkert noe med at hun vet at menneskene skal være i bilen og kjøre den. Og hun har jo bestemt seg for at menneskene skal være oppi. "Men hva mer kan jeg få plass til? Hva kan jeg få med meg på turen?"* (FG1:T8)
>
> Then there is probably something about the fact that she knows that people should be in the car and drive it. And she has decided that the people must be in it. "But what more can I fit in? What can I bring with me on the trip?"

As was the case in the Train Crash stories, the teachers considered that this child used a trial-and-error strategy, but this time, it was to modify the problem as well as to find different solutions to the same problem. According to the teachers, the child spent a lot of time playing with the car by herself. At one point, the teacher explained how the child placed a tower of plastic blocks in the car, but as the car drove off, the tower fell out. In reflecting on this story, FG1:T9 suggested that the child was learning about size relationships: *Så det er jo noe med forhold her, å lære seg om forhold.* (There is something about size relations here, learning about size relations). It is not completely clear how size relations were involved, but this comment suggests that this teacher saw a range of mathematical possibilities.

As was the case in the Goldilocks stories, the teachers' focus on relationships between the objects and the car's capacity suggests that they considered the child was explaining. As with the case of the other sets of stories, the child's explaining was generally done through body actions. However, the teachers also indicated that oral language was important for the children to describe what they were doing.

FG1:T7	*Det er mye læring, også for barn, når man holder på med det. Det er ikke bare en aktivitet uten at de erfarer mye, og det er sånn de lærer det, gjennom lek. Så kan vi gi dem noen benevnelser underveis*
FG1:T5	*Sette ord på det de gjør*
FG1:T7	*Så de får med seg noen ord og uttrykk som de kan ta med seg i skolen. Det er det vi tenker på, at de skal ha kjennskap til de forskjellige*
FG1:T5	*Begreper og navn*
FG1:T7	There's a lot of learning, also for children, when you're doing it. It's not just an activity, but they experience a lot and it's the way they learn it, through play. Then we can give them some terms along the way
FG1:T5	Put into words what they are doing
FG1:T7	So they get some words and expressions that they can bring to school. That's what we are thinking of, that they should know about the different
FG1:T5	Concepts and names

Although Explaining is not discussed explicitly, the teachers indicated that problem solving provided them with opportunities to develop the children's oral language. In doing so, they acknowledged that names and concepts were something needed for school, rather than for supporting their current problem posing or problem solving. Thus, although they recognised that the child's actions involved learning, they considered that the value of that learning was for the future, not for the present.

Similarly, Playing as a mathematical activity (Bishop, 1988) was discussed implicitly in regard to how the child adapted the problem, bringing "as-if" possibilities into focus. Bergen (2009) suggested that imaginative play, such as fitting things together and taking them apart, allowed children to become interested in "seeing what might happen" (p. 419). In this situation, the teachers considered that the child worked on solving the problem within the accepted rules—the objects must fit the space and not fall out when the car moved—and involved the child in testing different "as-if" scenarios, which fits the criteria for Playing as a mathematical activity.

21.4 Discussion

The stories that the teachers told about the children's engagement in mathematical situations raise a number of issues that connect problem posing and problem solving to Bishop's (1988) mathematical activities of Explaining and Playing. In our analysis, we have identified four interrelated components that we considered appeared in the complete set of stories and which we discussed in the four illustrative stories analysed in detail in this chapter. The components, shown in Fig. 21.5, that the teachers showed awareness of were: the routine or non-routine nature of the problems; known or unknown problem solving strategies; explaining through body actions or words; playing by exploring different scenarios or following rules.

In discussing the children's engagement with the Goldilocks box, the teachers seemed to consider the problems the children posed as being routine. They told that the children solved similar problems regularly. As Hiebert et al. (1996) found when children were able to problematise a situation, even if that situation seemed routine, the children were prepared to spend time working on the solution. In the teachers'

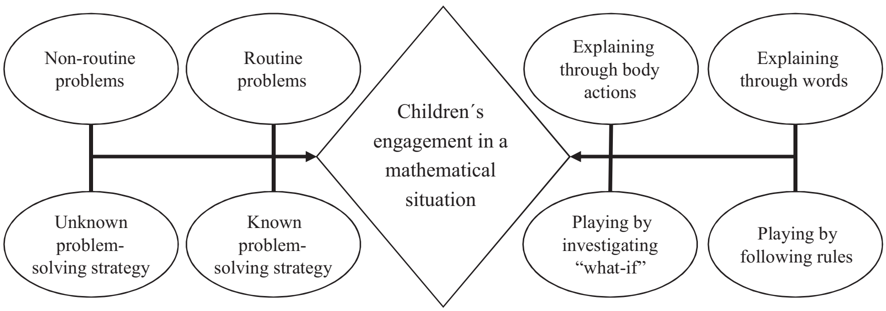

Fig. 21.5 Components of problem posing and problem solving from teachers' stories

stories about the children's engagement in Goldilocks, the children were not bored by the routine nature of the problem but had the possibility to recast it in different ways. This suggests that the routine nature of problems were linked to Bishop's (1988) description of rule following as part of Playing. The routine nature of the problem set the rules for engagement, but did not inhibit the children's desire to engage in the problem.

The teachers also indicated that the children engaged in non-routine, perplexing problems, for example in the Train Crash and Packing the Police Car stories. In both cases, the teachers considered that the children explored different "what-if" scenarios, where there was no expected solution. In the stories around Packing the Police Car, the teachers considered the child was exploring changes to the problem as well as the solutions. Thus, Bishop's (1988) description of Playing can be considered as being related to both routine and non-routine problems that the teachers identified the children engaging with.

The second component was whether the problem-solving strategy was known by the children as providing an appropriate answer, or whether the problem-solving strategy allowed the children to explore unexpected aspects of the problem as they solved it. For example, trial-and-error problem-solving strategies provided children with opportunities to explore different "what-if" scenarios. This strategy can, therefore, be considered part of Playing as a mathematical activity (Bishop, 1988). This was the predominant problem-solving strategy that the teachers described in their stories about Train Crash and Packing the Police Car. In contrast, the teachers noted that the children used problem-solving strategies in Goldilocks and the Three Bears and in Orange Boats, which both the children and the teachers expected would provide a specific result. However, using these solution strategies could still be challenging. For example, it was the teacher who proposed a routine problem (Orange Boats), where the problem-solving strategy, systematic counting, was expected to be used by both the teacher and the child, but which was something that the child found challenging and engaging.

The third component was to do with Explaining. Often, the teachers indicated that children explained their solutions and solving methods through their body actions (Train Crash and Packing the Police Car). As has been shown elsewhere, children's use of their bodies to solve problems (Meaney, 2016) and gestures (Johansson, Lange, Meaney, Riesbeck, & Wernberg, 2014) provides important insights into their mathematical thinking. Although the teachers, at least implicitly, recognised that actions could be interpreted as children's explanations, there may be some value in supporting the teachers to have a meta-language for discussing what they paid attention to in these mathematical situations, so that the discussions became explicit.

In the stories about Packing the Police Car, the need for verbal language was highlighted as being valuable for when the children went to school. Although children can learn from watching the actions of others (Johansson et al., 2014), verbal reflections might support these children to become better problem solvers who could learn from each other (Hiebert et al., 1996). Thus, the teachers could need help to support children to describe verbally their problem posing and problem solving.

The final component was about the kind of Playing (Bishop, 1988) that the children engaged in. The teachers identified two aspects of Playing which were either following a set of prescribed rules (Goldilocks and Orange Boats) or engaging with "what-if" scenarios (Train Crash and Packing the Police Car). As stated earlier, the kinds of engagement in the problem solving were related to the routine or non-routine nature of the problems that the children posed themselves or that were posed for them. The teachers generally considered that it was the children who posed problems which enabled them to engage in "what-if" scenarios. When the teachers posed problems, as in the case of the Orange Boats, there was an expectation about the kind of answers which would be found and the solution strategy for gaining these answers. We did not have any stories in which the teachers saw their role as supporting children to consider how they could pose different versions of the initial problem. This seems to be another area which could be developed through professional development.

21.5 Conclusion

According to our analyses, the teachers showed awareness of differences between problem situations that allowed us to posit four components connected to problem posing and problem solving. This is useful with the increased emphasis on problem solving in Norway, in the new curriculum (Kunnskapsdepartementet, 2017), as it can form the basis for professional development programmes that respect the play-based approach that underlies mathematics education in *barnehage*. Anthony et al. (2015) had raised a concern that teachers only documented easily recognisable mathematical learning opportunities, so it was not clear how problem solving could be discussed in the teachers' stories. Although the explicit mentioning of problem solving in the teachers' stories about the Train Crash photo showed that the teachers could discuss this, our analyses showed that the connections to problem posing and problem solving were more often implicit. It was by analysing the stories in relationship to Bishop's (1988) mathematical activities of Playing and Explaining that the nuances of how they interpreted the children's engagement in the different mathematical situations came forth.

Our analysis raised four components that the teachers paid attention to, which are illustrated in Fig. 21.5. These components showed both what the teachers were aware of and which areas could be developed further, in particular the development of children's verbal language to support their current problem posing and problem solving, as well as the teachers' possibilities for encouraging children to explore different variations of the problems or solution strategies. Nevertheless, this was a small study and more research is needed on how other *barnehage* teachers view and respond to mathematical problem posing and problem solving.

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Chapter 22
'You Are Already Bigger Than the Giraffe!'—The Use of Adjectives in Measurement Activities in Kindergarten

Sarah Keuch and Birgit Brandt

22.1 Introduction

In the paper presented at the CERME conference in 2017 titled 'The duck is the biggest' (Brandt & Keuch, 2017), we discussed, amongst other things, semantic deviations concerning the use of adjectives. This chapter aims at taking a deeper look at the role of adjectives and their function in small group interactions concerning the construction of an initial understanding of length and weight in kindergarten. The relevance of language for mathematical learning processes in early education has been repeatedly shown, and academic language proficiency is now widely acknowledged as an important factor (for example Prediger, Renk, Büchter, Gursoy, & Benholz, 2013). Unfortunately, the German school system is still in need for effective (pedagogic) approaches to support children with disadvantageous starting conditions like migration, socio-economic status or developmental speech disorder, in order to provide them with an equal chance to participate in (mathematic) education processes (Gogolin & Lange, 2010; Prediger et al., 2013).

Kindergarten teachers do not only provide language input, they also have the possibility to influence the child's language by deciding when and how to give feedback and therefore play an important role for the child's linguistic and mathematical development. While most German kindergarten teachers seem to be aware of their function as language role models, only a few have acquired a professional background that enables them to specifically support interactive language learning processes (Ritterfeld, 2000). Research projects from Germany and Switzerland investigated early years professionals' ability to support children's academic language development. Michel, Ofner, and Thoma (2014) examined German kindergarten teachers concerning their linguistic knowledge, their knowledge about children's language development and their ability to choose effective interventions.

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Only half of the items which experts see as relevant to foster the language development (in) young children were answered correctly. Isler, Künzli, and Wiesner (2014) analysed conversations between Swiss kindergarten teachers and children in order to investigate the potential for the acquisition and fostering of academic language skills. So far, their results show that kindergarten teachers have to be made more aware of the central meaning of their language acts and to support a setup of practical action patterns for the fostering of academic language skills. In our own research, we find similar results as kindergarten teachers show few approaches for supporting the children's language development in mathematic learning opportunities (Brandt & Keuch, 2017, 2018). Based on these results, the subordinate aim of our study is to raise kindergarten teachers' awareness of possible language hurdles and learning opportunities concerning a specific content, so that they are able to pay special attention to them in connection with supporting mathematical learning. Our objective is not to avoid these language structures, but to use them in a way that fosters the children's language as well as mathematic development. German and international experts agree that academic language education processes should start early, be designed age-appropriately and be oriented to a specific content (Prediger, 2015; Rudd, Satterwhite, & Lambert, 2010). With this project, we would like to make a contribution by having a closer look at special features in kindergarten teachers' interactions concerning measuring length and their potential for fostering mathematic as well as linguistic aspects with (a) focus on adjectives as an important component for differentiated language use. In this chapter, we concentrate on the two following questions:

– Which adjectives do kindergarten teachers and children use when talking about measuring length and mass?
– How do kindergarten teachers and children use these adjectives with respect to fostering mathematical as well as language aspects?

Therefore, we chose to have a closer look at measuring length in kindergarten for reasons explained in the following paragraph.

22.2 Measuring Length and Weight

Following Bishop‘s idea of mathematical enculturation, measurement is one of the six basic activities and is seen as a basis for the development of mathematics in different cultures (Bishop, 1988). Bishop claims that ‘measuring (...) is concerned with comparing, ordering, and with quantifying qualities' (p. 34). For these activities, an abstraction process is necessary, which results from a concentration on a quantifying characteristic. Real objects are compared regarding their length or weight, independent from their form, colour or other characteristics. The quantification of quality results from comparing with a unit, which is seen as a fundamental idea of all measuring activities independent from the magnitude. Here, it becomes clear that talking about comparing, ordering and quantifying qualities demands a differentiated language usage, including technical terms and specific grammatical structures and word classes, adjectives for example, that are needed, for example, to describe a comparison or quantification. Measurement, besides numbers, experiencing spatial relationships and geometric shapes, is a main content in many curricula for early mathematics education. It is of great significance for various reasons. On the one hand, it represents a link between mathematically abstract concepts and everyday life. On the other hand, measurement comprises multiple inner-mathematical relations (especially with numbers and geometry) (Barrett et al., 2011; Sarama, Clements, Barrett, van Dine, & McDonel, 2011; Skoumpourdi, 2015; Smith, van den Heuvel-Panhuizen, & Teppo, 2011). Beyond, the concept of measurement can be seen as a basis for further concepts, for example fractions and rational numbers (Barrett et al., 2011).

Our analyses concentrate on two different magnitudes, length and weight. Length, in contrast to weight, is directly perceivable, even for young children. However, length is not always easy to grasp because of the relation between length and area, and because children have difficulties in distinguishing between them (Barrett et al., 2011; Castle & Needham, 2007; Skoumpourdi, 2015). Although an integrated approach for different spatial magnitudes, especially in early education, is seen as reasonable (Barrett et al., 2011) in order to understand the differences and the fundamental idea of measuring as comparison with a unit, we only concentrate our linguistic analysis on length and weight. Length belongs to spatial measurement. Piaget, Inhelder, and Szeminska (1960) define the fundamental idea of spatial measurement this way: 'To measure (in Euclidean metrics) is to take out of a whole one element, taken as a unit, and to transpose this unit on the remainder of a whole: measurement is therefore a synthesis of sub-division and change of position' (p. 3). This change of position requires the understanding that (a) the size of the unit is conserved and (b) that the unit can be used iteratively. In doing so, the unit must be copied and repeated without a gap as well as without overlapping. Concrete objects become representations of length, and their mutual characteristic is constituted in their one-dimensional linearity (Nührenbörger, 2002). The activity of measuring length concentrates on the determination of a linear expansion. Therefore, one has to distinguish between objects with a rather clear linear characteristic, for example sticks or distances, and those objects with more than one dimension that can be measured (width, height, depth) (Nührenbörger, 2002; Skoumpourdi, 2015).

In contrast to length, weight is not directly perceivable. This means that even for direct comparisons and ordering, some kind of tool or mediator (other than one's eyes) is necessary. While hefting (holding objects in one's hands to make a judgement) comes most naturally and without the need of further devices, it is not the most precise method, for example because of different locating surfaces of differently shaped objects. Children might also have a problem with perceiving measuring weight with ordinary scales as some kind of comparison because the comparison process takes place within the scale, invisible for the weighing person, and only offers a number that needs to be interpreted. It might be helpful to focus on indirect comparisons with pan balances and Roberval balances, so that the comparison process becomes visible (Reuter, 2011). The question of how children acquire a (geometric) concept of magnitudes is dealt with in various research papers. Studies have also shown which milestones children have to master and where—from a mathematical point of view—they face special difficulties that they have to overcome (Sarama et al., 2011). We take this research as a background for our linguistic analyses, but will not discuss it in detail. Consequently, it becomes obvious that speaking about length and weight comes along with specific linguistic challenges, for example concerning the characteristic of linearity, the differentiation from area and the not always visually perceivable differentiation between light and heavy objects. The following paragraph looks at the linguistic features of adjectives and their role in talking about length and weight.

22.3 Learning Opportunities: Adjectives

The following remarks refer to the usage and characteristics of adjectives in German and English and are not in the least universal, which has to be kept in mind when analysing the language of learners of German as a second language. Adjectives are a part of speech which modify nouns concerning their quality or attribute. They can be divided into complementary (*dead* vs. *alive*) and gradable (*small* vs. *large*) adjectives. Complementary adjectives are characterised by an either–or relationship between the two members of such a pair. This means that the negation of one of the words is synonymous with the other—*not dead* is the same as *alive*. With gradable adjectives, however, this is not the case as *not big* does not automatically equal *small*, there might as well be something like 'medium' in between (Bieswanger & Becker, 2017). Further, these terms always have to be considered in relation. For example, a big mouse is most likely still smaller than a small elephant. This differentiation has to be taken into consideration when talking about measuring sizes with young children.

Gradable adjectives can be used to express comparisons. In English, there are two ways to express the comparative form depending on the number of syllables and final sound of an adjective (periphrastically *more adj* or synthetically *adj* + *-er*). In German, only the latter is possible. German uses only inflexional suffixes to form comparative and superlative forms. In German and English alike, '-er' added to a stem of a word forms the comparative form (*small-smaller/klein-kleiner*), '-est' and '-sten' in English and German, respectively, added to a word stem build the superlative form (*small-smallest/klein-kleinsten*) (Bieswanger & Becker, 2017). In German, adjectives can be used in three different ways, attributively, predicatively and adverbial. When an adjective like *slow* [*langsam*] is used attributively, it is placed between the article and the noun and in German it is declined according to the gender and number of the noun (*the slow mouse—die langsame Maus*). When used in its predicative function, the adjective is used with a copula verb like *to be* and is not adjusted to the noun (*the mouse is slow—die Maus ist langsam*). Adverbial adjectives appear with a verb and are used to describe it. Again, the adjective is not changed in German, but normally has an -ly suffix in

English (*the mouse runs slowly—die Maus rennt langsam*) (Habermann, 2009). Ninio (2004) found that children have problems analysing, understanding and producing attributive adjective–noun combinations.

When learning a new language, children (and people in general) deduce the meaning of new words from the input they receive. Information about the word class can be deduced from the syntactical structure in which it occurs. For the word class we are interested in, namely adjectives, this means: If a word occurs within a certain sentence frame, like 'this is an x one' (*this is a small one*), x is likely to be an adjective (*small*). Inferring the word class from the syntactical surrounding, however, does not automatically tell the meaning of the word. This is especially important with adjectives that describe non-perceptible characteristics (Corrigan, 2008). Even if they cannot immediately learn the meaning of a word, the information in the input can still be relevant for the acquisition process. Corrigan (2008) claims that explicit teaching might even be less important than exposing children to rich linguistic contexts.

By using adjectives in their conversations, adults in general and kindergarten teachers in particular provide information about its meaning, most often unintentionally but sometimes with full awareness. There are four different types of information that can be provided by adults to support vocabulary learning. In a more explicit way, they can give an explicit definition or synonym (1). In a less explicit way, adults can provide a semantically related word/phrase (2). Further, they can give comparing or contrasting information (3). In this context, comparative and superlative forms play an important role because they automatically provide comparative information if the morphological and syntactical structure is known to the hearer. Finally, adults can support understanding by providing evaluative information, expressing whether the target word is bad, good, weak or strong (4). In addition, the referring noun can help understand the meaning if it is provided either as a specific noun or as a personal pronoun. The use of pronouns that refer to things, general nouns like *thing* or *one* or even the complete omission of nouns might hinder the acquisition. In everyday situations, though, input often contains deviations from the standard language variation and possible irrelevant information (Corrigan, 2008).

While adjectives also play an important role in arithmetic contexts to describe the relation between numbers, their usage is crucial for measurement. As Bishop says, there is a clear cultural need for a language to be able to express qualities, comparisons and ordering before developing units for measurement (Bishop, 1988). A magnitude is marked by assigning joint characteristics to real existing objects in an abstraction process (Nührenbörger, 2002). To verbally express these characteristics, German and many other languages use adjectives. Lorenz mentions the comparative form as crucial for expressing relations and the development of an understanding of length transitivity (Lorenz, 2008). If you compare two objects regarding a magnitude, you put them in relation to each other. In this case, the equivalence relation and the order relation become important in a mathematical and linguistic sense.

The equivalence relation is defined by reflexivity and symmetry. Linguistically, this relation is expressed by '*is as long as*' for length and for mass, (that is)'*is as heavy as*'. Asymmetry and transitivity characterise the order relation, which is expressed by 'longer than' and 'heavier than'. While mathematically, the verbalisation of these relations seems clear and limited, our aim is to find out if the use of adjectives and syntactic constructions differs in oral face-to-face communication with small children, and if these variations might support or hinder the conceptual development.

Another difficulty might be found in the omission of reference objects. If the positive form is used without a reference object, it is called absolute positive (Albers, 2007) as in 'Tom is tall'. Presuming that Tom is a northern European male adult, the sentence means that Tom is taller than the normally expected body size of a northern European male adult (p. 11). This inference, however, stays implicit and might lead to difficulties understanding the sentence if you either do not know about this implication or if you have not (yet) established a benchmark for a northern European male adult or the respective reference norm. In the example 'Sally is three inches shorter than Bill' (p. 197), 'Bill' serves as the reference object. By the comparative adjective, a path away from Bill is specified along a certain scale of value. The distance can be further specified by the use of quantifiers or measure phrases, as in the example. Without a specific reference object, there cannot be a path and the distance—and therefore the meaning—stays unclear or at least vague (Jackendoff, 1983). In her literature review, Corrigan (2008) finds that there is a research gap in the kind of information adults provide about the meaning of adjectives when they are not explicitly teaching language. Her '[f]indings highlight the importance of looking at adult input in situations where teaching word meaning is not an explicit goal' (Corrigan, 2008, p. 159). Against this theoretical background, we will reconstruct the use of adjectives to detect aspects of language support and to show the connection to specific meanings and concepts that are negotiated in certain situations.

22.4 Research Design and Analysing Methods

The data basis for our analysis stems from the project erStMaL (early Steps in Mathematical Learning) (Acar Bayraktar, Hümmer, Huth, & Münz, 2011). Within this longitudinal project, there are 17 videotaped (15- to 51 min long) small group interactions with one kindergarten teacher and two to five children each, designed and prepared by the teachers themselves focusing on different magnitudes, namely length, weight, and volume, including combinations of two or three of them. For this project, the nine situations that mainly deal with length and the five situations that focus on mass were chosen for further analysis. Since some teachers and children participated in more than one setting, the sample includes data from 9 teachers and 35 children at 6 kindergartens. These 14 situations serve as a data basis for the presented project and were transcribed with EXMARaLDA and coded with MAXQDA.

In order to answer the first research question, each adjective used during a measuring process or while talking about measuring length or weight was coded in combination with their grammatical information concerning the speaker, the form (comparative, superlative) and function within an utterance (attributively, adverbial or predicatively).

In order to answer the second research question, we follow methods from interactional linguistics (Selting & Couper-Kuhlen, 2000). Interactional linguistics takes an interdisciplinary and cross-linguistic perspective on language. It looks at the structure and use of language, capturing it in its natural environment, the social interaction. Based on the linguistic element used in the utterance, we look at their role in the conversation. In our context, these linguistic elements are adjectives describing or accompanying measuring processes. We are especially interested in situations in which adjectives are used in ambiguous ways which might lead to different meanings within the child’s and the kindergarten teacher’s mind. Apart from that, we are interested in structures that deviate from a normatively correct way and which might therefore inhibit the construction of measuring concepts or at least make it harder for children to understand the concepts of length and mass.

22.5 Analysis and Interpretation of Empirical Data

In order to answer the first research question, which adjectives are used by kindergarten teachers and children to talk about length, every adjective used was coded and is displayed in the following table, independent from the grammatical information and function in the sentence:

While there is a total number of 11 adjectives that kindergarten teachers and children in our empirical examples use to talk about length within the nine situations, only about half of them are used more than five times. One can see at first glance that *big* [groß] and *long* [lang] seem to be popular adjectives while *thin*, *narrow* and *low* are less often used. However, the first-mentioned adjectives might cause confusion because of their various meanings. In German, *groß* has nine different meanings. Like *big*, it can be meant in terms of size, age or time, but also intensity or degree.[1] This also applies to the adjective *long*, which can be used to talk about a stretch of time or a geographic expansion.[2] When answering our second research question, how these adjectives are used in interactions, we will take a closer look at these ambiguities and see if and how these different meanings could lead to mathematic or linguistic problems. Each of the adjective’s antonyms (except for *medium*) is included in the situation, which could support to deduce the meaning

[1] https://www.duden.de/rechtschreibung/grosz

[2] https://www.duden.de/rechtschreibung/lang_Adjektiv_auch_raeumlich

from the context. If you look at two antonymic adjectives, for example *big* and *small*, it is true for each pair that the one with the higher degree is named more often. In total, 611 adjectives concerning measuring length can be found in the nine situations, which equals about 68 adjectives per situation. All adjectives are used more often by kindergarten teachers than by children, with five of them being exclusively used by the teachers.

Within the five situations that dealt with the magnitude weight, teachers and children used ten different adjectives. Seven of these ten adjectives were used more than five times. While *big* and *small* also occur in situations concerning length, the others seem to be specific for weight-situations. The adjective *heavy* is used most often, while *strong*, *long* and *full* appear less often. Only four adjectives (*heavy, much, light* and *small*) are used more often by kindergarten teachers than by children. *Strong* is the only adjective that is used by the children but not by the kindergarten teachers. Since it is semantically not directly associated with weight, we will have a look at the occurrences later. Quantity (*much*, *less*) also seems to be strongly connected with weight. This might have to do with the various occasions where non-standard units like bears or stones are used to determine the weight of certain objects or to compare these units with each other. The weight of the balance pan can be changed by changing the number of bears or stones within it. The high use of adjectives in weight-situations might also have to do with the fact that since the weight of an object is not visually perceivable, children and kindergarten teachers feel a higher need to verbally express these characteristics. Based on these results, it seems that talking about weight is accompanied with a greater use of adjectives, especially for children. If situations designed to learn about mass naturally offer more possibilities to use adjectives, they might be a good place to use these opportunities to foster semantic, morphologic and syntactic characteristics of adjectives. The following analyses will shed more light on the actual use of these adjectives within the situations.

The subsequent tables refer to the use of the comparative forms within the situations and their syntactical incorporation. Especially, these tables look at whether the utterance contains a reference object or not. It is further divided into 'complete' (the normal '*x is bigger than y*'- structure) and the post-positioned structure ('*which is bigger, x or y*'), which mostly occurred in questions, or no reference object. The analysis is conducted with three different adjectives, *big* as one adjective that is used in situations with length and weight, *long* as the adjective most occurring in length-situations after *big*, and *heavy* as the adjective used most often in weight-situations.

From a total of 242 coded occurrences of *big* (see Table 22.1), there are 98 tokens (40%) used as comparatives. When looking at the use of *bigger,* it is used without a reference object nearly twice as often as with its complete form. Although not present on the linguistic level, one might argue that the reference object is often clear within the situation because of non-verbal cues. Since we focus on verbal linguistic aspects and their relation to mathematical concepts, these kinds of utterances are also counted as incomplete, especially since they might still cause dissonances, as we will show in the following paragraph on examples for learning opportunities.

Table 22.1 Adjectives used when talking about length and weight sorted by quantity

Measuring length	Total[a] (teacher)	Measuring weight	Total (teacher)
Big [groß]	242 (78%)	Heavy [schwer]	318 (74%)
Long [lang]	114 (77%)	Much [viel]	178 (63%)
Small [klein]	100 (69%)	Light [leicht]	105 (58%)
High [hoch]	81 (61%)	Small [klein]	75 (52%)
Thick [dick]	30 (80%)	Big [groß]	73 (47%)
Short [kurz]	29 (93%)	Empty [leer]	18 (39%)
Medium [mittel]	5 (100%)	Few [wenig]	18 (50%)
Low [niedrig]	5 (100%)	Strong [stark]	5 (0%)
Wide [breit]	5 (100%)	Long [lang]	2 (50%)
Narrow [schmal]	1 (100%)	Full [voll]	2 (50%)
Thin [dünn]	1 (100%)		

[a]The first number in this column indicates the total number of times the adjective occurred within the teachers' and children's dialogues. The number in parentheses indicates the percentage of instances when the adjective is used by the teacher to give a better idea about the distribution of one adjective compared to another and of the teacher's and children's use

Children produce almost twice as many incomplete as complete structures, although the overall number is rather low. The post-positioned structure with *big* is only used by kindergarten children.

With a total number of 318 coded tokens, *heavy* is the most used adjective (Table 22.1). The comparative is used in 146 cases (45%). Hence, *heavy* is used as the comparative form most often, from the percentage and from the absolute perspective. The syntactical incorporation of *heavy* looks quite different from the one of *big*. Only a little over 10% of all utterances are complete, and all but one of them are produced by the kindergarten teacher. The frequent production of incomplete structures by kindergarten teachers might lead to imitations by the children and also to an only limited understanding.

The analysis of *long* again shows a completely different picture. While it is used 114 times in total (Table 22.1), the comparative form only occurs 25 times; thus, the comparative form is only used in 20% of all cases. Only once is it produced within a syntactically complete structure. Children only use the comparative form twice, both in an incomplete way. The infrequent use by children might be connected to the infrequent and incomplete use by the kindergarten teachers. Hence, the children do not have the linguistic model, and because of the incomplete usage, they might not be able to grasp the complete meaning of it.

Table 22.2 also shows that the adjectives used in the situation refer to different dimensions. This shall now exemplarily be shown for length (Table 22.3). *Thin*, *thick, wide* and *narrow* normally describe the distance between sides of a two- or three-dimensional object (Width). *Long* and *short* describe a clearly horizontal and also one-dimensional distance (Horizontal). With high and low, the expansion expressed is rather vertical (Vertical). *Small, medium* and *big*, amongst other definitions, refer to a rather three-dimensional expansion of a person or thing (Expansion). However, as mentioned above, *big* can also be used with other meanings. Together

Table 22.2 Reference objects used with 'bigger' (length- and weight-situations), 'heavier' (weight-situations) and 'longer' (length-situations)

	Complete	Post-positioned	Missing reference object
Bigger Example	Torben's bear is bigger than yours	Who is bigger, I or that?	Yellow is bigger, right
b-Total	36 (37%)	2 (2%)	60 (61%)
b-Teachers	28	0	46
Heavier Example	Ten little bears are heavier than two big ones	We don't know what is heavier, the sand or the stones	Yes right that is a little bit heavier
h-Total	17 (12%)	8 (5%)	121 (83%)
h-Teachers	16	7	85
Longer Example	Is this one longer than the other?	Whose woolen string is longer? Lorraine's or Nina's?	Look, this is longer!
l-Total	1 (4%)	5 (20%)	19 (76%)
l-Teachers	1	5	17

Table 22.3 Adjectives used in the different situations according to categories

Situation	Width	Horizontal	Vertical	Expansion	Sum
A (23 min)	8 (20%)	4	0	**29 (70%)**	41
B (15 min)	6 (13%)	9 (19%)	1	**30 (65%)**	46
C (36 min)	0	4	**36 (69%)**	12 (23%)	52
D (34 min)	0	2 (5%)	0	**42 (95%)**	44
E (29 min)	6 (3%)	6 (3%)	2	**137 (90%)**	151
F (24 min)	1	**27 (82%)**	1	4 (12%)	33
G (45 min)	4	**54 (60%)**	4	28 (31%)	90
H (30 min)	0	**27 (67%)**	0	13 (33%)	40
I (27 min)	0	3	**36 (55%)**		65
Sum	25	136	80		562

with *small* and *medium*, it can be used (in a rather imprecise way) to talk about other spatial dimensions. So while these adjectives originally describe some kind of expansion, they can be used as 'passe-partout' adjectives in the situations to describe different dimensions of length.

Most situations show a clear focus on one or two categories. Since all situations are of different length, their focus has to be considered in contrast to the other adjectives used within the situation and not compared to other situations. Three situations (D, E and F) show a clear focus on one dimension, with more than 80% of all adjectives used from one dimension. In all situations except for B, there is a focus on two dimensions with the two most often named dimensions adding up to at least 80%. Only situation B, which is also the shortest situation, shows percentages over 10% in three out of four dimensions. From a mathematical perspective, this means that when kindergarten teachers talk about length in kindergarten, they most often seem to concentrate on one or at most two different dimensions. From a linguistic perspective, it could also mean that kindergarten teachers and children align their language, especially concerning the 'passe-partout' adjectives in the category *Expansion*. While most situations do have a focus on one or two dimensions, these dimensions seem to vary a bit according to the situation. While F, G and H seem to mostly talk about a horizontal dimension of length, situations C and I are the only ones that focus on the vertical, linear aspect of length, at least in their speech. Four of the nine situations linguistically focus on the expansion of objects (A, B, D and E) or rather use these adjectives to talk (imprecisely) about other dimensions:

For the mathematical concept of measuring and length, this means that different aspects are highlighted depending on the use of adjectives. Using adjectives for measuring length that do not clearly indicate a one-dimensional distance might lead to a wrong or at least incomplete understanding of length, since the concept might be connected with wrong or unclear adjectives. If length is only talked about using adjectives that describe a horizontal distance, children might develop a concept that is only associated with this direction. On the other hand, reducing length to one dimension in the beginning might present a didactic reduction.

Adjectives that in their original meaning describe the three-dimensional expansion of persons or things, like big and small, however, might lead to a confusion of length and area or volume. In this case, the difference between the two cannot be made clear with language alone. In the next paragraph, we will have a closer look at some empirical examples which show how these adjectives are used in interactions between kindergarten teachers and children and how they foster or inhibit mathematical learning and language acquisition.

22.6 Empirical Examples for Learning Opportunities

In the following passage, we will first present some empirical insights into the use of adjectives in situations dealing with length, especially concerning the use of *big* and the question what dimension kindergarten teachers and children actually talk about when they use it. Then, we look at examples from situations that focus on weight. Here, we concentrate on the adjective *heavy* and the connection between weight and size.

As seen in Table 22.1 and discussed for the results of Table 22.3 above, *big* is the most used adjective in situations dealing with length. As mentioned above, it works as a passe-partout adjective and comes with many different definitions. Therefore, it is not much of a surprise that it is used in different contexts with different meanings within these situations:

Measuring the children's body length with different tools is a popular activity to introduce measuring length. Wooden measuring boards with animals, for example, are used in several situations to determine the children's body length. Most kindergarten teachers, who use these wooden measuring boards, use the animals to explain the children's body length without having to use numbers that exceed the child's actively mastered number range. However, this might lead to confusions:

Johanna (E) You're already bigger than the giraffe

While the child is not bigger than a real giraffe, it goes without saying that it is bigger than the drawing of the giraffe on the measuring board, especially in the sense of expansion. What the child needs to understand though, is that her body is longer than the distance between the floor and a special point of the giraffe on the measuring board. In this situation, the giraffe serves as a substitute for the omitted numerical scale value. Without further explanations, this might not be obvious for small children without any experience with measurement.

In other situations, kindergarten teachers use folding rules to measure the children. Since the declarations on the folding rule are made in centimetre, the numbers normally exceed the actively mastered number range of kindergarten children. So, the teachers try to give an understanding of these numbers in different ways. One kindergarten teacher points at number 120 on a folding rule where every tenth number is printed in red while all the others are black and explains:

Sabine (D) You are as big as this red number

Here as well, the child is not literally as big as the number on the folding rule, which is only a few millimetres in size. In contrast to the aforementioned example, it is not the location of the number which represents the child's body length, but a sign that symbolises the number of times you have to take a centimetre and put it one after another to get a stretch that is as long as the child's length. In both situations, the scale value is used within the sentence as an object for comparison. This means that in this and the example mentioned before, there is some kind of hidden vertical, one-dimensional view of length. This contemplation, however, is hidden by the use of expansion adjectives. Here, we have two examples where on the linguistic surface expansion 'passe-partout' adjectives are used to talk about an actual vertical dimension.

We also find occasions where kindergarten teachers use *big* in the sense of 'old'.

Johanna (E) Yes, your big sister can do that
Berna (G) Which topic do the big ones have today?

In situations planned for learning about length, the use of *big* in terms of age could be confusing, especially for young children. After being told that she is already bigger than the giraffe, Tamila replies:

Tamila (E) Four am I still big

Her utterance (in German '*Vier bin ich noch groß*') implies that in this situation, Tamila associates *big* with age. Since she was 4 years and 10 months old when the video was recorded, she is likely trying to express that she is still 4 years of age. Soon, she will turn five and will be 'bigger' then. While she seems to pick up the adjective and tries to verbally participate in the situation, her reaction assumes that she was not fully able to grasp the kindergarten teacher's former utterance concerning her length. The kindergarten teacher could have used Tamila's utterance to create a learning opportunity for her and the other child in this situation by directly or indirectly correcting her sentence and/or talk about the different meanings of *big*. Unfortunately, she is so obtained with the measuring activity that she might not even notice what Tamila wants to express.

Another problem with gradable adjectives like *big* is, that you have to have a reference object, at least in your mind. Something is always big in contrast to something else. Not all kindergarten teachers seem to have this problem in mind:

Dorothea (I) Square, yes. Good. And is that big or small?

In this example, it is almost impossible to answer the kindergarten teacher's question because she does not explicitly name the object that the square (which is actually a wooden cube) shall be compared with. Here, the children can only guess the right answer or, if they already have a concept of size at their disposal, ascertain a reference object from the context.

While the size or length of an object is visually perceivable, its weight is not. This might be the reason why it is sometimes difficult for children (Brandt & Keuch, 2017) and kindergarten teachers to use adjectives appropriately when weighing objects. Sometimes, not only the missing reference object inhibits the understanding of an adjective, but also the noun it refers to:

Bärbel Too heavy, we have to take another. The balance is too heavy.

Methodically, Bärbel is the only kindergarten teacher who seems to actively challenge the children's perception of associating big objects with high weight. She takes a big pillow and a small magnet and has the children first describe their sizes and subsequently feel their weight:

Bärbel And which is big and which is small?

Although in her utterance not exactly present, by presenting two objects, it might be quite clear to the children what they are supposed to compare. While this question would ask for a more complex answer, the children reply by saying 'big' or 'bigger' in one case, and pointing to the pillow.

Bärbel That's big and that's small, the magnet is small.
Gökhan But a little bit big.
Tobias But that's heavy *[points to the magnet]* this is not heavy *[points to the pillow]*
Bärbel Exactly, how do you know that?

Since Bärbel does not make verbally explicit, by using the comparative form, that in this case, she wants to only compare the magnet and the pillow concerning their size, Gökhan might refer to other magnets he knows that are smaller than the one presented. She could have used Gökhan's idea to clarify the meaning of *big* in this case. As Tobias says something that probably goes along with her script, she does not react to Gökhan but takes up Tobias' remark. Tobias only replies that he just knows it and that it is 'cuddly', so she asks all children to 'feel' if the pillow is 'light or heavy'. Again, she uses two contrary adjectives, which might help some children to understand the meaning, other children with other reference objects outside the actual situation in mind might again have problems with this task. Luckily, everyone agrees that the pillow is light and big. Then she hands the magnet to the children to feel its weight, again just by contrasting two antonymous adjectives in their positive form: 'Is it light or heavy?' While most children agree that the magnet is heavy, Zahide has a different opinion:

Zahide Is light.
Bärbel For you it's light? Feel both, feel both. Which is heavier?
Zahide Nothing.

Here again, a child might refer to a reference object outside the actual context. Further, the German word for *light* is the same as the German word for *easy* (*leicht*), so here Zahide might want to say that for her it is easy to hold the magnet (that she is strong enough to hold the magnet with ease). Bärbel then refers to the subjectivity of feeling weight. Instead of making verbally explicit that in this context, only the present pillow and the present magnet are the reference objects and that of course although the magnet is heavier than the pillow, it is still easy to hold it in your hand, she only hands the two objects to Zahide to have her feel them with both hands to have a direct comparison. Now, she finally uses the comparative, but only in an incomplete structure. Still, Zahide thinks that both things are not heavy or that neither of them is heavier than the other. Instead of clarifying possible language problems, Bärbel uses Zahide's apparent inability to determine the heavier object, to lead over to the next topic to introduce the pan balance. On the one hand, it might look like she is ignoring Zahide's problem. Since she still seems to have problems with direct comparisons, she is left behind. On the other hand, Bärbel might hope that, by making the magnitude weight more visually perceptible with pans that move into opposite directions, Zahide's problems will disappear.

22.7 Conclusion

With our analyses we are able to consolidate and refine our former assumptions concerning the use of adjectives for language and mathematic learning (Brandt & Keuch, 2017, 2018). The linguistic characteristics of adjectives offer various learning opportunities, not just language-wise but also mathematically. Comparing, ordering and quantifying qualities in German and English is impossible without the use of adjectives. Our analysis shows that a variety of adjectives are used when talking about length and weight in kindergarten. Situations with a focus on weight seem to offer many more possibilities to use adjectives, especially for children, than situations focusing on length. The inability to directly visually perceive weight differences might raise the need to verbally express them. The use of balance pans also seems to increase the (natural) speaking possibilities. In order to avoid talking about numbers that probably exceed the children's range of numbers, kindergarten teachers tend to use many comparisons with different objects in these situations. Hence, the invisibility of weight on the one hand leads to mathematical and conceptual difficulties. On the other hand, it opens various linguistic learning opportunities, if one knows about them and how to use them.

Although the comparative form plays an important role in the development of the ordering relation, it seems to be used rather carelessly by kindergarten teachers. At least half of the time they use it in an incomplete structure without explicitly expressing the reference object. In the empirical examples, we could show how this might lead to confusions. They also showed that some kindergarten teachers rather use antonyms than comparative forms, which also provides information about the meaning but might interfere with the children's imagined reference objects outside the situation.

Depending on the kind of adjectives used in the situation, different aspects of length are highlighted. However, the adjectives used most often—*big* and *long*—are ambiguous as they do not only describe a quality of size but also time or age. Additionally, they do not clearly hint to the one-dimensional characteristic of length, and therefore might lead to problems with the understanding of length and the construction of a solid concept.

The empirical examples show how some kindergarten teachers subconsciously reinforce difficulties concerning the emergence of solid concepts of length and weight. Especially, the use of passe-partout adjectives and incomplete syntactical structures seem to cause problems. While it is not always possible to avoid such constructions, it would offer an opportunity to talk about the different meanings which might lead to a deeper understanding of the concepts of length and weight—and offers opportunities for an academic language-oriented language use.

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Chapter 23
The Complexity of Teaching Mathematics in Kindergarten: A Case Study and Conceptualization

Per-Einar Sæbbe and Reidar Mosvold

23.1 Introduction

"[T]eaching mathematics to little children is as complex and challenging as is teaching it to older children" (Ginsburg & Amit, 2008, p. 284).

In their study of mathematics teaching in a U.S. kindergarten context, Ginsburg and Amit (2008) conclude that teaching mathematics to young children is "essentially the same as teaching it to older children" (p. 274). By this, they do not necessarily mean that mathematics teaching in kindergarten looks exactly like mathematics teaching in school, but they claim that the teaching of mathematics in kindergarten has the same complexity. Kindergartens differ between countries, however, and one might wonder how these differences influence teaching. In the U.S., school is preceded by a year of kindergarten class. Before entering kindergarten, children attend preschool and/or daycare. In other countries, like Belgium, children are enrolled in kindergarten at the age of 3. Before this, they attend childcare institutions. Yet, other countries have integrated kindergartens for children between the age of 0 and 5 years. Norway is an example of this. In addition to organizational and structural differences, the focus and contents of kindergarten also differ across countries. More than five decades ago, Sears and Dowley (1963) described a main distinction between systems that consider kindergarten and other early childhood institutions to be a downward extension of schools, as opposed to systems that consider such institutions to be upward extensions of the family. OECD (2006) makes a parallel distinction between a pre-primary and a social pedagogy kindergarten tradition. In the pre-primary tradition, kindergarten is considered to be a preparation for school; in the social pedagogy tradition, care and upbringing are emphasized more than learning and school preparation. The U.S. is an example of the former; Norway is an example of the latter. Studies from the Norwegian context illustrate how children

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might encounter mathematics through, for instance, fairy tales (e.g., Carlsen, 2013) and play situations (Fosse, 2016; Sæbbe & Mosvold, 2016), and how Norwegian kindergarten teachers refuse to even use the word “teaching” to describe the work they do (e.g., Sæbbe & Pramling Samuelsson, 2017). In Sweden, which has a similar kindergarten tradition to Norway, kindergartens also emphasize learning through play and everyday activities (e.g., Bäckman, 2016; Björklund, 2016; Helenius, 2018; Lange, Meaney, Riesbeck, & Wernberg, 2014). This differs from the pre-primary tradition, which primarily aims at preparing children for school (OECD, 2006). The present study does not aim to challenge claims about cultural differences in kindergarten traditions; neither does it argue that mathematics teaching differs across kindergarten traditions. Instead, our study aims to illustrate how the work of teaching mathematics in a Norwegian social pedagogy kindergarten tradition is complex and challenging work, similar to what Ginsburg and Amit (2008) found in a U.S. pre-primary kindergarten tradition.

The present study approaches the same general problem that Ginsburg and Amit (2008) considered: What might it mean to teach mathematics in kindergarten? Before presenting information about the design of our study and its considerations, we provide some theoretical background concerning research on teaching in early childhood education, and we clarify our use of core terms related to the work of teaching mathematics in kindergarten. Following a section where we elaborate on the design of the study, we present results from our analysis of challenges and tasks of teaching mathematics in kindergarten. Based on this analysis, we return to a discussion of how the work of teaching mathematics in a Norwegian kindergarten context entails a complexity that parallels the teaching of mathematics in a U.S. kindergarten context (e.g., Ginsburg & Amit, 2008), and how this work is mathematical.

23.2 Theoretical Background

It is a common impression that research studies in early childhood education tend to focus more on children than on teachers, and that more studies target learning than teaching. Yet, in the first edition of the *Handbook of research on teaching*, Sears and Dowley (1963) reported a considerable body of research on teaching in what they refer to as “nursery schools.” Back then, these were primarily experimental studies that investigated correlations between various personality variables or programs and outcomes. In the decades following the first Handbook, numerous programs for early childhood education were developed and analyzed. Stallings and Stipek (1986) evaluated the long-term effects of several of these programs in their chapter in the third edition of the Handbook. Some programs were based on cognitive theories, some were focused on “direct instruction,” whereas others were organized around so-called “mastery learning.” Most of these studies looked for correlations between isolated process variables and children’s achievement. This line of process-product research provides several interesting results. There is a tendency in these studies, however, to focus more on programs and their effects than on the actual work of teaching. When Genishi, Ryan, Ochsner, and Yarnall (2001) look back on several decades of research on teaching in early childhood education in their review for the fourth edition of *Handbook of research on teaching*, they state that, in the history of early childhood education research, “researchers have focused relatively little on teachers, teaching, or its effects” (p. 1176). Along the same lines, Ginsburg and Amit (2008) claim that, “little is known about early mathematics teaching” (p. 275).

In the decade following the study of Ginsburg and Amit, additional studies of early mathematics teaching have been conducted, and the field’s knowledge about mathematics teaching in kindergarten is growing. In the European context, the CERME (Congress of the European Society for Research in Mathematics Education) working group on early years mathematics and the POEM (Perspectives on Early Mathematics Learning) conferences that spun out from this group have been influential. A number of contributions from these conferences have come from the Nordic context. For instance, at the first POEM conference, Lange et al. (2014) made important contributions with their discussions of teachable moments in play situations. Whereas discussions on mathematics teaching in kindergarten often imply a tension between instruction and play, these authors contend that teaching can also occur in play situations. They conclude that respectful listening and posing of challenging questions are important teacher actions in this context. For the second POEM conference, Bäckman (2016) investigated everyday situations—she referred to them as “here-and-now” situations—and argued that such situations contain several teachable moments. Whereas Lange et al. (2014) and Bäckman (2016) discussed mathematics teaching in the Swedish kindergarten context, Carlsen, Erfjord, and Hundeland (2010) investigated kindergarten teachers’ orchestration of mathematical activities in a Norwegian kindergarten context. They emphasized the role of questions, and they suggested that questioning is prevalent in the work of teaching mathematics in the Norwegian kindergarten context. Other Norwegian studies also discussed the role of questions. In her analysis of features of mathematical conversations in a Norwegian kindergarten, Fosse (2016) observed how kindergarten teachers’ use of questions structured and organized the discussion in a situation where children engaged in building with Lego blocks. Similarly, Sæbbe and Mosvold (2016) also emphasized the role of questioning—and of affirmations—in their efforts to conceptualize the work of teaching mathematics in kindergarten. These are examples of studies that move beyond the early research on seeking correlations between variables related to teachers and teaching, and get deeper into the actual work of teaching and the challenges and tasks that are embedded in this work. Our study contributes to this research tradition.

A challenge within the social pedagogy kindergarten tradition is that the work of teaching is often unarticulated and invisible. Helenius (2018) aimed at making the acts of teaching in this kindergarten tradition more visible when he proposed “a conceptual framework for teaching mathematics in a play-based preschool practice” (p. 184). His conceptualization consists of three dimensions of practice: pedagogic explication, teacher participation, and situational planning. The first dimension explicates how consciousness of a mathematical object of learning is necessary for a situation to be "pedagogical." In a play-based kindergarten tradition, such consciousness cannot be taken for granted and often requires deliberate work from the kindergarten teacher. The dimension of teacher participation is important in a context where children are often engaged in play and everyday situations with little or no involvement by kindergarten teachers. Through more or less subtle ways, the kindergarten teacher can stimulate mathematical activities—also in more informal play situations. The third and final dimension relates to how kindergarten teachers might deliberately decide whether or not to plan certain activities and situations, and it also relates to flexibility and ability to act in the moment.

Up until now, we have used the terms "work of teaching" and "tasks of teaching" without defining them. Some consider teaching to be "one person's influence aimed at improving the learning of other persons" (Gage, 2009, p. 2). We agree with such a definition, but we follow Ball and Forzani (2009), who add the word "work" to emphasize two important aspects of teaching. First, the term "work" implies an emphasis on the deliberate, effortful, and dynamic doing of teaching (Ball, 2017). Second, we use "work" to indicate that we do not consider teaching as something teachers do, but as work to be done. Per definition, work is an activity that involves some kind of effort to achieve a result, and it is constituted by one or more tasks that are to be done (Oxford English Dictionary, n.d.). Throughout this study, we use the words "challenge" and "tasks" interchangeably. We use the word "task" to identify a piece of work that has to be done, and the phrase "tasks of teaching" signifies that the work of teaching can be decomposed into tasks that the teacher has to complete. Our distinction between the work of teaching and its constituent tasks of teaching is adopted from Ball and her colleagues at the University of Michigan (e.g., Ball & Forzani, 2009; Ball, Thames, & Phelps, 2008). Ball et al. (2008) identified a list of mathematical tasks of teaching that teachers are recurrently faced with in the work of teaching mathematics, and that require professional knowledge to undertake. For instance, they identified the tasks of asking productive mathematical questions, responding to students' questions, presenting ideas, and finding examples to illustrate a certain mathematical idea or point. Carrying out the work of teaching, which is constituted by such tasks, is also demanding, and we use the word "challenge" to emphasize the demanding nature of the work.

23.3 Design of the Study

In considering the problem of what mathematics teaching in kindergarten might look like, Ginsburg and Amit (2008) analyzed the work of an American kindergarten teacher (Daniela). We mimic their approach by considering the case of a Norwegian kindergarten teacher (Gunnar). Whereas Daniela worked in the preschool class of a parochial Catholic school in the U.S., Gunnar works in a Norwegian kindergarten that is located on a farm. Daniela's kindergarten followed the Big Math for Little Kids curriculum, whereas Gunnar follows the Norwegian framework plan for kindergartens (Directorate for Education and Training, 2017)—a framework plan that provides mostly general formulations of what kindergartens should enable children to experience. These differences are illustrative of the differences in kindergarten traditions between Norway and the U.S. Whereas Daniela taught a sequence of carefully planned lessons on mapping, Gunnar aimed at engaging children in mathematical discussions in an everyday activity that involved feeding the animals on the farm. In a context that differs significantly from that in Ginsburg and Amit's (2008) study, we follow a similar approach when we analyze Gunnar's work of teaching with the aim of revealing challenges and tasks that might be entailed in his work.

Like Daniela, Gunnar has more than 20 years of experience as a kindergarten teacher. He finished his education before mathematics was introduced as a compulsory course in the kindergarten teacher education program. At the time of the study, he worked with children aged 3–6 years in an outdoor kindergarten. We asked Gunnar to prepare an everyday activity—something he would normally do with the kindergarten children—that had a focus on mathematics. We video-recorded his interaction with a group of four children in the activity. The activity lasted for 20 min, and Gunnar planned and carried out the activity without any intervention or influence from the researchers. In the activity, Gunnar and the children were feeding the animals on the kindergarten farm. A follow-up interview was planned and carried out by the first author based on preliminary analysis of the video data, and the interview was also video-recorded. The purpose of this interview was to bring forth Gunnar's own reflections about the activity. These reflections were intended to indicate whether the tasks and challenges we identify from analyzing the activity were also considered challenging by Gunnar, and the reflections were meant to provide us with indications about the purpose of the choices that were made. Before the interview, Gunnar got a copy of the video-recording from the activity and he had seen the recording before the interview. The video from the activity served as a starting point for reflecting on his mathematical work of teaching in the interview. A similar use of video for stimulated recall has been applied in other studies (e.g., Jacobs & Morita, 2002), but unlike some of those studies—where videos of teaching in other countries were used—the kindergarten teacher in our study viewed recordings from his own practice and commented on that. The interviews focused on Gunnar's own stories and reflections about the work of teaching mathematics in kindergarten—elicited by the interviewer and the videos (Kvale, 2007).

The video-recording from the activity enabled careful analysis of Gunnar's communication with the children—verbal as well as non-verbal communication. Gunnar's reflections from the interview helped us understand the choices he made in his practice. Video recordings from the interview enabled further analysis of his reflections, and this also enabled the interviewer to concentrate without having to take notes and show the selected recordings at the same time.

The video-recordings were planned with Gunnar in advance to keep the room as quiet as possible, and to place the camera where it would be least obtrusive. The participants gave permissions, and the children were comfortable being videotaped. The video-recordings from the observation and interview were transcribed verbatim by the first author. Our analysis is empirically grounded, but we use these data as a starting point for developing conceptualizations of teaching mathematics in kindergarten as work to be done rather than to make claims about how Gunnar is teaching. The object of our study is thus the work of teaching mathematics in kindergarten—not Gunnar, or the population of Norwegian kindergarten teachers. For the purpose of conceptualizing the work of teaching mathematics in kindergarten, our analysis is inspired by the constant comparative method (Strauss & Corbin, 1998). First, we developed open coding and examined, compared, and conceptualized the data. The next step was the axial coding, where we made connections between the emerging categories from the open coding. The last phase was the selective coding, and we systematically related and filled in categories that needed development. During the process of analysis, subsequent theoretical sampling was applied (Strauss & Corbin, 1998).

23.4 Tasks of Teaching Mathematics in Kindergarten

Gunnar is working in a private outdoor kindergarten, which is located on a farm with animals. We consider an episode where Gunnar and four children are feeding the animals—a daily activity in this kindergarten. During this activity, Gunnar initiates a discussion on how to figure out the right amount of food for the sheep. He has prepared a form that they can use for documenting how many animals they have and how many cups of food each animal gets (see Fig. 23.1). The first row in this form represented the amounts with tally marks, whereas the second row represented the same amounts with numerals. Gunnar wanted to include both, since he thought tally marks were easier.

For the purpose of this study, we present our analysis of an episode that was selected to illustrate some of the complexity that might be involved in the work of teaching mathematics in this kindergarten context. We highlight the tasks that the kindergarten teacher has to solve when attending to the mathematics as well as to the development and needs of each individual child, and we discuss how these two perspectives are connected.

Gunnar:	(poking Mikkel) How many sheep do we have?
Frode:	Does that include the lambs?
Gunnar:	Yes, lambs are also included. Let's write a mark for them too.
[VOICEOVER/Gunnar:	I think this is a good thing about children, because this is how it is for them. A sheep is a sheep, right? And then you have the lamb. So they are, I mean, I didn't think about that. So, they correct me and then we agree that the lamb also counts for one; together they are two. All right. But, I mean, that is what makes it so interesting to work with children.]
Gunnar:	How many did the sheep get?
Mikkel:	Eh. Two (cups)?

ANIMAL	ALPACA	SHEEP	GOAT	SUM
HOW MANY ANIMALS DO WE HAVE?				
HOW MANY ANIMALS IN NUMBERS?				

ANIMAL	ALPACA	SHEEP	GOAT	SUM
HOW MUCH FOOD?				
HOW MUCH FOOD IN NUMBERS?				

Fig. 23.1 The form Gunnar used with the children

Gunnar: No. It only got? (Someone else responds) The sheep only got one. Then you make a mark there. And then the goat, how many did you give the goat? There were two, yes, so you write two there. And then the very last one. This is how many I got in total. All the cups we got, then you must count all of them, these and those (points at the sheet). (Mikkel does not seem to understand) How many are there? You have to count?

Nora: One, two, three, four, five, six, seven.

Frode: I got seven, too. I just counted one, two, three, four, five, six, seven. I didn't know it was seven there. I didn't calculate it.

Gunnar: (To Mikkel) How many did you count?

[VOICEOVER/Interviewer: You are in control with these children, and sometimes you let go of the control and just explain something to the one. Aren't you afraid the others start with completely different things?]

[VOICEOVER/Gunnar: [...] it is important to give a little extra to the one who needs something that the others do not need so much of. [...] I somehow see if I can practice a little extra. We use some time, and then I'm back with the others. It's kind of like that, and I think that's normal, I know the children from before because I've observed them in the same situations before. You still have to know in the activity what works and does not work [...] I notice it becomes more like this along the way in the activity, what are they struggling with and which children I have to follow up extra.]

23.4.1 Initiating Mathematical Discussions

Norwegian kindergartens do not have regular classrooms, lessons, or textbooks. Instead, kindergarten teachers are expected to facilitate children's learning in everyday situations and play. A common task of teaching is to initiate a mathematical discussion, or to direct the children's attention toward mathematics in an everyday situation. There are several ways in which Gunnar could have initiated a mathematical discussion with these children. He decides to start by asking one of the children, Mikkel, "How many sheep do we have?" This is a mathematical question that aims to have Mikkel consider quantity. Quantity is a core mathematical topic in kindergarten (Directorate for Education and Training, 2017), and the kindergarten teacher might expect a particular kind of response. For instance, Mikkel might have responded that there are "many" sheep, whereupon Gunnar could initiate a discussion about the concept of many. Mikkel might also have suggested that they can count, or that they might provide an approximation. None of this happened. Instead, another boy, Frode, interrupts and asks, "Does that include the lambs?" He is wondering whether the lambs should be counted with the sheep. In this particular situation, Frode's question makes a lot of sense, because it might easily be the case that lambs eat less than sheep. However, this question was unexpected, and Gunnar reveals in the interview that he did not anticipate this.

23.4.2 Responding to Unexpected Questions

Gunnar is now challenged with the task of dealing with this unexpected question from Frode. From our experience, dealing with unexpected questions and responses from children is a common task of teaching in the Norwegian kindergarten context. Deciding on how to respond implies a possible adjustment of the plan. Should Gunnar show flexibility and adjust his original plan (Ginsburg & Amit, 2008), or should he continue trying to establish the children's attention around the idea of quantity? Both alternatives might be viable. He could make use of Frode's question and instead switch the focus to classification of animals in relation to age. After all, comparison and sorting are also areas of mathematics that the framework plan signals that children should experience. Some children consider sheep and lambs to be different. They are different in terms of age, but also in terms of size and appearance. When young children think about quantity, it matters to them what objects they are dealing with. Kindergarten teachers know that this is a common aspect of children's emerging conception of number (Gelman & Gallistel, 1978). The challenge of responding to Frode's unexpected question is therefore not only a challenge of providing a sensible response, but it represents a task of deciding where the discussion should be allowed to go next.

In addition, the decision about how to deal with Frode's question requires careful attention to the children involved. Gunnar's initial question was directed toward Mikkel, and kindergarten teachers are often deliberate about whom to call on or ask questions. Mikkel might be a quiet boy that Gunnar wanted to include in the conversation. Encouraging verbal interaction is a common task of teaching (Ginsburg & Amit, 2008), and it can be challenging to involve quiet children in a discussion. When Frode interrupts by asking another question, Gunnar is not only faced with the challenge of deciding on the mathematical course of the discussion, but he also has to consider how to position the different children as participants in the discussion. Gunnar only gets a split second to make the decision.

23.4.3 Dealing with Wrong Answers

In this situation, Gunnar decides to respond that, "Yes, lambs are also included. Let's write a mark for them too." This response acknowledges Frode's contribution, and it maintains the mathematical focus of the discussion. After having responded to Frode's question, Gunnar makes another attempt to navigate the children's attention toward quantity. He asks, "How many did the sheep get?" Whereas his first question was related to the number of sheep, this next question asks how many cups of food the sheep get. Gunnar has still not received a response to the question about the number of sheep, and he could have decided to repeat this question. Instead, he poses a new and related question, which targets the number of cups of food each sheep is getting. This is another choice that has to be made in the moment, and it illustrates a task of following up on an unanswered question. Mikkel responds by suggesting that the sheep get two cups. Gunnar is now faced with a dilemma. He has succeeded in involving Mikkel in the discussion, but Mikkel's response is mathematically incorrect. On the one hand, kindergarten teachers want to encourage children's participation in the mathematical discussions, and to develop a sense that mistakes are essential for learning mathematics (Ginsburg & Amit, 2008). Given this, the kindergarten teacher might have decided to encourage Mikkel's continued participation and refrain from correcting him. On the other hand, the animals have to get the right amount of food. The question about quantity is not just an abstract mathematical question in this situation; it affects the animals' health. Gunnar has to make a quick decision on how to respond. Sometimes, a teacher decides to confirm or disprove a child's response. Other times, a teacher decides to ask more questions or call upon other children to initiate a discussion or argumentation to facilitate the children's own discoveries. In making a decision, the kindergarten teacher has to balance the attention between the mathematical content and the attention to the individual child. Correcting or disproving an incorrect response might be important to maintain the mathematical integrity—and, in this situation, to ensure that the animals get the right amount of food—but it can place the children in a negative position. Simultaneously, by confirming or rejecting a child's response, the kindergarten teacher puts himself in a position of authority and ownership of the mathematical content. In this case, Gunnar disproves Mikkel's response by saying, "No. It only got?" Someone else responds to this follow-up question, and Gunnar then affirms that, "The sheep only get one [cup]". This testifies to what Sæbbe and Mosvold (2016) identified, that questioning and affirmation of children's responses are core acts of teaching mathematics in kindergarten.

23.4.4 Using Representations

Following the correction of Mikkel's response, Gunnar directs the attention to the tally marks in the form. The tally marks are written in a table, one row with tally marks for the number of animals, and one row with tally marks for the number of cups (see Fig. 23.1). He explains, "Then you make a mark here. And then the goat, how many did you give the goat? There were two, yes, so you write two there." From an emphasis on the number of sheep and the number of cups that each animal gets, Gunnar now shifts attention to representations of number. This implies a task of using appropriate representations, and the tally marks represent the number of cups each animal gets. This situation thus involves use of representations, number conservation—since a mark represents the quantity of one, regardless of whether it refers to sheep or goats—and it involves counting the tally marks and understanding that the number word for the last tally mark counted signifies the amount, as in the principle of cardinality (Gelman & Gallistel, 1978). A number is an abstract mathematical idea, and this situation involves different representations of number. In Ginsburg and Amit's (2008) study, Daniela was also challenged to use representations, but in her case, it was the map as a representation of the physical world. Gunnar points to the sheet of paper and says, "All the cups we got, then you must count all of them, these and those." Mikkel does not seem to understand, and Gunnar is thus challenged to make another decision. He decides to repeat and reformulate the question, "How many are there? You have to count." In this episode, we see how Gunnar tries to involve Mikkel in the discussion, by posing new questions with more information regarding what Mikkel has to do to solve the task. First, Gunnar poses the question about how many the sheep got. When Mikkel fails to answer this correctly, Gunnar follows up by formulating the initial question in a different way, and he adds that Mikkel has to count. Finally, he asks Mikkel a third question: "How many did you count?" This work involves a delicate balance between attending to the mathematics and attending to each child, and we interpret the voiceover from the interview as an indication that Gunnar is conscious about this in the present situation. Mikkel's lack of response could indicate that he does not understand. Another possibility is that he is discouraged after having had his previous response corrected. Anyhow, Gunnar wants Mikkel to participate.

23.4.5 Positioning Children as Valuable Contributors

Following the question of how many and the call for counting, Nora replies by counting, “One, two, three, four, five, six, seven.” Young children might be able to count correctly without understanding that the last number word signifies the amount (Gelman & Gallistel, 1978), and a kindergarten teacher has to consider whether and how to react when children respond by counting. Again, the decision influences how the child is positioned in relation to the mathematical content as well as to the mathematics teacher. The kindergarten teacher might decide to commend Nora for counting correctly or even interpret her tone of voice when pronouncing “seven” as indication of her understanding of the cardinality principle (Gelman & Gallistel, 1978), and thus place Nora in a positive position. If the kindergarten teacher decides to repeat the question to indicate that her counting does not really answer the question, this might place Nora in a more negative position. Before the kindergarten teacher gets a chance to respond, Frode interrupts again, “I got seven too. I just counted one, two, three, four, five, six, seven. I didn’t know it was seven there. I didn’t calculate it.” From his response, Frode indicates that he understands the connection between counting and naming the quantity by repeating the last number word in the counting sequence (Gelman & Gallistel, 1978). Gunnar now has another decision to make. Should he return to Nora in order to see if she is on the same page as Frode? Or should he affirm Frode’s response? Or perhaps he should leave it open and ask some of the other children what they think? If the kindergarten teacher affirms the response that Frode gave, he could also use this to emphasize the important mathematical idea that the last number word in the counting sequence indicates the quantity of the objects counted. We refer to this as mathematical affirmation (cf. Sæbbe & Mosvold, 2016). However, Gunnar might decide that Frode has already gotten enough space in this discussion and deliberately try to let other children get a chance to contribute. By returning to Nora, the kindergarten teacher could position her as a valuable contributor to the evolving mathematical discussion (cf. Fosse, 2016). Such a decision would also make sense if the kindergarten teacher wanted to make sure that the girls were heard in the group, and the move could thus be motivated by a wish to disrupt patterns of gender inequity in mathematics. The kindergarten teacher has to figure out how to deal with this on the fly, without much time to think about and consider the alternatives. Gunnar decides to leave it open and approaches Mikkel, who previously did not seem to understand this, and asks him, “How many did you count?” This illustrates the delicate balance that kindergarten teachers always have to maintain when engaging in mathematical discussions with children. They want to see every child and let everyone get the opportunity to engage in mathematical thinking. In other words, they want to position every child as valuable contributors in the mathematical discussion—and they have to navigate all the different possible responses and questions that might come up in such a contingent moment without much time to think.

23.4.6 Asking Productive Mathematical Questions

Throughout this episode, we notice that Gunnar asks a lot of questions. In Ginsburg and Amit's (2008) study, Daniela also asked a lot of questions in her teaching. Ginsburg and Amit did not highlight the challenges of questioning, but others have identified asking productive mathematical questions as a core task in mathematics teaching (Ball et al., 2008)—also in the Norwegian kindergarten context (e.g., Sæbbe & Mosvold, 2016). In the interview, Gunnar explains that, "When I pose the question, I think that, that they regard it as their task." By asking questions, Gunnar not only wants to initiate a mathematical discussion, but he wants to offer the children ownership of the mathematics. Instead of solving the problem for them, "they are going to find the solution to this." As we have already noticed, however, there is complexity in the practice of asking questions. Questions might be asked to initiate a discussion or to engage the children in mathematical thinking. Questions might also serve to focus children's attention, or to put them in a position as owners of the mathematics. Sometimes questions are repeated—for various reasons—and sometimes questions are repeated with a slight variation. Thus, when considering the practice of asking mathematical questions, it seems important to consider the issue of purpose. There are different types of questions that kindergarten teachers might ask, and the purpose behind questions might differ. We see, from Gunnar's final reflections, that he considers the different purposes behind asking questions, and his reflections provide a glimpse into some of the considerations that have to be made in the dynamic and deliberate work of teaching mathematics.

23.5 Discussion

In the following discussion, we revisit the claims by Ginsburg and Amit (2008) about the complexity of mathematics teaching in kindergarten, and discuss them in light of our own analysis. In their analysis of mathematics teaching in a U.S. kindergarten context, Ginsburg and Amit (2008) set out to identify "the challenges that one teacher faced over time in teaching" (p. 284). Their analysis ends with a list of challenges that the preschool teacher—Daniela—was faced with in her work. Although Ginsburg and Amit describe these as challenges, we suggest that they are, in essence, tasks of teaching mathematics. For instance, Daniela engaged the children in a teacher-imposed task. She used an activity related to the children's everyday lives to motivate them, she sometimes provided explicit instruction and sometimes lectured to the children, she asked them to explain their thinking, and she created a classroom culture of learning (Ginsburg & Amit, 2008). The list goes on. Based on their analysis of this particular teacher, Ginsburg and Amit argue that teaching mathematics in this preschool context is similar to teaching mathematics in school. They do not suggest that teaching in kindergarten is identical, but they argue that it is "in essence the same" (Ginsburg & Amit, 2008, p. 284), because it is as complex and challenging as teaching mathematics to older children.

When we compare the analysis and arguments of Ginsburg and Amit with our own analysis of Gunnar's teaching of mathematics in a Norwegian kindergarten context, there are some apparent differences. For instance, Gunnar did not give a lecture to the children, and there was less direct instruction than in the case of Daniela. The classrooms were different, and so was the level of structure in the activities. Furthermore, differences in curriculum and kindergarten culture probably influenced the work. On the other hand, it is interesting to notice the similarities between the work of teaching in the cases of Gunnar and Daniela. Both Gunnar and Daniela had to act in the moment and solve challenges that appeared on the fly. Both engaged the children in a teacher-imposed task that spun from their experiences in everyday life. One might argue that the use of questions to guide children's learning is more prevalent in the Norwegian kindergarten context (cf. Carlsen et al., 2010), but that is not the main point of our study. Ginsburg and Amit (2008) did not argue that the case of Daniela was generalizable to the larger population of U.S. preschool and kindergarten teachers. Neither do we argue that the case of Gunnar is representative to all Norwegian kindergarten teachers, or even to Gunnar's own teaching. However, we claim that this case illustrates some of the complexity and challenges that kindergarten teachers might encounter in the work of teaching mathematics in a social pedagogy kindergarten tradition. Our point is to name some core components of mathematics teaching and thus contribute to a conceptualization of the work of teaching mathematics in kindergarten (cf. Sæbbe & Mosvold, 2016). Our approach considers teaching as a professional rather than a cultural practice, and we consider this distinction to be vital for the discussion of similarities or differences across cultures and levels.

In their investigation of mathematics teaching across countries, Stigler and Hiebert (1999) argued that the differences in teaching were more significant across than within countries. They identified several "cultural scripts" of teaching, and they argued that teaching is a cultural activity. We do not intend to disagree with or argue against Stigler and Hiebert, and we agree that it is indeed possible to consider teaching as a cultural activity. When considering teaching as a cultural activity, it is defensible and justifiable to argue that teaching mathematics is different across kindergarten contexts, and also—we posit—that teaching mathematics in kindergarten is different from teaching mathematics in school. On the other hand, it is also possible to view teaching as a professional activity that consists of "management of instructional interactions that are co-constructed by students and teacher around content" (Hoover, Mosvold, & Fauskanger, 2014, p. 11). Within this more general framing, common tasks of teaching can be identified that are defensible and justifiable across cultural contexts. For instance, both Gunnar and Daniela had to deal with the tasks of initiating and leading discussions, responding to children, encouraging verbal interaction, dealing with errors and wrong answers, using representations, and asking questions. These are examples of tasks that can be identified across cultural contexts, and they illustrate what it might mean for teaching to be a professional practice. In this sense, it is defensible and justifiable to argue that teaching mathematics is similar across kindergarten contexts.

So far, the discussion of our findings with those of Ginsburg and Amit (2008) has been on a level corresponding with how they presented their findings. As a result, our discussion of tasks of teaching might have appeared more pedagogical than mathematical. However, we want to emphasize—and our analysis shows this—that these are indeed *mathematical* rather than merely pedagogical tasks of teaching. For instance, the task of initiating mathematical discussions is a mathematical task that might involve asking questions to target a particular mathematical content. Responding to children's unexpected questions is a general task of teaching, but responding in a way that directs children's attention to the mathematical content is a mathematical task of teaching that requires mathematical competence. Dealing with mathematically wrong answers requires a careful balance between giving attention to mathematics and to the children, which can be particularly challenging. Representations are important in mathematics, and choosing and using representations is another mathematical task of teaching. Positioning children as valuable contributors might be considered to be a pedagogical task, but our analysis of Gunnar's interaction with the children showed how this is a mathematical task within the context of a mathematical discourse. Finally, the task of asking productive mathematical questions is a prominent mathematical task of teaching that was also identified by Ball et al. (2008).

Our analysis of the case of Gunnar thus illustrates what teaching mathematics in kindergarten might look like, and we contend that engaging with the various mathematical tasks of teaching constitutes a complex mathematical work of teaching. This work involves tasks of teaching that are similar to the tasks involved in the work of teaching mathematics in school (e.g., Ball et al., 2008). In this respect, and when viewing teaching as a professional practice, we suggest that the work of teaching mathematics in the Norwegian kindergarten context is similar to—or "in essence the same" as—teaching mathematics in the U.S. kindergarten context, and to teaching mathematics in school. By making this argument, we do not intend to argue against differences in kindergarten traditions (e.g., OECD, 2006). Neither do we want to propose that Norwegian kindergartens should become more like schools, or that teaching mathematics in the social pedagogy tradition should become more formalized (cf. Helenius, 2018). However, we want to highlight that the work of teaching mathematics in the Norwegian kindergarten context is a complex mathematical work that entails several pedagogical and mathematical tasks that the kindergarten teacher has to solve on the fly.

23.6 Conclusion

Our analysis contributes to the ongoing conceptualization of mathematics teaching in a social pedagogy kindergarten tradition by making the mathematical work of teaching more articulate and visible. Our conceptualization differs from many other studies (e.g., Helenius, 2018), in that it targets the challenges or tasks that are entailed in the work of teaching mathematics in kindergarten, rather than dimensions or features of teaching.

Recent research from the Nordic context indicates that play and everyday situations contain "teachable moments," which might be used to stimulate children's learning of mathematics (e.g., Bäckman, 2016; Lange et al., 2014). We agree, but we want to emphasize that transforming such teachable moments from possibilities to actual instances of learning is a deliberate work that requires effort and care. In our analysis, we have decomposed this work into several mathematical tasks of teaching. To solve these tasks and skillfully carry out the work of teaching mathematics in kindergarten, kindergarten teachers need professional competence (cf. Sæbbe, 2018). More efforts should be made to investigate the special *mathematical* work of teaching in kindergarten and the *mathematical* demands that are entailed by this work (cf. Ball, 2017). By investing in such efforts, we might contribute in making the teaching of mathematics in the social pedagogy kindergarten tradition more visible, to increase respect for the work of teaching mathematics in kindergarten, and provide grounds for further development of kindergarten teacher education.

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Part III
Parents' Role in Children's Mathematical Development

Chapter 24
Parents' Valuing of Mathematics for Young Children

Dorota Lembrér

24.1 Introduction

In this chapter, parents, as well as other family members, are recognised as young children's first educators who contribute to their learning of mathematics knowledge and skills (Phillipson, Gervasoni, & Sullivan, 2017). From this perspective, parents' views on mathematics education can be considered as assets that influence children's mathematical learning in their early years (Björklund & Pramling, 2017). However, there has been a limited amount of research that has taken parents' views seriously concerning young children's engagement in mathematical learning opportunities at home. In this chapter, I explore the narratives of nine Norwegian parents in order to understand their views on children's mathematics activities at home. From these views, I identify the values they hold about mathematics learning. I am interested in the values that are embedded in (LeFevre, Polyzoi, Skwarchuk, Fast, & Sowinski, 2010)—or emerge through (Aubrey, Bottle, & Godfrey, 2003)—the narratives they tell about the children's informal activities at home, rather than planned and goal-oriented mathematics activities in early childhood education institutions (Björklund, 2014).

Although research studies in mathematics education highlight parents' roles differently in regard to their children's mathematics learning, knowledge and skills, most situate young children as capable of showing adults (parents, teachers and researchers) their understanding of mathematics (Aubrey et al., 2003; Wager & Whyte, 2013). In fulfilling their roles as first educators, parents are considered to be active participants in the construction of their children's mathematics skills, which they interpret in many ways (Hawighorst, 2005). Some research has focused on *why* home environments are important for children's mathematics development and learning (Brenner, 1998). Other studies have sought to understand *how* home

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environments can contribute to children's school mathematics learning (Civil, Guevara, & Allexsaht-Snider, 2002). For example, Clarke and Robbins (2004) showed that parents were aware of the mathematics in everyday experiences that children engaged in at home, such as measuring ingredients for cooking. As a result of these studies, parents are often encouraged to use everyday experiences as tools to develop mathematical skills and knowledge with their children (Anderson & Anderson, 2018). When children attend preschool,[1] they bring with them experiences from outside the preschool that can form the basis for mathematical activities (Clarke & Robbins, 2004).

However, parents' views about mathematics activities that their children engage in at home may be different from those of teachers, but are important if children's transition to preschool is to be supported. Therefore, the focus of this study is on how parents describe their children's engagement with mathematics at home and the research question is: what do parents value in the mathematics activities that their children engage in at home?

24.2 Theoretical Perspective

In this study, parents' views about children's mathematics activities at home are investigated in photo-elicitated interviews (PEIs). In these interviews, the parents used narratives to reflect on the photos they took of their children engaging with mathematics at home. I consider parents' narratives to be socially constructed (Burton, 1996), in that they are formed by wider societal understandings of these specific situations. In an earlier study, I investigated the perspectives of Polish immigrant parents living in Sweden in regard to the mathematics that their children experienced in preschool (Lembrér, 2018). The findings suggested that the Polish parents had adopted Swedish societal norms and values about mathematics in preschools.

A narrative approach to research is a way of understanding how meaning is imposed on experiences (Burton, 1996, 1999). As Sfard and Prusak (2005) stated, the narratives are told by an author, about a person, to a listener: 'By foregrounding the "person's own narrativizations" and 'telling who one is", they link the notion of identity to the activity of communication, conceived broadly as including self-dialogue—that is, thinking' (Sfard & Prusak, 2005, p. 16).

[1] I adopt the word 'preschool' as a label for institutions for Early Childhood Education and Care (ECEC) in Norway, for 1–5 year-old children.

24.2.1 Narrative Approach to the Learning of Mathematics

For this study, there was a need for a framework for analysing parents' narratives in order to identify the values they held about the mathematics activities their children engaged in at home. Consequently, I adapted the narrative framework of Burton (1996, 1999). In work with children, Burton (1999) described a mathematical narrative as something that 'may be told and re-told in the style and with emphasis chosen by the agent(s) who author(s) the telling' (p. 24). Burton's (1996) narrative approach was designed to unpack students' learning of mathematics by interpreting their actions and interactions with others as narratives. The narrative approach provided insights into the children's understanding of the learning process. Burton (1996) stated that, 'with respect to the content of mathematics, instead of presenting it as "objective", independent and fixed, we can tell its socio-cultural story, seeing it as a solution to a social imperative of a particular culture' (p. 32).

Burton (2002) saw narratives, formed from the children's actions when doing mathematics, as having four aspects: authoring, sense-making, collaborating, and using non-verbal narratives. Authoring is when a person uses their experiences to reflect on and generate views about learning mathematics. Burton described authoring as a way of structuring the children's coming-to-know mathematics process. Burton (2002) gave an example of a boy counting from 1 to 11, 12 before jumping to 31, because he seemed to misread 13. He continued to count until he reached thirty ten and thirty twelve. Burton described the child as authoring his mathematical knowledge about counting beyond 10.

Burton indicated that by looking for differences between what the children did and how mathematics was taught, teachers (and other adults, including researchers) created possibilities for investigating the children's sense-making, which in this case was about counting. Burton suggested that this narrative gave information about the child's mathematical development, in that having learnt to count from 1 to 10, the boy then created a system to extend and use this pattern with larger numbers.

Burton (2002) exemplified the collaborating aspect by describing how one child used some of another child's response to explore number constructions together. The example came from a lesson when a teacher asked the children to state the biggest number they knew. One boy gave the response 252 thousand million, two thousand and, after a comment from the teacher, then changed his response to 252 thousand million. Another boy used this to suggest 252 and 20 million trillion as the biggest number, which was followed by the first boy saying 252 thousand million trillion. Burton classified this as collaborating, as the children played imaginatively with numbers by using each other's ideas.

An example of non-verbal communication was, 'Katy's partner asked her to make the calculator display 230. The constant had been set to +10. She approached this problem by stopping on 30 and staring at it, then stopping on 130 and commenting that there needed to be a 2 at the front before finally stopping on the correct symbolization' (Burton, 2002, p. 9). Katy used non-verbal communication when using the calculator and verbal communication to describe what the calculator was showing.

Burton recognised that people's actions and interactions provided information about mathematics, or what mathematics was about, what it could be and what it should be used for. Thus, she considered that narratives connect the social and the personal by showing how the social environment influences individuals' views of what it means to learn mathematics.

> Narrative is *a*, possibly *the*, way to explore the meaning of experience, narrating is participatory, involving a community in telling and responding to a story. Narrative starts from the personal and the particular, often encountering the general in its journey, and returns to the personal again. Narrative is a strategy for seeking possible answers to questions about our world. (Burton, 1996, p. 30).

Burton's (1996) narrative approach concentrates on the construction of personally meaningful mathematics, in which mathematics is viewed as a sociocultural artefact. Burton (1999) considered mathematics learning to be a narrative process in which mathematics knowledge and skills are validated by the adults, particularly teachers. She noted how this approach opened up possibilities for personal narratives to enhance and enrich children's possibilities to learn mathematics.

Burton (1999) identified two kinds of narratives that support people's understanding of mathematics and the learning processes. She stated that a paradigmatic narrative seeks 'to establish generalities out of particular examples' (p. 21) while an imaginative narrative is 'attempting to tell engaging and believable stories which become exemplifications' (p. 21). These two kinds of narratives impose coherent meanings on individual experiences and 'are personal in the degree to which they reflect a particular journey towards knowing, general (paradigmatic) where they develop mathematical generalities, that is where they turn from being imaginative to becoming recognizably paradigmatic knowledge' (p. 31).

For this study, I have adapted Burton's (1996, 1999) narrative approach to learning mathematics in order to identify, in the parents' discussions, the sets of values they hold about young children's mathematics learning at home. In the next section, I describe how Burton's (2002) four aspects of narratives are adapted for examining parents' narratives about their children engaging in mathematical activities at home and how this led to identifying insights into their values about mathematics learning in these activities.

24.3 Methodology

To gain parents' narratives, photo-elicitated interviews (PEIs) were used. In photo-elicitated interviews (PEI), participants are asked to take photos of a topic or issue and these photos are then used to gain personal views and to allow participants to influence the direction of the interview (Greenbaum, 1999). This methodology is considered more effective in gaining insider views, than information from exclusively verbal methods (Hurworth, 2004). PEIs can facilitate dialogues by engaging participants and are seen as enjoyable because participants can express their views and experiences (Torre & Murphy, 2015). The photos support participants to reflect on a moment or an action.

In the PEIs in this study, a set of photos taken by the parents were used to enhance collaborative and participatory data collection. The narratives were not in the photos, but emerged in parents' discussions as they talked together about the photos. The parents' narratives imposed meaning on their experiences about their children's engagement with mathematics at home and were endorsed or challenged in the discussions with others. By exploring parents' narratives, societal views about mathematics for young children could be identified, providing potential nuances to the views of teachers and policymakers.

Data were collected from nine Norwegian parents, after contact was made with preschool staff about the project. The parents received a letter asking them to participate in the study. The parents were made aware that they could withdraw from the study at any time and that all data would be anonymised. Those parents who agreed to participate received guidelines about photographing their children engaging in mathematical activities at home. No information was provided about what a mathematics activity was and it was left to the parents to decide what to photograph.

Each family sent 5–17 photos, taken during 1 week in May 2017. In June 2017, I conducted two PEIs, with five and four parents, respectively. To keep the interview to a reasonable length, a restricted number of photos were chosen. Choosing the photos was done by categorising the photos according to Bishop's (1988) six mathematical activities (playing, explaining, designing, locating, measuring and counting). Bishop's six activities allowed us to identify examples of different kinds of mathematics in these photos. Inspiration for categorising the photos was taken from the work of Hauge et al. (2018), who classified participants' photos using Bishop's (1988) six activities, by identifying the principal activity that the children were engaged in. A similar classification was done with the set of photos contributed by the parents, in collaboration with two other researchers who were part of the wider study (see Fosse, Lange & Meaney in this book). At least one photo representing each of Bishop's mathematical activities and at least one photo from each parent was chosen for each interview. However, it was not presumed that the participants would 'see' the same mathematics in the photographs as the researchers. Instead, the choice of photos was intended to provide parents with a range of possibilities to describe mathematics for young children. As this study is guided by a narrative approach, children engaging in mathematics activities at home were viewed from the perspective of the parents. It was their views and understanding of children's actions in these activities which were in focus.

For the two PEIs, 9 and 14 photos were chosen, respectively. In the PEIs, I initiated the discussions about each photo by asking: (1) Can you describe the story behind this photo? (2) What kinds of mathematics do you see your child doing in this photo? The PEIs lasted 55 min, and 1 h and 15 min, respectively, and were audio and video recorded. The parents are referred to as P1–P9, and in the case where both parents of one child were represented, they are referred to as P6a and P6b. The PEIs were transcribed, with some editing for clarity.

24.3.1 Analysis of the Data

The initial analysis began with identifying descriptions of mathematics. In this way, groups of similar activities were identified in the transcripts. These were: board games, counting, measuring, and using money. I then searched these groups for Burton's four aspects of narratives (see Table 24.1). I repeated this search twice to ensure that all aspects were identified. To identify the kinds of values the parents held about the mathematics learning of their children in home situations, I analysed the data by asking to what extent and in what way:

(a) parents used experiences to reflect on and generate their views about mathematics in children's activities (authoring)
(b) parents made sense of how their children engaged in mathematics activities, (sense-making)
(c) parents endorsed shared engagement and validated each other's views about what constitute mathematics in the activities (collaborating)

Table 24.1 The four aspects of narratives adapted from Burton, used to explore parents' views about mathematics activities of young children

Four aspects of narratives	Aspect about learning of mathematics identified in children's narratives	Aspect about learning of mathematics identified in parents' narratives
Authoring	Children's expressions and descriptions of mathematics in a particular activity/context	Parents' expressions and descriptions of how their children engaged with mathematics at home
Sense-making	Sense-making is about reflection of a particular journey towards certain mathematics knowledge and skills	Parents' views on how their children made sense of the mathematics they engaged with during an activity at home
Collaborating	Collaborating includes validating children's and other's reasoning and use of artefacts that encourage sharing amongst participants	Parents co-creating their understandings about their engagement with their children and by children engaging with each other and the use of artefacts that encourage sharing amongst family members
Non-verbal communication	This form of communication provides information about how children use artefacts to complement their verbal narratives about learning in mathematics It focuses on the non-verbal actions and children's use of artefacts	This is about how parents viewed their own and their children's use of artefacts or tools in mathematics activities at home It focuses on parents' view on the children's non-verbal actions and use of artefacts

(d) parents reflected on the artefacts their children engaged with when doing mathematical activities (non-verbal communication)

Table 24.1 describes the four aspects of narratives identified by Burton in children's narratives, and the adaptations I made to identify the values that the parents held about the mathematics their children did at home.

In the next section, I present the results from parents' narratives about young children's engagement in mathematics activities at home. From this analysis, I identify the components of mathematical activities for young children that the parents seemed to value the most.

24.4 Results

In this section, I present four groups of activities that emerged from the empirical data of two PEIs with parents. Each of the groups (board games, counting, measuring, and using money) is discussed in relation to the four aspects of narratives, and the parents' values about mathematics for young children.

24.4.1 Yahtzee and Ludo, the Value of Learning Numbers

The parents described their children playing the board games Yahtzee and Ludo in both PEIs. A photo of children playing Yahtzee was used as a stimulus for discussion in the first PEI group and a photo of children playing Ludo was discussed in the second PEI. The parents' narratives about board games indicated that they valued their children learning numbers, in addition to the use of particular pedagogical approaches for supporting their children to do this through playing Yahtzee and Ludo.

All the parents stated that these games were available at home or in holiday cottages. For example, P6b stated 'Ludo and Yahtzee were board games that were probably present in most homes'. One of the parents (P3) described that their children did not play Yahtzee at home and reflected over this:

> I have got a little guilty conscience, because we almost never played Yahtzee with our other child. So we must go home and do that.

In this example, P3 indicated that it was other parents' stories that made them reflect on the potential for mathematics learning when playing Yahtzee. This example showed how societal views, through the narratives of other parents, came to influence an individual's view on these kinds of activities. Although this explicit acknowledgement of the influence of others was not made again, it was clear that the social interaction in the PEI between the parents provided a possibility to collaborate in enlarging what was seen as possibilities for children to learn mathematics at home.

The parents' narratives suggested that there was potential for the children to engage in counting, adding and subtracting and, to a lesser extent, multiplying, when playing the board games. The parents considered that the children authored experiences about their coming to know number names and counting sequences. P8 provided a typical narrative about their child:

> We are playing Yahtzee with him (their younger son) in order to collect all the sixes, but it's also worth gathering all the sixes. We want to have six on the die because when playing Ludo we count how many dots there are on the die. We practice that with him, because he counts incorrectly. He could not count them, but he saw that there were six. He has just begun, so here we are a little patient with him.

The parents' valuing of learning numbers can be seen in the recognition that their child could not count to six so they used Ludo as a way of encouraging the child to count with them. This can be seen in the statements about 'gathering all the sixes' in order to follow the rules, 'we want to have six on the die'. P8 supported their child in noticing that there were six dots on a die, to learn a number word and to connect it to a specific quantity. In doing this, P8 showed that they also valued the collaboration and the sharing of experience as important in supporting the child's learning to count.

Other parents' narratives provided more detailed descriptions about the mathematics learning that children could gain from playing these games.

P5	It's good practice to add and subtract, to understand the relationships between numbers
P2	When you have four dice with five dots on each one, it will be four multiplied by five. But he takes all five dice and can really add up all the dice. He does that only because he has played a lot

In the last narrative, P2 reflected on how there was a possibility for the child to engage in multiplying, even though the child only used addition. This is an example of sense-making in that the parent identified how their child used addition, but could move towards using multiplication. Although P2 highlighted that the child could work out the amount on the dice, they also indicated that they saw understanding multiplication as being important for children. In P2's narrative, the child's learning is described as going from the particular context of multiplication and addition to their own experiences.

The parents' valuing of learning numbers seemed to lead them to facilitate their children playing Yahtzee. By collaborating through endorsing each other's experiences, the parents described their pedagogical decisions about providing opportunities for repeating the numbers and making connections to addition. As was the case in the other narratives, P5 explained how they used dice as artefacts to engage their young child in Yahtzee:

> We play with a die with the numbers 1 to 6, but with the younger child we use the die with dots. It is a little easier then.

In this narrative, the parent described their pedagogical choice about which representation of amounts on the die was needed so that their younger child could play Yahtzee. P5 situated their children's mathematics knowledge as not yet sufficient, in that P5's child could not yet recognise numerals. To support the children in learning the valued knowledge of counting, P5 made choices about the artefacts the children could engage with.

The child's use of dice to show their counting competence was an example of the non-verbal communication aspect of narratives and provided information about what the child was capable of doing. In another narrative, P8 stated that they played Ludo frequently and because their oldest son had learned to count very well, they now played with two dice. However, the additions that the child did were connected to the specific context of the mathematical activity:

> In other situations, I cannot say to him, 'What's 5 + 6', but when we play Ludo and he throws five and six then he knows that it is 11.

Like P5, P8 also recognised what their child could do and provided two dice to extend the child's mathematical understandings.

The parents' narratives showed that they valued their children learning numbers when playing board games and this valuing prompted the parents to seek out opportunities for the children to learn to count at home. If children are throwing a die as part of a game, then the question of 'how many dots are there' can be asked. A similar focus on the knowledge and skills linked to numbers was also found by LeFevre et al. (2010). In LeFevre's et al.'s study, the parents observed and paid attention to the numeracy activities that were linked to children's knowledge about counting sequences, with activities such as board games being related to learning numbers. However, in the Norwegian parents' narratives, there seemed to be a distinction between the value of learning numbers and the value of learning counting skills, as discussed in next section. The narratives about how children learned to count by themselves or in activities with family members did not include the parents describing their own engagement and pedagogical choices explicitly.

24.4.2 Everyday Activities, the Value of Learning Counting Skills

Several of the photos used in the PEIs showed children engaging in everyday activities, such as watching TV, eating, reading and free play. As was the case with the board games, in describing the photos in both the PEIs, the parents often focused on their children's development of counting skills, such as repeating number words and labelling specific quantities with the number words. The parents narrated how and in what sense children were encouraged to think of the world in terms of numbers and how they often spontaneously recognised numbers. The parents seemed to value counting strategies and understandings of cardinality.

The following examples of narratives illustrate the everyday activities that the parents saw as being connected to counting skills:

> Olle is watching TV, and they are counting. (P6a)
> He counts 1 and 2 because he is ready, and 3 (and he jumps in the water). (P7)
> He counts screws; he has a lot of them, and he counts those screws. He understands it. (P8)
> She has a book, this book tells counting sequences 1, 2, 3. […] She would count the objects and match them with a numeral on the screen. (P1)

In these narratives, parents described how their children authored their experiences as part of their sense making about learning to count. In making sense of their children's learning, the parents showed their understanding of the complexity of knowing how to count. For example, P7 stated that their 2-year-old child often had a toy in each hand:

> Two things—he tends to have one thing in each hand because he has two hands. For example, two cars. But, here there's not much counting on his part.

P7 viewed this activity as an introduction to the idea that one hand could represent one object. This was valued because it was seen as a beginning stage in developing counting skills.

Other narratives indicated that some parents valued their children learning one-to-one correspondence, in that a number name was connected to touching an object. P6a and P6b reflected on an activity that their son did frequently on his own in the family bathroom.

P6a	He is very keen on counting. He cannot count correctly, but counts 2, 4, 7, 8, 9, 10
P6b	1, 2, 4, 6, 7, 8, I think that's how he counts
P6a	He takes toilet paper rolls on and off again and counts. 'Many rolls!' So he has started to get a little interested in counting
P6b	He has found the most out of it himself. For usually children say '1, 2, 4', but he skips 3 and goes '1, 2, 4'. I do not know exactly how he's gotten into it; the toilet paper rolls are quite easy to move around, and he's able to place them on the toilet paper holder over and over again
P6a	He can count almost to 10, but he does not say all the numbers, so it's not a correct order, he's skipping some
P6b	He just practices saying counting words. Thus, it seems that he only practices in a way. He develops his language; I believe that's part of it

The parents seemed to indicate that touching each object and saying a numeral name aloud was a necessary step towards their child learning to place the correct number words into a counting sequence. This suggested that the parents valued counting but recognised that learning to count was made up of a number of different stages, each of which was important.

As had been the case with playing Ludo and Yahtzee, these narratives showed that the parents noticed that their children were counting. However, their interaction with their children in these activities was not discussed. This suggested that sometimes they chose to just watch and allow their children to explore counting in their own way. The manipulation of artefacts by the children was more prominent in the parents' narratives and seemed to act as an encouragement for the children to engage in activities by themselves. The parents paid attention to this use of non-verbal communication.

When the parents did mention that they interacted with their children, they seemed to situate themselves as supportive partners, who asked questions or provided opportunities to show that the last number said represented the total number of objects. P6b provided an example where his child asked the question 'how many', and to which the parent asked the child to find another set of objects which had a total of four items.

P6b	He often asks, 'How many fingers is 4?' He likes to count. I ask him what else 4 is, can you find something that is 4 of? We count together when he gets the wrong number of items and needs to add or subtract 1

These narratives are classified as illustrating collaboration between the parent and the child, in which the parent seemed aware that counting was related to the addition and subtraction of 1. Everyday activities also seem to provide opportunities for the children to learn about addition. For example:

P6b	He learns in many situations, such as when we sit down and eat waffles (the waffles can be divided into five pieces, each piece having the shape of a heart). We split the waffle and he has 3 + 2 pieces. Then he splits a 2-piece into 1 + 1, so it is 3 + 1 + 1. He has variants. So 3 + 1 + 1 and 3 + 2 is 5. We do a lot of mathematics everyday, rather randomly

P6b seemed to value the child's opportunity to visualise the decomposing of five in different ways and how this could contribute to the child developing their understanding of addition. This parent seemed to consider this kind of activity to be meaningful for their child.

P8 gave an example of the non-verbal communication aspect of narratives in relation to being able to recognise and write numerical symbols as important components for learning addition.

P8	At his age (their son is 5 years old) you are able to do mathematics when it is visual, if I would write 6 + 6 and ask him, 'How much is it,' I believe he would understand it

In this narrative, the symbolic mathematics was something that this parent considered would communicate meaning to the child and was something that a child would need to learn.

The parents highlighted the value of learning counting skills in everyday activities which their children were often engaged in by themselves. In these activities, the parents did not highlight collaborating with the children, as had been the case with playing board games. Other studies, such as by Aubrey et al. (2003), identified everyday activities and experiences of young children, initiated by both children themselves and by adults, as potential opportunities for mathematics learning. However, the Norwegian parents saw that everyday activities such as watching TV, eating, reading and free play did not always require them to facilitate their children's learning of counting. Although Aubrey et al. (2003) stated that it was difficult to know how parents' engagement or pedagogical knowledge about stages of learning counting influenced their children's counting skills development, the Norwegian parents showed that they understood that children needed to learn a range of different types of knowledge and skills in order for them to learn to count or do basic operations such as addition.

24.4.3 Length, Volume, Time, and the Value of Learning Measurement Skills

Like the counting activities, activities in which children were measuring items at home appeared frequently in the narratives. Parents highlighted that their children needed to become aware of the attributes of objects that were to be measured, with or without measuring tools. However, the parents also described the difficulties that children might have when learning about measurement.

P8	You cannot expect children to understand strategies in measuring

As was the case for learning to count, this view seemed to contribute to parents emphasising the need to support their children in developing specific, developmental, measuring skills. As such, the authoring of their children's experiences of learning measurement skills often included a reflection on the complexity of the learning process.

P4 stated in a narrative that children needed a lot of practical experiences to estimate quantities. They reflected on a shared experience with their child about determining how much food a fish needed. P4 said that the measuring was not about weight because at this point their son could not measure in grams. Instead, the child simply compared the size of the fish and the amount of food he had in his hand to determine if it was sufficient. P4 explained that the child had said, ‘Mum, this fish is quite small so he cannot get so much food’. In this narrative, the parent indicated that the child had identified that it was the amount of food, volume rather than weight, which was important. In doing so, they highlighted that the measuring process required an understanding that an object has attributes that can be measured and that there was a need to determine which attribute was most important in a specific situation. However, using standardised units was not considered essential as P4 described how the child used the measurement term ‘small’ and compared the fish with food, without using standardised measuring units.

The correct language for naming each unit was highlighted in the narratives about when children compared objects to determine which was longer or shorter or which was heavier or lighter. This suggested that the use of correct terms was valued by the parents.

P5	It is the same with height as well. When he (her son) comes with the yardstick and measures my height, he will say, 'You are sixty high'. But he does not understand what it means, so we can talk about it
	He is reading the numerals on the yardstick and can recognise the numeral 70
P5 told a story about an activity in which their son found objects that had a length of 70 cm:	
He goes around finding objects and comparing their length to the yardstick and then reports back to me that he found something that is 70 long	

In this narrative, the child identified objects that had a specific length by measuring with standard units, using a yardstick as a measuring tool. The parents authored narratives about children exploring the use of standardised units and direct and indirect comparisons. The parents used these narratives to make sense of what their children understood about the measuring process. The parents seemed to value the children exploring objects and their attributes as part of their learning. The collaborating and shared engagement, where children needed to use descriptive language, also seemed to be valued.

Measuring time was debated by the parents, especially in regard to whether young children could learn to read a clock. Parents seemed to agree that young children could not measure time precisely. Instead, reading a clock was a skill that was learned in progressive steps over time. Some parents indicated that an initial step was for children to show an interest in learning to tell time. Standard units of measurement, such as hours, seconds and minutes, were often introduced by the parents. For example, P5 explained how they helped their children make sense of a minute, by timing how long they took to brush their teeth.

> I try to explain a particular timeslot of 60 seconds, so I told my children that it takes 60 seconds to brush their teeth. So we can talk a little about the clock and how long one minutes takes.

The different aspects of learning mathematics, as exemplified by parents in this study is similar to what Meaney (2011) found in her study of a 6-year-old child engaged in a number of measurement activities at home. In Meaney's study, the parent understood and recognised that measuring time was a complex concept. Meaney suggested that the abstract nature of time is difficult to grasp, so there is a need to find ways to talk about it (for example, experiencing the timeslot of 60 s when brushing teeth). The Norwegian parents valued that the children need to want to know about learning time as the first stage, which is different from Meaney's study where the mother recognises a need to learn how to tell time for practical reasons.

Parents valued mathematics activities that drew children's attention to the attributes of objects and highlighted the descriptive language and comparison terms that are important for describing these attributes. This suggests that the parents valued the children's sense-making as they began to measure objects and their strategies for learning how to measure. There are some similarities to Clarke and Robbins (2004) in which children measuring ingredients and cooking at home were identified as illustrating measuring capabilities. According to these researchers, these kinds of activities offered general mathematical experiences for the children, while the narratives from the Norwegian parents gave more detailed justifications about learning measurement skills. The parents not only recognised opportunities for exploring attributes of objects or use of standardised measuring units but also valued the development stages of learning measurement skills when children engaged in activities such as feeding a fish.

24.4.4 Money and the Valuing of Equivalence

The last group of activities were to do with the equivalence of money, particularly between notes and coins. In both PEIs groups, there were photos in which children engaged with money, such as when they were saving for a toy or receiving money as a gift. The parents seemed to value their children understanding the equivalence of different representations of money as a prerequisite for children learning how to spend their money wisely. In the parents' narratives, there was evidence for valuing of this skill as something important and useful for children to develop the ability to use money in everyday circumstances. This is also validated in the work by Brenner (1998), where she found that activities with money, buying things or spending money, are among children's most common uses of mathematics outside of school. To gain this skill, parents indicated that they needed to collaborate with their children by using artefacts, such as money or objects. The parents authored narratives about how children came to know how to use money in everyday life.

One parent described going to a recycling centre and trying to help her child understand the value of the money gained from recycling a specific amount of cans by comparing it to an amount of Lego.

P3	The aim of going to the recycling centre with my son is to talk about the value of things. I explain to him how many bags of cans we need to recycle in order to have money to buy a small Lego set. So instead of talking a lot about numbers, I compare the number of bags we need to recycle to save for Legos. I am not good at it, but I try to help him relate to it

P3 considered that numbers alone would not help their child understand equivalence. Instead, P3 highlighted the need to make comparisons that made sense to the child. They supported the child in seeing the equivalence between a number of bags of cans that needed to be recycled and how much was needed to buy a Lego set. Yet, in the narrative, P3 highlighted their lack of pedagogical knowledge about explaining to their child the value of what was being recycled.

There were several other narratives in which the parents explained how their children did not recognise the value of notes, demanding to have coins instead because they were seen as more valuable.

P7	A 100 kroner note is 100 coins, and 10 coins with a value of 10 kroner are also 100 kroner. But my child would rather have 10 (lots) of 10 kroner coins than one 100 kroner note
P1	I can see that she has no understanding of how much 200 kroner is worth even when I would count 1, 2, and 3. She recognises the 50 kroner note, but she has little understanding about the value of 50 NOK, and she would rather have five 10 kroner coins. In terms of notes, she does not yet have an understanding
	My children got 200 kroner each from my mother, but the money was divided differently. I tried to explain to my older child that they both had the same 200 kroner, but she wanted to have more coins. So I decided to exchange her notes for coins. They were preoccupied with having equivalent amounts (volume) of money, not the value of the coins and notes

The parents' narratives seemed to highlight that they valued their children gaining an understanding of the equivalence of notes and coins so that their children could use banknotes in real-world exchanges. As such, the non-verbal communication aspect of the narratives illustrated the children's difficulty in valuing different coins and notes in any other way than simply counting them. These examples illustrated that the parents valued coming to know how to use money but realised that this was a difficult concept for young children to grasp, because of the need to understand the abstract value attached to notes and coins. The parents valued the everyday activities in which children engaged in experiences with money. This was also the case in the Clarke and Robbins (2004) study where shopping situations provided opportunities for parents to use examples of spending money to convey moral learning about not being able to buy everything children wanted to have. The Norwegian parents' narratives suggested that they valued the learning equivalence of money and considered the children's use of artefacts as important in developing this skill.

24.5 Discussion

In this chapter, I have investigated what parents valued in the mathematics activities that their children engaged in at home. These findings contribute to the research stating that parents and families play a role in children's learning and development (Phillipson et al., 2017), but provide a detailed justification for identifying the ways in which parents valued the mathematics activities their children engaged in at home. In considering parents' role as children's first educators, it was important to identify what they understood from their perspective. Yet, at the same time, it was clear that the parents' understanding did not arise in a vacuum, as it was through the discussion with others that their views were endorsed or challenged. In this way, the parents revealed some of the societal norms about mathematics for young children that they accepted. Through the narratives, these parents seemed to agree on the societal values of mathematics education for young children. As had been the case with the Polish parents (Lembrér, 2018), these values were affected by the wider views, but in this study, it was possible to see how the discussions endorsed particular norms and values through their discussions.

By analysing the authoring, sense-making, collaborating and non-verbal communicating aspects of Burton's (1999) narrative approach, the parents' values became apparent to do with: learning numbers, counting and measurement skills, and the use of money in everyday life. The parents also indicated that they valued activities which supported the children at their different developmental stages.

The parents authored narratives in which they indicated that they were aware of how their children engaged with mathematics learning at home by identifying a variety of knowledge and actions that their children could currently do and what they would like them to be able to do in the future. For example, they recognised that their children had a range of different counting skills but were not yet able to do multiplication or recognise the equivalence of coins and notes.

As the activities were undertaken both by the children themselves and with family members, the parents were able to discuss when and why they would interact with their children during the activity as the collaborating aspect of the narratives. They also collectively explored their role in their children's mathematics learning and its impact on how their children were engaging with mathematics. The parents discussed their pedagogical skills, used different methods to support their children when engaging in mathematics activities, and were aware of what their children could potentially learn in a specific situation. For some parents, playing board games allowed them to identify a range of mathematics skills that were part of the children's everyday experiences.

In regard to the non-verbal aspect of narratives, the parents showed awareness of how artefacts, such as dice and measuring tools, could be used to support children's engagement with particular mathematical ideas. They indicated that they understood that some artefacts, such as money, required the children to understand abstract ideas that were too advanced for them. Mathematical symbols were seen by one parent as possibly being understood by their child, but parents of other, younger children might have seen this as being too advanced. This suggests that the parents were able to identify what artefacts would help children to make sense of different mathematical ideas. The parents negotiated their own understandings of children's actions as something valuable in terms of mathematics learning. This confirms what Anderson and Anderson (2018) stated, that parents see children's home experiences as an important source for mathematics learning.

24.6 Conclusion

As parents are children's first educators, it is important to understand what parents value in young children engaging in mathematics activities at home. To do this, I used a narrative approach to identify the ways in which parents' views reveal what they value about doing mathematics at home. In particular, I argue that mathematics learning is something that emerges between parents and children at home, building on the pedagogical choices made by parents. For example, depending on the rules applied by the parents in board games, the children can be supported in different ways, creating opportunities of development of basic numerical skills. Parents also recognised and valued some unexpected aspects of children's engagement in mathematics activities at home. This provides an opportunity for learning between parents and teachers of young children to become a two way-street with knowledge going between both groups. In earlier research (Whyte & Karabon, 2016), communication has often been situated as a one-way street with parents being told what is important by teachers. Parents' experiences of, and their focus, on children's actions could be considered as potential resources and an example of partnership for shared responsibilities in education between preschool staff and parents (Hujala, Turja, Gaspar, Veisson, & Waniganayake, 2009).

This research continues to expand early mathematics research with evidence for, and understanding of, young children's expressions to learn mathematics, and to widen our perspectives about providing children with rich, meaningful mathematics learning experiences. Although this study confirms findings from other studies in early childhood mathematics, the nine Norwegian parents provided detailed and more nuanced information about the mathematics learning they valued in young children's activities at home. They highlighted how the authoring of children's mathematics skills by engaging in concrete experiences contributed to sense-making, as well as how they promoted dialogues to help their children make use of artefacts.

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Chapter 25
"Pedagogical" Mathematics During Play at Home: An Exploratory Study

Ann Anderson and Jim Anderson

25.1 Introduction

To improve equitable access to school mathematics learning, teachers and researchers need to recognize and value the mathematics each child carries with them from the socially and culturally diverse experiences in which they engage every day, prior to coming to school. Similarly, researchers and teachers need to view parents and significant others from a strengths-based, rather than deficit, perspective, in full recognition of the "funds of knowledge" (Moll, Amanti, Neff, & Gonzalez, 1992) family and community members possess in terms of scaffolding young children's mathematics development. Yet, claims regarding parents' limited ability to support their children's mathematics learning (e.g., Milner-Bolotin & Marotto, 2018) continue to proliferate, and are at times, class or race based. In this chapter, we report on two "pedagogical" at-home, play-based activities (i.e., Playdoh: Pizza and Toys: Cars) to both familiarize readers with two mothers' capacity to "teach" mathematics and to critically examine the privilege we seem to afford "pedagogical" mathematics as the way forward.

25.2 Theoretical Framework

Informed by socio-historical theory (Rogoff, 2003; Vygotsky, 1978; Wertsch, 1998), our research of parent–child engagement with mathematics prior to school is premised on our recognition and valuing of the social, cultural, and individual ways in which children learn mathematics and parents and significant others mediate that learning. As such, we argue that children's mathematical experiences "cannot be

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understood out of the context of the immediate practical goals being sought and the enveloping socio-cultural goals into which they fit." (Rogoff, 1990, p.139). In addition, we believe that parents' perceptions of their role in their children's learning and their conceptions of the task-at-hand influence the types of support they provide. In turn, Vygotsky's *zone of proximal development* provides us with a conception of children's learning when adults structure activities that engage children in more complex behaviors and thinking than they would on their own. Finally, we argue that parents' contributions to their children's mathematical development are such that, as Bjorklund and Pramling (2017) suggest, the adult "is sensitive to what the child expresses" (p. 76) and "mathematics is, in Vygotsky's terms, cultivated rather than imposed" (p. 77).

25.3 Background Literature

Research into young children's engagement with mathematics at home tends to fall into three categories. A number of studies (e.g., LeFevre et al., 2009; Skwarchuk & LeFevre, 2015) rely on parent reports captured through questionnaires and/or interviews. Other studies (e.g., Anderson, 1997; Anderson, Anderson, & Shapiro, 2004, 2005; Vandermaas-Peeler, Nelson, Bumpass, & Sassine, 2009) investigate parent–child interactions during tasks for which the researchers provide the materials in clinical, childcare, or at-home settings. Finally, there are observational studies (Anderson & Anderson, 2014, 2018; Aubrey, Bottle, & Godfrey, 2003; Bjorklund & Pramling, 2017; Tudge & Doucet, 2004; Walkerdine, 1988) of parent–child interactions during "naturally occurring" events at home. Overall, this research captures considerable diversity across families in terms of the frequency and types of activity, and the mathematics inherent in them, although the literature appears to suggest that families emphasize counting and number.[1] Likewise, overall findings, from parent reports and "naturalistic" observations, suggest that children's engagement with mathematics prior to school tends not to occur during explicitly didactic interactions (Benigno & Ellis, 2008). For instance, Tudge and Doucet (2004) concluded that children from middle- and working-class families were minimally engaged in either "academic lessons" or "play with academic objects" pertaining to mathematics, although they found significant variation across the families.

While educators acknowledge the importance of young children's mathematics learning prior to school, we appear to know less about the impact of the different types of mathematical activity that occur at home. In the 1980s, researchers (Tizard & Hughes, 1984; Walkerdine, 1988) classified certain at-home tasks as either "pedagogic" or "instrumental," the former referring to occasions where "the focus was

[1] In Anderson and Anderson (2018), we argue that "the prevalence of number previously associated with mathematics in the home, … needs further consideration and research" (p. 196).

predominantly the teaching and practice of counting" (Walkerdine, 1988, p. 81). In our longitudinal study (Anderson & Anderson, 2014) of six middle-class homes, we found that the role of mathematics within joint adult–child activity fell along a continuum where mathematics was the goal, a major emphasis, an equal emphasis as another goal, a minor focus, or incidental. Hence, pedagogical and instrumental categories were viewed as endpoints of such a continuum, corresponding to "math as a goal" and "math as incidental," respectively. Considering the varied activities within these families, we further argued that young children's mathematical experiences prior to school are likely more eclectic than research to date has been able to illustrate. While Aubrey et al. (2003) provided evidence of the maximum variation in parental mediation styles by describing one mother's pedagogical and another mother's instrumental style, they argued that neither style of parent mediation could be considered "better" than the other, since teachers reported that both children were performing equally well in mathematics at school. Likewise, Skwarchuk and LeFevre's (2015) results from their longitudinal project showed that "both formal and informal experiences are related to children's numeracy acquisition but through different mechanisms" (p. 109 [emphasis added]).

Our interpretation of the literature is that educators continue to dismiss or undervalue activities and experiences where mathematics is incidental, including those found in preschool and Kindergarten classrooms. Similarly, likely due to the limited amount of didactical events captured in "naturalistic" studies to date, there appear to be increasing calls for educators and researchers to tell parents, and in turn early childhood educators, who emphasize a play-based curriculum, how to make mathematics more explicit or pedagogical. However, to date, the qualitative nature of "pedagogical" mathematics, and parent–child engagement with it, without outside intervention, is largely unknown. To begin to address this gap in the literature, the current study explores the research question: In what ways do mothers of preschool children engage their children with mathematics during "pedagogical" tasks at home?

25.4 Method

Periodically over the course of 2.5 years, six preschoolers were videotaped as they participated with family members in at-home activities, such as baking cookies or reading a storybook. On the day of video recording, the mothers, knowing that the focus of the study was on mathematics, designated what to have taped. A research assistant recorded four of the families, and two mothers elected to do their own recording. Of the 45 activities collected (Anderson & Anderson, 2014, p. 8), 18 were classified as those in which "math is the goal" (see Table 25.1).

Table 25.1 Activities, mothers chose to videotape, previously classified as "Math is a goal"

Activity	Each family's "pedagogical" activity ADAM (9)	Liu (8)	PENN (4)	Star (7)	Beet (7)	Pimm (10)
Puzzles	Number	Number				
Joint play	Store			Stickers		
Board game	Snakes and ladders		Bingo			
Story time	Number and shapes					
Family time					Baking	
TOYS		Pop up animals	CARS			
PLAYDOH	PIZZA					
Physical game	Hopscotch					
Matching game	Numerals dots and words	Rods: Ten pairs				
School like	Word problems	Computer game				
Songs	Number					
Other games		Macaroni				

25.4.1 Data Sources

In this chapter, we focus on two "pedagogical" activities, namely Playdoh: Pizza and Toys: Cars, which the same research assistant recorded in the Adam and Penn (pseudonyms) homes,[2] respectively. In the Adam family, the daughter (aged 3.5 years) made Playdoh pizza,[3] while she and her mother chatted about imaginary visitors who were coming to dinner. In the Penn family, the son (aged 3 years) played with his toy cars and a wooden "town" set, while he and his mother chatted about his cars and nearby neighborhoods (see Appendix A, Figures 25.2 and 25.3). In Anderson and Anderson (2014), while both Playdoh: Pizza and Toys: Cars were designated as play-based activities, they were also deemed to be pedagogical since doing mathematics appeared to be each mothers' goal from the outset and they verbalized mathematics throughout. However, the extent to which the Adam and Penn mothers pre-planned these "pedagogical" tasks is unknown. For analysis purposes, we repeatedly viewed the videotaped sessions of Playdoh: Pizza and Toys: Cars and augmented previous verbatim transcriptions with descriptive details regarding the non-verbal and verbal interactions. We then analyzed each of these augmented transcriptions for trends, within and across the two activities, regarding the mothers'

[2] The Adam mother was an elementary teacher on maternity leave for her second child at the time of study; the Penn mother was an international graduate student (her program was unrelated to math & education). Both homes were considered middle class (based on values and earning power), although family income at time of study may have situated them otherwise.

[3] Playdoh is the trade name of a non-toxic modeling compound sold by Hasbro Toys (https://playdoh.hasbro.com/en-ca)

pedagogical approaches. In turn, we categorized the pedagogical moves we found according to their role in mathematics engagement.

25.5 Results

From the outset, both mothers use several pedagogical moves (see Table 25.2), to varying degrees, to make and keep mathematics as the goal of these two play-based activities.

25.5.1 Establishing the Mathematics Goal

Both mothers joined their children in their play, sitting at the child’s level and in close proximity to the play materials. While the Adam dyad opened with the idea of making a Playdoh pizza, the Penn dyad played briefly with plastic animals before turning to the toy cars. Both mothers drew from the unfolding dialogue to set the stage and pose the mathematical problem to their children. The Adam mother used child-like language (e.g., “let’s pretend”) to invite her daughter to consider the

Table 25.2 Summary of pedagogical “moves” in Playdoh: Pizza (Adam) and Toys: Cars (Penn)

Mathematics engagement	Pedagogical “moves”	Adam mother—daughter	Penn mother—son
Establishing goal	Setting the stage	Four visitors … cut pizza so each gets same	Line cars in a row and then take some away
	Changing plans (slightly)	How many slices do we have/need?	How many “orange” cars are there?
	Refocusing on original	Cutting & counting “equal” slices for visitors	Adding cars to a group and counting (one more)
Sustaining goal	Scaffolding (correcting) perceived error	Is this piece equal to others?	Can you count them [to check]?
	Connecting math with previous experiences	Fractions with money	Numbers with age
	Responding to child’s interest in math	D: Let’s pretend we have six people	S: And now you can count all of these cars
Addressing child’s role	Engaging with child’s storying of task	D: Let’s pretend we’re sisters, I’m 7, you’re 8	S: This is downtown; … daddy’s red truck
	Supporting child’s problem solving	Child proposes alternate cuts; Tic tac toe	Child counts cars of both color
	Encouraging child’s independence	Little circles of pepperoni	Different parking spaces

Table 25.3 Setting the stage: mothers pose original problem

Playdoh Pizza: Adam mother (M)—daughter (D)	Toy Cars: Penn mother (M)—son (S)
M and D are seated at child's table (adjacent) 1 M: Are you going to make us a pizza? 5 M: … What shape is it going to be? 6 D: A heart 7 M: … What shapes are pizza usually? 8 D: Triangle. 9 M: Yes, when you cut the pieces they are. Let's see if we can make a regular shape. A big circle right? A big pizza 18 M: … How is the pizza? OK … Let's pretend we are going to have four people over for supper, four people are coming. We want to cut this pizza so everybody gets the same. So the first thing we need to do is cut it in half	*M and S seated on floor (facing); pairs of cars near* 25 M: What do you want to play with the cars? 26 S: A broken car. (inaudible) 27 M: Do you want to play the counting game with the cars? 28 S: Yeah 33 M:…look almost the same. What kind are they? 34 S: Tow trucks 35 M: Two tow trucks 39 M: Shall we play the game. Shall I start? 40 S: Counting game 41 M: …do you want to line them in a row *(M picks up car)* and then take some away and see how many there are?

Table 25.4 Changing plans (slightly): mothers ask "how many" questions directly

Playdoh Pizza: Adam mother (M)—daughter (D)	Toy Cars: Penn mother (M)—son (S)
24 M: So you cut it in half first. (*D cuts circle horizontally in the middle*) Good girl. *(M separates semi-circles)* And there is how many pieces? 25 D: Two 26 M: Two. But we need more than two, don't we? How many people are coming? *D holds up four fingers*	58 S: Only one red car 59 M: Hum, only one—how many, … 61 M: How many orange cars are there? 62 S: One, two, three.*(points toward (above) the cars)* 63 M: Three orange cars and … how many brown cars are there? 64 S: Only one, two. … Only two!

problem of "sharing a pizza equally when four visitors come to supper." The Penn mother referenced "a counting game" they had played before and invited her son to consider the problem of "how many cars remain, when a number of them are removed" (Table 25.3).

Thus, both mothers' problems emanate from the child's play while making Playdoh pizza or sorting toy cars, and redirect the child's efforts toward fulfilling a mathematical goal, namely, cutting a pizza into equal slices and finding a numerical difference with cars. Interestingly, as both mothers engaged the child in solving the problem, plans seem to change slightly, as they asked their child about small sets, as an intermediary step (Table 25.4). In doing so, the Adam daughter checks the problem's parameters and the Penn son attends to the number of cars already sorted.

Table 25.5 Refocusing on (revised) original: mothers pose similar problems as the original

Playdoh Pizza: Adam mother (M)—daughter (D)	Toy Cars: Penn mother (M)—son (S)
27 M: Four people are coming for supper so can you cut it in half this way? *(M gestures to cut in middle vertically; D gestures a horizontal cut)* 28 M: This way sweety? *(gestures vertically)* Yeah 28 D: … Gotta go this way. Let's pretend it's a … 29 M: OK, but let's do this first; so you already cut it in half this way so let's cut it in half this way. Yeah. Let's see what happens. *(D cuts vertically at middle)* That's a girl. So we cut it in half that way. Now how many pieces do we have? One, two 30 D: One, two, three, four. *(M moves each)* 31 M: Four	67 M: And can you see any more orange cars? … Do you want to put it with the other orange cars? 68 S: Another blue car. *(picks up a blue car)* 69 M: Another blue car, good one. *(S places it near other blue cars)* Now how many blue cars are there? 70 S: Only three 71 M: Three. You've got two and one and you've got three. And now you've got three yellow ones and look is that another yellow one? 73 S: Yeah (picks up a yellow car) 74 M: You've got three and now you are going to add another one. Now how many …?

Immediately following these simpler "how many" questions, both mothers refocus their child's attention onto the original problem. However, the problem has been revised somewhat, either consciously or inadvertently (Table 25.5).

For the Penn dyad, the revised problem involved adding "one more" to small sets of a known quantity and for the Adam dyad, the revised problem involved one-to-one correspondence between the number of visitors named and the slices needed. As a result of these small and subtle shifts, the Penn mother encouraged a "number operations" goal as planned but included the child's interest in sorting, and adding, cars into color groups. Likewise, while the Adam mother continued to encourage the "dissecting" goal, the focus was more on counting slices to match a given number and less on "sharing equally."

25.5.2 Sustaining the Mathematics Goal

At times, both mothers appear to scaffold their child toward a "correct" answer by asking them to try a different strategy (Table 25.6). The Penn mother encouraged her son to count to check his solution, while the Adam mother suggested using a whole circle, not four parts, to cut for six visitors.

Although both mothers offer the alternate strategy as a choice, both children agree with their mother's idea. Interestingly, each recommended strategy seemed to "backfire" for the child, if the mothers' interpretations, in the moment, accurately reflect the child's intentions.[4] Both mothers also connect the mathematics in the

[4] While our data are inconclusive, it is plausible that both mothers "misinterpret" the child's "solutions"; the sliver may simply be residual from Adam child's inability to retrace a previous cut, and the Penn child may be "counting" the entire row of cars (browns and yellows).

Table 25.6 Scaffolding (correcting) perceived error: mothers name checking strategy

Playdoh Pizza: Adam mother (M)—daughter (D)	Toy Cars: Penn mother (M)—son (S)
33 M: … How can we divide it for six people?	74 M: You've got three and … another one. Now how many do you have?
34 D: You go like that. Go like that and then you go like that. (*gestures a middle cut for each slice*)	75 S: Five
35 M: Should I build it back together or do you want to keep it like this?	76 M: Can you count them?
36 D: Build it back	77 S: One, two, three
44 M: OK you show me	78 M: Count all of the yellow ones, try. *(M spaces brown cars further from yellow ones)*
D cuts horizontally near middle and again just slightly below. D tries to separate semi-circles	79 S: (*points to each car*) One, two, three, (taps 4th car, shouts) nine
46 M: … if one person gets this piece (*holds up a very thin strip*) is that the same size as this …?	80 M: One, two three—how many. One, two, three—*(M points at each car and keeps finger on last car)*
47 D: No. *(D shakes head)*	81 S: (shouts) Eight
48 M: This guy will be so hungry. We need to make them so that they are the same size. So let's put it back together. How can we do that?	82 M: …What comes after three?
	83 S: Five

Table 25.7 Connecting math with previous experiences: mothers' associate with everyday use

Playdoh Pizza: Adam mother (M)—daughter (D)	Toy Cars: Penn mother (M)—son (S)
76 D: One, two, three, four. *(touches each piece)*	82 M: …What comes after three?
77 M: Right. So four of these pieces make	83 S: Five
78 D: Have seven	84 M: What is it going to be when it's your birthday? How old are you going to be after three?
79 M: The whole thing right? Make one. Remember when we went to buy Sara your doggy? (*puts small toy dog on table)*	85 S: Four
80 D: Yeah	86 M: Four
81 M: And you helped mommy count up all the change and I told you to put quarters in a pile to make one dollar. Remember how many quarters you had to put to make one dollar?	87 S: It's four *(S looking at cars M has organized)*
82 D: Four	88 M: It's four, yes
83 M: Four quarters just like this	90 M: And what is it going to be after your next birthday when there is one more? *(puts car next to 4 others)*
	91 S: *(looks down at the row of cars)* five

activity with previous experiences involving everyday use of number to scaffold the child's understanding (Table 25.7). When the Penn son seems to "confuse" counting beyond 3, the Penn mother points to "his age on his next birthday" to help him count the "fourth" car and beyond. To explain the idea of quarters, the Adam mother reminds her daughter of when she saved 4 quarters[5] to buy a stuffed dog.

While the mothers initiated the mathematics problems during the play activities, and both children cooperated with their mothers, each child also explicitly signaled their own interest in the mathematics as a goal. For instance, the Adam daughter

[5] Canadian coins, valued at 25 cents, are commonly called quarters.

Table 25.8 Responding to child's interest in math: mothers acknowledge child's suggestion

Playdoh Pizza: Adam mother (M)—daughter (D)	Toy Cars: Penn mother (M)—son (S)
32 D: Let's pretend we have (pause) six people 33 M: Six people coming? OK. How can we divide it for six people? 34 D: You go like that. Go like that and then you go like that. (*gestures a middle cut for each of the 4 slices of pizza*) 	129 S: And now you can count all of these cars 130 M: Count all of these cars? 131 S: All of them right here. *(adds cars to row)* 132 M: Hum 133 S: Another one—(inaudible) 134 M: … Do you want to put white ones together?

suggested a number of visitors to consider, and the Penn son invited his mother to count cars he had assembled (Table 25.8).

While both mothers acknowledge their child's suggestions, the Adam mother follows through on her daughter's suggestion by posing the problem of sharing the pizza with six people. The Penn mother, on the other hand, does not "count the cars" as the child asks but rather redirects his attention to grouping more cars.

25.5.3 Addressing Each Child's Role in Play-Based Mathematics

While both children readily engage with the mathematics their mothers set out as a goal, each child wove components of a story throughout the conversations (Table 25.9), thereby infusing characteristics into the task, which appear to maintain the playfulness of the activity. For example, the Adam daughter constructs the activity as two sisters of different ages (i.e., she and her mother) attending a birthday party. The Penn child contextualized the cars on the carpet as a short distance from Downtown, represented by a building placed on the floor nearby, and inserted references to "specific" cars, such as his dad's red car and an ice cream truck, as well as actual places, such as Granville Island, into the play context.

Both mothers "play along" with the storyline, at times adding contextual information, like when the Adam mother giggles as an older sister might or the Penn mother recites Granville Island's parking rules, thereby keeping the tenor of the activity playful.

Also, when supporting their child's problem-solving approaches, both mothers are receptive to their suggestions. The Adam mother repeatedly listens, and watches her child "propose" solutions, and the Penn mother watches her son repeatedly make his own shapes, which he called "parking lots," or count unintended groupings (Table 25.10). On occasion, the Adam daughter does not have the opportunity to follow through on her proposed ideas.

Table 25.9 Engaging with child's extension of task: mothers play along with the storyline

Playdoh Pizza: Adam mother (M)—daughter (D)	Toy Cars: Penn mother (M)—son (S)
37 M: So you want to start fresh for six people? OK? *(M & D pat playdoh back into circle)* 38 D: Let's pretend we're sisters and I'm 7 and your 8. 39 M: I am seven. Oh you are seven and I am eight? *(M lifts and flips the circle onto table)* 40 D: Let's switch. I am seven and you are 14 41 M: Oh I am fourteen. So I am the oldest sister? (M giggles) OK… 52 M: …first show mommy how you do it in half 53. D: You're not my mother 54 M: Oh show your sister 65 D: Perhaps I can be the birthday girl 66 M: You're the birthday girl? Ok you can pick 67 D: Ok, this one (picks up one of the 6 slices)	41 M: … take some away and see how many …? 42 S: This is the downtown. (S *places a tall building a short distance from the line of cars*) 43 M: That is the town 44 S: That is the downtown 53 M: We should have a red car, why? 54 S: For daddy's car. … 56 S: This could be daddy's red truck … 195 S: We have to make a Burnaby 196 M: a Burnaby 288 M: Yeah, it could be Granville Island 289 S: It is Granville Island 297 S: There is no many parking 300 M:…after three hours you have to leave Granville Island… or the tow truck comes…

Table 25.10 Supporting a child's problem-solving approach: mothers encourage child's actions

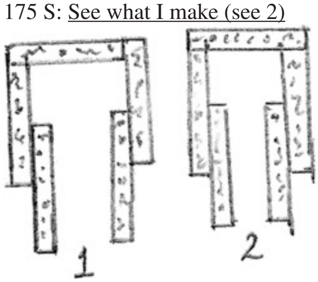
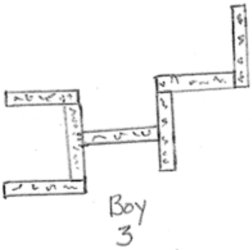
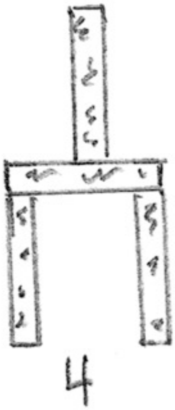
Playdoh Pizza: Adam mother (M)—daughter (D)	Toy Cars: Penn mother (M)—son (S)
49 D: This is how you do it. I will show you, like that. *(D moves plastic knife as if to make a horizontal cut near the middle)* and that (*D moves knife as if to make a vertical cut)* and that *(moves knife as if to make another horizontal cut below the previous one)* 50 M: Let's try it 51 D: OK. (*D cuts across lower half of circle)* 52 M: *(places hand as if to chop pizza in middle)* right in the middle 53 D: *(D continues making a second horizontal cut closer to middle of the circle*) Tic, Tack, Toe 52 M: Tic, Tack, Toe. (*M giggles; smooths out pizza*)	108 M: … how many … of this color? … 109 S: One, two—one, two, five, six, eight. It's three 110 M: It's three. Would you like to count to eight? I think there could be eight along here. *(points across 3 brown and 5 yellow cars in a row*) Well let's see how many …one 111 S: Red 112 M: Two 168 M: … You can set the city up now … 169 S: Yeah. Houses. We can make a real … 171 S *selects some flat rectangular "sidewalks"*) 173 S: See, look what I make (see 1) 174 M: What did you make? 175 S: See what I make (see 2) 

Table 25.11 Encouraging child's independence

<table>
<tr><th>Playdoh Pizza: Adam mother (M)—daughter (D)</th><th>Toy cars: Penn mother (M)—son (S)</th></tr>
<tr><td>92 M: … want me to make a little yellow one?
93 D: Yes. (D is flattening pink playdoh.) Do you want to put the pepperoni in (inaudible)?
99 M: I will make little circles of pepperoni and you can put them on how you want OK?
100 D: No you can put them on
101 M: Each piece should get the same pepperoni
102 D: Cut it in half. Pretend this one is going to be dog food for Sara
103 M: OK, so half of our pizza is dog food and what about this half?
104 D: This half is going to be our food
105 M: OK
108 D: I need some more pepperoni</td><td>178 M: How about you just make it. How about you tell me what you want to make
179 S: A parking space
180 M: Parking spaces. Then you can drive the car into park. Hum, good idea
183 S: A different parking spot. Mommy [see 3]


203 S: … this is a parking spot, mommy [see 3]
204 M: This is a parking lot
205 S: I am making a different kind of parking spot. Looks good. I want to do it like that [see 4]


M: Is the car going to go inside that space you made? [S makes a U shape parking lot]
S: I'm making a parking spot … to fit your van, mommy. … (Continues to move sticks)</td></tr>
</table>

Indeed, nurturing the child's independent actions or thinking varied within the activities, with both children afforded more autonomy toward each episode's latter portion (Table 25.11). Certainly, the Penn son's independent activity increased when the focus turned to building the town, and the Adam daughter took the lead when the focus shifted to adding pepperoni to a pizza. While the Adam daughter speaks of "halves" and "more pepperoni," her attention to mathematics appears to wane. On the other hand, the Penn son's unspoken attention to shape and space, while making parking spots and later parking cars, seems to escalate.

Thus, it is evident that these two mothers behave in teacher-like ways when engaging their children with mathematics within these two play-based activities. However, it is important to note that these "pedagogical moves" were not static, single occurrences arising chronologically, as the excerpted examples inadvertently portray. Rather, as each activity unfolded, each mother repeated certain "pedagogical moves" and thus, the frequency with which they arose, and the order in which they occurred, varied.

25.5.4 Beyond Pedagogical Moves: Mother-as-Teacher and Funds of Knowledge

In addition to the commonalities found in their pedagogical moves, these two mothers' teaching of mathematics during play at home carried other similarities, as well as differences. For instance, both mothers tended to use a Question-Answer-Evaluation discourse to elicit or ascertain their child's knowledge, especially when number, more specifically counting, was the focus. When either asking, or directing, their child to reconsider their answers when they made errors, each mother tended to point toward strategies for checking. For example, the Penn mother directed her son to count to check a total she perceived to be incorrect, and the Adam mother asked her daughter to compare slices to check their sizes (see Table 25.6).

However, during the Adam daughter's exploration of shape while dissecting a circle, and the Penn son's exploration of space through creating parking spots, the mothers seemed to position themselves differently. For instance, the Adam mother-as-teacher appeared more directive, telling and showing her daughter where to cut the circle, and at times making the cuts herself. That said, this mother-as-teacher repeatedly had her daughter suggest solutions, verbally predicting where she might cut, gesturing those cuts and on occasion, making one or two cuts before the mother intervened. The Penn mother-as-teacher, however, appeared more facilitative, acknowledging her son's efforts and encouraging his experimentation. For instance, she praises her son for creating parking spaces where "[a] car [can] enter and exit a spot without blocking another" and supports his efforts to generate various configurations (e.g., "you are making all interesting designs out of the same pieces"). In addition, this mother-as-teacher extends the son's making of parking lots to include the parking of cars in the spots created (e.g., "It is a parking lot ... Can you park four cars in there for me?"), culminating in a parking lot that accommodates about 16 cars, in albeit atypical ways (see Fig. 25.1). Interestingly, this mother-as-teacher responds to her son's "creative" parking plan by checking if all cars can move freely, thereby reviewing spatial properties of such a configuration.

M: If you put that one there, how is this one going to get out though?
What if that one comes to the car before this one and wants to go home?

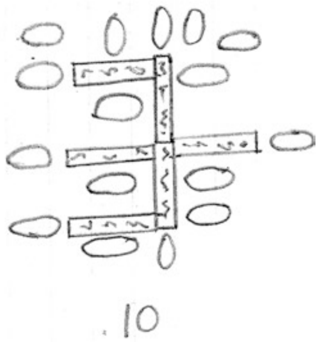


Fig. 25.1 Penn son’s parking lot with 16 parked cars

S: It can go right here.
M: You have to put this one right here so that this one has room …
now there, very good. Now this one can go out this way,
and … this one can go that way.

To observers, these two pedagogical activities appear reminiscent of a “lesson” which, in its implementation, tends to broaden or diverge from the original goal in unexpected ways, but still allows mathematics to be done and talked about. As such, these two mothers demonstrated funds of knowledge for “teaching” and “doing” mathematics with their preschoolers during shared play activity. What remains unclear from the data is what funds of knowledge informed what decisions as each mother-as-teacher worked to support each of their child’s mathematics learning. For instance, why does the Adam mother repeatedly model the cutting of the pizza for her child, while the Penn mother repeatedly encourages her son’s building of parking spaces? Is one mother drawing on her funds of knowledge gained from formal pedagogical education and teacher practice, while the other one is drawing on her funds of knowledge of mothering this particular child? Or, why does the Penn mother invoke an “abstract” number sequence, such as the child’s ages on upcoming birthdays, to assist her son in counting “concrete objects,” and why does the Adam mother invoke the money analogy (quarters) to explain the number of fourths in a whole? Is it that their funds of knowledge of their particular child’s lived experiences take precedence? Likewise, when the Adam mother supports her daughter’s desire to cut sixths of a circle or thirds of a semi-circle, it remains unclear whether she is using her funds of knowledge of school math regarding dissecting circles or her funds of knowledge of everyday math regarding slicing pizzas. Whereas, the Penn mother’s use of her funds of knowledge of everyday parking experiences and rules (e.g., “That is what you have to do when you do a parking lot, you have to make sure they can all get in and out and not be blocked by another one”) seems more evident.

Overall, when we take a strengths-based perspective, we see these mothers as competent teachers of mathematics. Like other teachers in school-based or in early childhood settings, they enact their pedagogical practice with both the content and the child in mind. Whereas a deficit perspective would highlight what they cannot or did not do, we prefer to take note of what they can do, and indeed, did during the activity. As Oughton (2010) suggests, we wish to use "funds of knowledge" as a metaphor that allows us to "explore an understanding of cultures as diverse and dynamic" (p. 71) and that positions "learning as participation, within which the learning of a subject is regarded as the process of becoming a member of a certain community (Lave and Wenger, 1991)." (p. 71). Hence, we see these mothers-as-teachers drawing on funds of knowledge that are growing and not static, developed from their participation in many communities, and as Oughton (2010) cautions, we are mindful not to "arbitrate what counts as valid or useable funds of knowledge" (p. 73).

25.6 Conclusion

Both these mothers co-constructed a pedagogical task within a play-based context, in such a way that both preschoolers engaged willingly in mathematics related to the child-friendly materials. Indeed, each mother was able to establish and sustain mathematics as the goal of the activity, while respectfully addressing each child's role in the play. What more would we want a mother to do? What more would we have expected of a teacher during such a task? Advocates who seek to design "parent training" so they might "show or tell" parents how to make math more explicit more often, may need to question if their view of parents-as-teachers is a deficit one. Those seeking to offer parents support or education likewise need to recognize that giving time and space for parents to share what they currently do is likely to deepen both the parents' and educators' reflections on important taken-for-granted assumptions.

While research on more didactical events at-home, and across diverse families, is needed, the current study points to the viability of learning more about such practices in homes, in more qualitative and nuanced ways. In contrast to previous research where time sampling during observations (Tudge & Doucet, 2004) or analysis (Aubrey et al., 2003), or parent reports on questionnaires (Skwarchuk & LeFevre, 2015), led to a characterization of infrequent attention to mathematics in homes, observing and analyzing specific in-home activity from start to finish as we have done here permitted us to document mathematical engagement underreported to this date. For example, if we were to analyze 30 s snippets every 6 min of the Playdoh: Pizza activity, similar to how Tudge and Doucet time-sampled and live-coded their observations, a very different picture of the activity emerges. Accordingly, we see a mother watching as a child rolls out playdoh (first 30 s), a mother watching as her child cuts a "pizza" (6 min later) and a mother making small balls of playdoh to use for 'pepperoni' as the child flattens the 'pizza' (6 min later), leaving the impression that minimal attention was paid to mathematics. Instead, our holistic analysis of Playdoh: Pizza and Toys: Cars provided ample evidence of the ways in which these two mothers attend to and expand on their children's mathematical experiences as they play together. Thus, as we collectively continue to “flesh out” what is happening within and across families, we argue for more in-depth examination of “naturally occurring” pedagogy.

In addition, to broaden our current understandings of parents’ capacity for, and propensity toward, pedagogical tasks at home, this case study points to the value of finding expansive contexts in which to investigate parents’ pedagogical practices. In the current study, we deliberately examined play-based activity, which we deemed pedagogical. In doing so, we argue that our sense of pedagogical (i.e., a task in which “math is the main goal” for the parent) aligns well with other researchers’ definitions of “pedagogical tasks” (Aubrey et al., 2003; Walkerdine, 1988), “formal activities” (Skwarchuk & LeFevre, 2015), and “academic lessons” (Tudge & Doucet, 2004). Yet, the tenor of these two play-based activities also resembles the “participative parental interaction” (Aubrey et al., 2003, p. 94), “play with non-academic objects” (Tudge & Doucet, 2004) and “more collaborative approaches” (Skwarchuk & LeFevre, 2015) that, although mentioned, remained unexamined, or at least unreported, within previous studies. Interestingly, our findings with regard to these two mothers’ pedagogical moves during such play-based activity suggest that contexts in which parents “teach” their young children mathematics vary and likely go beyond those activities which are stereotypically identified as “mathematical” (i.e., oftentimes, school-like content narrowly based in number). Thus, while it is likely that most readers would identify these mothers’ “how many” questions as pedagogical, we argue that the efforts the Adam mother expends to “teach” her child how to dissect a circle and the attention the Penn mother gives to the spatial properties of her son’s parking lots, are equally intentional, mathematical and pedagogical, although less verbal. Consequently, we believe an openness to hybridity (i.e., play + pedagogy) and a willingness to seek out less conventional contexts, in addition to block play, board or card games, storybooks and so forth, serves to enhance the types of pedagogical acts we document, and in turn, validate and understand.

Regardless of methods or the contexts in which we examine mathematics pedagogy in out-of-school settings, our findings here raise the importance of finding authentic ways to investigate the funds of knowledge parents bring to their pedagogical practices. Like teachers and early childhood educators in more institutional settings, each of these individuals enacted the role of mother-as-teacher differently, even though common pedagogical moves framed their endeavors.

In what ways do parents’ schooling experiences or personal out-of-school experiences with mathematics influence their mother-as-teacher stances? What strengths do they bring to the shared activity which permits them to scaffold their child? While research valuing both pedagogical and instrumental tasks (Aubrey et al., 2003) and formal and informal numeracy activity (Skwarchuk & LeFevre, 2015) is a step in the right direction, we continue to find what appears to be deficit-based claims that “parents often do not know how to do this [*make math talk happen at home*] or how to support their children in mathematics in general” (Lee & Kotsopoulos, 2016, p. 154 [author added]). We wonder to what extent the use of a teacher–child lens or possibly a math expert lens to view parent–child interactions informs what we believe may be a bias toward schooling pedagogy [e.g., expecting homes to mimic schools]. This inevitably places most parents with a deficit. Yet, if we position parents-as-teachers as experts in what they do, and genuinely seek to learn from them, we argue much is yet to be discovered regarding at-home mathematics experiences of young children. We therefore call for future research into pedagogical practices in the home, which values and validates parents' myriad funds of knowledge.

Finally, we end with a cautionary note, that this exploratory study of two well-educated mothers, with the privilege to sit and play with their child, for an extended period, must not be the norm to which we expect all mothers to aspire. Rather, it must inspire us to seek out differences, to learn from mothers in other circumstances, in equally respectful ways.

Appendix

As the RA video records, Adam's mother and her preschool daughter are seated at adjacent sides of the child's table. With her mother's help, the daughter empties a "lump" of Play-Doh onto the table and begins to roll it flat with a wooden rolling pin. At times the mother helps flatten the dough to make it easier for the roller and converses with her daughter as the child concentrates on rolling out the pizza. Early in the conversation, when the mother asks what shape she is trying to make, the daughter suggests "a heart"; when the mother asks, "what's the shape of a Pizza?" She's heard qualifying her daughter's response of "triangle" with "when it's sliced"; she then encourages her child to make a "big circle" like regular pizzas. As the child continues to roll the Play-Doh into a circle, the mother turns their attention to sharing the pizza with imaginary friends who visit for dinner. For the bulk of the episode, the child and mother discuss how to share the pizza fairly among various numbers of visitors, as the child, and at times the mother, cuts the pizza into slices. In addition to cutting and counting the slices, on occasion, they talk about whether the slices are fair, namely the "same size."

As the RA video records, Penn's mother joins her preschool son, who is seated on the floor with some plastic toys (e.g., a whale, a rat) and a basket of miniature cars nearby. After attending briefly to the plastic animals, the son begins to sort through the cars in the basket, looking for any broken ones that he might discard. After the mother watches the boy line up several of his cars in pairs, one pair behind another pair, near the edge of the carpet, she asks "if he'd like to play the counting game" with his cars. While the son agrees, they at first talk about how some of his cars are the same (e.g., "two tow trucks" and "a black one and another black one"). Almost simultaneously, the mother begins to set up the game, by relocating a small number of the same colored cars in a horizontal line close to her, and a small

18 Mother: How is the pizza? OK you know what? Let's pretend we are going to have four people over for supper, four people are coming. We want to cut this pizza so everybody gets the same. So the first thing we need to do is cut it in half.

19 Daughter: Like this? (*gestures cutting middle of circle*)

20 M: Yeah right down the middle so there is two pieces the same size.

21 D: Cut it that way?

22 M: Yeah, good.

23 D: And that way? (*gestures to cut parallel and to right of previous cut*)

24 M: So you cut it in half first, good girl. And there is how many pieces?

25 D: Two.

26 M: Two. But we need more than two, don't we? How many people are coming?

Daughter holds up four fingers.

27 M: Four people are coming for supper so can you cut it in half this way? Or this way? Yeah.

28 D: I am going to go this way (inaudible) (*vertical cut gestured*)

29 M: OK but let's do this first so you already cut it in half this way so let's cut it in half this way. Yeah. Let's see what happens. That's a girl. So we cut it in half that way. Now how many pieces do we have? One, two---

30 D: One, two, three, four.

31 M: Four.

32 D: Let's pretend we have (pause) six people.

33 M: Six people coming? OK How can we divide it for six people?

34 D: You go like that. Go like that and then you go like that (*gestures a middle cut for each slice*).

35 M: Should I build it back together or do you want to keep it like this?

36 D: Build it back.

Fig. 25.2 Mother-daughter dyad "making Play-Doh pizza" (Adam family)

distance away from the pairs of cars the child had created. While she had intended to play a "take-away" game, they instead talk about the number of cars of similar color, which she, or the child, gathers together, and to which the child adds more. When the mother and child decide to "build" the town, the conversation shifts towards the "parking spots" the child makes, and later, back to the cars he parks in them.

25M: What do you want to play with the cars?

26 S: A broken car (inaudible).

27 M: Do you want to play the counting game with the cars?

28 S: Yeah

33 M: You've got some matching cars, don't you? You've got two that look almost the same---looks like two the same. What kind are they? *(M points to a pair of trucks near S; when S moves to the side we see he has lined up ten cars in pairs*, and *each pair is behind the other; and then a small space separates those cars from the two trucks which were one behind the other; M moves the trucks so that they are side by side, but still spaced from the longer line)*

34 S: Tow trucks.

39M: Shall we play the game. Shall I start?

40 S: Counting game.

41 M: We can line them in a row on the carpet and then we can---do you want to line them in a row *(M picks up another car from the lineup and moves it closer to her, between her and S)* and then take some away and see how many are there?

61 M: How many orange cars are there? *(moves a car, so orange cars in her row are together)*

62 S: One *(S points towards (but above) the cars)*, two *(moves finger closer to face)*, three *(extends arm and points towards cars again)*

63 M: Three orange cars and how many and how many brown cars are there? *(M pushes the brown cars closer together and away from orange cars)*

Fig. 25.3 Mother-son dyad "playing with cars" (Penn family)

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