ε^2-Order Normal Form Analysis for a Two-Degree-of-Freedom Nonlinear Coupled Oscillator

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Abstract In this paper, we describe an ε^2-order normal form decomposition for a two-degree-of-freedom oscillator system that has a mass supported with horizontal and vertical support springs. This system has nonlinear terms that are not necessarily ε^1-order small when compared to the linear terms. As a result, analytical approximate methods based on an ε expansion would typically need to include higher-order components in order to capture the nonlinear dynamic behaviour. In this paper we show how this can be achieved using a direct normal form transformation up to order ε^2. However, we will show that the requirement for including ε^2 components is primarily due to the way the direct normal form method deals with quadratic coupling terms rather than the relative size of the coefficients.

Keywords Nonlinear oscillator · Normal form · ε^2-order

1 Introduction

Normal form transformations are a classical method for studying dynamical systems first introduced by Poincaré [1]. The historical background of normal form transformations can be found in a number of texts including [2–4]. This work is motivated by vibration problems involving coupled nonlinear oscillators, where the objective of a normal form transformation is to both simplify the system, but also to identify potential nonlinear resonances that might occur. For vibration problems, Jezequel and Lamarque [5] proposed a normal form decomposition for a system of two coupled oscillators with cubic nonlinearities and both forcing and damping. The relationship between the normal form transformation and nonlinear normal modes was established by Touzé and co-workers [6, 7], based on examples of coupled oscillator systems that included both quadratic and cubic nonlinear terms.

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In this paper, we will consider an oscillator system consisting of a mass supported by vertical and horizontal springs that are attached to solid supports. This system is shown schematically in Fig. 1. The equations of motions of this example system, as derived by Touzé et al. [6], are taken to be

$$\ddot{x}_1 + 2\zeta_1\omega_1\dot{x}_1 + \omega_1^2 x_1 + a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 + a_4 x_1^3 + a_5 x_1 x_2^2 = f_1 \cos(\Omega t),$$
$$\ddot{x}_2 + 2\zeta_2\omega_2\dot{x}_2 + \omega_2^2 x_2 + b_1 x_1^2 + b_2 x_1 x_2 + b_3 x_2^2 + b_4 x_1^2 x_2 + b_5 x_2^3 = f_2 \cos(\Omega t), \quad (1)$$

where the coefficients of the nonlinear terms a_i, b_i for $i, j = 1, 2, \ldots, 5$ are of the same size order as the natural frequencies ω_1 and ω_2, respectively. The other coefficients are damping ratios ζ_i, and forcing amplitudes f_i, for each degree of freedom $i = 1, 2$ and the external forcing frequency is Ω.

This system has been studied in depth by several previous authors [6–8]. In particular, Touzé and Amabili [7] showed how a single-linear-mode approximation to the system dynamics would predict hardening instead of softening behaviour for a specific set of parameter values, whereas a nonlinear normal mode type analysis predicts the correct softening resonance, behaviour. Furthermore, in Touzé and Amabili [7] backbone curves for the system were computed, and these curves were compared with forced-damped simulations of the system. In [8] a detailed study of methods for computing backbone curves was carried out. As part of their study Breunung and Haller, [8], used the current example to make a comparison between a spectral sub-manifold method and the methods of Touzé and Amabili [7] and Neild and Wagg [9]. This comparison showed that the ε^1 direct normal form proposed by Neild and Wagg [9] gave the incorrect approximations for this example. In fact, using the ε^1 version gave a result similar to the linear-mode approximation first discussed by Touzé and Amabili [7]—predicting hardening instead of softening behaviour. In this paper, we will show that the ε^2 terms are required in the direct

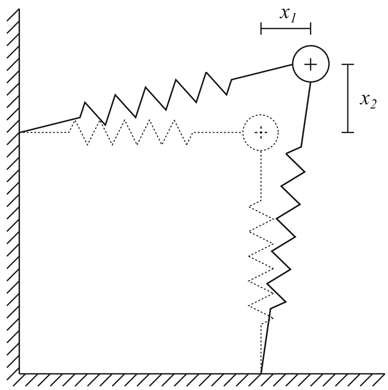

Fig. 1 The example system considered in this paper

normal form method of Neild and Wagg [9] to give the correct solutions. Typically the direct normal form method, [9], is applied to systems where the nonlinear, damping and forcing terms are assumed to be of order ε^1 small (or higher orders of ε) when compared to the linear terms [10–14]. The linear terms are the natural frequencies, taken to be of order ε^0, meaning that the ε^1 nonlinear terms are typically an order smaller than the natural frequencies. In Eq. (1) this is not the case, and it is possible for the nonlinear coefficients to be of the same size order as the natural frequencies. As a result, the normal form approximation would typically need to be extended to include higher-order terms. Here, we show that an ε^2-order analysis is sufficient to capture the required behaviour, although in fact the need for the ε^2-order terms is actually because of the quadratic coupling terms, as will be explained below.

2 ε^2-Order Normal Form Analysis

We follow the method set out in Chapter 4 of [15] for a ε^2 direct normal form method. The coefficients of the nonlinear terms in Eq. (1) are taken to be

$$\begin{aligned} &a_1 = \tfrac{3}{2}\omega_1^2 \quad a_2 = \omega_2^2 \quad a_3 = \tfrac{1}{2}\omega_1^2 \quad a_4 = \tfrac{1}{2}(\omega_1^2 + \omega_2^2) \quad a_5 = \tfrac{1}{2}(\omega_1^2 + \omega_2^2) \\ &b_1 = \tfrac{1}{2}\omega_2^2 \quad b_2 = \omega_1^2 \quad b_3 = \tfrac{3}{2}\omega_2^2 \quad b_4 = \tfrac{1}{2}(\omega_1^2 + \omega_2^2) \quad b_5 = \tfrac{1}{2}(\omega_1^2 + \omega_2^2). \end{aligned} \tag{2}$$

As the conservative form of Eq. (1) is naturally linearly decoupled, it can be described in the matrix form as $\ddot{\mathbf{q}} + \boldsymbol{\Lambda}\mathbf{q} + \mathbf{N}_q(\mathbf{q}) = \mathbf{0}$ by setting $\mathbf{q} = [q_1,\ q_2]^\mathsf{T} = [x_1,\ x_2]^\mathsf{T}$, where

$$\boldsymbol{\Lambda} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}, \quad \text{and} \quad \mathbf{N}_q(\mathbf{q}) = \begin{pmatrix} a_1 q_1^2 + a_2 q_1 q_2 + a_3 q_2^2 + a_4 q_1^3 + a_5 q_1 q_2^2 \\ b_1 q_1^2 + b_2 q_1 q_2 + b_3 q_2^2 + b_4 q_1^2 q_2 + b_5 q_2^3 \end{pmatrix}, \tag{3}$$

although as noted above $\mathbf{N}_q(\mathbf{q})$ is not ε^1 small in this example. Here the non-internal-resonant case is considered, such that the detuned response frequencies $\omega_{ri} \neq n\omega_{rj}$ for $i, j = 1, 2$ with $i \neq j$ and $n = 1, 2, \cdots$. Note that other rational resonances, such as $n = 3/5$ are not considered here. The exact detuning mechanism is explained in detail in [15].

Next we carry out a ε^2 near identity transformation $\mathbf{q} = \mathbf{u} + \varepsilon\mathbf{h}_{(1)}(\mathbf{u}) + \varepsilon^2\mathbf{h}_{(2)}(\mathbf{u})$. The first step in this process is to substitute $\mathbf{q} = [q_1,\ q_2]^\mathsf{T} = [u_{1p} + u_{1m},\ u_{2p} + u_{2m}]^\mathsf{T}$ into Eq. (3). This then leads to a $[30 \times 1]$ dimension $\mathbf{u}^*$ vector, which is used to redefine $\mathbf{N}_q(\mathbf{u}) = \mathbf{n}_1\mathbf{u}^*$ and $\mathbf{h}_{(1)}(\mathbf{u}) = \mathbf{h}_1\mathbf{u}^*$, such that $\mathbf{n}_1$ and $\mathbf{h}_1$ are coefficient matrices for the ε^1 terms. The objective is to obtain a normal form of $\ddot{\mathbf{u}} + \mathbf{\Lambda u} + \mathbf{N}_u(\mathbf{u}) = \mathbf{0}$, with $\mathbf{N}_u = \varepsilon \mathbf{n}_{u(1)} + \varepsilon^2 \mathbf{n}_{u(2)}$. To find the transformed vectors $\mathbf{n}_{u(1)}$ and $\mathbf{n}_{u(2)}$, solutions to the following equations are required

$$\varepsilon^1 : \qquad \ddot{\mathbf{h}}_{(1)}(\mathbf{u}) + \mathbf{\Upsilon h}_{(1)}(\mathbf{u}) + \mathbf{n}_{(1)}(\mathbf{u}) = \mathbf{n}_{u(1)}(\mathbf{u}), \tag{4a}$$

$$\varepsilon^2 : \qquad \ddot{\mathbf{h}}_{(2)}(\mathbf{u}) + \mathbf{\Upsilon h}_{(2)}(\mathbf{u}) + \mathbf{n}_{(2)}(\mathbf{u}) = \mathbf{n}_{u(2)}(\mathbf{u}), \tag{4b}$$

where $\mathbf{\Upsilon}$ is a $\{N \times N\}$ diagonal matrix of the square of the response frequencies, ω_{ri}^2 such that $\mathbf{\Lambda} = \mathbf{\Upsilon} + \varepsilon \mathbf{\Delta}$, and

$$\mathbf{n}_{(1)}(\mathbf{u}) = \mathbf{N}_q(\mathbf{q} = \mathbf{u}), \tag{5a}$$

$$\mathbf{n}_{(2)}(\mathbf{u}) = \left(\mathbf{\Delta} + \left. \frac{\partial \mathbf{N}_q(\mathbf{q})}{\partial \mathbf{q}} \right|_{\mathbf{q}=\mathbf{u}} \right) \mathbf{h}_{(1)}(\mathbf{u}). \tag{5b}$$

Solving Eq. (4a) we can first obtain the ε^1 terms as

$$\mathbf{n}_{u(1)} = \begin{pmatrix} 3a_4(u_{1p}^2 u_{1m} + u_{1p} u_{1m}^2) + 2a_5(u_{1p}u_{2p}u_{2m} + u_{1m}u_{2p}u_{2m}) \\ 2b_4(u_{1p}u_{1m}u_{2p} + u_{1p}u_{1m}u_{2m}) + 3b_5(u_{2p}^2 u_{2m} + u_{2p}u_{2m}^2) \end{pmatrix}. \tag{6}$$

For the ε^2 terms, we must determine Eq. (5b) up to cubic order which should provide an accurate solution for this example, and thus the nonlinear terms vector $\mathbf{n}_{(2)}$ is truncated at $\mathcal{O}(\mathbf{u}^4)$. As a result we can simplify $\mathbf{N}_q$ because we only need terms up to order $\mathcal{O}(\mathbf{u}^2)$ in the partial derivative, and so we write $\mathbf{N}_q(\mathbf{u}) = \tilde{\mathbf{N}}_q(\mathbf{u}) + \mathcal{O}(\mathbf{u}^3)$. Then we have

$$\mathbf{\Delta} = \begin{bmatrix} \omega_{n1}^2 - \omega_{r1}^2 & 0 \\ 0 & \omega_{n2}^2 - \omega_{r2}^2 \end{bmatrix} = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}, \quad \text{and} \tag{7a}$$

$$\frac{\partial \tilde{\mathbf{N}}_q(\mathbf{u})}{\partial \mathbf{u}} = \begin{bmatrix} 2a_1(u_{1p} + u_{1m}) + a_2(u_{2p} + u_{2m}) & a_2(u_{1p} + u_{1m}) + 2a_3(u_{2p} + u_{2m}) \\ 2b_1(u_{1p} + u_{1m}) + b_2(u_{2p} + u_{2m}) & b_2(u_{1p} + u_{1m}) + 2b_3(u_{2p} + u_{2m}) \end{bmatrix}. \tag{7b}$$

Therefore we can compute $\mathbf{n}_{(2)}$ using

$$\mathbf{n}_{(2)} = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \mathbf{h}_1 \mathbf{u}^* + \frac{\partial \tilde{\mathbf{N}}_q(\mathbf{u})}{\partial \mathbf{u}} \tilde{\mathbf{h}}_1 \tilde{\mathbf{u}}^* + \mathcal{O}(\mathbf{u}^4), \tag{8}$$

where $\tilde{\mathbf{h}}_1$ and $\tilde{\mathbf{u}}^*$ are the respective projections of $\mathbf{h}_1$ and $\mathbf{u}^*$ to $\mathcal{O}(\mathbf{u}^2)$. This allows the vector of nonlinear terms up to order ε^2 to be obtained as

$$\mathbf{n}_{u(2)} = \begin{array}{l} \left(-\frac{10}{3\omega_{r1}^2}a_1^2 + \frac{3\omega_{r2}^2 - 8\omega_{r1}^2}{\left(4\omega_{r1}^2 - \omega_{r2}^2\right)\omega_{r2}^2}a_2 b_1\right)\left(u_{1p}^2 u_{1m} + u_{1p}u_{1m}^2\right) \\ \left(-\frac{10}{3\omega_{r2}^2}b_3^2 + \frac{3\omega_{r1}^2 - 8\omega_{r2}^2}{(4\omega_{r2}^2 - \omega_{r1}^2)\omega_{r1}^2}a_3 b_2\right)\left(u_{2p}^2 u_{2m} + u_{2p}u_{2m}^2\right) \\ + \left(\frac{2}{\omega_{r2}^3 - 4\omega_{r1}^2}a_2^2 - \frac{4}{\omega_{r1}^2}a_1 a_3 + \frac{4}{\omega_{r1}^2 - 4\omega_{r2}^2}a_3 b_2 - \frac{2}{\omega_{r2}^2}a_2 b_3\right) \\ \quad \times \left(u_{1p}u_{2p}u_{2m} + u_{1m}u_{2p}u_{2m}\right) \\ + \left(\frac{2}{\omega_{r1}^3 - 4\omega_{r2}^2}b_2^2 - \frac{4}{\omega_{r2}^2}b_1 b_3 + \frac{4}{\omega_{r2}^2 - 4\omega_{r1}^2}a_2 b_1 - \frac{2}{\omega_{r1}^2}a_1 b_2\right) \\ \quad \times \left(u_{1p}u_{1m}u_{2p} + u_{1p}u_{1m}u_{2m}\right) \end{array} . \tag{9}$$

Now using $\mathbf{N}_u = \mathbf{n}_{u(1)} + \mathbf{n}_{u(2)}$, the direct normal form for the system (for the non-internally resonant case) is given by

$$\begin{aligned} \ddot{u}_1 + \omega_1^2 u_1 + A(u_{1p}^2 u_{1m} + u_{1p}u_{1m}^2) + B(u_{1p}u_{2p}u_{2m} + u_{1m}u_{2p}u_{2m}) = 0, \\ \ddot{u}_2 + \omega_2^2 u_2 + C(u_{2p}^2 u_{2m} + u_{2p}u_{2m}^2) + D(u_{1p}u_{1m}u_{2p} + u_{1p}u_{1m}u_{2m}) = 0, \end{aligned} \tag{10}$$

where

$$A = 3a_4 - \frac{10}{3\omega_{r1}^2}a_1^2 + \frac{3\omega_{r2}^2 - 8\omega_{r1}^2}{(4\omega_{r1}^2 - \omega_{r2}^2)\omega_{r2}^2}a_2 b_1, \tag{11a}$$

$$B = 2a_5 + \frac{2}{\omega_{r2}^3 - 4\omega_{r1}^2}a_2^2 - \frac{4}{\omega_{r1}^2}a_1 a_3 + \frac{4}{\omega_{r1}^2 - 4\omega_{r2}^2}a_3 b_2 - \frac{2}{\omega_{r2}^2}a_2 b_3, \tag{11b}$$

$$C = 3b_5 - \frac{10}{3\omega_{r2}^2}b_3^2 + \frac{3\omega_{r1}^2 - 8\omega_{r2}^2}{(4\omega_{r2}^2 - \omega_{r1}^2)\omega_{r1}^2}a_3 b_2, \tag{11c}$$

$$D = 2b_4 + \frac{2}{\omega_{r1}^3 - 4\omega_{r2}^2}b_2^2 - \frac{4}{\omega_{r2}^2}b_1 b_3 + \frac{4}{\omega_{r2}^2 - 4\omega_{r1}^2}a_2 b_1 - \frac{2}{\omega_{r1}^2}a_1 b_2. \tag{11d}$$

Substituting $u_{1p} = (\frac{U_1}{2}\mathrm{e}^{-\mathrm{i}\phi_1})\mathrm{e}^{\mathrm{i}\omega_{r1}t}$ and $u_{1m} = (\frac{U_1}{2}\mathrm{e}^{\mathrm{i}\phi_1})\mathrm{e}^{-\mathrm{i}\omega_{r1}t}$ into Eq. (10) enables expressions for the backbone curves to be obtained as

$$\left[-\omega_{r1}^2 + \omega_1^2 + \frac{1}{4}AU_1^2 + \frac{1}{4}BU_2^2\right]\frac{U_1^2}{2} = 0, \tag{12a}$$

$$\left[-\omega_{r2}^2 + \omega_2^2 + \frac{1}{4}CU_1^2 + \frac{1}{4}DU_2^2\right]\frac{U_2^2}{2} = 0, \tag{12b}$$

where U_i is the displacement amplitude of u_i, for $i = 1, 2$. Successively setting U_2 and U_1 to zero will give the S_1 and S_2 backbone curves

$$S1: \quad \omega_{r1}^2 = \omega_1^2 + \frac{1}{4}AU_1^2, \tag{13a}$$

$$S2: \quad \omega_{r2}^2 = \omega_2^2 + \frac{1}{4}CU_2^2. \tag{13b}$$

Note that these are now implicit expressions for ω_{r1}^2 and ω_{r2}^2, respectively, which can be solved numerically to find the backbone curves.

Finally, the physical displacement responses may be computed using the corresponding reverse transform $u_1 \to q_1 = x_1$, and $u_2 \to q_2 = x_2$ such that

$$\begin{aligned} x_1 &= q_1 = u_1 + h_{1,1}\mathbf{u}^* + h_{2,1}^+\mathbf{u}^+, \\ x_2 &= q_2 = u_2 + h_{1,2}\mathbf{u}^* + h_{2,2}^+\mathbf{u}^+, \end{aligned} \tag{14}$$

where $h_{i,j}$ are row vectors taken from the $\mathbf{h}_1$ and $\mathbf{h}_2^+$ coefficient matrices based on the fact that $\mathbf{h}_{(2)}$ has been redefined as $\mathbf{h}_{(2)} = \mathbf{h}_2^+\mathbf{u}^+$—see Chapter 4 of [15] for full details of this procedure.

3 Numerical Results

The simulation uses the parameters $\omega_1 = 2$, $\omega_2 = 4.5$, $\zeta_1 = 0.001$, $\zeta_2 = 0.001$, $f_k = 0.0015$ and $f_\ell = 0$ for the two different forcing cases $k = 1, \ell = 2$ and $k = 2, \ell = 1$. The results for the $S1$ and $S2$ backbone curves computed using Eqs. (13) are shown as the red lines in Figs. 2 and 3. For comparison, the order ε^1 backbone curves are shown as blue lines in the figures. In order to verify the analytically approximated ε^2 backbone results, resonance response curves for the corresponding forced, damped case are computed using the continuation Matlab toolbox—COCO [16]. These are shown as black lines in Figs. 2 and 3.

The plots in Fig. 2 are presented in the projection of the response amplitude of the physical coordinates, X_i, against the forcing frequency, Ω (or ω_r for the undamped backbone curves). In each figure X_1 against Ω is shown in plot (a) and X_2 against Ω is shown in plot (b). Consequently, in Fig. 2 where the forcing is applied to the x_1 equation, the dominant response is in the X_1 amplitude (plot Fig. 2a), and the response in plot (b) of X_2 vs Ω is primarily due to the harmonic terms via Eq. (14).

Values of ω_i are chosen as they are exactly the same as those used by previous studies [6, 8] to demonstrate the non-internally resonant dynamics of the system.

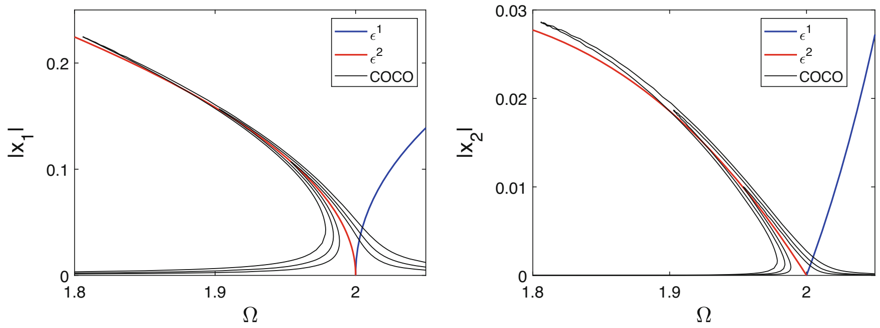

Fig. 2 The backbone curves S_1, and resonance response curves of the two-degree-of-freedom example system described in Eq.(1) for the case where its horizontal mode is dominant. The red and black lines denote the backbone curves and numerically computed forced response curves using COCO, respectively. Parameters: $\omega_1 = 2$, $\omega_2 = 4.5$, $\zeta_1 = 0.001$, $\zeta_2 = 0.001$. There are three different forcing amplitude curves $f_1 = 0.001, 0.0016, 0.0025$ and $f_2 = 0$. Note that the stability of the solution curves is not indicated on this figure

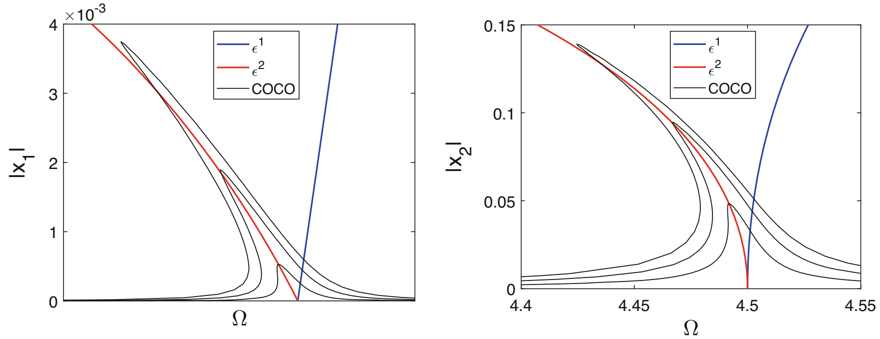

Fig. 3 The backbone curves S_2, and resonance response curves of the two-degree-of-freedom example system described in Eq. 1 for the case where its horizontal mode is dominant. The red and black lines denote the backbone curves and numerically computed forced response curves using COCO, respectively. Parameters: $\omega_1 = 2$, $\omega_2 = 4.5$, $\zeta_1 = 0.001$, $\zeta_2 = 0.001$. Here there are three different forcing amplitude curves shown $f_2 = 0.002, 0.004, 0.006$, and $f_1 = 0$. Note that the stability of the solution curves is not indicated on this figure

For the damping values, $\zeta_1 = 0.001$ was used previously by Touzé et al. [6], but here we have used $\zeta_2 = 0.001$ as well so that the COCO continuation curves are very close to the undamped case. It can be seen that the analytical backbone curves correctly predict the softening dynamics of the example system which is consistent with the findings in [6]. However, it is important to note that the backbone curve expression computed with just the ε^1 terms gives a hardening response, which does not match the system behaviour correctly, as shown by the blue lines in Figs. 2 and 3 (and also the comparison presented by Breunung and Haller [8]).

The specific reason for this can be seen in Eq. (11) which gives the coefficients for the S_1 and S_2 backbone curves in Eqs. (13). Specifically for the S_1 backbone the coefficient producing curvature is A. In the ε^1 case, $A = 3a_4$, which will give a hardening S_1 curve. However in the ε^2 case, A is given by Eq. (11a) and there are two additional terms that reverse the curvature of S_1, for the given parameters, to produce a softening backbone curve. In fact reducing the ω_2 value to a value of 3.8rad/s (whilst keeping all other parameters the same) results in the backbone curve switching to hardening.

This is consistent with the finding of [7] that the quadratic terms of the type found in this example will generate cubic terms in the nonlinear coordinate transformation. As we have shown, in the direct normal form method of Neild and Wagg [9], these generated terms from the quadratics are only captured in the ε^2 expansion not the ε^1 version. This explains why the ε^1 version of the direct normal form will not show the correct softening nonlinear behaviour—as also shown in the comparison by Breunung and Haller [8]. It is also clear from the results presented above that this can be rectified by the inclusion of the ε^2 terms.

Although not the specific cause (and therefore less important) we note that the direct normal form method does rely on the nonlinear terms being small in the sense that they should be significantly smaller than the ω_{ni}^2 values. However, in this example the nonlinear coefficients are of the same order as the ω_{ni}^2 values, and yet despite this, by adding ε^2 terms, the direct normal form method gives a very good approximation to the solution. Specifically, the maximum response position of the COCO curves is very close to the backbone curves for both S_1 and S_2.

4 Conclusions

In this paper, ε^2-order approximate analytical expressions for the backbone curves of a coupled two-degree-of-freedom system have been obtained using the direct normal form method proposed by Neild and Wagg. The motivation for this study was the observation that the ε^1 version of the direct normal form method did not predict the correct softening type of behaviour for this example. In fact, we have shown in this paper that the primary cause of this discrepancy is due to how the direct normal form treats the quadratic coupling terms of the type found in this example.

This is because during the backbone curve approximation process quadratic terms actually generate terms up to cubic order. These terms are significant in obtaining a representative model for the backbone curve. In the method proposed by Neild and Wagg, these additional cubic terms are captured only in the ε^2 part of the approximation. As a result, if using this method for a system with quadratic nonlinearities, then the ε^2 version is needed to fully capture the relevant dynamic behaviour.

In addition to this, and despite the fact that the direct normal form assumes small nonlinear terms, which are not the case in this example, the results obtained from the ε^2 version and the numerical method agree well.

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