The Occurrence of Zero-Hopf Bifurcation in a Generalized Sprott A System

Marcelo Messias and Alisson C. Reinol

Abstract From the normal form of polynomial differential systems in $\mathbb{R}^3$ having a sphere as invariant algebraic surface, we obtain a class of quadratic systems depending on ten real parameters, which encompasses the well-known Sprott A system. For this reason, we call them *generalized Sprott A systems*. In this paper, we study the dynamics and bifurcations of these systems as the parameters are varied. We prove that, for certain parameter values, the z-axis is a line of equilibria, the origin is a non-isolated zero-Hopf equilibrium point, and the phase space is foliated by concentric invariant spheres. By using the averaging theory we prove that a small linearly stable periodic orbit bifurcates from the zero-Hopf equilibrium point at the origin. Finally, we numerically show the existence of nested invariant tori around the bifurcating periodic orbit.

Keywords Sprott A system · Invariant sphere · Zero-Hopf bifurcation · Linearly stable periodic orbit · Invariant torus

1 Introduction

Consider the polynomial differential system in $\mathbb{R}^3$ defined by

$$\dot{x} = P(x, y, z), \quad \dot{y} = Q(x, y, z), \quad \dot{z} = R(x, y, z), \tag{1}$$

where P, Q, and R are relatively prime real polynomials in the variables x, y, z and the dot denotes derivative with respect to the independent variable t. We say that

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$m = \max\{\deg(P), \deg(Q), \deg(R)\}$ is the *degree* of system (1). If $m = 2$, it is a *quadratic* polynomial differential system.

For $m \geq 2$, the dynamical behavior of system (1) is in general very difficult to be studied, especially when it exhibits chaos. In the last decades, chaotic differential systems have been intensively studied, as the Lorenz system, Chen system, Lü system, and many others [1]. Recently, there is an increasing interest in finding and studying three categories of chaotic systems: without equilibrium points, with an infinity of equilibria, and with only stable equilibrium points. In these systems are frequently found attractors whose basin of attraction does not intercept with points in small neighborhoods of equilibrium points. These kinds of attractors are called *hidden attractors* and have theoretical and applied interests. They allow, for instance, unexpected and potentially disastrous responses to perturbations in structures like bridges and airplane wings, for details see [2, 3] and references therein.

The oldest and best-known chaotic differential system having no equilibrium points is the *Sprott A system* [4], given by

$$\dot{x} = y, \quad \dot{y} = -x - yz, \quad \dot{z} = y^2 - a, \tag{2}$$

where $a \in \mathbb{R}$. This system was shown to be chaotic for $a = 1$, even without having equilibrium points for this parameter value. From the physical point of view, the Sprott A system is a special case of the well-known and widely studied Nosé–Hoover oscillator [5, 6] as pointed out in [7]. Moreover, it plays an important role in nonlinear dynamics studies, since its structure became source of inspiration for the study of many new quadratic chaotic differential systems in $\mathbb{R}^3$. Chaotic systems without equilibrium points, as the Sprott A system, appear naturally in the mathematical modeling of some electromechanical problems with rotation and in electrical circuits with cylindrical phase space, as presented for instance in [8]. In this way, the bifurcation analysis of these kind of systems helps to better understand the phenomena described by them.

In this context and motivated by the studies developed in [9, 10], we propose and study a more general class of quadratic polynomial differential systems which contains the Sprott A system and have similar, and even richer, dynamical behavior. Here we call these systems *generalized Sprott A systems*, which are given by

$$\dot{x} = -y\, P_1 + z\, P_2, \quad \dot{y} = x\, P_1 - z\, P_3, \quad \dot{z} = -x\, P_2 + y\, P_3 - \alpha, \tag{3}$$

where $P_i = P_i(x, y, z) = a_i\, x + b_i\, y + c_i\, z + d_i$ with $\alpha, a_i, b_i, c_i, d_i \in \mathbb{R}$, for $i = 1, 2, 3$. Taking $d_1 = -1$, $b_3 = 1$, $\alpha = a$, and the other parameters equal to zero into system (3), we obtain the Sprott A system (2).

In this paper, we study the dynamical behavior of system (3) as the parameter value α varies and under certain conditions on the other parameters. The paper is organized as follows. In Sect. 2 we prove that system (3) is the most general quadratic differential system in $\mathbb{R}^3$ having a family of concentric invariant spheres and give some additional properties of its phase space. In Sect. 3, for the sake of completeness and to fix the notation we present a result of the averaging theory of first order, then we use it to prove the existence of a small linearly stable periodic orbit bifurcating from the origin of system (3). In Sect. 4 we present some numerical simulations from which we show the existence of nested invariant tori around the bifurcating periodic orbit. Finally, in Sect. 5 some concluding remarks are given.

2 Invariant Spheres of Generalized Sprott A System

In [11], the authors determined the normal form of all polynomial differential systems in $\mathbb{R}^3$ having a sphere as an invariant algebraic surface. More precisely, they proved the following result.

Theorem 1 *Assume that a sphere $\mathcal{S} = 0$ is an invariant algebraic surface of the polynomial differential system (1). Then, after an affine change of coordinates, system (1) can be written as*

$$\dot{x} = \mathcal{S}\, Q_1 - y\, P_1 + z\, P_2, \quad \dot{y} = \mathcal{S}\, Q_2 + x\, P_1 - z\, P_3, \quad \dot{z} = \mathcal{S}\, Q_3 - x\, P_2 + y\, P_3, \tag{4}$$

where $Q_i = Q_i(x, y, z)$ and $P_i = P_i(x, y, z)$, for $i = 1, 2, 3$, are arbitrary real polynomials and $\mathcal{S} = x^2 + y^2 + z^2 - 1 = 0$ is the invariant sphere of system (4).

The following result holds.

Theorem 2 *For $\alpha = 0$, system (3) is the most general class of quadratic polynomial differential systems whose phase space is foliated by concentric invariant spheres. In this case, if $d_1 \neq 0$ and $c_i = d_i = 0$, for $i = 2, 3$, then the z-axis is a line of equilibrium points, the origin is a non-isolated zero-Hopf equilibrium point and it is the center of the invariant spheres.*

Proof Suppose that system (1) has degree $m = 2$ and $\mathcal{S} = 0$ is an invariant sphere of this system. By Theorem 1, after an affine change of coordinates, system (1) can be written as (4), with $Q_i = q_i$ and $P_i = a_i\, x + b_i\, y + c_i\, z + d_i$, where $a_i, b_i, c_i, d_i, q_i \in \mathbb{R}$, for $i = 1, 2, 3$, and the equation of the invariant sphere is $\mathcal{S} = x^2 + y^2 + z^2 - 1 = 0$, with cofactor $K = 2q_1\, x + 2q_2\, y + 2q_3\, z$. If $K \equiv 0$ then the phase space of system (4) is foliated by concentric invariant spheres. Observe that it is equivalent to take $q_1 = q_2 = q_3 = 0$, from which we obtain system (3) with $\alpha = 0$.

Now considering the flow of system (3), with $\alpha = 0$, restricted to the z-axis, we obtain

$$\dot{x} = z\,(c_2\, z + d_2), \quad \dot{y} = -z\,(c_3\, z + d_3), \quad \dot{z} = 0.$$

Then taking $c_i = d_i = 0$, for $i = 2, 3$, the z-axis is a line of equilibrium points of system (3). The eigenvalues of the linear part of system (3) at the origin are

$\lambda_1 = 0$ and $\lambda_{2,3} = \pm i\, d_1$. Hence, for $d_1 \neq 0$, the origin is a non-isolated zero-Hopf equilibrium point. This proves Theorem 2. □

Remark 1 The eigenvalues of the linear part of system (3) at the equilibrium points in the z-axis are $\lambda_1 = 0$ and $\lambda_{2,3} = \beta\, z \pm \sqrt{\gamma}$, where

$$\beta = \frac{1}{2}\,(a_2 - b_3) \quad \text{and}$$

$$\gamma = \frac{1}{4}\,z^2\,(a_2 + b_3)^2 - (c_1\,z + d_1)^2 + z\,[(a_3 + b_2)\,(z\,c_1 + d_1) - a_3\,z\,b_2].$$

Hence, if $\gamma \neq 0$ and $\beta^2\,z^2 > |\gamma|$, then the equilibrium points $(0, 0, z)$ with $z < 0$ and $z > 0$ have opposite stability, as it occurs in the Sprott A system, as shown in [9, 10].

3 Zero-Hopf Bifurcation via Averaging Theory

Recall that an equilibrium point of a differential system in $\mathbb{R}^3$ is a *zero-Hopf equilibrium* if the Jacobian matrix at this point has one zero and a pair of purely imaginary eigenvalues. It is known that, generically, a zero-Hopf bifurcation takes place in this kind of equilibrium point and, in some cases, this type of bifurcation can imply a local birth of chaos [12]. For $\alpha = 0$, $d_i \neq 0$ and $c_i = d_i = 0$, $i = 2, 3$, the origin is a zero-Hopf equilibrium point of system (3), as stated in Theorem 2. In this section we will use the averaging theory to prove the existence of a periodic orbit bifurcating from the origin of system (3). For the sake of completeness and to fix the notation which will be used ahead, in the next subsection we present the main result from averaging theory, whose proof can be found in [13].

3.1 Averaging Theory of First Order

Consider the initial value problems

$$\dot{\mathbf{x}} = \varepsilon\,F_1(t, \mathbf{x}) + \varepsilon^2\,F_2(t, \mathbf{x}, \varepsilon), \qquad \mathbf{x}(0) = \mathbf{x}_0, \tag{5}$$

and

$$\dot{\mathbf{y}} = \varepsilon\,g(\mathbf{y}), \qquad \mathbf{y}(0) = \mathbf{x}_0, \tag{6}$$

with $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x}_0$ in some open subset Ω of $\mathbb{R}^n$, $t \in [0, \infty)$, and $\varepsilon \in (0, \varepsilon_0]$, for some fixed $\varepsilon_0 > 0$ small enough. Suppose that F_1 and F_2 are periodic functions of period T in the variable t, and set

$$g(\mathbf{y}) = \frac{1}{T}\int_0^T F_1(t, \mathbf{y})dt.$$

Denote by $D_{\mathbf{x}}g$ and $D_{\mathbf{xx}}g$ all the first and second derivatives of g, respectively. Under these assumptions, the following result is proved in [13].

Theorem 3 *Let F_1, $D_{\mathbf{x}}F_1$, $D_{\mathbf{xx}}F_1$, and $D_{\mathbf{x}}F_2$ be continuous and bounded by a constant, which does not depend on ε, in $[0,\infty) \times \Omega \times (0,\varepsilon_0]$ and assume that $\mathbf{y}(t) \in \Omega$ for $t \in [0, 1/\varepsilon]$. Then, the following statements hold.*

1. *For $t \in [0, 1/\varepsilon]$, we have $\mathbf{x}(t) - \mathbf{y}(t) = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$.*
2. *If $p \neq 0$ is an equilibrium point of system (6) such that $\det[D_{\mathbf{y}}g(p)] \neq 0$, then system (5) has a periodic solution $\phi(t,\varepsilon)$ of period T, which is close to p and such that $\phi(0,\varepsilon) - p = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$.*
3. *The stability of the periodic solution $\phi(t,\varepsilon)$ is given by the stability of the equilibrium point p.*

Based on Theorem 3, in the next subsection we provide necessary conditions under the parameters of system (3) for the existence of a periodic orbit bifurcating from the origin.

3.2 Existence of a Periodic Orbit

Theorem 4 *Consider system (3) with $d_1 \neq 0$ and $c_i = d_i = 0$, for $i = 2, 3$. If $b_2 = a_3$ and $b_3 - a_2 \neq 0$, then, for $\alpha > 0$ sufficiently small, there exists a periodic orbit γ_α in the phase space of system (3), which tends to the non-isolated zero-Hopf equilibrium point at the origin as $\alpha \to 0$. Moreover, γ_α is linearly stable if $a_2 < b_3$ and it is unstable (of saddle type) if $a_2 > b_3$.*

Proof Consider system (3) with $c_i = d_i = 0$, for $i = 2, 3$, $b_2 = a_3$, and $d_1 \neq 0$. Without loss of generality, take $d_1 = 1$. In order to apply Theorem 3, we write the obtained system in cylindrical coordinates (r, θ, z), where $x = r\cos\theta$ and $y = r\sin\theta$. Then it becomes

$$\begin{aligned}
\dot{r} &= [-b_3 + (a_2 + b_3)\cos^2\theta]\, r\, z,\\
\dot{\theta} &= 1 - (a_3 - c_1)\, z + b_1\, r\sin\theta + [a_1\, r - z\,(a_2 + b_3)\sin\theta]\cos\theta,\\
\dot{z} &= b_3\, r^2 - (a_2 + b_3)\, r^2\cos^2\theta - \alpha.
\end{aligned} \tag{7}$$

Introduce the variable $\varepsilon > 0$ into system (7) considering $\alpha = \varepsilon^2$ and doing the change of coordinates $(r, \theta, z) \to (R, \theta, Z)$, where $r = \varepsilon R$ and $z = \varepsilon Z$. Then, taking θ as the independent variable and doing the Taylor expansion of order 2 of the obtained equations at $\varepsilon = 0$, we get

$$\frac{dR}{d\theta} = -R\,Z\,[b_3 - (a_2 + b_3)\,\cos^2\theta]\,\varepsilon + \mathcal{O}(\varepsilon^2),$$
$$\frac{dZ}{d\theta} = -[1 - b_3\,R^2 + (a_2 + b_3)\,R^2\,\cos^2\theta]\,\varepsilon + \mathcal{O}(\varepsilon^2). \tag{8}$$

Using the notation of Theorem 3, consider

$$\mathbf{x} = \begin{pmatrix} R \\ Z \end{pmatrix}, \quad t = \theta, \quad T = 2\pi,$$
$$F_1(\theta, \mathbf{x}) = \begin{pmatrix} -[b_3 - (a_2 + b_3)\,\cos^2\theta]\,R\,Z \\ -[1 - b_3\,R^2 + (a_2 + b_3)\,R^2\,\cos^2\theta] \end{pmatrix}.$$

In this way we have

$$g(\mathbf{y}) = \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta, \mathbf{x})\,d\theta = \frac{1}{2}\begin{pmatrix} -(b_3 - a_2)\,RZ \\ -2 + (b_3 - a_2)\,R^2 \end{pmatrix}.$$

Hence, $g(\mathbf{y}) = 0$ has the unique real solution

$$p = (R, Z) = \left(\sqrt{\frac{2}{b_3 - a_2}}, 0\right),$$

which satisfies $\det[D_{\mathbf{y}}g(p)] = b_3 - a_2 \neq 0$. Then, by Theorem 3, for $\varepsilon > 0$ sufficiently small, system (8) has a periodic solution $\phi(\theta, \varepsilon) = (R(\theta, \varepsilon), Z(\theta, \varepsilon))$ such that $\phi(0, \varepsilon) \to p$ as $\varepsilon \to 0$. Moreover the eigenvalues of the matrix $[D_{\mathbf{y}}g(p)]$ are $\pm\sqrt{a_2 - b_3}$. Thus, the obtained periodic orbit is linearly stable if $a_2 - b_3 < 0$ and unstable (of saddle type) if $a_2 - b_3 > 0$.

Changing back the coordinates to system (3), we have that, for $\alpha > 0$ sufficiently small, such system has a periodic solution γ_α of period approximately 2π given by

$$x_\alpha(t) = \sqrt{\frac{2\alpha}{b_3 - a_2}}\,\cos t + \mathcal{O}(\alpha), \quad y_\alpha(t) = \sqrt{\frac{2\alpha}{b_3 - a_2}}\,\sin t + \mathcal{O}(\alpha), \quad z_\alpha(t) = \mathcal{O}(\alpha).$$

Note that γ_α tends to the origin, which is a non-isolated zero-Hopf equilibrium point, as $\alpha \to 0$. □

4 Existence of Nested Invariant Tori

Under generic assumptions, the presence of a linearly stable periodic orbit implies the occurrence of rich dynamics: it forces, for example, the existence of a subset of positive measure in the phase space filled by invariant tori, as stated for instance in

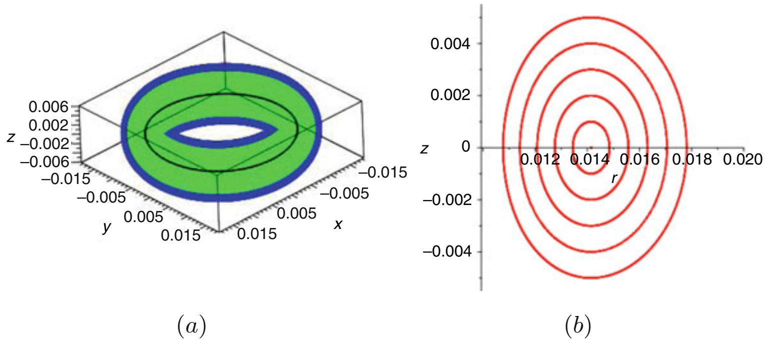

Fig. 1 (**a**) Periodic orbit and nested invariant tori of system (3) with $c_2 = c_3 = d_2 = d_3 = 0$, $a_1 = a_2 = a_3 = b_1 = b_2 = c_1 = d_1 = 1$ and $b_3 = 2$. (**b**) Respective 2π–Poincaré map of system (3). In both cases $\alpha = 10^{-4}$

[12]. In this context, in [9, 10] the authors showed the existence of nested invariant tori surrounding a bifurcating linearly stable periodic orbit in the phase space of the Sprott A system. Performing a detailed numerical study, we obtained the following similar result for the more general differential system (3).

Numerical Result Around the linearly stable periodic orbit γ_α of differential system (3) with $\alpha > 0$ small, described in Theorem 4, there exist nested invariant tori, as shown in Fig. 1.

In Fig. 1a is shown the nested invariant tori around the periodic orbit γ_α of system (3) for $\alpha > 0$ small and, in Fig. 1b, its respective 2π–Poincaré map, which was obtained as follows. We consider system (3) in cylindrical coordinates (r, θ, z) and take θ as the independent variable, obtaining a system in the variables $(r(\theta), z(\theta))$, 2π-periodic in θ. We compute the solutions of the obtained system, taking initial conditions near of the periodic orbit γ_α, for discrete values $\theta = 2k\pi$, where $k = 0, 1, \ldots, N$, with N sufficiently large, obtaining Fig. 1b. The fixed point representing the periodic orbit γ_α, given in Theorem 4, is surrounded by concentric circles, suggesting the existence of nested invariant tori around it.

It is also possible to explain the existence of invariant tori as a deformation of invariant spheres. Indeed, for $\alpha = 0$, $d_i \neq 0$ and $c_i = d_i = 0$, $i = 2, 3$, we proved in Theorem 2 that the phase space of system (3) is foliated by invariant spheres and the z-axis is a line of equilibria. For suitable choices of the parameter values (see Remark 1) and for z small, the equilibrium points in the z-axis are foci with opposite stability for $z < 0$ and $z > 0$. In this case, there exist heteroclinic orbits on each invariant sphere, connecting the (unstable) south pole to the (stable) north pole, see Fig. 2a. For $\alpha > 0$ sufficiently small, a linearly stable periodic orbit γ_α bifurcates from the origin of system (3), as stated in Theorem 4 and shown in black color in Fig. 2b. Based on the numerical simulations performed, it is possible to observe that the concentric invariant spheres evolve to nested invariant tori around the periodic

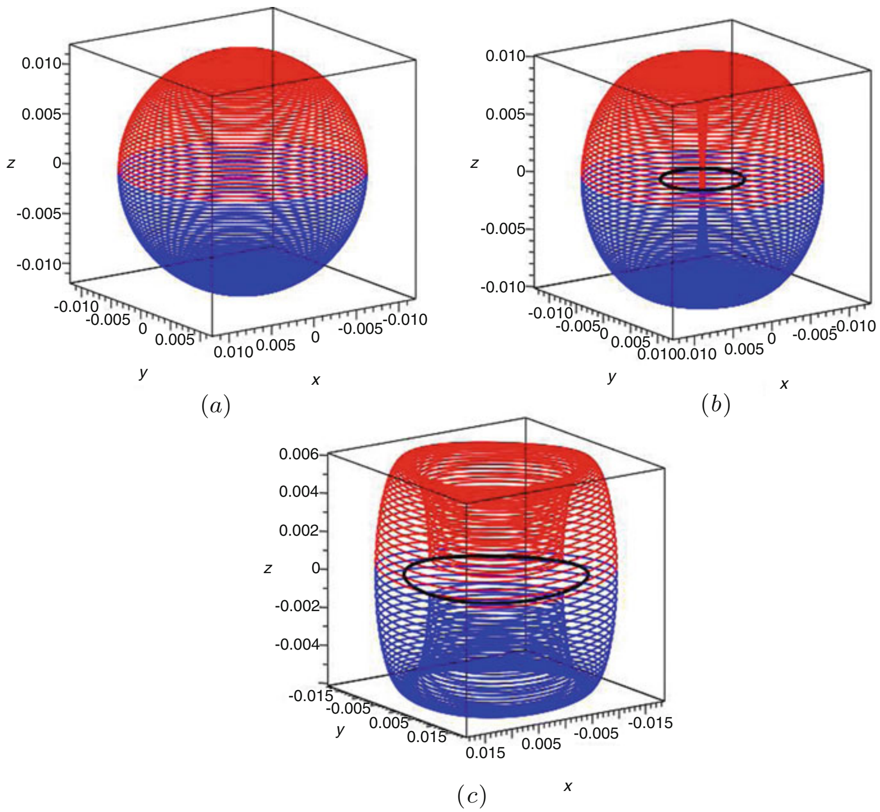

Fig. 2 Periodic orbit γ_α of system (3) (black) and orbit with initial condition $(\sqrt{2\alpha/(b_3 - a_2)}, 1.2 \cdot 10^{-2}, 0)$ for $t < 0$ (blue) and $t > 0$ (red), with $c_2 = c_3 = d_2 = d_3 = 0$, $a_1 = a_2 = a_3 = b_1 = b_2 = c_1 = d_1 = 1$ and $b_3 = 2$. For the parameter α we consider: (**a**) $\alpha = 0$, (**b**) $\alpha = 10^{-5}$, and (**c**) $\alpha = 10^{-4}$

orbit γ_α, as the parameter α increases, as shown in Fig. 2a–c, considering one of the concentric invariant spheres.

5 Concluding Remarks

Whereas the Sprott A system is claimed to be the simplest conservative differential system presenting chaotic behavior, in this paper we consider system (3), which is a more general and comprehensive class of differential systems containing and presenting similar dynamical behavior to the Sprott A system, for small values of the parameter α. In this way, system (3) can contain other nonlinear oscillators besides the Nosé–Hoover oscillator (or the Sprott A system). For appropriate choices of parameters values in system (3), we proved that for $\alpha > 0$ small enough a linearly stable periodic orbit bifurcates from a zero-Hopf equilibrium point located at the origin and we numerically show the existence of nested invariant tori around the periodic orbit. These elements can generate complex and interesting dynamical behavior, as shown in [4, 6, 9, 10] for Sprott A system, where the existence of a linearly stable periodic orbit and nested invariant tori around it play an important role in the formation of chaotic behavior in that system. We believe that the same is true for the more general system (3). It will be studied in future works.

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