Asymmetry in the Basin Stability of Oscillation Death States Under Variation of Environment-Oscillator Links

Manish Yadav, Sudhanshu Shekhar Chaurasia, and Sudeshna Sinha

Abstract We explore the effect of a common external system, which may be considered as a common environment, on the oscillation death(OD) states of a group of Stuart–Landau(SL) oscillators. It was found in Chaurasia et al. (Phys Rev E 98:032223, 2018), that the group of oscillators, when uncoupled to the common environment, yield a completely symmetric oscillation death state, i.e. there is an equal probability of occurrence of positive and negative oscillation death states. However, remarkably, this symmetry is significantly broken, when coupled to a common external system. For exponentially decaying common environment, the symmetry breaking of the OD states was found to be very pronounced for low environmental damping and strong oscillator-environment coupling. Here we consider the effect of disconnections of the oscillator-environment links on this asymmetry in the basin stability of the OD states. Interestingly, we find that the asymmetry induced by environmental coupling decreases with increase in fraction of such disconnections, and at some intermediate fraction close to half the symmetry is restored. However, further increase in disconnections induces asymmetry in the OD state again, until all oscillator-environment links are switched off. This suggests that a balance of on-off oscillator-environment links restores the symmetry of the OD state, and when half of the environmental connections are switched off one obtains the positive and negative OD states with almost equal probability.

Keywords Symmetry breaking · Oscillation death · Basin stability · Steady states

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1 Introduction

In the context of many real world systems, interactions can occur through a common medium. For instance, chemical oscillations of catalyst-loaded reactants have been found in a medium of catalyst-free solution, where the coupling is through exchange of chemicals with the surrounding medium [1]. Similarly, in the context of genetic oscillators coupling occurs by diffusion of chemicals between cells and extracellular medium [2]. Further, in a collection of circadian oscillators, the concentration of neurotransmitter released by each cell can induce collective behaviour [3]. In general, such cases occur due to the common medium, referred to as a common environment, interacting with the dynamical systems. In this work we will investigate a generic model that unifies many specific models of particular systems such as biochemical oscillators coupled through an environment, and allows us to obtain some basic general results which potentially apply to all of them.

It was found in [4] that coupling a group of oscillators to a common external medium [5] destroyed the symmetry in occurrence of the oscillation death (OD) states [6–8], also known as inhomogeneous steady states (IHSS), in the system. That is, the oscillator death states in the presence of environmental coupling are no longer symmetrically distributed. Rather the distribution of the emergent OD states is significantly skewed. This implies that the basin stability of the OD states is no longer the same. Instead there is coupling induced *asymmetry in the basin stability* [9] which leads to the system evolving preferentially towards one of the OD states. This manifests as a remarkable asymmetric distribution of the OD states, though both states are linearly stable. So one observes that the average fraction of oscillators going to a particular OD state is not the same.

Specifically, here we will study a group of globally coupled oscillators. The oscillators will be considered in the oscillation death regime, by setting appropriate values of the control parameters of the individual oscillators [10]. Each oscillator is also connected with the common external medium. This environmental coupling effectively pushes all the oscillators towards negative OD-state. We then go on to investigate the dynamics as the environment-oscillator links are disconnected one by one. We will show how cutting off the environment-oscillator links leads to a restoration of symmetry in the distribution of OD states. We will further demonstrate that one can use the external medium coupling strength and the environmental damping constant to control the distribution of oscillators in the different OD-states for a given fraction of environment-oscillator links.

2 Coupling via Common Environment

Our representative model is described by the generalized equation given below in Eq. (1) and schematically elucidated with Fig. 1:

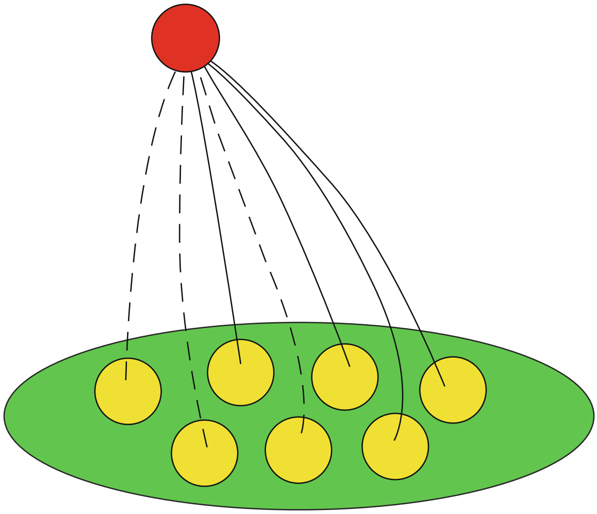

Fig. 1 Schematic diagram of group of oscillators connected to a common external environment

$$\dot{X}_i = F(X_i) + \varepsilon_{intra}\alpha(q\bar{X} - X_i) + \varepsilon_{ext}\beta u,$$
$$\dot{u} = -ku + \frac{\varepsilon_{ext}}{N}\beta^T \sum_{i=1}^{N} X_i \quad (1)$$

X_i is the column vector representing an m-dimensional nonlinear oscillator, each of the oscillators are connected within the group with mean-field diffusive coupling and with an external common environment (u). All the oscillators are connected with each other with the given global mean-field diffusive coupling with coupling parameter ε_{intra} and the mean diffusion is controlled by the diffusion coefficient q within this group. α is the $(m \times m)$ matrix with elements 0 and 1 to represent the components of m-dimensional oscillator taking part in the intra-group coupling. Here, we are taking α to be a diagonal matrix; $\alpha = diag(\alpha_1, \alpha_2, \ldots, \alpha_m)$. For external coupling of each of the oscillators with common environment we use ε_{ext} as coupling parameter and again to decide the component of oscillator to receive the external coupling we use β as the m-dimensional column matrix.

In this paper, we use Stuart–Landau oscillators ($m = 2$; x and y variables) as the unit component of the group and a damped environment as the external common medium (with damping constant k). We are taking α to be $\alpha = diag(1, 0)$ such that only x-variable will take part in within the group coupling. We take $\beta = (0, 1)$, i.e. the y-variable of each Stuart–Landau oscillator gives and receives signals from the damped common medium. So the dynamics of the full system comprised of the group of Stuart–Landau oscillators and the common external medium is:

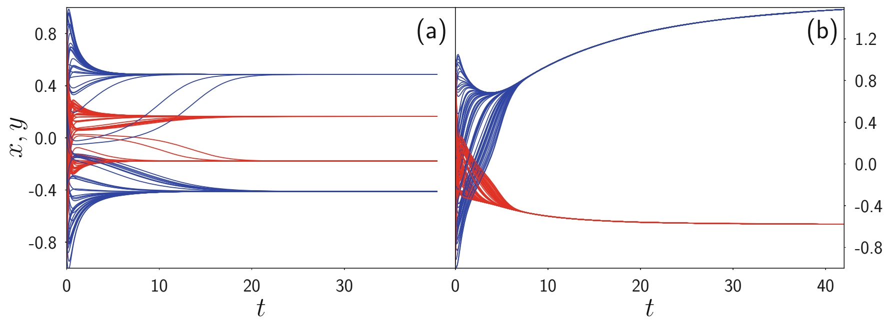

Fig. 2 Time series of x variable (red) and y variable (blue) with $\varepsilon_{intra} = 6$, $q = 0.4$ (**a**) without external coupling and (**b**) with externally coupled with $\varepsilon_{ext} = 0.5$, $k = 0.1$ for $N = 64$ oscillators in the group

$$\begin{aligned}
\dot{x}_i &= (1 - x_i^2 - y_i^2)x_i - \omega y_i + \varepsilon_{intra}(q\bar{x} - x_i) \\
\dot{y}_i &= (1 - x_i^2 - y_i^2)y_i + \omega x_i + \varepsilon_{ext} u \\
\dot{u} &= -ku + \varepsilon_{ext}\bar{y}
\end{aligned} \tag{2}$$

The mean-field diffusive coupling has been observed to show oscillation death (OD) states in limit cycle oscillators in the parameter space of coupling parameter ε and control parameter q. Here we consider the group in the OD-state, with $\varepsilon = 6$ and $q = 0.4$ (cf. time series shown in Fig. 2a). Our aim here is to analyse the effects of the common external environment on the stable OD-states. So we connect each oscillator of the group to the common external medium according to Eq. (1). The time series of the group in the presence of the common environment is shown in Fig. 2b, with $\varepsilon_{ext} = 0.5$ and the intrinsic environment damping constant $k = 0.1$. It is clearly evident that the presence of such a damped common medium results in all the oscillators of the group evolving to one of the OD states. This gives rise to a very asymmetric distribution of the oscillators among the two OD states as most oscillators now preferentially go to one particular state. Now, in the subsequent sections we will analyse mechanisms that restore this broken symmetry in the probability of obtaining an OD state in the oscillator group induced by the common medium.

3 Fractionally Disconnected Links

Previously, in Fig. 2 we saw that coupling to an external damped environment lead all the oscillators to one particular OD state, thereby breaking the symmetry of the distribution of the OD states. Now, we will disconnect the oscillator-environment links one at a time till all of the oscillators in the group are uncoupled to the external

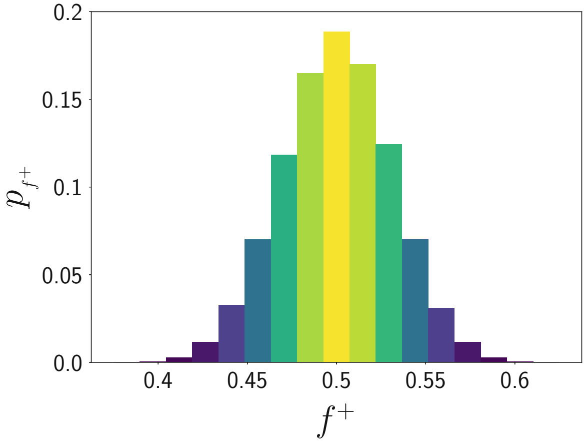

Fig. 3 Histogram showing the probability of the fraction of oscillators in positive state when the coupling of the oscillator group to the environment $\varepsilon_{ext} = 0$

medium and look for the changes in distribution of oscillators in the oscillation death states. In our case, we have two OD states on either side of the origin i.e. positive x which we will call x^+ and another on the negative side which we name x^- from now on. Without any external environment the oscillators occur almost equally (statistically speaking) in the positive and negative OD states. To quantify this observation we show in Fig. 3 the probability distribution of the oscillators in the positive x^+ steady state, obtained by sampling over 50,000 initial conditions, uniformly distributed over phase space volume $[-1, 1]$, of globally coupled SL oscillators without external environment.

In Fig. 4 we plot the fraction of oscillators that go to the positive OD state (x^+) with respect to varying number of oscillator-environment links. In particular, we disconnect one environment-oscillator link at a time, and we denote the fraction of disconnected links by f_{disc}. So $f_{disc} = 0$ corresponds to the case where all oscillators in the group are connected to the external environment, while $f_{disc} = 1$ corresponds to the limit of a group of SL oscillators having no interactions with the common environment. We observe changes in the fraction of oscillators in the positive OD-state, averaged over different initial conditions, denoted by $\langle f^+ \rangle$, as a function of f_{disc}. That is, we investigate how the distribution of the oscillators between the two available OD states changes as the number of environment-oscillator links changes. The results of the dependence of $\langle f^+ \rangle$ on f_{disc} for different values of external environment coupling ($\varepsilon_{ext} = 0.2$ and 0.5) are displayed. It is clear that this dependence is *non-monotonic* and has several non-trivial features. For instance, if we consider the case of $\varepsilon = 0.5$ in Fig. 4, we find that at $f_{disc} \simeq 0.2$ the oscillators are predominately in the negative OD state and $\langle f^+ \rangle$ is $\simeq 0.2$, i.e. around 20% of the oscillators in the group go to the positive OD state, while the rest are attracted to the negative OD state. As we change f_{disc} the probability of being in the

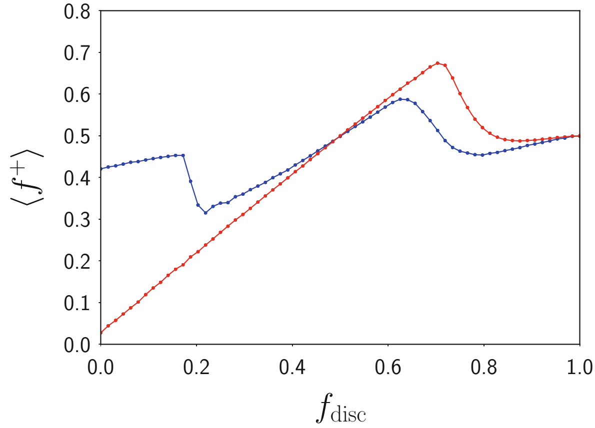

Fig. 4 Average fraction of oscillators in positive OD-state ($\langle f^+ \rangle$), obtained by sampling over 10,000 initial conditions, with respect to the fraction of oscillators-environment links for $\varepsilon_{ext} = 0.2$ (blue) and 0.5 (red). Here $N = 64$ and $k = 0.1$

positive OD state increases to a maximum of $\langle f^+ \rangle \simeq 0.7$ at $f_{disc} \simeq 0.7$. After that, $\langle f^+ \rangle$ decreases again and reaches 0.5, namely the completely symmetric situation, in the limit of $f_{disc} = 1$ where we have completely disconnected the oscillator group from the environment. One remarkable observation is that at $f_{disc} \simeq 0.5$ the value of $\langle f^+ \rangle$ is 0.5. This implies that when half of the oscillators are connected with the external common medium the statistical symmetry of the OD states returns i.e. both the positive and negative OD states are equally occupied by the oscillators. So when half of the oscillators in the group are connected to the environment ($f_{disc} = 0.5$) we obtain a dynamical outcome that is equivalent to the case of the oscillator group being completely unconnected to the external environment ($f_{disc} = 1$).

Further, we examine the effect of the damping constant k of the external environment on the distribution of the oscillators between the positive and negative OD-states, i.e. the dependence of $\langle f^+ \rangle$ and $\langle f^- \rangle$ on k for different values of f_{disc}. To illustrate this, we show results for three values of external coupling ($\varepsilon_{ext} = 0.25, 0.5$, and 0.7) in Fig. 5. For $f_{disc} = 0.25$ (blue), the fraction of positive OD-state ($\langle f^+ \rangle$) always remains less than 50% for the entire range of k sampled, and it slowly increases to $\sim$ 50% for $k \geq 0.85$. The oscillator distribution tends to maintain its symmetry (i.e. $\langle f^+ \rangle \sim 0.5$) for all values of k when only half of the oscillators are connected/disconnected with the external environment (i.e. $f_{disc} = 0.5$). For $f_{disc} = 0.7$ (green) the oscillator distribution reaches its most skewed position when $\langle f^+ \rangle$ becomes maximum at ~ 0.7 (cf. Fig. 4). On increasing k, this again approaches $\langle f^+ \rangle \sim 0.5$ as k approaches 1. This suggests that the environmental damping constant can be utilized as a parameter to *control the distribution of oscillators in the positive and negative OD-states.*

The coupling strength between the external medium and the oscillators in the group (ε_{ext}) is vital in controlling the flow of information between the group of

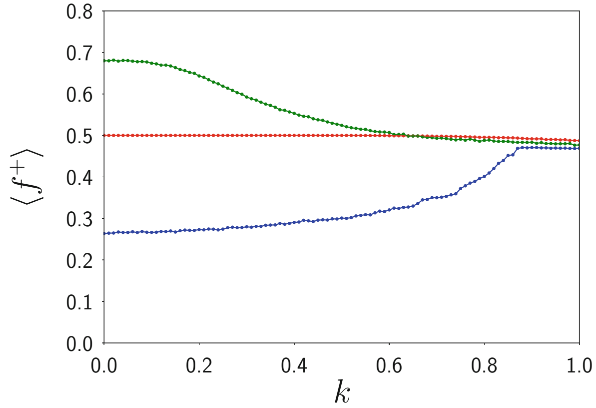

Fig. 5 Average fraction of oscillators in positive OD-state ($\langle f^+ \rangle$), obtained by sampling over 10,000 initial conditions, with respect the damping constant of the common external environment k, for $f_{disc} = 0.25$ (blue), 0.5 (red), and 0.7 (green). Here $\varepsilon_{ext} = 0.5$ and $N = 64$

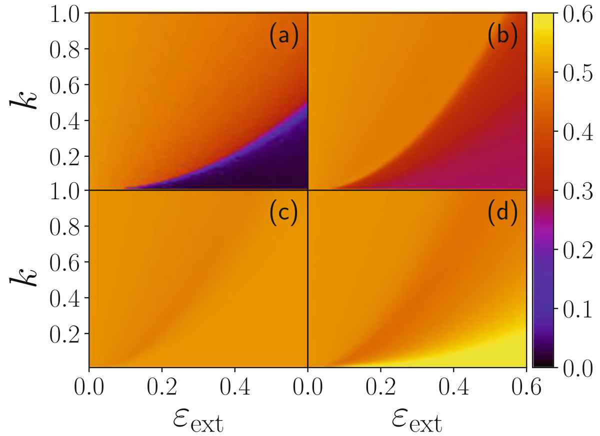

Fig. 6 Average fraction of oscillators in positive OD-state ($\langle f^+ \rangle$), estimated by sampling over 10,000 initial conditions, in the parameter space of $\varepsilon_{ext} - k$ for f_{disc} (**a**) 0.0, (**b**) 0.25, (**c**) 0.5 and (**d**) 0.75, with $N = 64$

oscillators. So we look for changes in $\langle f^+ \rangle$ in the parameter space of $\varepsilon_{ext} - k$. Figure 6a shows $\langle f^+ \rangle$ when all oscillators are connected to the environment (i.e. $f_{disc} = 0.0$). This will act as a reference for comparison with the case where some fraction of environment-oscillator links are disconnected. For higher ε_{ext} values and lower damping constant (k) the fraction $\langle f^+ \rangle$ is almost 0 (or $\langle f^- \rangle \simeq 1$). Interestingly, at this particular region of the $\varepsilon_{ext} - k$ parameter space, $\langle f^+ \rangle$ increases for increasing f_{disc}. This demonstrates that as increasing number of disconnections of the environment-oscillator links, the number of oscillators going to the positive

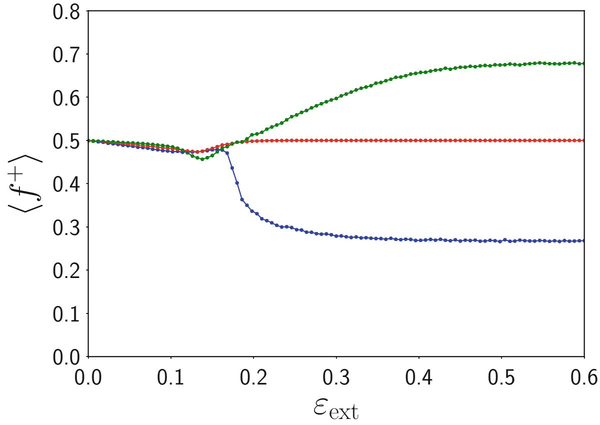

Fig. 7 Average fraction of oscillators in positive OD-state ($\langle f^+ \rangle$), estimated by sampling over 10,000 initial conditions, with respect to ε_{ext}, for $f_{disc} = 0.25$ (blue), 0.5 (red), and 0.7 (green), with $k = 0.1$ and $N = 64$

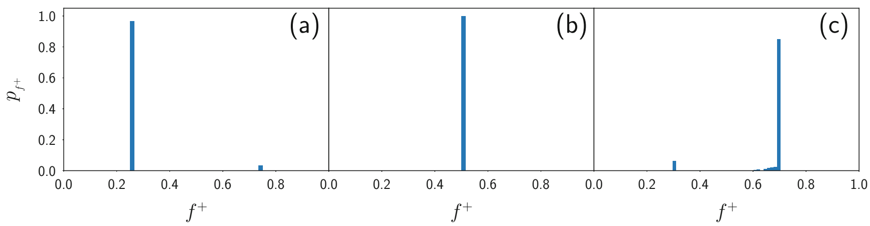

Fig. 8 Histogram showing the probability of the fraction of oscillators in positive state when the coupling of the oscillator group to the environment $\varepsilon_{ext} = 0.5$ and damping constant of the environment $k = 0.1$, with fraction of disconnected links (**a**) $f_{disc} = 0.25$, (**b**) $f_{disc} = 0.5$, (**c**) $f_{disc} = 0.7$

OD state increases, leading to a more symmetric distribution of oscillators among the two OD states. Further we consider the variation of $\langle f^+ \rangle$ with respect to ε_{ext} at fixed k in Fig. 7, for $k = 0.1$. Three scenarios become clearly evident from the figure, corresponding to three different fractions of disconnected links. So we can conclude that along with environmental damping constant (k), oscillator-environment coupling strength (ε_{ext}) is also an important parameter controlling the distribution of oscillators between the positive and negative OD-states.

We had shown the histogram of the probability of obtaining fraction f^+ in the positive OD state, in Fig. 3, for globally coupled SL oscillators in OD-state without environmental coupling, and seen a symmetric distribution of the oscillators around 0.5. Similarly, we now estimate the distribution of oscillators in the positive OD-state in the presence of a common environment. We explore cases with different fractions of disconnected environment-oscillator links. In Fig. 8a we show the distribution for $f_{disc} = 0.25$, with $\varepsilon_{ext} = 0.5$ and $k = 0.1$. Interestingly, there is no spread in the distribution of oscillators, as is clearly seen from the single pronounced peak in the distribution at 0.5 for Fig. 8b and around 0.7 in Fig. 8c. This sharp localization of oscillators in one of the two available stable states is in contradistinction to the usual statistical spread observed in Fig. 3. This is especially remarkable for the case of the symmetric distribution that arises in Fig. 8b, vis-a-vis the statistically symmetric case seen in Fig. 3. So we can infer that one can tailor the distribution of oscillators in positive and negative OD-states by disconnecting a suitable number of environment-oscillator links (f_{disc}) and adjusting the control parameters ε_{ext} and k.

4 Discussion

We investigated the impact of a common environment, which acts as a common external system, on a group of Stuart–Landau oscillators. First, we considered the group of oscillators completely disconnected from the external environment. When there is no coupling to an external system, and the group of Stuart–Landau oscillators are only coupled to each other via mean-field interaction, one obtains a completely symmetric distribution of oscillation death states, i.e. half of the oscillators attain positive OD states and other half attain negative OD states. This symmetry is significantly broken, when the same group of oscillators are connected to an external common environment. The symmetry breaking depends on damping constant of the external system k, environment-oscillator coupling strength ε_{ext} and the fraction of oscillators connected to the external system. When very few oscillators are connected to the environment, the OD states are almost symmetrically distributed. On the other hand when a large fraction of oscillators are coupled to the environment, the symmetry is broken to a very high extent, for high environment-oscillator coupling strengths and low environmental damping constants. Interestingly, when half the environment-oscillator links are disconnected, the symmetry is restored, independent of the damping constant of the environment and the environment-oscillator coupling strength. In fact in this case, exactly half of the oscillators attain positive OD states and the other half attain negative OD states. So our work here suggests a potent method to control the basin stability of the oscillation death states.

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