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# Twistor Geometry and Gauge Fields

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In our course we have presented the basics of twistor theory and its applications to the solution of Yang–Mills duality equations. The first part describes the twistor correspondence between geometric objects in Minkowski space and their counterparts in twistor space. In the second part we apply twistor theory to the study of Yang–Mills duality equations on $\mathbb{R}^4$. We include a list of references for further study.

## 1. Twistor model of Minkowski space

We start with the geometry of Minkowski space $M$ provided with the action of the Lorentz group. The main geometric objects are the light lines (light rays) and light cones together with their complex analogues. Complexified Minkowski space $\mathbb{C}M$ contains both $M$ and its Euclidean counterpart $E$. We also make use of the future tube $\mathbb{C}M_+ = M + iV_+$ ($V_+$ is the future light cone) which is an open subset in $\mathbb{C}M$.

The Pauli map associating with a vector $x = (x^0, x^1, x^2, x^3) \in M$ the Hermitian matrix

$$X := \sum_{\mu=0}^{3} x^\mu \sigma_\mu,$$

where $\sigma_0 = I$, $\sigma_i$, $i = 1, 2, 3$, are Pauli matrices, realizes $M$ as the space $\mathrm{Herm}(2)$ of Hermitian $2\times 2$-matrices and $\mathbb{C}M$ as the space $\mathbb{C}[2\times 2]$ of complex $2\times 2$-matrices. Under this map the Lorentz norm of $x \in M$ is sent to $\det X$. The group $\mathrm{SL}(2,\mathbb{C})$ acts naturally on $\mathrm{Herm}(2)$ and is a double cover of the Lorentz group.

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The future tube $\mathbb{C}M_+$ under Pauli map is transformed into the matrix upper halfplane

$$H_+ = \left\{ Z \in \mathbb{C}[2 \times 2] : \ \mathrm{Im}\ Z := \frac{1}{2i}(Z - Z^*) > 0 \right\}$$

where the inequality Im $Z > 0$ means that the Hermitian matrix Im $Z$ is positive definite. The space $\mathbb{C}^2$, provided with the action of the group $\mathrm{SL}(2, \mathbb{C})$, is called the spinor space.

The twistor space $\mathbb{T}$ is the 4-dimensional complex vector space with coordinates written in the form $\zeta = (\omega, \pi)$ with $\omega, \pi \in \mathbb{C}^2$. Associate with a matrix $Z \in \mathbb{C}[2 \times 2]$ the 2-dimensional complex subspace in $\mathbb{T}$ determined by the system of two complex homogeneous equations: $\omega = Z\pi$. This map defines an embedding of the space $\mathbb{C}[2 \times 2]$ into the Grassmann manifold $G_2(\mathbb{T})$ of 2-dimensional complex subspaces in $\mathbb{T}$. Taking its composition with the Pauli map we obtain the embedding

$$\mathbb{C}M \longrightarrow \mathbb{C}[2 \times 2] \longrightarrow G_2(\mathbb{T})$$

of the complexified Minkowski space $\mathbb{C}M$ into the Grassmannian $G_2(\mathbb{T})$. Since $G_2(\mathbb{T})$ is compact it is natural to consider it as a model of compactified complexified Minkowski space $\mathbb{CM}$. The projectivization $\mathbb{PT}$ of the twistor space $\mathbb{T}$ is called the space of projective twistors. We can also consider the Grassmannian manifold $G_2(\mathbb{T})$ as the space $G_1(\mathbb{PT})$ of projective lines in $\mathbb{PT}$. The composite map $\mathbb{C}M \to G_2(\mathbb{T}) = G_1(\mathbb{PT})$ is called the twistor transform or Penrose correspondence.

## 2. Twistor correspondence

Consider first the properties of twistor correspondence in the case of complex Minkowski space. By twistor transform a point of $\mathbb{C}M$ is sent to a projective line in $\mathbb{PT}$. On the other hand, a point in $\mathbb{PT}$ corresponds to a light plane in $\mathbb{C}M$ called $\alpha$-plane (light plane is the plane generated by the pair of light lines). In dual way, a projective plane in $\mathbb{PT}$ corresponds to a light plane in $\mathbb{C}M$ called $\beta$-plane. It implies that a complex light line (which is the complexification of light line) is sent to a $(0, 2)$-flag in $\mathbb{PT}$ consisting of a point in $\mathbb{PT}$ and projective plane containing this point.

Switch now to the case of real Minkowski space $M$. Denote by $\Phi(\zeta)$ the norm of a twistor $\zeta = (\omega, \pi) \in \mathbb{T}$ given by $\Phi(\zeta) = \mathrm{Im} < \omega, \pi >$ where $< \omega, \pi >$ is the Hermitian product of vectors $\omega, \pi \in \mathbb{C}^2$. Denote by $\mathbb{N}$ the quadric in $\mathbb{T}$ given by the equation $\mathbb{N} = \{\zeta \in \mathbb{T} : \ \Phi(\zeta) = 0\}$ and by $\mathbb{PN}$ the associated projective quadric. The points of $M$ under twistor transform are sent to the projective lines lying in $\mathbb{PN}$. On the other hand, a light line in $M$ corresponds to a point of $\mathbb{PN}$. So in the case of $M$ we have the following duality: points of $M$ correspond to projective lines in $\mathbb{PN}$ and light lines in $M$ correspond to points of $\mathbb{PN}$. We see that the light lines, which can intersect in $M$, split into separate points of $\mathbb{PN}$. This fact is of fundamental importance for the twistor theory.

The quadric $\mathbb{N}$ divides the twistor space $\mathbb{T}$ into two parts. Denote them by $\mathbb{T}_\pm = \{\zeta \in \mathbb{T} :\ (\pm 1)\Phi(\zeta) > 0\}$ and by $\mathbb{PT}_\pm$ the corresponding projective subsets. A point of the future tube $\mathbb{C}M_+$ under twistor transform is sent to a projective line contained in $\mathbb{PT}_+$. The quadric $\mathbb{N}$ has the signature (2,2) and the group $\mathrm{SU}(2,2)$ of linear transformations of $\mathbb{T}$, preserving this quadric, is a 4:1 covering of the group of conformal transformations of $M$.

We turn now to the case of Euclidean space $E$. A point of $E$ under twistor transform is sent to the projective line in $\mathbb{PT}$ which is invariant under the map $j : [\zeta_1 : \zeta_2 : \zeta_3 : \zeta_4] \longmapsto [-\zeta_2 : \zeta_1 : -\zeta_4 : \zeta_3]$. In the Euclidean case the twistor transform coincides with the Hopf bundle

$$\pi : \mathbb{CP}^3 \xrightarrow{\mathbb{CP}^1} \mathbb{E}$$

where $\mathbb{E}$ is the *compactified Euclidean space* equal to the sphere $S^4$ and the fibers of $\pi$ are precisely the $j$-invariant projective lines.

The main idea of Penrose twistor program is that under twistor transform solutions of conformally invariant equations of field theory in $\mathbb{M}$ should correspond to the objects of complex algebraic geometry in $\mathbb{PN}$.

## 3. Instantons and Yang–Mills fields

Let $X$ be a compact oriented Riemannian 4-manifold and $G$ is the gauge group being a compact Lie group (e.g. $G = \mathrm{SU}(2)$) with Lie algebra $\mathfrak{g}$. Let $P \to X$ is a principal $G$-bundle on $X$ and $A$ is a gauge potential on $X$ given a 1-form $A \in \Omega^1(X, \mathrm{ad}\, P)$ with values in the adjoint bundle $\mathrm{ad}\, P = P \times_G \mathfrak{g}$. Denote by $D$ the exterior covariant derivative associated with $A$. Then $F = DA$ is the gauge field generated by $A$.

The Yang–Mills action is given by the formula

$$S(A) = \frac{1}{2} \int_X \|F\|^2 \mathrm{vol}$$

where the norm $\|\cdot\|^2$ is the inner product on differential forms with values in $\mathfrak{g}$, generated by the Riemannian metric on $X$ and an invariant inner product on $\mathfrak{g}$, vol is the volume element on $X$. The Yang–Mills field is a critical point of the functional $S$ being a solution of the Euler–Lagrange equations. They have the form $D^*F = 0$ ($D^*$ is the adjoint operator of $D$) and are called the Yang–Mills equations. They can be also written in the form $D(\star F) = 0$, where $\star$ is the Hodge $\star$-operator.

A gauge field $F$ is called selfdual (resp. anti-selfdual) if $*F = F$ (resp. $*F = -F$). Due to Bianchi identity $DF = 0$, solutions of the duality equations $*F = \pm F$ satisfy automatically the Yang–Mills equations. By writing $F$ in the form $F = F_+ + F_-$ where $F_\pm = \frac{1}{2}(*F \pm F)$ we can rewrite Yang–Mills functional as

$$S(A) = \frac{1}{2} \int_X \|F_+\|^2 + \|F_-\|^2)\mathrm{vol}.$$

The topological charge of $F$ is given by the formula

$$S(A) = \frac{1}{8\pi^2}\int_X \|F_+\|^2 - \|F_-\|^2)\mathrm{vol}.$$

Comparing the last two formulas we see that

$$S(A) \geq 4\pi^2|k|$$

and the equality here is attained precisely on solutions of the duality equations. In other words, solutions of the duality equations yield local minima of $S$. Instantons (resp. anti-instantons) are anti-selfdual (ASD)(resp. selfdual) solutions of duality equations with finite Yang–Mills action. The moduli space of instantons is the quotient of the space of instantons modulo gauge transformations.

## 4. Atiyah–Ward theorem

We specify now to the case when $X = S^4$ and $G = \mathrm{SU}(2)$. We have a principal SU(2)-bundle $P \to S^4$ and associated complex vector bundle $E \to S^4$ of rank 2. Consider an instanton given by an ASD solution $A$ of the duality equations and denote by $\nabla = \nabla_A$ the covariant derivative associated with $A$.

Consider the twistor bundle $\pi : \mathbb{CP}^3 \to S^4$ and denote by $\tilde{E} := \pi^*E$ the pull-back of the bundle $E$ to $\mathbb{CP}^3$ via the map $\pi$. The anti-selfduality of $A$ implies that its pullback $\tilde{A}$ to the bundle $\tilde{E}$ defines a holomorphic structure on $\tilde{E}$. The obtained holomorphic bundle $\tilde{E} \to \mathbb{CP}^3$ is by construction holomorphically trivial on $j$-invariant projective lines in $\mathbb{CP}^3$ being the fibers of the map $\pi$.

**Atiyah–Ward theorem**. There exists a bijective correspondence between

$$\left\{\begin{matrix}\text{moduli space of}\\ \text{instantons on } S^4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{holomorphic vector bundles over } \mathbb{CP}^3\\ \text{which are holomorphically trivial on } \pi\text{-}\\ \text{fibers}\end{matrix}\right\}.$$

There is also a purely complex version of this theorem. Consider it first for the future tube $\mathbb{C}M_+$. Let $E$ be a holomorphic vector bundle over $\mathbb{C}M_+$ and $\nabla = \nabla_A$ is the holomorphic covariant derivative acting on sections of $E$ generated by a holomorphic connection $A$. This connection is called anti-selfdual (ASD) if its curvature vanishes on all $\alpha$-planes. The complex variant of Atiyah–Ward theorem asserts that there exists a bijective correspondence between

$$\left\{\begin{matrix}\text{moduli space of holomorphic}\\ \text{ASD-connections on } \mathbb{C}M_+\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{holomorphic vector bundles on}\\ \mathbb{PT}_+ \text{ holomorphically trivial on}\\ \text{projective lines lying in } \mathbb{PT}_+\end{matrix}\right\}.$$

This theorem is based on the following Ward construction. Let $\tilde{E}$ be a holomorphic vector bundle over $\mathbb{PT}_+$ which is holomorphically trivial on projective lines in $\mathbb{PT}_+$. The fiber $E_z$ of the corresponding holomorphic vector bundle $E \to \mathbb{C}M_+$ at $z \in \mathbb{C}M_+$ consists by definition of holomorphic sections of $\tilde{E}$ over the projective line $\mathbb{CP}^1_z$ corresponding to the point $z$. If two projective lines $\mathbb{CP}^1_z$ and $\mathbb{CP}^1_{z'}$ intersect, i.e. the points $z$ and $z'$ lie on the same complex light line, we can identify the fibers $E_z$ and $E_{z'}$. In this way we define a parallel transport on $E$ along complex light lines in $\mathbb{C}M_+$ generating a holomorphic connection in $E$. By construction this connection is anti-selfdual.

For the inverse construction (from $E$ to $\tilde{E}$) it is convenient to use the double diagram

$$\begin{tikzcd} & \mathbb{F}_+ \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathbb{PT}_+ & & \mathbb{C}M_+ \end{tikzcd}$$

where $\mathbb{F}_+$ is the space of $(0,1)$-flags in $\mathbb{PT}_+$, i.e. pairs (point of $\mathbb{PT}_+$, projective line in $\mathbb{PT}_+$ containing this point). The space $\mathbb{C}M_+$ is identified with the Grassmann manifold $G_1(\mathbb{PT}_+)$ of projective lines lying in $\mathbb{PT}_+$, and $\mu$, $\nu$ are natural projections. Denote by $E'$ the pull-back of $E$ to a bundle over $\mathbb{F}_+$ via the map $\nu$ and by $\nabla'$ the pull-back of the connection $\nabla$ to the bundle $E'$. Define the fibre of the bundle $\tilde{E} \to \mathbb{PT}_+$ at $\zeta \in \mathbb{PT}_+$ as the space of holomorphic sections $s' \in \Gamma(\mu^{-1}(\zeta), E')$ satisfying the equation $\nabla'_\mu s' = 0$ ($\nabla'_\mu$ is the component of $\nabla'$ acting along the fibers of $\mu$). In other words, the fibre $\tilde{E}_\zeta$ consists of horizontal holomorphic sections of $E'$ over $\mu^{-1}(\zeta)$. This definition is correct due to the anti-selfduality of $\nabla$.

The given complex version of Atiyah–Ward theorem remains true if we replace $\mathbb{PT}_+$ by a domain $\tilde{D}$ in $\mathbb{CP}^3$ such that projective lines in $\tilde{D}$ correspond to the points of some domain $D$ in $\mathbb{C}M$. This domain should have an additional property that the intersection of any complex light line with $D$ is connected and simply connected.

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