Quantitative Methods in Economics and Finance

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Abstract *Evaluation of Financial and Actuarial Risk*. The research group has focused on several different aspects of quantitative finance. As a first research subject, we faced the problem of pricing credit default swaps (CDSs), which entails the calculation of the risk of default. As a second research subject, we considered the problem of pricing complex derivatives. Precisely, we took into account barrier options on an underlying described by either the geometric fractional Brownian motion or a time-changed Brownian motion. Finally, we dealt with the problem of pricing real options in the presence of stochastic interest rates. *Nonlinear Dynamics in Economic and Financial Models*. Nonlinear Dynamics is an interdisciplinary area characterized by a rapid and extensive development in recent years, which has proved to be very useful in explaining some important facts in Economics and Finance (such as endogenous fluctuations). In this contribution, we provide an overview of our research on this topic, both in continuous and in discrete time.

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1 Evaluation of Financial and Actuarial Risk[1]

The activity of the research group in Mathematical Methods for Economics, Actuarial Science and Finance of the University of Ancona focuses on several different aspects of quantitative finance. Special emphasis is put on the development of new theoretical models and new numerical and analytical methods for computing the default risk, for pricing financial derivatives and for conducting empirical tests to validate the models. These contributions are on one hand theoretically relevant and on the other hand useful for financial practitioners. In the following, we present and discuss the main topics covered by our researches.

A relevant issue tackled in our researches is the assessment of the risk of default of companies. Pricing the risk of default is a fundamental task for several financial market players such as corporate bond investors, credit derivative traders, banks, mortgage suppliers and insurance companies. To this aim, various models of credit risk have been developed which are based on two different approaches: the structural approach and the reduced-form approach.

Structural models describe the default event by means of one or more variables related to the capital structure of the firm issuing the debt. For example, according to the first proposed structural model, which has been developed in Merton (1974), a firm defaults if at the debt maturity the value of its assets is lower than the value of its obligations. An improvement of this model is presented in Black and Cox (1976), where the possible occurrence of premature bankruptcy as well as the debt seniority are taken into account. Other more sophisticated structural models are also available which incorporate variables such as the interest rate Longstaff and Schwartz (1995), Briys and de Varenne (1997), Bernard et al. (2005), tax benefits Anderson and Sundaresan (1996), debt restructuring Abínzano et al. (2009), liquidation costs Leland and Toft (1996) and downgrade-triggered termination clauses Feng and Volkmer (2012). As revealed by several empirical studies, see, e.g, Jones et al. (1984), Franks and Torous (1989), such a kind of models have a major issue: if the firm's assets value is specified as a continuous-time stochastic process, then for short debt maturities the probability of default that the models predict turns out to be very close to zero, contrary to what happens in reality, see, e.g, Crouhy et al. (2000), Bäuerle (2002). In order to account for high short-term spreads, some authors, see, e.g. Zhou (2001), Chen and Panjer (2003), have proposed structural models with unexpected jumps in the firm asset value. Nevertheless, this approach lacks analytical tractability, which makes it difficult to calibrate the model parameters to observed credit spreads.

To overcome the issues of the structural models, reduced form models have been proposed. According to the reduced-form approach, see, e.g, Duffee (1999), Duffie and Singleton (1999), Madan and Schoutens (2008), Schoutens and Cariboni (2009), Fontana and Montes (2014) the default event is modeled as the first jump of a counting process whose intensity, termed intensity of default, is not assumed to be firm-specific but is prescribed exogenously. This allows one to take into account the possible occurrence of a sudden (unpredictable) default event and henceforth the high credit

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spreads that are often experienced for short debt maturities can be recovered. In addition, reduced-form models are relatively simple from a mathematical standpoint and thus they usually offer a large amount of analytical tractability.

However, reduced-form models have the heavy drawback of not taking into account any information about the capital structure of the firm. Such an issue has prompted some authors to develop hybrid models of credit risk in which the reduced-form approach is combined with some structural variable, see, e.g, Madan and Unal (1998), Madan and Unal (2000), Duffie and Lando (2001), Cathcart and El-Jahel (2003), Cathcart and El-Jahel (2006), Giesecke (2006), Ballestra and Pacelli (2014).

Among these models, the one proposed in Madan and Unal (1998) deserves a special attention as it is a parsimonious hybrid model. In particular, it is mainly developed based on the reduced-form approach, but the default intensity, instead of being prescribed exogenously, is specified as a convenient function of the firm's equity value. This allows us to recover the desirable features of both the structural and the reduced-form models, but at the same time the parameters involved are only one more than the parameters of Merton's model, see Merton (1974). Therefore, the approach by Madan and Unal turns out to be particularly appealing and suitable for practical uses.

The model in Madan and Unal (1998) does not have an analytical closed-form solution. To be more precise, Madan and Unal have provided a closed-form expression for the survival probability, but, as pointed out in Grundke and Riedel (2004), the procedure used to derive this formula is not mathematically correct. Consequently, as shown in Grundke and Riedel (2004), where the default probability is computed by finite difference approximation, the closed-form solution obtained by Madan and Unal yields a survival probability that can also differ substantially from the true survival probability of the model. In order to compensate for the lack of a closed-form solution, we developed two alternative methods to approximate it.

The first one is proposed in Ballestra and Pacelli (2009), and it is an analytical approximation of the survival probability of the model by Madan and Unal. This formula is fairly accurate and computationally fast, but it is applicable only if one of the model parameters is sufficiently small (as it is based on a perturbation approach, see Ballestra and Pacelli 2009).

To provide a more general approximation of the survival probability in the Madan and Unal framework, which can also be useful for more general purposes, e.g. pricing financial derivatives such as credit default swaps (CDS, hereafter), a second method is proposed by Ballestra et al. (2017). It conducts to a quasi-analytical approximation of the survival probability in the model by Madan and Unal (1998). Such a formula, which is based on a Laplace-transform approach, turns out to be very accurate and computationally fast. Remarkably, this analytical expression for the survival probability allows us to price credit default swaps (CDSs) very easily. Specifically, a quasi-analytical formula to compute CDS par spreads is derived which is used in the aforementioned manuscript to calibrate the Madan-Unal credit risk model by fitting realized CDS par spreads. In particular, CDS names with different Moody's ratings are considered and the agreement between theoretical and empirical data is rather satisfactory, especially if we think that the stochastic differential equations on which the model by Madan and Unal stands involve only two unknown parameters. The pricing of CDSs has also been considered in Andreoli et al. (2016a), where the risk of default is specified according to a reduced-from model in high dimension. In particular, the interest rate and the survival probability are modeled by using a system of stochastic differential equations with either four or six factors, which generalizes an earlier model developed in Andreoli et al. (2015) and leads to a partial differential equation that is solved by means of a high-order finite difference scheme. Finally, a method to value CDOs, i.e. large portfolios of credit instruments, has been proposed in Andreoli et al. (2016b). Specifically, to deal with the incompleteness of the CDO market (due by the non-tradability of the survival probability), a Sharpe ratio approach is developed which generalizes the actuarial model presented in Bayraktar and Young (2007), Young (2008) and allows one to take into account non-hedgeable risk as well. This technique yields a set of partial differential equation in two independent space variables, which is efficiently solved by finite difference approximation.

Another relevant problem in quantitative finance is the empirical behavior of stock returns. In classical models for assessing the risk of default, stock returns are assumed to follow a geometric Brownian motion. This implies that many empirically observed features of the stock returns are neglected when assessing the solvency probability of a company. The neglected features of the logarithmic returns of the stocks include stylized facts such as self-similarity, heavy tails, long-range dependence and volatility clustering, see, e.g., Lo (1991), Ding et al. (1993) and Zhang et al. (2014). To overcome the mismatch between real stock returns and their theoretical representation, the fractional Brownian motion is often considered as an alternative formulation. It is a generalization of the Brownian motion that allows to replicate the stylized facts of the financial markets such as distribution of stock returns characterized by self-similarity, heavy tails, long-range dependence and volatility clustering, see, e.g., Lo (1991), Ding et al. (1993) and Zhang et al. (2014).

This modeling framework is however problematic for risk-neutral pricing as it does not allow one to construct a self-financing strategy yielding the risk-neutral price of financial options, see, e.g., Cheridito (2003) and Bender and Elliott (2004).

To include these striking features of the distribution of the stock returns in models for pricing financial derivatives under the risk-neutral approach, we introduce a mixed fractional Brownian motion (MFBM, hereafter) in a classical Merton modeling framework. The MFBM is a generalization of the fractional Brownian motion obtained as a linear combination of the fractional Brownian motion itself, see, e.g., Mandelbrot and Van Ness (1968), Duncan et al. (2000), Wang et al. (2001), Lim and Muniandy (2002, 2003), Longjin et al. (2010), Rostek and Schöbel (2013) and Hao et al. (2014), and of the standard Brownian motion, see, e.g., Øksendal (2003) (for other possible generalizations of the fractional Brownian motion the interested reader is referred to Wang et al. 2003, 2006, 2012; Liang et al. 2010; Gu et al. 2012). This stochastic process is particularly useful for our purposes because it allows to replicate the stylized facts of the distribution of the stock returns and, at the same time, it allows to construct a self-financing strategy. This makes it suitable for risk-neutral pricing.

In a recent contribution, see, e.g. Ballestra et al. (2016), we employed the MFBM to construct models for assessing the risk of default of a company and to price financial options. When pricing financial derivatives under the MFBM, the mathematical formulas to use are often more complicated than the classical ones developed assuming a Black-Scholes market. Moreover, some numerical approximations and techniques developed ad-hoc are often required. In this respect, we dealt with the problem of pricing barrier options on an underlying described by the mixed fractional Brownian model. To this aim, we considered the initial-boundary value partial differential problem that yields the option price and we derived an integral representation of it in which the integrand functions must be obtained solving Volterra equations of the first kind. In addition, we developed an ad-hoc numerical procedure to solve the integral equations obtained. Numerical simulations reveal that the proposed method is extremely accurate and fast, and performs significantly better than the finite difference method.

The models developed in finance are often used in management science to evaluate the opportunities of investments, the so-called real option analysis. Managers use real option analysis, also termed real option valuation, to decide about investment projects that can be undertaken at a future time, see, e.g., McDonald and Siegel (1986), Myers and Majd (1990), Dixit and Pindyck (1994), Charalampopoulos et al. (2001), Baldi and Trigeorgis (2009), Manley and Niquidet (2010), Fernandes et al. (2011b), Fernandes et al. (2011a), Bernardo et al. (2012), Santos et al. (2014), Nadarajah et al. (2015), Loureiro et al. (2015) and Nadarajah et al. (2017). Such an approach is particularly appealing because it allows one to take into account both the uncertainty and the flexibility related to the project to be valued. In this respect, we investigate in a recent contribution, see Ballestra et al. (2014), the problem of assessing investment projects under non-constant interest rates, see also Ingersoll and Ross (1992), Alvarez and Koskela (2006) and Schulmerich (2010). To this aim, a mathematical model is proposed where the revenue generated by investment projects is modeled as a geometric Brownian motion, and the interest rate is specified as a stochastic differential equation of Vasicek type Vasicek (1977). Under these assumptions, the problem of assessing the value of an investment opportunity is similar to the problem of pricing European vanilla options which can be solved in closed-form, see, e.g. Rabinovitch (1989).

The empirical analysis is done by considering several companies belonging to different production sectors and operating in the euro area. The results obtained show that investment projects are overvalued if the interest rate is assumed to be constant rather than stochastic. Nevertheless, the interest rate uncertainty does not have a substantial impact on the evaluation of firms' investments as confirmed by an empirical investigation underlying that the difference between the values of the investment projects under constant and stochastic interest rates is not statistically significant.

2 Nonlinear Dynamics in Economic and Financial Models[2]

It is well-known that economic and financial variables show fluctuations, even without exogenous shocks. In fact, in spite of the absence of external forces, the Economy can be unstable. For this reason, it is very useful to model complex economic systems as *Nonlinear Dynamical Systems*, which are able to explain endogenous fluctuations (see e.g. Dieci et al. 2014; Hommes 2013). In the following, we deal with Nonlinear Dynamical Systems both in discrete and continuous time, as natural words for studying important stylized facts in Economics and Finance.

2.1 Discrete Dynamical Systems

Many theoretical problems in Economics and Finance are formalized as dynamical systems in discrete time, also thanks to the large variety of methods and techniques (analytical and numerical) that this area offers. Concerning this subject, we shared in several researches, some of them are described in the following (Brianzoni et al. 2007, 2009, 2010a, b, c, 2011, 2012, 2015a, b, 2018).

2.1.1 Models with Differential Savings and Non Constant Population Growth Rate

A first line of research in discrete time concerns neoclassical one-sector growth models with differential savings, in order to investigate the possibility of complex dynamics. Moreover, for several production functions, the case of non constant population growth rate has been considered. In fact, when labor force is assumed to grow at a constant rate, population grows exponentially, which is clearly unrealistic given the carrying capacity of the environment.

The paper by Brianzoni, Mammana and Michetti, *Complex dynamics in the neoclassical growth model with differential savings and non-constant labor force growth*, analyzes the dynamics shown by the neoclassical one-sector growth model with differential savings, CES production function and the labour force dynamics described by the Beverton Holt equation. The resulting dynamical system is bidimensional, autonomous and triangular. The study of its qualitative and quantitative dynamic properties confirms that the system can exhibit cycles or even a chaotic dynamic pattern, if shareholders save more than workers, when the elasticity of substitution drops below one (so that capital income declines).

In order to take into account that the population growth rate can exhibit more complex dynamics, the paper by Brianzoni, Mammana and Michetti, *Non-linear dynamics in a business-cycle model with logistic population growth*, makes use of the logistic map. The resulting model is a bidimensional triangular dynamic system.

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The study of its long run behaviour is developed as regards the parameter values. More precisely, the existence of the compact global attractor is proved. Moreover, the shape of the chaotic attractor changes in consequence of some global bifurcations, as some parameters vary. The analysis of these bifurcations is performed by using the critical curves method. The results confirm the central role of the production function's elasticity of substitution in the creation and propagation of complicated dynamics as in models with explicitly dynamic optimizing behaviour by private agents. Another parameter responsible for chaotic dynamics is the one in the logistic map, related to the amplitude of fluctuations in the population growth rate.

Differently, the paper by Brianzoni, Mammana and Michetti, *Variable elasticity of substitution in a discrete time Solow-Swan growth model with differential saving*, assumes that the technology is described by a variable elasticity of substitution (VES) production function. Differently from CES, VES production function allows to consider that the elasticity of substitution between capital and labor can be affected by a change in the level of the capital per-capita within the economic system and also the capital accumulation and output depend on such a change. Multiple equilibria are likely to emerge, according to the parameter values. Moreover, the model can exhibit unbounded endogenous growth when the elasticity of substitution between labour and capital is greater than one, as it is quite natural while the variable elasticity of substitution is assumed (and differently from CES). Being the final system defined by a continuous piecewise map, the study involves further mathematical methods in order to assess the possibility of complex dynamics (cycles of high periods or chaotic patterns) to be exhibited. In fact, a different type of bifurcations can occur, i.e. border-collision bifurcations, which are related to the contact of an invariant set with the border separating the regions of different definition of the map.

In order to generalize the previous results, the paper by Brianzoni, Mammana and Michetti, *Local and global dynamics in a neoclassical growth model with non concave production function and non constant population growth rate*, considers sigmoidal production function. The final two-dimensional dynamic system, describing the capital per capita and the population growth rate evolution, admits two coexisting attractors, whose structure becomes more complicated when the elasticity of substitution between production factors is low enough.

2.1.2 Growth Models with Corruption in Public Procurement

Another field of research focuses on the relationship between corruption in public procurement and economic growth within the Solow framework in discrete time. To be more precise, in the paper by Brianzoni, Coppier and Michetti, *Complex dynamics in a growth model with corruption in public procurement*, the public good is an input in the productive process and the State fixes a monitoring level on corruption. After solving a one-shot game via the backward induction method, a triangular piecewise smooth dynamic system is obtained. The resulting system admits multiple equilibria with nonconnected basins, moreover the existence of a global compact attractor is proved. The analysis of the local and global bifurcations (such as border collision and contact bifurcations) shows that the model is able to exhibit chaotic fluctuations. Moreover, long run equilibria without corruption cannot exist. Finally, corruption implies endogenous instability in the economic growth, due to periodic or aperiodic fluctuations. A further paper Brianzoni et al. (2010c) studies in depth the map from a mathematical point of view and describes the bifurcation curves of the superstable cycles.

The paper by Brianzoni, Coppier and Michetti, *Multiple equilibria in a discrete time growth model with corruption in public procurement*, makes the corruption level endogenous. More in detail, firms producing the public good differ with respect to their "reputation cost" deriving the fraction of firms which produce the low–quality public good by solving a one-shot game via the backward induction method. The resulting dynamic system describes the evolution of the capital per capita and of the corruption ratio. It admits multiple equilibria. The analysis of their stability and of the structure of their basins proves that stable equilibria with positive corruption may exist (according to empirical evidence), even though the State may reduce corruption by increasing the wage of the bureaucrat or by increasing the amount of tax revenues used to monitor corruption.

More recently, the paper by Brianzoni, Campisi and Russo, *Corruption and economic growth with non constant labour force growth*, extends the previous model by introducing endogenous labour force growth (described by the logistic equation). This leads to a three dimensional continuous piecewise system, where the state variables are the capital per capita, the corruption ratio and the population growth rate. The existence of an attractor (which may be strange) is proved. Moreover, numerical simulations are performed in order to obtain measures for the policy maker to fight corruption. To this regard, the role of several parameters is analyzed. A further research inherent to growth models with corruption, which is in progress, considers the evolution of non-compliant behaviour in public procurement.

2.1.3 Asset Pricing Models with Heterogeneous Expectations

Following a different line of research, asset pricing models with heterogeneous agents have been considered. As typical in the literature, different groups of agents have different expectations about future variables. For example, the paper by Brianzoni, Cerqueti and Michetti, *A dynamic stochastic model of asset pricing with heterogeneous beliefs*, studies an asset pricing model with heterogeneous expectations where the prevision rules depend on the proportion of agents belonging to the same group, weighted by a *confidence parameter*. The new ingredient has a stabilizing effect in the dynamic behaviour, since the system is globally stable if the confidence parameter is great enough. Differently, for small values of the confidence parameter, the system shows complex dynamics. In this last case, there exists a stability region which is analyzed both in the deterministic and in the stochastic framework.

The paper by Brianzoni, Mammana and Michetti, *Updating wealth in an asset pricing model with heterogeneous agents*, focuses on the wealth dynamics when two groups of agents (with different beliefs) populate the market. In fact, at all times, the wealth of each group is updated as a consequence of the switching mechanism. As a result, the final system is defined by a nonlinear, three-dimensional and continuous piecewise map, describing the evolution of the difference in the fractions of agents, the difference in the relative wealths and the fundamental price ratio. Although the complexity of the resulting system, the authors prove the existence of two types of steady states (fundamental and non fundamental equilibria) and perform the stability analysis of the fundamental steady state. Moreover, the paper shows how complexity is mainly due to the wealth dynamics. In the framework of heterogeneous agents, as a future development, we are interested in considering different ways to model heterogeneous expectations, taking into account empirical evidences that the related literature offers.

2.2 Continuous Dynamical Systems with Delays

The introduction of time delay in differential equations has been shown to be an efficient method for the modeling of nonlinear dynamics appearing in many complex phenomena of the applied sciences. What defines a time delay system is the feature that the system's future evolution depends not only on its present value but also on its past history. In recent times there has been a renewed interest in the development and analysis of delayed mathematical models in economics due to the role played by time delays in capturing more complex dynamics, thus enriching the description of the whole system. Time delays can be modeled in many different ways. The choice of the type has situation-dependency and implies the use of different analytical methods and techniques. There are two main types of delay: fixed time delay and continuously distributed time delay. The models that require fixed time delays make use of delay differential equations, whose characteristic equation is a mixed polynomial-exponential equation with infinitely many eigenvalues. The models with continuously distributed time delays have instead dynamic equations that are Volterra type integro-differential equations, where the characteristic equation is a polynomial equation with finitely many eigenvalues. In the following, some of our contributions on the subject are briefly reviewed (see Refs. Caraballo et al. 2018; Guerrini et al. 2018, 2019; Gori et al. 2018). For future research, we propose the introduction of stochastic terms in our models.

The paper by Caraballo, Colucci and Guerrini, *Dynamics of a continuous Hénon model*, deals with a continuous time version of the Hénon map, which is one of the most studied examples of discrete dynamical systems that exhibit chaotic behavior. In absence of time delays, the model's solutions either converge to the stable steady state or diverge. Hence, the complex dynamics of the discrete model is not displayed by the continuous one. On the other hand, if time delays are introduced in the system, then the complexity of the discrete model may be recovered. Through Hopf bifurcation and stability switches analysis, the authors show the appearance of stable limit cycles with increasing period and the presence of a strange attractor that resembles the famous Hénon attractor. The last part of the work concentrates on this attractor. In particular, the existence of a trapping region (positively invariant set) and of an absorbing set is proved. For future research, it is proposed the problem of multistability, i.e. the coexistence of several local attractors for the system.

The paper by Caraballo, Colucci and Guerrini, *On a predator prey model with nonlinear harvesting and distributed delay*, analyses a two predator prey model with nonlinear harvesting (Holling type II) with both constant and distributed delay. The aim is to study the impact of harvesting on a two species community in presence of delays. The authors consider the cases of a single delay and continuously distributed delays, respectively, and then show a variety of dynamics ranging from simple cyclic oscillations to complex behavior involving chaos. Following the tradition of the study on fixed delay equations, the authors start from the local stability analysis of the steady state and then consider the question of stability switching of the fixed delay. It is found that the model may become unstable with an increase of time delay, the number of stability switches is finite, and Hopf bifurcations may emerge. The authors then turn their attention to similarity and dissimilarity of dynamics generated under the fixed time delay to those generated under the continuously distributed time delay. It is found that if the delay kernel is a weak kernel the model exhibits an attracting fixed point or an attracting limit cycle (periodic orbit) generated by Hopf bifurcation, while if the delay kernel is a strong kernel then cycles with increasing periods occur.

The paper by Guerrini, Matsumoto and Szidarovszky, *Neoclassical growth model with multiple distributed delays*, extends the neoclassical model of Solow and Swan, which has been a prototype model for analyzing long-run economic growth, by assuming two independent and distributed delays: one in the nonlinear capital accumulation through savings and the other one in the capital depreciation. This modeling of time delays allows reduction of the dynamics to a set of ordinary differential equations. According to the Routh-Hurwitz stability criterion, it is found a condition under which a stationary state loses stability and bifurcates to a cyclic oscillation. In addition, by combining analytical methods and numerical experiments, the authors reveal the dual role of time delay in destabilizing or stabilizing the economy, depending on the combination of two delays. Such duality of time delay does not appear in one delay models.

The paper by Guerrini, Matsumoto and Szidarovszky, *Delay Cournot duopoly models revisited*, generalizes a duopoly model with best reply dynamics, special case of the Cournot oligopoly model proposed in Howroyd and Russel, where sufficient condition for stability are provided in cases each firm experiences delays in implementing information on its own output (implementation delay) and in collecting information on its competitors' outputs (information delay). Two Cournot duopoly with different delays are built in a continuous time framework. The first model (model I) includes equal implementation and information delays. The second model (model II) possesses instead only the information delays, i.e. the implementation delays are assumed to be zero. The stability of these models is investigated by modern analytical techniques, such as the method of stability switches curves, and by means of sophisticated numerical methods. It emerges that the delays have the dual roles of destabilizer and stabilizer in model I, while they do not affect stability in model II.

Therefore, these delay models may explain various dynamics ranging from simple to complex behavior under Cournot competition. The following model extensions are also proposed: the case of more than two firms in the model, the introduction in the duopoly case of more alternative specifications for the delays, the assumption of nonlinear price and/or cost functions in order to make the system nonlinear.

The paper by Gori, Guerrini and Sodini, *Time delays, population, and economic development*, proposes a growth model à la Solow augmented with time delays in technology and population dynamics, where population grow evolves according to a logistic-type equation with carrying capacity positively correlated with the accumulation of physical capital. The resulting model is a system of delay differential equations with at most three steady state equilibria. The stability and bifurcations of these equilibria are analyzed and the emergence of complex behaviors is demonstrated. More specifically, by making use of a mixture between analytical and numerical tools, the authors illustrate how the Solow model becomes able to explain the convergence towards a high equilibrium or a Malthusian trap as well as long-term fluctuations in income and population. This paper thus illustrates how the Solow model may offer insight into the understanding of some transmission mechanisms between economic and demographic variables, that are often difficult to identify in growth models belonging to the Unified Growth Theory due to the complexity of their structure.

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