

Chapter 12

A Socio-Constructivist Elaboration of Realistic Mathematics Education



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Abstract This chapter describes a socio-constructivist elaboration of Realistic Mathematics Education (RME) that emerged from my collaboration with Paul Cobb and Erna Yackel. It is argued that RME and socio-constructivism are compatible and complement each other. Socio-constructivism points to the critical role of the classroom culture, while RME offers a theory on supporting students in (re-)constructing mathematics. Furthermore, the role of symbols and models is discussed, which was considered problematic in constructivist circles, while being central in RME. The emergent modelling design heuristic is presented as a solution to this puzzle. Together, guided reinvention, didactical phenomenology, and emergent modelling, are combined to delineate RME as an instructional design theory. This is complemented by a discussion of pedagogical content tools as counter parts of the emergent modelling and guided reinvention design heuristics at the level of classroom instruction. Finally, research on student learning and enactment of RME in Dutch classrooms is discussed.

12.1 Introduction

When Freudenthal (1971) coined his adage of mathematics as a human activity, concrete elaborations of what that would mean in practice still had to be worked out. This became one of the main tasks of the IOWO,¹ the predecessor of the current Freudenthal Institute. In the 1980s, Treffers took stock of what had been developed up to then and construed the Realistic Mathematics Education (RME) theory by generalising over the characteristics the prototypical instructional sequences and local instruction theories that were available had in common. This resulted in the publication of a

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framework for a domain-specific instruction theory for RME (Treffers, 1987). The RME approach did fit very well in the broader trend in the international mathematics education community which emerged around that time. This concerned the general recognition of the importance of students' active constructive role in appropriating new mathematics, and an emphasis on applicability. RME did not only share similar starting points, it also offered a fitting theory on how to support students in constructing mathematics; moreover, this was accompanied by various concrete elaborations. In this respect RME arrived at the right time. This undoubtedly contributed to the international interest in this approach—which in turn led to collaborative projects in various countries. This was, however, not a one-way stream; international collaborations influenced RME as well. Collaborative projects in the United States, for instance, brought researchers from the Freudenthal Institute in contact with (socio-)constructivism, which influenced their thinking about RME to a greater or smaller extent. Especially my ten-year collaboration with Paul Cobb, Erna Yackel and colleagues was marked by a mutual influence (see also Cobb, with Gravemeijer & Yackel, 2011). This resulted in a new elaboration of RME, which we will denote as, 'a constructivist elaboration of RME'.

This constructivist version, which emerged next to the original RME theory, will be the theme of this chapter. We will start by looking into the compatibility of the underlying conceptual positions of RME and socio-constructivism. This will be followed by a discussion of the implications of the socio-constructivist collective perspective as elaborated by Cobb and Yackel (1996) for RME. Next, we will discuss the apparent contrariness of the views on the role of symbols and models, and how those positions were reconciled in the emergent modelling design heuristic. Subsequently we will discuss how the socio-constructivist perspective can illuminate the complexity of enacting RME in everyday classrooms. We will complement this with recent investigations of the state of affairs in Dutch classrooms.

12.2 Conceptual Compatibility of (Socio-)Constructivism and Realistic Mathematics Education

In the early stages, some protagonists of RME quickly acknowledged the compatibility of constructivism and RME. Some, however, were more reluctant. Freudenthal (1991) was actually very negative about constructivism. He rejected Von Glasersfeld's critique on the status of objective scientific knowledge by pointing out the achievements of science. He argued that we should not focus on the philosophy of science, but on the philosophy of education, and stated:

I cannot see any bond between mathematics instruction on the one hand and an alleged or assumed lack of faith in objective mathematical knowledge on the other hand, whether it is called constructivism or anything else (Freudenthal, 1991, pp. 146–147).

At the time that Freudenthal expressed this critique, there were various ideas around on what the radical constructivist position would mean for education. Including the

position that teachers should not interfere because of the risk of endangering the students' own constructive activity. Gradually, however, a more pragmatic standpoint won out. In this process, Paul Cobb was very influential. While his theoretical perspective evolved towards a more pragmatic stance; from radical constructivism towards socio-constructivism. He saw great value in what Putnam (1987, cited by Cobb, 2001) denoted 'pragmatic realism' (Cobb, 2001). And he contended that, whereas radical constructivism claims that it is impossible to bridge the gap between one's own knowledge and some pre-given external reality, pragmatic realism questions this focus on the dichotomy between this external reality and our personal knowledge. Instead of focusing on an unknowable pre-given external reality, we should focus on the realities which people experience. This pragmatic realism is clearly compatible with Freudenthal's conception of reality, which he does not link to some pre-existing external reality, but to one's self-constructed experiential reality: "I prefer to apply the term 'reality' to what at a certain stage common sense experiences as real" (Freudenthal, 1991, p. 17).

Cobb (1994a) himself made a nice connection between the two views when pointing out that (socio-)constructivism is not a pedagogy. He argued that if it is true that people always construct their own knowledge, then students will do so in every classroom—even with direct instruction. The issue, he went on to say, is not *whether* they construct, but *how* and *what* they construct. The question therefore is: What do we want mathematics to be for the students? Cobb (1994a) concluded that a potential answer to that question was in Freudenthal's notion of mathematics as a human activity.

12.3 A Socio-Constructivist Perspective on Teaching and Learning

A shared belief in the compatibility of (socio-)constructivism and RME formed the basis for a ten-year collaboration between Cobb cum suis and me, in which we further elaborated RME theory while working on a series of classroom design experiments. The starting point was that RME and socio-constructivism are not only compatible, but also complemented each other. On the one hand, socio-constructivism offers a background theory for RME, and, more importantly, adds a collective perspective. On the other hand, RME offers an instructional design theory that aims to support students in constructing mathematics.

Adopting a socio-constructivist view implies a collectivist perspective on teaching and learning, which situates the students' activity within the classroom community, whereas RME originally tended to a more individual, psychological, perspective, even though the roles of interaction and collaboration between students were of course acknowledged. Socio-constructivism offers an important addition in that it focuses attention on the crucial role of the classroom culture in the enactment of RME in the classroom. To analyse the situated activity of students, Cobb and Yackel

(1996) developed an interpretative framework, denoted ‘emergent perspective’, in which they try to coordinate a social and a psychological perspective. The former involves the norms and practices of the classroom community. The latter focusses on individual students’ reasoning, and concerns beliefs of students and teacher. Cobb and Yackel (1996) discern classroom social norms, socio-mathematical norms, and practices. The social norms describe the expected ways of acting and explaining in a given classroom. They elucidate that the social norms are reflexively related to the students’ and the teacher’s beliefs about their obligations, which are shaped by the classroom’s history. Typical social norms of the traditional mathematics classroom are that students are expected to try to come to grips with the knowledge and procedures presented by the teacher and the textbook. The teacher’s role is to explain and clarify, and the students’ role is to try to figure out what the teacher has in mind and act accordingly. RME asks for different social norms, which in line with Cobb and Yackel (1996) encompass the obligations for students to come up with their own solutions, explain and justify their solutions, to try to understand the explanations and solutions of their peers, to ask for clarification when needed, and eventually to challenge the ways of thinking with which they do not agree. The teacher’s role is not to explain, but to pose tasks, and ask questions that may foster the students’ thinking, and help them in this manner to build on their current understanding and to construe more advanced mathematical insights.

This recognition of the need for fitting social norms has significant consequences for putting RME into practice. It signals the need for changing the social norms, which in turn asks for changing the individual beliefs of the students. It also highlights that students’ beliefs about their role and that of the teacher are formed by experience. In traditional classrooms students are used to being rewarded for reproducing the teacher’s reasoning and procedures, and the belief that this is what is expected from them will not change unless they gain compelling new experiences. This takes some conscious effort (Cobb & Yackel, 1996). To establish new social norms, the teacher has to show that what is valued and what is rewarded has changed.

In addition to the general classroom social norms, Cobb and Yackel (1996) discern socio-mathematical norms and mathematical practices. The socio-mathematical norms refer to what mathematics is and what it means to do mathematics in a given classroom. For example, what counts as a mathematical problem, what counts as a mathematical solution, and what counts as a mathematical argument. We may link those socio-mathematical norms with the notion of ‘mathematical interest’ (Gravemeijer & Cobb, 2013). To engage in the activity of mathematising vertically, students will have to develop an interest in mathematical aspects of their solutions. Teachers may cultivate mathematical interest by asking questions such as: What is the general principle here? Why does this work? Does it always work? Can we describe it in a more precise manner?

With mathematical practices Cobb and Yackel (1996) refer to taken-as-shared ways of acting and reasoning, which may evolve over time. The mathematical practices are reflexively related to individual students’ mathematical conceptions. They

speak of an established mathematical practice when certain ways of acting and reasoning are no longer challenged by individual students. This does not necessarily mean that all student's conceptions and actions correspond with that practice. Mathematical practices do, however, offer a means of identifying and describing the progress of the classroom as a whole.

The emergent perspective offers an important addition to RME in that it reveals that a certain classroom culture has to be put in place in order to allow for guided reinvention; an aspect that had not yet been articulated in Treffers' (1987) theory. Moreover, it highlights the reflexive relation between the individual's interpretations and constructions and the norms and practices of the classroom community.

Mark that we may look at the relation between Cobb and Yackel's perspective and RME theory in two ways: We may consider the emergent perspective an integral part of a socio-constructivist take on RME, or conceive of the emergent perspective as describing a necessary requirement—as enacting RME is not possible if the students adhere to traditional school-mathematics social norms. Whichever one chooses, socio-constructivism offers a significant expansion of or addition to RME theory. We may note, however, that conversely RME offers a significant addition to socio-constructivism by offering an instruction theory for supporting students in constructing mathematical knowledge. Further, building on both an RME and a socio-constructivist perspective proved especially fruitful in the domain of symbols and tools, a development we will discuss in the following.

12.4 Symbolising and Modelling

Initially there was a strong wariness among socio-constructivist scholars regarding the use of external representations. This was supported by research in contemporary mathematics classrooms, which had shown that students often could not make sense of the symbolic representations introduced by the teacher (see, e.g., Cobb, 1994b). Broadly speaking, the use of tactile models and visual representations was associated with the transmission model of teaching, in which tacit and visual models were treated as powerful means of supporting learning for understanding. By acting with well-designed concrete models, students were expected to discover the mathematics that was embedded in the models. In relation to the latter, Cobb, Yackel, and Wood (1992) speak of a representational view. They argue that mathematics educators, who use tactile models and visual representations in this manner, implicitly or explicitly hold the view that learning is characterised as a process in which students construct mental representations that mirror the mathematical features of external representations. The problem with this approach, however, is that the meaning of external representations is dependent on the knowledge and understanding of the interpreter. This creates the problem, known as 'the learning paradox' (Bereiter, 1985), that can be captured by the following question: How is it possible to learn the symbolisations, which you need to come to grips with new mathematics, if you have to have mastered this new mathematics to be able to understand those very symbolisations?

The underlying problem, Cobb et al. (1992) argue, is that mathematics educators experience mathematics as an objective body of knowledge, which is mirrored by the external representations they use to make the corresponding mathematics accessible for students. This presupposes an objective body of knowledge that exists independently of some agent. According to socio-constructivist theory, however, knowledge has to be constructed by an actor, and cannot be separated from the knowing individual. Thus, for those who have not yet constructed the more sophisticated mathematical knowledge that has to be learned, this body of more sophisticated mathematical knowledge, literally, does not exist, and thus cannot be conveyed by external representations.

12.4.1 Emergent Modelling

However, while constructivist scholars were wary of symbols and models and pointed to the learning paradox, RME relied heavily on the use of models, model situations, and schemata, as is indicated for instance in Treffers' (1987) characterisation of progressive mathematisation. Consequently, the need arose to reconcile the two conceptions of the role of symbols and models. A beginning of an answer could be found in Treffers' (1987) elucidation that in the RME approach, models etcetera, rather than being offered right away, arise from problem-solving activities. In this manner, Treffers' model characteristics (1987) pointed to a dynamic aspect that could be explicated and elaborated as an explicit design heuristic offering a way to circumvent the learning paradox. We should also refer to Streefland (1985) who argued that by modelling reality you create a model of that reality—which he calls an 'after-image' ('nabeeld' in Dutch). This after image may foster reflection, which in turn may lead to the insight that the model can be used for other problem situations. The model has become a 'pre-image' ('voorbeeld' in Dutch) that is used for reasoning about other situations (which he in later publications expands with supporting abstracting and level raising (Streefland, 1992, 1993).

The constructivist concerns about the role of models and the associated learning paradox are eventually addressed by the design heuristic that originated from noticing a shift in the thinking of students using the empty number line (Gravemeijer, 1991). It showed that the students initially used calculations that closely matched the situation in the contextual problem, but later on started to come up with solutions that were based on number relations and were only indirectly connected with the context. This implied that the number line had acquired a new meaning for the students; it started to signify number relations. This insight led to the rationale underlying the emergent modelling heuristic, that the learning paradox dissolves when one adopts a more dynamic view of learning in which mathematical symbols and models are developed in a bottom-up manner. The latter appeared to agree with Meira's (1995) observation that in the history of mathematics, symbols did not suddenly appear in their full-fledged form. Instead, these symbols grew out of informal, situated, forms

of symbolising that developed over time in a reflexive process in which symbolisations and meaning co-evolved. Following Meira (1995), we may envision a dynamic process in which symbolisations and meaning co-evolve, and in which the ways that symbols are used and the meanings they come to have, are seen to be mutually constitutive. It showed that a similar pattern could be found in many prototypical RME instructional sequences, such as Van den Brink's (1989) design for addition and subtraction, Streefland's (1990) work on fractions, and the various sequences for the written algorithms (Treffers, 1987). The idea of a dynamic process in which symbolisations and meaning co-evolve has been elaborated in the emergent modelling design heuristic. Here the label 'emergent' refers both to the character of the process by which models emerge, and to the process by which these models support the emergence of more formal mathematical conceptions.

According to the emergent-modelling design heuristic, the model first comes to the fore as a *model of* the students' situated informal strategies. In subsequent activities, the role of the model begins to change. As the students gather more experience with similar problems, their attention may be directed to the mathematical relations involved. In this manner, the students start to develop a network of mathematical relations. This changes what the model signifies for the students. Instead of deriving its meaning from activity in the context in which the problem is situated, the model starts to derive meaning from the mathematical relations involved. Consequently, the model becomes more a base for more formal mathematical reasoning than a way of representing a contextual problem. In other words: the *model of* informal mathematical activity develops into a *model for* more formal mathematical reasoning. We should add that, although we speak of 'the model', the model we are referring to is more an overarching conception, than one specific model. In practice, 'the model' in the emergent-modelling heuristic is actually shaped as a series of consecutive *sub-models* that can be described as a cascade of inscriptions (Latour, 1990) or a chain of signification (Roth & McGinn, 1998). Key here is that acting with each new inscription signifies the earlier activity with the preceding inscriptions for the students. Mark, however, that the series of symbolisations is invented by the instructional designer, not by the students. To adjust for this, one may try to ensure that each new tool/symbolisation emerges as a solution to a problem that has its roots in activity with the earlier symbolisation. In this manner, the history of working with the earlier symbolisation may provide the imagery underlying the new tool. Whether this is the case, may be inferred from whether or not the new symbolisation is used flexibly by the students.

From a more global perspective, the symbolisations can be seen as various manifestations of some overarching model that evolves from a 'model of' situated activity to a 'model for' more formal mathematical reasoning. In relation to this, we may discern four different types or levels of activity (Gravemeijer, 1999):

- (1) *Situated activity* in a task setting that is experientially real for the students
- (2) *Referential activity*, in which models refer to activity in the task setting
- (3) *General activity*, in which models refer to a framework of mathematical relations

- (4) *Formal mathematical reasoning* which is no longer dependent on the support of models-for mathematical activity.

These four levels of activity illustrate that the students' understanding of models is grounded in their understandings of paradigmatic, experientially real settings. At the level of referential activity, the models are meaningful to the students because they refer to situated activity in the task setting. General activity begins to emerge when the students start to reason about the mathematical relations that are involved. In this manner the students develop a network of mathematical relations. Consequently, the model starts to lose its dependency on situation-specific imagery, and gradually develops into a model that derives its meaning from the emerging framework of mathematical relations. In this manner the model starts to function as a model for more formal mathematical reasoning.

The transition from model-of to model-for coincides with a progression from informal to more formal mathematical reasoning that is interwoven with the creation of some new mathematical reality—consisting of mathematical objects (Sfard, 1991) within a framework of mathematical relations. Thus, the model-of/model-for transition is not tied to specific manifestations of the model, instead, it relates to the student's thinking, within which 'model-of' refers to an activity in a specific setting or context, and 'model for' to a framework of mathematical relations. Mark that the constitution of a framework of mathematical relations—and thus some new mathematical reality—is an essential element of the emergent modelling design heuristic. In this respect, it differs from a modelling conception in which a model of a contextual problem is generalised in order to function as a model for solving similar problems in other contexts. We may add that model-of/model-for transition in the emergent modelling design heuristic has to be understood in a metaphorical sense. Central is the series of symbolisations or sub-models, which together constitute 'the model', which may or may not be placed under one label—such as the notion of a ruler as the overarching model in the measurement annex number-line sequence (Stephan, Bowers, & Cobb, with Gravemeijer, 2003).

Let us briefly return to the aim of supporting students in developing a framework of mathematical relations and the corresponding mathematical objects, which is experienced as some new mathematical reality. This experienced reality corresponds with the perceived body of mathematical knowledge that we identified as the central problem when discussing the learning paradox. Thus, instead of trying to help students to make connections with a mathematical reality that does not exist for them; the emergent modelling approach helps students in constructing such a mathematical reality by themselves. This focus on the constitution of mathematical objects and a framework of mathematical relations also signifies a deviation from Treffers' conception of RME theory, since he tends to characterise students' mathematical development in terms of students' development of increasingly sophisticated solution methods (see, e.g., Treffers, 1991).

12.5 RME in Terms of Instructional Design Heuristics

The conception of emergent modelling as an instructional design heuristic allowed for an alternative description of RME theory in terms of instructional design heuristics by combining it with guided reinvention and didactical phenomenology (Gravemeijer, 2004).

12.5.1 *Emergent Modelling Heuristic*

We already discussed the emergent-modelling design heuristic above, but we may add that this heuristic has been used in design research projects on a variety of topics, such as addition and subtraction up to 20 (Gravemeijer, Cobb, Bowers, & Whitenack, 2000), addition and subtraction up to 100 (Stephan et al., 2003), written algorithms for addition and subtraction (Bowers, 1995), integers (Stephan, & Akyuz, 2012), data analysis (Gravemeijer & Cobb, 2013), algebraic functions (Doorman, Drijvers, Gravemeijer, Boon, & Reed, 2012), calculus (Doorman, 2005), and differential equations (Rasmussen, 1999).

12.5.2 *Guided Reinvention Heuristic*

When elucidating the principle of guided reinvention, Freudenthal (1973) suggested the instructional designer should look at the history of mathematics to see how certain mathematical practices developed over time. The designer is advised to especially look for potential conceptual barriers, dead ends, and breakthroughs. These may be taken into account when designing a potential reinvention route. Streefland (1990) developed a second guideline, which suggests that the informal interpretations and solutions of students who do not know the applicable mathematics might ‘anticipate’ more formal mathematical practices. If so, students’ initial informal reasoning can be used as a starting point for the reinvention process. In summary, the designer may take both the history of mathematics and students’ informal interpretations as sources of inspiration for delineating a tentative, potential route along which reinvention might evolve.

As a special point of attention, we may note that reinvention has both an individual and a collective aspect; it is especially the interaction between students that is to function as a catalyst. The designer has to develop instructional activities that are bound to give rise to a variety of student responses. What is aimed for is a variety in responses that to some extent mirrors the reinvention route. When some students come up with more advanced forms of reasoning than others, teachers can exploit these differences. They can try to frame the mathematical issue that underlies those differences as a topic for discussion (Cobb, 1997). In orchestrating such a discussion,

they can then advance the reinvention process. Mark that without such differences, the teacher will not have a basis for organising a productive classroom discussion, and will have to refer to soliciting preferred responses by asking leading questions. We may further observe that in line with the emergent modelling heuristic, the end points of a guided reinvention process are typically cast in terms of mathematical objects and frameworks of mathematical relations in the context of a constructivist elaboration of RME.

12.5.3 Didactical Phenomenology Heuristic

The third RME design heuristic concerns the didactical phenomenological analysis, or didactical phenomenology for short (Freudenthal, 1983). Here the word ‘phenomenological’ refers to a phenomenology of mathematics. In this phenomenology, the focus is on how mathematical ‘thought-things’ (which may be concepts, procedures, or tools) organise—as Freudenthal (1983) puts it—certain phenomena. Knowing how certain phenomena are organised by the thought thing under consideration, one can envision how a task setting in which students are to mathematise those phenomena may create the need to develop the intended thought thing. In this manner, problem situations may be identified, which may be used as starting points for a reinvention process. Note that such starting-point-situations may also be used to explore students’ informal strategies as Streefland (1990) suggests. To find the phenomena that may constitute starting-point-situations, we may look at applications of the concept, procedure or tool under consideration. Assuming that mathematics has emerged as a result of solving practical problems, we may presume that the present-day applications encompass the phenomena which originally had to be organised. Consequently, the designer is advised to analyse present-day applications in order to find starting points for a reinvention route. Mark, however, that, as the students progress further in mathematics, applications may concern mathematics itself. Essential for valuable starting points is that they are experientially real for the students, that they concern situations in which the students know how to act and reason sensibly. An additional function of a phenomenological analysis is that it allows for construing a broad phenomenological base, which may both strengthen and enrich the experiential real foundation and foster the applicability of the concepts, procedures, or tools under consideration.

12.6 Pedagogical Content Tools

We may complete this exposition on instructional design heuristics with a discussion of the ‘pedagogical content tools’ (PCTs) that have been put forwards by Rasmussen and Marongelle (2006) as instructional counter parts of the design heuristics of emergent modelling and guided reinvention. They define a pedagogical content tool as, “a

device, such as a graph, diagram, equation, or verbal statement that a teacher intentionally uses to connect student thinking while moving the mathematical agenda forward” (Rasmussen & Marongelle, 2006, p. 388). They describe two PCTs, ‘transformational records’ and ‘generative alternatives’, which in their view address the problem of how teachers can proactively support their students’ learning. Sometimes the design heuristics are too general in their view. Transformational records, which are seen as the instructional counter part of the emergent modelling heuristic, are defined as graphical representations that emerge as ways to record student thinking, which are later used by students to solve new problems. As an example, they discuss an episode of a classroom on differential equations, which starts with the task of making predictions about the shape of a population versus time (P versus t) graph for a single species that reproduces continuously and has unlimited resources. The teacher started the discussion by asking whether the initial slope at $P = 10$ and $t = 0$ should be zero or positive. During this discussion—in which the students adhered to the classroom social norms that they were expected to explain and justify their solutions, and try to understand their peers—most of the students began to realise that the slope had to be positive. Thereupon the teacher drew a tangent line vector with a positive slope as “a notational record of the taken-as-shared reasoning of the classroom community” (Rasmussen & Marongelle, 2006, p. 396). In a similar manner, the notational record was supplemented with some more vectors. This extended notational record was used as a means of support by the classroom community, when sorting out whether the rate of change function would depend only on the size of the population, or also on the time. The teacher, in short, took a proactive role in reshaping the initial record, while supporting the students in developing a line of reasoning that corresponded with what an expert in the subject would recognise as an emerging tangent-vector field. He did so in such a manner that he at the same time cultivated the social norms of an inquiry classroom by initiating, and building on, whole class discussions.

The generative alternatives are linked to the notion that guided reinvention tries to find a position between too much and too little guidance. Here one of the examples concerns a problem about salt water—containing 1 lb salt per gallon—that is pumped into a tank at a rate of 2 gallons a minute. The students came up with two different ideas about the rate of change, which should be 2, according to some, or $2t$, according to others. By framing the justification of one of both as topic for a whole-class discussion, the teacher fostered the social norms, as the students were expected to explain and justify their reasoning and try to make sense of others’ reasoning. When the students started to lean towards $2t$, the teacher realised that the students were not making a conceptual distinction between rate of change in the amount of salt and the amount of salt. He then assumed more responsibility for the content and the direction of the discussion by pointing out that after t minutes $2t$ pounds of salt are flowing into the tank, and asking: Is that the rate of change? In the then unfolding discussion the students start to realise that the *amount* of salt after 2 min is $2t$ (pounds), whereas the *rate of change* is 2 (pounds per minute). The authors point out that what makes this an example of a generative alternative is not just that two alternatives, t and $2t$, were discussed. Key here is that this discussion advanced the mathematical idea

of the explicit distinction between the rate of change in a quantity and the quantity itself.

With the pedagogical content tools, we have moved from RME theory to RME in the classroom. We will discuss the latter more extensively in the following.

12.7 RME and Classroom Practice

As is already noted above, the socio-constructivist perspective reveals the complexity of enacting RME in everyday classrooms. And we may conclude that this is more difficult than the initiators in the Netherlands were aware of. One of the hurdles concerns classroom culture. There was, and to a large extent still is, a lack of awareness of the need to invest in changing the classroom social norms. Another difficulty concerns the need to anticipate and build on the students' thinking. Following Simon's (1995) line of reasoning, teachers have to ascertain the students' level of reasoning and design or choose instructional activities that support students in expanding, and building on, their current ways of thinking. They have to develop hypothetical learning trajectories (HLTs), which involve anticipating the mental activities of the students when they engage in the envisioned tasks, and considering how these relate to the learning goals. This requires teachers to have a sound understanding of the rationale that underlies the instructional sequences they are working with. Usually, however, teachers are insufficiently informed about the local instructional theories that underpin the instructional sequences. Moreover, they are not schooled in thinking about the mental activities of students.

We may add that the students have to play their part as well. We already mentioned the classroom norms, but knowing that they are expected to think for themselves, explicate their thinking, etc., does not necessarily mean that they are willing to do so. An inhibiting factor may be the ego-orientation (Nicholls, Cobb, Wood, Yackel, & Patashnick, 1990) of some students. This includes being more concerned about how one looks in the eyes of one's peers, than about solving the task at hand. Fear of failure may keep those students from starting to work at a challenging task. Teachers, therefore have to invest in fostering a task-orientation (*ibid.*), the willingness to work on mathematical tasks. Important in this respect is that teachers refrain from judging students by external standards, or comparing them with their classmates, and instead promote that students take their personal progress as an evaluation criterion. This is of course hard to achieve with the current emphasis on testing.

12.8 Recent Research on Instructional Practice in the Netherlands

Following on the discussion of theory on enacting RME we may ask ourselves how RME actually works out in Dutch classrooms. This question is actually in line with a discussion that is going on in the Netherlands about the quality of mathematics education. This discussion was evoked by the results of the national assessments, known as PPON.² The PPON survey of 2004 showed a significant decline in the mastery of (procedures for handling) whole number addition, subtraction, multiplication, and division. However, the results did not decline across the board; the results on various other topics showed improvement instead. A comparison of consecutive PPON surveys (Janssen, Van der Schoot, & Hemker, 2005) showed a positive effect on a number of topics that RME innovators deem important (Van den Heuvel-Panhuizen, 2010). We may further argue that national assessments, and also international assessments—on which the Netherlands were, and are, still doing very well—are too crude instruments to come to grips with what is going on in mathematics education. This kind of considerations gave rise to three independent Ph.D. studies, which investigated the proficiency of Dutch students on specific topics, respectively addition and subtraction up to 100 (Kraemer, 2011), fractions (Bruin-Muurling, 2010), and algebra (Van Stiphout, 2011). Analysing the results of those three Ph.D. studies, Gravemeijer, Bruin-Muurling, Kraemer and Van Stiphout (2016) found that Dutch students' proficiency fell short of what might be expected of reform in mathematics education that targets conceptual understanding. In each of those three cases this appeared to be caused by a deviation from the original intentions of the reform. Firstly, the textbooks capitalised on procedures that can quickly generate correct answers, instead of investing in the underlying mathematics while accepting that fluency may come later. In relation to this, the authors speak of “task propensity”, “the tendency to think of instruction in terms of individual tasks that have to be mastered by students” (ibid., p. 26). Secondly, there was an overall lack of attention for more advanced conceptual mathematical understandings in Dutch textbooks. Instructional sequences in the textbooks end too early, before the more advanced conceptual goals are reached. What is missing from the instructional sequences is the phase that Sfard (1991) denotes as reification. The students are not supported in constructing mathematical objects. The other reason they bring to the fore is that more advanced conceptual mathematical understandings are not formulated as instructional goals, not in the textbooks, nor in official curriculum documents. They plead for changing the usual goal descriptions in curriculum documents by identifying more advanced conceptual mathematical understandings as key curriculum goals.

²Periodieke Peiling van het Onderwijsniveau (Periodic Assessment of the Education Level).

12.9 Conclusion

We started our discussion of the socio-constructivist elaboration of RME with the question of the compatibility of RME and socio-constructivism, and we concluded that on a meta level both positions are well compatible. We showed that adopting the collectivist perspective that is inherent to socio-constructivism especially has consequences for how we think about enacting RME in the classroom. Establishing social norms that encompass student responsibility for coming up with their own solutions and discussing these and those of other students, is a prerequisite for enacting RME. We further found that a potential irreconcilable difference concerned the different views on the role of symbolising and modelling, which created the need to reconcile the two positions. In relation to this we discussed the emergent modelling design heuristic, which is designed to circumvent the so-called learning paradox. The emergent modelling heuristic tackles the concern of socio-constructivists that symbols do not come with an inherent meaning, by ensuring that symbolisations and meaning co-evolve in a reflexive process, while at the same time supporting the construction of some new mathematical reality, which may be thought of as consisting of mathematical objects that derive their meaning from a network of mathematical relations. The heuristic may be characterised as a transition from a model of the students' situated informal strategies to a model for more formal mathematical reasoning. But the pith of the matter concerns (a) the sequence of sub-models that together form a chain of signification, in which activity with each new sub-model signifies activity with the earlier sub-model, and (b) the construction of a framework of mathematical relations by the students. We observed that by acknowledging that guided reinvention and didactical phenomenology also can be seen as instructional design heuristics, allows an alternative manner of describing RME theory—in which RME theory is described in terms of instructional design heuristics.

When turning to the classroom practice we of course reiterated the importance of the classroom culture. We also highlighted that the constructivist elaboration of RME entails a shift in attention from the instructional sequence with a rationale or local instruction theory that underpins it, to the local instruction theory with a series of instructional activities that can be used a resource. For the constructivist position that students construct their own knowledge implies that teachers have to adjust their teaching to the students' thinking. This means that teachers have less use for ready-made instructional sequences, but instead need at their disposal knowledge about the intended learning process and the possible means of supporting that learning process, or about local instruction theories. On the basis of this, teachers may develop hypothetical learning theories (Simon, 1995), which put the mental activities of the students at the centre of the teachers thinking. Given the results we reported in the last section, we may argue that the local instruction theories the teachers are to be provided with have to target more advanced conceptual mathematical understandings. The latter should also be worked as goals in national curriculum documents.

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