Chapter 3
Matter Multiplets

Abstract After the Weyl multiplet is introduced, we can now define matter multiplets whose transformations respect the algebra with structure functions that depend on the fields of the Weyl multiplet. We treat here vector multiplets and hypermultiplets. We define them for $D = 4$, 5 and 6, first for rigid supersymmetry and then for the superconformal theory. In the second part of this chapter we define actions for these multiplets, which will be the basis for the further chapters.

The goal of this chapter is to construct local superconformal actions for the matter multiplets, exploiting our knowledge of the Weyl multiplet. In principle there are many representations of the superconformal algebra that define matter multiplets. The physical theories for $D = 4$ can all be obtained with vector multiplets and hypermultiplets. For $D = 5$ and $D = 6$ tensor multiplets can lead to inequivalent theories. For $D = 5$, this has been included in the treatments of [1–5], to which we will come back to this in Sect. 6.2. One might also prefer formulations in terms of other multiplets to make connections with other descriptions, e.g. in string theory. We will briefly discuss the $D = 4$ tensor multiplet in a superconformal background [6] in Sect. 3.2.5. The action with one tensor multiplet was given in [7] and extended to more multiplets in [8]. On-shell matter couplings using different formalisms have been given in [9–11]. Recently [12], also the (off-shell) coupling of one tensor multiplet to an arbitrary number of vector multiplets has been obtained.

The main focus of this chapter will be on vector and hypermultiplets. Importantly, the latter will be used not only as physical multiplets, but also as compensating multiplets to describe super-Poincaré theories with matter couplings. This is in the spirit of the general strategy outlined in Sect. 1.2.2 that we review in Sect. 3.1.

The remainder of this chapter is split in two parts. In Sect. 3.2 we explain the structure of first vector and then hypermultiplets and their embedding in the superconformal algebra. The construction of actions is postponed to the second part, Sect. 3.3. We explicitly construct the superconformal invariant actions for sets of these multiplets, which will be combined in Chap. 4 by the gauge fixing to Poincaré supergravity. Many parts of this chapter, especially for the case of $D = 4$, have been obtained in the context of the master thesis of De Rydt and Vercnocke [13].

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3.1 Review of the Strategy

In Sect. 1.2.2, we already outlined the general idea of the superconformal construction for actions with super-Poincaré invariance. At that time, we had not yet explained the gauging of the conformal algebra. Now we can be more precise. For this example, we will still restrict to the bosonic case. Consider a scalar field ϕ with Weyl weight w and no intrinsic special conformal transformations: $k_\mu(\phi) = 0$. Its superconformal covariant derivative is

$$\mathcal{D}_\mu \phi = \left(\partial_\mu - w b_\mu\right) \phi \,. \tag{3.1}$$

The transformation of the covariant derivative $\mathcal{D}_a \phi$ can be easily obtained from the 'easy method' (Sect. 2.3.4). One takes into account (2.25) to find that there is a K transformation. The transformation law of a covariant derivative determines the covariant box

$$\begin{aligned} \square^C \phi \equiv \eta^{ab} \mathcal{D}_b \mathcal{D}_a \phi &= e^{a\mu} \left(\partial_\mu \mathcal{D}_a \phi - (w+1) b_\mu \mathcal{D}_a \phi + \omega_{\mu\, ab} \mathcal{D}^b \phi + 2w f_{\mu a} \phi\right) \\ &= e^{-1} \left(\partial_\mu - (w + 2 - D) b_\mu\right) e g^{\mu\nu} \left(\partial_\nu - w b_\nu\right) \phi - \frac{w}{2(D-1)} R\, \phi . \end{aligned} \tag{3.2}$$

We use here the constraint (2.72) (without matter for the pure bosonic case). The last term is the well-known $R/6$ term in $D = 4$. In fact, choosing $w = \frac{D}{2} - 1$, one has a conformal invariant scalar action

$$S = \int \mathrm{d}^D x\, e \phi \square^C \phi \,. \tag{3.3}$$

Exercise 3.1 Show that $\int \mathrm{d}^D x\, e \mathcal{D}_a \phi\, \mathcal{D}^a \phi$ is not a special conformal invariant, while $\square \phi$ is invariant under K iff $w = \frac{D}{2} - 1$. □

In order to obtain a Poincaré invariant action, we have to break dilatations and special conformal transformations (as these are not part of the Poincaré algebra). Considering (2.25), it is clear that the latter can be broken by a gauge choice

$$K-\text{gauge}: \qquad b_\mu = 0 \,. \tag{3.4}$$

One could take as gauge choice for dilatations a fixed value of a scalar ϕ. As a consequence, the action (3.3) reduces to the Poincaré gravity action: only the frame field of the 'Weyl multiplet' (which was in the background) remains.

The lesson to learn is: once the gauge for the superfluous symmetries in the matter action is fixed, without considering any action for the Weyl multiplet, we find kinetic terms for the gravity sector.

We can schematically summarize this procedure in the following diagram:

$$\begin{array}{ll} \text{Weyl multiplet: } e_\mu{}^a,\ b_\mu \text{ (Background)} & \\ + & \\ \text{matter field: } \phi & \\ \downarrow & \text{gauge fixing } K_a,\ D \\ \text{Poincaré gravity } e_\mu{}^a\,, & \end{array} \tag{3.5}$$

namely we introduce, in the background of the Weyl multiplet, the conformally invariant action of a matter field ϕ and we fix the gauge to get the action of Poincaré gravity. In the above scheme, the field ϕ provides the *compensating field* degree of freedom that makes the combined field gauge equivalent to an irreducible multiplet of Poincaré gravity. We remark that, at the classical level, every gauge fixing is equivalent to redefinitions of the fields. In this case, defining (the conformal invariant)

$$\tilde{g}_{\mu\nu} = g_{\mu\nu}\phi^{4/(D-2)}\,, \tag{3.6}$$

and writing the action in terms of $R(\tilde{g})$, the field ϕ disappears from the action

$$S = -\frac{D-2}{4(D-1)}\int \mathrm{d}^D x\,\sqrt{\tilde{g}}\,R(\tilde{g})\,. \tag{3.7}$$

The absence of ϕ from the action above is just a consequence of dilatational invariance

$$\int \mathrm{d}^D x\,\left[\frac{\delta S(\tilde{g},\phi)}{\delta \tilde{g}_{\mu\nu}(x)}\delta_{\mathrm{D}}\tilde{g}_{\mu\nu}(x) + \frac{\delta S(\tilde{g},\phi)}{\delta\phi(x)}\delta_{\mathrm{D}}\phi(x)\right] = 0\,, \tag{3.8}$$

which, together with $\delta_{\mathrm{D}}\tilde{g}_{\mu\nu} = 0$, implies $S(\tilde{g},\phi) \equiv S(\tilde{g})$.

3.2 Conformal Properties of the Multiplets

Having the Weyl multiplet, the further step now is to introduce other multiplets in the background of the Weyl multiplet. The resulting algebra, which depends for part on the fields of the Weyl multiplet, is fixed for what concerns the superconformal transformations. On the other hand, extra terms with gauge transformations of extra vectors or antisymmetric tensors may still appear in the algebra. As long as the fields of the Weyl multiplet are inert under these transformations (as we will impose by hypothesis), these extra transformations do not modify our previous results.

A first modification of this structure is obtained by the introduction of a gauge vector multiplet. The commutator of the supersymmetries can still be modified by a gauge transformation that depends on fields of this vector multiplet. For this structure to make sense, the algebra of the Weyl multiplet had to close without using an equation of motion. Furthermore, as long as the vector multiplet is well defined off-shell, a matter multiplet (in the background of both the vector and Weyl multiplet) may now be introduced whose algebra closes only modulo equations of motion.

All fields in 'matter multiplets' will now have to obey the same 'soft' algebra defined by the Weyl multiplet. A first step is to define their transformations under the bosonic symmetries. We assume the rules (2.87) and (2.88) under Weyl and chiral transformations, where the weights will be given in Table 3.1. The R-symmetry SU(2) transformation is implicit in the index structure of the fields.

Table 3.1 Fields in some superconformal matter multiplets

D = 4				**D = 5**			**D = 6**				
field	w	c	#	field	w	#	field	w	#	SU(2)	γ_*
Off-shell vector multiplet											
X	1	1	2	σ	1	1				1	
W_μ	0	0	3	W_μ	0	4	W_μ	0	5	1	
Y_{ij}	2	0	3	Y_{ij}	2	3	Y_{ij}	2	3	3	
Ω_i	3/2	1/2	8	ψ_i	3/2	8	λ_i	3/2	8	2	+
				On-shell tensor multiplet							
				$B_{\mu\nu}$	0	3	$B_{\mu\nu}$	0	3	1	
				ϕ	1	1	σ	2	1	1	
				λ^i	3/2	4	ψ^i	5/2	4	2	
On-shell hypermultiplet											
q^X	1	0	4	q^X	3/2	4	q^X	2	4	2	
ζ^A	3/2	−1/2	4	ζ^A	2	4	ζ^A	5/2	4	1	+
Off-shell chiral multiplet											
A	w	w	2							1	
B_{ij}	$w+1$	$w-1$	6							3	
G^-_{ab}	$w+1$	$w-1$	6							1	
C	$w+2$	$w-2$	2							1	
Ψ_i	$w+\frac{1}{2}$	$w-\frac{1}{2}$	8							2	+
Λ_i	$w+\frac{3}{2}$	$w-\frac{3}{2}$	8							2	−
Off-shell linear multiplet											
L_{ij}	2	0	3	L_{ij}	3	3	L_{ij}	4	3	3	
E_a	3	0	3	E_a	4	4	E_a	5	5	1	
G	3	−1	2	N	4	1				1	
φ_i	5/2	1/2	8	φ^i	7/2	8	φ^i	9/2	8	2	−

We indicate for each dimension the Weyl weight (and for $D = 4$ chiral weight), the number of real degrees of freedom, the SU(2) representations, which is the same in any dimension, and the chirality for $D = 4$ and $D = 6$. For each multiplet we give first the bosonic fields, and then the fermionic fields (below the line)

3.2.1 Vector Multiplets

Vector multiplets can first be defined in 6 dimensions, and then reduced to 5 or 4 dimensions.

3.2.1.1 Vector Multiplet in 6 Dimensions (Abelian Case)

Consider the vector multiplet in $D = 6$, which has already been introduced in Sect. 2.3.2. It has been shown in (2.59) that the supersymmetry transformations do not close. The solution to this issue is well-known: the 5 bosonic components of the gauge vector, and the 8 components of the spinor, need an SU(2)-triplet of real scalars, $Y^{(ij)}$. The latter will appear in the transformation law of the fermion.

As an illustrative example, let us show how the transformation laws of the $D = 6$ vector multiplet have been determined with methods that can be used in general. In general, it is useful to first consider the Weyl weights of the fields. One useful principle is that gauge fields (beyond the Weyl multiplet) should have Weyl weight 0, as all transformations beyond the superconformal group must commute with the conformal generators. Equivalently, all the parameters beyond the superconformal group have to be considered[1] as Weyl weight 0.

For the U(1) gauge vector W_μ, whose abelian gauge transformation is $\delta_G W_\mu = \partial_\mu\theta$, the previous argument implies that W_μ has Weyl weight 0.[2] The same argument holds in fact for any gauge field, or gauge two-form, ... Then the associated curvature F_{ab} has Weyl weight 2 (due to the frame fields involved in $F_{ab} = e_a{}^\mu e_b{}^\nu F_{\mu\nu}$). As we have explained, these are the covariant quantities that should appear in the transformations of other matter fields. The supersymmetry parameter ϵ should be considered to be of Weyl weight $-\frac{1}{2}$, identical to its gauge field ψ_μ. Thus the supersymmetry transformation of the gaugino to the field strength of the gauge field determines that the conformal weight of λ is indeed $\frac{3}{2}$.

Exercise 3.2 Determine the same result from the transformation of the gauge field to the gaugino. □

The auxiliary field Y^{ij} can appear in the transformation of the fermion via an extra term $\delta\lambda^i = Y^{ij}\epsilon_j$, hence the auxiliary field should be of Weyl weight 2. In its supersymmetry transformation law can appear a covariant fermionic object of Weyl weight $\frac{5}{2}$. This is consistent with a transformation to the covariant derivative of the

[1] In principle parameters do not transform, but the commutators of symmetries can be stated in these terms.

[2] We could straightforwardly have generalized to a non-abelian algebra. We will do this below for $D = 5$ and $D = 4$.

gaugino, in order to cancel (2.59). The full transformation laws are

$$\begin{aligned}\delta W_\mu &= \partial_\mu\theta + \tfrac{1}{2}\bar{\epsilon}\gamma_\mu\lambda\,,\\ \delta\lambda^i &= \left(\tfrac{3}{2}\lambda_{\rm D} - \tfrac{1}{4}\gamma^{ab}\lambda_{ab}\right)\lambda^i + \lambda^{ij}\lambda_j - \tfrac{1}{4}\gamma^{ab}\widehat{F}_{ab}\epsilon^i - Y^{ij}\epsilon_j\,,\\ \delta Y^{ij} &= 2\lambda_{\rm D}Y^{ij} + 2\lambda^{k(i}Y^{j)}{}_k - \tfrac{1}{2}\bar{\epsilon}^{(i}\not{\mathcal{D}}\lambda^{j)} + \bar{\eta}^{(i}\lambda^{j)}\,.\end{aligned} \tag{3.9}$$

Starting from the rigid transformations, we replaced F_{ab} by the covariant expression $\widehat{F}_{ab}$ and the derivative of λ^j has been replaced by a covariant derivative.

$$\begin{aligned}\widehat{F}_{\mu\nu} =& F_{\mu\nu} - \bar{\psi}_{[\mu}\gamma_{\nu]}\,,\\ \mathcal{D}_\mu\lambda^i =& \left(\partial_\mu - \tfrac{3}{2}b_\mu + \tfrac{1}{4}\gamma^{ab}\omega_{\mu ab}\right)\lambda^i - V_\mu{}^{ij}\lambda_j + \tfrac{1}{4}\bar{\psi}^i_\mu\gamma^{ab}\widehat{F}_{ab} + Y^{ij}\psi_{\mu\,j}\,.\end{aligned} \tag{3.10}$$

The consistency with Weyl weights does not leave place for other terms in the Q-transformations. Since the S-supersymmetry parameter η has to be considered as having Weyl weight $\frac{1}{2}$, the only S-transformation that can occur consistent with Weyl weights is the last term in (3.9). Its coefficient has to be fixed from calculating the $[\delta_Q(\epsilon), \delta_Q(\eta)]$ commutator on the gaugino or from the method in item (3) in Sect. 2.6.1. One can check that the extra terms from Y^{ij} cancel the non-closure terms (2.59).

Exercise 3.3 Check that all the transformation laws determine (and are consistent with) λ to be a left-chiral spinor, in accordance with Table 1.1. □

3.2.1.2 Vector Multiplet in 5 Dimensions

The transformations of the vector multiplet in 5 dimensions can be obtained from dimensional reduction[3] of the transformations for $D = 6$. Note that one component of the $D = 6$ vector is a real scalar σ in $D = 5$.

We will introduce here the vector multiplet in a non-abelian group, based on matrix representations with $[t_I, t_J] = f_{IJ}{}^K t_K$. Note that we will use the index I from now on to enumerate the vector multiplets, and thus the generators of the non-abelian algebra that can be gauged. We hope that this does not lead to confusion with the index I that was used so far to denote all standard gauge transformations as it was done in Chap. 2.

[3] The reader can easily find the linearized transformations from those in (3.9) using the rules in Appendix A.4. It may be more difficult to find the nonlinear transformations, since there are redefinitions such as $W_\mu(D = 6) = W_\mu(D = 5) + e^5_\mu\sigma$. It is easier to obtain the nonlinear transformations from directly imposing the supersymmetry algebra in $D = 5$.

The full rules can be found in [14, 5] for a generalization containing also tensor multiplets. For simplicity, we give here the supersymmetry transformations for only vector multiplets:

$$\begin{aligned}
\delta W_\mu^I &= \partial_\mu \theta^I - \theta^J W_\mu^K f_{JK}{}^I + \tfrac{1}{2}\bar{\epsilon}\gamma_\mu \psi^I - \tfrac{1}{2}\mathrm{i}\sigma^I \bar{\epsilon}\psi_\mu, \\
\delta Y^{ijI} &= -\tfrac{1}{2}\bar{\epsilon}^{(i}\not{\mathcal{D}}\psi^{j)I} + \tfrac{1}{2}\,\mathrm{i}\bar{\epsilon}^{(i}\gamma\cdot T\psi^{j)I} - 4\mathrm{i}\sigma^I\bar{\epsilon}^{(i}\chi^{j)}, \\
&\quad + \tfrac{1}{2}\mathrm{i}\bar{\epsilon}^{(i} f_{JK}{}^I \sigma^J \psi^{j)K} + \tfrac{1}{2}\mathrm{i}\bar{\eta}^{(i}\psi^{j)I}, \\
\delta\psi^{iI} &= -\tfrac{1}{4}\gamma\cdot \widehat{F}^I \epsilon^i - \tfrac{1}{2}\mathrm{i}\not{\mathcal{D}}\sigma^I\epsilon^i - Y^{ijI}\epsilon_j + \sigma^I\gamma\cdot T\epsilon^i + \sigma^I\eta^i, \\
\delta\sigma^I &= \tfrac{1}{2}\mathrm{i}\bar{\epsilon}\psi^I.
\end{aligned} \tag{3.11}$$

The (superconformal) covariant derivatives are given by

$$\begin{aligned}
\mathcal{D}_\mu \sigma^I &= D_\mu\sigma^I - \tfrac{1}{2}\mathrm{i}\bar{\psi}_\mu\psi^I, \\
D_\mu\sigma^I &= (\partial_\mu - b_\mu)\sigma^I - f_{JK}{}^I W_\mu^K \sigma^J, \\
\mathcal{D}_\mu\psi^{iI} &= D_\mu\psi^{iI} + \tfrac{1}{4}\gamma\cdot\widehat{F}^I\psi_\mu^i + \tfrac{1}{2}\mathrm{i}\not{\mathcal{D}}\sigma^I\psi_\mu^i + Y^{ijI}\psi_{\mu j} - \sigma^I\gamma\cdot T\psi_\mu^i, \\
&\quad + \tfrac{1}{2} f_{JK}{}^I\sigma^J\sigma^K\psi_\mu^i - \sigma^I\phi_\mu^i, \\
D_\mu\psi^{iI} &= \left(\partial_\mu - \tfrac{3}{2}b_\mu + \tfrac{1}{4}\gamma_{ab}\hat{\omega}_\mu{}^{ab}\right)\psi^{iI} - V_\mu^{ij}\psi_j^I - f_{JK}{}^I W_\mu^K\psi^{iJ}.
\end{aligned} \tag{3.12}$$

with $\widehat{F}_{\mu\nu}^I$ given by

$$\widehat{F}_{\mu\nu}^I = F_{\mu\nu}{}^I - \bar{\psi}_{[\mu}\gamma_{\nu]}\psi^I + \tfrac{1}{2}\mathrm{i}\sigma^I\bar{\psi}_{[\mu}\psi_{\nu]}, \qquad F_{\mu\nu}{}^I = 2\partial_{[\mu}W_{\nu]}^I + W_\mu^J W_\nu^K f_{JK}{}^I, \tag{3.13}$$

There is one more aspect in the dimensional reduction (whether the multiplet is abelian or not). Remember that the covariant general coordinate transformations contain a linear combination of all gauge symmetries. That involves also the gauge transformation of the vector. Thus in the commutator of two supersymmetry transformations in $D = 6$ is a term $\bar{\epsilon}_2\gamma^\mu\epsilon_1 W_\mu$. When reduced to 5 dimensions (and below also to 4 dimensions), some components of W_μ are replaced by the scalars σ. This is the origin of a new term in the supersymmetry commutator involving structure functions depending on the scalars, which is implicit in the form of the last term in $\widehat{F}_{\mu\nu}^I$, which is of the form of the last term in (2.5) for gravitini as gauge fields.

3.2.1.3 Vector Multiplet in 4 Dimensions

Further dimensional reduction leads to the vector multiplet in 4 dimensions. As mentioned already in Sect. 1.2.1, it has then a complex scalar, built from the fourth and fifth components of the vector of 6 dimensions. To get the right behaviour of gauge and general coordinate transformations, one has to consider the reduction of the vector with tangent spacetime indices (see [15, 16] and a useful general introduction to dimensional reduction is [17]). In other words, the object from where the scalars originate in the dimensional reduction should be a world scalar, $e_a{}^\mu W_\mu$, which has Weyl weight 1. Therefore the complex[4] scalar X of the $D = 4$ vector multiplets has $w = 1$.

Before giving the supersymmetry transformations, we have to translate the reality of the triplet Y_{ij} in appropriate notation for 4 dimensions. In 6 dimensions the reality is $Y = Y^* = \sigma_2 Y^C \sigma_2$. It is in the form with Y^C that we have to translate it, thus giving rise to

$$Y_{ij} = \varepsilon_{ik}\varepsilon_{j\ell}Y^{k\ell}\,, \qquad Y^{ij} = (Y_{ij})^*\,. \tag{3.14}$$

As for $D = 5$, we write the transformations for the non-abelian vector multiplet. The transformations under dilatations and chiral U(1) transformations follow from Table 3.1, with the general rules (2.87) and (2.88). The supersymmetry (Q and S), and the gauge transformations with parameter θ in 4 dimensions are[5]

$$\begin{aligned}
\delta X^I &= \tfrac12 \bar\epsilon^i \Omega_i^I - \theta^J X^K f_{JK}{}^I\,,\\
\delta \Omega_i^I &= \not{D} X^I \epsilon_i + \tfrac14 \gamma^{ab}\mathcal{F}_{ab}^{I-}\varepsilon_{ij}\epsilon^j + Y_{ij}^I\epsilon^j + X^J \bar X^K f_{JK}{}^I \varepsilon_{ij}\epsilon^j\\
&\quad + 2X^I\eta_i - \theta^J \Omega_i^K f_{JK}{}^I\,,\\
\delta W_\mu^I &= \tfrac12 \varepsilon^{ij}\bar\epsilon_i\gamma_\mu\Omega_j^I + \varepsilon^{ij}\bar\epsilon_i\psi_{\mu j}X^I + \text{h.c.} + \partial_\mu\theta^I - \theta^J W_\mu^K f_{JK}{}^I\,,\\
\delta Y_{ij}^I &= \tfrac12 \bar\epsilon_{(i}\not{D}\Omega_{j)}^I + \tfrac12 \varepsilon_{ik}\varepsilon_{j\ell}\bar\epsilon^{(k}\not{D}\Omega^{\ell)I} + \varepsilon_{k(i}\left(\bar\epsilon_{j)}X^J\Omega^{kK} - \bar\epsilon^k \bar X^J \Omega_{j)K}\right) f_{JK}{}^I\\
&\quad - \theta^I Y_{ij}^K f_{JK}{}^I\,,
\end{aligned} \tag{3.15}$$

where

$$\mathcal{F}_{ab}^{I-} \equiv \widehat{F}_{ab}^{I-} - \tfrac12 \bar X^I T_{ab}^-\,. \tag{3.16}$$

In the latter expression $\widehat{F}_{ab}^{I-}$ denotes the anti-self-dual part of $\widehat{F}_{ab}$, which is covariant with the new structure functions, as dictated by definitions given in Chap. 2 and

[4]To be in accordance with common practice here, we denote the complex conjugates of the scalar fields by $\bar X$ rather than X^*.

[5]For the translation from $D = 5$, we use $X^I = \frac12(W_4^I - \mathrm{i}\sigma^I)$, and Ω has been defined with the opposite sign as would straightforwardly follow from Appendix A.4: $\psi^{iI} = -\Omega^{iI} - \Omega_j^I\varepsilon^{ji}$.

reported here for convenience

$$\widehat{F}_{\mu\nu}{}^{I} = F_{\mu\nu}{}^{I} + \left(-\varepsilon_{ij}\bar{\psi}^{i}_{[\mu}\gamma_{\nu]}\Omega^{Ij} - \varepsilon_{ij}\bar{\psi}^{i}_{\mu}\psi^{j}_{\nu}\bar{X}^{I} + \text{h.c.}\right),$$
$$F^{I}_{\mu\nu} = \partial_\mu W^{I}_{\nu} - \partial_\nu W_\mu{}^{I} + W_\mu{}^{J} W_\nu{}^{K} f_{JK}{}^{I}. \tag{3.17}$$

Indeed, the second term of the transformation of the vector reflects the presence of the new term in the commutator of two supersymmetries, as already discussed for $D = 5$, and modifies (2.96) to

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_P\left(\xi^a_3\right) + \delta_M\left(\lambda^{ab}_3\right) + \delta_{\rm K}\left(\lambda^a_{\rm K3}\right) + \delta_S\left(\eta_3\right)$$
$$+\delta_G\left(\theta^I_3(\epsilon_1, \epsilon_2) = \varepsilon^{ij}\bar{\epsilon}_{2i}\epsilon_{1j}X^I + \text{h.c.}\right), \tag{3.18}$$

where δ_G is the (non-abelian) gauge transformation parameterized by θ^I.

Exercise 3.4 Check that this leads to the form of $\widehat{F}_{\mu\nu}{}^{I}$ as given in (3.17). □

The covariant derivatives are

$$\begin{aligned}
\mathcal{D}_\mu X^I &= D_\mu X^I - \tfrac{1}{2}\bar{\psi}^{i}_{\mu}\Omega^{I}_{i},\\
D_\mu X^I &= \left(\partial_\mu - b_\mu - \mathrm{i}A_\mu\right) X^I + W^{J}_{\mu} X^K f_{JK}{}^{I},\\
\mathcal{D}_\mu \Omega^{I}_{i} &= D_\mu\Omega^{I}_{i} - \not{\mathcal{D}}X^I\psi_{\mu i} - \tfrac{1}{4}\gamma^{ab}\mathcal{F}^{I-}_{ab}\varepsilon_{ij}\psi^{j}_{\mu}\\
&\quad -Y^{I}_{ij}\psi^{j}_{\mu} - X^J\bar{X}^K f^{I}_{JK}\varepsilon_{ij}\psi^{j}_{\mu} - 2X^I\phi_{\mu i},\\
D_\mu\Omega^{I}_{i} &= \left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \tfrac{3}{2}b_\mu - \tfrac{1}{2}\mathrm{i}A_\mu\right)\Omega^{I}_{i} + V_{\mu i}{}^{j}\Omega_j{}^{I} + W^{J}_{\mu}\Omega^{K}_{i} f_{JK}{}^{I}.
\end{aligned} \tag{3.19}$$

As will become clear in the following section, the vector multiplet is a constrained chiral multiplet. This observation becomes relevant when constructing actions for the vector multiplet (Sect. 3.3).

3.2.2 *Intermezzo: Chiral Multiplet*

A multiplet corresponds to a superfield in superspace. A multiplet or a superfield can be real or chiral, or carry a Lorentz representation, or be in a non-trivial representations of the R-symmetry, ... For the multiplet, this just reflects the property of its 'lowest component'.[6] For example, a chiral multiplet is characterized by the fact that its lowest component transforms 'chirally', i.e. only under the left-handed supersymmetry and not under the right-handed one. In superspace this means that one chiral superspace derivative vanishes on the field. Furthermore multiplets or

[6] 'Lowest' refers here to the Weyl weight in superconformal language (or to the engineering dimensions, if we do not discuss the superconformal properties).

superfields can be constrained. In this section we explain, first in the context of rigid supersymmetry, how further constraints on a chiral multiplet lead to the vector multiplet, which is smaller. A generalization to the rigid superconformal case follows.

Let us consider a general scalar multiplet, whose 'lowest' component is a complex scalar A. In general, a complex scalar can transform under Q to arbitrary spinors

$$\delta_Q(\epsilon)A = \tfrac{1}{2}\bar{\epsilon}^i\Psi_i + \tfrac{1}{2}\bar{\epsilon}_i\Lambda^i\,. \tag{3.20}$$

Then the transformations of these arbitrary spinors Ψ_i and Λ^i can have arbitrary expressions containing new fields, as long as it is consistent with the algebra. See e.g. [18, Sect. 14.1.1] for the example of $\mathcal{N}=1$ chiral multiplets, and in Sect. 2.2 of [19] this is worked out for the chiral multiplet of $\mathcal{N}=2$, which we consider here.

If $\Lambda^i=0$, then the lowest component only transforms under left supersymmetry:

$$\delta_Q(\epsilon)A = \tfrac{1}{2}\bar{\epsilon}^i\Psi_i\,, \tag{3.21}$$

and the multiplet is called *chiral*. Imposing the rigid supersymmetry algebra leads to the following general expressions:

$$\begin{aligned}
\delta_Q(\epsilon)A &= \tfrac{1}{2}\bar{\epsilon}^i\Psi_i\,,\\
\delta_Q(\epsilon)\Psi_i &= \partial\!\!\!/ A\epsilon_i + B_{ij}\epsilon^j + \tfrac{1}{4}\gamma_{ab}G^{-ab}\varepsilon_{ij}\epsilon^j\,,\\
\delta_Q(\epsilon)B_{ij} &= \tfrac{1}{2}\bar{\epsilon}_{(i}\partial\!\!\!/\Psi_{j)} - \tfrac{1}{2}\bar{\epsilon}^k\Lambda_{(i}\varepsilon_{j)k}\,,\\
\delta_Q(\epsilon)G^-_{ab} &= \tfrac{1}{4}\varepsilon^{ij}\bar{\epsilon}_i\partial\!\!\!/\gamma_{ab}\Psi_i + \tfrac{1}{4}\bar{\epsilon}^i\gamma_{ab}\Lambda_i\,,\\
\delta_Q(\epsilon)\Lambda_i &= -\tfrac{1}{4}\gamma^{ab}G^-_{ab}\overleftarrow{\partial\!\!\!/}\epsilon_i - \partial\!\!\!/ B_{ij}\varepsilon^{jk}\epsilon_k + \tfrac{1}{2}C\varepsilon_{ij}\epsilon^j\,,\\
\delta_Q(\epsilon)C &= -\varepsilon^{ij}\bar{\epsilon}_i\partial\!\!\!/\Lambda_j\,.
\end{aligned} \tag{3.22}$$

The reader can count that this is a $16+16$ multiplet counted as real components. In fact it is reducible, since one can impose the following *consistent constraints*[7]:

$$\begin{aligned}
B_{ij} - \varepsilon_{ik}\varepsilon_{j\ell}B^{k\ell} &= 0\,,\\
\partial\!\!\!/\Psi^i - \varepsilon^{ij}\Lambda_j &= 0\,,
\end{aligned}$$

[7]There is an extension possible that the first of these expressions is not zero [20] but a constant. This leads to magnetic couplings in rigid supersymmetry, and possibilities for partial breaking to $\mathcal{N}=1$ supersymmetry. Recently [21], it has been shown how to generate these constants dynamically using multiplets with 3-form gauge fields, and in [22] this has been related to deformations in Dirac–Born–Infeld actions. It is not clear how to generalize this to supergravity, and hence we will not further discuss this.

$$\partial_b(G^{+ab} - G^{-ab}) = 0\,,$$
$$C - 2\partial_a\partial^a \bar{A} = 0\,, \tag{3.23}$$

where $B^{k\ell}$ is, as usual, defined by the complex conjugate of $B_{k\ell}$, and similarly G^+ is the complex conjugate of G^-, and thus self-dual as G^- is anti-self-dual. These constraints are *consistent* in the sense that a supersymmetry variation of one of them leads to the other equations, and this is a complete set in that sense.

The third equation is a Bianchi identity that can be solved by interpreting G_{ab} as the field strength of a vector. To conclude, the independent components are then those of the vector multiplet, with the following identifications:

$$X = A\,, \qquad \Omega_i = \Psi_i\,, \qquad F_{ab} = G_{ab}\,, \qquad Y_{ij} = B_{ij}\,. \tag{3.24}$$

Indeed the linear part of (3.15) corresponds to (3.22). We have thus identified the vector multiplet as a *constrained* chiral multiplet.

To define the chiral multiplet in the conformal algebra, one first allows an arbitrary Weyl weight for A, say that this is w. Then consistency with Weyl weights imposes that a general S-supersymmetry transformation for Ψ_i should be proportional to A. Imposing the $\{Q, S\}$ anticommutator immediately shows that the chiral U(1) weight of A should be related to its Weyl weight. In fact, to avoid the ϵ_i terms in this anticommutator, one should impose that under dilatations and U(1),

$$\delta_{\mathrm{D},T}(\lambda_\mathrm{D}, \lambda_T)A = w\,(\lambda_\mathrm{D} + \mathrm{i}\lambda_T)\,A\,. \tag{3.25}$$

The same transformations for the other fields can be obtained by requiring compatibility with Q-transformations, to obtain

$$\begin{aligned}
\delta_{\mathrm{D},T}(\lambda_\mathrm{D}, \lambda_T)\Psi_i &= \left(\left(w + \tfrac{1}{2}\right)\lambda_\mathrm{D} + \mathrm{i}\left(w - \tfrac{1}{2}\right)\lambda_T\right)\Psi_i\,,\\
\delta_{\mathrm{D},T}(\lambda_\mathrm{D}, \lambda_T)B_{ij} &= ((w+1)\lambda_\mathrm{D} + \mathrm{i}(w-1)\lambda_T)\,B_{ij}\,,\\
\delta_{\mathrm{D},T}(\lambda_\mathrm{D}, \lambda_T)G^-_{ab} &= ((w+1)\lambda_\mathrm{D} + \mathrm{i}(w-1)\lambda_T)\,G^-_{ab}\,,\\
\delta_{\mathrm{D},T}(\lambda_\mathrm{D}, \lambda_T)\Lambda_i &= \left(\left(w + \tfrac{3}{2}\right)\lambda_\mathrm{D} + \mathrm{i}\left(w - \tfrac{3}{2}\right)\lambda_T\right)\Lambda_i\,,\\
\delta_{\mathrm{D},T}(\lambda_\mathrm{D}, \lambda_T)C &= ((w+2)\lambda_\mathrm{D} + \mathrm{i}(w-2)\lambda_T)\,C\,.
\end{aligned} \tag{3.26}$$

To complete the superconformal multiplet, one has to add S-transformations, and there are nonlinear transformations involving the matter fields of the Weyl multiplet χ_i and T_{ab}, necessary in order to represent the anticommutators (2.96). The full result was found in [23]:

$$\delta_{Q,S}(\epsilon,\eta)A = \tfrac{1}{2}\bar{\epsilon}^i\Psi_i\,,$$

$$\delta_{Q,S}(\epsilon,\eta)\Psi_i = \not{D}A\epsilon_i + B_{ij}\epsilon^j + \tfrac{1}{4}\gamma\cdot G^-\varepsilon_{ij}\epsilon^j + 2wA\eta_i\,,$$

$$
\begin{aligned}
\delta_{Q,S}(\epsilon,\eta)B_{ij} &= \tfrac{1}{2}\bar\epsilon_{(i}\not{\mathcal{D}}\Psi_{j)} - \tfrac{1}{2}\bar\epsilon^{k}\Lambda_{(i}\varepsilon_{j)k} + (1-w)\bar\eta_{(i}\Psi_{j)}\,,\\
\delta_{Q,S}(\epsilon,\eta)G^{-}_{ab} &= \tfrac{1}{4}\varepsilon^{ij}\bar\epsilon_{i}\not{\mathcal{D}}\gamma_{ab}\Psi_{j} + \tfrac{1}{4}\bar\epsilon^{i}\gamma_{ab}\Lambda_{i} - \tfrac{1}{2}\varepsilon^{ij}(1+w)\bar\eta_{i}\gamma_{ab}\Psi_{j}\,,\\
\delta_{Q,S}(\epsilon,\eta)\Lambda_{i} &= -\tfrac{1}{4}\gamma\cdot G^{-}\overleftarrow{\not{\mathcal{D}}}\,\epsilon_{i} - \not{\mathcal{D}}B_{ij}\epsilon_{k}\varepsilon^{jk} + \tfrac{1}{2}C\epsilon^{j}\varepsilon_{ij}\\
&\quad -\tfrac{1}{8}(\not{\mathcal{D}}A)T\cdot\gamma)\epsilon_{i} - \tfrac{1}{8}wA(\not{\mathcal{D}}T)\cdot\gamma\epsilon_{i} - \tfrac{3}{4}(\bar\chi_{[i}\gamma_{a}\Psi_{j]})\gamma^{a}\epsilon_{k}\varepsilon^{jk}\\
&\quad -2(1+w)B_{ij}\varepsilon^{jk}\eta_{k} + \tfrac{1}{2}(1-w)\gamma\cdot G^{-}\eta_{i}\,,\\
\delta_{Q,S}(\epsilon,\eta)C &= -\varepsilon^{ij}\bar\epsilon_{i}\not{\mathcal{D}}\Lambda_{j} - 6\bar\epsilon_{i}\chi_{j}B_{k\ell}\varepsilon^{ik}\varepsilon^{j\ell}\\
&\quad +\tfrac{1}{8}(w-1)\bar\epsilon_{i}\gamma\cdot T\,\overleftarrow{\not{\mathcal{D}}}\,\Psi_{j}\varepsilon^{ij} + \tfrac{1}{8}\bar\epsilon_{i}\gamma\cdot T\not{\mathcal{D}}\Psi_{j}\varepsilon^{ij} + 2w\varepsilon^{ij}\bar\eta_{i}\Lambda_{j}\,.
\end{aligned}
\tag{3.27}
$$

This time, the set of consistent constraints is[8]

$$
\begin{aligned}
0 &= B_{ij} - \varepsilon_{ik}\varepsilon_{jl}B^{kl}\,,\\
0 &= \not{\mathcal{D}}\Psi^{i} - \varepsilon^{ij}\Lambda_{j}\,,\\
0 &= \mathcal{D}^{a}\left(G^{+}_{ab} - G^{-}_{ab} + \tfrac{1}{2}AT_{ab} - \tfrac{1}{2}\bar A T_{ab}\right) - \tfrac{3}{4}\left(\varepsilon^{ij}\bar\chi_{i}\gamma_{b}\Psi_{j} - \text{h.c.}\right)\,,\\
0 &= -2\Box\bar A - \tfrac{1}{2}G^{+}_{\mu\nu}T^{\mu\nu} - 3\bar\chi_{i}\Psi^{i} - C\,.
\end{aligned}
\tag{3.28}
$$

Interestingly, the constraints above are consistent only for a specific choice of w. For example, the first constraint is a reality condition, and it is easy to check that this is only consistent if the chiral weight of B_{ij} is zero. This fixes $w = 1$, which in turn is the appropriate value also to interpret G_{ab} as a covariant field strength. Note that the Bianchi identity in the third line of (3.28) shows the shift between the pure covariant field strength and the G. Compare this with (3.16).

The chiral multiplet plays an important role in the construction of the actions in rigid supersymmetry, as its highest component C is a scalar transforming to a total derivative. That action corresponds in superspace to take the full chiral superspace integral of the chiral superfield. However, in local supersymmetry, as in the superconformal transformations in (3.27), the transformation of C is not a pure derivative. Therefore in order to have an invariant action, one has to include more terms, i.e. something of the form

$$
I = \int \mathrm{d}^4x\, e\, C + \cdots + \text{h.c.}\,. \tag{3.29}
$$

The $+\cdots$ in (3.29) are terms that should be such that the transformation of the integrand is a total derivative.

[8]For rigid supersymmetry, an imaginary constant in B_{ij} would be possible, describing magnetic charges.

But we can first make a few general observations. The integrand should be invariant under all superconformal transformations. Let us start with the Weyl transformations. The Weyl weight of the determinant of the frame field is -4, so C should have Weyl weight 4. It should also be invariant under T-transformations, which means that the chiral weight should be zero. We see from (3.26) that these two requirements are consistent with a requirement that the chiral multiplet should have Weyl weight 2. Note that this implies that it will not be a constrained chiral (i.e. vector) multiplet. We found above that these have Weyl weight 1. But if we start from a vector multiplet, any holomorphic function of X still transforms only under 1 chirality of Q. Hence any $F(X)$ is a chiral multiplet. If we take a homogeneous function of second degree in X, this gives us a chiral multiplet with $w = 2$ on which we can use the action formula.

To determine the full expression in (3.29) one considers other terms that have Weyl weight 4 and chiral weight 0, and imposes the condition of invariance of the action. In practice, imposing S-supersymmetry is easiest to determine all the coefficients of these terms. For local superconformal symmetry the result is [23]

$$
\begin{aligned}
e^{-1}\mathcal{L} = C &- \bar{\psi}_i \cdot \gamma \Lambda_j \varepsilon^{ij} + \frac{1}{8}\bar{\psi}_{\mu i}\gamma \cdot T^{+}\gamma^{\mu}\Psi_j \varepsilon^{ij} - \frac{1}{4} A T^{+}_{ab} T^{+ab} \\
&- \frac{1}{2}\bar{\psi}_{\mu i}\gamma^{\mu\nu}\psi_{\nu j} B_{kl}\varepsilon^{ik}\varepsilon^{jl} + \bar{\psi}_{\mu i}\psi_{\nu j}\varepsilon^{ij}\left(G^{-\mu\nu} - A T^{+\mu\nu}\right) \\
&+ \frac{1}{2}\mathrm{i}\varepsilon^{ij}\varepsilon^{k\ell} e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu i}\psi_{\nu j}\bar{\psi}_{\rho k}\left(\gamma_{\sigma}\Psi_{\ell} + \psi_{\sigma\ell} A\right) + \text{h.c.}\,. \qquad (3.30)
\end{aligned}
$$

This is called the *chiral density formula*.

3.2.3 Rigid Hypermultiplets

Hypermultiplets are the analogues of the chiral multiplets of $\mathcal{N} = 1$ supersymmetry. They contain four scalars and two spin- $1/2$ fields. In supergravity, they are defined in the background of the Weyl multiplet and possibly also in the background of the vector multiplet (i.e. they can transform non-trivially under the gauge transformations of the vector multiplets). One can further introduce auxiliary fields to close the algebra for the simplest quaternionic manifolds. The methods of harmonic or projective superspace mentioned in the introduction [24–30] are also equivalent to introducing an infinite number of auxiliary fields. However, we do not need auxiliary fields any more at this point because the hypermultiplets are at the end of the hierarchy line.[9]

[9] We are not going to introduce any further multiplet in the background of the hypermultiplets, as these do not introduce new gauge symmetries. This is to be confronted to when we considered the

The closure of the supersymmetry algebra will impose equations that we will interpret as equations of motion, even though we have not defined an action yet. Later we will see how an action can be constructed that gives precisely these equations as Euler–Lagrange equations.

Although our interest is the local case, the present section is mostly devoted to rigid super(conformal) symmetry. This choice has been made since the rigid case provides already simpler and explicative examples of the story. We remark that since dimensional reduction for scalars and spin-1/2 fermions leads to the same type of particles in lower dimension, the properties of the hypermultiplets do not depend on whether we consider $D = 6$, $D = 5$, or $D = 4$ (or even $D = 3$). There is a technical difference since the four on-shell (or eight off-shell) degrees of freedom are captured in symplectic Weyl, symplectic or Majorana spinors, respectively. In practice, we mostly report formulae in $D = 5$. These can be translated to $D = 6$ and $D = 4$ by the rules in Appendix A.4.

Before starting the mathematical formulation, we still want to point out how massive hypermultiplets can be described in this context, since the readers will mainly see equations of motion that describe only massless hypermultiplets. This is of course also related to the fact that we are mainly interested in conformal theories. Massive hypermultiplets in rigid supersymmetry are obtained in this setting by adding a coupling to a vector multiplet that has just a first scalar component equal to the mass, and all other components zero. The reader can glimpse at (3.88) for $D = 5$ with σ^I equal to a mass, or to (3.93) and (3.95) for X^I providing the mass to see that with a suitable choice of the Killing vectors these are massive field equations. In supergravity this will be natural for the σ^I or X^I referring to the compensating multiplet.

3.2.3.1 Rigid Supersymmetry

We consider a set of n_H hypermultiplets. The real scalars are denoted as q^X, with $X = 1, \ldots, 4n_H$, and the fermions are indicated by $\zeta^{\mathcal{A}}$, where the indices $\mathcal{A} = 1, \ldots, 2n_H$ will indicate a fundamental representation of $\mathrm{Sp}(2n_H)$. Imposing the supersymmetry transformations on the bosons lead to the identification of a hypercomplex manifold[10] parameterized by these bosons q^X. The structure is determined by frame fields $f^{i\mathcal{A}}{}_X$, connections $\omega_{X\mathcal{A}}{}^{\mathcal{B}}$ and Γ^Z_{XY} (the latter symmetric in its lower indices) such that

$$f^{i\mathcal{A}}{}_Y f^X{}_{i\mathcal{A}} = \delta^X_Y\,, \qquad f^{i\mathcal{A}}{}_X f^X{}_{j\mathcal{B}} = \delta^i_j \delta^{\mathcal{A}}_{\mathcal{B}}\,. \tag{3.31}$$

vector multiplets. The construction of the latter had to take into account that the multiplets can be used for various possible actions (including hypermultiplets or not).

[10] In supergravity the scalars span a quaternionic manifold, see Sect. 5.6.

and

$$\nabla_Y f^X{}_{i\mathcal{A}} \equiv \partial_Y f^X{}_{i\mathcal{A}} - \omega_{Y\mathcal{A}}{}^{\mathcal{B}}(q) f^X{}_{i\mathcal{B}} + \Gamma^X_{YZ}(q) f^Z{}_{i\mathcal{A}} = 0\,,$$
$$\nabla_Y f^{i\mathcal{A}}{}_X \equiv \partial_Y f^{i\mathcal{A}}{}_X + f^{i\mathcal{B}}{}_X \omega_{Y\mathcal{B}}{}^{\mathcal{A}}(q) - \Gamma^Z_{YX}(q) f^{i\mathcal{A}}{}_Z = 0\,, \tag{3.32}$$

are satisfied. The frame field satisfies a reality condition, for which we will also introduce indices $\bar{\mathcal{A}}$:

$$\left(f^{i\mathcal{A}}{}_X\right)^* = f^{j\mathcal{B}}{}_X \varepsilon_{ji} \rho_{\mathcal{B}\bar{\mathcal{A}}}\,, \qquad \left(f^X{}_{i\mathcal{A}}\right)^* = \varepsilon^{ij} \rho^{\bar{\mathcal{A}}\mathcal{B}} f^X{}_{j\mathcal{B}}\,, \tag{3.33}$$

in terms of a non-degenerate covariantly constant tensor $\rho_{\mathcal{A}\bar{\mathcal{B}}}$ that satisfies

$$\rho_{\mathcal{A}\bar{\mathcal{B}}} \rho^{\bar{\mathcal{B}}\mathcal{C}} = -\delta^{\mathcal{C}}_{\mathcal{A}}\,, \qquad \rho^{\bar{\mathcal{A}}\mathcal{B}} = \left(\rho_{\mathcal{A}\bar{\mathcal{B}}}\right)^*\,. \tag{3.34}$$

By field redefinitions, we could bring it in the standard antisymmetric form

$$\rho_{\mathcal{A}\bar{\mathcal{B}}} = \begin{pmatrix} 0 & \mathbb{1}_{n_H} \\ -\mathbb{1}_{n_H} & 0 \end{pmatrix} = \rho^{\bar{\mathcal{A}}\mathcal{B}}\,. \tag{3.35}$$

We will not impose this basis choice in general. In Sect. 3.3.4 we will show how such a basis could be implemented.

The complex conjugate of $\omega_{X\mathcal{A}}{}^{\mathcal{B}}$ is

$$\left(\omega_{X\mathcal{A}}{}^{\mathcal{B}}\right)^* \equiv \bar{\omega}_X{}^{\bar{\mathcal{A}}}{}_{\bar{\mathcal{B}}} = -\rho^{\bar{\mathcal{A}}\mathcal{C}} \omega_{X\mathcal{C}}{}^{\mathcal{D}} \rho_{\mathcal{D}\bar{\mathcal{B}}}. \tag{3.36}$$

The above conditions lead to the identification of almost quaternionic structures

$$2 f^{i\mathcal{A}}{}_X f^Y{}_{j\mathcal{A}} = \delta^Y_X \delta^i_j + J_X{}^Y{}_j{}^i\,, \qquad J_X{}^Y{}_j{}^i = \boldsymbol{\tau}_j{}^i \cdot \mathbf{J}_X{}^Y\,,$$
$$\mathbf{J}_X{}^Y = \left(\mathbf{J}_X{}^Y\right)^* = -f^{i\mathcal{A}}{}_X f^Y{}_{j\mathcal{A}} \boldsymbol{\tau}_i{}^j\,. \tag{3.37}$$

We use here the 3-vectors notation and $\boldsymbol{\tau}_i{}^j = \mathrm{i}\boldsymbol{\sigma}_i{}^j$ in terms of the three Pauli-matrices $\boldsymbol{\sigma}_i{}^j$ as in (1.52), (1.54). Related formulas are given in Appendix A.2.2. The three matrices **J** satisfy the quaternionic algebra, i.e. for any vectors **A**, **B**

$$\mathbf{A} \cdot \mathbf{J}_X{}^Z \mathbf{B} \cdot \mathbf{J}_Z{}^Y = -\delta_X{}^Y \mathbf{A} \cdot \mathbf{B} + (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{J}_X{}^Y\,. \tag{3.38}$$

In passing, we note that we can solve (3.32) for $\omega_{X\mathcal{A}}{}^{\mathcal{B}}$, such that the independent connection is $\Gamma_{XY}{}^Z$. The latter is the unique connection on the scalar manifold respect to which

$$\nabla_Z \mathbf{J}_X{}^Y \equiv \partial_Z \mathbf{J}_X{}^Y - \Gamma_{ZX}{}^U \mathbf{J}_U{}^Y + \Gamma_{ZU}{}^Y \mathbf{J}_X{}^U = 0. \tag{3.39}$$

This last condition promotes $\mathbf{J}_X{}^Y$ to be quaternionic structures.

The integrability condition of (3.32) relates the curvatures defined by the two connections:

$$
\begin{aligned}
&R_{XY}{}^W{}_Z \equiv 2\partial_{[X}\Gamma^W_{Y]Z} + 2\Gamma^W_{V[X}\Gamma^V_{Y]Z}\,,\\
&\mathcal{R}_{XY\mathcal{B}}{}^{\mathcal{A}} \equiv 2\partial_{[X}\omega_{Y]\mathcal{B}}{}^{\mathcal{A}} + 2\omega_{[X|\mathcal{C}|}{}^{\mathcal{A}}\omega_{Y]\mathcal{B}}{}^{\mathcal{C}}\,,\\
&R_{XY}{}^W{}_Z = f^W{}_{i\mathcal{A}} f^{i\mathcal{B}}{}_Z \mathcal{R}_{XY\mathcal{B}}{}^{\mathcal{A}}\,, \qquad \mathcal{R}_{XY\mathcal{B}}{}^{\mathcal{A}} = \tfrac{1}{2} f^W{}_{i\mathcal{B}} f^{i\mathcal{A}}{}_Z R_{XY}{}^Z{}_W\,.
\end{aligned}
\tag{3.40}
$$

In order to work with these tensors, it can be useful to introduce also tensors L that are orthogonal to the complex structures:

$$
\begin{aligned}
&L_Y{}^Z{}_{\mathcal{A}}{}^{\mathcal{B}} \equiv f^Z{}_{i\mathcal{A}} f^{i\mathcal{B}}{}_Y\,, \quad \mathbf{J}_Z{}^Y L_Y{}^Z{}_{\mathcal{A}}{}^{\mathcal{B}} = 0\,,\\
&L_X{}^Y{}_{\mathcal{A}}{}^{\mathcal{B}} L_Y{}^Z{}_{\mathcal{C}}{}^{\mathcal{D}} = L_X{}^Z{}_{\mathcal{C}}{}^{\mathcal{B}} \delta_{\mathcal{A}}{}^{\mathcal{D}}\,,\\
&L_X{}^X{}_{\mathcal{A}}{}^{\mathcal{B}} = 2\delta_{\mathcal{A}}{}^{\mathcal{B}}\,, \quad L_X{}^Y{}_{\mathcal{A}}{}^{\mathcal{B}} L_Y{}^X{}_{\mathcal{C}}{}^{\mathcal{D}} = 2\delta_{\mathcal{C}}{}^{\mathcal{B}}\delta_{\mathcal{A}}{}^{\mathcal{D}}\,.
\end{aligned}
\tag{3.41}
$$

If the affine connections is the Levi-Civita connection of a metric, then the curvatures satisfy the cyclicity properties $R_{(XY}{}^W{}_{Z)} = 0$, and one can show that

$$
f^X{}_{i\mathcal{C}} f^Y{}_{j\mathcal{D}} \mathcal{R}_{XY\mathcal{B}}{}^{\mathcal{A}} = -\tfrac{1}{2}\varepsilon_{ij} W_{\mathcal{C}\mathcal{D}\mathcal{B}}{}^{\mathcal{A}}\,, \qquad W_{\mathcal{A}\mathcal{B}\mathcal{C}}{}^{\mathcal{D}} \equiv -\varepsilon^{ij} f^X{}_{i\mathcal{A}} f^Y{}_{j\mathcal{B}} \mathcal{R}_{XY\mathcal{C}}{}^{\mathcal{D}}\,.
\tag{3.42}
$$

The tensor $W_{\mathcal{A}\mathcal{B}\mathcal{C}}{}^{\mathcal{D}}$ is symmetric in its lower indices, and the other curvatures can be expressed in function of this one as

$$
R_{XY}{}^W{}_Z = L_Z{}^W{}_{\mathcal{D}}{}^{\mathcal{C}} \mathcal{R}_{XY\mathcal{C}}{}^{\mathcal{D}} = -\varepsilon_{ij}\tfrac{1}{2} L_Z{}^W{}_{\mathcal{C}}{}^{\mathcal{D}} f^{i\mathcal{A}}{}_X f^{j\mathcal{B}}{}_Y W_{\mathcal{A}\mathcal{B}\mathcal{C}}{}^{\mathcal{D}}\,.
\tag{3.43}
$$

The Bianchi identity on $\mathcal{R}_{XY\mathcal{A}}{}^{\mathcal{B}}$ implies also a symmetry of the covariant derivative of W:

$$
f^X{}_{i\mathcal{A}} \nabla_X W_{\mathcal{B}\mathcal{C}\mathcal{D}}{}^{\mathcal{E}} = f^X{}_{i(\mathcal{A}|} \nabla_X W_{|\mathcal{B}\mathcal{C}\mathcal{D})}{}^{\mathcal{E}}\,.
\tag{3.44}
$$

When a metric will be defined on the manifold, the W-tensor will become symmetric in the 4 indices. As a consequence, the manifold will be *Ricci flat*:

$$
R_{YZ} = R_{XY}{}^X{}_Z = 0\,.
\tag{3.45}
$$

3.2.3.2 Reparameterizations and Covariant Quantities

The hypermultiplet is defined in terms of the scalars q^X, which form a parameterization of a $4n_H$-dimensional manifold, and the fermions $\zeta^{\mathcal{A}}$, which are a parameterization of a $2n_H$-dimensional manifold of fermions. Both these basic parameterizations can be changed [14]. There are thus two kinds of reparameterizations. The first ones are the target space diffeomorphisms, $q^X \to \widetilde{q}^X(q)$, under which $f^X{}_{i\mathcal{A}}$ transforms as a vector, $\omega_{X\mathcal{A}}{}^{\mathcal{B}}$ as a one-form, and $\Gamma_{XY}{}^Z$ as a connection. The second set are the reparameterizations, which act on the tangent space indices $\mathcal{A}$, $\mathcal{B}$ etc. On the fermions, they act as

$$\zeta^{\mathcal{A}} \to \widetilde{\zeta}^{\mathcal{A}}(q) = \zeta^{\mathcal{B}} U_{\mathcal{B}}{}^{\mathcal{A}}(q)\,, \tag{3.46}$$

where $U_{\mathcal{A}}{}^{\mathcal{B}}(q)$ is an invertible matrix, and the reality conditions impose $U^* = \rho^{-1}U\rho$, defining $\mathrm{G}\ell(r,\mathbb{H})$. In general, the right-hand side of (3.46) depends on the $\zeta^{\mathcal{A}}$ and on the scalars. Thus the new basis $\widetilde{\zeta}^{\mathcal{A}}$ is a basis where the fermions depend on the scalars q^X. In this sense, the hypermultiplet is written in a special basis where q^X and $\zeta^{\mathcal{A}}$ are independent fields. We will develop a covariant formalism which also takes into account these reparameterizations.

The supersymmetry transformations in $D = 5$ are

$$\begin{aligned}
\delta q^X &= -\mathrm{i}\bar{\epsilon}^i \zeta^{\mathcal{A}} f^X{}_{i\mathcal{A}}\,,\\
\delta\zeta^{\mathcal{A}} &= \tfrac{1}{2}\mathrm{i} f^{i\mathcal{A}}{}_X \partial\!\!\!/ q^X \epsilon_i - \zeta^{\mathcal{B}}\omega_{X\mathcal{B}}{}^{\mathcal{A}}\delta q^X\,.
\end{aligned} \tag{3.47}$$

They are covariant under (3.46) if we transform $f^{i\mathcal{A}}{}_X(q)$ as a vector and $\omega_{X\mathcal{A}}{}^{\mathcal{B}}$ as a connection,

$$\omega_{X\mathcal{A}}{}^{\mathcal{B}} \to \widetilde{\omega}_{X\mathcal{A}}{}^{\mathcal{B}} = \left[\left(\partial_X U^{-1}\right)U + U^{-1}\omega_X U\right]_{\mathcal{A}}{}^{\mathcal{B}}. \tag{3.48}$$

These considerations lead us to define the covariant variation of vectors (see [18, Appendix 14B]) with indices in the tangent space, as $\zeta^{\mathcal{A}}$, or a quantity Δ^X with coordinate indices:

$$\widehat{\delta}\zeta^{\mathcal{A}} \equiv \delta\zeta^{\mathcal{A}} + \zeta^{\mathcal{B}}\omega_{X\mathcal{B}}{}^{\mathcal{A}}\delta q^X\,, \qquad \widehat{\delta}\Delta^X \equiv \delta\Delta^X + \Delta^Y\Gamma_{YZ}{}^X\delta q^Z\,, \tag{3.49}$$

for any transformation δ (as e.g. supersymmetry, conformal transformations,...).

Two models related by either target space diffeomorphisms or fermion reparameterizations of the form (3.46) are equivalent; they are different coordinate descriptions of the same system. We usually work in a basis where the fermions and the bosons are independent, i.e. $\partial_X\zeta^{\mathcal{A}} = 0$. But in a covariant formalism, after a transformation (3.46), this is not anymore valid. This shows that the expression $\partial_X\zeta^{\mathcal{A}}$ has no basis-independent meaning. It makes only sense if one compares a transformed basis, like the $\tilde{\zeta}^A$ with the original basis where $\partial_X\zeta^{\mathcal{A}} = 0$. But in the same way also the expression $\zeta^{\mathcal{B}}\omega_{X\mathcal{B}}{}^{\mathcal{A}}$ makes only sense if one compares different bases, as the connection has no absolute value. The only object that has a coordinate-invariant meaning is the covariant derivative

$$\nabla_X \zeta^{\mathcal{A}} \equiv \partial_X \zeta^{\mathcal{A}} + \zeta^{\mathcal{B}} \omega_{X\mathcal{B}}{}^{\mathcal{A}}. \tag{3.50}$$

In the basis where the fermions $\zeta^{\mathcal{A}}$ are considered independent of the bosons, i.e. $\partial_X \zeta^{\mathcal{A}} = 0$, which is the basis used to write down the transformation rules (3.47), only the second term in the covariant derivative above remains, and thus (3.49) becomes

$$\widehat{\delta}\zeta^{\mathcal{A}} = \delta\zeta^{\mathcal{A}} + \nabla_X \zeta^{\mathcal{A}} \delta q^X. \tag{3.51}$$

We will always consider independent bosons and fermions when we write variations.

On any covariant coordinate quantity that depends only on the coordinates q^X, covariant transformations act by covariant derivatives, e.g. for some vectors $V^X(q)$, $W_{\mathcal{A}}$ or $W^{\mathcal{A}}$:

$$\begin{aligned}
\widehat{\delta} V^X(q) &= \delta q^Y \nabla_Y V^X(q) = \delta q^Y \left(\partial_Y V^X(q) + \Gamma^X_{YZ} V^Z(q) \right) , \\
\widehat{\delta} W^{\mathcal{A}}(q) &= \delta q^Y \nabla_Y W^{\mathcal{A}}(q) = \delta q^Y \left(\partial_Y W^{\mathcal{A}}(q) + W^{\mathcal{B}}(q) \omega_{Y\mathcal{B}}{}^{\mathcal{A}} \right) , \\
\widehat{\delta} W_{\mathcal{A}}(q) &= \delta q^Y \nabla_Y W_{\mathcal{A}}(q) = \delta q^Y \left(\partial_Y W^{\mathcal{A}}(q) - \omega_{Y\mathcal{A}}{}^{\mathcal{B}} W_{\mathcal{B}}(q) \right) .
\end{aligned} \tag{3.52}$$

In particular, $\widehat{\delta}$ of any covariantly constant object (like the frame fields $f^{i\mathcal{A}}{}_X$) is zero.

Note that we can exploit covariant transformations to calculate any transformation on e.g. a quantity $W_{\mathcal{A}}(q)\zeta^{\mathcal{A}}$:

$$\delta \left(W_{\mathcal{A}}(q) \zeta^{\mathcal{A}} \right) = \widehat{\delta} \left(W_{\mathcal{A}}(q) \zeta^{\mathcal{A}} \right) = \delta q^X \nabla_X W_{\mathcal{A}} \zeta^{\mathcal{A}} + W_{\mathcal{A}} \widehat{\delta} \zeta^{\mathcal{A}}. \tag{3.53}$$

Coordinates are not covariant, but their derivatives are, and e.g. the Laplacian[11]

$$\Box q^X = \nabla^\mu \partial_\mu q^X = \partial^\mu \partial_\mu q^X + \Gamma_{YZ}{}^X \left(\partial_\mu q^Y \right) \left(\partial^\mu q^Z \right) , \tag{3.54}$$

is covariant for target space transformations.

[11] In the local (gravity) theory, the first term should be $(\sqrt{g})^{-1} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu$.

Another interesting relation is that the commutator of $\widehat{\delta}$ and ∇ gives rise to curvature terms:

$$\widehat{\delta}\nabla_\mu V^X = \nabla_\mu\widehat{\delta}V^X + R_{ZW}{}^X{}_Y V^Y \left(\delta q^Z\right)\left(\partial_\mu q^W\right) . \tag{3.55}$$

Similarly the commutator gets adapted by curvature terms:

$$\begin{aligned}
&[\delta(\epsilon_1), \delta(\epsilon_2)]\, V^X = \delta(\epsilon_3) V^X \;\rightarrow \\
&\left[\hat{\delta}(\epsilon_1), \hat{\delta}(\epsilon_2)\right] V^X = \widehat{\delta}(\epsilon_3) V^X + R_{ZW}{}^X{}_Y V^Y \left(\delta(\epsilon_1) q^Z\right)\left(\delta(\epsilon_2) q^W\right) ,
\end{aligned} \tag{3.56}$$

where ϵ_3 is the function of ϵ_1 and ϵ_2 determined by the structure functions. With these methods, it is easy to compute the commutator of two covariant derivatives. E.g. in $D = 5$ with (3.47) for the fermions

$$\begin{aligned}
\left[\widehat{\delta}(\epsilon_1), \widehat{\delta}(\epsilon_2)\right] \zeta^{\mathcal{A}} =& \tfrac{1}{2}\gamma^\mu \epsilon_{2i} f^{i\mathcal{A}}{}_X \bar{\epsilon}_1^j \nabla_\mu \zeta^B f^X{}_{j\mathcal{B}} - \;(1 \leftrightarrow 2) \\
=& \tfrac{1}{4}\gamma^\mu \left[(\bar{\epsilon}_2\epsilon_1) + \gamma^\nu(\bar{\epsilon}_2\gamma_\nu\epsilon_1)\right] \nabla_\mu \zeta^{\mathcal{A}} \\
=& \tfrac{1}{2}\nabla_\mu \zeta^{\mathcal{A}}(\bar{\epsilon}_2\gamma^\mu\epsilon_1) + \tfrac{1}{4}\left[(\bar{\epsilon}_2\epsilon_1) - \gamma^\nu(\bar{\epsilon}_2\gamma_\nu\epsilon_1)\right] \not{\nabla}\zeta^{\mathcal{A}} ,
\end{aligned} \tag{3.57}$$

with the definition

$$\nabla_\mu \zeta^{\mathcal{A}} \equiv \partial_\mu \zeta^{\mathcal{A}} + \left(\partial_\mu q^X\right) \zeta^{\mathcal{B}} \omega_{X\mathcal{B}}{}^{\mathcal{A}}. \tag{3.58}$$

Indices i, j are raised and contracted as in Appendix A.3.2. This result shows that the algebra does not close: we will interpret the extra parts as equations of motions of a putative action, see Sect. 3.3.3.

3.2.3.3 Non-closure Relations for Fermions and Bosons

From the result (3.57), using (3.56), we can obtain the following commutator of transformations:

$$\begin{aligned}
[\delta(\epsilon_1), \delta(\epsilon_2)]\, \zeta^{\mathcal{A}} =& \tfrac{1}{2}\partial_\mu \zeta^{\mathcal{A}}(\bar{\epsilon}_2\gamma^\mu\epsilon_1) + \tfrac{1}{4}\left[(\bar{\epsilon}_2\epsilon_1) - \gamma^\nu(\bar{\epsilon}_2\gamma_\nu\epsilon_1)\right] \not{\nabla}\zeta^{\mathcal{A}} \\
&+ \zeta^B \mathcal{R}_{XY\mathcal{B}}{}^{\mathcal{A}} \bar{\epsilon}_1^i \zeta^C f^X{}_{iC} \bar{\epsilon}_2^j \zeta^{\mathcal{D}} f^Y{}_{j\mathcal{D}} .
\end{aligned} \tag{3.59}$$

With (3.42) and a Fierz transformation, we obtain that the non-closure terms (the last term on the first line and the second line) are

$$+\tfrac{1}{4}\left[(\bar{\epsilon}_2\epsilon_1) - \gamma^\nu(\bar{\epsilon}_2\gamma_\nu\epsilon_1)\right] \not{\nabla}\zeta^{\mathcal{A}} + \tfrac{1}{8} W_{\mathcal{CDB}}{}^{\mathcal{A}} \zeta^B \bar{\zeta}^C \left[(\bar{\epsilon}_2\epsilon_1) + \gamma^\nu(\bar{\epsilon}_2\gamma_\nu\epsilon_1)\right] \zeta^{\mathcal{D}} . \tag{3.60}$$

Using some $D = 5$ Fierz identities:

$$5\zeta^{(\mathcal{B}}\bar{\zeta}^{\mathcal{C}}\zeta^{\mathcal{D})} = -\gamma^\mu\zeta^{(\mathcal{B}}\bar{\zeta}^{\mathcal{C}}\gamma_\mu\zeta^{\mathcal{D})}\,,$$
$$\zeta^{(\mathcal{B}}\bar{\zeta}^{\mathcal{C}}\gamma^\nu\zeta^{\mathcal{D})} = -\gamma^\nu\zeta^{(\mathcal{B}}\bar{\zeta}^{\mathcal{C}}\zeta^{\mathcal{D})}\,, \tag{3.61}$$

we find

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\,\zeta^{\mathcal{A}} = \xi^\mu\partial_\mu\zeta^{\mathcal{A}} + \tfrac{1}{4}\left[\left(\bar{\epsilon}_2^i\epsilon_1^j\right) - \gamma^\nu\left(\bar{\epsilon}_2^i\gamma_\nu\epsilon_1^j\right)\right]\varepsilon_{ji}\mathrm{i}\Gamma^{\mathcal{A}}\,, \tag{3.62}$$

i.e., the non-closure terms are proportional to[12]

$$\mathrm{i}\Gamma^{\mathcal{A}} \equiv \not\nabla\zeta^{\mathcal{A}} + \frac{1}{2}W_{\mathcal{BCD}}{}^{\mathcal{A}}\zeta^{\mathcal{B}}\bar{\zeta}^{\mathcal{C}}\zeta^{\mathcal{D}}\,. \tag{3.63}$$

The expression above must be interpreted as an equation of motion for the fermions. The supersymmetry transformation of (3.63) gives then also an equation of motion for the scalar fields:

$$\widehat{\delta}(\epsilon)\Gamma^{\mathcal{A}} = \frac{1}{2}f^{i\mathcal{A}}{}_X\epsilon_i\Delta^X\,, \tag{3.64}$$

where

$$\Delta^X = \Box q^X - \frac{1}{2}\bar{\zeta}^{\mathcal{B}}\gamma_a\zeta^{\mathcal{D}}\partial^a q^Y f^{i\mathcal{C}}{}_Y f^X{}_{i\mathcal{A}}W_{\mathcal{BCD}}{}^{\mathcal{A}}$$
$$-\frac{1}{4}\nabla_Y W_{\mathcal{BCD}}{}^{\mathcal{A}}\bar{\zeta}^{\mathcal{E}}\zeta^{\mathcal{D}}\bar{\zeta}^{\mathcal{C}}\zeta^{\mathcal{B}}f^{iY}{}_{\mathcal{E}}f^X{}_{i\mathcal{A}}\,. \tag{3.65}$$

The equations of motion given by (3.63) and (3.65) form a multiplet, since (3.64) has the counterpart

$$\widehat{\delta}(\epsilon)\Delta^X = \bar{\epsilon}^i\not\nabla\Gamma^{\mathcal{A}}f^X{}_{i\mathcal{A}} - \bar{\epsilon}^i\Gamma^{\mathcal{B}}\bar{\zeta}^{\mathcal{C}}\zeta^{\mathcal{D}}f^X{}_{i\mathcal{A}}W_{\mathcal{BCD}}{}^{\mathcal{A}}\,, \tag{3.66}$$

where the covariant derivative of $\Gamma^{\mathcal{A}}$ is defined similar to (3.58). As announced before, we thus have already physical equations despite the absence of an action.

3.2.3.4 Rigid Superconformal

To allow the generalization to superconformal couplings, the essential question is whether the manifold has dilatational symmetry. This means, according to (1.30),

[12] We inserted a factor i in order that Γ^A is symplectic Majorana.

that there is a 'closed homothetic Killing vector' [31] (see also [18, Sect. 15.7]). The dilatations act as[13]

$$\delta_{\rm D}(\lambda_{\rm D})q^X = \lambda_{\rm D}k_{\rm D}{}^X(q), \tag{3.67}$$

where $k_{\rm D}{}^X$ satisfies (we generalize here already to D dimensions, as the modifications involve only a normalization factor)

$$\nabla_Y k_{\rm D}{}^X \equiv \partial_Y k_{\rm D}{}^X + \Gamma^X_{YZ}k_{\rm D}{}^Z = \frac{D-2}{2}\delta_Y{}^X. \tag{3.68}$$

On a flat manifold, the fields q^X have thus Weyl weight $(D-2)/2$. The presence of this vector allows one to extend the transformations of rigid supersymmetry to the superconformal group [31, 32, 14], with e.g. transformations under the SU(2) R-symmetry group:

$$\delta_{\rm SU(2)}(\boldsymbol{\lambda})q^X = \mp 2\,\boldsymbol{\lambda}\cdot\mathbf{k}^X\,, \qquad \mathbf{k}^X \equiv \frac{1}{D-2}k_{\rm D}{}^Y\mathbf{J}_Y{}^X\,. \tag{3.69}$$

Note the sign difference between $D=4$, upper sign, and $D=5,6$, lower sign, as in (A.24).

In general, one can introduce the sections

$$A^{i\mathcal{A}} = k^X_{\rm D} f^{i\mathcal{A}}{}_X\,, \tag{3.70}$$

and in terms of these

$$\mathbf{k}^X = -\frac{1}{D-2}A^{i\mathcal{A}}\boldsymbol{\tau}_i{}^j f^X{}_{j\mathcal{A}}\,. \tag{3.71}$$

Using the rules of covariant transformations (and in particular that $\nabla_Y f^{i\mathcal{A}}{}_X$ implies $\widehat{\delta} f^{i\mathcal{A}}{}_X = 0$), the $A^{i\mathcal{A}}$ transform as

$$\begin{aligned}\widehat{\delta}A^{i\mathcal{A}} &= f^{i\mathcal{A}}{}_X\nabla_Y k^X_{\rm D}\delta q^Y = \frac{D-2}{2}f^{i\mathcal{A}}{}_X\delta q^X\\ &= \frac{D-2}{2}\left(-{\rm i}\bar\epsilon^i\zeta^{\mathcal{A}} + \lambda_{\rm D}A^{i\mathcal{A}}\right) + A^{j\mathcal{A}}\lambda_j{}^i\,,\end{aligned} \tag{3.72}$$

[13] Note that we give here only the intrinsic part of the dilatations, i.e. the $\lambda_{\rm D}$ term in (1.24), and not the 'orbital' part included in the general coordinate transformation $\xi^\mu(x)$. Similarly for special conformal transformations, we will write here only the intrinsic part represented as $(k_\mu\phi)$ in that equation and also the 'orbital' S-supersymmetry part (1.60) is not mentioned explicitly.

using (3.68) and (3.69). Note that the supersymmetry transformation in this equation is written for the symplectic spinors of $D = 5, 6$. Below, we will write them for $D = 4$.

We can then derive the other (super)conformal transformations using the algebra. The intrinsic special conformal transformations on q^X and $\zeta^{\mathcal{A}}$ vanish. They have only the 'orbital' parts as follows from (1.24). The latter imply e.g. that $\delta_K(\lambda_K)\partial\!\!\!/ q^X \neq 0$. The algebra gives then for the intrinsic S-supersymmetry

$$\delta_S(\eta^i)\zeta^{\mathcal{A}} = -A^{i\mathcal{A}}\eta_i\,. \tag{3.73}$$

The (intrinsic) bosonic conformal symmetries act as

$$\widehat{\delta}_D\zeta^{\mathcal{A}} = \frac{D-1}{2}\lambda_D\zeta^{\mathcal{A}}\,, \qquad \widehat{\delta}_{SU(2)}\zeta^{\mathcal{A}} = 0\,. \tag{3.74}$$

The fermions are inert under SU(2) R-symmetries group.

3.2.3.5 Isometries and Coupling to Vector Multiplets

So far we considered the hypermultiplet with ungauged isometries. A more general situation includes couplings to vector multiplets and in this case one has to define the hypermultiplet in the algebra including the vector multiplet with its gauge transformations. Let us consider general isometries (not necessarily gauged) of the hypermultiplet:

$$\delta_G(\theta)q^X = \theta^I k_I{}^X(q)\,, \tag{3.75}$$

where θ^I are constant parameters and the $k_I{}^X(q)$ represent the transformations. The index I identifies the different generators of the isometry group. Then a subgroup of these could be gauged, identified by an embedding tensor [33–36] projecting from all the symmetries to those that are gauged.[14] When we have a metric, $k_I{}^X(q)$ should be Killing vectors in order to define symmetries of the action. As we have not discussed a metric yet, we could define here some generalization of symmetries, but we just refer the interested reader to [14]. The transformations (3.75) constitute an algebra with structure constants $f_{IJ}{}^K$,

$$k_I{}^Y\partial_Y k_J{}^X - k_J{}^Y\partial_Y k_I{}^X = f_{IJ}{}^K\, k_K{}^X\,. \tag{3.76}$$

[14]However, we will here soon gauge the symmetries, and thus restrict the index I to the gauged symmetries.

We consider symmetries that respect the hypercomplex structure. This is the requirement that $k_I{}^X(q)$ is tri-holomorphic:

$$\left(\nabla_X k_I^Y\right) \mathbf{J}_Y{}^Z = \mathbf{J}_X{}^Y \left(\nabla_Y k_I^Z\right) . \tag{3.77}$$

Extracting affine connections from this equation, it can be written as

$$\left(\mathcal{L}_{k_I}\mathbf{J}\right)_X{}^Y \equiv k_I{}^Z\partial_Z\mathbf{J}_X{}^Y - \partial_Z k_I{}^Y\mathbf{J}_X{}^Z + \partial_X k_I{}^Z\mathbf{J}_Z{}^Y = 0 . \tag{3.78}$$

This is the Lie derivative of the complex structure in the direction of the vector k_I.

Multiplying (3.77) with $f^X{}_{i\mathcal{A}} f^{j\mathcal{B}}{}_Y$ proves that $f^Y{}_{i\mathcal{A}}\nabla_Y k_I{}^X f^{j\mathcal{B}}{}_X$ should be proportional to δ_i^j. This leads to the definition of the matrices

$$t_{I\mathcal{A}}{}^{\mathcal{B}} = \frac{1}{2} f^Y{}_{i\mathcal{A}}\nabla_Y k_I{}^X f^{i\mathcal{B}}{}_X , \qquad f^Y{}_{i\mathcal{A}}\nabla_Y k_I{}^X f^{j\mathcal{B}}{}_X = \delta_i^j\, t_{I\mathcal{A}}{}^{\mathcal{B}} . \tag{3.79}$$

These matrices satisfy a reality and an almost covariant constancy equation[15]

$$\left(t_{I\mathcal{A}}{}^{\mathcal{B}}\right)^* = -\rho^{\bar{\mathcal{A}}\mathcal{C}} t_{I\mathcal{C}}{}^{\mathcal{D}} \rho_{\mathcal{D}\bar{\mathcal{B}}} = -t_I{}^{\bar{\mathcal{A}}}{}_{\bar{\mathcal{B}}} , \qquad \nabla_X t_{I\mathcal{A}}{}^{\mathcal{B}} = k_I^Y R_{XYA}{}^B , \tag{3.80}$$

as well as the commutation relations

$$[t_I, t_J]_{\mathcal{B}}{}^{\mathcal{A}} = f_{IJ}{}^K t_{K\mathcal{B}}{}^{\mathcal{A}} - k_I^X k_J^Y R_{XY\mathcal{B}}{}^{\mathcal{A}} , \tag{3.81}$$

which are consistent with (3.56).

The transformation of the fermions under the gauge group follows from the requirement that the commutator of supersymmetry and Killing symmetries vanishes. It is given by the above-defined matrices:

$$\widehat{\delta}_G(\theta)\zeta^{\mathcal{A}} = \theta^I t_{I\mathcal{B}}{}^{\mathcal{A}}(q)\zeta^{\mathcal{B}} . \tag{3.82}$$

For the coupling of the hypermultiplet to the vector gauge multiplets in the presence of the superconformal algebra, these isometries should be consistent with the conformal structure. The requirement that dilatations commute with the isometries is the equation

$$0 = k_{\rm D}{}^Y\partial_Y k_I{}^X - k_I{}^Y\partial_Y k_{\rm D}{}^X = k_{\rm D}{}^Y\nabla_Y k_I{}^X - \frac{D-2}{2} k_I{}^X . \tag{3.83}$$

[15]Note that we defined $t_I{}^{\bar{\mathcal{A}}}{}_{\bar{\mathcal{B}}}$ using the common NW–SE convention for raising and lowering indices, and that the equation implies in this sense that t_I is imaginary.

This implies that the dilatations also commute with the SU(2) transformations generated by $\mathbf{k}^X$, defined in (3.69). This equation can also be written as

$$A^{i\mathcal{B}}t_{I\mathcal{B}}{}^{\mathcal{A}} = \frac{D-2}{2} f^{i\mathcal{A}}{}_X k_I{}^X \,. \tag{3.84}$$

One can also obtain the covariant transformation of $A^{i\mathcal{A}}$ (as for the other transformations in (3.72)), using (3.77) and (3.83)

$$\widehat{\delta}_G(\theta) A^{i\mathcal{A}} = A^{i\mathcal{B}}\theta^I t_{I\mathcal{B}}{}^{\mathcal{A}} \,. \tag{3.85}$$

3.2.3.6 Non-closure Relations in $D = 5$

We now have all the ingredients to understand the case when the isometry with index I is coupled to the gauge symmetry of the vector multiplet (label by index I)—see Sect. 3.2.1.2. The full form of (3.47) is now

$$\begin{aligned} \delta_Q(\epsilon)q^X &= -\mathrm{i}\bar{\epsilon}^i\zeta^{\mathcal{A}} f^X{}_{i\mathcal{A}} \,, \\ \widehat{\delta}_Q(\epsilon)\zeta^{\mathcal{A}} &= \tfrac{1}{2}\mathrm{i}\not{D}q^X f^{iA}{}_X\epsilon_i + \tfrac{1}{2}\sigma^I k_I{}^X f^{i\mathcal{A}}{}_X\epsilon_i \,, \end{aligned} \tag{3.86}$$

with covariant derivatives defined as follows:

$$\begin{aligned} D_\mu q^X &= \partial_\mu q^X - W_\mu^I k_I{}^X \,, \\ \nabla_\mu \zeta^{\mathcal{A}} &\equiv \partial_\mu\zeta^{\mathcal{A}} + \left(\partial_\mu q^X\right)\zeta^{\mathcal{B}}\omega_{X\mathcal{B}}{}^{\mathcal{A}} - W_\mu^I\zeta^{\mathcal{B}}t_{I\mathcal{B}}{}^{\mathcal{A}}. \end{aligned} \tag{3.87}$$

Due to the gaugings, there are extra terms in the supersymmetry transformation of the fermions and the non-closure functions (3.63) and (3.65) are now modified to [14]

$$\begin{aligned} \mathrm{i}\Gamma^{\mathcal{A}} &= \not{\nabla}\zeta^{\mathcal{A}} + \frac{1}{2}W_{\mathcal{BCD}}{}^{\mathcal{A}}\zeta^{\mathcal{B}}\bar{\zeta}^{C}\zeta^{\mathcal{D}} - \mathrm{i}k_I{}^X f^{i\mathcal{A}}{}_X\psi_i{}^I + \mathrm{i}\zeta^{\mathcal{B}}\sigma^I t_{I\mathcal{B}}{}^{\mathcal{A}} \,, \\ \Delta^X &= \Box q^X - \frac{1}{2}\bar{\zeta}^{\mathcal{A}}\gamma_a\zeta^{\mathcal{B}}D^a q^Y W_Y{}^X{}_{\mathcal{AB}} - \frac{1}{4}f^X{}_{i\mathcal{A}}\varepsilon^{ij}f^Y{}_{j\mathcal{E}}\nabla_Y W_{\mathcal{BCD}}{}^{\mathcal{A}}\bar{\zeta}^{\mathcal{E}}\zeta^{\mathcal{D}}\bar{\zeta}^{C}\zeta^{\mathcal{B}} \\ &\quad - k_I^Y \mathbf{J}_Y{}^X\cdot\mathbf{Y}^I + \sigma^I\sigma^J k_J^Y\nabla_Y k_I^X \\ &\quad + 2\mathrm{i}\bar{\psi}^{iI}\zeta^B t_{I\mathcal{B}}{}^{\mathcal{A}} f^X{}_{i\mathcal{A}} - \tfrac{1}{2}\sigma^I k_I^Y W_Y{}^X{}_{\mathcal{AB}}\bar{\zeta}^{\mathcal{A}}\zeta^{\mathcal{B}} \,, \end{aligned} \tag{3.88}$$

where $\Box q^X$ is now also covariant for gauge transformations:

$$\Box q^X = \partial_a D^a q^X - D_a q^Y\nabla_Y k_I^X W^{aI} + D_a q^Y D^a q^Z \Gamma^X_{YZ} \,, \tag{3.89}$$

and we introduced the notation, using (3.41),

$$W_X{}^Y{}_{\mathcal{A}\mathcal{B}} = L_X{}^Y{}_C{}^{\mathcal{D}} W_{\mathcal{A}\mathcal{B}C}{}^{\mathcal{D}} \,. \tag{3.90}$$

3.2.3.7 Rigid Superconformal Case in $D = 4$

To formulate the results in 4 dimensions, we consider the same bosonic fields q^X. The fermionic formulae have to be translated using the rules explained in Appendix A.4. This leads again to $2n_H$ spinors, whose left-handed part is $\zeta^{\mathcal{A}}$, with $\mathcal{A} = 1, \ldots, 2n_H$ and the left-handed ones (C-conjugates of the former) are $\zeta_{\bar{\mathcal{A}}}$. Thus, in absence of an SU(2) index on these spinors, the chirality is indicated by the fact that it has the index $\mathcal{A}$ up or down. One can start again by allowing arbitrary transformations for the scalars, and transformations of the spinors to derivatives of the scalars and deduce again the conditions on quantities that appear in these transformations. We would arrive again at (3.31) and (3.32). But as we have already done all the work for $D = 5$ (for which in fact the formalism is easier) we can also translate the results from what we already know.

This leads in 4 dimensions to the transformations [18, (20.33)]

$$\begin{aligned}
\delta_Q(\epsilon) q^X &= -\mathrm{i} f^X{}_{i\mathcal{A}} \bar{\epsilon}^i \zeta^{\mathcal{A}} + \mathrm{i} f^{Xi\bar{\mathcal{A}}} \bar{\epsilon}_i \zeta_{\bar{\mathcal{A}}} \,, \\
\widehat{\delta}_Q(\epsilon) \zeta^{\mathcal{A}} &= \tfrac{1}{2} \mathrm{i} f^{i\mathcal{A}}{}_X \not{D} q^X \epsilon_i + \mathrm{i} \bar{X}^I k_I{}^X f^{i\mathcal{A}}{}_X \varepsilon_{ij} \epsilon^j \,, \\
\delta_Q(\epsilon) \zeta_{\bar{\mathcal{A}}} &= -\tfrac{1}{2} \mathrm{i} f_{i\bar{\mathcal{A}}X} \not{D} q^X \epsilon^i - \mathrm{i} X^I k_I{}^X f_{i\mathcal{A}X} \varepsilon^{ij} \epsilon_j \,,
\end{aligned} \tag{3.91}$$

where the complex conjugates of the frame fields are denoted as $f^{Xi\bar{\mathcal{A}}} = (f^X{}_{i\mathcal{A}})^*$ and $f_{i\bar{\mathcal{A}}X} = (f^{i\mathcal{A}}{}_X)^*$, see e.g. (A.31). $D_\mu q^X$ is given in (3.87).

The non-closure of the supersymmetries on the fermions is obtained in Appendix A.4 as an example of the translation rules from $D = 5$ to $D = 4$. The result is

$$\left[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)\right] \zeta^{\mathcal{A}} = \xi^\mu \partial_\mu \zeta^{\mathcal{A}} - \tfrac{1}{2} \varepsilon^{ij} \rho^{\bar{\mathcal{B}}\mathcal{A}} \Gamma_{\bar{\mathcal{B}}} \bar{\epsilon}_{1i} \epsilon_{2j} - \tfrac{1}{2} \gamma_\mu \bar{\epsilon}^i_{[1} \gamma^\mu \epsilon_{2]i} \Gamma^{\mathcal{A}} \,, \tag{3.92}$$

with ξ^μ as in (1.6). The non-closure functions are

$$\begin{aligned}
\Gamma^{\mathcal{A}} &\equiv -\not{\nabla} \zeta^{\mathcal{A}} + \tfrac{1}{2} W_{\mathcal{B}C}{}^{\bar{\mathcal{D}}\mathcal{A}} \zeta_{\bar{\mathcal{D}}} \bar{\zeta}^{\mathcal{B}} \zeta^C + 2\bar{X}^I t_I{}^{\bar{\mathcal{B}}\mathcal{A}} \zeta_{\bar{\mathcal{B}}} + \mathrm{i} f^{i\mathcal{A}}{}_X k_I{}^X \varepsilon_{ij} \Omega^{Ij} \,, \\
\Gamma_{\bar{\mathcal{A}}} &\equiv -\not{\nabla} \zeta_{\bar{\mathcal{A}}} + \tfrac{1}{2} W^{\bar{\mathcal{B}}\bar{C}}{}_{\mathcal{D}\bar{\mathcal{A}}} \bar{\zeta}_{\bar{\mathcal{B}}} \zeta_{\bar{C}} \zeta^{\mathcal{D}} + 2X^I t_{I\mathcal{B}\bar{\mathcal{A}}} \zeta^{\mathcal{B}} + \mathrm{i} k_I{}^X f^{i\mathcal{B}}{}_X \Omega^I_i \rho_{\mathcal{B}\bar{\mathcal{A}}} \,,
\end{aligned} \tag{3.93}$$

where $W_{\mathcal{B}C}{}^{\bar{\mathcal{D}}\mathcal{A}} = \rho^{\bar{\mathcal{D}}\mathcal{E}} W_{\mathcal{B}C\mathcal{E}}{}^{\mathcal{A}}$ and $W^{\bar{\mathcal{B}}\bar{C}}{}_{\mathcal{D}\bar{\mathcal{A}}}$ is its complex conjugate. We will raise or lower indices changing the holomorphicity with the tensors $\rho_{\mathcal{A}\bar{\mathcal{B}}}$ in NE–SW convention, e.g.

$$t_I{}^{\bar{\mathcal{A}}\mathcal{B}} = \rho^{\bar{\mathcal{A}}\mathcal{A}} t_{I\mathcal{A}}{}^{\mathcal{B}}\,, \qquad t_{I\mathcal{A}\bar{\mathcal{B}}} = t_{I\mathcal{A}}{}^{\mathcal{B}} \rho_{\mathcal{B}\bar{\mathcal{B}}} = \left(t_I{}^{\bar{\mathcal{A}}\mathcal{B}}\right)^* \,. \tag{3.94}$$

These fermionic non-closure functions transform in real bosonic quantities Δ^X as in (3.64)[16]:

$$\begin{aligned}
\delta(\epsilon)\Gamma^{\mathcal{A}} &= -\tfrac{1}{2}\mathrm{i} f^{i\mathcal{A}}{}_X \epsilon_i \Delta^X \,,\\
\Delta^X &= \Box q^X + 2\left(X^I \bar{X}^J + X^J \bar{X}^I\right) k_I^Y \nabla_Y k_J{}^X - 2k_I{}^Y \mathbf{J}_Y{}^X \cdot \mathbf{Y}^I\\
&\quad + X^I k_I{}^Y W_Y{}^X_{AB} \bar{\zeta}^{\mathcal{A}} \zeta^{\mathcal{B}} + \bar{X}^I k_I{}^Y W_Y{}^{X\bar{\mathcal{A}}\bar{\mathcal{B}}} \bar{\zeta}_{\bar{\mathcal{A}}} \zeta_{\bar{\mathcal{A}}} + \bar{\zeta}^A \gamma_a \zeta_{\bar{\mathcal{B}}} D_a q^Y W_Y{}^X{}_{\mathcal{A}}{}^{\bar{\mathcal{B}}}\\
&\quad + \tfrac{1}{2} f^X{}_{i\mathcal{A}} \varepsilon^{ij} f^Y{}_{j\mathcal{B}} \nabla_Y W^{\bar{\mathcal{D}}\bar{\mathcal{E}}}{}_C{}^A \bar{\zeta}^{\mathcal{B}} \zeta^C \bar{\zeta}_{\bar{\mathcal{D}}} \zeta_{\bar{\mathcal{E}}}\\
&\quad - 2\mathrm{i} f^X{}_{i\mathcal{A}} \bar{\Omega}^{Ii} \zeta_{\bar{\mathcal{B}}} t_I{}^{\bar{\mathcal{B}}\mathcal{A}} + 2\mathrm{i} f^{Xi\bar{\mathcal{A}}} \bar{\Omega}_i^I \zeta^{\mathcal{B}} t_{I\mathcal{B}\bar{\mathcal{A}}} \,.
\end{aligned} \tag{3.95}$$

Finally, for the remaining U(1) factor in the R-symmetry group we find

$$\begin{aligned}
\widehat{\delta}_{\mathrm{U}(1)} q^X &= 0,\\
\widehat{\delta}_{\mathrm{U}(1)} \zeta^{\mathcal{A}} &= \tfrac{1}{2}\mathrm{i}\lambda_T \zeta^{\mathcal{A}}.
\end{aligned} \tag{3.96}$$

3.2.4 Hypermultiplets in Superconformal Gravity

The previous results (Sect. 3.2.3) for rigid hypermultiplets can be generalized to local superconformal invariant theories by properly 'covariantizing' the previous expressions with respect to the superconformal algebra.

3.2.4.1 Case $D = 5$

The supersymmetry rules for the hypermultiplet coupled to the $D = 5$ standard Weyl multiplet and the gauge symmetry of the vector multiplet were found to be [14][17]

$$\begin{aligned}
\delta q^X &= -\mathrm{i}\bar{\epsilon}^i \zeta^{\mathcal{A}} f^X{}_{i\mathcal{A}} \,,\\
\widehat{\delta}\zeta^{\mathcal{A}} &= \tfrac{1}{2}\,\mathrm{i}\not{D} q^X f^{i\mathcal{A}}{}_X \epsilon_i + \tfrac{1}{2}\sigma^I k_I{}^X f^{i\mathcal{A}}{}_X \epsilon_i - A^{i\mathcal{A}} \eta_i \,.
\end{aligned} \tag{3.97}$$

[16] Note that we use here the translation between Y^{ij} and $\mathbf{Y}$ from (A.21), which will be used a lot further on.

[17] A few changes of notation can be found in (C.3).

The new ingredients with respect to (3.86) are the 'matter terms' of the Weyl multiplets and the S-supersymmetry. These transformations and the conformal and R-symmetry transformations determine the superconformal covariant derivatives

$$\begin{aligned}
\mathcal{D}_\mu q^X &= D_\mu q^X + \mathrm{i}\bar\psi^i_\mu \zeta^{\mathcal{A}} f^X{}_{i\mathcal{A}},\\
D_\mu q^X &= \partial_\mu q^X - b_\mu k^X_{\mathrm{D}} - 2\mathbf{V}_\mu \cdot \mathbf{k}^X - W^I_\mu k_I{}^X,\\
\widehat{\mathcal{D}}_\mu \zeta^{\mathcal{A}} &= \widehat{D}_\mu \zeta^{\mathcal{A}} - \tfrac12 \mathrm{i}\not{\mathcal{D}} q^X f^{i\mathcal{A}}{}_X \psi_{\mu i} - \tfrac13 \gamma\cdot T k_{\mathrm{D}}{}^X f^{i\mathcal{A}}{}_X \psi_{\mu i} - \tfrac12 \sigma^I k^X_I f^{i\mathcal{A}}{}_X \psi_{\mu i}\\
&\quad + A^{i\mathcal{A}} \phi_{\mu i}\,, \qquad (3.98)\\
\widehat{D}_\mu \zeta^{\mathcal{A}} &= \partial_\mu \zeta^{\mathcal{A}} + \tfrac14 \omega_\mu{}^{bc}\gamma_{bc}\zeta^{\mathcal{A}} - 2b_\mu \zeta^{\mathcal{A}} - W^I_\mu \zeta^{\mathcal{B}} t_{I\mathcal{B}}{}^{\mathcal{A}} + \partial_\mu q^X \omega_{X\mathcal{B}}{}^{\mathcal{A}} \zeta^{\mathcal{B}}\,.
\end{aligned}$$

The equations of motion for $\zeta^{\mathcal{A}}$ can be obtained by imposing the closure of the superconformal algebra

$$\begin{aligned}
\mathrm{i}\Gamma^{\mathcal{A}} \equiv{}& \not{\mathcal{D}}\zeta^{\mathcal{A}} + \frac12 W_{\mathcal{BCD}}{}^{\mathcal{A}} \zeta^{\mathcal{B}} \bar\zeta^{C} \zeta^{\mathcal{D}} + 2\mathrm{i}\gamma^{ab} T_{ab} \zeta^{\mathcal{A}}\\
& - \mathrm{i}k_I{}^X f^{i\mathcal{A}}{}_X \psi^I_i + \mathrm{i}\zeta^{\mathcal{B}} \sigma^I t_{I\mathcal{B}}{}^{\mathcal{A}} + \tfrac83 \mathrm{i}k_{\mathrm{D}}{}^X f^{i\mathcal{A}}{}_X \chi_i\,. \qquad (3.99)
\end{aligned}$$

3.2.4.2 Case $D = 4$

The covariant supersymmetry transformations are those from (3.91) with only a replacement of D_μ by the fully covariant $\mathcal{D}_\mu$, which are

$$\begin{aligned}
\mathcal{D}_\mu q^X &= D_\mu q^X + \mathrm{i}\bar\psi^i_\mu \zeta^{\mathcal{A}} f^X{}_{i\mathcal{A}} - \mathrm{i}\varepsilon^{ij}\rho^{\bar{\mathcal{A}}\mathcal{B}} \bar\psi_{\mu i} \zeta_{\bar{\mathcal{A}}} f^X{}_{j\mathcal{B}}\,,\\
D_\mu q^X &= \partial_\mu q^X - b_\mu k_{\mathrm{D}}{}^X + 2\mathbf{V}_\mu \cdot \mathbf{k}^X - W_\mu{}^I k_I{}^X\,, \qquad (3.100)\\
\widehat{\mathcal{D}}_\mu \zeta^{\mathcal{A}} &= \hat{D}_\mu \zeta^{\mathcal{A}} - \tfrac12 \mathrm{i} f^{i\mathcal{A}}{}_X \not{\mathcal{D}} q^X \psi_{\mu i} - \mathrm{i}\bar X^I k_I{}^X f^{i\mathcal{A}}{}_X \varepsilon_{ij} \psi^j_\mu - \mathrm{i}A^{i\mathcal{A}} \phi_{\mu i}\,,\\
\hat{D}_\mu \zeta^{\mathcal{A}} &= \left(\partial_\mu + \tfrac14 \omega_\mu{}^{ab}\gamma_{ab} - \tfrac32 b_\mu + \tfrac12 \mathrm{i}A_\mu\right)\zeta^{\mathcal{A}} - W^I_\mu t_{I\mathcal{B}}{}^{\mathcal{A}} \zeta^{\mathcal{B}} + \partial_\mu q^X \omega_{X\mathcal{B}}{}^{\mathcal{A}} \zeta^{\mathcal{B}}\,.
\end{aligned}$$

Note that the hatted covariant derivatives are covariant for target space transformations as well and that $\partial_\mu q^X$ in the last term should not be covariantized to obtain this covariant expression $\widehat{D}_\mu \zeta^{\mathcal{A}}$. Because of central-charge like terms, the algebra does not close on the spinors. The new non-closure functions Γ^A will be used to derive the action for the hypermultiplet, as we will explain in Sect. 3.3.3.

In terms of $A^{i\mathcal{A}}$ (3.70), the covariant transformations are

$$\begin{aligned}
\widehat{\delta} A^{i\mathcal{A}} &\equiv \delta A^{i\mathcal{A}} + A^{i\mathcal{B}} \omega_{X\mathcal{B}}{}^{\mathcal{A}} \delta q^X = -\mathrm{i}\bar\epsilon^i \zeta^{\mathcal{A}} + \mathrm{i}\bar\epsilon_j \zeta_{\bar{\mathcal{B}}} \varepsilon^{ji} \rho^{\bar{\mathcal{B}}\mathcal{A}}\,,\\
\widehat{\delta}\zeta^{\mathcal{A}} &= \tfrac12 \mathrm{i}\widehat{\not{\mathcal{D}}} A^{i\mathcal{A}} \epsilon_i + \mathrm{i}\bar X^I k_I{}^X f^{i\mathcal{A}}{}_X \varepsilon_{ij} \epsilon^j + \mathrm{i}A^{i\mathcal{A}} \eta_i\,, \qquad (3.101)
\end{aligned}$$

where we used

$$\begin{aligned}\widehat{\mathcal{D}}_\mu A^{i\mathcal{A}} &= f^{i\mathcal{A}}{}_X \nabla_Y k_{\mathrm{D}}{}^X \mathcal{D}_\mu q^Y = f^{i\mathcal{A}}{}_X \mathcal{D}_\mu q^X \\ &= f^{i\mathcal{A}}{}_X \partial_\mu q^X - b_\mu A^{i\mathcal{A}} - A^{j\mathcal{A}} V_{\mu j}{}^i - W_\mu^I A^{i\mathcal{B}} t_{I\mathcal{B}}{}^{\mathcal{A}} \\ &\quad + \mathrm{i}\bar{\psi}_\mu^i \zeta^{\mathcal{A}} - \mathrm{i}\bar{\psi}_{\mu j} \zeta_{\bar{\mathcal{B}}} \varepsilon^{ji} \rho^{\bar{\mathcal{B}}\mathcal{A}} .\end{aligned} \tag{3.102}$$

Note that the $\widehat{\delta}$ used in (3.101) has no SU(2) connection, similar as in (3.32).

3.2.5 Tensor Multiplet in $D = 4$ Local Superconformal Case

The tensor multiplet in $D = 4$ dimensions was obtained in [7]. It is in fact the multiplet of the constraints (3.28). We can name these constraints, respectively, as L_{ij}, φ^i, E_b (satisfying a differential constraint) and G. These transform in each other and thus form a multiplet. It starts from an SU(2) triplet L_{ij} (hence satisfying the reality property as in (A.21)). The constrained E_a implies that the multiplet has a gauge tensor $E_{\mu\nu}$ (3 degrees of freedom) and a complex auxiliary G, to balance the 8 fermionic degrees of freedom in φ_i. The transformation rules in the background of conformal supergravity are[18]

$$\begin{aligned}\delta L_{ij} &= \bar{\epsilon}_{(i}\varphi_{j)} + \varepsilon_{ik}\varepsilon_{j\ell}\,\bar{\epsilon}^{(k}\varphi^{\ell)} + 2\lambda_{\mathrm{D}} L_{ij} , \\ \delta\varphi^i &= \tfrac{1}{2}\not{\mathcal{D}} L^{ij}\,\epsilon_j + \tfrac{1}{2}\varepsilon^{ij}\,\not{E}\,\epsilon_j - \tfrac{1}{2} G\,\epsilon^i + 2L^{ij}\,\eta_j + \left(\tfrac{5}{2}\lambda_{\mathrm{D}} + \tfrac{1}{2}\mathrm{i}\lambda_T\right)\varphi^i , \\ \delta G &= -\bar{\epsilon}_i \not{\mathcal{D}}\varphi^i - 3\bar{\epsilon}_i L^{ij}\,\chi_j + \tfrac{1}{8}\bar{\epsilon}_i \gamma^{ab} T^+_{ab}\varphi_j \varepsilon^{ij} + 2\bar{\eta}_i\varphi^i + (3\lambda_{\mathrm{D}} - \mathrm{i}\lambda_T)\, G , \\ \delta E_{\mu\nu} &= \tfrac{1}{4}\mathrm{i}\bar{\epsilon}^i\gamma_{\mu\nu}\varphi^j\,\varepsilon_{ij} - \tfrac{1}{4}\mathrm{i}\bar{\epsilon}_i\gamma_{\mu\nu}\varphi_j\,\varepsilon^{ij} + \tfrac{1}{2}\mathrm{i}L_{ij}\varepsilon^{jk}\bar{\epsilon}^i\gamma_{[\mu}\psi_{\nu]k} - \tfrac{1}{2}\mathrm{i}L^{ij}\varepsilon_{jk}\bar{\epsilon}_i\gamma_{[\mu}\psi_{\nu]}{}^k ,\end{aligned} \tag{3.103}$$

where

$$E^\mu = e^{-1}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu E_{\rho\sigma} - \tfrac{1}{2}\left(\bar{\psi}_\nu^i\gamma^{\mu\nu}\varphi^j\varepsilon_{ij} + \mathrm{h.c.}\right) - \tfrac{1}{2}\mathrm{i}e^{-1}\varepsilon^{\mu\nu\rho\sigma} L_{ij}\varepsilon^{jk}\bar{\psi}_\nu^i\gamma_\rho\psi_{\sigma k} . \tag{3.104}$$

A first step in building actions from this multiplet has been set in [7], but more applications can be found in [8].

[18] Of course the tensor multiplet for rigid supersymmetry can be obtained from (3.103) by setting to zero the fields of the Weyl multiplet (T and ψ_μ) and replacing the covariant derivatives by ordinary derivatives.

3.3 Construction of the Superconformal Actions

This section is devoted to the construction of local superconformally invariant actions for the vector and the hypermultiplet. As we have shown in the example of Sect. 3.1, later one gauge-fixes the extra symmetry such that the remaining theory has just the super-Poincaré invariance. Crucially, as we will explain in Chap. 4, for these last steps one needs to include compensating multiplets. Besides interacting matter, the resulting action from the gauge fixing will contain also the pure gravity sector.

3.3.1 Action for Vector Multiplets in D = 4

Let us consider the basic supergravity multiplet coupled to n vector multiplets. The physical content that one should have (from representation theory of the super-Poincaré group) can be represented in terms of particles with spin as follows:

$$
\begin{array}{cccccccc}
 & \text{SUGRA} & & & & \text{vector multiplet} & & \\
 & 2 & & & & & & \\
\frac{3}{2} & & \frac{3}{2} & & & & & \\
 & 1 & & & & 1 & & \to n+1 \\
 & & & +n\,* & \frac{1}{2} & & \frac{1}{2} & \\
 & & & & 0 & & 0 &
\end{array}
\tag{3.105}
$$

The supergravity sector contains the graviton, 2 gravitini and a so-called graviphoton, W_μ (that is a spin-1 field). When coupled to n vector multiplets, W_μ gets part of a set of $n+1$ vectors, which will be uniformly described by the special Kähler geometry. The scalars inside these vector multiplets appear as n complex fields z^α, with $\alpha = 1, \ldots, n$.

In the framework of superconformal calculus, we consider $n+1$ superconformal vector multiplets with scalars X^I $(I = 0, \ldots, n)$ in the background of the Weyl multiplet (main formulae can be found in Sect. 3.2.1). One of these multiplets should contain the graviphoton, while we will use the missing fermions and scalars to fix superfluous gauge symmetries of the superconformal algebra.

Exploiting the fact that vector multiplets are constrained chiral multiplets (Sect. 3.2.2), we can build an action for the vector multiplet from an action for a chiral multiplet. The lowest component of the chiral multiplet should be $A = \frac{1}{2}\mathrm{i}F(X)$,[19] being then A a new chiral superfield, given by an arbitrary holomorphic function of the scalars in vector multiplets. This function $F(X)$ will determine the

[19] The overall normalization is for later convenience to get a result with the normalization that is most used in the literature

action, and is called the prepotential. The further components are then defined by the transformation laws, which give, comparing with (3.21), $\Psi_i = \frac{1}{2}\mathrm{i}F_I\Omega_i^I$, where we defined

$$F_I(X) = \frac{\partial}{\partial X^I}F(X)\,, \qquad \bar{F}_I(\bar{X}) = \frac{\partial}{\partial \bar{X}^I}\bar{F}(\bar{X})\,,$$
$$F_{IJ} = \frac{\partial}{\partial X^I}\frac{\partial}{\partial X^J}F(X) \qquad \dots\,. \tag{3.106}$$

Calculating the transformation of Ψ_i one finds B_{ij}, G^-_{ab}, ...

$$\begin{aligned}
A &= \tfrac{1}{2}\mathrm{i}F\\
\Psi_i &= \tfrac{1}{2}\mathrm{i}F_I\Omega_i^I\\
B_{ij} &= \tfrac{1}{2}\mathrm{i}F_IY_{ij}^I - \tfrac{1}{8}\mathrm{i}F_{IJ}\bar{\Omega}_i^I\Omega_j^J\\
G^-_{ab} &= \tfrac{1}{2}\mathrm{i}F_I\mathcal{F}^{-I}_{ab} - \tfrac{1}{16}\mathrm{i}F_{IJ}\bar{\Omega}_i^I\gamma_{ab}\Omega_j^J\varepsilon^{ij}\\
\Lambda_i &= -\tfrac{1}{2}\mathrm{i}F_I\not{D}\Omega^{jI}\varepsilon_{ij} - \tfrac{1}{2}\mathrm{i}F_If_{JK}^I\bar{X}^J\Omega_i^K - \tfrac{1}{8}\mathrm{i}F_{IJ}\gamma^{ab}\mathcal{F}^{-I}_{ab}\Omega_i^J\\
&\quad -\tfrac{1}{2}\mathrm{i}F_{IJ}\Omega_k^JY_{ij}^I\varepsilon^{jk} + \tfrac{1}{96}\mathrm{i}F_{IJK}\gamma^{ab}\Omega_i^I\bar{\Omega}_j^J\gamma_{ab}\Omega_k^K\varepsilon^{jk}\\
C &= -\mathrm{i}F_ID_aD^a\bar{X}^I - \tfrac{1}{4}\mathrm{i}F_I\mathcal{F}^{+I}_{ab}T^{+ab} - \tfrac{3}{2}\mathrm{i}F_I\bar{\chi}_i\Omega^{iI} + \tfrac{1}{2}\mathrm{i}F_If_{JK}^I\bar{\Omega}^{iJ}\Omega^{jK}\varepsilon_{ij}\\
&\quad -\mathrm{i}F_If_{JK}^If_{LM}^J\bar{X}^K\bar{X}^LX^M - \tfrac{1}{2}\mathrm{i}F_{IJ}Y^{ijI}Y_{ij}^J + \tfrac{1}{4}\mathrm{i}F_{IJ}\mathcal{F}^{-I}_{ab}\mathcal{F}^{-abJ}\\
&\quad +\tfrac{1}{2}\mathrm{i}F_{IJ}\bar{\Omega}_i^I\not{D}\Omega^{iJ} - \tfrac{1}{2}\mathrm{i}F_{IJ}f_{KL}^I\bar{X}^K\bar{\Omega}_i^J\Omega_j^L\varepsilon^{ij} + \tfrac{1}{4}\mathrm{i}F_{IJK}Y^{ijI}\bar{\Omega}_i^J\Omega_j^K\\
&\quad -\tfrac{1}{16}\mathrm{i}F_{IJK}\varepsilon^{ij}\bar{\Omega}_i^I\gamma^{ab}\mathcal{F}^{-J}_{ab}\Omega_j^K + \tfrac{1}{48}\mathrm{i}F_{IJKL}\bar{\Omega}_i^I\Omega_l^J\bar{\Omega}_j^K\Omega_k^L\varepsilon^{ij}\varepsilon^{kl}.
\end{aligned} \tag{3.107}$$

This is the composite chiral multiplet that we discussed at the end of Sect. 3.2.2, and on which we can apply the 'density formula' (3.30). As mentioned, $F(X)$ must be homogeneous of weight 2, where the X fields carry weight 1. This implies the following relations for the derivatives of F:

$$2F = F_IX^I\,, \qquad F_{IJ}X^J = F_I\,, \qquad F_{IJK}X^K = 0. \tag{3.108}$$

Inserting (3.107) in (3.30) leads to

$$\begin{aligned}
e^{-1}\mathcal{L}_g &= -\mathrm{i}F_ID_aD^a\bar{X}^I + \tfrac{1}{4}\mathrm{i}F_{IJ}\mathcal{F}^{-I}_{ab}\mathcal{F}^{-abJ} + \tfrac{1}{2}\mathrm{i}F_{IJ}\bar{\Omega}_i^I\not{D}\Omega^{iJ}\\
&\quad -\tfrac{1}{2}\mathrm{i}F_{IJ}Y^{ijI}Y_{ij}^J + \tfrac{1}{4}\mathrm{i}F_{IJK}Y^{ijI}\bar{\Omega}_i^J\Omega_j^K\\
&\quad -\tfrac{1}{16}\mathrm{i}F_{IJK}\varepsilon^{ij}\bar{\Omega}_i^I\gamma^{ab}\mathcal{F}^{-J}_{ab}\Omega_j^K + \tfrac{1}{48}\mathrm{i}F_{IJKL}\bar{\Omega}_i^I\Omega_\ell^J\bar{\Omega}_j^K\Omega_k^L\varepsilon^{ij}\varepsilon^{k\ell}\\
&\quad +\tfrac{1}{2}\mathrm{i}F_If_{JK}^I\bar{\Omega}^{iJ}\Omega^{jK}\varepsilon_{ij} - \tfrac{1}{2}\mathrm{i}F_{IJ}f_{KL}^I\bar{X}^K\bar{\Omega}_i^J\Omega_j^L\varepsilon^{ij}
\end{aligned}$$

$$
\begin{aligned}
&-\mathrm{i}F_I f_{JK}^I f_{LM}^J \bar{X}^K \bar{X}^L X^M \\
&-\tfrac{1}{4}\mathrm{i}F_I \mathcal{F}_{ab}^{+I} T^{+ab} - \tfrac{3}{2}\mathrm{i}F_I \bar{\chi}_i \Omega^{iI} - \tfrac{1}{2}\mathrm{i}F_{IJ}\bar{\psi}_i \cdot \gamma \Omega_j^I Y^{ijJ} \\
&+\tfrac{1}{2}\mathrm{i}F_I f_{JK}^I \bar{X}^J \bar{\psi}_i \cdot \gamma \Omega_j^K \varepsilon^{ij} - \tfrac{1}{2}\mathrm{i}F_I \bar{\psi}_i \cdot \gamma \not{D}\Omega^{iI} \\
&+\tfrac{1}{8}\mathrm{i}F_{IJ}\mathcal{F}_{ab}^{-I}\bar{\psi}_i \cdot \gamma \gamma^{ab}\Omega_j^J \varepsilon^{ij} \\
&+\tfrac{1}{12}\mathrm{i}F_{IJK}\bar{\Omega}_\ell^J \Omega_j^K \bar{\psi}_i \cdot \gamma \Omega_k^I \varepsilon^{ij}\varepsilon^{k\ell} + \tfrac{1}{16}\mathrm{i}F_I \bar{\psi}_{\mu i}\gamma \cdot T^+ \gamma^\mu \Omega_j^I \varepsilon^{ij} \\
&-\tfrac{1}{8}\mathrm{i}F T_{ab}^+ T^{+ab} \\
&-\tfrac{1}{2}\mathrm{i}F_I \bar{\psi}_{\mu i}\gamma^{\mu\nu}\psi_{\nu j} Y^{ijI} - \tfrac{1}{2}\mathrm{i}F T^{+\mu\nu}\bar{\psi}_{\mu i}\psi_{\nu j}\varepsilon^{ij} + \tfrac{1}{2}\mathrm{i}F_I \mathcal{F}^{-\mu\nu I}\bar{\psi}_{\mu i}\psi_{\nu j}\varepsilon^{ij} \\
&-\tfrac{1}{16}\mathrm{i}F_{IJ}\bar{\psi}_{\mu i}\psi_{\nu j}\bar{\Omega}_k^I \gamma^{\mu\nu}\Omega_\ell^J \varepsilon^{ij}\varepsilon^{k\ell} + \tfrac{1}{8}\mathrm{i}F_{IJ}\bar{\Omega}_k^I \Omega_\ell^J \bar{\psi}_{\mu i}\gamma^{\mu\nu}\psi_{\nu j}\varepsilon^{ik}\varepsilon^{j\ell} \\
&-\tfrac{1}{4}\varepsilon^{ij}\varepsilon^{k\ell}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu i}\psi_{\nu j}\bar{\psi}_{\rho k}\left(\gamma_\sigma F_I \Omega_\ell^I + F\psi_{\sigma\ell}\right) + \text{h.c.}
\end{aligned} \tag{3.109}
$$

The first terms of the action (3.109) are kinetic terms for the scalars X, the vectors and the fermions Ω. The following term says that Y_{ij} is an auxiliary field that can be eliminated by its field equation. The first 5 lines are the ones that we would encounter also in rigid supersymmetry, see [18, (20.15)]. For these terms, the relations (3.108) have not been used, and this part is thus the general result for rigid supersymmetry. The other lines are due to the local superconformal symmetry. For those interested in rigid symmetry, we repeat that in that case the covariant derivatives (3.19) reduce to, e.g.,

$$
\begin{aligned}
D_a X^I &= \partial_a X^I - W_a^K X^J f_{JK}{}^I , \\
D_a \Omega_i^I &= \partial_a \Omega^I - W_a^K \Omega_i^J f_{JK}{}^I , \\
\mathcal{F}_{ab}^I &= 2\partial_{[a} W_{b]}^I + W_b^K W_a^J f_{JK}{}^I .
\end{aligned} \tag{3.110}
$$

Note that the Lagrangian is a total derivative if $F(X)$ is a quadratic function of X^I with real coefficients:

$$
F(X) = C_{IJ} X^I X^J , \qquad C_{IJ} \in \mathbb{R} \qquad \to \qquad S = \int \mathrm{d}^4 x\, \mathcal{L}_g = 0 . \tag{3.111}
$$

In deriving the above formulae, we assumed for simplicity that F is a gauge-invariant function such that $\mathcal{L}_g$ is invariant under gauge transformations. However, the property (3.111) suggests that there is a more general situation [37, 38] in which F transforms under the gauge transformations as

$$
\delta_G(\theta) F \equiv F_I \theta^K X^J f_{JK}{}^I = -\theta^I C_{I,JK} X^J X^K , \tag{3.112}
$$

where $C_{I,JK}$ are real constants. In fact, due to (3.111), the action is invariant for rigid transformations that satisfy (3.112), transforming to

$$\delta_G \mathcal{L}_g = \tfrac{1}{3} C_{I,JK} \varepsilon^{\mu\nu\rho\sigma} \theta^I F_{\mu\nu}^J F_{\rho\sigma}^K \,. \tag{3.113}$$

In order to allow this extra possibility with local θ^I, one has to add to (3.109) a Chern–Simons term

$$\mathcal{L}_{CS} = \tfrac{2}{3} C_{I,JK} \varepsilon^{\mu\nu\rho\sigma} W_\mu{}^I W_\nu{}^J \left(\partial_\rho W_\sigma{}^K + \tfrac{3}{8} f_{LM}{}^K W_\rho{}^L W_\sigma{}^M \right) . \tag{3.114}$$

To prove the supersymmetry invariance of $\mathcal{L}_g + \mathcal{L}_{CS}$ one needs a few more relations that follow from (3.113). Replacing the arbitrary θ^K by X^K the variation vanishes, and thus for the consistency of (3.112) we should have

$$C_{(I,JK)} X^I X^J X^K = 0 \,, \tag{3.115}$$

namely the completely symmetric part of $C_{I,JK}$ must vanish.

By taking two derivatives of (3.112) we obtain

$$C_{K,IJ} = f_{K(I}{}^L F_{J)L} - \tfrac{1}{2} F_{IJL} X^M f_{MK}{}^L = f_{K(I}{}^L \bar{F}_{J)L} - \tfrac{1}{2} \bar{F}_{IJL} \bar{X}^M f_{MK}{}^L \,. \tag{3.116}$$

To prove the invariance of the sum of (3.109) and (3.114) one needs an identity [37]

$$f_{KL}{}^M C_{M,IJ} = 2 f_{J[K}{}^M C_{L],IM} + 2 f_{I[K}{}^M C_{L],JM} \,, \tag{3.117}$$

which follows from the requirement that the gauge group closes on $F(X)$. A simple example of the occurrence of a Chern–Simons term is given in [37, (3.21)].

3.3.1.1 Simplifications

In order to get a more useful form of the action, one has to make the conformal covariant derivatives explicit. The principle is explained for the bosonic case in (3.2). This leads here to

$$\begin{aligned} \Box^C \bar{X}^I = {} & \widehat{\partial}_\mu D^\mu \bar{X}^I - \omega_\mu{}^{\mu\nu} D_\nu \bar{X}^I - \mathrm{i} W_\mu D^\mu \bar{X}^I + 2 f_\mu{}^\mu \bar{X}^I - \tfrac{1}{2} \bar{\psi}_{\mu i} D^\mu \Omega^{iI} \\ & + \tfrac{1}{32} \bar{\psi}_\mu^i \gamma^\mu \gamma \cdot T^+ \Omega^{jI} \varepsilon_{ij} - \tfrac{1}{2} \bar{\Omega}^{iI} \gamma \cdot \phi_i - \tfrac{3}{4} \bar{\psi}^i \cdot \gamma \chi_i \bar{X}^I \\ & - \tfrac{1}{2} \varepsilon^{ij} \bar{\psi}_i \cdot \gamma \Omega_j^J \bar{X}^K f_{JK}^I - \tfrac{1}{2} \varepsilon_{ij} \bar{\psi}^i \cdot \gamma \Omega^{jJ} \bar{X}^K f_{JK}^I . \end{aligned} \tag{3.118}$$

Hence the first term of (3.109) after adding a total derivative, is

$$\begin{aligned}
-\mathrm{i}F_I\square^C\bar{X}^I =& \mathrm{i}F_{IJ}\mathcal{D}_\mu X^I\left(\mathcal{D}^\mu\bar{X}^J-\tfrac{1}{2}\bar{\psi}_i^\mu\Omega^{iJ}\right)-2\mathrm{i}F_I f_\mu{}^\mu\bar{X}^I+\tfrac{1}{2}\mathrm{i}F_I\bar{\psi}_{\mu i}D^\mu\Omega^{iI}\\
&-\tfrac{1}{32}\mathrm{i}F_I\bar{\psi}_\mu^i\gamma^\mu\gamma\cdot T^-\tfrac{1}{2}\Omega^{jI}\varepsilon_{ij}+\tfrac{1}{2}\mathrm{i}F_I\bar{\Omega}^{iI}\gamma\cdot\phi_i+\tfrac{3}{4}\mathrm{i}F_I\bar{\psi}^i\cdot\gamma\chi_i\bar{X}^I\\
&+\tfrac{1}{2}\mathrm{i}F_I\varepsilon^{ij}\bar{\psi}_i\cdot\gamma\Omega_j^J\bar{X}^K f_{JK}^I+\tfrac{1}{2}\mathrm{i}F_I\varepsilon_{ij}\bar{\psi}^i\cdot\gamma\Omega^{jJ}\bar{X}^K f_{JK}^I\\
&+\mathrm{i}F_I\bar{\psi}_{[\mu}^i\gamma^\nu\psi_{\nu]i}\left(\mathcal{D}^\mu\bar{X}^I-\tfrac{1}{2}\bar{\psi}_i^\mu\Omega^{iI}\right)\\
&+\text{ total derivative.}
\end{aligned}\tag{3.119}$$

The other term that has to be written explicitly is the covariant derivative of the fermions

$$\begin{aligned}
\not{D}\Omega^{iJ} =& \not{\mathcal{D}}\Omega^{iJ}-\gamma^\mu\gamma^\nu\psi_\mu^i\left(\mathcal{D}_\nu\bar{X}^J-\tfrac{1}{2}\bar{\psi}_{\nu j}\Omega^{jJ}\right)\\
&-\tfrac{1}{4}\gamma_\mu\gamma\cdot\mathcal{F}^{+J}\psi_j^\mu\varepsilon^{ij}-\gamma\cdot\psi_j\left(Y^{ijJ}+\varepsilon^{ij}\bar{X}^K X^L f_{KL}^J\right)-2\bar{X}^J\gamma\cdot\phi^i\,.
\end{aligned}\tag{3.120}$$

Deleting total derivatives, the action is at this point (adding also (3.114))

$$\begin{aligned}
e^{-1}\mathcal{L}_g =& \mathrm{i}F_{IJ}\mathcal{D}_\mu X^I\mathcal{D}^\mu\bar{X}^J-2\mathrm{i}F_I f_\mu{}^\mu\bar{X}^I+\tfrac{1}{4}\mathrm{i}F_{IJ}\mathcal{F}_{ab}^{-I}\mathcal{F}^{-abJ}-\tfrac{1}{8}\mathrm{i}F T_{ab}^+T^{+ab}\\
&-\tfrac{1}{4}\mathrm{i}F_I\mathcal{F}_{ab}^{+I}T^{+ab}-\tfrac{1}{2}\mathrm{i}F_{IJ}Y^{ijI}Y_{ij}^J-\mathrm{i}F_I f_{JK}^I f_{LM}^J\bar{X}^K\bar{X}^L X^M+e^{-1}\mathcal{L}_{CS}\\
&+\mathrm{i}F_I\bar{X}^I\bar{\psi}_{\mu i}\gamma^{\mu\nu}\phi_\nu^i+\tfrac{1}{2}\mathrm{i}F_{IJ}\bar{\Omega}_i^I\not{\mathcal{D}}\Omega^{iJ}+\tfrac{1}{2}\mathrm{i}F_I\bar{\psi}_{\mu i}\gamma^{\mu\nu}\gamma^\rho\mathcal{D}_\rho\bar{X}^I\psi_\nu^i\\
&+\tfrac{1}{2}\mathrm{i}F_I\bar{\Omega}^{iI}\gamma\cdot\phi_i\\
&+\tfrac{1}{8}\mathrm{i}F_I\bar{\psi}_{\mu i}\gamma^{\mu\nu}\gamma\cdot\mathcal{F}^{+I}\varepsilon^{ij}\psi_{\nu j}-\tfrac{1}{32}\mathrm{i}F_I\bar{\psi}_\mu^i\gamma^\mu\gamma\cdot T^+\Omega^{jI}\varepsilon_{ij}\\
&+\tfrac{3}{4}\mathrm{i}F_I\bar{\psi}^i\cdot\gamma\chi_i\bar{X}^I\\
&+\tfrac{1}{2}\mathrm{i}F_I\varepsilon^{ij}\bar{\psi}_i\cdot\gamma\Omega_j^J\bar{X}^K f_{JK}^I+\tfrac{1}{2}\mathrm{i}F_I\varepsilon_{ij}\bar{\psi}^i\cdot\gamma\Omega^{jJ}\bar{X}^K f_{JK}^I\\
&+\mathrm{i}F_I\bar{\psi}_{[\mu}^i\gamma^\nu\psi_{\nu]i}\left(\mathcal{D}^\mu\bar{X}^I-\tfrac{1}{2}\bar{\psi}_i^\mu\Omega^{iI}\right)-\tfrac{3}{2}\mathrm{i}F_I\bar{\chi}_i\Omega^{iI}\\
&+\tfrac{1}{2}\mathrm{i}F_I f_{JK}^I\bar{\Omega}^{iJ}\Omega^{jK}\varepsilon_{ij}\\
&-\tfrac{1}{2}\mathrm{i}F_{IJ}\mathcal{D}_\mu X^J\bar{\psi}_i^\mu\Omega^{iI}-\tfrac{1}{2}\mathrm{i}F_I\bar{\psi}_{\mu i}\gamma^{\mu\nu}\mathcal{D}_\nu\Omega^{iI}\\
&-\tfrac{1}{2}\mathrm{i}F_{IJ}\bar{\Omega}_i^I\gamma^\mu\gamma^\nu\psi_\mu^i\mathcal{D}_\nu\bar{X}^J+\tfrac{1}{2}\mathrm{i}F_I\mathcal{F}^{-\mu\nu I}\bar{\psi}_{\mu i}\psi_{\nu j}\varepsilon^{ij}\\
&-\tfrac{1}{8}\mathrm{i}F_{IJ}\bar{\Omega}_i^I\gamma_\mu\gamma\cdot\mathcal{F}^{+J}\psi_j^\mu\varepsilon^{ij}-\tfrac{1}{2}\mathrm{i}F_{IJ}\bar{\Omega}_i^I\gamma\cdot\psi_j\varepsilon^{ij}\bar{X}^K X^L f_{KL}^J\\
&-\mathrm{i}F_{IJ}\bar{X}^J\bar{\Omega}_i^I\gamma\cdot\phi^i-\tfrac{1}{2}\mathrm{i}F_{IJ}f_{KL}^I\bar{X}^K\bar{\Omega}_i^J\Omega_j^L\varepsilon^{ij}+\tfrac{1}{4}\mathrm{i}F_{IJK}Y^{ijI}\bar{\Omega}_i^J\Omega_j^K\\
&-\tfrac{1}{16}\mathrm{i}F_{IJK}\varepsilon^{ij}\bar{\Omega}_i^I\gamma^{ab}\mathcal{F}_{ab}^{-J}\Omega_j^K
\end{aligned}$$

$$
\begin{aligned}
&-\tfrac{1}{2}\mathrm{i}\varepsilon^{ij}\bar{\psi}_{\mu i}\left(FT^{+\mu\nu}\psi_{\nu j}-\tfrac{1}{8}\gamma\cdot T^{+}\gamma^{\mu}F_I\Omega_j^I\right)\\
&+\tfrac{1}{2}\mathrm{i}F_I f_{JK}^I\bar{X}^J\bar{\psi}_i\cdot\gamma\Omega_j^K\varepsilon^{ij}+\tfrac{1}{8}\mathrm{i}F_{IJ}\mathcal{F}_{ab}^{-I}\bar{\psi}_i\cdot\gamma\gamma^{ab}\Omega_j^J\varepsilon^{ij}\\
&+\tfrac{1}{12}\mathrm{i}F_{IJK}\bar{\Omega}_\ell^J\Omega_j^K\bar{\psi}_i\cdot\gamma\Omega_k^I\varepsilon^{ij}\varepsilon^{k\ell}-\tfrac{1}{4}\mathrm{i}F_I\bar{\psi}_{\mu i}\gamma^{\mu\nu}\gamma^{\rho}\psi_\nu^i\bar{\psi}_{\rho k}\Omega^{kI}\\
&+\tfrac{1}{8}\mathrm{i}F_{IJ}\bar{\Omega}_k^I\Omega_\ell^J\bar{\psi}_{\mu i}\gamma^{\mu\nu}\psi_{\nu j}\varepsilon^{ik}\varepsilon^{j\ell}+\tfrac{1}{4}\mathrm{i}F_{IJ}\bar{\Omega}_i^I\gamma^{\mu}\gamma^{\nu}\psi_\mu^i\bar{\psi}_{\nu j}\Omega^{jJ}\\
&+\tfrac{1}{48}\mathrm{i}F_{IJKL}\bar{\Omega}_i^I\Omega_\ell^J\bar{\Omega}_j^K\Omega_k^L\varepsilon^{ij}\varepsilon^{k\ell}-\tfrac{1}{16}\mathrm{i}F_{IJ}\bar{\psi}_{\mu i}\psi_{\nu j}\bar{\Omega}_k^I\gamma^{\mu\nu}\Omega_\ell^J\varepsilon^{ij}\varepsilon^{k\ell}\\
&-\tfrac{1}{4}\varepsilon^{ij}\varepsilon^{k\ell}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu i}\psi_{\nu j}\bar{\psi}_{\rho k}\left(\gamma_\sigma F_I\Omega_\ell^I+F\psi_{\sigma\ell}\right)+\text{h.c.}
\end{aligned}
\tag{3.121}
$$

We use then (3.16) and the values of the conformal gauge fields that follow from the constraints:

$$
\begin{aligned}
f_\mu{}^\mu &= -\tfrac{1}{12}R-\tfrac{1}{2}D\\
&\quad+\left\{\tfrac{1}{8}\bar{\psi}^i\cdot\gamma\chi_i+\tfrac{1}{24}\mathrm{i}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu^i\gamma_\nu\mathcal{D}_\rho\psi_{\sigma i}+\tfrac{1}{24}\bar{\psi}_\mu^i\psi_\nu^j\varepsilon_{ij}T^{+\mu\nu}+\text{h.c.}\right\},\\
\phi_\mu^i &= \tfrac{1}{4}\gamma_\mu\chi^i+\tfrac{1}{4}\left(\gamma^{\nu\rho}\gamma_\mu-\tfrac{1}{3}\gamma_\mu\gamma^{\nu\rho}\right)\left(\mathcal{D}_\nu\psi_\rho^i-\tfrac{1}{16}\gamma\cdot T^{-}\varepsilon^{ij}\gamma_\nu\psi_{\rho j}\right).
\end{aligned}
\tag{3.122}
$$

This leads to various simplifications, after which the vector action reduces to [18, (20.89)]:

$$
\begin{aligned}
e^{-1}\mathcal{L}_g &= -\tfrac{1}{6}N\,R-N\,D-N_{IJ}D_\mu X^I D^\mu\bar{X}^J+N_{IJ}\mathbf{Y}^I\cdot\mathbf{Y}^J\\
&\quad+N_{IJ}f_{KL}{}^I\bar{X}^K X^L f_{MN}{}^J\bar{X}^M X^N+e^{-1}\mathcal{L}_{CS}\\
&\quad+\Big\{-\tfrac{1}{4}\mathrm{i}\bar{F}_{IJ}\widehat{F}_{\mu\nu}^{+I}\widehat{F}^{+\mu\nu J}-\tfrac{1}{16}N_{IJ}X^IX^JT_{ab}^{+}T^{+ab}+\tfrac{1}{4}N_{IJ}X^I\widehat{F}_{ab}^{+J}T^{+ab}\\
&\quad-\tfrac{1}{4}N_{IJ}\bar{\Omega}^{iI}\not{D}\Omega_i^J+\tfrac{1}{6}N\,\bar{\psi}_{i\mu}\gamma^{\mu\nu\rho}D_\nu\psi_\rho^i\\
&\quad-\tfrac{1}{2}N\,\bar{\psi}_{ia}\gamma^a\chi^i+N_{IJ}X^I\bar{\Omega}^{iJ}\chi_i-\tfrac{1}{3}N_{IJ}X^J\bar{\Omega}^{iI}\gamma^{\mu\nu}D_\mu\psi_{\nu i}\\
&\quad+\tfrac{1}{2}N_{IJ}\bar{\psi}_\mu^i\not{D}\bar{X}^I\gamma^\mu\Omega_i^J+\tfrac{1}{4}N_{IJ}\bar{X}^I\bar{\psi}_{ai}\gamma^{abc}\psi_b^iD_cX^J\\
&\quad+\tfrac{1}{2}\mathrm{i}\bar{F}_{IJ}\varepsilon_{ij}\left(\bar{\Omega}^{iI}\gamma_\mu-\bar{X}^I\bar{\psi}_\mu^i\right)\psi_\nu^j\tilde{\hat{F}}^{\mu\nu J}-\tfrac{1}{16}\mathrm{i}F_{IJK}\bar{\Omega}_i^I\gamma^{\mu\nu}\Omega_j^J\varepsilon^{ij}F_{\mu\nu}^{-K}\\
&\quad+\tfrac{1}{2}N_{IJ}\bar{\Omega}_i^I f_{KL}{}^J\left(\Omega_j^L+\gamma^a\psi_{aj}X^L\right)\bar{X}^K\varepsilon^{ij}\\
&\quad+\left(\tfrac{1}{12}N\,\bar{\psi}_i^a\psi_j^b-\tfrac{1}{6}N_{IJ}\bar{X}^I\bar{\Omega}_i^J\gamma^a\psi_j^b+\tfrac{1}{32}\mathrm{i}F_{IJK}\bar{\Omega}_i^I\gamma^{ab}\Omega_j^J\bar{X}^K\right)T_{ab}^{-}\varepsilon^{ij}\\
&\quad-\tfrac{1}{4}\mathrm{i}F_{IJK}D_\mu X^I\bar{\Omega}_i^J\gamma^\mu\Omega^{iK}+\tfrac{1}{4}\mathrm{i}F_{IJK}Y^{ijI}\bar{\Omega}_i^J\Omega_j^K+\text{h.c.}\Big\}\\
&\quad+\text{4-fermion terms}\,.
\end{aligned}
\tag{3.123}
$$

Here, important quantities are introduced, which will often be used below:

$$N_{IJ} = N_{IJ}(X,\bar{X}) \equiv 2\,\mathrm{Im}\,F_{IJ} = -\mathrm{i}F_{IJ} + \mathrm{i}\bar{F}_{IJ}\,, \qquad N \equiv N_{IJ}X^I\bar{X}^J\,. \tag{3.124}$$

Since F_{IJ} is a function of X^K, we have the chain rule for the gauge transformations:

$$\delta F_{IJ} = F_{IJK}\delta X^K = F_{IJK}\theta^L X^M f_{ML}{}^K\,, \tag{3.125}$$

and therefore the gauge transformation of N_{IJ} is by (3.116)

$$\delta N_{IJ} = 2\theta^K f_{K(I}{}^L N_{J)L}\,. \tag{3.126}$$

Covariant derivatives are presented in (3.19) together with

$$D_\mu\psi_{\nu i} = \left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \tfrac{1}{2}b_\mu + \tfrac{1}{2}\mathrm{i}A_\mu\right)\psi_{\nu i} + V_{\mu i}{}^j\psi_{\nu j}\,. \tag{3.127}$$

3.3.2 Action for Vector Multiplets in D = 5

The scheme for $D = 5$ is similar to (3.105) with the only exception that vector multiplets[20] have only one real scalar, and thus at the end we will have n real scalars, which are the σ^I that we saw already in Sect. 3.2.1. For $D = 6$, which we will not treat in detail here, it is also be similar but without scalars in the vector multiplets.

The rigid superconformally invariant action $D = 5$ is determined by a prepotential $C_{IJK}\sigma^I\sigma^J\sigma^K$ cubic in the scalars σ^I [39, 40]. Since the vectors W_μ^I can gauge a group, I is also the index of the adjoint of the gauge group. As a consequence, the local superconformal action is determined by a gauge-invariant symmetric tensor C_{IJK}

$$f_{I(J}{}^M C_{KL)M} = 0. \tag{3.128}$$

[20] We use here and below freely the terminology 'spin 1' for vectors, spin-$\frac{1}{2}$ for spinors, ..., though of course only in 4 dimensions the representations of the little group of the Lorentz group can be characterized by just one number, which is called 'spin'. In higher dimensions, the representations should be characterized by more numbers, but often the same fields, like graviton as a symmetric tensor, vectors, ... occur, and we denote them freely with the terminology that is appropriate for the 4-dimensional fields.

Note that this tensor has no relation to the tensor $C_{I,JK}$ introduced for $D = 4$ in (3.112). The presence of this tensor allows Chern–Simons terms in the action. The kinetic term of the scalars defines the 'very special real geometry' (see [18, Sect. 20.3.2]). The full superconformal invariant action takes the form [5]

$$\begin{aligned}
e^{-1}\mathcal{L}_g = C_{IJK}\Big[&\Big(-\tfrac{1}{4}\widehat{F}^I_{\mu\nu}\widehat{F}^{\mu\nu J} - \tfrac{1}{2}\bar{\psi}^I\not{\mathcal{D}}\psi^J + \tfrac{1}{3}\sigma^I\Box^c\sigma^J \\
&+\tfrac{1}{6}\mathcal{D}_a\sigma^I\mathcal{D}^a\sigma^J + 2\mathbf{Y}^I\cdot\mathbf{Y}^J\Big)\sigma^K \\
&-\tfrac{4}{3}\sigma^I\sigma^J\sigma^K\left(D + \tfrac{26}{3}T_{ab}T^{ab}\right) + 4\sigma^I\sigma^J\widehat{F}^K_{ab}T^{ab} \\
&-\tfrac{1}{8}\mathrm{i}\bar{\psi}^I\gamma\cdot\widehat{F}^J\psi^K - \tfrac{1}{2}\mathrm{i}\bar{\psi}^{iI}\psi^{jJ}Y^K_{ij} + \mathrm{i}\sigma^I\bar{\psi}^J\gamma\cdot T\psi^K - 8\mathrm{i}\sigma^I\sigma^J\bar{\psi}^K\chi \\
&+\tfrac{1}{6}\sigma^I\bar{\psi}_\mu\gamma^\mu\Big(\mathrm{i}\sigma^J\not{\mathcal{D}}\psi^K + \tfrac{1}{2}\mathrm{i}(\not{\mathcal{D}}\sigma^J)\psi^K - \tfrac{1}{4}\gamma\cdot\widehat{F}^J\psi^K + 2\sigma^J\gamma\cdot T\psi^K \\
&-8\sigma^J\sigma^K\chi\Big) \\
&-\tfrac{1}{6}\bar{\psi}_a\gamma_b\psi^I\left(\sigma^J\widehat{F}^{abK} - 8\sigma^J\sigma^K T^{ab}\right) - \tfrac{1}{12}\sigma^I\bar{\psi}_\lambda\gamma^{\mu\nu\lambda}\psi^J\widehat{F}^K_{\mu\nu} \\
&+\tfrac{1}{12}\mathrm{i}\sigma^I\bar{\psi}_a\psi_b\left(\sigma^J\widehat{F}^{abK} - 8\sigma^J\sigma^K T^{ab}\right) + \tfrac{1}{48}\mathrm{i}\sigma^I\sigma^J\bar{\psi}_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho\widehat{F}^K_{\mu\nu} \\
&-\tfrac{1}{2}\sigma^I\bar{\psi}^i_\mu\gamma^\mu\psi^{jJ}Y^K_{ij} + \tfrac{1}{6}\mathrm{i}\sigma^I\sigma^J\bar{\psi}^i_\mu\gamma^{\mu\nu}\psi^j_\nu Y^K_{ij} - \tfrac{1}{24}\mathrm{i}\bar{\psi}_\mu\gamma_\nu\psi^I\bar{\psi}^J\gamma^{\mu\nu}\psi^K \\
&+\tfrac{1}{12}\mathrm{i}\bar{\psi}^i_\mu\gamma^\mu\psi^{jI}\bar{\psi}^J_i\psi^K_j - \tfrac{1}{48}\sigma^I\bar{\psi}_\mu\psi_\nu\bar{\psi}^J\gamma^{\mu\nu}\psi^K + \tfrac{1}{24}\sigma^I\bar{\psi}^i_\mu\gamma^{\mu\nu}\psi^j_\nu\bar{\psi}^J_i\psi^K_j \\
&-\tfrac{1}{12}\sigma^I\bar{\psi}_\lambda\gamma^{\mu\nu\lambda}\psi^J\bar{\psi}_\mu\gamma_\nu\psi^K + \tfrac{1}{24}\mathrm{i}\sigma^I\sigma^J\bar{\psi}_\lambda\gamma^{\mu\nu\lambda}\psi^K\bar{\psi}_\mu\psi_\nu \\
&+\tfrac{1}{48}\mathrm{i}\sigma^I\sigma^J\bar{\psi}_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho\bar{\psi}_\mu\gamma_\nu\psi^K + \tfrac{1}{96}\sigma^I\sigma^J\sigma^K\bar{\psi}_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho\bar{\psi}_\mu\psi_\nu \\
&-\tfrac{1}{24}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}W^I_\mu\Big(F^J_{\nu\lambda}F^K_{\rho\sigma} - f_{FG}{}^J W^F_\nu W^G_\lambda\left(\tfrac{1}{2}F^K_{\rho\sigma} - \tfrac{1}{10}f_{HL}{}^K W^H_\rho W^L_\sigma\right)\Big) \\
&+\tfrac{1}{4}\mathrm{i}\sigma^I\sigma^J f_{LM}{}^K\bar{\psi}^L\psi^M\Big],
\end{aligned} \tag{3.129}$$

where covariant derivatives and $\widehat{F}^I_{\mu\nu}$ are given in (3.12) and (3.13), and the superconformal d'Alembertian is defined as

$$\begin{aligned}
\Box^c\sigma^I &= \mathcal{D}^a\mathcal{D}_a\sigma^I \\
&= \left(\partial^a - 2b^a + \omega_b{}^{ba}\right)D_a\sigma^I + f_{JK}{}^I W^J_a\mathcal{D}^a\sigma^K - \tfrac{1}{2}\mathrm{i}\bar{\psi}_\mu D^\mu\psi^I - 2\sigma^I\bar{\psi}_\mu\gamma^\mu\chi \\
&\quad+\tfrac{1}{2}\bar{\psi}_\mu\gamma^\mu\gamma\cdot T\psi^I + \tfrac{1}{2}\bar{\phi}_\mu\gamma^\mu\psi^I + 2f_\mu{}^\mu\sigma^I - \tfrac{1}{2}\bar{\psi}_\mu\gamma^\mu f_{JK}{}^I\psi^J\sigma^K.
\end{aligned} \tag{3.130}$$

The dependent gauge fields are given in (2.100).

3.3.3 *Action for Hypermultiplets*

While the actions of vector multiplets were constructed using tensor calculus manipulations, for the hypermultiplets we use another procedure. The main difference is that we have already the equations of motion from the non-closure relations, e.g. (3.99) in $D = 5$, and we can therefore infer the action from the latter. To this end, we need a few ingredients that we are going to introduce in the following.

3.3.3.1 Ingredients

We first define a covariantly constant antisymmetric tensor $C_{\mathcal{A}\mathcal{B}}(q)$ that describes the proportionality between the field equations for the fermions $\zeta^{\mathcal{A}}$ and the non-closure functions. For example, in $D = 5$,

$$\frac{\delta S_{\text{hyper}}}{\delta \bar{\zeta}^{\mathcal{A}}} = 2C_{\mathcal{A}\mathcal{B}}\mathrm{i}\Gamma^{\mathcal{B}} . \tag{3.131}$$

Then, once the right-hand side of (3.131) is known, one can functionally integrate the above equation in order to obtain the action. The properties of the tensor are (independent whether we consider $D = 5$ or $D = 4$):

$$\begin{aligned} \nabla_X C_{\mathcal{A}\mathcal{B}} &\equiv \partial_X C_{\mathcal{A}\mathcal{B}} + 2\omega_{X[\mathcal{A}}{}^{\mathcal{C}} C_{\mathcal{B}]\mathcal{C}} = 0 , \\ C_{\mathcal{A}\mathcal{B}} &= -C_{\mathcal{B}\mathcal{A}} , \\ C^{\bar{\mathcal{A}}\bar{\mathcal{B}}} &\equiv (C_{\mathcal{A}\mathcal{B}})^* = \rho^{\bar{\mathcal{A}}\mathcal{C}} \rho^{\bar{\mathcal{B}}\mathcal{D}} C_{\mathcal{C}\mathcal{D}} . \end{aligned} \tag{3.132}$$

As will become clear below, the kinetic terms involve the Hermitian metric in tangent space

$$\begin{aligned} d^{\bar{\mathcal{A}}}{}_{\mathcal{B}} &\equiv -\rho^{\bar{\mathcal{A}}\mathcal{C}} C_{\mathcal{C}\mathcal{B}} , \\ d^{\bar{\mathcal{A}}}{}_{\mathcal{B}} &= (d^{\bar{\mathcal{B}}}{}_{\mathcal{A}})^* = \rho^{\bar{\mathcal{A}}\mathcal{C}} d^{\bar{\mathcal{D}}}{}_{\mathcal{C}} \rho_{\mathcal{B}\bar{\mathcal{D}}} , \end{aligned} \tag{3.133}$$

such that

$$C_{\mathcal{A}\mathcal{B}} = \rho_{\mathcal{A}\bar{\mathcal{C}}} d^{\bar{\mathcal{C}}}{}_{\mathcal{B}} . \tag{3.134}$$

We also define an inverse

$$C^{\mathcal{A}\mathcal{C}} C_{\mathcal{B}\mathcal{C}} = \delta^{\mathcal{A}}{}_{\mathcal{B}} , \tag{3.135}$$

so that we can use these matrices to raise and lower $\mathcal{A}$ indices, using the common NE–SW convention

$$V_{\mathcal{A}} = V^{\mathcal{B}} C_{\mathcal{B}\mathcal{A}}\,, \qquad V^{\mathcal{A}} = C^{\mathcal{A}\mathcal{B}} V_{\mathcal{B}}\,. \tag{3.136}$$

On the other hand, we raise or lower indices changing the holomorphicity as in (3.94). This is then consistent with changing the holomorphicity using $d^{\bar{\mathcal{A}}}{}_{\mathcal{B}}$. For example, for the gauge-transformation matrices in (3.94):

$$t_{I\mathcal{A}\mathcal{B}} = t_{I\mathcal{A}}{}^{\mathcal{C}} C_{\mathcal{C}\mathcal{B}} = t_{I\mathcal{A}\bar{\mathcal{B}}} d^{\bar{\mathcal{B}}}{}_{\mathcal{B}}\,. \tag{3.137}$$

Consistency of the transformations of the left- and right-hand side of (3.131) under the isometry group, determined by (3.82), implies that this matrix should be symmetric:

$$t_{I\mathcal{A}\mathcal{B}} = t_{I\mathcal{B}\mathcal{A}}\,. \tag{3.138}$$

This equation is, using (3.80), equivalent to

$$t_I{}^{\bar{\mathcal{A}}}{}_{\bar{\mathcal{B}}} d^{\bar{\mathcal{B}}}{}_{\mathcal{C}} = t_{I\mathcal{C}}{}^{\mathcal{B}} d^{\bar{\mathcal{A}}}{}_{\mathcal{B}}\,, \tag{3.139}$$

which shows more clearly that it is related to the invariance of the action with signature matrix $d^{\bar{\mathcal{A}}}{}_{\mathcal{B}}$.

With the above conditions, $d^{\bar{\mathcal{A}}}{}_{\mathcal{B}}$ respects the quaternionic structure. It has been proven in [37], using the theorems of [41], that at any point one can choose a basis such that ρ is in the form (3.35) and at the same time

$$d^{\bar{\mathcal{A}}}{}_{\mathcal{B}} = \begin{pmatrix} \eta & \\ & \eta \end{pmatrix} = \begin{pmatrix} -\mathbb{1}_p & & & \\ & \mathbb{1}_q & & \\ & & -\mathbb{1}_p & \\ & & & \mathbb{1}_q \end{pmatrix}, \qquad p+q = n_H\,. \tag{3.140}$$

For rigid supersymmetry, positive kinetic terms will be obtained for $p = 0$ and $q = n_H$. For supergravity we need one compensating multiplet and will use $p = 1$. These matrices should be covariantly constant. As we use a basis where they are actually constant, this implies from (3.132) that (using the lowering of indices as in (3.136))

$$2\omega_{X[\mathcal{A}}{}^{\mathcal{C}} C_{\mathcal{B}]\mathcal{C}} = -\omega_{X\mathcal{A}\mathcal{B}} + \omega_{X\mathcal{B}\mathcal{A}} = 0\,. \tag{3.141}$$

Thus the USp-connection is symmetric in such bases. From

$$\nabla_X d^{\bar{\mathcal{A}}}{}_{\mathcal{C}} = -\bar{\omega}_X{}^{\bar{\mathcal{A}}}{}_{\bar{\mathcal{B}}} d^{\bar{\mathcal{B}}}{}_{\mathcal{C}} - \omega_{X\mathcal{C}}{}^{\mathcal{D}} d^{\bar{\mathcal{A}}}{}_{\mathcal{D}} = 0\,, \tag{3.142}$$

one finds

$$d^{\bar{\mathcal{A}}}{}_{C}\omega_{X\mathcal{B}}{}^{C} = -\bar{\omega}_X{}^{\bar{\mathcal{A}}}{}_{\bar{C}}d^{\bar{C}}{}_{\mathcal{B}}\,. \tag{3.143}$$

When $d = \mathbb{1}$ the above condition is the antihermiticity of ω. In the preferred basis with (3.35) and (3.140) we can also write

$$C_{\mathcal{A}\mathcal{B}} = \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix}, \qquad t_{I\mathcal{A}}{}^{\mathcal{B}} = \begin{pmatrix} U_I & V_I \\ W_I & U_I{}^* \end{pmatrix}, \qquad \begin{array}{l} V_I{}^T = \eta V_I \eta\,,\ W_I{}^T = \eta W_I \eta\,, \\ U^\dagger = -\eta U \eta\,,\ V^* = -W\,. \end{array} \tag{3.144}$$

This expresses that the transformations are in the subgroup of $G\ell(n_H, \mathbb{H})$ that preserves the antisymmetric metric $C_{\mathcal{A}\mathcal{B}}$ and the metric $d^{\bar{\mathcal{A}}}{}_{\mathcal{B}}$, which is $\mathrm{USp}(2p, 2q)$.

We define then the metric of the manifold to be

$$g_{XY} = \left(f^{i\bar{\mathcal{A}}}{}_X\right)^* d^{\bar{\mathcal{A}}}{}_{\mathcal{B}} f^{i\mathcal{B}}{}_Y = f^{i\mathcal{A}}{}_X \varepsilon_{ij} C_{\mathcal{A}\mathcal{B}} f^{j\mathcal{B}}{}_Y\,, \tag{3.145}$$

such that the holonomy associated to g_{XY} is indeed $\mathrm{USp}(2p, 2q)$.

The curvature tensor on the scalar manifold is determined in terms of a 4-index symmetric tensor in $\mathrm{Sp}(2n_H)$, denoted by $W_{\mathcal{A}\mathcal{B}\mathcal{C}\mathcal{D}}$:

$$\begin{aligned} W_{\mathcal{A}\mathcal{B}\mathcal{C}\mathcal{D}} \equiv W_{\mathcal{A}\mathcal{B}\mathcal{C}}{}^{\mathcal{E}} C_{\mathcal{E}\mathcal{D}} &= -\,\varepsilon^{ij} f^X{}_{i\mathcal{A}} f^Y{}_{j\mathcal{B}} \mathcal{R}_{XY\mathcal{C}\mathcal{D}} \\ &= \tfrac{1}{2} f^{Xi}{}_{\mathcal{A}} f^Y{}_{i\mathcal{B}} f^{Zk}{}_{\mathcal{C}} f^W{}_{k\mathcal{D}} R_{XYZW}\,, \end{aligned} \tag{3.146}$$

where we used the metric g_{XY} (3.145) to lower the indices.

3.3.3.2 Remark on the Conformal Symmetry

Due to the fact that we have now a metric available, we can invoke the homothetic Killing equation (3.68) and, similarly as in (1.45), introduce a scalar function $\tilde{k}_{\mathrm{D}}$ such that

$$k_{\mathrm{D}X} = g_{XY} k_{\mathrm{D}}{}^Y = \partial_X \tilde{k}_{\mathrm{D}}\,. \tag{3.147}$$

It is also possible to start from this scalar function, and generate the metric from

$$g_{XY} = \frac{2}{D-2} \nabla_X \partial_Y \tilde{k}_{\mathrm{D}}\,. \tag{3.148}$$

We also define $k_{\mathrm{D}}{}^2$ using the metric (3.145)

$$k_{\mathrm{D}}{}^2 \equiv g_{XY} k_{\mathrm{D}}^X k_{\mathrm{D}}^Y\,. \tag{3.149}$$

It will be also useful to express ${k_{\rm D}}^2$ in terms of the sections introduced in (3.70) and their complex conjugates

$$A_{i\bar{\mathcal{A}}} = \left(A^{i\mathcal{A}}\right)^* = A^{j\mathcal{B}}\rho_{\mathcal{B}\bar{\mathcal{A}}}\varepsilon_{ji}\,, \qquad A^{i\mathcal{A}} = -\varepsilon^{ij}\rho^{\bar{\mathcal{B}}\mathcal{A}}A_{j\bar{\mathcal{B}}}\,. \tag{3.150}$$

To do so, we note that the matrix

$$M^i{}_j \equiv A^{i\mathcal{A}}d^{\bar{\mathcal{B}}}{}_{\mathcal{A}}A_{j\bar{\mathcal{B}}}\,, \tag{3.151}$$

is Hermitian and equal to $\varepsilon^{ik}\varepsilon_{j\ell}M^\ell{}_k$, i.e. $\sigma_2 M^T\sigma^2$. Therefore it should be proportional to the unit matrix. Indeed, using (3.70) and (3.145)

$$M^i{}_j = \tfrac{1}{2}\delta^i_j A^{k\mathcal{A}}d^{\bar{\mathcal{B}}}{}_{\mathcal{A}}A_{k\bar{\mathcal{B}}} = \tfrac{1}{2}\delta^i_j {k_{\rm D}}^2\,. \tag{3.152}$$

Another way in which ${k_{\rm D}}^2$ appears is in terms of an inner product of the SU(2) Killing vectors introduced in (3.69):

$${k_{\rm D}}^2 = \tfrac{1}{3}(D-2)^2\mathbf{k}_X\cdot\mathbf{k}^X\,. \tag{3.153}$$

It is useful to record the relation between these quantities for arbitrary vectors $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{A}\cdot\mathbf{k}_X\,\mathbf{B}\cdot\mathbf{k}^X = \frac{1}{(D-2)^2}{k_{\rm D}}^2\mathbf{A}\cdot\mathbf{B}\,. \tag{3.154}$$

3.3.3.3 Moment Maps

The isometries defined in (3.75) can be expressed in terms of moment maps. The definition of the latter depends on the theory. As we will discuss in Sect. 5.4.1, isometries for Kähler manifolds can be generically generated from a real moment map function using the complex structure and the metric. The hypermultiplet geometry has three complex structures, and as such have a triplet moment map for any isometry $\mathbf{P}_I$. They should satisfy

$$\partial_X\mathbf{P}_I = \mathbf{J}_X{}^Y k_{IY}\,. \tag{3.155}$$

Furthermore, they satisfy an ‘equivariance relation’, which is necessary to build supersymmetric actions with these symmetries:

$$k_I{}^X\mathbf{J}_{XY}k_J{}^Y = f_{IJ}{}^K\mathbf{P}_K\,. \tag{3.156}$$

With conformal symmetry, the solution of (3.155) is determined to be[21]

$$\mathbf{P}_I = \mathbf{k}^X k_{IX} = \frac{1}{D-2} k_{\rm D}{}^Y \mathbf{J}_Y{}^X k_{IX} = \frac{2}{(D-2)^2} A^{i\mathcal{A}} t_{I\mathcal{A}\mathcal{B}} \boldsymbol{\tau}_{ij} A^{j\mathcal{B}} . \tag{3.157}$$

In this context, it is also convenient to rewrite an expression that appears in the potential that occurs in these theories

$$k_I^X k_{JX} = \frac{4}{(D-2)^2} \varepsilon_{ij} A^{i\mathcal{A}} A^{j\mathcal{B}} t_{I\mathcal{A}}{}^{\mathcal{C}} C_{\mathcal{C}\mathcal{D}} t_{J\mathcal{B}}{}^{\mathcal{D}} . \tag{3.158}$$

3.3.3.4 Action for Hypermultiplets in *D* = 5

The resulting action is [5]

$$\begin{aligned} e^{-1}\mathcal{L}_h =& -\tfrac12 g_{XY} D_a q^X D^a q^Y + \bar\zeta_A \not{D}\zeta^A + \tfrac49 D k_{\rm D}{}^2 + \tfrac{8}{27} T^{ab} T_{ab} k_{\rm D}{}^2 \\ & + \tfrac{16}{3}{\rm i}\bar\zeta_A \chi_i k_{\rm D}{}^X f^{iA}{}_X + 2{\rm i}\bar\zeta_A \gamma\cdot T\zeta^A - \tfrac14 W_{ABCD}\bar\zeta^A\zeta^B\bar\zeta^C\zeta^D \\ & - \tfrac29 \bar\psi_a\gamma^a\chi k_{\rm D}{}^2 - \tfrac13\bar\zeta_A\gamma^a\gamma\cdot T\psi_{ai} k^X f^{iA}{}_X - \tfrac12 {\rm i}\bar\zeta_A\gamma^a\gamma^b\psi_{ai} D_b q^X f^{iA}{}_X \\ & + \tfrac23 f_a{}^a k_{\rm D}{}^2 - \tfrac16 {\rm i}\bar\psi_a\gamma^{ab}\phi_b k_{\rm D}{}^2 + \bar\zeta_A\gamma^a\phi_{ai} k_{\rm D}{}^X f^{iA}{}_X \\ & + \tfrac{1}{12}\bar\psi_a^i\gamma^{abc}\psi_b^j D_c q^Y J_{YX\,ij} k_{\rm D}{}^X - \tfrac19\,{\rm i}k_{\rm D}{}^2\bar\psi^a\left(\psi^b T_{ab} - \tfrac12\gamma^{abcd}\psi_b T_{cd}\right) \\ & + {\rm i}\sigma^I t_{IB}{}^A\bar\zeta_A\zeta^B - 2{\rm i}k_I^X f^{iA}{}_X\bar\zeta_A\psi_i^I - \tfrac12\sigma^I k_I^X f^{iA}{}_X\bar\zeta_A\gamma^a\psi_{ai} \\ & - \tfrac12\bar\psi_a^i\gamma^a\psi^{jI} P_{Iij} + \tfrac14{\rm i}\bar\psi_a^i\gamma^{ab}\psi_b^j\sigma^I P_{Iij} + Y_{ij}^I P_I^{ij} - \tfrac12\sigma^I\sigma^J k_I^X k_{JX} , \end{aligned} \tag{3.159}$$

with covariant derivatives given in (3.98).

3.3.3.5 Action for Hypermultiplets in *D* = 4

When we discuss $D = 4$, we can multiply (3.131) at both sides with a chiral projection P_R. Using the rules (A.63) we should now impose for the action $S_{\rm hyper}$

$$\frac{\delta S_{\rm hyper}}{\delta\bar\zeta_{\bar{\mathcal{A}}}} = 2 d^{\bar{\mathcal{A}}}{}_{\mathcal{B}} \Gamma^{\mathcal{B}} . \tag{3.160}$$

[21] It is a nice exercise to prove that (3.155) is solved by (3.157). You may replace the ∂_X by covariant derivatives and use (3.69), (3.39), (3.68), (3.77) and (3.83).

We also want the action to generate the field equations for the scalars that we have seen in (3.95). This leads in rigid supersymmetry to

$$\begin{aligned}\mathcal{L}_h =& -\tfrac{1}{2}g_{XY}D_\mu q^X D^\mu q^Y - \left(\bar{\zeta}_{\bar{\mathcal{A}}}\not{\nabla}\zeta^{\mathcal{B}}d^{\bar{\mathcal{A}}}{}_{\mathcal{B}} + \text{h.c.}\right)\\ &+ \tfrac{1}{2}W_{\mathcal{A}\mathcal{B}}{}^{\mathcal{E}\mathcal{F}}d^{\bar{\mathcal{C}}}{}_{\mathcal{E}}d^{\bar{\mathcal{D}}}{}_{\mathcal{F}}\bar{\zeta}_{\bar{\mathcal{C}}}\zeta_{\bar{\mathcal{D}}}\bar{\zeta}^{\mathcal{A}}\zeta^{\mathcal{B}}\\ &+ \left(2X^I t_{I\mathcal{A}\mathcal{B}}\bar{\zeta}^{\mathcal{A}}\zeta^{\mathcal{B}} + 2\mathrm{i} f_X^{i\mathcal{A}}k_I^X\bar{\zeta}_{\bar{\mathcal{B}}}\Omega^{jI}\varepsilon_{ij}d^{\bar{\mathcal{B}}}{}_{\mathcal{A}} + \text{h.c.}\right)\\ &+ 2\mathbf{P}_I\cdot\mathbf{Y}^I - 2\bar{X}^I X^J k_I{}^X k_{JX}\,, \end{aligned}\tag{3.161}$$

with the covariant derivatives in (3.87), which satisfies (3.160), and also

$$\frac{\delta S_{\text{hyper}}}{\delta q^X} = g_{XY}\Delta^Y - \left(2\bar{\zeta}_{\bar{\mathcal{A}}}\Gamma^{\mathcal{B}}\omega_{X\mathcal{B}}{}^{\mathcal{C}}d^{\bar{\mathcal{A}}}{}_{\mathcal{C}} + \text{h.c.}\right)\,. \tag{3.162}$$

See [18, Exercises 20.8 and 20.9] for a concrete example.

After gauge covariantization and using the values of the conformal gauge fields as in (3.122) and the covariant derivatives (3.100), the superconformal hypermultiplet action with gauged isometries in $D = 4$ is [18, (20.93)]

$$\begin{aligned}e^{-1}\mathcal{L}_h =& -\tfrac{1}{12}k_{\text{D}}{}^2R + \tfrac{1}{4}k_{\text{D}}{}^2D - \tfrac{1}{2}g_{XY}D_\mu q^X D^\mu q^Y - 2\bar{X}^I X^J k_I{}^X k_{JX} + 2\mathbf{P}_I\cdot\mathbf{Y}^I\\ &+\Big\{ -\bar{\zeta}_{\bar{\mathcal{A}}}\widehat{\not{D}}\zeta^{\mathcal{B}}d^{\bar{\mathcal{A}}}{}_{\mathcal{B}} + \tfrac{1}{12}k_{\text{D}}{}^2\bar{\psi}_{i\mu}\gamma^{\mu\nu\rho}D_\nu\psi_\rho^i\\ &+ \tfrac{1}{8}k_{\text{D}}{}^2\bar{\psi}_{ia}\gamma^a\chi^i - 2\mathrm{i}d^{\bar{\mathcal{A}}}{}_{\mathcal{B}}A^{i\mathcal{B}}\bar{\zeta}_{\bar{\mathcal{A}}}\chi_i\\ &+ \tfrac{1}{2}\mathrm{i}\bar{\zeta}_{\bar{\mathcal{A}}}\gamma^a\not{D}q^X\psi_{ai}f^{i\mathcal{B}}{}_X d^{\bar{\mathcal{A}}}{}_{\mathcal{B}} - \tfrac{1}{3}\mathrm{i}d^{\bar{\mathcal{A}}}{}_{\mathcal{B}}A^{i\mathcal{B}}\bar{\zeta}_{\bar{\mathcal{A}}}\gamma^{\mu\nu}D_\mu\psi_{\nu i}\\ &+ \left(\tfrac{1}{12}\mathrm{i}d^{\bar{\mathcal{A}}}{}_{\mathcal{B}}A^{i\mathcal{B}}\bar{\zeta}_{\bar{\mathcal{A}}}\gamma_a\psi_b^j - \tfrac{1}{48}k_{\text{D}}{}^2\bar{\psi}_a^i\psi_b^j\right)T^{+ab}\varepsilon_{ij}\\ &- \tfrac{1}{8}\bar{\zeta}_{\bar{\mathcal{A}}}\gamma^{ab}T_{ab}^+\zeta_{\bar{\mathcal{B}}}C^{\bar{\mathcal{A}}\bar{\mathcal{B}}} + 2\mathrm{i}\bar{X}^I k_I^X\bar{\zeta}_{\bar{\mathcal{A}}}\gamma^a\psi_a^j\varepsilon_{ij}d^{\bar{\mathcal{A}}}{}_{\mathcal{B}}f^{i\mathcal{B}}{}_X\\ &+ 2X^I\bar{\zeta}^{\mathcal{A}}\zeta^{\mathcal{B}}t_{I\mathcal{A}\mathcal{B}} + 2\mathrm{i}k_I^X f^{i\mathcal{A}}{}_X\bar{\zeta}_{\bar{\mathcal{B}}}\Omega^{jI}\varepsilon_{ij}d^{\bar{\mathcal{B}}}{}_{\mathcal{A}}\\ &+ \tfrac{1}{2}\bar{\psi}_{aj}\gamma^a\Omega_i^I P_I{}^{ij} + \tfrac{1}{2}\bar{X}^I\bar{\psi}_a^i\gamma^{ab}\psi_b^j P_{Iij} + \text{h.c.}\Big\}\\ &+ \tfrac{1}{2}\bar{\psi}_a^i\gamma^{abc}\psi_{bj}D_c q^X\mathbf{k}_X\cdot\boldsymbol{\tau}_i{}^j + 4\text{-fermion terms}\,. \end{aligned}\tag{3.163}$$

We can rewrite the kinetic terms for the scalars q^X in terms of the sections (3.70) using the bosonic part of (3.102)

$$g_{XY}D_\mu q^X D^\mu q^Y = \varepsilon_{ij}C_{\mathcal{A}\mathcal{B}}\left(\widehat{D}_\mu A^{i\mathcal{A}}\right)\left(\widehat{D}^\mu A^{j\mathcal{B}}\right)\,. \tag{3.164}$$

3.3.4 Splitting the Hypermultiplets and Example

In general we did not use the basis (3.35). Sometimes, however, it will be convenient to use such basis in examples. To do this, we split the index $\mathcal{A} = 1, \ldots, 2n_H$ into $\mathcal{A} = (\alpha a)$, with $\alpha = 1, 2$ and $a = 1, \ldots, n_H$. The index $\bar{\mathcal{A}}$ will then have the same form, but with a in the opposite (up-down) position. We can then write the canonical basis with (3.35) and (3.140) as

$$\rho_{\mathcal{A}\bar{\mathcal{B}}} = \varepsilon_{\alpha\beta}\delta_a^b\,, \qquad d^{\bar{\mathcal{A}}}{}_{\mathcal{B}} = \eta_{ab}\delta_\beta^\alpha\,, \qquad C_{\mathcal{A}\mathcal{B}} = \eta_{ab}\varepsilon_{\alpha\beta}\,. \tag{3.165}$$

The components of $A^{i\mathcal{A}}$ can then be written as $A^{i\alpha a}$ and $(A^{i\alpha a})^* = A^{j\beta b}\varepsilon_{ji}\varepsilon_{\beta\alpha}$. Upon this splitting the action (3.163) starts with

$$\begin{aligned} e^{-1}\mathcal{L}_h =& -\tfrac{1}{12}\kappa_{\mathrm{D}}{}^2 R + \tfrac{1}{4}\kappa_{\mathrm{D}}{}^2 D - \tfrac{1}{2}\widehat{D}_\mu A^{i\alpha a}\widehat{D}^\mu A^{j\beta b}\varepsilon_{ij}\varepsilon_{\alpha\beta}\eta_{ab} + \cdots\,, \\ \kappa_{\mathrm{D}}{}^2 =& A^{i\alpha a}A^{j\beta b}\varepsilon_{ij}\varepsilon_{\alpha\beta}\eta_{ab} = A^{i\alpha a}\left(A^{i\beta b}\right)^*\eta_{ab}\,. \end{aligned} \tag{3.166}$$

The conditions on the symmetry matrices $t_{I\mathcal{A}}{}^{\mathcal{B}}$ (see (3.144)) are such that they can be decomposed as

$$\begin{aligned} t_{I\alpha a}{}^{\beta b} &= t_{I0a}{}^b\delta_\alpha{}^\beta + \mathbf{t}_{Ia}{}^b\tau_\alpha{}^\beta\,, \quad t_{I0a}{}^b,\ \mathbf{t}_{Ia}{}^b \in \mathbb{R}\,, \\ t_{I0a}{}^b &= -\eta_{ac}t_{I0d}{}^c\eta^{db}\,, \quad \mathbf{t}_{Ia}{}^b = \eta_{ac}\mathbf{t}_{Id}{}^c\eta^{db}\,. \end{aligned} \tag{3.167}$$

As an example, we may consider

$$t_{I\alpha a}{}^{\beta b} = \mathrm{i}Q_{Ia}{}^b(\sigma_3)_\alpha{}^\beta\,, \qquad Q_{Ia}{}^b \in \mathbb{R}\,, \qquad Q_{Ia}{}^b = \eta^{bc}Q_{Ic}{}^d\eta_{da}\,. \tag{3.168}$$

Then from (3.84) and (3.157) we have

$$\begin{aligned} k_I{}^X &= \mathrm{i}f^X{}_{i(\alpha a)}A^{i\beta b}Q_{Ib}{}^a(\sigma_3)_\beta{}^\alpha\,, \\ \mathbf{P}_I &= \tfrac{1}{2}\mathrm{i}A^{i\alpha a}Q_{Iab}(A^{j\beta b})^*\boldsymbol{\tau}_i{}^j(\sigma_3)_\alpha{}^\beta\,, \end{aligned} \tag{3.169}$$

with $Q_{Iab} = Q_{Ia}{}^c\eta_{cb} = Q_{Iba}$.

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