Schopenhauer on Diagrammatic Proof

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Abstract The present paper discusses the treatment of diagrammatic proof in Schopenhauer's philosophy of mathematics. 'Picture proofs' have been the subject of some scattered contemporary debate, and my aim here is to see whether Schopenhauer's treatment might prove fruitful in the context of recent discussion. In particular I argue that Schopenhauer's remarks on diagrammatic proof, though few and far between, might be able to provide conceptual tools adequate to meet some of the broader challenges facing the legitimacy of such proof. In § 1 the notion of a picture proof is introduced and two general objections to its legitimacy are formulated. In § 2 I set out what I take to be the substance of Schopenhauer's advocacy of picture proofs and in § 3 I formulate replies to these challenges based on the Schopenhauerian distinction between a proposition's ground of knowledge (*Erkenntnißgrund*) and its ground of being (*Seynsgrund*).

Keywords Schopenhauer · Diagram · Picture · Proof

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1 Two Challenges to The Legitimacy of Picture Proofs

A quite ubiquitous way of characterising proof in mathematics is as a certain species of argumentation—for example, as sound, deductive argumentation which is non-circular, commits no fallacies, etc. Some might emphasise additional criteria:

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intuitionists will add that a proof must be constructive, relevance logicians will say that a proof must have a conclusion relevant to its premises, and so on. The general assumption that demonstration is a species of argumentation can be detected as far back as Aristotle, for whom the study of proof taken up in the *Prior Analytics* [1] consists of the study of categorical premises and their syllogistic consequences.[1] Yet if one takes up the assumption that mathematical demonstration in particular must be argumentative, the diagram below (Fig. 1) presents a problem.

This hexagon is the very one supplied by Schopenhauer in the *World as Will and Representation* ([10], Vol. 1, § 15) and in the *Fourfold Root* ([11], § 39). What is most notable about it is that it seems sufficient, by itself and unaccompanied by words, to establish a general geometrical proposition about right triangles—namely, that those with two equal sides satisfy the Pythagorean theorem.

'Picture proofs' have been the subject of scattered debate in contemporary philosophy of mathematics. They cast the aforementioned argumentative assumption about mathematical proof into doubt, for pictures are not arguments—in fact they seem to be a radically different sort of thing: arguments can be stated and defended, deemed cogent, persuasive, circular or fallacious, and in what sense can any of these descriptions be ascribed to pictures? What can it mean to assert or deny that a picture is question-begging, or otherwise circular? Moreover all arguments must have premises and yet, evidently, if we were to ask someone who offered us a picture proof what the *premises* of their picture were we would be making a category error.[2] If pictures are not arguments, then we have two options in proceeding. Either we reject the assumption that all demonstration is argumentation, or we somehow account for our receiving general mathematical knowledge from such pictures in such a way as to steer just clear of calling them *proofs*. One way to cash out the latter option is to say that, rather than being a proof, a diagram instead *represents* or in some way *encodes* an argument, and that this argument is what establishes the proposition.[3] Now there might appear to be an element

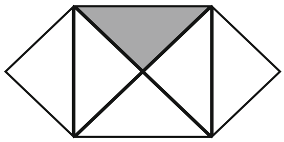

Fig. 1 Schopenhauer's Pythagorean Hexagon

[1]The historical emphasis on linguistic-argumentative proof and the corresponding marginalisation of diagrammatic proof is a theme explored in detail by Greaves [4].

[2]I should note that Norton [9], contrary to the preceding considerations, idiosyncratically *does* hold both thought experiments and picture proofs to be arguments. Under such a position, the legitimacy of picture proofs is rendered entirely unproblematic, and the present discussion is entirely uninteresting.

[3]This offloading of epistemic work, as it were, onto represented or encoded arguments can be seen implicitly even in very sympathetic treatments of diagrammatic proof. To my mind, the most conspicuous example is the recent tradition of diagrammatic proof theory (cf. Shin [12]; Mumma [7]; Shin et al. [13, § 2ff.]), wherein proof-theoretic techniques are applied to precisely defined

of stubbornness in this response—why *not* allow diagrams to be proofs in their own right? Why should we treat the assumption that proofs must be arguments as anything more than a historically entrenched prejudice? But in fact, there are some grounds for caution—challenges to the legitimacy of diagrammatic proof as such. Two particularly pressing challenges are what I will term the *objection from particularity* and the *objection from misleading pictures*.[4]

The objection from particularity is the descendant of an old family of reservations against diagrammatic reasoning, one ancestor of which we find addressed by Proclus in his commentary on Euclid [6, p. 162]. The problem is this: where there is a geometrical diagram, there is a diagram of a particular geometric figure. Hence in reasoning with a diagram, one is reasoning and making judgement about a single figure. How then can a geometer ever be justified in arriving at general results by the use of diagrams? Now when diagrams are used in tandem with arguments, there is a natural way of answering this which Proclus adopts: provided that the accompanying argument only makes reference to the relevant features of the figure, the result will apply to all figures which share these features and thus one may come, via the proof, to knowledge of a general proposition. Such a strategy is not open to the advocate of picture proofs however, for a picture proof is by hypothesis an unaccompanied diagram. There is no argument accompanying Schopenhauer's hexagon for example, and so we cannot talk about it 'making use of', in the sense of referring to, this or that feature of the right triangle pictured.

The second objection is based on the thought that some pictures can be misleading—they can suggest the truth of a proposition which is false. The existence of such pictures gives rise to a problem of epistemic luck: if there is no inherent difference between misleading pictures and picture proofs, then even in cases where one happens to gain a true belief through a diagram, one will merely have been lucky that they were not actually looking at a misleading one. To contrast this with arguments: an argument is misleading (in the sense of seeming sound but being really unsound) only if it is either invalid or has a false premise. Thus, prior to knowing the truth-value of the conclusion, it is possible in principle to see whether or not an argument might be misleading by checking the truth of its premises, or its form. But if the only way to know that a picture is not misleading is to have an independent proof which establishes the proposition in question, then how can a picture by itself ever serve to provide mathematical knowledge?

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systems of drawing and manipulating diagrams. Such, in effect, treat diagrams as another sort of mathematical notation, and thus their mode of proof as largely discursive (rather than purely intuitive).

[4]These names are not widespread, but the objections are. Each is highlighted in various ways by, for example, Shin (cf. [12, p. 3ff.]), Brown (cf. [2, p. 161ff.]), Norman (cf. [8, p. 144]), Starikova (cf. [14, p. 85]), Mumma (cf. [7, pp. 255–262]).

2 Schopenhauer's Advocacy

Despite the fact that Schopenhauer provides a paradigm case of diagrammatic proof in his Pythagorean hexagon (Fig. 1) his advocacy of such has been, to the best of my knowledge, entirely neglected in the contemporary literature on the topic.[5] My hope is to rectify this by consulting his remarks on the matter, and then to formulate responses based on these to the challenges just outlined.[6]

Because Schopenhauer's advocacy of diagrammatic proof has been so neglected, I think it is prudent that I should first say a little to establish its existence beyond doubt, before summarising what I take to be its main thrust. For perhaps one might think it a leap to call Schopenhauer an advocate of diagrammatic *proof*—after all, he never explicitly refers to his own example as a proof (*Beweis*). He comes close in a number of places, seemingly within a hair when he says that "[t]he mere sight of it without any words conveys twenty times more conviction than does Euclid's mousetrap proof" [11, p. 205], and apparent near-misses like this might lead one to suspect that he withholds the term on purpose. Moreover one might feel as if Schopenhauer has some reason to withhold the term of 'proof': consider the following passage taken from the middle of the same section of the *Fourfold Root*. Commenting on Euclid's proof that in any triangle, sides subtending equal angles are equal, he says that

> [w]hen we have the ground of being, our conviction of the truth of the proposition is based solely thereon, and certainly no longer on that of [the ground of] knowledge which is given by demonstration. [11, p. 201]

As it relates to geometry, Schopenhauer's distinction between the ground of knowledge (*Erkenntnißgrund*) and ground of being (*Seynsgrund*) of a proposition applies in the following way. For Schopenhauer, 'demonstrations', with particular reference to those of Euclid, give one insight into the ground of knowledge, but not that of being: they give knowledge *that* the theorems are true, compel one to assume their truth on pain of contradiction, but rarely do they grant insight into *why* the theorems are true (cf. [11, pp. 200–202]). This being so, perhaps Schopenhauer avoids calling unaided diagrams 'proofs' so as to avoid suggesting that they supply knowledge in any way analogous to deductive demonstrations. That is to say, unlike the 'mousetrap proof', Schopenhauer's hexagon gives one knowledge via direct

[5]In the process of review, it was brought to my attention by Dr Lemanski that towards the end of the twentieth century, Schopenhauer's remarks on mathematics yet enjoyed something of a new advocacy amongst a number of German and Swiss mathematicians (cf. [5, pp. 333–334]). Regrettably, this advocacy seems not to have had any detectible interaction with the wider literature on picture proofs and the like.

[6]These remarks are contained in Vol. 1, §15 and Vol. 2, Ch. 13 of the *World as Will and Representation* and, in particular, §39 of the *Fourfold Root*.

intuition into the ground of being of the theorem,[7] and thus supplies knowledge in a radically different way to that in which proof does.

This would be the best case I can think to make in favour of denying Schopenhauer's advocacy of diagrammatic proof, but we can reply to it adequately with two points. Firstly, one may well recognise Schopenhauer's withholding of the term 'proof' when it comes to pictures, and yet doubt that it reflects anything further than a stylistic consideration. Certainly one can see a stylistic reason Schopenhauer might have had in withholding the term 'proof' from pictures since, as was admitted, this helps to avoid the suggestion that pictures and deductive demonstrations supply knowledge in anything like the same way. But this being so, the absence of the term in certain passages cannot then establish deeper philosophical motivation; style would be explanation enough. Indeed, for our second point, we may contradict the thought that Schopenhauer had *philosophical* motivation to withhold the term 'proof' by citing the following passage.

> The whole of geometry also rests on the nexus of the position of the parts in space. It would thus be an insight into that nexus; but, as I have said, as such an insight is not possible through mere concepts, but only through intuition, every geometrical proposition would have to be reduced to this, *and the proof* (Beweis) *would consist merely in our clearly bringing out the nexus whose intuition is required; more we could not do.* ([11, p. 198]; my italics)

Here is a use of the term not in reference to Euclidean demonstrations, but rather to the act of evoking intuition into the ground of being, as occurs in the case of his hexagon. Given this, it is quite clear that he takes his diagram as proving its proposition in the requisitely strong sense.

Establishing this much has allowed me to introduce the crucial distinction between the ground of being and ground of knowledge of a proposition. With this to hand, the core of Schopenhauer's stance on diagrammatic proof can, I think, be stated succinctly as follows. A diagrammatic proof, like Schopenhauer's hexagon, proves a proposition to be the case by displaying the ground of being of this proposition, such that by contemplating the picture, we are able to intuit this ground and come to immediate knowledge both *that* the proposition is true and *why* it is true. Insofar as a picture is able to do this, it is in fact superior to those purely argumentative proofs which present only the grounds of knowledge of their conclusion—that is, the latter do not grant understanding as to *why* the conclusion holds. Such understanding can only be received through intuition of the ground of being, which diagrams are particularly well suited to supply.

[7]—Albeit, the theorem as restricted to right triangles with two equal sides. It seems as if Schopenhauer took his diagram to establish the general Pythagorean theorem; whether or not it does is irrelevant to our discussion.

3 Applying Schopenhauer's Remarks

We turn back now to the two objections directed at diagrammatic proof that were set out previously. Recall that these were the following.

Particularity If a diagram only represents a particular geometrical figure (a particular triangle or rectangle, etc.), then it seems no conclusions of a general nature can be drawn justifiably by its use—particularly in the case of picture proofs, which are *unaccompanied* diagrams.

Misleading Pictures Some pictures suggest the truth of a proposition that is false. If it is not possible for one to tell merely by looking at a picture whether or not it is misleading, then it is always a matter of luck whether or not one gets a true belief from such pictures. This being so, one must never be able to gain knowledge of a mathematical proposition from a picture alone.

3.1 Particularity and Generality

Starting with the objection from particularity, our task is to offer an account of how general mathematical knowledge can be drawn from an unaccompanied picture. To make the matter concrete, we consider Schopenhauer's hexagon (Fig. 1) as an example, which is supposed to prove the restricted form of the Pythagorean theorem—that is, as restricted to triangles with two equal sides. Call the grey-shaded triangle in the picture T. I distinguish the following propositions:

(P_T) The square of T's hypotenuse is equal to the sum of the squares of T's other sides.

(P) Any right triangle with two equal sides is such that the square of the hypotenuse is equal to the sum of the squares of the other sides

P_T is just the instance of P in the case of T. As I hope to show now, granted that Schopenhauer's hexagon establishes P_T, then on the back of the remarks set out in §2, we can also show it to establish P. The assumption that the hexagon can establish the particular proposition P_T is not so problematic, since the possibility of diagrams establishing *particular* geometrical propositions is not what the objection from particularity calls into question. I also take the assumption to be plausible in itself.

Our reply runs as follows. Supposing that the diagram establishes P_T, on Schopenhauer's account we say that a diagram displays the ground of being of P_T—that is, it shows us *in virtue of what* it is the case that the square of T's hypotenuse is equal to the sum of the squares of T's other two sides. Now if one allows that it shows this to be so *in virtue of* T's being a right triangle with two sides equal, this then serves as the link between the particular and general propositions: since T's being a right triangle with two sides equal *makes it* such that the square of its hypotenuse is equal to the sum of the squares of the other two sides, we can legitimately infer that any other right triangle with two sides equal will also be so. These two features are shown in this instance to stand in a relationship of grounding, and so we see that one is a sufficient condition for the other. Put schematically for the sake of clarity, my thought is that Schopenhauer has the resources to say that a general mathematical proposition of the form ‘all Fs are Gs’ may be proven by a picture if this picture indeed shows that a figure x is G merely in virtue of being F.

This reply functions in a similar way to Proclus’ mentioned previously. Both emphasise that in proofs which work with (for us: are identical to) a diagram, only certain general features of the figure depicted should be considered salient. This notion of salience is, however, cashed out in different ways. For Proclus, it means that the mathematician, in giving an argumentative proof, only “make[s] use of” [6, p. 162] the relevant general features of the figure in formulating his premises. For us, it means that the picture shows that the particular figure has some property in virtue of possessing the relevant general features. In the case of Schopenhauer’s hexagon, the proof shows T’s satisfaction of the Pythagorean theorem as grounded in its being a right triangle with two sides equal.

3.2 Misleading Pictures

Passing to the objection from misleading pictures, I adapt an example from Brown [2], pp. 178–179, cf. [3] to make things concrete. On the Euclidean plane, draw four circles centred at the points $(\pm1, \pm1)$ and a fifth at the origin just large enough to touch the other four (Fig 2a). Note that the centre circle is contained in the box $\{(x, y)| -2 \le x, y \le +2\}$. Again for three-dimensional Euclidean space: draw eight spheres centred at $(\pm1, \pm1, \pm1)$, and a ninth sphere at the origin touching the other four (Fig. 2b). Note that the centre sphere is entirely contained within the enclosing box $\{(x, y, z)| -2 \le x, y, z \le +2\}$.

By drawing these diagrams, it seems as if we now see why the results hold in the two- and three-dimensional cases, and so we may consequently accept the following generalisation.

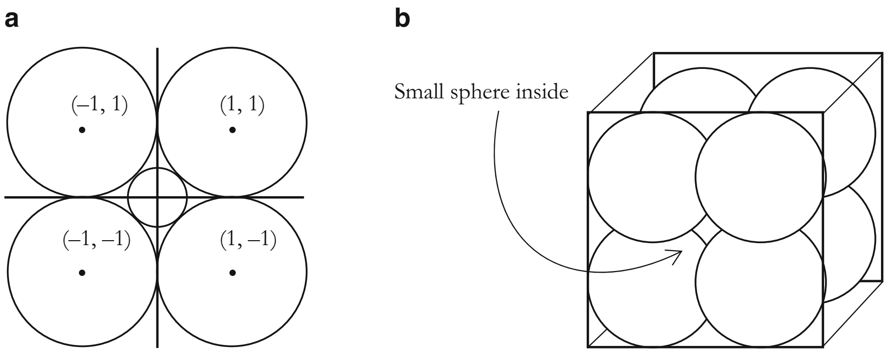

Fig. 2 Brown’s Examples

> For every natural number n: suppose that in n-dimensional Euclidean space we have 2^n $(n-1)$-spheres each of radius 1 centred at $(\pm 1, \pm 1, \ldots, \pm 1)$, and an additional $(n-1)$-sphere centred at the origin which just touches the other spheres. Then the $(n-1)$-sphere at the origin is contained within $\{(x_1, \ldots, x_n) | -2 \leq x_1, \ldots, x_n \leq +2\}$.

But this proposition fails first at n = 10 (Ibid., 178). This is a clear case in which pictures mislead us. We can now set out the general objection in detail. Provisionally, say that a candidate picture proof of the proposition *p* is misleading when *p* is false. Now, if our only way of telling apart genuine picture proofs from other pictures is a kind of general feeling (i.e., that this or that picture just seems to show that *p*), and if these feelings are unreliable, then we have no way of knowing whether or not a picture is misleading prior to establishing the truth or falsity of the proposition in question by other means. But if we cannot independently know whether a given picture is misleading without already knowing the truth-value of the proposition it is supposed to prove, then it does not seem as if the picture itself can establish mathematical knowledge.

If the premise of this objection is correct—if, without knowledge of the truth-value of the supposed theorem, our only way of distinguishing misleading pictures from genuine picture proofs is a kind of gut-feeling—then it is clear that the objection is devastating. For we see here, and know from experience, that in mathematics such feelings are often mistaken, and cannot be taken as evidence. Therefore our task must be to find a way to deny this premise, and identify some other way by which one might identify genuine picture proofs independently of prior knowledge of the truth-value of the would-be theorem.

I take it that a Schopenhauerian can reply to this as follows—though I am more tentative as regards success than against the previous objection. In the case of misleading cases such as Brown's, I agree with the objector that the truth of the general proposition is merely suggested by the pictures. Perhaps, as here, it is suggested in such a way as to make the generalisation very plausible, but I say that the suggestion of plausibility is *all* that occurs. In the case of a genuine picture proof, I say to the contrary that an entirely different event takes place, and that this difference must be detectable by introspection. That is, rather than being suggested or made plausible to the subject, I say that the truth of the proposition in such cases is instead *seen* or *grasped*. Unlike in Brown's example, one is not merely given good evidence of the proposition's truth on which to base a justified induction to the general case ('induction', that is, as in the empirical sense). With picture proofs, as in Schopenhauer's case for instance, one instead comes to *see* immediately the truth of the proposition in question, in that they come to observe why it is the case. In doing so, that the (potentially quite complex) proposition is true becomes as immediate a realisation as that two and two are four, or that a ball cannot be red and green all over. This introspective difference can be illustrated by the empirical fact that, when we are told by a mathematician that Brown's generalisation fails at n = 10, we may likely do little more than raise an eyebrow. For even though the diagrams make it plausible that Euclidean spaces of any dimension are such that the centre sphere does not escape the box, we can hardly be said to *see* that *being a Euclidean space* is what makes it true in the space that the centre ball does not escape the box—not only because it cannot be this alone which makes it so, else the generalisation would in fact hold, but also because it is not clear that we intuit the general property of *being an n-dimensional Euclidean space* at all,[8] let alone intuit that this property makes anything to be thus and so. Whereas on intuiting its ground of being as displayed in Fig. 1, if one were to tell us that the restricted Pythagorean theorem were false, we would be as certain of their error as if they had told us that 7 and 5 did not make 12.

Unlike the attitude one has to a proposition which is merely very plausible, the attitudes of grasping-that and seeing-that are factive and imply certitude. That is, one cannot grasp that p—see that it is the case that p—without it actually *being* the case that p and without one being certain that p. I therefore say that the inherent difference between picture proofs and misleading pictures—that is, that the former but not the latter display the ground of being of their respective propositions—reflects a detectable difference in the attitudes of, on the one hand, merely being persuaded of some proposition, and on the other, grasping its truth. This being so, it now appears far more difficult to claim that in a genuine case of picture proof, one's belief in the theorem is merely fortunately in accordance with facts. When one *grasps* the truth of a proposition, one does not gain a merely fortunately true belief, but one must instead gain knowledge. Such can only occur when a picture is in fact a proof.

4 Conclusion

I had set out to show that, despite their scarcity and neglect, Schopenhauer's remarks on diagrammatic proof are very fruitfully applicable to the contemporary debate over the legitimacy of picture proofs—in particular, that they provide us with conceptual tools adequate to address the most pressing and general concerns about such proofs. I considered that Schopenhauer's conception of a proposition's ground-of-being as intuited via a picture could be drawn out and adapted so as first to address how pictures might establish general geometrical propositions and, second, to defend picture proofs against sceptical worries stemming from the existence of misleading pictures. I hope that this helps towards a corrective to the neglect which Schopenhauer's remarks on the matter have suffered, and that it might thereby open new avenues of discussion.

[8]That is, as opposed to the specific properties of *being a three-dimensional Euclidean space* or of *being a two-dimensional Euclidean space*, of which it is far more plausible that we have intuitions. If there *were* a general intuitive grasping of the property of being an n-dimensional Euclidean space, I suspect the often counter-intuitive results of higher-dimensional geometry (Brown's $n = 10$ case being just one example) would be far less so.

References

1. Aristotle: Categories. On Interpretation. Prior Analytics. Transl. by H. P. Cooke, H. Tredennick. Harvard University Press, Cambridge, MA (1938)
2. Brown, J.R.: Proofs and Pictures. British Journal for the Philosophy of Science **48**, 161–180 (1997)
3. Brown, J.R.: Philosophy of Mathematics. Routledge, London (1999)
4. Greaves, M.: The Philosophical Status of Diagrams. CSLI Publications: Stanford (2002)
5. Lemanski, J.: Geometrie. In Schubbe, D., Koßler, M. (eds.) Schopenhauer-Handbuch: Leben – Werk – Wirkung. 2nd ed. Metzler, Stuttgart, 331–335 (2018)
6. Morrow, G.R.: Proclus: A Commentary on the First Book of Euclid's Elements. Princeton University Press, G. Princeton (1970)
7. Mumma, J.: Proofs, Pictures and Euclid. Synthese **175**, 255–287 (2010)
8. Norman, A.J.: Visual Reasoning in Euclid's Geometry: An Epistemology of Diagrams. Ph.D Thesis. University College London (2003)
9. Norton, J. D.: Are Thought Experiments Just What You Always Thought?. Canadian Journal of Philosophy **26**, 333–366 (1996)
10. Schopenhauer, A.: The World as Will and Representation, 2 vols. Dover Publications, New York (1969)
11. Schopenhauer, A.: The Fourfold Root of the Principle of Sufficient Reason. Transl. by Payne, E. F. J. Open Court Publishing (1974)
12. Shin, S-J.: The Logical Status of Diagrams. Cambridge University Press, Cambridge (1994)
13. Shin, S-J., Lemon, O., Mumma, J.: Diagrams. In Zalta, E. N. (ed.) Stanford Encyclopedia of Philosophy. Summer 2018 Edition. <plato.stanford.edu/archives/sum2018/entries/diagrams/>. Accessed 7.10.18.
14. Starikova, I.: Picture-Proofs and Platonism. Croatian Journal of Philosophy **7**(19), 81–92 (2007)