Schopenhauer and the Mathematical Intuition as the Foundation of Geometry

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Abstract Schopenhauer did not write extensively on mathematics, but he discussed the subject in almost all of his works. His thesis about the superiority of intuition in establishing the truth of geometrical theorems became a battle against the traditional demonstrative procedure in geometry. Commentators have generally provided internal readings of Schopenhauer's texts on mathematics but have neglected their context.

This paper examines Schopenhauer's philosophy of mathematics by discussing its relationship with both his views on the acquisition of knowledge and his familiarity with the contemporary British discussions of mathematics. An overview of his ideas on the primacy of intuition in both mathematics and its teaching is the basis of this inquiry into the connection of those ideas with both his conception of the role of mathematics in natural philosophy and his encounter with the 1830s British texts on mathematics, which he quoted in the second volume of *The World as Will and Representation*. By making him aware that Euclidean geometry required a thorough scrutiny of its foundations—notwithstanding its undisputed reputation—these texts contributed to the hitherto unappreciated modifications in his mathematical considerations.

Schopenhauer participated in an early phase of the debate on the foundations of geometry by taking a fresh look at intuition: not only as an alternative to demonstration, but also as the ground of truth and certainty in the Euclidean system.

Keywords Schopenhauer · British philosophy · Intuition · Philosophy of mathematics · Foundations of geometry

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1 Introduction

Schopenhauer did not write extensively on mathematics, but he discussed the subject in almost all of his works, from the 1813 Dissertation *On the Fourfold Root of the Principle of Sufficient Reason* until *Parerga and Paralipomena* (1851). His thesis about the superiority of intuition in establishing the truth of geometrical theorems is not unique in the history of philosophy but is certainly noticeable. It battled against the traditional demonstrative procedure in geometry—"as akin to someone cutting off his legs so that he can go on crutches" [WI, § 15, p. 95/83]—that leads "to the obvious detriment of the science" [WI, § 15, p. 95/83] and breaks the unity of mathematics: "arithmetic and algebra are not taken up with the kind of proofs that fill geometry; rather, their whole content simply amounts to an abbreviated way of counting" [WI, §15, p. 101/90].

Interpreters and scholars have generally provided internal readings of Schopenhauer's texts,[1] with the exception of François Rostand—who traces similar views on the importance of intuition in mathematics back to Descartes, Locke, Pascal, Malebranche, Leibniz and recalls Kant [43]. But the philosopher's provocative stance has also called forth resolute response, as reminded by Jens Lemanski: criticism—especially by mathematicians—around 1900 [32, pp. 330–331] and appreciation—in the second half of the twentieth century—of the view of intuition in geometry either as an alternative approach to demonstration or as an essential pedagogical instrument [32, pp. 331–333].

Yet, what still lacks in the analysis of Schopenhauer's views on mathematics is an attention to their context—with respect to both Schopenhauer's conception of philosophical knowledge and his familiarity with contemporary discussions of mathematics. A severe judgement like Cajori's ("Schopenhauer attacked mainly the logic of mathematics as found in Euclid. As a critique of the logic as used by Euclid the attack is childish and has no value for us")[2] is based on an inadequate appreciation of Schopenhauer's inquiry into the role of mathematics in philosophical and scientific knowledge—an inquiry that derived from a thoughtful assessment of contemporary discussions, and not just from the internal exigencies of his philosophy. Generally, commentators do not delve into the role that Abraham Gotthelf Kästner's approach to the "Parallelenproblem"—and Kant's reception of it—played in Schopenhauer's criticism of Euclid[3]; they do not assess Schopenhauer's divergence from Herbart and Fries in qualifying the importance of mathematics in metaphysics and philosophy of nature[4]; they neglect the importance of the 1830s British debate on mathematics, despite Schopenhauer's reference to it. It is not even mentioned that Schopenhauer was no stranger to mathematics' new

[1] See [4, 38, 41], [2, pp. 60–63].

[2] [13, p. 371]. See also p. 368: "his criticism is focused directly upon questions of logic, of mode of argumentation and of sufficiency of proof".

[3] See [39, pp. 141–153]. On Kant's philosophy of mathematics and its context, see [19, 29, 34].

[4] On the affinities and the important differences between Herbart and Fries, see [8].

course of the nineteenth century, with the affirmation of non-geometrical analysis—and yet his library included a book of Ernst Gottfried Fischer (cf. [17]), his professor of physics at the University of Berlin in the winter semester 1812–1813, introducing a logico-philosophical interpretation of analysis that sustained and encouraged the Lagrangian approach.[5]

This paper analyses Schopenhauer's philosophy of mathematics by discussing its relationship with both his views on the acquisition of knowledge and his reading of mathematical-related publications—focusing on the intellectual context provided by the British discussions. After an overview of his ideas on the primacy of intuition in mathematics—based on the Dissertation and *The World as Will and Representation*—the second section explores how those ideas were connected to his conception of the role of mathematics in natural philosophy. The third section deals with the 1830s British texts dedicated to mathematics that Schopenhauer read and quoted in the second volume of *The World as Will and Representation*. It appears that those publications contributed to hitherto unappreciated modifications in Schopenhauer's mathematical consideration: on the one hand, he expressed negative judgements on mathematical formalism and mathematical-physics that are not present in the works preceding *Parerga and Paralipomena*; on the other hand, his treatment of intuition in mathematics developed in a new form. As argued in section four, he appreciated that the authors of his British readings were debating on the very foundations of the Euclidean geometry—and not only on the "Parallelenproblem"—and he developed the notion that intuition could have been the answer to their questions.

It is generally maintained that Schopenhauer's philosophical theses did not change, if not marginally, after their first version in the system of 1819. This paper takes care to underscore mathematical-related ideas and contents that changed over time—as shown in his publications but even referring to the manuscripts when it is relevant. It aims to demonstrate that Schopenhauer participated in an early phase of the debate on the foundations of geometry by taking a fresh look at intuition: not only as an alternative to demonstration—something that was clearly unpopular among the mathematicians—but also as the ground of truth and certainty in the Euclidean system.

2 Intuition in Mathematics

Schopenhauer's philosophy of mathematics was mainly focused on geometry. His criticism started from the psychological observation that the logical method of proof in geometry provides "the conviction that the demonstrated proposition is true, but in no way does one see why what the proposition asserts is as it is" [Diss, § 40,

[5]See [HNV, p. 285]. About Fischer's non-peripherical role in the analytic movement in Germany, see [44, p. 562].

p. 135]. He blamed “the Euclidean method” for this separation of the *what* from the *why* that lets us “know only the former, not the latter” [WI, § 15, p. 98/86]. The consequence was that Euclid’s system provided a conceptual knowledge, like that of medical theories: “a mere empirical and non-scientific knowledge” [VorI, p. 457].

A significant consequence was the challenge to the distinction between axioms and theorems in Euclid’s *Elements*. It was probably connected to the question of the axiomatic nature of the fifth postulate: admitting that “the axioms themselves are no more immediately evident than any other geometrical theorems; they are simply less complicated because they have less content” [WI, § 15, p. 100/89] was a simple solution of the “Parallelenproblem”. He sustained that “every theorem introduces a new spatial construction that is in itself independent of its predecessors” and can be demonstrated “through pure spatial intuition, in which even the most involved construction is actually as immediately evident as an axiom” [WI, § 14, p. 88/75]. On the contrary, the Euclidean method required that theorems “are proven logically, that is, by presupposing the axioms and then by means of consistency with the assumptions made in a theorem or with a prior theorem, or by means of the inconsistency of the negation of a theorem with the assumptions, with the axioms, with prior theorems or even with itself” [WI, § 15, p. 100/89].

Another important dissent concerned the use of the *reductio ad absurdum* in demonstrations, for it is the principle of non-contradiction that obliges to accept a conclusion—not the content of the theorem: “the truth almost always emerges through a back door, the accidental result of some peripheral fact. An apagogic proof often closes every door in turn, leaving open only one, through which we are forced simply because it is the only way to go” [§ 15, p. 96/84]. Thirty years later, in the 1847 edition of *On the Fourfold Root of the Principle of Sufficient Reason* [G, § 39, p. 139], his denunciation of the demonstrative method as blind and forced was expressed by the metaphorical designation of Euclid’s proof of the Pythagorean theorem as a “mousetrap”—an image with a certain appeal.

Schopenhauer first published his views in his 1813 Dissertation *On the Fourfold Root of the Principle of Sufficient Reason*. He established his theses on the premise that mathematics—traditionally articulated in arithmetic and geometry—pertains to space and time as intuited a priori, “just as the infinite extension and infinite divisibility of space and time are objects only of pure intuition and are foreign to empirical intuition” [Diss, § 36, p. 130]. Succession and position define the relations within, respectively, portions of time and space; they “are intelligible to us simply and solely by means of pure, a priori intuition”—never by concepts. The law governing those relations is the *principle of sufficient reason of being* and the geometrical example of “the connection between the sides and angles of a triangle” shows that it “is completely different both from that between cause and effect and from that between cognitive ground and consequence” [Diss, § 37, p. 131].

According to these notions, he defined arithmetic and geometry. As it conveys the “nexus of the parts of time”, the former “is the basis of all counting” and “teaches absolutely nothing but methodical abbreviations of counting” [Diss, §

39, p. 133]. The latter is intuitive, non-conceptual "insight" into "the nexus of the positions of the parts of space"; this brings to the famous notion that "every geometrical proposition would have to be reduced to this intuition, and the proof would merely consist in clearly bringing out the nexus whose intuition is at issue" [Diss, § 40, p. 133]. The long § 40 of the 1813 Dissertation develops these ideas by analysing the intuitive nature of Euclid's 12 axioms as distinguished from the demonstrative character of the theorems. Demonstrations compel to accept the truth of theorems, but "thus, the logical truth, not the transcendental truth of the theorem, is demonstrated" [Diss, § 40, p. 135]. The former "produces mere conviction (*convictio*), not insight (*cognitio*)" and "leaves behind an unpleasant feeling" [Diss, § 40, p. 135], while "the ground of being of a geometric proposition recognized through intuition gives satisfaction" [Diss, § 40, p. 136].

To substantiate his point, he offered alternative, intuitive demonstrations of Euclid's 6th and 16th propositions and concluded his exploration with a caveat: "through all of this I have in no way proposed a new method of mathematical demonstration, no more than my proof will take the place of Euclid's" [Diss, § 40, p. 138]. More modestly he intended to underline how the lack of insight and satisfaction in demonstrative geometry might contribute to disliking mathematics.

In the first edition of *The World as Will and Representation* (1819) Schopenhauer revisited the discrepancy between immediate, intuitive truth and "truth that is grounded in proof" [WI, § 14, p. 89/77] and refined his notions on mathematics within a wider discourse. He emphasised the epistemic value of "feeling" geometrical truths by drawings [WI, § 11] and explained that intuition, as an immediate apprehension of truth, is more convincing than reasoning. Once again, however, a caveat clarifies that even if not immediately connected to truth, nonetheless abstraction and demonstration are necessary for precise communication and reliable application of knowledge: "in pure intuition we are perfectly acquainted with the essence and lawlike nature of a parabola, a hyperbola or a spiral. [. . .] Differential calculus does not really extend our cognition of curves in any way. [. . .] But it does change the kind of cognition we have: it converts intuitive cognition into an abstract cognition that is so rich in consequences for practical application" [WI, § 12, p. 78/63]. Arithmetic can really benefit from conceptualisation because numbers "can be expressed in abstract concepts that correspond exactly to them" [WI, § 12, p. 79/64]. It is not the case of geometry, where abstract cognition cannot precisely express spatial relations: it is easier to *see* "how the cosine decreases as the sine increases" [WI, § 12, p. 79/64] than to explain it conceptually. Schopenhauer's thesis is that geometry must "be translated" into numbers "if it is to be communicable, precisely determined, and applicable in practice" [WI, § 12, p. 79/64]. But such a translation is unnatural: the three dimensions of space must be expressed by numbers, which conceptualise the single dimension of time. Schopenhauer commented: "how the single dimension of time must suffer, as it were, to reproduce the three dimensions of space" [WI, § 12, p. 79/65]. The conclusion derived from these premises is that "a Euclidean proof, or an arithmetic solution to a spatial problem" [WI, § 12, p. 80/66] cannot acquiesce the mind looking for *real* comprehension.[6]

In 1819 Schopenhauer was able to elaborate a radical philosophy of geometry where theorems are nothing more than complex axioms, the *reductio ad absurdum* should be banned, and the logical demonstration is judged as useless or, worse, detrimental. He stated not only that "every truth discovered through inferences and communicated through proofs could also, somehow, have been recognized directly, without inferences or proofs" [WI, § 14, p. 91/78] but also that mathematics could gain from such a radical change in perspective: "abandoning the prejudice that a proven truth is at all preferable to one that we have intuitive cognition of" can lead to "an improved method in mathematics" [WI, § 15, p. 99/87]. Geometry

> never relies on the stilted march of a logical proof, since such a proof always misses the point and is usually soon forgotten without affecting anyone's conviction; we could even dispense with proof entirely and geometry would remain just as evident because it is quite independent of such proof, which only ever demonstrates something that we were already completely convinced of beforehand by a different kind of cognition. So logical proof is like a cowardly soldier who inflicts another wound on the corpse of an enemy already killed by someone else, but then boasts of finishing him off [WI, § 15, p. 102/90–91].

To strengthen his point, Schopenhauer introduced a visual demonstration of Pythagoras' theorem [WI, § 15, p. 98/87] and recalled that an implicit confirmation of his theses could be found in Kant's doctrine of space and time in the Transcendental Aesthetic of *Critique of pure reason*. According to his reading, Kant

> did not finish his train of thought, since he did not reject the whole Euclidean method of demonstration, even after saying [. . .] that all geometric knowledge is immediately evident in intuition. It is quite remarkable that even one of his opponents, and in fact the most astute of them all, G. E. Schulze (Critique of Theoretical Philosophy, II, 241), drew the conclusion that Kant's doctrine would give rise to an entirely different treatment of geometry than the usual one. He meant this to be an apagogic proof against Kant, but in fact he unwittingly began a war against the Euclidean method [WI, Appendix, pp. 465–6/519].

In the following years, Schopenhauer reiterated some aspects of his views on the primacy of intuition. "On the method of mathematics", chapter 13 of the second volume of *The World as Will and Representation* (1844), made explicit the criticality of the "Parallelenproblem" while emphasising the necessity of reform in the standard model of demonstration. The chapter on mathematics in the second edition of *On the Fourfold Root of the Principle of Sufficient Reason* (1847) explicitly referred to § 15 of the *The World as Will and Representation*, reproduced the visual demonstration of Pythagoras' theorem, and introduced the notion of "mousetrap" [G, § 39].

It is worth noting that in the second edition of the *The World as Will and Representation* (1844) he added this sentence to § 14: "all ultimate, i.e. original *evidentness* is *intuitive*: as the word already indicates" [1844, § 14, p. 78; 1859,

[6] Such a stance implied a negative judgement of both analytic geometry and mathematical analysis.

§ 14, p. 91/78]. Another addition is in the third edition (1859): "it is only on this sort of a geometrical basis (i.e. by means of *a priori* intuition) [. . .] that significant progress can be made with inferences" [1859, § 14, p. 92/79]. He clearly intended to strengthen the intuitive approach to mathematics by a more incisive praise of immediateness.

There are analogous remarks in *Parerga and Paralipomena* (1851). It is recalled that mathematics is not analytical: "the synthetical nature of geometrical propositions can be demonstrated by the fact that they contain no tautology. This is not so obvious in the case of arithmetic, but yet it is so" [PII, On logic and dialectic, § 23, p. 22/20]. And a passage from the chapter on the history of philosophy summarises: "mathematics is based on *intuitive perceptions* on which its proofs are supported; yet because such perceptions are not empirical but *a priori,* its theories are apodictic. [. . .] Accordingly, philosophy is now a science from mere *concepts,* whereas mathematics is a science from the *construction* (intuitive presentation) of its concepts". [PI, *Fragments for the history of philosophy*, § 13, pp. 79/74–75].

3 Role and Purpose of Mathematics

It is debatable whether Schopenhauer's belligerent attitude toward Euclid was really aimed at rewriting the traditional *corpus* of the geometry. Some passages in *The World as Will and Representation*, likewise the concluding remarks in § 39 of the 1813 Dissertation, suggest a concern for pedagogy in mathematics instead. He complained that Euclid's model deprives

> students of any insight into the laws of space, indeed, it gets them quite out of the habit of investigating the ground and inner nexus of things, and teaches them instead to let themselves to be satisfied with the historical knowledge *that* it is so. The exercise of acumen that wins Euclid's method such incessant praise amounts to no more than this: schoolchildren practise making inferences (i.e. applying the principle of non-contradiction), but more particularly they strain their memories remembering all the data whose mutual agreements have to be compared [WI, §15, p. 101/89].

For this reason, he admitted that "for teaching mathematics, I altogether prefer the analytical method to Euclid's synthetic method, even though it runs into very serious—if not insuperable—problems in the case of complicated mathematical truths" [WI, § 15, p. 99/87]. He also specified: "the most decisive step in this direction has been taken by Herr *Kosack*, a teacher of physics and mathematics at the Nordhausen Gymnasium, who has added a thoroughgoing attempt to treat geometry according to my principles to the schedule for school examination on the 6th of April 1852" [WI, § 15, p. 99/87]. Such a reference was not disinterested: as a matter of fact, Carl Rudolph Kosack mentioned Kant and Schopenhauer as his sources of the idea that demonstration in geometry requires eminently intuition (cf. [30, p. 10]; see [32]).

It would be limiting, however, to insist on the primacy of intuition and the pedagogical issue as the only relevant claims of Schopenhauer's philosophy of mathematics. He held a more complex view of mathematics and its role in the construction of knowledge that is not easily noticed in his texts—even because it was partially expunged from the pages of the *The World as Will and Representation* after its first edition. It had to do with the excess of abstraction and formalism not only in demonstrations but even in mathematical content. It timidly emerged in the manuscripts of 1813 and in the first edition of *The World as Will and Representation*, where he referred to Abel Bürja, Ferdinand Schweins and Bernhard Friedrich Thibaut, who had been his mathematics professor at the University of Göttingen (1809–1811).

Abel Bürja was the author of two treatises on autodidacticism in arithmetic and geometry [9, 10] which Schopenhauer borrowed from the Weimar Library in summer 1809, just before leaving for Göttingen, and in summer 1813, while writing the Dissertation.[7] Bürja's observations on the explanation of geometrical theorems were later mentioned in an 1813 manuscript regarding Kant's third *Critique* [HNI, pp. 63–64/83–84: § 95]. We cannot establish whether Bürja was a source of Schopenhauer's views on mathematics, but it is worth noting that after reading his books Schopenhauer chose Thibaut's mathematical course at Göttingen, whose manual of mathematics mentioned intuition as grounding geometrical notions [50, pp. 187–188, 310–311]. It was likely through Thibaut that Schopenhauer heard about Schweins, who had studied and taken his doctoral degree at Göttingen in 1807. Before moving to Heidelberg in 1810, where he became full professor in 1816, Schweins had taught mathematics at Darmstadt, where in 1810 he published the book later mentioned by Schopenhauer [45].

In 1817 Thibaut's manual was briefly discussed in the manuscripts [HNI, p. 447/602: § 655]. The reference was enriched by comments on Schweins in the first edition of *The World as Will and Representation*:

> Professor Thibaut in Göttingen has performed a great service in his Outline of Pure Mathematics [Grundriß der reinen Mathematik], although I would like a much more decisive and thorough substitution of the evidentness of intuition in place of logical proof. Professor Schweins in Heidelberg (Mathematics for primary scientific instruction [Mathematik für den ersten wissenschaftlichen Unterricht] 1810) has also declared himself against the Euclidean treatment of mathematics and attempted to move away from it. Only I find that his improvement reaches only as far as the presentation and not the method of treating mathematics itself, which still remains wholly Euclidean. He has certainly adopted a more coherent, more pragmatic approach rather than the fragmentary approach of Euclid, and that is definitely praiseworthy; but then he has abandoned Euclid's strict form without in the least moving away from his method as such, that is, logical proof in places where immediate evidentness would have been available [W1, pp. 571–72/109–110].

Schopenhauer praised the "pragmatic" approaches of those books to mathematics, but it is evident that he was not satisfied by their notions and methods; this is probably the reason why they were expunged from the 1844 and 1859 editions of the *The World as Will and Representation*. He looked for clarity and visibility of

[7] About the loans, see [HNV, p. 284]. In summer 1813 he also borrowed the 1800 German edition of Euclid's *Elementa* (see [DSW, vol. 16, p. 108]).

the truth, like in intuition, but also for concreteness against formalism, because the validity of theorems must not "reveals itself accidentally [*per accidens*]" [W1, p. 572/109].

His ideal of mathematics was related to his philosophy of science. He sought a philosophy of nature as a synthesis of the natural sciences and metaphysics—whose grounding, by the way, was in intuition. The former would make available empirical content and exhibit the effectiveness of metaphysics of will in providing knowledge of the world [46]. He praised factual and verifiable content as the solid foundation of scientific knowledge and the main source of progress. Instead, mathematics was abstraction, and even if he had accepted Kant's view of mathematical truths as synthetic, nevertheless he did not consider them as contributing to the advancement of learning. In an unpublished manuscript written in 1832, he clearly expressed the view that logic and mathematics "do not teach anything more than what we already *apriori* know" [Pandectae, p. 39].[8] The project of reinstalling intuition in mathematical demonstration was the way to preserve the connection between mathematics and knowledge.

On the contrary, the pernicious logical demonstrative procedure in mathematics had contaminated philosophy and contributed to widening the gap between metaphysics and reality. His criticism of Spinoza's *more geometrico* [WI, p. 102/91 footnote] is a clear example of his low esteem of the benefits of mathematics to philosophy. Something similar, even if inverted, could be observed in Schelling's philosophical procedure of "construction": here philosophy aimed to ground the mathematical demonstration.[9] In one way or the other, mathematics had widened its detachment from reality.

On the front of the sciences, things were not better. Abstraction and logical demonstrations had become values of the mathematised sciences. Schopenhauer's penchant for Goethe was probably related to his polemics against Newton and the mathematical description of the world. Melanchthon's famous acclamation of arithmetic and geometry as "the wings of human minds" [36, p. 288]—which had contributed to the boosting of the scientific revolution in the Reformed lands—never persuaded Schopenhauer. Notwithstanding Melanchthon's explicit reference to Plato, Schopenhauer was deeply convinced that the concrete truth about the world cannot derive from the abstractions of mathematics.

As a consequence, Schopenhauer was generally reluctant to consider mathematics as philosophically and epistemically relevant. When assessing scientific knowledge, Schopenhauer valued the role that empirical truth plays in establishing a sound theory. Precision and certainty of mathematics (and logic), on the contrary, do not provide content and knowledge. He certainly recognised the profound impression of Euclid's model of explanation on metaphysics and natural philosophy in the modern era, and his criticism was both a response to the undue honour paid

[8] "[. . .] sie uns eben nichts weiter lehren, als was wir schon vorher (a priori) wußten".

[9] On the relationship between construction, demonstration, and the project of transcendental philosophy in Schelling, see [55, pp. 188–193]. See also [7, 25].

to the traditional deductive procedure and a reminder of the privileged access to knowledge provided by intuition. Besides, he was aware that history had indelibly marked the fate of mathematics and a reversal would be implausible. He was not pursuing a quixotic dream, rather he reflected upon mathematics as a concrete form of knowledge, something intrinsically useful in everyday life, schools, the sciences, and even philosophy.

4 A British Debate

An explicit expression of those ideas appeared in print at the end of chapter 13 of the second volume of *The World as Will and Representation*, by referring to "the sense in which Plato recommended geometry to philosophers [. . .] as a preliminary exercise, by which the mind of the pupils became accustomed to dealing with incorporeal objects, after this mind had hitherto in practical life had to do only with corporeal things" [WII, 13, p. 131/144]. Schopenhauer pointed out that an interesting perspective had emerged in Britain, in the review of a book of William Whewell by the Scottish philosopher William Hamilton [21].[10] Described as "an investigation of the influence of mathematics on our mental powers and of its use for scientific and literary education in general", Hamilton's review was interpreted by Schopenhauer as assessing that "the value of mathematics is only indirect, and is found to be in the application to ends that are attainable only through it; it is by no means necessary; in fact, it is a positive hindrance to the general formation and development of the mind. [. . .] The only immediate use left to mathematics is that it can accustom fickle and unstable minds to fix their attention" [WII, 13, pp. 131/144–5]. Hamilton's "fine" essay was later mentioned again in the chapter "On learning and the learned" of *Parerga and Paralipomena*, where Schopenhauer acknowledged the peculiarity of the "aptitude for mathematics", which "does not by any means run parallel to the other mental faculties, and in fact has nothing in common with them" [PII, §256, p. 489/409].

If we want to understand Schopenhauer's convinced reference to Hamilton, we should consider the context that stimulated both Whewell's intervention about mathematical education in relationships with higher learning and Hamilton's response to it. The starting point was the so-called 'analytic revolution', around 1800, when Lagrange's seminal work and its dissemination by Lacroix showed the superiority of analysis over synthetic-geometric mathematics to pursue generality.[11] In a few decades, mathematicians would acknowledge that geometry had become inadequate to scientific investigation. Notwithstanding the "peculiar excellence" of the "method of synthesis", "the very circumstances, which cause its perspicuity and

[10] Schopenhauer added a reference to [22], the German translation of Hamilton's review.

[11] On that seminal moment, see [20, Chap. 2].

evidence, render it unfit for the deduction of truths that are remote and intricate".[12] As a consequence, not only mathematics underwent an inevitable transformation: it became clear that mathematical education required substantial reformation, too.[13] Above all, it was questioned whether learning mathematics should still be part of a general education because skills and the talent required to be a proficient mathematical analyst were peculiar and rare.

4.1 Whewell on the Study of Mathematics

Whewell reflected upon these events from the extraordinary point of view of tutor and professor at the University of Cambridge from 1818. According to him, "the object of a liberal education is to develop the whole mental system of man, and thus to bring it into consistency with itself; to make his speculative inferences coincide with his practical convictions; to enable him to render a reason for the belief that is in him".[14] The analytic revolution, as recalled by Harvey Becher, "challenged the entire Cambridge educational system, for mathematics formed the core of the liberal education that was Cambridge's *raison d'être*" [3, p. 3]. Synthetic-geometric mathematics functioned as trainer of logical and open minds, necessary to ground culture and the intellectual abilities of an elite which would pursue professional and clerical careers. Instead, pure analysis's vocation was abstraction and formalism, which dismissed geometry and its intuitive foundation: "there exist certain modes of treating the study of mathematics, and certain views concerning its foundations, which must diminish its benefits as a mental discipline and a preparation for all other branches of philosophical speculation" [52, p. 168].

In the 1830s Whewell had already developed critical views against a privileged role of analysis in Cambridge education. He considered analysis as having limited or even pernicious effects on the mind: "analysis too often merely gives us results which exercise no intellectual faculty, nor convey any satisfactory knowledge" [52, p. vi]. To ground his stance, Whewell embarked on a series of inquiries in the area of pedagogy: *Thoughts on the Study of Mathematics as a part of a Liberal Education* (1835), *On the Principles of English University Education* (1837), *Of a Liberal Education in General, and with Particular Reference to the Leading Studies of the University of Cambridge* (1845). The last one offered harsh criticism like the following: "the destructive effect of mere analysis upon the mind"; "so far as the analytical method has superseded the geometrical, I am obliged to say [. . .], the result has been very unfortunate"; analysis is "of little value as a discipline

[12] These were the words of a British reviewer of Lacroix's *Traité du calcul différentiel et du calcul integral* (1797–1798) in *Monthly Review* (see [1, p. 492]).

[13] At this time the challenge of non-Euclidean geometries was not present yet: Euclidean geometry was still the cornerstone of the English liberal education. See [42].

[14] Whewell, *Thoughts on the Study of Mathematics as a part of a Liberal Education*, in [52, p. 139].

of the reason for general purposes. [. . . It] belongs to a class of intellectual habits which it is the business of a good education to counteract, correct, and eradicate, not confirm, aggravate, and extend".[15] The good education could be found in the old curriculum of Euclid and Newton, whose *Principia* contained "beautiful examples of mathematical combination and invention, following the course of the ancient geometry". A person educated according to the traditional programmes "had commonly acquired a command of certain mathematical methods, and a love of mathematics, which he retained through life" [51, pp. 35, 185]. It is worth noting that Whewell was quite candid about the aim of mathematical education: "the use of mathematical study [. . .] is not to produce a school of eminent mathematicians, but to contribute to a Liberal Education of the highest kind" [51, p. 77].

The primacy of "liberal education" and the protection of the youngsters' minds from the aridity of formalism was at first defended in the brief pamphlet (less than 50 pages) *Thoughts on the Study of Mathematics as a part of a Liberal Education* (1835). After maintaining the educational superiority of the study of mathematics ("teaching of reasoning by practice") over the study of logic (teaching of reasoning "by rule"),[16] Whewell asserted that mathematics can train minds to deal "with other kinds of truth" and "on any particular subject" [*Thoughts*, p. 141, 142] only if conventional or empirical views of its first principles are banished and excessive formalism and generalisation are avoided [*Thoughts*, p. 142]. Otherwise, "we not only sow the seeds of endless obscurity and perplexity [. . .], but we also weaken his [the student's] reasoning habits and disturb his perception of speculative truths; and thus make our mathematical discipline produce, not a wholesome and invigorating, but a deleterious and perverting effect upon the mind" [*Thoughts*, p. 156]. He was adamant that "the foundation of all geometrical truth resides in our general conception of space" and that the teaching of differential calculus according to the new course of analysis was misleading [*Thoughts*, pp. 149–153]. In order to learn at best geometry and calculus, then, the sources were still Euclid and Newton's *Principia*, notwithstanding all of modern mathematics.

In conclusion, to be part of a liberal education, mathematics must be rigorous, not abstract, and grounded in the notions of geometrical space and arithmetic number: "I believe that the mathematical study to which men are led by our present requisitions has an effect, and a very beneficial effect, on their minds: but I conceive that the benefit of this effect would be greatly increased, if the mathematics thus communicated were such as to dissipate the impression, that mathematical reasoning is applicable only to such abstractions as space and number" [*Thoughts*, p. 174].

[15] [51: dedicatory letter to Airy; p. 204; p. 45].

[16] Whewell, *Thoughts on the Study of Mathematics as a part of a Liberal Education*, in [52, p. 141]: mathematics, then, is to be considered "as a means of forming logical habits better than logic itself".

4.2 Hamilton's Review

One year after Whewell's *Thoughts on the Study of Mathematics*, *The Edinburgh Review* published a long review by William Hamilton—in fact as long as Whewell's pamphlet. Together with Dugald Stewart, Hamilton (1788–1856) was the most influent interpreter of Thomas Reid's common sense realism and pillar of the Scottish philosophical movement—at least until John Stuart Mill would demolish his philosophy in the memorable *Examination of Sir William Hamilton's Philosophy* (1865).[17] He visited Germany in 1817 and 1820 and contributed to the diffusion of Kantian and post-Kantian philosophy in Britain. His fame in the second quarter of the century was certainly related to his extensive knowledge of Continental philosophy; besides, he was a brilliant philosopher, a talented logician,[18] and a respected reviewer in influent journals like *The Edinburgh Review*. To be reviewed by Hamilton could be crucial for the success of a book—as acknowledged by Mill, who expected his forthcoming *System of Logic* (1843) would be reviewed by the "hostile, but intelligent" Scottish philosopher.[19]

Tackling Whewell over the subject of mathematics as a means of liberal education brought Hamilton to discussing the nature of mathematical principles, the notion of liberal education itself, and the comparison between mathematical and philosophical knowledge—while expressing opinions, critiques and strong dissent that would stimulate Whewell's reaction.[20] His conclusions were that the primacy of mathematics at Cambridge was "indirectly discouraging the other branches of liberal education", tended "positively to incapacitate and to deform the mind", was worthless "for the conduct of the business, or for the enjoyment of the leisure" [21, pp. 453–454], and was not serving the cause of mathematics, as no Cambridge mathematician had ever gained recognition in the field [21, p. 410].

Hamilton's analysis started from a different view about what a "liberal education" should be: "we speak not now of *professional*, but of *liberal* education; not of that, which makes a mind an instrument for the improvement of science, but of this, which makes science an instrument for the improvement of the mind" [21, p. 411]. Such a perspective, it is evident, would not admit the curricular primacy of

[17]On Hamilton (1788–1856) and his fame at the time of Schopenhauer's reference, see [35, pp. 113–114, 120–133].

[18]His decennial (1846–1856) controversy with Augustus De Morgan about the priority in theorising the quantification of the predicate was also famous. See [18, 31, 40].

[19]"If you do not review the book it will probably fall into the hands either as you suggest, of Sir W. Hamilton, or of Brewster. The first would be hostile, but intelligent, the second, I believe, favourable, but shallow": John Stuart Mill to John Austin, July 7, 1842, in [37, p. 528].

[20]On January 23rd 1836, Whewell wrote a letter to *The Edinburgh Review* (vol. XLIII, n. 127, 1836, pp. 270–272; then reprinted in [52, pp. 186–189]) making clear that his pamphlet was about "what kind of mathematics is most beneficial as a part of a liberal education" and not "a vindication of mathematical study" as Hamilton had suggested—"having thus made me work at a task of his own devising" [52, pp. 186–187]. Such a casual missive was nevertheless followed by the more committed works of 1837 and 1845.

mathematics. But it was its "utility as an intellectual exercise" that he essentially contested: instead of "its importance as a logical exercise", the "evidence" speaks "of its contracted and partial cultivation of the faculties"; besides, "the most competent judges" and "the authorities" of the philosophical tradition have generally sustained "that the tendency of a too exclusive study of these sciences, is, absolutely, to disqualify the mind for observation and common reasoning" and, even more precisely, that "none of our intellectual studies tend to cultivate a smaller number of the faculties, in a more partial manner, than mathematics" [21, pp. 411, 412, 419].

Amongst those authorities Hamilton quoted Aristotle and the notion of virtuous man as educated through a varieties of disciplines, German pedagogic books, Goethe ("the cultivation afforded by the mathematics is, in the highest degree, one-sided and contracted"), Voltaire ("j'ai toujours remarqué que la geometrie laisse l'esprit ou elle le trouve"), Franklin and even first-rank mathematicians like D'Alembert and Descartes.[21] Other authors were recalled to support the view that geometry—as based on imagination and senses—does not reinforce understanding or the capacity of generalisation: Mersenne, Digby, Coleridge, Kant, Duhamel, Pestalozzi and Warburton ("the routine of demonstration [is] the easiest exercise of reason, where much less of the vigour than of the attention of mind is required to excel" [21, pp. 425–429]. He also reproduced long passages from mathematicians like Pascal, Berkeley, s'Gravesande, D'Alembert and from other illustrious intellectual and philosophers in order to support his argument about narrowness and proneness to error of the mathematical mind [21, pp. 434–441]. The conclusion that mathematicians "are disposed to one or other of two opposite extremes—credulity and skepticism" gave Hamilton the opportunity, on the one hand, to express his Reidian anti-metaphysical stance and condemn as bad philosophers (because too credulous) the mathematicians "Pythagoras, Plato, Cardan, Descartes, Mallebranche, and Leibnitz"[22]; on the other hand, to denounce mathematicians' inclination towards atheism, negation of moral freedom and denial of the soul.[23]

Hamilton's criticism of Whewell's arguments was strong. Firstly, he demolished Whewell's idea that mathematics is more apt than logic to ground a liberal education. Hamilton reproached Whewell of having overlooked the distinctions between *theoretical* and *practical* logic and between *practical* logic "as specially applied to Necessary Matter=Mathematical reasoning" and "as specially applied

[21] [21, pp. 417–421]. See [21, p. 421] for the quotations. Here Hamilton was amply using his first-hand knowledge of German philosophy and literature.

[22] [21, p. 443]: "Conversant, in their mathematics, only about the relations of ideal objects, and exclusively accustomed to the passive recognition of absolute certainty, they seem in their metaphysics almost to have lost the capacity of real observation, and of critically appreciating comparative degrees of probability. In their systems, accordingly, hypothesis is seen to take the place of fact; and reason, from the mistress, is degraded to the handmaid, of imagination."

[23] [21, pp. 445–450]. On this subject, Hamilton quoted Patristic authors, philosophers like Berkeley, Kant, Fries and added a long passage (without any reference) from Jacobi's 1815 Preface to *David Hume on Faith, or Idealism and Realism, a Dialogue* (1787), in [26, pp. 51–55]. It is worth noting that the same passage from Fries was in a footnote of Jacobi's text [26, pp. 52–53].

to Contingent Matter=Philosophy and General reasoning". It is the latter, stated Hamilton, that helps to "cultivate the reasoning faculty for its employment on contingent matter". On the contrary, Whewell ignored practical logic and erroneously concluded for the primacy of mathematics [21, p. 413].

Secondly, he attacked Whewell on the nature of mathematical first principles. According to Hamilton, Whewell was addressing a question of philosophy of mathematics without actually referring to philosophical notions or authors. On this subject, Hamilton the philosopher showed pertinence and precision—and made it evident that Whewell had offered an interpretation of the foundations of mathematics that misinterpreted Kant's views [21, pp. 414–417].

Thirdly, he defended the superiority of philosophical education over the mathematical by considering their different objects, ends and "modes of considering their objects". While mathematics "take no account of things",[24] "philosophy, on the other hand, is mainly occupied with realities; it is the science of a real existence, not merely of a conceived existence" [21, p. 422]. As to the ends, they tend to two different kinds of knowledge: in mathematics the whole science is contained in the principles—which "afford at once the conditions of the construction of the science, and of our knowledge of that construction (*principia essendi et cognoscendi*)"—and "it is only the evolution of a potential knowledge into an actual, and its procedure is thus merely explicative". Philosophy is quite different: "its principles are merely the rules for our conduct in the quest, the proof, and the arrangement of knowledge: it is a transition from absolute ignorance to science, and its procedure is therefore ampliative" [21, p. 423]. But even more relevant is the difference in the modes of considering their objects: mathematical science "contemplates the general in the particular", while philosophy "views the particular in the general"]; mathematics is perfectly expressed by its own language, while philosophy struggles with common linguistic expressions of concepts which do not mirror its notions[25]; in mathematics deductions are "apodictic or demonstrative", while in philosophy "such demonstrative certainty is rarely to be attained" [21, p. 424]. All this considered, "it will easily be seen how an excessive study of the mathematical sciences not only does not prepare, but absolutely incapacitates the mind, for those intellectual energies which philosophy and life require. We are thus disqualified for observation either internal or external—for abstraction and generalization—and for common reasoning; and disposed to the alternative of blind *credulity* or irrational *scepticism*" [21, p. 424].

All in all, Hamilton denied any positive effect of studying mathematics while pursuing a liberal education: mathematical demonstration is counterproductive "as a practice of reasoning in general"; it "educates to no sagacity" and "allows no room for any sophistry of thought". Against Whewell's convinced view that mathematics establishes "logical habits better than logic itself", Hamilton rebutted that the very

[24] Hamilton used "mathematics" as a plural noun.

[25] [21, p. 424]. Hamilton speaks of "the absolute equivalence of mathematical thought and mathematical expression".

perfection of mathematical reasoning makes it useless: the "art of reasoning *right* is assuredly not to be taught by a process in which there is no reasoning *wrong*" [21, pp. 426–427]. He also explained why mathematics appears extremely easy to the inclined and acutely painful for many other students: the simplicity and monotony of demonstrations require an unbearable attention from minds "endowed with the most varied and vigorous capacities". Paradoxically, "to minds of any talent, mathematics are only difficult because they are too easy", because in "mathematics dullness is thus elevated into talent, and talent degraded into incapacity" [21, p. 430].

The only way to benefit from the study of mathematics, Hamilton concluded, was under restricted conditions: "if pursued in moderation and efficiently counteracted, [it] may be beneficial in the correction of a certain vice, and in the formation of its corresponding virtue. The vice is the habit of mental distraction; the virtue the habit of continuous attention" [21, p. 450]. Such a benefit, however, would not redeem mathematics from the disadvantage of narrowness—while the mind needs "an extensive, a comprehensive, or an intensive application of thought"—and in any case it cannot train students without inclination to attention: "after all, we are afraid that D'Alembert is right; mathematics may distort, but can never rectify the mind" [21, pp. 452–453].

5 Intuition and the Foundations of Geometry

Generally, historians have assessed Hamilton's attack on Whewell by looking at the battle of the latter against the former's common sense philosophy and in particular at the dispute about the philosophy of mathematics—the nature of axioms and definitions in geometry and the general interpretation of mathematical truth.[26] Schopenhauer did not miss this foundational controversy—that offered a new perspective over the "Parallelenproblem"—and was inspired by Hamilton's treatment of the role of mathematics in education and the production of knowledge. We can discern what he appreciated in Hamilton's observations, comments and sources.

Firstly, the vast and informed quotations from authors (many of them from German sources) who supported his argument certainly impressed Schopenhauer: he, too, was used to this kind of justification in his writings—in the manuscripts even more than in publications. It is worth noting that in the 1859 edition of *The World as Will and Representation* he added the same quotation from Baillet's *Life of Descartes* which Hamilton had translated in his review.[27] Secondly, the Scottish philosopher expressed knowledge of and admiration for Kant's views on space, time

[26] [11, 12, 15, 16, 47], [48, pp. 86–89].

[27] [WII, p. 132/145], [21, p. 421].

and mathematics—something that certainly captivated Schopenhauer.[28] Besides, even if cursorily the review mentioned intuition as essential to the understanding of mathematics: "the principles of mathematics are self-evident; [. . .] every step in mathematical demonstration is intuitive" [21, p. 428]. Thirdly, Hamilton insisted that mathematical truth and knowledge had nothing in common with the same notions in natural philosophy: "the truth of mathematics is the harmony of thought and thought; the truth of philosophy is the harmony of thought and existence" [21, p. 423]. His observation that in philosophy "demonstrative certainty is rarely to be attained" and is not comparable to the apodictic truth of mathematics [21, p. 424] was similar to Schopenhauer's remark on Spinoza [WI, p. 102/91 footnote], who had mixed the two of them—a remark that was introduced in the second edition of *The World as Will and Representation*, i.e. after having read Hamilton.

Schopenhauer agreed with Hamilton about the distance between mathematical formalism and natural philosophy. It is often sustained, claimed Hamilton, that the mathematics is "the passport to other important branches of knowledge. In this respect mathematical sciences (pure and applied) stand alone: to the other branches of knowledge they conduce —to none directly, and if indirectly to any, the advantage they afford is small, contingent, and dispensable" [21, p. 453]. Schopenhauer's distrust of mathematics as contributing to knowledge was evidently strengthened by Hamilton. In 1851, he attacked vehemently arithmetic as a tool for arid calculations:

> that the lowest of all mental activities is arithmetic is proved by the fact that it is the only one that can be performed even by a machine. In England at the present time, calculating machines are frequently used for the sake of convenience. Now all analysis *finitorum et infinitorum* ultimately amounts to repeated reckoning. It is on these lines that we should gauge the 'mathematical profundity', about which Lichtenberg is very amusing when he says: 'The so-called professional mathematicians, supported by the childish immaturity of the rest of mankind, have earned a reputation for profundity of thought that bears a strong resemblance to that for godliness which the theologians claim for themselves' [PII, Psychological remarks, § 356, p. 610/493].

Such a diminishing appreciation helped to build a case against Newton—"the great mathematician" who enjoyed "ludicrous veneration" [PII, On philosophy and natural science, § 80, p. 126/99]—and his theory of colours: "Goethe had the true objective insight into the nature of things, a view that is given up entirely to this. Newton was a mere mathematician, always anxious to measure and calculate and taking as the basis of this purpose a theory that was pieced together from the superficially understood phenomenon" (PII, On the theory of colours, §107, p. 197–

[28][21, p. 423]: "without entering on the metaphysical nature of Space and Time, as the basis of concrete and discrete quantities, of geometry and arithmetic, it is sufficient to say that Space and Time, as the necessary conditions of thought, are, severally, to us absolutely one; and each of their modifications, though apprehended as singular in the act of consciousness, is, at the same time, recognised as virtually, and in effect, universal. Mathematical science, therefore, whose conceptions (as number, figure, motion) are exclusively modifications of these fundamental forms, separately or in combination, does not establish their universality on any a posteriori process of abstraction and generalization; but at once contemplates the general in the particular."

8/211].[29] Similar harsh criticism was levelled at Laplace and the French Newtonian physics around 1800: "le calcul! le calcul! This is their battle-cry. But I say: ou le calcul commence, l'intelligence des phénomènes cesse".[30]

Schopenhauer's reference to Hamilton should not be overlooked. He was the most important author in orienting Schopenhauer towards an utter devaluation of mathematics as a source of actual knowledge. Before reading that iconoclastic text, Schopenhauer's was simply considering the vindication of intuition against demonstration and the reasons of pedagogy—but he had never questioned that mathematics was essential in culture and education. The attacks against mathematical-physics were a substantial leap from the previous position but it should be misleading viewing them as motivated by incomprehension of advanced mathematics. Hamilton had offered several arguments to sustain the idea that mathematics was not only useless to education and knowledge but even harmful. He had provided examples of how good mathematician had turned into bad (natural) philosophers. In his long quotation from Jacobi there was the same kind of criticism of the mathematical-physics tradition as barren and empty later exploited by Schopenhauer: "he [the mathematical-physicist] no longer marvels at the object, infinite as it always is, but at the human intellect alone, which, in a Copernicus, Kepler, Gassendi, Newton, and Laplace, was able to transcend the object, by science to conclude the miracle, to reave the heaven of its divinities, and to disenchant the universe" [21, p. 449].[31]

Hamilton's review also engaged Schopenhauer's attention to the question of the foundations of mathematics—and specifically of geometry—as debated in Britain. In particular, he acknowledged that the generalised perplexities about the fifth postulate could be related to deeper questions. He could fully appreciate them in 1838, when *The Edinburgh Review* published another review on Whewell: Thomas Flower Ellis analysing the *Mechanical Euclid* [14, 53].[32] Two years younger than Whewell, Ellis (1796–1861) had graduated from Trinity College, Cambridge, in the 1810s and was acquainted with the philosopher.[33] Yet his review was not sympathetic: it discussed the question of the foundations of geometrical certainty and Ellis confronted Whewell's position on the absolute necessity of mathematical truths from Dugald Stewart's point of view, who had maintained Euclid's axioms and theorems "to consist, in truth, of definitions and of propositions requiring proof" [14, p. 87].[34] While Whewell asserted that "deductive proofs consist of many steps, in each of which we apply known general propositions in particular cases;— 'all triangles have their angles equal to two right angles, therefore this triangle has; therefore, &c.'" [53, p. 182], Ellis countered:

[29] The quotation comes from Deussen's posthumous edition of *Parerga* [DSW, vols. 3–4].

[30] Originally in [Sen, p. 32], the quotation was included in [F, p. 90] by Frauenstädt.

[31] The original text in [26, p. 52].

[32] On Whewell's work, see [27].

[33] On Ellis, see [33].

[34] Ellis referred to Stewart's *Elements of the Philosophy of the Human Mind* [49, p. 43, p. 40, p. 527].

> the reception of one truth does not precede the reception of the other in the order of reasoning. These axioms are, in truth, practical laws of thought; they are a part of the machinery by which the reason works, not of the material from which it obtains its results. Again, it is not possible for human ingenuity to deduce a single geometrical inference from these axioms. [. . .] The science therefore does require the definitions, but does not require the axioms [14, p. 88].

Overtly relying on Stewart, who had been an object of Whewell's criticism, Ellis defended the notion that geometry could be founded only on self-evident truths: "the definition requires the possibility of the thing as defined. The possibility should therefore be presented as a self-evident proposition, that is, as an axiom" [14, p. 92]. According to him, the difference between Stewart and Whewell "appears to be on the question, merely, whether what we have here called the second class of axioms be truly axioms. Mr Stewart thinks that they consist of definitions and propositions requiring proof; while Mr Whewell considers them to be truly axioms" [14, p. 94].[35] If an axiom were not self-evident and required a definition to be understood, it was not an actual axiom, but rather a theorem whose truth benefitted of the definition's self-evidence. He rhetorically asked: is it correct to consider

> as an axiom which merely supplies the incompleteness of the definition? Is that properly called an axiom, which adds to the properties given in the definition, or explains the meaning of the words? Is that properly called a definition, which conveys an incomplete or indefinite (or 'vague') conception, till an explanation be added, or an addition supplied, by an axiom? [14, p. 96].

If an axiom is an addition, Ellis concluded, and is neither a definition nor a self-evident proposition, "we protest against founding any argument, respecting mathematical reasoning, on a part of the system which is acknowledged to be a violation of the principles of such reasoning" [14, p. 97].

Schopenhauer appreciated Ellis's contribution, which had distinctly expressed criticism of the diffuse praise of Euclid's geometry. He immediately registered some passages from the review in his manuscripts and later he elaborated them in chapter 13 of the second volume of *The World as Will and Representation*.[36] Ellis had developed arguments that supported Schopenhauer's own denunciation of the Euclidean system—with its "futile attempts to demonstrate the *directly* certain as merely *indirectly* certain" [WII, 13, p. 130/144]—and his unconventional view of the equivalence between postulates and theorems.[37] Both of the themes were recapitulated in the first paragraph of chapter 13, but without considering the

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[35] Ellis's "second class of axioms" corresponds to Euclid's five postulates. At the time, they called "axioms" all the fundamental propositions of Euclidean geometry, the seven axioms and the five postulates.

[36] [HNIV(1), pp. 289–290/254–255: Spicilegia, § 36 (1838)], [WII, 13, pp. 130–131/144].

[37] As an example, Ellis had insisted that Euclid's fifth postulate required a demonstration and had advanced a general consideration that appealed to Schopenhauer: "the proposition has, universally we believe, been allowed to require demonstration, and to be improperly termed an axiom. It is surely not correct to assert that a chain of truths owes its peculiar certainty to its resting upon that which itself requires, and has not received, a demonstration. Euclid's twelfth axiom is, indeed,

reference to Ellis it seems that Schopenhauer was simply reasserting the primacy of intuition over demonstration in geometry. Instead, such an insistence reveals that Schopenhauer had acquired full awareness of the debate on the foundations of geometry—and of its importance and width in Britain[38]—and thus reshaped the notion of intuition as foundational. He took the similarity of his views to those of Ellis (and Stewart), and specifically their thesis that geometry's only ground was in definitions——and consequently there was not any difference between axioms and theorems—as the occasion to reconsider demonstration and *reduction ad absurdum* as symptoms of a more serious problem: the lack of foundations. Even the notion of intuition as an alternative to demonstration—like he had presented it in the 1810s—did not confront the real question. The point at issue was neither the "Parallelenproblem" nor substituting the (overly complicated) demonstration of theorems with the immediate vision of their truth; instead it was the promise of certain and definitive truth of geometry itself. Rather than for the (better) procedure, the quest was for the foundation.

Hamilton's and Ellis's reviews of Whewell had shown Schopenhauer a lively and heated debate that gave new meaning to his own view of intuition in mathematics—as the foundation of geometry. Whereas in the 1810s he had developed his philosophy of mathematics as Kant's follower and as a consequence of his praise for the fundamental role of intuition in metaphysics,[39] in the 1830s he explored the possibility of intuition as the ultimate foundation of geometry. The British debate had demonstrated that the traditional interpretation of the Euclidean geometry brought to both inconclusive discussions about the demonstration of the fifth postulate and the denigration of mathematics as a part of education. Schopenhauer realised that the relevant philosophical questions concerned the foundations of mathematics and that the entire mathematical structure was less firm than believed. His harsh judgement of the calculus's abstraction derived from a more radical view encouraged by the British discussion: if intuition was the foundation, abstraction became an actual perversion.

Overlooking Schopenhauer's reading of *The Edinburgh Review* has persuaded commentators that he was far from the mainstream of mathematics; thus, some may have considered his views as vitiated by a substantial incomprehension of the new course of mathematics in the nineteenth century.[40] Certainly, he never exhibited an aptitude for the exact sciences, but this is not the point. Intuition as both an

merely an indication of the point at which geometry fails to perform that which it undertakes to perform" [14, p. 91].

[38]Whewell continued the discussion with Ellis in the second book of [54].

[39]In an annotation of spring 1820 he celebrated "the joy of *conceiving directly* and intuitively, correctly and sharply, the *universal and essential aspect of the world*" [HNIII, p. 23/19: Reisebuch, § 61].

[40]Cajori bluntly commented: "Schopenhauer discloses no acquaintance with such modern mathematical concepts as that of a function, of a variable, of coordinate representation, and the use of graphic methods. With him Euclid and mathematics are largely synonymous. Because of this one-sided and limited vision we can hardly look upon Schopenhauer as a competent judge of the

alternative to logical demonstration and a pedagogical aid was Schopenhauer's starting point; but it evolved, enriched by the encounter with the British debate. At the time geometry was still solidly Euclidean and it represented the model for any pursuit of truth and certainty—within and without mathematics.[41] Doubts were typically related to the fifth postulate, but until the acceptance of the non-Euclidean geometries—thanks to Eugenio Beltrami [5], Hermann von Helmholtz [23, 24] and Felix Klein [28]—there was not a real interest in discussing the foundations of geometry. The British debate stimulated Schopenhauer to reconsider his approach based on intuition as an answer to the questions raised by Stewart, Whewell, Hamilton and Ellis on the foundations of truth in mathematics and geometry—especially after calculus had abandoned the geometrical model, and abstraction and conceptualisation had gained centrality in analysis. Schopenhauer was certainly not equipped to discuss analysis, but he lucidly saw that if logical deduction had become the only guarantee of mathematics, the synthetical character of mathematics would have been lost. In 1844 and 1851, condemnation of both abstraction in analysis and deduction in geometry appeared as consequences of Schopenhauer's development of his treatment of intuition—after reflecting on the more stringent question of the foundations of mathematics in general and geometry in particular.

A few years later, unfortunately, the non-Euclidean geometries revolutionised the philosophy of geometry: that precocious British debate on the Euclidean system inexorably aged and Schopenhauer's participation was easily forgotten.

Abbreviations

Arthur Schopenhauer's Works

Sämtliche Werke, hrsg. von Arthur Hübscher, dritte Aufl., 7 Bände, Wiesbaden, Brockhaus, 1972

Diss *Ueber die vierfache Wurzel des Satzes vom zureichenden Grunde. Eine philosophische Abhandlung*, Rudolstadt, in Commission der Hof-Buch und Kunsthandlung, 1813, in Band VII, pp. 1–94 (tr.: *On the Fourfold Root of the Principle of Sufficient Reason and Other Writings*, ed. by D. Cartwright, E. Erdmann, C. Janaway, Cambridge, Cambridge UP, 2015)

G *Ueber die vierfache Wurzel des Satzes vom zureichenden Grunde. Eine Philosophische Abhandlung*, zweite Auflage, Frankfurt a.M., Hermann'sche Buchhandlung, Suchsland, 1847, in Band I (tr.: *On the Fourfold Root of the Principle of Sufficient Reason and Other Writings*, ed. by D. Cartwright, E. Erdmann, C. Janaway, Cambridge, Cambridge UP, 2015)

educational value of modern mathematics" [13, p. 367]. Similar accusations around 1900, when Schopenhauer was even called "enemy" of mathematics, are mentioned by [32, pp. 330–331].

[41] The only exception, Bolzano's *Beyträge zu einer begründeteren Darstellung der Mathematik*, [6], which proposed a logico-mathematical foundation of geometry. Bolzano admitted that his first source and guide to such an approach had been Kästner's *Anfangsgründe der Arithmetik* (1758). Like most of Bolzano's works, the *Beyträge* were ignored until the second half of the nineteenth century.

W1 *Die Welt als Wille und Vorstellung*, Leipzig, Brockhaus, 1819 (tr.: *The World as Will and Representation*, ed. by C. Janaway, J. Norman, A. Welchman, Cambridge, Cambridge UP, 2014)

WI *Die Welt als Wille und Vorstellung*, 1. Band, *Vier Bücher nebst einem Anhange, der die Kritik der Kantischen Philosophie enthält*, dritte Auflage, Leipzig, Brockhaus, 1859, in Band II (tr.: *The World as Will and Representation*, ed. by C. Janaway, J. Norman, A. Welchman, Cambridge, Cambridge UP, 2014)

WII *Die Welt als Wille und Vorstellung*, 2. Band, *welcher die Ergänzungen zu den vier Büchern des ersten Bandes enthält*, dritte Auflage, Leipzig, Brockhaus, 1859, in Band III (tr.: *The World as Will and Representation*, vol. 2, ed. by E.F.J. Payne, New York, Dover, 1966)

PI *Parerga und Paralipomena: kleine philosophische Schriften*, 1. Band, Berlin, Hayn, 1851, in Band V, (tr.: *Parerga and Paralipomena*, vol. 1, ed. by E.F.J. Payne, Oxford, Oxford UP, 1974)

PII *Parerga und Paralipomena: kleine philosophische Schriften*, 2. Band, Berlin, Hayn, 1851, in Band VI, (tr.: *Parerga and Paralipomena*, vol. 2, ed. by E.F.J. Payne, Oxford, Oxford UP, 1974)

F *Ueber das Sehen und die Farben*, hrsg. von Julius Frauenstädt, Leipzig, F.A. Brockhaus, 1870

DSW *Arthur Schopenhauers sämtliche Werke*, hrsg. von Paul Deussen, München, Piper, 1911–1942

Arthur Schopenhauer's Manuscripts

Der handschriftliche Nachlaß in fünf Bänden, hrsg. von Arthur Hübscher, München, Deutscher Taschenbuch Verlag, 1985 (tr.: *Manuscript Remains*, 4 vols., ed. by E.F.J. Payne, Oxford, Berg, 1988)

HNI *Frühe Manuskripte (1804–1818)*, in Band I (tr.: vol. 1)

HNII *Kritische Auseinandersetzungen (1809–1818)*, in Band II (tr.:vol. 2)

HNIII *Berliner Manuskripte (1818–1830)*, in Band III (tr.:vol. 3)

HNIV(1) *Die Manuskriptbücher der Jahre l830–l852*, in Band IV.1 (tr.:vol. 4)

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