Arthur Schopenhauer on Naturalness in Logic

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Abstract The question of naturalness in logic is widely discussed in today's research literature. On the one hand, naturalness in the systems of natural deduction is intensively discussed on the basis of Aristotelian syllogistics. On the other hand, research on "natural logic" is concerned with the implicitly existing logical laws of natural language, and is therefore also interested in the naturalness of syllogistics. In both research areas, the question arises what naturalness exactly means, in logic as well as in language. We show, however, that this question is not entirely new: In his *Berlin Lectures* of the 1820s, Arthur Schopenhauer already discussed in depths what is natural and unnatural in logic. In particular, he anticipates two criteria for the naturalness of deduction that meet current trends in research: (1) Naturalness is what corresponds to the actual practice of argumentation in everyday language or scientific proof; (2) Naturalness of deduction is particularly evident in the form of Euler-type diagrams.

Keywords Natural deduction · Natural logic · Natural language · Syllogistics · Euler diagrams

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1 Introduction

Naturalness is a widely discussed topic in logic today, especially in relation to (1) systems of natural deduction and in the field of (2) natural logic. (1) The first systems of natural deduction were invented in the 1920s and 1930s by Gerhard Gentzen and Stanisław Jaśkowski. Gentzen intended to set up a calculus "which comes as

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close as possible to actual reasoning" [9, p. 68]. Also Jaśkowski expected that his system "will be more suited to the purposes of formalizing practical proofs" [11, p. 32]. Both, Gentzen and Jaśkowski initially oriented towards natural reasoning in mathematical practices. Both had noticed that axiomatic calculi in the tradition of Gottlob Frege, Bertrand Russell, and David Hilbert did not do justice to the actual reasoning of mathematicians.

This is an essential criterion for distinguishing between axiomatic and natural systems. (For more criteria see [21, chap. 2].) Another criterion is mentioned by Danielle Macbeth:

> In an axiomatic-system, a list of axioms is provided (perhaps with a long explicitly stated rule or rules of inference) on the basis of which to deduce theorems. Axioms are judgements furnishing premises for inferences. In a natural deduction system one is provided not with axiom but instead with a variety of rules of inference governing the sorts of inferential moves from premises to conclusions that are legitimate in the system. In natural deduction, one must furnish the premises oneself; the rules only tell you how to go on. [19, p. 75]

The difference between axiomatic and natural deduction also concerned the interpretation of Aristotelian syllogistics. The axiomatic interpretation of Aristotle was mainly represented by Jan Łukasiewicz and Günter Patzig (cf. [17, 25]). From the 1970s, however, the interpretation of John Corcoran and Timothy Smiley prevailed in research. This interpretation of the Aristotelian *Organon* comes closer to a system of natural deduction as provided by Gentzen. For Corcoran "Aristotle's syllogistic is an underlying logic which includes a natural deductive system and [...] is not an axiomatic theory [...]" [6, p. 85]. According to John N. Martin, Corcoran argues against two main theses of Łukasiewicz "that syllogisms should be construed as conditional sentences in an object-language [...] and that Aristotle's reduction of the valid syllogisms to the 'perfect syllogisms' should be viewed as axiomatic theory in which Barbara and Celarent serve as axioms" [20, p. 1]. In recent years, Neil Tennant interprets syllogistics in a Gentzen–Prawitz system and "aim[s] to show that fresh logical insights are afforded by a proof-theoretically more systematic account of all four figures" [28, p. 120].

(2) Whereas systems of natural deductions are oriented towards the mathematical practice of proof, the so-called natural logic is more focused on the linguistic structure of everyday argumentation. In 1970, George Lakoff [13, p. 254] defined natural logic as "the empirical study of the nature of human language and thought." And Johan van Benthem interprets natural logic as a "system of reasoning based directly on linguistic form, rather than logical artefacts." [3, p. 109]. Therefore, natural logic is based on the conviction that natural language implies logical laws that do not have to be artificially formalized, but can be made explicit in a regimented language (cf. [12]). After all, there are always disadvantages both in artificial formalization and in natural regimentation of everyday language: Whereas formalization always requires interpretation, regimentation is difficult to calculate with.

Since the 2000s natural logic took on a Janus-faced character: on the one hand, it looked backward, because it revisited linguistic form in the same way as pre-Fregean philosophy had done, and on the other hand, it looked forward, because its methods and principles are increasingly being used for neural networks and artificial intelligence (cf., e.g., [5, 24]). However, similar to the research on natural deduction systems, the question of what naturalness exactly means also arises in the area of natural logic. At present, various ancient, medieval, or early modern logicians are being discussed in research as precursors of natural logic [2]. Although these logicians are considered to be the source of ideas for today's research, the naturalness of logic is not explicitly addressed as a topic in their texts.

Surprisingly, however, already in the early nineteenth century a discussion about the naturalness of deduction in syllogistics can be found: In his *Berlin Lectures* of the 1820s, Arthur Schopenhauer distinguished between a natural and an unnatural kind of deduction within traditional syllogistics. But what are the criteria for natural and unnatural reasoning in syllogistics?

Schopenhauer discusses this question on almost 40 printed pages of his *Berlin Lectures* [26, pp. 293–331] including many different arguments. Since there has not been research on these text passages so far, we can here only take a first step. Thus we would like to concentrate here mainly on two criteria for naturalness given by Schopenhauer, for which we think that they are particularly interesting from today's point of view: (1) For Schopenhauer, the naturalness of deduction depends on the practice of actual reasoning and proving; (2) the naturalness of deduction is promoted by Euler-type diagrams. (1) is a mental-linguistic criterion, (2) a diagrammatic one.

In Sect. 2, we will first discuss how Schopenhauer distinguishes between unnaturalness and naturalness in logic. We will first examine the concept of unnaturalness (2.1) and then outline which inferences are natural in logical reasoning (2.2–2.4). Section 3 will show that the unnaturalness of reasoning results from the completeness of natural deduction in syllogistic. One could thus say that Sect. 2 presents a positive, whereas Sect. 3 is a negative approach towards the question of naturalness in logic. Finally, in Sect. 4, we will argue that Schopenhauer's criteria for naturalness and unnaturalness often include arguments that can be still relevant to today's research discussion: First, Schopenhauer argues that certain inferences are natural because they correspond to our current use in scientific and everyday reasoning. Second, he argues that diagrams play a specific role for the naturalness of deduction.

2 Unnatural and Natural Deduction in Syllogistics

In this section, we will first refer to a relevant quote from the section *On Inferences* in Schopenhauer's *Berlin Lectures*, which makes his criticism of unnatural deduction in syllogistic explicit. Here, Schopenhauer labels all inferences of the fourth syllogistic figure—the so-called Galenic figure—as unnatural. We will then show in each of the following sections (Sects. 2.2–2.4) why Schopenhauer considers the first three figures to be natural. This will reveal a criterion of completeness, which we will examine later in Sect. 3.

2.1 Unnaturalness of $\mathcal{F}_{IV}$

In syllogistics we usually distinguish four figures for up to 24 valid types of inferences (modi). Depending on the position of the three terms (termini major = M, minor = m, and medius = μ) in the premises, the 24 modi are classified into one of the four figures ($\mathcal{F}$).

$$\mathcal{F}_I \begin{bmatrix} \mu & M \\ m & \mu \\ m & M \end{bmatrix} \mathcal{F}_{II} \begin{bmatrix} M & \mu \\ m & \mu \\ m & M \end{bmatrix} \mathcal{F}_{III} \begin{bmatrix} \mu & M \\ \mu & m \\ m & M \end{bmatrix} \mathcal{F}_{IV} \begin{bmatrix} M & \mu \\ \mu & m \\ m & M \end{bmatrix}$$

In Aristotle we find only $\mathcal{F}_{I-III}$; $\mathcal{F}_{IV}$ is often attributed to the Greek physician Galen of Pergamon (∼129–216) (cf. [18]). This is one of the reasons why Schopenhauer claims that there are basically not four, but rather only three figures since $\mathcal{F}_{IV}$ only turns $\mathcal{F}_I$ upside down. Since only the position of μ is relevant for the assignment of $\mathcal{F}$, one can argue that $\mathcal{F}_I = \mathcal{F}_{IV}$ or $\binom{\mu M}{m\mu} = \binom{M\mu}{\mu m}$, in which μ is in a diagonal position in both cases. For Schopenhauer, the change from $\mathcal{F}_I$ to $\mathcal{F}_{IV}$ results in grammatical and diagrammatical problems that do not meet the requirements of natural logic and deduction. A first relevant quote of Schopenhauer on this topic is given in $Q1$:

> ($Q1$) Aristotle has only the first three: the fourth is (according to a legend of the Arabs) invented by Galen. It's just the reversed first. Actually, there is no unique relation between concepts: it is quite unnatural and really only the very first figure turned upside down: therefore Aristotle intentionally did not set it up. [26, p. 305]
> Aristoteles hat nur die drei ersten: die vierte soll (nach einer Sage der Araber) von Galen erfunden seyn. Sie ist bloß die umgekehrte erste. Ihr liegt eigentlich kein besondres Verhältniß der Begriffe zum Grunde: sie ist ganz unnatürlich und wirklich nur die ganz auf den Kopf gestellte 1ste Figur: daher Aristoteles sie absichtlich nicht aufstellte.

$Q1$ does not explicitly answer the question of what naturalness of deductive inferences mean and why $\mathcal{F}_{I-III}$ can be considered as natural; however, $Q1$ clearly states that $\mathcal{F}_{IV}$ and associated inferences can be characterized as unnatural. What is given in $Q1$ only as an unjustified claim will be discussed in Sect. 3 in more detail. But before we come to the arguments of Schopenhauer which deal with the naturalness of $\mathcal{F}_{I-III}$, let us examine what role Aristotle played for Schopenhauer ($Q1.1$) what "unique relation between concepts" in $Q1$ mean ($Q1.2$).

($Q1.1$) Schopenhauer claimed that Aristotle did not introduced $\mathcal{F}_{IV}$ because he recognized it as unnatural. This is not an untypical argument raised by Schopenhauer since it can already be found in the *Port-Royal Logic* (III 4) in a similar way. From the perspective of today's research, however, it is not uncontroversial. Theodor Ebert and Günther Patzig have pointed out that there are of course only three figures in ancient syllogistics, but Aristotle treats the modes of the fourth figure "implicitly as additional (indirect) modes of the first figure" [7, p. 13]:

> The Aristotelian indirect modes of the first figure and the modes of our fourth figure differ only in the arrangement of the premises, a difference that becomes recognizable only in the definition of a standard formulation, and which is irrelevant to the question of its validity.

> That is why all the judgements about an alleged inferiority of the fourth figure, as they were given by Aristotle's lack of this figure by former logicians and logic historians, lie on the one hand on an inaccurate reading of the Aristotelian texts, and on the other on a misunderstanding of the underlying logical facts. [7, p. 14]

Of course, Schopenhauer does not explicitly equate unnaturalness with inferiority, but the Schopenhauer/Port-Royal argument that suggests that Aristotle did not attach the fourth figure intentionally is nonetheless problematic. Most modern interpretators of Aristotle agree that there is no text passage of the *Organon* from which one can read without a hitch that Aristotle intentionally considered $\mathcal{F}_{IV}$ to be unnatural. However, before Schopenhauer there were already several authors who characterized $\mathcal{F}_{IV}$ as unnatural (cf. [18, sect. 2], [27]); and, as far as we know, nobody discusses the topic of naturalness and unnaturalness in as detailed a way as Schopenhauer does. Furthermore, one fact seems to be certain for Schopenhauer and all his precursors: Unnaturalness does not necessarily have anything to do with invalidity.

($Q1.2$) In $Q1$, the lack of unique relation is an important characteristic for the unnaturalness of $\mathcal{F}_{IV}$. Thus we have to clarify the question of what Schopenhauer means with "unique relation between concepts" in $Q1$. An essential feature of Schopenhauer's lectures on logic is that he represents the relation of concepts with the help of Euler-type diagrams (cf. [23]). Even though he designates the respective diagrams by scholastic mnemonics, he mainly uses a regimented language based on rules to explain the inferences. These rules are listed in [26, pp. 324–325], but cannot discussed here in detail. In the following quote ($Q2$), Schopenhauer names the point of view that and why the regimented inferences should best be represented by Euler-type diagrams:

> ($Q2$) Namely, between the possible relations that concepts can have to each other and the positions in which circles can be put together is a very precise and absolutely consistent analogy. ([26, p. 269])
> Nämlich zwischen den möglichen Verhältnissen, die Begriffe zu einander haben können, und den Lagen in welch[en] man Kreise zusammenstellen kann ist eine ganz genaue und schlechthin durchgängige Analogie.

The relationship between concepts and Euler diagrams can therefore basically be understood as an analogy: In the same way in which two concepts behave towards each other in logic, two circles can behave towards each other in geometry. Euler diagrams thus graphically depict the conceptual relations expressed in linguistic inferences. From the analogy-thesis of $Q2$ and the reference to the peculiarity of the conceptual relations of $Q1$, it can be concluded that there are specific Euler-type diagrams for $\mathcal{F}_{I-III}$, but not for $\mathcal{F}_{IV}$. Thus one can also conclude that the naturalness of $\mathcal{F}_{I-III}$ can be demonstrated with the help of geometric forms and that unnaturalness is given by the fact that no unique or autonomous Euler-type diagram of $\mathcal{F}_{IV}$ can be constructed.

Since the central objective of his chapters *On inferences* is to show the naturalness of the first three and the unnaturalness of the fourth figure, Schopenhauer gives his audience the following instruction ($Q3$):

($Q3$) Draw them [sc. the diagrams] down in order to be able to follow my remarks about them: it is precisely in our reflections on the various combinations of concepts underlying the various syllogistic figures that you will receive deep insight into the essence of concepts in general, into the mechanism of thought, and thus into the nature of our reason itself. [26, p. 306]
Zeichnen Sie solche [sc. die Diagramme] auf, um nachher meinen Bemerkungen darüber folgen zu können: eben an unsern Betrachtungen über die verschied[nen] Kombinationen der Begriffe welche den verschied[nen] syllogistischen Figuren zum Grunde liegen, werden Sie tiefre Einsicht erhalten in das Wesen der Begriffe überhaupt, in den Mechanismus des Denkens und somit in die Natur unsrer Vernunft selbst.

This instruction ("Draw them down...") which was originally meant for students discloses not only a didactic procedure, but it also opens up a kind of characterization concerning the question of how logic works. Those who draw Euler diagrams

(3.1) will achieve theoretical knowledge of *conceptual relations* between spheres of concepts because concepts resp. circles either partially overlap or are completely included or excluded,
(3.2) will achieve a deeper understanding of the *essence of concepts* because concepts behave to each other like geometrical circles behave to each other,
(3.3) will achieve knowledge of the *mechanism of thought* because by applying the Eulerian rules of construction one does consciously what reason otherwise automatically does (cf. [26, p. 306]),
(3.4) will gain "*insight*" [26, p. 306]—and we think this is meant literally—*into the nature of reason in general*. In other words: At the moment we see a diagram, we simultaneously see the validity of reasoning via an analogy.

In $Q2$ we have found out that Schopenhauer considers $\mathcal{F}_{IV}$ to be unnatural: It does not have a unique conceptual relationship that can be depicted with the help of logic diagrams. As seen in $Q3$, this unique relationship illustrated by Euler diagrams is important since the diagrams give a deep insight into the essence of natural language. From this consideration one could conclude that $\mathcal{F}_{I-III}$ are natural in so far as each provides a unique relationship of concepts resp. circles. We will now focus on the quotes in which Schopenhauer describes the naturalness of the first three figures. In Sect. 3, we will focus again on Schopenhauer's arguments concerning the unnaturalness of $\mathcal{F}_{IV}$.

2.2 Naturalness of $\mathcal{F}_I$

But let us now come to the arguments for the naturalness of the inferences in $\mathcal{F}_{I-III}$. Schopenhauer characterizes $\mathcal{F}_I$ as "the simplest and most natural" form of reasoning [26, p. 301]. The following quote ($Q4$) repeats this judgment in even more detail:

($Q4$) [T]he 1st figure is the most perfect, because every thought can finally take its form: very natural: just subsumption of one concept under a wider one and this again under a wider one is the simplest and most essential operation of reason: it is the mere retrospective view of a wider abstraction to the narrower [...]. [26, pp. 302–303]

> Sodann beweist dies [. . .] daß die 1[ste] Figur die vollkommenste wäre, indem jeder Gedanke zuletzt ihre Gestalt annehmen kann: sehr natürlich: grade Subsumtion eines Begriffs unter einen weitern und dieses wieder unter einen weitern ist die einfachste und wesentlichste Vernunftoperation: es ist der bloße Rückblick von einer weitren Abstraktion, auf die enger[n] [. . .].

In $Q4$ we already find two arguments for the naturalness of $\mathcal{F}_I$, of which the following is to be significantly emphasized: The first argument is a mental-linguistic one ($Q4.1$), the second argument a diagrammatic one ($Q4.2$).

($Q4.1$) We characterize logical thinking as natural, if a thought can be expressed in a unique syllogistic form. In order to express a thought, an intentional state, a propositional attitude, an inferential context, etc., one needs a certain linguistic form. This linguistic form can correspond to one of the three syllogistic figures, i.e., $\mathcal{F}_{I-III}$. The choice of $\mathcal{F}$ depends on which thought one intends to articulate in an inferential context. If every thought can be expressed in only one $\mathcal{F}$, this naturalness would be perfect. Following Kant, Schopenhauer claims that in $\mathcal{F}_I$ this is the case.

The cognitive activity that one can express with the help of $\mathcal{F}_I$ is the decision or resolution ("Entscheidung," [26, p. 326]). $\mathcal{F}_I$, containing four modes, always expresses a resolution about the relationship between M and m (in the conclusion). In each mode the conclusion shows one of the four possible categorical propositions in syllogistics (A, E, I, O). Therefore, any possible resolution between m and M (within the conclusion) can be represented by one of the four modes:

Barbar$A$	`All`	mM
Celar$E$nt	`No`	mM
Dari$I$	`Some`	mM
Feri$O$	`Some...not`	mM

This high expressivity of $\mathcal{F}_I$ is one reason why Schopenhauer says that this one is the "most perfect" and also a "very natural" figure above all others [26, p. 302].

($Q4.2$) The diagrammatic criterion is difficult to deal with in brief, since it depends strongly on the logical interpretation of the diagrams and this in turn depends exceedingly on the diagrammatic interpretations of the editors. (In Schopenhauer's manuscripts, the diagrams are often drawn very inaccurately.) However, Schopenhauer emphasizes especially the role of μ in the respective figure as a diagrammatic criterion. In $\mathcal{F}_I$, μ is the "mediator" ("Vermittler") between M and m: what does (not) belong to μ, also does (not) belong to m. Due to this function, Schopenhauer also speaks metaphorically of μ as a "Handhabe" ("manipulator," [26, p. 309]): μ grasps one concept and passes it on to the other. Schopenhauer alludes here to the metaphor of the concept (cf. [14]).

For example, in the modus Barbara (Fig. 1), the middle-sized concept (i.e. μ) grasps the narrower concept (i.e., m) and subsumes it under the widest one (i.e. M)

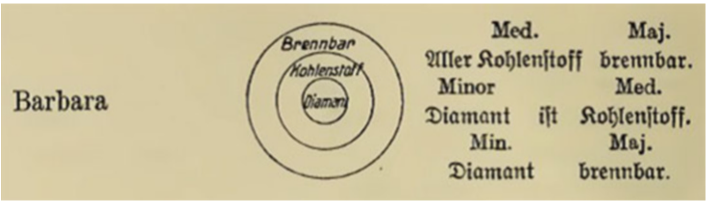

Fig. 1 Barbara [26, p. 304]

(cf. [26, pp. 297, 301, 321]). This function also explains the origin of the names "medius," "major," and "minor."

`All`	carbon$^{\mu}$		flammable^{M}
	diamond^{m}	`is`	carbon$^{\mu}$
	diamond^{m}		flammable^{M}

As we can see in this example (Fig. 1) naturalness is given, because of the specific function of μ. The conclusion ("All diamonds are flammable.") consists in its diagrammatic representation of a wider sphere of M ("flammable") which contains an entirely narrower sphere of m ("diamonds") with the help of μ which mediates between M and m. The expression "the mere retrospective view" in $Q4$ means nothing else than the fact that the conclusion goes back to the first premise. In other words, the resolution about the relationship between M and m in the conclusion depends on the relationship between μ and M in the first premise. Due to these and other functions of μ not discussed here, $\mathcal{F}_I$ in the Euler-type diagrams proves to be natural.

2.3 Naturalness of $\mathcal{F}_{II}$

For Schopenhauer, the inferences resp. modi of the $\mathcal{F}_{II}$ also belong to a system of natural deduction and natural logic. This can be seen in the following quote ($Q5$):

> ($Q5$) The 2nd figure is therefore the natural form of our thought process, when we want to separate concepts, to distinguish things, and to establish characteristic features of their difference. So we use the second figure mainly when we want to avoid misunderstandings and confusion of concepts. In all such cases it is the natural form of thought, not the first.
> Die 2te Figur ist daher die natürliche Form unsers Gedankenganges, wann wir Begriffe trennen, Dinge unterscheiden wollen, und hiezu karakteristische Merkmale ihres Unterschiedes aufstellen. Wir gebrauchen also die 2te Figur hauptsächlich wenn wir Verwechselung und Konfusion der Begriffe verhüten wollen. In allen solchen Fällen ist sie die natürliche Form des Gedankens, nicht die erste. [26, p. 310]

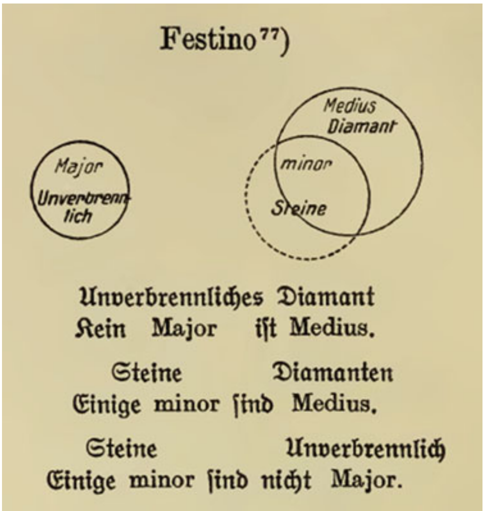

Fig. 2 Festino [26, p. 308]

Even if we get more references to the mental-linguistic criterion ($Q5.1$) and less references to a diagrammatic one ($Q5.2$), we can, with reference to the context of $Q5$, indicate some essential criteria for the naturalness of $\mathcal{F}_{II}$:

($Q5.1$) Negative propositions can be represented in $\mathcal{F}_I$ by Celar$E$nt and Feri$O$. But if one does not want to emphasize one's own resolution, but rather intend to express a difference between the concepts of the conclusion, i.e., m and M, the linguistic form of all inferences within $\mathcal{F}_{II}$ is better suited than the structure of $\mathcal{F}_I$. With the help of $\mathcal{F}_{II}$ a speaker can make clear to his audience that his intention is to avoid a misunderstanding or to prevent a confusion of concepts. Regarding such intentions, $\mathcal{F}_{II}$ is more natural than $\mathcal{F}_I$.

($Q5.2$**)** A peculiarity of Schopenhauer's logic is that he only gives one single diagram for each mode. As Schopenhauer should have known from Euler [8, l. CIV] (cf. also [10] or [4, p. 45 et seq.]), in many cases only one diagram is not sufficient in order to decide whether a mode is valid or not. However, Schopenhauer is more concerned with the question of naturalness than with the question of the validity of modes. This can be seen, for example, by Festino (Fig. 2) and Baroco (Fig. 3).

No	incombustible^M	is a	diamond$^\mu$
Some	stones^m	are	diamonds$^\mu$
Some	stones^m	are not	incombustible^M

All	meerschaum bowls^M	are	turning brown$^\mu$
Some	pipe bowls^m	will not	turn brown$^\mu$
Some	pipe bowls^m	are not	meerschaum bowls^M

Fig. 3 Baroco [26, p. 308]

These modes require several different diagrams since the conclusion in both is particular and negative (Festin$O$, Baroc$O$). However, Schopenhauer only gives the Euler-type diagrams in which the O-prop. is interpreted as a shortened variant of the E-prop. (universal negative). It is precisely through this interpretation that he emphasizes the mental-linguistic criterion of $\mathcal{F}_{II}$.

In all diagrammatic modes of $\mathcal{F}_{II}$, μ functions as a "Scheidewand" ("septum," [26, pp. 308–309]) between M and m. In relation to μ, m does the opposite of what M, in relation to μ, does. With the help of the diagrams, Schopenhauer tries to show that μ "grasps" one of the two other concepts (either M or m) and thereby separates the other remaining one from it. The naturalness of the mental-linguistic criterion, namely separation and differentiation, is thus shown in the separation of the Euler-type circles for M and m. (Whether this thesis is coherent or not, cannot be discussed here.)

2.4 Naturalness of $\mathcal{F}_{III}$

Schopenhauer discusses many arguments for the naturalness of $\mathcal{F}_{III}$. The following quote ($Q6$) offers a relevant text passage:

> ($Q6$) The 3rd figure is thus the natural form of thinking when noting an exception, only it is rarely pronounced in extenso, but almost always contracted: the medius is briefly given as the example of the exception, which is the argument of the proposition. ([26, p. 316])
> Die 3$^{\text{te}}$ Figur ist also die natürliche Form des Denkens beim Anmerken einer Ausnahme: nur wird sie selten in extenso ausgesprochen, sondern fast immer kontrahirt: der medius wird kurz angegeben als das Beispiel der Ausnahme, welches das Argument des Satzes ist.

Also in $Q6$ we again see two criteria for naturalness, whereby the mental-linguistic criterion ($Q6.1$) is emphasized more clearly than the diagrammatic criterion ($Q6.2$):

($Q6.1$) If it is the intention of a speaker to represent an anomaly or exception, she naturally applies $\mathcal{F}_{III}$, even if she is often unaware of the theoretical function. Schopenhauer explains that we use contracted forms of $\mathcal{F}_{III}$ "countless times" ("unzählige Mal," [26, p. 312]) in everyday reasoning and argumentation. The structure of the contracted form is not very complicated and can be summarized by a simple formula:

$$\text{Some } m \text{ are (not) } M, \text{ such as } \mu.$$

This formula already shows a conspicuous characteristic of $\mathcal{F}_{III}$: The conclusion in $\mathcal{F}_{III}$ is always a particular proposition (I-prop. or O-prop.: "Some m are (not) M") and is justified by adding the exception to M ("such as μ"). Furthermore, another mental component comes into play: Schopenhauer repeatedly speaks of the fact that $\mathcal{F}_{III}$ expresses something unexpected ("unerwartet"). The contracted form only works to the extent that we can already assume the acquaintance of a certain rule (given in the second premise). But if there is an exception (positive or negative) to this well-known rule, the exception is expressed by adding μ to the conclusion. This is illustrated by the enthymeme "Some water dwellers are not fish, such as dolphins." However, the explicit form of this enthymeme is given in Fig. 4:

Fig. 4 Felapton [26, p. 313]

`No`	dolphin$^{\mu}$	`is a`	fish^{M}
`All`	dolphins$^{\mu}$	`are`	water dweller^{m}
`Some`	water dweller^{m}	`[are] not`	fish^{M}

(Q6.2) Similar to (Q4.2) and (Q5.2), the diagrammatic criterion for naturalness of $\mathcal{F}_{III}$ depends on μ. In $\mathcal{F}_{III}$, μ is called an "Anzeiger" ("indicator," [26, p. 316]). In the case of a particular positive conclusion (I-prop.), μ is the indicator of an unexpected difference between the normally homogeneous concepts m and M. In the case of a particular negative conclusion (O-prop.), μ is the indicator of an unexpected congruence between the normally heterogeneous concepts m and M. In Felapt$O$n, Bocard$O$, and Feris$O$n, μ indicates an unexpected difference; in Darapt$I$, Disam$I$s, and Datis$I$, μ indicates an unexpected congruence. This is well illustrated by the diagram of the example for contracted forms in Q6.1 (cf. Fig. 4): "water dweller" and "fish" are very homogeneous terms, but μ makes the difference between both obvious. Due to the well-known rule (second premise), "All dolphins are water dwellers," it is expected that the circle for dolphins also at least intersects the circle for fish. However, the conclusion makes explicit that this expectation does not always have to come true. In the diagram the circles for dolphins and fish are completely separated, but both are completely within the circle for water dwellers.

3 Natural Completeness and Unnatural Redundancy

In Sects. 2.2–2.4 we have given a mental-linguistic as well as a diagrammatic criterion in order to clarify Schopenhauer's thesis that only $\mathcal{F}_{I-III}$ can be part of a system of natural deduction and natural logic. A justification to the claim that $\mathcal{F}_{IV}$ is unnatural, however, was only insufficiently addressed in Sect. 2.1. There, Schopenhauer explained that $\mathcal{F}_{IV}$ turns $\mathcal{F}_{I}$ upside down. But this fact alone is no argument for declaring $\mathcal{F}_{IV}$ as being unnatural.

In this section, we would like to further strengthen Schopenhauer's argument that $\mathcal{F}_{IV}$ is unnatural. He himself already envisages at least three central backings for his argument in only one short text passage [26, p. 329]. But since we have already selectively emphasized criteria in Sect. 2, we concentrate here again only on some mental-linguistic (3.1) and some diagrammatic aspects (3.2). In both criteria, Schopenhauer's argumentative strategy is as follows: whereas $\mathcal{F}_{I-III}$ already form a complete system of natural deduction and logic, $\mathcal{F}_{IV}$ is mentally linguistic as well as diagrammatically redundant.

3.1 Mental-Linguistic Completeness and Redundancy

One criterion that Schopenhauer repeatedly used as an argument for the naturalness of $\mathcal{F}_{I-III}$ concerned the relationship between intentional states and their linguistic expressions: If I want to express a decision or resolution, I should use $\mathcal{F}_I$. But if it is my intention to make a difference clear, $\mathcal{F}_{II}$ is usually considered. If, however, an unexpected exception is to be expressed, it is clever to formulate this in the form of $\mathcal{F}_{III}$. Schopenhauer leaves open whether the choice of $\mathcal{F}$ is conscious or unconscious.

However, Schopenhauer claims that with $\mathcal{F}_{I-III}$ the expressivity of natural logic is exhausted. $\mathcal{F}_{IV}$ is redundant for several reasons: (1) On the one hand, it is unnatural because its syntax is not in accordance with our natural feeling for language. (2) On the other hand, $\mathcal{F}_{IV}$ does not expand our expressiveness, since no thought or intentional state is expressed more clearly in it than in one of the other $\mathcal{F}$. Both cases are confirmed for Schopenhauer by the fact that no examples for $\mathcal{F}_{IV}$ can be found in ordinary or scientific language. All typical examples appear to be artificially constructed.

Let us first look at what Schopenhauer says in the following quote ($Q7$) about the unnatural syntax in $\mathcal{F}_{IV}$:

> ($Q7$) Therefore, this figure [sc. $\mathcal{F}_{IV}$] is always unnatural, and one will never think in it. Most naturally it still appears in Fesapo: but apparently the upper sentence of the same is a Conversio: One will never originally think “No Christian is a Bashkir”: but rather “No Bashkir is a Christian”: for one always takes the narrower term to the subject, the further to the predicate.
>
> Daher ist diese [sc. die vierte] Schlußart immer unnatürlich, und nie wird man in ihr denken. Am natürlichsten erscheint sie noch in Fesapo: aber doch ist offenbar der Ober-Satz desselben eine Conversio: Man wird nie ursprünglich denken “Kein Christ ist ein Baschkire”: sondern “Kein Baschkir ist ein Christ”: denn man nimmt immer den engern Begriff zum Subjekt, den weitern zum Prädikat. [26, p. 323]

In $Q7$, Schopenhauer alludes to the following modus Fesapo given in Fig. 5:

`No`	Christian^M	`is a`	Bashkir$^\mu$
`All`	Bashkirs$^\mu$	`are`	Russians^m
`Some`	Russians^m	`are not`	Christians^M

For Schopenhauer, it would be more natural if the intended resolution would have been expressed in a mode of $\mathcal{F}_I$. But in contrast to a $\mathcal{F}_I$ inference the first premise of Fesapo was changed by a conversio simplex and the second one by a conversio per accidens. Schopenhauer claimed that the unnaturalness is already apparent in the first premise: One would usually not think or express oneself in this way, since it is more natural to place the wider concept in the subject position and the narrower concept in the predicate position, and not vice versa (as in Fesapo).

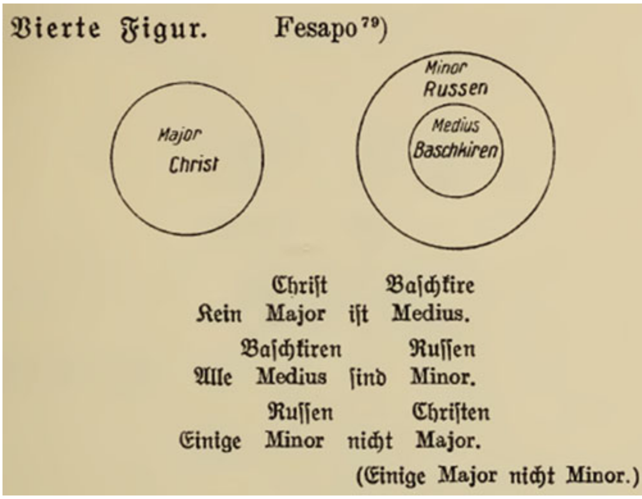

Fig. 5 Fesapo [26, p. 319]

Let us now come to the argument that $\mathcal{F}_{IV}$ does not expand our logical expressiveness. This argument is reflected in the following quote ($Q8$):

> (Q8) If one meanwhile also wants to give a rational basic thought to the 4th figure, it would be the following one: the 1st figure always has the purpose to *decide* a case by a general rule: therefore it subsumes the case under the rule: the 4th, which is its straight conversion, also has the opposite purpose: it wants to *confirm* a rule by a case, the case should be the proof of the rule[.]
> Will man inzwischen auch der 4ten Figur einen vernünftigen Grundgedanken unterlegen; so wäre es dieser: die 1ste Figur hat immer den Zweck einen Fall durch eine allgemeine Regel zu *entscheiden*: daher sie den Fall der Regel subsumirt: die 4te, welche ihre grade Umkehrung ist, hat auch den umgekehrten Zweck: sie will nämlich eine Regel durch einen Fall *bestätigen,* der Fall soll der Beleg der Regel seyn[.] ([26, p. 323]; emphasized by H.M.S. & J.L)

$Q8$ points out that Schopenhauer also regards $\mathcal{F}_{IV}$ as redundant insofar as it cannot express a thought that could not already be expressed by $\mathcal{F}_{I-III}$. The only intention that can be expressed by using $\mathcal{F}_{IV}$, namely confirmation, is already contained in $\mathcal{F}_I$ as one aspect of the resolution: In the conclusion, $\mathcal{F}_I$ decides whether the relationship between M and m is positive or negative. A certain case can thus be confirmed or rejected by a rule. However, some inferences of $\mathcal{F}_{IV}$ (which are more complicated than Fesapo, e.g., Calemes) reverse this relationship and can only confirm the rule. Consequently, its expressivity is already covered by $\mathcal{F}_I$, which also has a much more natural syntax. In summary, it can be said that $\mathcal{F}_{IV}$ is mentally linguistically superfluous and thus unnatural, since it does no more than $\mathcal{F}_{I-III}$.

3.2 Diagrammatic Completeness and Redundancy

According to Schopenhauer, the completeness of $\mathcal{F}_{I-III}$ and thus the redundancy of $\mathcal{F}_{IV}$ can also be shown diagrammatically. The following ($Q9$) is a relevant quote for this thesis and deals mainly with the diagrammatic possibilities of μ:

> ($Q9$) Thus, three times we find the possibility [sc. of μ] consistently exhausted. 1) Three ways in which the medius can be the reason of the judgment in the conclusion: every possible inference corresponds to one of it. 2) Three possible positions of the medius in the premises [. . .]. 3) In the spheres: The medius is either the widest, or the narrowest, or the middle sphere. [26, pp. 329–330]
> Wir finden also die Möglichkeit [sc. von μ] drei Mal übereinstimmend erschöpft. 1) Drei Arten wie der Medius Grund des Urtheils der conclusio seyn kann: in jedem möglichen Schluß ist ers auf eine dieser drei Arten. 2) Drei mögliche Stellungen des Medius in de[n] Prämissen [. . .]. 3) In den Sphären: der Medius ist entweder die weiteste, oder die engste, oder die mittlere Sphäre.

In his chapter *On inferences* Schopenhauer treats all three points of $Q9$ in detail. We can only briefly discuss the three points in the following:

($Q9.1$) We have already shown in Sects. 2.2–2.4 that μ has its own diagrammatic function, which Schopenhauer describes metaphorically:

1. In $\mathcal{F}_I$, μ functions as the “Handhabe” (manipulator)
2. In $\mathcal{F}_{II}$, μ functions as a “Scheidewand” (septum)
3. In $\mathcal{F}_{III}$, μ functions as an “Anzeiger” (indicator)

As the argument of the whole inference, μ is always a kind of reason (cf. [26, pp. 297, 325]). According to Schopenhauer, the diagrammatic completeness of $\mathcal{F}_{I-III}$ can be found in the fact that there are exactly three and no more than three reasons. Depending on how μ interacts with the other terms in the Euler-type diagram, it becomes one of the following reasons:

1. “Entscheidungsgrund,” the reason for resolution [26, p. 326];
2. “Unterscheidungsgrund,” the reason for distinction [26, p. 327];
3. “Ausscheidungsgrund,” the reason for inclusion and difference [26, p. 327].

($Q9.2$) The easiest way to interpret the second point is to illustrate the position of μ in $\mathcal{F}_{I-IV}$ (cf. above, Sect. 2.1): Schopenhauer argues that μ can have only three possible positions in the premises, namely either

($P1$) diagonally (sometimes as subject, sometimes as predicate),
($P2$) or completely right (always as predicate),
($P3$) or completely left (always as subject).

If one now look at the position of μ, one can see that $\mathcal{F}_{I-III}$ corresponds to P1–3, but $\mathcal{F}_{IV}$ repeats P1:

$$(P1) = \begin{bmatrix} \mu & M \\ m & \mu \\ m & M \end{bmatrix} (P2) = \begin{bmatrix} M & \mu \\ m & \mu \\ m & M \end{bmatrix} (P3) = \begin{bmatrix} \mu & M \\ \mu & m \\ m & M \end{bmatrix} (P1) = \begin{bmatrix} M & \mu \\ \mu & m \\ m & M \end{bmatrix}$$

In $\mathcal{F}_I$, μ is always diagonal $(P1)$, $\mathcal{F}_{II}$ completely right $(P2)$, and $\mathcal{F}_{III}$ completely left $(P3)$. $\mathcal{F}_{IV}$ is thus diagrammatically a repetition of the diagonal position $(P1)$, already fulfilled by $\mathcal{F}_I$, and thus redundant.

$(Q9.3)$ The third point is the most interesting one, but it is difficult to explain if Schopenhauer's Euler-type diagrams are used as a diagrammatic criterion. It is easier to explain, however, if one uses $P1 - P3$ already discussed in $Q9.2$. Additionally, one has to assume the rule that the predicate is always wider than the subject, as discussed in $Q7$. In regards to these rules, Schopenhauer argues for diagrammatic completeness and redundancy:

$(D1)$ If $(P1)$, as given in $\mathcal{F}_I$, μ represents a medium-sized sphere.
$(D2)$ If $(P2)$, as given in $\mathcal{F}_{II}$, μ represents the widest sphere.
$(D3)$ If $(P3)$, as given in $\mathcal{F}_{III}$, μ represents the narrowest sphere.

This, however, exhausts all possibilities of the relationship between μ and the other concepts m and M [26, pp. 327–328]. Moreover, in $\mathcal{F}_{IV}$ as well as in $\mathcal{F}_I$, μ represents a medium-sized sphere, since in both μ is sometimes in the position of the subject and sometimes in the position of the predicate (i.e., $(P1)$). Thus $\mathcal{F}_{IV}$ is only a repetition of $\mathcal{F}_I$.

4 Conclusion and Outlook

In this section we would like to summarize the central criteria and arguments from Sects. 2 and 3 and then examine some aspects of Schopenhauer's approach from today's perspective.

4.1 Summary

We have seen in Sects. 2 and 3 that Schopenhauer indicates a mental-linguistic criterion in order to separate the natural from the unnatural inferences. Inferences are natural (1) if the position of the concepts in the judgment and (2) the position of the judgments in the inference meet our ordinary linguistic usage and thus (3) express a specific mental intent. Inferences are unnatural if they have an unusual syntax and if they only allow us to express something that can be better expressed by other inferences.

Regarding the classification into four figures in syllogistics, Schopenhauer can say that all valid inferences of the first three figures are natural, whereas the inferences of the fourth figure are unnatural. With the first of the three figures all forms of resolutions can be expressed, with the second figure confusions and misunderstandings can be clarified, and with the third figure exceptions and paradoxes are proved. However, the fourth figure has an unnatural syntax and no autonomous or unique expressivity.

For Schopenhauer there is a complete analogy between the logical relations of the concepts in the judgment and the spatial relation of circles in geometry. For this reason, all differences regarding inferences can also be represented with the help of Euler-type diagrams. If one looks only at the medius (at the argument of the inference), one can see that it always clarifies the relationship between the major and the minor term. In the first figure it decides whether the termini major and minor completely or partially overlap or exclude each other. In the second figure it always separates the major term from the minor one and in the third figure it shows exceptions between the termini major and minor, which are usually expected to be in correspondence.

4.2 Schopenhauer's Logic in the Context of Current Debates

We are of the opinion that Schopenhauer puts forward many arguments which are still worth discussing today or which are still being discussed in current research. Since we have only focused on two criteria in Sects. 2 and 3, we will concentrate here only on (1) the mental-linguistic criterion that might be of interest to natural logic and (2) the diagrammatic criterion which is in line with current trends in systems of natural deductions.

1. Schopenhauer's criterion for naturalness and unnaturalness in syllogistics does not depend on the method of mathematical proof, but on a mental-linguistic criterion. It therefore does not have much to do with the approach that became known through Gentzen and Jaśkowski (cf. Sect. 1) for which reasoning and proving were mainly limited to the activity of the mathematician. For Schopenhauer, logic is not an *ancilla mathematicae*. The naturalness of logic is not a criterion that can only be oriented towards the scientific practice of mathematical proof. This does not mean, of course, that his logic based on criteria of naturalness ignores or even excludes mathematical proofs. On the contrary, Schopenhauer explains several times how proofs fit into his logical approach. However, the decisive criterion for him concerns the question of whether his logic can represent the naturalness of both scientific and everyday thinking and reasoning or not. His mental-linguistic criterion says that naturalness is given if we actually think this way ("countless times," *Q*6.1) and that unnaturalness is given if we would not think in this or that certain way. Similar to current trends in "natural logic" (cf. [1, 12]), Schopenhauer's criteria for naturalness lead to a regimented fragment of natural language and argues against the formalization of Aristotelian scholasticism. However, Schopenhauer sees the basis of a natural logic not in a regimented

language, but in the description of the corresponding intuition with the help of Euler diagrams.

2. As we have seen in Sect. 1 of Danielle Macbeth's quote, a criterion for the difference between axiomatic and natural systems concerns the role of axioms, assumptions, and rules. Schopenhauer, however, proves the validity of inferences by their correspondence to the Aristotelian rules. We have not discussed this in detail here, since Schopenhauer separates the question of validity from the question of naturalness (cf. above Sect. 2.1). The naturalness of inferences is not based on their validity, but on their actual application in everyday argumentation and can be illustrated with the help of the construction rules of Euler diagrams. In these diagrams Schopenhauer sees not only reasoning aids, but the actual foundation of logic (cf. [15]). Furthermore, Schopenhauer sees another advantage of the diagrams in the fact they can automatically make implicit information explicit (see above $Q3.3$ and $Q3.4$). With these arguments Schopenhauer supports current developments in the information sciences in which various systems of natural deduction are presented more and more by using Venn- and Euler-type diagrams, e.g., in [16, 22], etc.

In this paper, we have only presented a few ideas on naturalness in Schopenhauer and given some hints as to how they might be of relevance for current research on natural logic and natural deduction. Detailed investigations on Schopenhauer's logic, but also on his predecessors, seem to be necessary in order to reach a verdict on the subject of naturalness in pre-Fregian logic. So far, it seems to be unique that Schopenhauer combines a mental-linguistic criterion with a diagrammatic one in dealing with the question of the naturalness of logic.

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