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Jens Lemanski
Editor

Language, Logic, and Mathematics in Schopenhauer

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Contents

An Introduction to *Language, Logic and Mathematics in Schopenhauer*

Jens Lemanski

Abstract This paper is an introduction to the volume *Language, Logic and Mathematics in Schopenhauer*. It shows the basic interpretations discussed in Schopenhauer's research, explains the aims and tasks of Schopenhauer's philosophy and shows the importance of language, logic and mathematics in Schopenhauer's system.

Keywords Schopenhauer · System · World · Diagrams

Mathematics Subject Classification (2000) Primary 99Z99, Secondary 00A00

1 Interpretations and Contradictions

It is probably not often the case that all three terms given in the title of the volume are mentioned in the same breath, and then also in connection with Arthur Schopenhauer. At first glance, the philosopher, who was born in Gdansk in 1788 and died in Frankfurt in 1860, seems typically not to be associated with any of the three topics mentioned, not with language, nor with mathematics, and certainly not with logic. Many philosophers seem to have a prejudiced view of Schopenhauer as a philosopher of the Irrational and of contradiction: His philosophical system—so the view might go—is determined by the irrational principle of the will, which man should finally overcome in a mystical way and with the help of self-knowledge. In such a system, themes such as language, mathematics or logic, which are considered rational and exact, do not seem to play a significant role.

Serious scholarly opinions testify to this prejudice: The Romanist Eugenio Coseriu, for example, already judged as follows in the title of his 1979 essay: *The Case of Schopenhauer—A Dark Chapter in German Language Philosophy*

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[2]. The mathematician Alfred Pringsheim wrote in 1904: "What Schopenhauer says about elementary geometry can only be considered for our purposes to the extent that his lack of any deeper mathematical insight is already clearly expressed on this occasion" [13, p. 359]. In 1911, Richard Groeper gave a lecture in Berlin entitled *Is Schopenhauer a Man of the Past or a Man of the Future?* and claimed: "Schopenhauer was not a systematist or dogmatist, and above all not a logician." [4, p. 431]

But these extreme positions have not remained without opposition: Gerold Ungeheuer, a famous communication scientist in Bonn has vehemently defended Schopenhauer's philosophy of language against Coseriu: "According to my judgment [. . .] Coseriu's claims are mistaken, and according to my insight Coseriu's methods are contestable" [24, p. 119]. Knut Radbruch, professor of mathematics at the TU Kaiserslautern, stressed in 1988 against the prejudice of Pringsheim that despite the temporal difference to a nineteenth century philosopher "certain questions, insights and perspectives of Schopenhauer on mathematics are of astonishing actuality." [14, p. 119] And 4 years before Groeper's lecture, Adolf Kewe wrote a dissertation which contradicted Groeper's opinion already in the title: *Schopenhauer as Logician* [6].

Scholarly opinion, then, testifies to the fact that Schopenhauer is a thinker who produces contradictions. But such does not come as the result of any intensive research into the three topics mentioned. There is no continuous research on Schopenhauer's language, logic or mathematics—although there have been several papers and one or two monographs in each of the three areas, in most cases the authors have either failed to react to previously published works or else the studies themselves have not been subsequently well-received.[1] Due to this lack of continuous research, it is not surprising that one encounters contradictions when comparing these isolated studies.

Yet, in all this, one problem which has remained untouched is that of whether Schopenhauer's oeuvre as a whole contains any contradictions. This is a question which has remained topical for over a century: in 1906, Otto Jenson compiled fourteen fundamental and 52 marginal incongruences and contradictions from Schopenhauer's work in tabular form in his dissertation ([5], cf. also [12, 20, 21]). And considering the breadth of this problem, one should not expect the areas of language, logic and mathematics to remain unaffected. It is worth nothing that Schopenhauer himself weighs in on this question, repeatedly and explicitly emphasizing that his philosophical system is a single thought, and that his writings contain no contradictions (cf.[1, p. 11], [5, p. 8]).

[1]The state of research on philosophy of language can be found in [3]. (For the special field of linguistics and etymology, see [23].)—On the state of research on logic see [9].—For the state of mathematics see [8].

2 System and World

But what exactly is Schopenhauer's system and what role do language, mathematics and logic play in it? Schopenhauer presented his philosophical system in his major work entitled *The World as Will and Presentation* (= *WWP*), first published in 1819. During the author's lifetime, the work was published in two slightly modified editions (1844, 1859) together with a supplementary volume (i.e. *WWP*, vol. II). This major work is divided into four books, the first of which contains a treatise on the faculties of knowledge, the second on philosophy of nature, the third on aesthetics and the fourth on ethics. Language, logic and mathematics are topics that are clearly dealt with in the first book. In addition, Schopenhauer recommends various of his works as supplements to the system presented in *WWP*, esp. his dissertation entitled *On the Fourfold Root of the Principle of Sufficient Reason* (1st edition: 1813, 2nd edition: 1847).

The interpretations of this system are heterogeneous.[2] Until the 1990s the four books of *WWP* were mainly interpreted according to the above-mentioned prejudice: in the first three books Schopenhauer developed a doctrine based on the principle of irrationalism, i.e. the all-encompassing and purposeless will, which led to a pessimistic view of the world—a world of pain and struggle. It is in the fourth book that Schopenhauer recommended to deny the world in order to reach nothingness and thereby salvation. Even though there were always authors who dealt with Schopenhauer's language, logic and mathematics, this normative as well as nihilistic interpretation prevented a clear preoccupation with these topics. For in comparison with the central book on ethics, all other topics of *WWP* appeared marginal, only as a means to an end. The slogan of this interpretation is simply: "salvation through knowledge" ("Erlösung durch Erkenntnis", cf. [16]).

At the end of the twentieth century, however, a turning point in Schopenhauer scholarship began. Matthias Koßler convincingly demonstrated that Schopenhauer's ethics must be interpreted descriptively and not normatively (cf. [7]): Schopenhauer does not recommend the negation of the will to live, but he describes it. Also Dieter Birnbacher [1] focused more on the descriptive and interpretive function of Schopenhauer's metaphysics. Schopenhauer himself confirmed this interpretation against false interpretations, which already existed during his lifetime: "I [. . .] generally do not expect anything of anybody, but rather reflect upon the world and show what everything is, and how it is connected" ([18, p. 343, no. 332], cf. [10]). Schopenhauer also presents religious behavior that renounces the non-rational principle, but this topic is as important as language, logic or mathematics, for example.

What Schopenhauer often emphasizes as the primary goal of his philosophy is the representation of the world. The aim of the *WWP* is to deliver "all of the manifold

[2] A detailed overview of the opinions in Schopenhauer research can be found in [22]. An English translation of [22] was published in *Voluntas: Revista Internacional de Filosofia* 10(1), 199–210 (2019).

things in the world, incorporated into a few abstract concepts according to their essence", so that the one single thought presented in only one book becomes a "complete replication, as it were mirroring, of the world in abstract concepts" [19, p. 119]. For Schopenhauer, the precursor of this representationalist approach is Francis Bacon. At relevant text passages (cf. [19, pp. 119, 445], he cites him as the guarantor of a philosophy that makes use of abstract concepts in order to represent all intuitive phenomena. And *WWP* is intended to reflect the entire (subjectively and objectively given) world with the help of a system of abstract concepts. Since language, logic and mathematics are components of the world, they play just as important a role in it as, for example, themes of aesthetics, ethics or religion.

3 Language, Logic and Mathematics

The structure of *WWP* clearly shows the position that language, logic and mathematics occupy in the system, and that they can claim a thematic relationship due to their position.[3] Schopenhauer first divides *the world* into *presentation* and *will*. This is already stated in the title of his main work. The first book of *WWP* deals with presentation and with all the cognitive faculties associated with it, and it is again divided into two parts. If one considers the paragraph numbering, which is only given in the second and third edition of *WWP*, one can say that the purely intuitively working *cognizance* is dealt with up to §7, the discursively working *reason* up to §16. Schopenhauer in turn divides the faculty of reason into three subbranches: *Language* (§§9–13), (science of) *knowledge* (§§13–15) and *practical reason* (§§16). Thus a first complex of topics of the present volume has already been determined in Schopenhauer's system: language. After some general remarks on language, Schopenhauer differentiates §9 into two sub-areas: *Logic* and *dialectics*. Thus a second complex of topics of the present book is determined in the system of Schopenhauer: Logic, which is a certain faculty for reason that makes use of language. The last topic still to be determined here, namely *mathematics*, can be found in §15: Among all other sciences listed in §14 mathematics (together with *philosophy*, especially Schopenhauer's own view on philosophy) has a special status, which also connects it with the subject of language and logic: it is essentially based on intuition (see below, Sect. 4).

Before we take the common ground of language, logic and mathematics into consideration, we should point out the structure of the system again. So far we have only located language, logic and mathematics in the system of *WWP* and this can be illustrated in Fig. 1. Schopenhauer never substantially revised the positioning of these three within his system. Though there are many other of his writings in which language, logic and mathematics are dealt with. The large number of relevant text passages on these topics can be found in the papers collected in this volume; only

[3] A detailed interpretation of the structure of *WWP* can be found in [11].

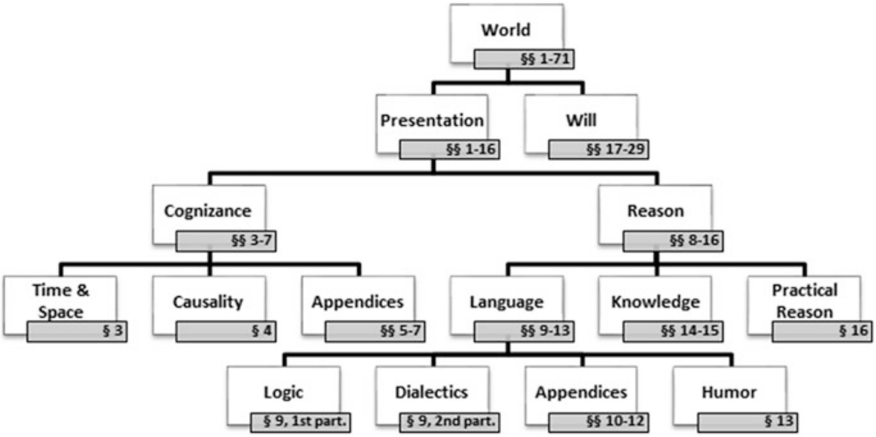

Fig. 1 System of *WWP*, Book I

the main sources are referred to here, which compile the essential basic ideas on the three subject areas as a supplement to the *WWP*:

Language: *Berlin Lectures*, vol. I, Cap. 3; *WWP*, vol. II, chap. 6–8; *The Fourfold Root of the Principle of Sufficient Reason*, chap. 5 (esp. §§26–29); *Parerga and Paralipomena*, vol II, chap. 25.

Logic: *Berlin Lectures*, vol. I, Cap. 3; *WWP*, vol. II, chap. 9, 10; *The Fourfold Root of the Principle of Sufficient Reason*, chap. 5 (esp. §§29–34); *Parerga and Paralipomena*, vol. II, chap. 2.

Mathematics: *Berlin Lectures* I, Cap. 4; *WWP*, vol. II, chap. 13; *The Fourfold Root of the Principle of Sufficient Reason*, chap. 6.

In this compilation of texts I would like to pay special attention to the *Berlin Lectures* that Schopenhauer gave in the 1820s and which were published in 1913 in complete form [17].[4] On the one hand they present only the views of Schopenhauer until the beginning of the 1830s, but as one can see in many text passages of *WWP*, vol. II and *Parerga and Paralipomena*, Schopenhauer resorted to the text of the lecture many times and published fragments of them in later writings.

Furthermore, the reference (e.g. of the *Berlin Lectures*) show that the topics of language and logic often overlap in one chapter. This is of course not surprising, as Schopenhauer subordinates logic under the branch of language (see Fig. 1). At first

[4]We currently have a complete edition (published in 1913 by Franz Mockrauer), a slightly modified reprint (by Volker Spierling) and a new edition (by Daniel Schubbe in collaboration with Judith Werntgen-Schmidt and Daniel Elon) of these *Berlin Lectures*. Unfortunately, there is no foreign language translation available at the moment. For detailed information on the *Berlin Lectures* and the corresponding editions see [15].

glance, however, it is surprising that Schopenhauer worked out the philosophy of language and logic much further in his *Berlin Lectures*: For example, the chapter on logic contains about 130 pages, in contrast to less than 10 pages within *WWP*, vol I, §9. This may be due to the fact that Schopenhauer conceived the publication of *WWP* for a broad audience, but considered the lectures to academics and students. For example, he writes in *WWP* that it is "unnecessary to burden our memory" with logic here [19, p. 77], but that it should "be taught in universities" [19, p. 79]. In his *Berlin Lectures* presented to academics and students, Schopenhauer says that he will intensively "present the basis, the essence, the main doctrines of logic" [17, p. 72]. However, since Schopenhauer scholarship has rarely taken the *Berlin Lectures* into account, the main texts on Schopenhauer's philosophy of language and logic are only rarely known today. Furthermore, the main theses on mathematics can also be found in *Berlin Lectures*, collected in only one chapter and not distributed over several writings (cf. [14]).

4 Intuition and Diagrams

So far we have found many reasons why language, logic and mathematics have rarely been mentioned in connection with Arthur Schopenhauer. Nevertheless, the question remains of why there ought to be a volume which treats of the three together. Language, logic and mathematics are indeed linked in their subordination under the faculty of reason—but then why not a volume on language, dialectic and astronomy? After all, dialectic is also dealt with in §9 and astronomy in §14.

Language, logic and mathematics have long been in an extraordinary state of tension. Particularly since the twentieth century, numerous philosophical questions have been discussed in various scientific fields, which concern all three topics: Is mathematics a language? Can language be formalized or arithmetized? Is logic the basis of mathematics or should mathematical ways of thinking determine the nature and extent of our logics? Is there a universal logic, just as there is universal algebra (or maybe a *mathesis universalis* or even a *characteristica universalis*)? These and many other similar topics seem to provide at least an extrinsic motivation for linking them in only one book.

There is also an intrinsic reason to Schopenhauer's writings: although all three branches belong to the discursive faculty, they refer in an extraordinary way to *intuition*. But this is not only a pure thesis whose discursivity is in turn so irreductible and necessary that the claim of this thesis would be a *contradictio in adjecto*. Rather, Schopenhauer uses a method in all branches that is in line with the tendencies in many scientific fields of the twenty-first century: Schopenhauer uses *diagrams* in order to depict the intuitive fundaments of our discursive faculties. He already recognizes essential aspects of diagrams that are still being investigated and discussed in various disciplines: Diagrams can display more information than they are supposed to show; diagrams correspond between intuitive cognizance and discursive reason; diagrams represent ways out of explanatory and proof-theoretic problems.

As in all areas of his system, Schopenhauer is concerned with the abstract representation of the mirrored phenomena of the world, that is, with the fundamental essences, with the philosophical core: diagrams can simply represent complex facts; they can show our basic patterns of thought; and they can explain the metaphors we live and think by. Nevertheless, Schopenhauer must not be interpreted as the last advocate of intuition-based tyranny, as some opponents of intuitionism claimed in the late nineteenth and early twentieth century (cf. [8]). As I said, he is for illustrating fundamental philosophical questions, not about refocusing our sciences with the methods of intuition. Schopenhauer, for example, knows—even if he does not underline the point—the benefits of arithmetization in the field of practical geometrical application (cf. [19, p. 88]).

5 Overview of the Volume

The contributions in this volume are united by their thematization of intuition and by many controversial views on Schopenhauer's work. Furthermore, many of the contributions also refer to Schopenhauer's *Berlin Lectures*, whose chapters on language and logic are here analyzed in-depth for the first time. The three sections of the present volume are arranged according to the positioning of the respective topics within Schopenhauer's system (see above, Sect. 3): first language in general, then logic in detail and finally mathematics.

5.1 Part I: Language

The volume begins with a section on philosophy of language that deals with individual aspects of Schopenhauer's semantics, semiotics, translation theory, language criticism and communication theory. Central to this is the dispute that has arisen in recent years concerning Schopenhauer's anticipation of modern contextualism or compositionalism. In particular, the question arises as to whether Schopenhauer anticipated several key aspects of Ludwig Wittgenstein's use-theory of language and other semantic principles of modern analytic philosophy and linguistics. *Matthias Koßler* argues against a mere instrumental interpretation of Schopenhauer's philosophy of language. Although these aspects are present in Schopenhauer's work, there is also a clear use-oriented theory of language. *Michał Dobrzański* goes into more detail on the question currently being discussed in research as to whether Schopenhauer advocates contextualism or representationalism. He argues that both theories can be found in Schopenhauer's texts: Representation theory is used for concrete concepts and contextualism for abstract ones. Based on Schopenhauer's *Berlin Lectures*, *Jens Lemanski* also argues for a combination of contextualism and representationalism: Whereas contextualism is important for understanding, representationalism is crucial for explaining concepts. In this context, an analogy between

Schopenhauer's semantics and modern concept diagrams in information science and knowledge representation is shown. *Sascha Dümig* contradicts the current semantic discussions in Schopenhauer scholarship: In his view, the theories of representationalism and contextualism based on Wittgenstein are merely misinterpreted e-language theories. In Schopenhauer, however, Dümig sees rather a representative of modern i-language theories which are in accordance with Noam Chomsky and Jerry Fodor. In defending his thesis he refers, as the preceding contributions, to the chapter *On Concepts* from the *Berlin Lectures*. *Gunnar Schumann* also disagrees with the approaches of the Schopenhauer-Wittgenstein research, but from a completely different perspective. For him, use theory and representation theory are incompatible, which is similar in late Wittgenstein. In addition to several problems in detail, he argues that Schopenhauer is not a forerunner of Wittgenstein and modern philosophy of language. *Michel-Antoine Xhignesse* takes up the consensus from this debate, namely that Schopenhauer makes a strict distinction between intuition and concepts, as well as between concrete and abstract concepts. He transfers this distinction to Schopenhauer's 'Perceptive Invectives' and thus explains what the criterion is for his language criticism.

5.2 Part II: Logic

The second part of the volume deals with logic, in particular with proof theory, metalogic, natural logic, systems of natural deduction, conversion theory, logical geometry and the history of logic. A central research question here is what role the diagrams play in Schopenhauer's logic. Schopenhauer uses numerous diagrams in his *Berlin Lectures* on over 200 pages. *Amirouche Moktefi* begins with a paper dealing with the question of whether these diagrams of Schopenhauer correspond to Leonhard Euler's or not. Moktefi notes that in some places Schopenhauer's diagrams show some interesting innovations. *Valentin Pluder* examines Schopenhauer's logic in the context of his time. He orients on the chapter *On the Origin and Development of Logic*, which Schopenhauer himself presents in his *Berlin Lectures*. Pluder comes to the result that although Schopenhauer's logic (as a whole) is typical for the paradigm of the early nineteenth century, Schopenhauer himself does not notice that he himself presents groundbreaking innovations in detail. One of these previously unnoticed highlights is Schopenhauer's distinction between natural and unnatural reasoning, which *Hubert Martin Schüler* and *Jens Lemanski* examine in the section *On Inferences* of the *Berlin Lectures*. Both show that Schopenhauer can make an important contribution to the question of what naturalness in logic means exactly. Thus one can place his arguments in the context of current research, such as natural logic or systems of natural deduction. *Anna-Sophie Heinemann* deals with the equation form in Schopenhauer's chapter *The Opposition and Conversion of Judgments* of the *Berlin Lectures*. She claims that Schopenhauer's use of the equality sign "=" can be interpreted as a step towards the 'mathematization' of logic in the nineteenth century. *Lorenz Demey* comes back to the circle diagrams given in §9 of *WWP*. Unlike Moktefi, however, he does not see them as a reference back to Leonhard Euler, but rather as a connection towards current logical geometry. *Jean-Yves Béziau* gives an overview of the modern history of the concept of metalogic. He explains the origin of various expressions with the prefix '-meta' and leads the reader from the expression 'metamathematics' of the Hilbert School to the current project of Universal Logic. Although the term 'metalogica' was not invented by Schopenhauer, he is in some sense the first author to be associated with modern metalogic and Universal Logic.

5.3 Part III: Mathematics

The third part of the volume deals with the philosophy of mathematics of Schopenhauer. In this section, the topics of the previous sections culminate, since here the difference between intuition and concepts as well as the question concerning the foundations of mathematics and reason become crucial. Schopenhauer deals mainly with Euclidean geometry and comes to the conclusion that mathematics, such as language and logic, is based in its foundations on intuition, which is then represented by diagrams. *Marco Segala* argues that Schopenhauer has reworked and elaborated his philosophy of mathematics several times over many decades. He thus refutes the myth of the unmodified system, which persists in Schopenhauer research, and shows to what extent Schopenhauer's philosophy of mathematics reacts to the Hamilton-Whewell debate of the 1830s in Great Britain. *Jason Costanzo* asks less about the historical but more about the systematic validity of Schopenhauer's philosophy of geometry. He sees a strong divide between intuition and abstraction in Schopenhauer and points out three difficulties concerning this divide: misleading intuitions, particularity of diagrams, and epistemic vacuity of argumentative proofs. In contrast to this viewpoint, *Michael Bevan* argues for the advantages of diagrammatic proofs. Whereas argumentative proofs often have explanatory gaps, diagrammatic proofs can show why a geometric proposition is true. Finally, with reference to Schopenhauer, Bevan rejects the objections that diagrams must always be particular or misleading. *Laura Follesa* shows the difference, but also some similarities, between Leibniz's rationalist approach and Schopenhauer's more intuitionist theory. Both philosophers are initially interested in the topic of mathematical truth, but from different points of view: Whereas Leibniz emphasizes the domination of necessary truths that exist independently of the senses, Schopenhauer rather speaks of a feeling of truth in mathematics.

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contributed to this conference and to this volume come from various fields of science: philosophy, logic, mathematics, history, linguistics, cultural studies, etc. Hopefully, the contributions collected here will prove that Schopenhauer research can benefit from interdisciplinary research just as much as these many varied disciplines can themselves benefit from the study of Schopenhauer's theses.

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Part I
Language

Language as an "Indispensable Tool and Organ" of Reason: Intuition, Concept and Word in Schopenhauer

Matthias Koßler

Abstract On the first sight, Schopenhauer's theory of language seems to be a rather simple instrumental one: Language is a means to convey information to others by sensual, vocal or written signs. However, Schopenhauer also argues that the real empirical object is the basis of a concept, which is abstracted from the former leaving out most of its properties and keeping the "essential" ones. In this paper, it is shown that Schopenhauer's view on language cannot be reduced to an instrumental theory of language. Such a reduction would be really surprising in view of Schopenhauer's style of writing and his critique of language.

Keywords Schopenhauer · Philosophy of language · Linguistic abstraction

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1 Introduction

On first sight, Schopenhauer's theory of language seems to be a rather simple instrumental one. Language is a means to convey information to others by sensual, vocal or written signs. In the last instance the content leads back to empirical reality. The other way round, the real empirical object is the basis of a concept, which is abstracted from the former leaving out most of its properties and keeping the "essential" ones. Words are sensual signs for concepts. By means of words we are able to communicate with others about empirical objects and their relations. Thus, language is a "tool" provided by the faculty of reason to fix and share information acquired by empirical experience. This interpretation of Schopenhauer's language

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theory is supported by his proto-evolutionary theory according to which reason and language emerged as a tool in the struggle for life.[1]

However, a closer look at the process leading from empirical objects through concepts to words and a more precise investigation of the instances of this process make obvious that Schopenhauer's theory of language is much more complicated than it seems on first sight, and also that the pure instrumental character of knowledge and language is no more tenable. In the following, I will first explain the three elements of his theory of language, namely real empirical object, concept and word. Afterwards I will try to draw some conclusions from these explanations regarding a theory of language and in the end make a remark on the question whether it can be characterized as instrumental.

2 Real Empirical Objects

Looking for the meaning of empirical reality in Schopenhauer we first have to pay attention to his "basic idealistic view" [FR, p. 35]: There are no objects independent of the cognition of a subject that in the case of empirical objects is an understanding animal. "Object" is synonymous with "representation" and "real empirical object" means nothing else but "complete intuitive representation". It is remarkable that you cannot find any definition of "intuitive representation" in Schopenhauer, even if it is one of the most important and frequently used expressions in his philosophy. The place where one expects such a definition is the dissertation because here Schopenhauer presents as the first class of objects the "intuitive, complete, empirical representations" [FR, p. 33]. But in the definition of this kind of object there is only said that they are complete in the sense that they "contain not merely what is formal, but also what is material in appearances" [ibid.]. Regarding the form of those objects one has to refer to Schopenhauer's explanation of the constitution of intuitive empirical reality as a whole.

Before we do so we should recall a crucial aspect of the object in general. We have already seen that "object" means nothing else than a representation of a subject. Moreover, no object can be apprehended alone but stands "in a relation of interconnectedness with other objects that in respect of form is governed by a rule determinable a priori" [EFR, p. 13]. In other words: Being an object means being in relation to others. In the case of empirical objects, this becomes particularly obvious, because intuition is the result of the activity of understanding, which connects space and time to a "complex held together held together by the forms

[1] Sascha Dümig [1, p. 159] denies the evolutionary and neurophysiologic conditions for reason. He might be right that there is no place where Schopenhauer explicitly puts reason in connection with the evolution of the brain, but as a part of the intellect, reason is included whenever Schopenhauer claims a connection between intellect and the evolution of brain. As well for instance he talks about "brain activity that rises to the level of reason" [WWR II, p. 361]. Nevertheless this is an interesting observation.

of the principle of reason, although with problematic limits" [FR, p. 34]. Any individual representation, i.e. any individual empirical object is regarded as "a part of this complete representation" [ibid.]. However, since representation formally is constituted by the connection of time and space, which are continua, parts of the "all-comprehensive complex of reality" [FR, p. 36] cannot be distinct but must be continuous. This is also obvious from Schopenhauer's identification of reality (Wirklichkeit), matter and causality. In his table of "praedicabilia a priori" in the second volume of the *World as Will and Representation*, Schopenhauer writes about matter:

> Matter is homogenous and a continuum: i.e. it consists neither of originally different (homoiomeries) nor of originally distinct parts (atoms); consequently it is not composed of parts that are essentially separated by anything, which is no matter [WWR II, p. 52; slightly modified translation].

When Schopenhauer, in his Dissertation, distinguishes concepts from intuitive representations by the feature that they "are not thoroughly determinate in the same degree" [EFR, p. 37; modified translation] as the latter, the determination of an individual representation must be in a way open, because as continual it has no fixed limits within the complex of reality. It is, in our context, interesting that Schopenhauer here only concedes a comparative thoroughly determination in intuitive representations and deletes this passage in the second edition of FR instead of G; in his later writings the expression "thoroughly determinate" is restricted to the Platonic Ideas. As thoroughly determinate representations Ideas include not only all qualities of an object but also all of its relations to others in past and future. They must be considered as something like the monads of Leibniz, living mirrors of the universe: Thoroughly determination leads to the whole of the complex of reality by the principle of sufficient reason, grasped in the Idea intuitively as a totality. In the supplements to his aesthetics in the second volume of the *World as Will and Representation,* Schopenhauer writes that the Idea is "the complete expression of the essence as it presents itself to intuition as an object; it is not grasped in relation to an individual will, but rather as it expresses itself spontaneously, so that it determines all of its relations [. . .]" [WWR II, p. 381; slightly modified translation].

What does it mean that empirical objects are more determined than concepts but less than Platonic Ideas? The role of imagination in the production of art gives a hint: The genius of art has need of imagination "in order to see in things not what nature actually created but rather what it was trying to create" [WWR I, p. 210]. Nature, here understood as the sum of empirical objects, had striven to bring to pass the thoroughly determination of the reality but it failed a real totality since it achieved only comparative determinations in individual appearances. Nevertheless, in any intuitive present "the essential aspect of all things in general is contained and represented in a virtual form" [WWR II, p. 80]. Every single empirical object has a more or less extensive quantity of determinations that is neither limited nor ever complete. This is the material from which concepts are abstracted by dropping most of its determinations.

3 Concepts

In the second edition of the *Fourfold Root of the Principle of Sufficient Reason*, Schopenhauer seems to advocate such a nominalist theory of concepts: “The formation of concepts occurs generally by dropping much from what is given intuitively in order to be able to think in isolation of what remains. A concept is thus a reduction in thought [Wenigerdenken] of that which is intuited. When various intuitive objects are considered, if something different is dropped from each and still something the same remains among all this is the genus of any species” [FR, p. 94]. Like in any other nominalist theory the question arises if not that “what remains” must be taken for granted before the act of dropping the rest. In the same passage, Schopenhauer suggests that concepts are condition for that act, when he adds to the identification of what remains with the genus of any species, that “any possible concept can be thought of as a genus”. Indeed, in the *World as Will and Representation*, he denies that concepts are produced by abstraction from various intuitive objects: “A concept does not possess universality because it is abstracted from many objects, but the other way around: it is because universality, that is, indeterminacy with respect to the individual, is essential to the concept as an abstract rational representation, that different things can be thought through the same concept” [WWR I, p. 65].

Universality is the distinguishing quality of concepts by which they are “entirely different in kind” [WWR I, p. 62] from intuitive representations. Thus, concepts are “representations of representations” in the sense that they are copies of representations “of a very special kind in a completely heterogeneous material” [WWR I, p. 63]. Leaving aside for a moment the problem of the completely heterogeneous material, universality is defined as “indeterminacy” with respect to the individual empirical object. On the other hand, as we have seen, the empirical object is in a sense indeterminate with respect to the Idea, the essence of a thing. So we have (1) the Idea as the thoroughly determined representation of a thing, (2) the empirical representation that is less determined because it is fixed by the understanding in a defined connection of time and space and therefore lacks determinations in respect as well to other individuals as to its own past and future and (3) the concept, which is even less determined to a degree where it grasps the genus of a species. Since neither Ideas nor concepts are individual, Schopenhauer calls the former by the scholastic expression “unitas ante rem (unity before the fact)”, the latter “unitas post rem (unity after the fact)” [WWR I, p. 261]. The question now arises and pushes us forward to the problem of the complete heterogeneity of concepts: What is the relation between Ideas and concepts? Is the heterogeneity to such an extent that “unitas” in both cases is only an equivocal expression?

In the passage that explains the use of these expressions, it seems to be not:

> The *Idea* is unity shattered into simplicity through the temporal and spatial forms of our intuitive apprehension: the *concept* on the other hand is unity reassembled from plurality by means of the abstraction of our reason [WWR I, p. 261].

Here the difference between Idea and concept seems to lie merely in the number of determinations, while the unity as reassembled in a concept is the same as the original in the Idea. But if we look back to what we have learned about Ideas up to now we find that unity of an Idea is the complex of reality without fixed limits grasped as totality. It is, as Schopenhauer points out at the transition from the second to the third book of his work "representation as a whole, the entire intuitive world" [WWR I, p. 189]. And since the world is infinite the Idea as well is inexhaustible in its determinations, although thoroughly determined. In contrast, the concept seems not to be a totality but a collection of a certain number of determinations that becomes a unity just by drawing a sharp line, which is the definition of the concept. Schopenhauer gives an illuminating image of the difference between both unities: "*concepts* are like dead receptacles; what we place inside actually lies next to each other, and we cannot take out more (through analytic judgements) than we have put in (through synthetic reflection): in those who have grasped them, on the other hand, *Ideas* develop representations that are novel with respect to concepts sharing the same name: the Idea is like a living and developing organism endowed with generative powers, an organism that produces what not already was packaged up inside it" [ibid., slightly modified translation].

In regard of the complete heterogeneity of concepts the meaning of "synthetic reflection" is crucial, because by this activity reason constitutes concepts placing inside its limits a certain number of determinations. In the second edition of the *Fourfold Root* Schopenhauer explains this activity: With the formation of concepts "the faculty of abstraction reduces complete and hence, intuitive representations [...] to their constituents in order to be able to think about these separately, each in itself, as different properties or relations of things" [FR, p. 94]. The reduction, or rather dissection, of the intuitive representation is the first step of synthetic reflection. Since intuitive representations are continuous and indefinite, the separation of their properties, i.e. abstraction is not just an analytic act like taking apples and peaches out of a basket full of different kinds of fruit, but as well the creation of something totally different from the properties existing in the intuition. The connecting, living tie between the latter is lost and representations "lose their intuitive quality, just as water loses its fluidity and visibility when reduced to its constituents" [ibid.]. This is what makes concepts completely heterogeneous with intuitive representations, similar to the heterogeneity of the water of a fountain and synthetic oxygen and hydrogen. However, what is really synthetic in the reflection we consider requires a second step that is scarcely explained by Schopenhauer but nevertheless important for the understanding of his theory of concepts. The separated properties must be put into a unity, the concept, under which then different intuitive representations can be thought. When concepts are used to convey intuitive knowledge to someone or to recall it at times, they must be translated back to intuitive representations. The mediation between intuitive representations and concepts is the work of the power of judgement, while the combination and comparison of concepts themselves is the field of logic. The latter does not contribute to the improvement of knowledge, but the operating of judgement is "the real core of all cognition” [FR, p. 99], since any series of cognitive grounds in logic “must terminate with a concept that has its ground in intuitive cognition” [WWR I, p. 64].

The role of judgement in the conveyance of intuitive knowledge is thus two-fold: For an intuitively given case, it “seeks either the concept or the rule to which the case belongs; or instead, for a given concept, it seeks the case which verifies it [. . .] in the first case it is a reflecting activity and in the second, a subsuming activity” [FR 98 sq.]. With respect to the formation of concepts the reflecting activity of the power of judgement is at least part of what has been explained as the synthetic reflection of reason. One may say that reason is the faculty of forming concepts in general, while power of judgement is the faculty of “appropriate concepts” [WWR I, p. 90]. Only the latter is “able to really advance the progress of the sciences” [ibid.]. According to what is said above, a concept is “appropriate” in the degree in which it reassembles the unity of the Idea. As we have seen, the unity of the Idea is a totality like a living organism, in which the properties and relations are standing in an infinite continuous connection with each other. An appropriate concept therefore should contain a set of properties that are able to stimulate the imagination in a case that it must be translated back to an intuitive representation. When Schopenhauer writes that a concept is an “adequate representative” of what is intuited, if it preserves “what is essential” to this, leaving out “many non essential aspects” of it [WWR II, p. 70], that what is “essential” must refer to the essence, the Idea. Even if Schopenhauer himself does not explain what he here means by “essential” this interpretation is obvious. While an Idea as the thoroughly determined representation is the “complete expression of the essence” and thus, as a totality “the true character of the thing” [WWR II, p. 381], the concept must be something that characterizes the essence in the way that it consists of some traits to describe its complete expression. As long as it is used only in combination and comparison with other concepts, i.e. in its pure logical use by reason, the concept is indeed like a receptacle filled with few properties in a sharp line. But when the question is about the reference to real objects, the meaning of concepts, it must be considered like a sketch as something that not only collects properties but as well shows the principles of their connection and development and thus imitates or indicates the unity of the Idea. This feature of concepts becomes clearer if we now turn to its use in language.

4 Words

In a well-known picture, Schopenhauer compares speech with a telegraph: “As an object of outer experience, speech is clearly nothing other than a highly perfected telegraph that communicates arbitrary signs with the greatest speed and finest nuance” [WWR I, p. 62]. The arbitrary signs in speech are words. This picture leads to a mechanistic reading of Schopenhauer’s theory of language, in which concepts as “abstract codes” are transmitted by words as signs in the way “SOS” is transmitted by the sequence of three long, three short and three long tones (cf. [1, p. 154 sq.]). Apart from the question what exactly is meant here by “code” this interpretation blurs the “complete heterogeneity” between concepts and intuitive representations. A Word in Schopenhauer’s opinion is a kind of the latter, namely the “sensuous sign of the concept and as such the necessary means of *fixing* it, i.e. of making it present to consciousness (bound up as this is with the form of time), and hence of creating a connection between reason (whose objects are merely general universals that know neither time nor place) and consciousness (which is sensible, bound up with time, and to this extent merely animal)” [WWR II, p. 72]. In this passage the heterogeneity of concept and word is reaching so far that reason seems to be something outside of consciousness. However, words as intuitive representations are not considered in regard to their own empirical qualities like sound or ink but they are used as a tool, only in regard to their general quality of being in time and space. In this sense, language is a tool, namely a tool for making concepts accessible to animal consciousness. But that does not mean that the language is nothing more than a tool and that communicating through language can be characterized as instrumental. This becomes obvious if we take into consideration the use of language.

Firstly, Schopenhauer mentions a case in which the relation between words and concepts cannot be reduced of that of a sign to the designated. Especially when we compare different languages in regard to the same concept it turns out that there is also an influence of the used signs on the concept. In that case we have no fixed concepts just replaced by different sensual signs but the use of different signs modifies the concept of an object. “Therefore you don’t learn the true value of words of a foreign language through a dictionary but only ex usu [. . .] we don’t learn mere words but acquire concepts”. In the *Parerga* Schopenhauer even claims that in learning a foreign language “new concepts form” [PP II, p. 510].

The second case in which words have an influence on concepts is poetry and philosophy: “Poetry and Philosophy are constantly trying to use intuitions in order to enrich concepts” [WWR II, p. 80]. At this place, Schopenhauer gives no further explications of the way how poetry and philosophy achieve the enrichment of concepts. However, it must be done by the use of language. In connection with his philosophy of arts he describes how poetry is able to present Ideas by arranging concepts “in such a way that the pattern of intersection of their spheres ensures that no concept can persist in its abstract generality; instead an intuitive representative appears before the imagination and the poet’s words continually modify this in keeping with his intentions” [WWR I, p. 269]. In this passage words deviate totally from their purpose as signs of concepts, on the contrary, they are able to use concepts in order to give rise to an intuitive representation, namely to an Idea. The poet “understands how to connect the abstract, transparent generality of concepts in order to precipitate out, as it were, what is concrete and individual, the intuitive representation” [ibid.]. Even if Schopenhauer talks about the arrangement and connection of concepts, it is obvious that this can happen only by the use of words.[2] We have no explanations from Schopenhauer if the philosopher proceeds in the same way, or if not, how he or she achieves an enrichment of a concept. When he compares an effective style of writing with a “completed oil painting” [PP II, p. 488], he suggests a congruence between philosophical and poetic use of language, particularly because the comparison is connected with the claim for of such an “objective” style to “place the words such that they actually compel the reader to think the same thing that the author thought” [ibid.]. On the other hand, Schopenhauer emphasizes as a main difference between philosophy and art that the former presents the Idea not intuitively but “presents it in abstracto” [MR I, p. 533 sq.]. However, there is a prominent example for how in philosophy the use of a word modifies the concerning concept: When Schopenhauer introduces the “will” as thing in itself in the second book of the *World as Will and Representation*, he demands a “broadening” [WWR I, p. 136] of the concept of will from the reader in the way that he uses the word “will” as a “denomination a potiori”, i.e. as a sign for its concept with “a broader scope than it had before” [WWR I, p. 135]. Nevertheless he emphasizes that this is not an arbitrary naming which could be substituted by another word, e.g. by “force”. “[. . .] I will be misunderstood by anyone who thinks it is ultimately a matter of indifference whether the word will or some other is used to designate the essence in itself of all appearance” [WWR I, p. 136]. It is remarkable that here the ultimate reference of the word as sign is not an intuitive representation but the essence in itself. In this particularity the philosophical use of language differs from ordinary speaking as well as from poetry where the essence is not designated directly by a word but indirectly by an arrangement of different words.[3]

Is it possible to reconcile these different relationships between word, concept and intuition with one theory of language? It is, considering the interpretation of concept from the end of the last chapter. If we regard concepts not only as receptacles of a few properties of an intuitive representation but as well as an outline of the character of the respective object, the use of words as sensuous signs of concepts can be more than attaching a name to a mental object (cf. [2, p. 40]) and it can be of various kinds. Looking not for a pure logical use but in respect to the expressed intuition, a concept characterizes the essence of a thing by indicating the unity of its Idea. The difference in the function of concepts in both cases is a bit like the difference between the

[2]Schopenhauer does not keep concepts and words apart clearly. On the blurring of the difference between both cf. [2, pp. 39–41].

[3]Music in relation philosophy differs from other arts. Referring directly to the essence, music structurally comes closer to the philosophical use of language than to the poetic. It “uses a highly universal language to express the inner essence, the in-itself of the world (which we think through the concept of will, after its clearest expression) and does so in a distinctive material, namely pure tones [. . .] moreover, in my view and according to my endeavours, philosophy is nothing other than a complete and correct repetition and expression of the essence of the world in very general concepts [. . .]” [WWR I, p. 292] On the other hand, and different to philosophy, the universality of music“is in no way the empty universality of abstraction; rather, it is [. . .] united with thorough and clear-cut determinateness” [WWR I, p. 289]. It could be instructive to elaborate on the relationship between music and philosophical speech but would go beyond the scope of this essay. For an attempt in this direction cf. [3, p. 66 sq.].

coordinates of a geometrical function and the line that is drawn in order to give the function, e.g. a hyperbola, as a whole for intuition. More appropriate in regard to the expression “characterizing” and to the basis of Schopenhauer’s philosophy seems to be a comparison with the human character[4]: From a few personal traits “one can obtain a correct knowledge of someone’s character” or, as Schopenhauer puts it exaggerated, “from one characteristic deed thereby constructing it, as it were [. . .]” [PP II, p. 209; slightly modified translation]. In a similar way a well-chosen use of language can evoke the ability of a concept which is an “adequate representative” of an intuition to make the listener construct the essence or Idea of it. Ordinary use of language only leads to the imagination of an intuitive representation that is inadequate with respect to the concept and will always differ more or less from the intuition of the speaker (cf. [2, pp. 84 sq.]). Without reference to the intuition, finally, words are used as arbitrary names for concepts working like receptacles of data placed inside. As words without reference to intuition could be substituted by variables this may be called formal or logical use of language. Only regarding the latter the picture of a telegraph applies to language but not in regard to the comparison of encoding and decoding with the activity of the power of judgement. Note that in the quotation Schopenhauer restricts the picture to speech “as an object of outer experience”. As objects of outer experience we have only the sounds of words and the real objects to which they refer, but neither concepts nor meanings.

5 Conclusive Remarks

After all it is obvious that Schopenhauer’s view on language cannot be reduced to an instrumental theory of language (cf. [4]). Such a reduction would be really surprising in view of Schopenhauer’s style of writing and his critique of language (cf. [5]). Nevertheless Schopenhauer talks about language as a tool and uses mechanical images in order to illustrate the function of speech. He does not reject these aspects of language but places them into a hierarchic order of different uses of language. There is no elaborate theory of language in Schopenhauer but he offers an approach that could contribute to current debates with further development. His explanations of intuition, concept and word have the capacity for a theory by which different uses of language including nominalist and realistic (of universals) views are reconciled in a way that they are not indifferent to each other but get their place in one comprehensive task, namely the communication of essential truth. The relationship of intuitive representations and concepts to the Idea as the complete expression of the essence ensure that words have a meaning that can be understood by anyone even if everybody connects it with a different imagination when he or she verifies the concept by intuition. Thus, Schopenhauer’s theory of language fits in the main purpose of his entire philosophy to discover “the true meaning of

[4]For a more detailed explanation of this kind of characterizing cf. [6, p. 100 sq.].

intuitive representation, which ensures that the images do not pass by us strange and meaningless as they would otherwise necessarily have done; rather, they directly speak to us and are understood and have an interest that engages our entire being" [WWR I, p. 119; slightly modified translation].

Abbreviations
The writings of Schopenhauer are quoted in English. We refrained from giving the original German text in order to avoid an excessive amount of notes. In some cases we thought it necessary to modify the translation. This is marked in the references. The following English editions are used with standard abbreviations:

EFR	Schopenhauer's Early Fourfold Root. Translated by F. C. White. Avebury Aldershot (1997)
FR	On the Fourfold Root of the Principle of Sufficient Reason and Other Writings. Translated by D. Cartwright, E. E. Erdmann, C. Janaway. University Press, Cambridge (2012)
MR I	Manuscript Remains in Four Volumes: vol. I: Early Manuscripts. Translated by E. F. J. Payne. Berg, Oxford (1988)
PP I/II	Parerga and Paralipomena, translated by A. del Caro, S. Roehr, C. Janaway, vol. I/II. Cambridge University Press, Cambridge (2014/2015)
WWR I/II	The World as Will and Representation, translated by J. Norman, A. Welchman, C. Janaway, vol. I/II. Cambridge University Press, Cambridge (2010/2018)

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Problems in Reconstructing Schopenhauer's Theory of Meaning: With Reference to His Influence on Wittgenstein

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Abstract The article contributes to the discussion of Schopenhauer's possible anticipation of both the representational theory of language and the use theory of meaning and the reception of his philosophy by early and late Wittgenstein. Schopenhauer's theory of language is presented and brought into the context of these two theories. His use of the terms "word," "concept," and "meaning" is analyzed and it is shown that he applies them ambivalently. The article's main findings include a demonstration of how Schopenhauer's ambivalent terminology enables the twofold interpretation of meaning: as representation-based and use-based.

Keywords Schopenhauer · Wittgenstein · Meaning · Word · Concept · Sense · Reference · Semantic theory · Representational theory of language · Use theory of meaning

Mathematics Subject Classification (2000) Primary 03A03, Secondary 03A02

1 Introduction

With the following article I seek to make a contribution to the discussion regarding Arthur Schopenhauer's theory of language and especially his possible anticipation of questions later elaborated upon by Ludwig Wittgenstein. Given the fact that the latter is known to have been a reader of the former, this discussion also includes questions about Schopenhauer's influence on Wittgenstein.

In my article, I refer to Jens Lemanski's article *Schopenhauers Gebrauchstheorie der Bedeutung und das Kontextprinzip* (*Schopenhauer's Use Theory of Meaning and the Context Principle*) in the *Schopenhauer-Jahrbuch* of 2016 (cf. [5]). In it, Lemanski argues that there are significant parallels between Schopenhauer's theory

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of language as presented in his *Philosophical Lectures* [13] and the one presented by late Wittgenstein in *Philosophical Investigations* [17], even though comparative research until now has focused mainly on similarities between Schopenhauer's *The World as Will and Representation* [11] and early Wittgenstein's *Tractatus Logico-Philosophicus*[1] [16].

I consider it crucial for the further investigation of the question to point out some problems in Schopenhauer's theory of language that have not yet garnered much attention. To do so, in Sect. 2, I start with a concise reconstruction of the main features of Schopenhauer's theory of language, based on which I show its parallels with the representational and use theories. Then, in Sect. 3, I point out some fundamental problems with Schopenhauer's formulation of his theory of language. I do this in reference to what I first formulated in my article from 2015 (cf. [2]) and then elaborated more extensively in Chapter 1 of my book *Begriff und Methode bei Arthur Schopenhauer* (*Concept and Method in Arthur Schopenhauer's Philosophy*) (cf. [1])—neither of which is available in English. I specifically point out the ambivalent use of the terms "concept" ("Begriff") and "word" ("Wort") in Schopenhauer's writings. In Sect. 4 I proceed to demonstrate how this problem influences the interpretation of which theory of meaning Schopenhauer actually supports. I show that both the representational theory and use theory can be founded upon this ambivalence. In Sect. 5 I analyze some additional issues that appear in connection with his use of the term "meaning"—namely, his anticipation of the separation of sense and reference. In Sect. 6 I show that even though two theories of meaning might be ascribed to him, he strongly favors one over another. In the last section I provide a brief summary of the findings and propose some future research considerations.

2 Schopenhauer's Theory of Language and its Links to the Representational and Use Theories

Even though Schopenhauer's theory of language has been discussed by a few scholars, unlike some topics appearing in his philosophy, it has never been the main focus of Schopenhauerian research. Apart from one article by Rudolf Malter in which the most crucial elements are discussed (cf. [7]), there is little literature dedicated solely to the problem of language. A significant motivation for investigating Schopenhauer's views on language seems to have come from the analysis of his impact on Wittgenstein and the analytical tradition in philosophy. However, these analyses were comparative and the need for a holistic approach to Schopenhauer's philosophy of language has remained unaddressed by researchers. For example, Lemanski points out that Wittgenstein scholars mainly refer to para. 9 of *The World as Will and Representation* when discussing Schopenhauer's views on

[1]For literature on the impact on Wittgenstein, see the extensive analysis prepared by Lemanski [5, pp. 174–176]. For the interest in Schopenhauer from analytical philosophy, see Weimer [15].

language [5, p. 183]. However, this paragraph is not sufficient for a definite account of what Schopenhauer had to say on this matter. A planned new German edition of Schopenhauer's *Philosophical Lectures* (*Philosophische Vorlesungen*), a work which contains several long passages referring to logic, language, and concepts, may stimulate new research.[2]

To understand the position of the language problem within Schopenhauer's philosophy, it has to be pointed out that in his philosophical investigation of what the world is he assumes the stance that our experience is made up of two different epistemological dimensions: (1) what we experience by our senses and (2) what we experience by our minds.[3]

We find this division of human experience into two classes in para 9. of *The World as Will and Representation*. The first class, which is referred to as "the real external world," is made up of "representations of perception" ("anschauliche Vorstellung") [11, p. 39]. The second class, which is referred to as "reflection," consists of "non-perceptual" or "abstract" representations ("nichtanschauliche Vorstellungen") [11, p. 40]. Notice that Schopenhauer refers to these two classes of experience using the same term "representations," and differentiates them into (1) perceptual and (2) non-perceptual. The reason for this is his adoption of the Kantian point of view wherein all experience is phenomenal as it is constructed by the subject. In Schopenhauer's terminology, what we experience as the real external world and as our thoughts are, from the metaphysical perspective, different modes of how we represent the world to ourselves. Thus both the external and abstract dimensions of the world can be understood as forms of representation.

In this context it should also be mentioned that Schopenhauer's system includes the metaphysics of will by which the world as representation is reduced to a mere appearance of the will. From this perspective it is difficult to talk about a real external world. However, as a starting point for the presentation of his philosophy, Schopenhauer uses the perspective of the world as representation, not as will. In the first book of volume one of *The World as Will and Representation* he assumes the perspective that the "world is my representation" [11, p. 3]. It is significant for the reconstruction of his theory of language that para. 9, which is dedicated solely to the problem of concepts and can be considered the most coherent presentation of his theory of language from works published during his life, is also included in this book and within this perspective. Therefore, this seems to be the main perspective from which he analyzes the problem of language.

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[2]As of August 9, 2018, the last volume, entitled *Metaphysik der Sitten*, had already been published [9], and the first three volumes are expected to be published before 2019.

[3]These correspond with the first two classes of objects distinguished by Schopenhauer in *On the Fourfold Root of Sufficient Reason*. About the first class he says: "The first class of objects possible to our representative faculty, is that of *intuitive* ('anschaulichen'), *complete, empirical* representations. They are *intuitive* ('anschauliche') as opposed to mere thoughts, *i.e.,* abstract conceptions ('Begriffe'); [...]" [14, p. 31]. The second class "are *conceptions* ('Begriffe'), therefore *abstract*, as opposed to *intuitive* ('anschaulichen'), representations, from which they are nevertheless derived" [14, p. 114]. However, in his investigations on language the other two classes distinguished in the work, i.e., the *a priori* forms of space and time and the subject in volition ("Subjekt des Wollens"), are not mentioned.

As has been said, Schopenhauer identifies the class of representations of perception with what we might call the real external world, i.e., with the world of sensory objects. On the other hand, the second class of representations, which are non-perceptual, is identified with "concepts." These "form a peculiar class existing only in the mind of man, and differing entirely from the representations of perception"[4] [11, p. 39]. This means that the two classes of representation differ strongly from each other. However, Schopenhauer also remarks that abstract representations "stand in a necessary relation" to the representations of perception, without which "they would be nothing" [11, p. 40]. This stance is based on the assumption that "the abstract representation has its whole nature simply and solely in its relation to another representation" [11, pp. 40–41]. Such a representation might also be an abstract representation or a representation of perception. Additionally, Schopenhauer makes a remark that strictly determines the relation of reflection and perception: "the whole world of reflection rests on the world of perception as its ground of knowledge" [11, p. 41].

What this means for the relation between abstract and perceptual representations, or, accordingly, the relation between conceptual and sensory knowledge, is summarized in the second volume of *The World as Will and Representation*:

> It has been shown that concepts borrow their material from knowledge of perception, and that therefore the whole structure of our world of thought rests on the world of perceptions. It must therefore be possible for us to go back from every concept, even if through intermediate stages, to the perceptions from which it has itself been directly drawn, or from which have been drawn the concepts of which it is in turn an abstraction [12, p. 71].

Succinctly, Schopenhauer seems to state here that to understand the concepts we have in our minds we need to refer them to what we consider to be objects of the real world. He thus gives a clear epistemological priority to knowledge gained from the senses and subordinates conceptual knowledge to experience.

This stance strongly resembles Wittgenstein's formulation of what Lemanski refers to as the representational theory of language (cf. [5, p. 173]). Its manifesto can be found in the *Tractatus Logico-Philosophicus*: "3.203. The name means the object. The object is its meaning [. . .]" [16]. Another formulation, the one used by Lemanski in his article, appears in the first paragraph of the *Philosophical Investigations*. There, Wittgenstein argues with this theory and formulates it with the following words: "the individual words in language name objects ('Gegenstände')" [17]. In both formulations this theory assumes that elements of language (names *resp.* words) represent real objects.

Schopenhauer's understanding of concepts as presented above appears to share the same thought with the representational theory—this is even reinforced by

[4] Actually this division is even more complicated. Schopenhauer also identifies objects in the mind that are perceptual and calls them "phantasms." They are something like mental images and are opposed to concepts. However, it does not seem necessary to include this problem in the current investigation for the sake of brevity. For the problem of phantasms see Chapter 1, Section 3.2.3 of [1].

the fact that he refers to concepts as "representations of representations" [11, p. 40]. However, it has to be remarked that there is a significant difference between Wittgenstein and Schopenhauer's account of the relationship between reality and language. Schopenhauer makes the additional distinction between words and concepts. It can be found throughout his main philosophical writings and is well explained in a passage from *On the Fourfold Root of Sufficient Reason*:

> Now as representations, thus sublimated and analysed to form abstract conceptions, have, as we have said, forfeited all perceptibility, they would entirely escape our consciousness, and be of no avail to it for the thinking processes to which they are destined, were they not fixed and retained in our senses by arbitrary signs. These signs are words. In as far as they constitute the contents of dictionaries and therefore of language, words always designate general representations, conceptions, never perceptible objects; [...] [14, p. 116].

Based on this quotation, the difference between words and concepts seems quite obvious: words are *sensory* symbols of concepts and concepts are *mental* objects that are represented by words. A very similar account of their difference can also be found in Schopenhauer's *Lectures*, where he says that a word is "the sensory sign of the concept" ("das Wort: es ist das sinnliche Zeichen des Begriffs")[5] [13, p. 243]. This claim in fact states that there is an ontological difference between words and concepts: the first belong to the dimension of sensory reality (i.e., perceptual representation), whereas the second belong to the dimension of reflection (i.e., non-perceptual representation).

Thus, there is at least one significant difference between the representational theory of language as presented by Wittgenstein and Schopenhauer's stance. It can be summed up in the following way.

- Wittgenstein: individual words in language name objects.
- Schopenhauer: individual words in language name concepts, which represent objects.

The crucial difference between these two stances is that in Wittgenstein's formulation of the representational theory words are considered representations of real objects, whereas in Schopenhauer's theory words are not representations of real objects, but of mental objects which he calls "concepts." These mental objects are representations of real objects.

This difference brings up the problem of the carrier of meaning. In Wittgenstein's account the meaning of a word is explicitly identified with an object: "In this picture of language we find the roots of the following idea: Every word has a meaning. This meaning is correlated with the word. It is the object for which the word stands" [17, para. 1]. In Schopenhauer's theory we have to ask the question whether it is a word or a concept that has meaning. If it is the word, then its meaning will be understood as the thought of a specific person, a mental object called "concept." If, on the other

[5] All translations from German sources for which there were no English translations available were done by myself. This is indicated wherever the original German quotation is in parentheses.

hand, concepts are found to be the carriers of meaning, then the meaning will be the real object that is represented by a concept in the mind of a person.

Therefore, at this stage of our investigation, in order to answer the question of what kind of language theory Schopenhauer formulates it should suffice to determine whether he considers words (i.e., sensory signs of language) or concepts (i.e., mental representations of real objects) to be the carriers of meaning. If concepts are the carriers of meaning, we could say that his theory of language is similar to the representational theory in Wittgenstein.

However, if words are the carriers of meaning, we would have to underline that it does not accord with this theory. Such a view would instead resemble what is called the use theory of meaning. Its formulation, according to Lemanski (cf. [5, p. 172]), can also be found in late Wittgenstein's *Philosophical Investigations*. Arguing against his own earlier view on language, Wittgenstein proposes a new theory of meaning, as an alternative to the representational one: "the meaning of a word is its use in the language" [17, para. 43]. In this account, the meaning of a word is not identified with objects, i.e., elements of reality, but with its use in different contexts. Or, we might perhaps say, with how people use it. If we consider the possibility that, for Schopenhauer, the word is the carrier of meaning, then meaning will have to be understood as the mental state of a subject using it. This already bears some similarity with late Wittgenstein's account. In both cases meaning is not an element of reality but something in the minds of people using language.[6] Thus, if we were able to determine that Schopenhauer proposes that words are the carriers of meaning, we would find substantial evidence for his influence on late Wittgenstein's understanding of language.

However, it is difficult to determine whether Schopenhauer holds that words or concepts are the carriers of meaning, because, contrary to his claims that words and concepts differ ontologically and that he strictly distinguishes them, he often uses the terms "word" and "concept" as names for what seems to be the same designate.

3 The Ambivalent Use of the Terms "Concept" and "Word"

Schopenhauer's claims on the difference between words and concepts have encouraged scholars to treat this distinction as an important and indisputable element of his theory of language. Lemanski, although being somewhat careful, says that in almost all of his texts Schopenhauer strictly distinguishes between the terms "word" and "concept" (cf. [5, p. 187]). This is understandable. In a number of

[6]However, there is a difference that should be mentioned. Schopenhauer, when talking about such mental states, concludes that meaning is in fact private: "If perceptions were communicable, there would then be a communication worth the trouble; but in the end everyone must remain within his own skin and his own skull, and no man can help another" [12, p. 74]. Wittgenstein also reflects upon this problem in his famous beetle-in-the-box metaphor, but apparently comes to the conclusion that this problem is irrelevant (cf. [17, para. 293]).

passages Schopenhauer indeed reassures us that he clearly distinguishes these terms as shown in the section above (cf. [14, p. 116], [13, p. 243]). However, we cannot rely on this distinction when discussing his theory of meaning. His sharp theoretical distinction between words and concepts is not reflected in his use of the terms when writing about the problems of language. A number of passages can be found in which he breaks away from self-imposed definitions of both words and concepts. Significantly, this happens when he dedicates his attention to examining what concepts actually are and how they function.

A hint that the distinction between words and concepts is in fact rather blurred can be found in some articles on Schopenhauer's theory of language. For example, Gerhard Mollowitz obviously struggles to keep them apart when he writes about the "difficult border between abstract creation of words and concepts" ("schwierige Grenzpunkt der abstrakten Wort- und Begriffsbildung"). He later coins the term "conceptual words" ("Begriffs-Worte") to adequately describe Schopenhauer's theory of how thoughts are expressed in language [8, p. 53]. If the difference between words and concepts were as clear as claimed by Schopenhauer, there would be no such problem at all.

To find out the reason for this problem it is necessary to enhance his claims with an analysis of how he actually uses these two terms. Above all, he clearly states that words and concepts are connected to each other so strongly that we usually fail to notice the difference between them:

> Thus when we read or listen, we receive mere words, but from these we pass over to the concepts denoted by them so immediately, that it is as if we received *the concepts immediately*; for we are in no way conscious of the transition to them [12, p. 23].

In other words, the claim that there is an ontological difference between words (sensory objects) and concepts (mental objects) is relativized here by the observation that from the perspective of the subject it is difficult to tell them apart. It is also expressed in the following remark from the *Lectures* which states that we are actually unable to grasp our concepts without words that represent them: "I should give you example of a concept for which there is no word: this is impossible" ("denn ich soll Ihnen als Beispiel einen Begriff mittheilen, für den es kein Wort giebt: das geht nicht") [13, p. 244]. Thus, it seems that the claimed ontological difference between words and concepts is enhanced here by the claim that there is almost no epistemological difference between them. Or, in other words, their difference cannot be experienced by the subject.

This is further supported by the position that a person cannot have intellectual access to concepts (which are, by the above definition, mental objects of some kind), unless they are represented by words:

> In this property they [i.e. concepts] have, to a certain extent, an objective existence that yet does not belong to any time-series. Therefore, to enter the immediate present of an individual consciousness, and consequently to be capable of insertion into a time-series, they must be to a certain extent brought down again to the nature of particular things, individualized, and thus linked to a representation of the senses; this is the *word*. Accordingly, this is the sensible sign of the concept, and as such is the necessary means of *fixing* it, in other words, of presenting it vividly to the consciousness that is tied to the form

> of time, and thus of establishing a connexion between our faculty of reason, whose objects are merely general *universalia* knowing neither place nor time, and consciousness which is tied to time, sensuous, and to this extent merely animal [12, p. 66].

Yet another interesting feature of concepts is also contained in this argument. They are presented as kinds of “objective” entities, i.e., something outside of the subject’s consciousness. If it were not for the words that fix them in consciousness, we would be unable to grasp them. If we stick to the definition of words as sensory signs which Schopenhauer gives in the above passages from the *Fourfold Root* and the *Lectures*, we might conclude that conceptual thinking is possible mainly when we are speaking aloud or writing, i.e., using sensory signs. However, here he is apparently speaking of mental processes when he refers to using words for bringing concepts into the subject’s consciousness. Taking the matter strictly, this is inconsistent with his claim that words are sensory. There are other passages where he does this. For example, here he directly states that we can also have words in our *thoughts*:

> Of course, it sometimes happens that concepts occupy consciousness even without their signs, since occasionally we run through a chain of reasoning so rapidly that we could not have thought of the words [nicht hätten die Worte denken können] in so short a time. But such cases are exceptions that assume great exercise of the faculty of reason, which it could have attained only by means of language [12, p. 66].

What Schopenhauer means here is that sometimes, when reasoning very quickly, our thinking does not include the use of words. However, this should be treated as an exception. He seems to assume that the thinking process usually consists in *thinking* words, which obviously is not a sensory but a mental process. This opens up a whole new dimension of interpretation of what can be understood by the term “word.” Words are not merely sensory signs but also something like *mental* signs. They are used for “fixing” concepts. Not only does this modify Schopenhauer’s initial claim about the sensory ontology of words, but it also brings up new questions about what the term “concept” refers to. Concepts can no longer be defined simply as the mental objects that words refer to. The ontological *differentia specifica* between words and concepts described in the preceding section (i.e., their sensory or mental nature) cannot be upheld. Now both words and concepts have to be understood as kinds of mental objects, as elements of our mental processes. Given this, a new way of differentiating between these two types of mental objects needs to be found.

Solutions can be found in several passages of Schopenhauer’s works where he indicates that concepts are something which we are not *aware of* as long as they have not been represented by words. This can already be seen in the quotation above where he claims that concepts have a certain kind of “objective existence” (cf. [12, p. 66]). In the *Lectures* he similarly states: “Usually we become aware of a concept only together with its sign, the word.” (“In der Regel werden wir uns des Begriffs immer nur mit seinem Zeichen, dem Wort, zugleich bewusst”) [13, p. 244]. This means that words and concepts understood as mental objects differ in the following way: words are within the scope of our consciousness and concepts are, at least in most cases, outside of it, since we are usually *not* aware of them. In this picture conceptual thinking becomes something that occurs without our willful participation and, in order to become aware of our thoughts, we need to represent them with symbols in our minds, to which Schopenhauer refers using the term "words."

Therefore, we can distinguish at least three types of objects with different ontologies that Schopenhauer indicates when reflecting upon the nature of language. For the sake of clarity I am going to use non-Schopenhauerian terminology to name them and from now on I will refer to them by the terms in parentheses. These objects are as follows:

(i) sensory signs that represent mental objects (*sensory signs*),
(ii) mental signs within the scope of our consciousness (*mental signs*),
(iii) mental states outside the scope of our consciousness (*notions*).

Having made this distinction, we are able to formulate the main problem with Schopenhauer's use of the terms "word" and "concept": he uses these *two* terms to refer to the *three* types of entities listed above.

The ambivalent use of the term "word" has already been shown above. At one time Schopenhauer claims that words are sensory signs, and later speaks of words as mental signs. Thus, let us now investigate the ambivalent use of the term "concept."

In the *Lectures* Schopenhauer reflects upon a possible situation: struggling to find the proper word to express a concept. From this he concludes that concepts might be grasped without words, however, we need words if we want to grasp them on demand:

> However necessary words are for thinking and however much a concept needs a sign; the necessity of a sign does not consist in the fact that without it the concept could not be grasped at all, that it could not be thought of (because it can in and for itself, as we often lack a word for expressing a concept we have). It rather consists in the fact that the willful [willkürlich], arbitrary evocation of the concept is possible only through the sign: the sign does not serve for thinking it, but for making it present at any time. Thus, it would be false to argue with the necessity of the signs for concepts for the assumption that during thinking and speaking we actually solely operate with signs and they completely represent [vertreten] concepts; [...]
>
> (So nothwendig auch zum Denken die Worte sind und so sehr auch der Begriff eines Zeichens bedarf; so beruht dennoch die Nothwendigkeit des Zeichens nicht darauf daß ohne dasselbe der Begriff überhaupt gar nicht gefaßt, gar nicht gedacht werden könnte (denn das kann er an und für sich, da oft uns ein Wort fehlt unsern Begriff auszudrücken), sondern darauf, daß die willkürliche, beliebige Hervorrufung des Begriffs nur durch das Zeichen möglich ist: das Zeichen dient nicht ihn zu denken, sondern ihn jederzeit zu vergegenwärtigen. Darum wäre es falsch wenn man aus der Nothwendigkeit der Zeichen für die Begriffe die Annahme begründen wollte, daß wir beim Denken und Reden eigentlich ganz allein mit den Zeichen operirten, und sie völlig die Begriffe vertreten; [...]) [13, p. 247].

Here "concepts" seems to refer to entities of type (iii) which might perhaps be described by the English term "notions"—i.e., cognitions the subject has but cannot grasp with full clarity on demand. This understanding of concepts also appears on page 244 of the *Lectures* where Schopenhauer clearly states that, in the case of abstract concepts such as "justice" or "power," it is impossible for the subject to fully grasp them if they are not mediated by specific words (cf. [13]). The general idea that thinking happens with notions that are not fully evident to the subject itself unless they are represented by mental signs probably finds its fullest expression in the metaphor of the surface of water:

> To make the matter clear, let us compare our consciousness to a sheet of water of some depth. Then the distinctly conscious ideas [Gedanken] are merely the surface; on the other hand, the mass of the water is the indistinct, the feelings, the after-sensation of perceptions and intuitions and what is experienced in general, mingled with the disposition of our own will that is the kernel of our inner nature. Now this mass of the whole consciousness is more or less, in proportion to intellectual liveliness, in constant motion, and the clear pictures of the imagination, or the distinct, conscious ideas [Gedanken] expressed in words, and the resolves of the will are what comes to the surface in consequence of this motion. The whole process of our thinking and resolving seldom lies on the surface, that is to say, seldom consists in a concatenation of clearly conceived judgements; although we aspire to this, in order to be able to give an account of it to ourselves and others. But usually the rumination of material from outside, by which it is recast into ideas [Gedanken], takes place in the obscure depths of the mind. This rumination goes on almost as unconsciously as the conversion of nourishment into the humours and substance of the body. Hence it is that we are often unable to give any account of the origin of our deepest thoughts; they are the offspring of our mysterious inner being [12, pp. 135–136].

This quotation gives a picture of how the process of thinking functions according to Schopenhauer. It can, however, also be seen in the context of Schopenhauer's theory of language by referring to his definition of "thinking": "What is properly called thinking, in its narrowest sense, is the occupation of the intellect with conceptions ('Begriffen')" [14, p. 119]. In other words, for Schopenhauer, thinking is being intellectually concerned with concepts. Consequently, the passage above might be interpreted as a metaphor for how concepts are dealt with by our minds. It shows that Schopenhauer assumes the existence of some kind of conceptual thinking that is not fully within the scope of the consciousness of the subject, from which we might conclude that concepts are to be understood as mental states (notions) outside of our consciousness.

In the quotations above it therefore seems that Schopenhauer is presenting a theory wherein conceptual thinking is a process that takes place *independently* from language. Words are in turn elements of language necessary for giving an account, to others and ourselves, of the effects of conceptual thinking. Using the chosen metaphor, we might say that words and language are on the water's surface, whereas concepts are in its depths. Interestingly enough, I have come across only one scholar who actually realizes this problem. Only Jankowitz describes concepts in Schopenhauer's theory as "extralingual reality categories" ("übersprachliche Wirklichkeitskategorien") [4, p. 65].

However, this understanding of concepts is not consistently represented by Schopenhauer. This can be seen in the following passage:

> In other words, it must be possible for us to verify the concept with perceptions that stand to abstractions in the relation of examples. Therefore these perceptions furnish us with the real content of all our thinking, and wherever they are missing we have had in our heads not concepts, but mere words [12, p. 71].

Here "concepts" are juxtaposed against "mere words," with their *differentia specifica* being the reference to perception. Schopenhauer seems to assume that "mere words" and "concepts" are experienced by the subject in the same way - as mental objects so similar to each other that we might not be able to distinguish them from one another at all. He seems to be speaking of mental objects (the ontology of type ii above) that are either "mere words" or "concepts," depending on whether they have or do not have reference in the real world. The same way of understanding concepts can be found in several other passages. For example, when Schopenhauer criticizes the use of very abstract concepts he is obviously referring to elements of language that the subject is conscious of:

> Every philosophy which [...] takes as its starting-point arbitrarily chosen abstract concepts such as, for example, the absolute, absolute substance, God, infinite, finite, absolute identity, being, essence, and so on, floats in air without any support, and so can never lead to a real result. However, philosophers have at all times attempted it with such material; [12, pp. 82–83].

According to the picture contained in the water metaphor, a subject has no influence on what happens in the depths of its unconsciousness, i.e., what notions are created there. Thus, Schopenhauer's intention here is obviously to criticize the use of elements of language that refer to very general concepts. If by "concepts" he meant unconscious notions, his criticism would become nonsensical, as it would refer to something the criticized author had no impact upon.

Another example is found where Schopenhauer says that "the most special concept is almost the individual and thus almost real; and the most universal concept, e.g., Being (the infinitive of the copula) is scarcely anything but a word" [12, p. 64]. Again, by using the terms "word" and "concept" he seems to be referring to signs of language and he treats both of them as ontologically equal.

In summary, it seems that Schopenhauer uses the terms "word" and "concept" to refer to the three types of entities listed above in the following way:

- By the term "word" he sometimes refers to entities of type (i), i.e., sensory signs, and sometimes to entities of type (ii), i.e., mental signs,
- By the term "concept" he sometimes refers to entities of type (ii), i.e., mental signs, and sometimes to entities of type (iii), i.e., notions.

In other words, by "words" Schopenhauer might mean both sensory and mental signs for the subject's cognition. By "concepts" he might sometimes mean mental signs for such cognition, and sometimes the content of cognition, i.e., a notion that cannot be grasped clearly and voluntarily by the subject without a sign. Additionally, he differentiates the entities of type (ii) into mental signs with or without semantics, which he calls "mere words" and "concepts," respectively. This means that he might refer to entities of type (ii) as "words," and elsewhere talk of ontologically similar entities as "concepts," when he will intend to stress that *signs with semantics* are meant. And yet, elsewhere he will underline that they are completely different: "Yet the *concept* is entirely different not only from the word to which it is tied, but also from the perceptions from which it originates" [12, p. 63].

All this brings up a serious problem in discussing Schopenhauer's theory of meaning. His own terminology lacks a clear distinction between words and concepts. He sometimes uses these terms as synonyms, when referring to mental signs with semantics, but sometimes as antonyms, when distinguishing between signs and their semantic content.

Consequently, within the framework of Schopenhauer's own terminology, it is almost impossible to answer the question whether it is words or concepts that are the carriers of meaning, at least without a precise analysis of the specific context of each passage in which the terms "words" and "concepts" are used. Even though Schopenhauer introduces an apparently strict ontological difference between words and concepts (i.e., sensory and mental objects) it is lost in his further elaborations where he also seems to distinguish signs of language (both sensory and mental) and the extralingual notions they refer to using the same terminology. Significantly, none of these uses can be considered final.

4 Consequences of Ambivalence for Understanding Schopenhauer's Theory of Meaning

In Sect. 2 I showed that the classification of Schopenhauer's theory of language depends on what we determine to be the carrier of meaning in his theory of language: words or concepts. In Sect. 3 I demonstrated that the definitions of words and concepts are not strict. Consequently, settling on a definition of meaning within his theory becomes difficult.

These difficulties are iterated in Lemanski's above-mentioned article from 2016. Referring to Schopenhauer's *Lectures*, Lemanski points out that what Schopenhauer means when using the expression the "real value of words" ("wahren Werth der Wörter") is precisely the same as the "meaning" of the word, i.e., the concept signified by the word, which makes up the semantics of the word [5, p. 187]. On the other hand, a few sentences later, Lemanski clearly states that the distinction between words and concepts lies in the fact that *concepts* have meanings, whereas words are sensory symbols of concepts, just like digits are symbols of numbers [5, p. 187]. This happens again when Lemanski discusses the theory of foreign language acquisition and states that Schopenhauer describes this process as representing a semantically empty word from a foreign language with a meaningful concept of one's own language [5, pp. 187–188]. From this it might be concluded that, on the one hand, a concept makes up the semantics of a word (i.e., a word can have or not have meaning depending on whether it represents a concept or not) and, on the other hand, it is also possible for concepts to either *have* or *not have* meaning (when he is speaking of a *meaningful concept*, it seems that it is also possible for a concept to be meaning*less*). Of course the question also has to be answered whether the semantics of a word, i.e., a concept, is the same as its meaning. As has been shown in Sect. 3, Schopenhauer sometimes speaks of meaningless words in distinction to meaningful concepts. By "words" and "concepts" he would then mean the same type of mental objects present in the consciousness of the subject. In other words, in his terminology, a meaningless word is just an "empty word," whereas a meaningful word is called a "concept" (cf. [12, p. 71]).

Let us sum up the claims that result from reading Lemanski's account of Schopenhauer's theory of meaning:

(C1) words have meanings which consist of the concepts they represent,
(C2) concepts have meanings.

Claim (C1) can be found directly in Schopenhauer's quotation from the *Lectures* which Lemanski refers to:

> [...] only from the different context in which a word is found do we abstract its true meaning, do we find the concept which the word describes.
>
> ([...] erst aus dem verschiednen Zusammenhang in dem man das Wort findet abstrahirt man sich dessen wahre Bedeutung, findet den Begriff aus, den das Wort bezeichnet) [13, p. 246].

Claim (C2) is made by Lemanski based on the argument provided by Schopenhauer on p. 243 of the *Lectures* where he says that words are merely sensory symbols of concepts. Significantly, Schopenhauer himself does not clearly state that concepts have meaning (in fact he does not even use the term "meaning"/"Bedeutung" on this page).

This does not mean that Lemanski is misinterpreting Schopenhauer's intention. In other quotations Schopenhauer speaks directly about concepts, not words, as entities having meaning. For example: "I wish to trace back to their proper meaning these concepts of *good* and *bad*" [11, p. 395]. However, in this quotation, by the term "concept" he seems to be referring to a mental sign whose semantics he is about to examine. As shown above, in other passages he is referring to mental signs with semantics as "words." Putting it simply, Schopenhauer's intention here is to examine the true value of the sign "good," i.e., he wants to analyze the notion which the word "good" signifies. Several additional passages can be found where Schopenhauer discusses the meaning of concepts in a similar sense, i.e., he expresses his intention to analyze the semantics of a sign of language. In the following example the meaning of a concept is made explicit:

> It is true that, so far as the abstract representation, the concept, is concerned, we also obtained a knowledge of it according to its content [Gehalt], in so far as it has all content [Gehalt] and meaning [Bedeutung] only through its relation to the representation of perception, without which it would be worthless and empty [11, p. 95].

Here he refers to the semantics of a concept by a number of terms that seem to be synonymous: "content" ("Gehalt"), "meaning" ("Bedeutung"), and "relation to the representation of perception" ("Beziehung auf die anschauliche Vorstellung"). He also indicates that concepts might have value (as they might also be "worthless") and content (as they might be "empty"). The last expression is a clear reference to his axiomatic stance that concepts are "representations of representations," i.e., they are abstract/mental representations of other representations and eventually can be traced back to representations of perception. It therefore seems that, here, Schopenhauer is presenting the stance that concepts have meaning that consists of their connection to perception.

Based on the above selections, we might therefore distinguish two different theories of meaning formulated by Schopenhauer by enhancing the previous list:

(T1) words have meanings which consist of the concepts they refer to,
(T2) concepts have meanings which consist of the sensory objects they refer to.

These two theories of meaning are based on two fundamental assumptions about language made by Schopenhauer:

(A1) signs of language (either sensory or mental) signify mental states (notions),
(A2) mental states (notions) draw their content from the perceptions that are the foundation for them.

This is the core of the problem with determining which theory of meaning Schopenhauer is actually formulating. By the term "meaning" he sometimes means mental states (notions), and he sometimes means perceptions (elements of sensory reality).

Additionally, none of these theories of meaning refers strictly to what has been described here as notions, i.e., concepts understood as extralingual elements of unconscious mental processes. Rather, both theories result from the fact that Schopenhauer tends to interchangeably use the terms "words" and "concepts" when referring to signs of language. Simultaneously, the second theory is apparently formulated in reference to Schopenhauer's investigation of the nature of concepts understood as notions.

These two theories are connected with what was described in Sect. 2 as the representational and use theories of language. The first one anticipates the use theory: signs of language signify mental states, which are private, and therefore the only way to find out their meaning will be through the analysis of how they are used by the speaker. The second one anticipates the representational theory: all our notions are founded in perception and their meaning can be drawn only in reference to real objects.

It thus seems that depending on whether the term "words" or "concepts" is discussed, Schopenhauer presents one theory or another. In Sect. 6 I am going to show that he also puts them into a hierarchy. However, before this is done, the question of the meaning of concepts requires further investigation.

5 Meaning as Sense and Reference

In the preceding sections I tried to show that Schopenhauer uses the term "meaning" in reference to two different things: mental states, when he is talking about the meaning of the signs of language, and empirical objects, when he is talking about the meaning of notions. This ambivalent use of the term is connected with his blurred and inconsequential distinction between words and concepts.

However, it has to be underlined that in most cases Schopenhauer seems to avoid the term "meaning" when he speaks of "concepts." Instead, as shown in the quotation in the section above (cf. [11, p. 95]), he often uses the German term "Gehalt," translated by Payne as "content." When he gets more specific, he clearly distinguishes between the "extent" ("Umfang") and "content" ("Inhalt") of a concept, a distinction that is lost in the English translation in which both the German terms "Gehalt" and "Inhalt" are translated as "content." This can be seen in the following passage which we are somewhat familiar with from above:

> Further, since the content [Inhalt] and extent [Umfang] of concepts are in inverse relation to each other, and thus the more that is thought *under* a concept, the less is thought *in* it, concepts form a sequence, a hierarchy, from the most special to the most universal, at the lower end of which scholastic realism, and at the upper end nominalism, are almost right. For the most special concept is almost the individual and thus almost real; and the most universal concept, e.g., Being (the infinitive of the copula) is scarcely anything but a word [12, p. 64].

He also elaborates on this issue in the *Lectures* when discussing the use of spherical diagrams for representing concepts.[7] Here we gain further insight into what is meant by these two terms:

> The relative size of the spheres refers consequently not to the size of the content [Inhalt] of the concepts, but to the size of the extent [Umfang]: not the concept in which we think the most (the most qualities) has the broadest sphere, that is not the concept richest in thoughts; but the one through which we think most things: that is the one which is the quality of many things.
>
> (Die verhältnißmäßige Größe der Sphären bezieht sich also nicht auf die Größe des Inhalts der Begriffe, sondern auf die Größe des Umfangs: nicht der Begriff, in welchem wir das meiste (die meisten Eigenschaften) denken, hat die weitere Sphäre, also nicht der gedankenreichste Begriff; sondern der durch den wir die meisten Dinge denken: also der welcher eine Eigenschaft sehr vieler Dinge ist) [13, p. 271].

From these two passages we can see that Schopenhauer clearly distinguishes between the two following semantic capacities of a concept: (1) the capacity to refer to specific qualities and (2) the capacity to refer to empirical objects. He also says that these two capacities are indirectly proportional to each other, i.e., the more specific qualities a concept refers to, the fewer empirical objects it refers to. Or, conversely, the higher the number of empirical objects the concept can refer to, the lower the number of specific qualities of the objects it can refer to. This is strongly connected with Schopenhauer's stated claim that the most abstract concepts are the "emptiest and poorest," [14, p. 116] whereas "all truth and all wisdom ultimately lie in *perception*" [12, p. 74].

[7] For Schopenhauer's use of spherical diagrams, see Jens Lemanski's article *Concept Diagrams and the Context Principle* in this volume.

This distinction has much in common with Gottlob Frege's distinction of "sense" and "reference." Frege, talking about a sign defined as "name, combination of words, letter," understands its reference as "that to which the sign refers," and its sense as "the mode of [its] presentation." He then gives the famous example: "The reference of 'evening star' would be the same as that of 'morning star,' but not the sense" [3, p. 57]. In other words, he expresses the thought that although "evening star" and "morning star" refer to the same object, the use of each of these terms brings up different associations for the subject. Schopenhauer seems to understand the content ("Inhalt") of a concept as a bundle of features of an object that the subject has grasped and associates with the object. Frege says that the "sense of a proper name is grasped by everybody who is sufficiently familiar with the language or totality of designations to which it belongs; [...]" [3, pp. 57–58]. The similarity to Schopenhauer consists of the realization that even though we might think that a sign of language refers to specific objects, this reference does not give an account of its full meaning—which is also constituted by something else, something that depends on the subject using the sign.

A question remains regarding Schopenhauer's distinction of content and extent: which of these two, if any, should be understood as the "meaning" of a concept? It has already been remarked that Schopenhauer avoids using the term "meaning" when referring to concepts. In the above quotation he does not say that abstract concepts are meaningless but instead uses the figurative expression "emptiest and poorest." However, some passages can be found where Schopenhauer seems to imply that it is the content ("Inhalt") of a concept that constitutes its meaning: "concepts obtain all meaning ('Bedeutung'), all content ('Inhalt'), only from their reference to representations of perception, from which they have been abstracted, drawn off, in other words, formed by the dropping of everything inessential" [11, p. 474]. Here he seems to treat both terms as synonyms. Still, the possibility should be noted that in this passage Schopenhauer does not rigidly use the terminological differentiation of content ("Inhalt") and extent ("Umfang") he established in other passages.

Notwithstanding the uncertainty as to whether the content ("Inhalt") of a concept can be unhesitatingly identified with its meaning, it has to be underlined that, for Schopenhauer, both the extent ("Umfang") and the content ("Inhalt") of concepts are rooted in perception. Thus, his distinction is founded upon a representational theory—both the qualities a concept connotes as well as the objects it denotes are representations of something that can be found in empirical reality. Otherwise the concept is empty.

In addition, it has to be noted that when Schopenhauer mentions concepts and refers to spherical diagrams indicating their extension it is difficult to make out whether he means signs of language or notions. It appears that this is irrelevant for him since he claims that it is impossible to think of a specific notion without having a sign for it (cf. [13, p. 244], quoted in Sect. 3). What can be theoretically distinguished—sign and notion signified by it—is almost incomprehensible from the point of view of a phenomenological analysis of the content of one's mental processes—a perspective Schopenhauer assumes when analyzing conceptual thinking.

6 Use Theory of Meaning as Unnatural Way of Language Acquisition

Let us sum up the findings of this article. In Sect. 2 I showed that the classification of Schopenhauer's theory of language and meaning depends on whether we take words or concepts as the carriers of meaning. In Sect. 3 I showed that the distinction between words and concepts is not strict in Schopenhauer's philosophy and that under "concepts" he sometimes understands notions which cannot be considered elements of language. In Sect. 4 I showed that this ambivalence results in traces of both the representational theory of language and use theory of meaning appearing in Schopenhauer's theory of language. In Sect. 5 I also showed that within the scope of the representational theory of language Schopenhauer seems to differentiate between sense and reference. Finally, in this section I would like to show that even though the representational and use theories seem to co-occur in Schopenhauer's reflections on language, he himself clearly gives precedence to the representational theory.

As has been pointed out in Sect. 5, by the term "meaning" Schopenhauer sometimes means notion-like extralingual mental states of the subject that are represented by signs of language. Let us once again go back to Lemanski's article in which he points out that Schopenhauer anticipates the use theory of meaning in a quotation from his *Lectures* (cf. [5, pp. 185–190]). In this quotation Schopenhauer clearly states that the "true meaning" ("wahre Bedeutung") of a word, i.e., the "concept the word signifies" ("Begriff [...], den das Wort bezeichnet") can be abstracted from the "different context in which the word is found" ("aus dem verschiednen Zusammenhang[,] in dem man das Wort findet"). This is preceded by the statement that the "true value of words of a foreign language" ("den wahren Werth der Wörter einer fremden Sprache") is acquired "*ex usu*" [5, p. 186] (cf. [13, p. 246]).

From these statements Lemanski infers that Schopenhauer hereby formulates the main thought of the use theory of meaning, which later will be formulated by late Wittgenstein as follows: "the meaning of a word is its use in the language" [5, p. 190] (cf. [17, para. 43]). Eventually, analysis of the quotation from the *Lectures* and other statements of Schopenhauer regarding meaning (that we have already discussed above) leads him to the thesis that the use theory of meaning and the representational theory of language need not be understood as opposing semantic theories given the fact that they appear simultaneously in Schopenhauer's analysis of language and meaning [5, p. 190].

I hope to have shown in the preceding sections that the co-occurrence of these two semantic theories in Schopenhauer's writings is strongly stimulated by his terminology, which does not strictly separate signs of language from their meanings. Consequently, this leads to his formulation of two different theories of meaning: (T1) the meanings of words are mental states and (T2) the meanings of concepts are the real objects they represent. The first theory enables the formulation of the use theory of meaning, just as Lemanski has shown, by providing the assumption that meanings are actually some kind of subjective mental states that can be inferred by the subject from observing how other people use certain words. Conversely, theory (T2) is a representational theory of language since in it meaning is understood as objects represented by concepts.

Now I would like to enhance this by showing that Schopenhauer brings the two theories into a hierarchy and clearly states which one he considers more important, again with reference to the quotation from the *Lectures* discussed by Lemanski. In this passage Schopenhauer clearly speaks about the acquisition of a *foreign* language, i.e., he refers to a situation in which the subject has already developed its concepts and has generally acquired the ability to use them. This situation is quite different from that of a child learning his or her first language. The child does not know any language and starts conceptualizing the world and referring to it with language for the first time, i.e., he or she develops what is to become the meaning of the words by using his or her native language. Interestingly enough, Schopenhauer actually elaborates on this issue in his late work *Parerga und Paralipomena*, in para. 372 at the beginning of Chapter 28 entitled *On education*, in which he distinguishes between two ways of acquiring concepts by a child. He calls the first one "natural education" ("natürliche Erziehung") and describes it as the process by which a child comes in contact with empirical reality and as a consequence develops concepts that refer to facts from reality. He calls the second one "artificial education" ("künstliche Erziehung") and describes it as the process by which a child is taught a bundle of concepts (and by "concepts" he means signs of language) that are not and cannot be immediately brought into direct reference with the empirical world. This can occur only after the signs of language have already been acquired. It is presented as a lengthy and tiresome process which, until finished, enables a lot of misinterpretations, mistakes, and misunderstandings[8] [10, pp. 562–563].

It seems that in describing these two modes of language acquisition Schopenhauer actually refers to two semantic theories. Natural education is founded on the representational theory of meaning, whereas artificial education consists in acquiring concepts by means of the use theory. What Schopenhauer does here is in fact making a descriptive statement that

(D) concepts *are* acquired both by means of empirical interaction with the world and by the interaction with the language others use,

and making a normative statement that

(N) concepts *should* be acquired by means of empirical interaction with the world.

---

[8]A similar example already appears in the first volume of *The World as Will and Representation*, in the renowned para. 9. There, Schopenhauer discusses the concept of a town we might know only from geography that in fact might be applied to different real towns (cf. [11, p. 42]). Probably he intends to imply that a concept known only from geography is less specific than a concept developed by the subject's empirical examination of the object, but it has to be said that this example is difficult to interpret and the passage from *Parerga and Paralipomena* gives better insight into the nature of the problem.

Obviously, (N) implies that the representational theory *should* be applied, whereas (D) presents the use theory as the one that *is* used, but *should not*.

It seems that Schopenhauer is exploring artificial education according to (D). Given the additional fact that the subject learning a foreign language already has its own concepts, it needs to infer the concepts behind the foreign words (i.e., their meanings) from how they are used in the foreign language. During this process the subject also realizes that in the foreign language there are signs that signify notions different from those for which there are words in its native language. Thus, it is neither sufficient nor even possible to simply learn the equivalent terms from the foreign language by using a dictionary since the signs of different languages signify different notions (cf. [13, p. 246]).

However, keeping in mind the interpretation of concepts as extralingual notions, we might also conclude that Schopenhauer is referring to two completely different processes: gaining the capacity to use *any* language and learning a *second* language. The first process, which we might call the conceptualization of the world, is presented with reference to the representational theory, whereas the second one refers to the use theory.

7 Conclusions and Prospects

Schopenhauer's theory of language can be summed up with this sentence: signs of language represent private notions of subjects, which in turn represent real objects. This grounds the interpretation that it anticipates both the representational and use theory of language. Perhaps the most interesting thing to learn here is that these theories do not have to be understood as opposing. In reference to Schopenhauer's influence on Wittgenstein, this provides an additional argument in favor of Bryan Magee's stance that Schopenhauer's philosophy might be interpreted as the common framework for early and late Wittgenstein, which would support the theory of continuity in his philosophy (cf. [6, pp. 324f.]). At least we can say that both theories of language represented by Wittgenstein can be traced to Schopenhauer's observations on language—regardless of the fact that they might stem from Schopenhauer's ambivalent use of terminology.

Based on Schopenhauer's elaborations on education, we might conclude that language in the broader sense is a mode of representing the empirical world, but we acquire many of its elements by observing how other people use it; we infer the meanings of words from context and finally might refer them to reality. This way of acquiring signs of language is especially present when we are learning a foreign language. It seems that Schopenhauer realized the difference between the process of acquiring the ability to use language (i.e., establishing the connection between language and world and being able to express one's thoughts in language) and the process of learning a foreign language—where the ability to use language is already apparent, but new signs of language and constructions have to be acquired.

Let us now synthetically sum up the article's main findings.

1. Schopenhauer uses the terms "word" and "concept" ambivalently. In his investigation of meaning the anticipation of two different semantic theories can be traced: the representational theory of meaning and the use theory of language. The main reason for this is that he defines the meaning of words and concepts differently, and, at the same time, he occasionally uses the terms "word" and "concept" as synonyms.
2. These two semantic theories, although they are both present in his system, are not treated equally. He gives obvious and strong privilege to the representational theory of meaning. This can be seen in his claim that the entire content of concepts is derived from empirical experience as well as his normative claim that natural language acquisition should be rooted in empirical experience. The use theory of language is only discussed by Schopenhauer when he enumerates situations of unnatural language acquisition, such as when a child is forced to learn concepts it does not understand and when the phenomenon of foreign language acquisition is explained. Therefore, we might treat the use theory in Schopenhauer's writings as an *auxiliary* theory.
3. In certain contexts, Schopenhauer seems to use the term "concept" in the sense of something that cannot be considered a sign of language at all (an unconscious notion).

Finally, there are a number of questions that could not be investigated in this article and need further research.

1. Schopenhauer assumes that every person's use of language is based on a different conceptualization, since everyone gains their concepts from dealing with their own personal empirical experience, which, in fact, is *private* (cf. footnote 6, Sect. 2). This should consequently lead him to claim that the meaning/content of concepts is private. Instead, he seems to assume that each *language community* has a different conceptualization of the empirical world which can be realized when we learn a foreign language and see that the concepts behind its words are not equal with the concepts we hitherto used when speaking our native language. This means that he assumes that concepts have an intersubjective content within one language community. Thus, the question of whether and how he explains the possibility of this should be investigated.
2. In Sect. 5 it was shown that when discussing the content of concepts Schopenhauer distinguishes two dimensions of how concepts refer to reality, the content ("Inhalt") and the extent ("Umfang"). Also the claim has been made that these resemble the distinction of sense and reference by Frege. This claim needs further investigation regarding both the systematic similarities between these philosophical terms, as well as the possible influence of Schopenhauer's theory of language on Frege and his philosophical successors.

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Concept Diagrams and the Context Principle

Jens Lemanski

Abstract What is the primacy of logic? Concepts, judgments, or inferences? Whereas representationalists traditionally argue for a primacy of the conceptual, rationalists, referring to the context principle and the use theory of meaning, consider judgments and inferences to be primary. This dispute also seems to be applicable to logic diagrams: Whereas "Euler-type diagrams" are actually only for judgments and inferences, "concept diagrams" represent ontologies by using concepts. With reference to Schopenhauer, the paper develops a position called "rational representationalism." According to this point of view, the question of primacy is decided by analyzing the functions of the logic principles: For the explanation of logic and language, concepts are primary, but for understanding it is judgments. The mediation between intuitive representation and logical rationality is ensured by concept diagrams.

Keywords Knowledge representation · Ontology · Philosophy of language · Logic diagrams

Mathematics Subject Classification (2000) Primary 03A05; Secondary 68T30, 00A66

1 Introduction

Traditional logic is divided into three parts: Concepts, judgments, and inferences. This arrangement gives rise to the primacy question: what part of logic has priority, and why must logic begin with this primacy? Traditionally, logicians argue for a bottom-up structure. This begins atomistically with concepts and then composes judgments and finally joins together inferences. However, this structure was turned

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upside down in modern times. At the latest since Ludwig Wittgenstein's use theory of meaning, rationalists have used the context principle to justify a primacy of the propositional. This primacy results in a top-down structure: from a holistic point of view, judgments or inferences have priority, and words and their meanings are derived from their contextual use. Rationalists who advocate the top-down structure refer to authors such as Immanuel Kant, Gottlob Frege, and the late Wittgenstein as the founders of the primacy of judgment, the context principle, and the use theory of meaning. Representationalists who support the bottom-up structure refer to the traditional arrangement of the Aristotelian organ, but also to Kant, Frege, or modern model-theoretic semantics.

In what follows, I would like to argue that Arthur Schopenhauer's lectures on logic of the 1820s already presented concept diagrams, but also the context principle and the use theory of meaning. Thus, Schopenhauer's Euler-type diagrams are historical forerunners of two different theories: on the one hand, they anticipate modern concept diagrams and the primacy of concepts, and on the other hand, they anticipate the context principle, the use theory of meaning, and the primacy of the propositional. Thus Schopenhauer offers us a third position, in contrast to a form of representationalism, in which objects are represented naively by concepts, as well as a form of rationalism, in which judgments and inferences are overloaded if they alone have to explain the semantics of words and their relation to objects. In delimiting both the naive representationalism and the overloaded rationalism, the third position will be called "rational representationalism." It takes the view that representationalism is responsible for the explanation and that rationalism for the understanding of concepts. By dividing the functions of the logic principles, it is possible to provide a tertium quid between rationalism and representationalism.

Sections 2 and 3 take up the previously mentioned theses of rationalists and representationalists in detail. Sections 4 and 5, however, provide the main arguments of the paper. In Sect. 2 I will show the difference between the so-called Euler-type diagrams and concept diagrams (cf. [25, 26]). It will become apparent that Euler-type diagrams are closer to a rationalist approach, whereas concept diagrams are patently representationalist in character. In Sect. 3 I will refer this difference to the primacy question in logic. Whereas representationalists defend the primacy of the conceptual, rationalists urge the primacy of judgment. Section 4 deals with Schopenhauer's Euler-type diagrams, which he introduces in form of concept diagrams. It will be shown that Schopenhauer already distinguishes between the explanation and the understanding of concepts (in the sense of rational representationalism). Finally, in Sect. 5 I will present the position of rational representationalism, which is based on the theses of Schopenhauer.

2 Euler-Type Diagrams and Concept Diagrams

In this section, I will discuss the primacy question in connection with two related but different types of logic diagrams: Euler-type and concept diagrams. Section 2.1 deals with the fact that Euler himself was a representationalist, but his diagrams show an affinity to rationalism. This affinity is demonstrated by the fact that Euler's diagrams have traditionally been used to support the doctrines of judgments and inferences only. In contrast to this specific type of application, I will show in Sect. 2.2 that in recent years there have also been attempts to extend Euler-type diagrams to the area of conceptual logic and formal ontology. Here Euler-type diagrams are first used to represent concepts or classes and only then applied to propositions.

2.1 Euler and Euler Diagrams

If one chose to interpret the history of philosophy as a debate between the conflicting positions of representationalism and rationalism, then one has to assign Leonhard Euler to the former one: He particularly criticized the rational innatism as well as the monad theory of Gottfried Wilhelm Leibniz (cf. [14]) and advocated a theory of language and representation that followed John Locke. For Euler, concepts are not based on an innate semantics, but are abstractions of real intuitions. In this respect, Euler's popular presentation of logic begins with a treatise on concepts explaining how they are abstracted from intuition.

(E1) Though the impressions which occasion these sensations are made in the brain, they present, then, to the soul, a species of image similar to the object which the soul perceives, and which is called the sensible idea, because it is excited by the senses [. . .] [and] the foundation of all our attainments in knowledge. [8, p. 377; L 98]

(E2) [G]eneral notions, formed by abstraction, are the source of all our judgments and all our reasonings. [8, p. 394; L 102].

(E1) offers a representationalist theory, in which the senses give data to the brain or to the soul in the form of ideas, which are then processed into concepts. (E2) reflects on the ability of abstraction, which forms concepts on the basis of ideas. In other words, (E1) shows a primacy of the sensual experience in philosophy of language and (E2) a primacy of the conceptual in logic. We will see later (Sect. 4.1.3) that Euler's theory is consistent with Locke's representationalism that rationalists often dismiss as naive (cf. [28]). For rationalists, this naivety is already evident in the fact that representationalists usually do not even notice that their theory has always had to presuppose rational abilities in order to refer to intuitive representations or that concepts are already operative in intuitive representations. But from the representationalist point of view, rationalism carries the danger of overloading inferential rules and the use of judgment. For finally, it is only from the inferences and judgments that the meanings of the concepts are derived, which make it possible to refer to an object. Without a presupposed semantics or reference, even the criterion that declares a judgment to be true or false becomes problematic.

Despite his representationalism and the thesis of conceptual primacy given in (E2), Euler's famous diagrams only support the doctrines of judgment and inference. In the doctrine of judgment, he first establishes the four categorical propositions and clarifies these with the help of four diagrams, each of which forms a cross

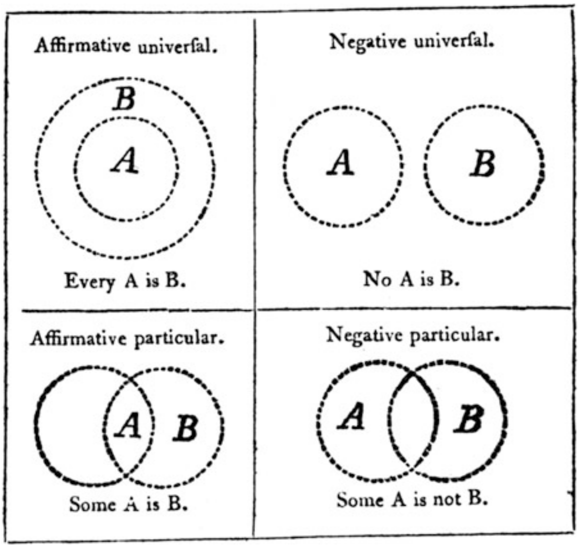

Fig. 1 Euler diagrams [8, p. 398]

classification in the order of the "square of opposition," as shown in Fig. 1 (cf. [1]): In the upper row are the universal, in the lower row the particular judgments. In the left column are the affirmative, in the right column are the negative judgments. Euler thus wants to show "how all propositions may be represented by figures" [8, L 103, p. 399]). Diagrams are therefore used to verify whether all judgments in an inference are illustrated correctly or not. Thus, Euler diagrams never display autonomous or isolated terms, but always concepts and classes in relation with other, i.e., within the context of a judgment or within the inferential relationship of judgments in syllogistics.

Euler's position in the history of logic is not without difficulties: On the one hand, he characterizes himself as a steady opponent of rationalists such as Leibniz as well as a follower of representationalist theories such as that of Locke. On the other hand, his diagrammatic method is restricted to the doctrine of judgments and inferences, which makes him, at first glance, compatible with rationalist theories that advocate the primacy of judgment and the context principle. Unintentionally, he is thus often included in the history of rationalism among authors ranging from Leibniz to Frege.

2.2 *Concept Diagrams*

Due to Euler's application of his diagrams to the doctrine of judgments and inferences, an ontological use of circle diagrams has not been established for a long time. In the nineteenth and twentieth century, logicians who wanted to adopt a diagrammatic theory of concept used tree diagrams known since ancient times, the so-called arbor porphyriana (cf. [10]). It was precisely this that strengthened the impression that Euler-type diagrams would only play a certain role in the doctrine of judgments and inferences.

Only recently computer scientists at the University of Brighton made a different use of Euler-type diagrams popular. By referring to Euler, they use circle diagrams in order to describe ontologies and hierarchies of concepts independent of an application in the doctrine of judgments and inferences. The aim was to develop the so-called concept diagrams, which played a similar role to tree diagrams also known in formal ontology. It quickly becomes apparent that many forms of tree diagrams can be translated into concept diagrams, but that circle diagrams have strong advantages [5].

In order to clarify this possibility of translation between circle and tree diagrams, I will use the following example of an ontology, which can be read from the first book of Schopenhauer's *The World as Will and Representation I* (cf. [19]). If one reads top-down, the ontology can be presented as follows: At first, "world" is the widest or highest concept. This concept is divided into the world as "will" and the world as "representation." "Representation" is a higher or wider concept in relation to "cognizance" and "reason." Both structure the two parts of the first book of *The World as Will and Representation I*. The concept "reason" is divided in three sub-classes: "language," "science," and "practical reasoning." And in the section on language, Schopenhauer deals with "logic" and "eristic" in §9. How "logic" is finally structured, I will discuss in Sect. 4.

In Fig. 2 one can see the user interface of an ontology editor called Protégé. A so-called indented list [5, p. 51] on the left side of Fig. 2 represents the hierarchy, taxonomy, or ontology of the main concepts of *The World as Will and Representation I*, which have just been mentioned. Since the end of the twentieth century, these indented lists are well known from several graphical operating systems for personal computers. For example, some files are in a folder and that folder is in a parent folder, and so on. If one expands this structure, it is usually

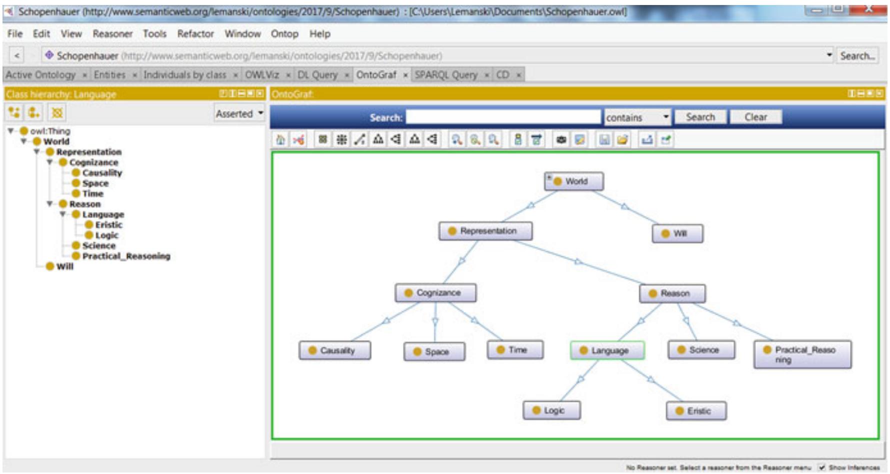

Fig. 2 Protégé [35]

displayed as an indented list in those graphical operating systems. On the right side one can see a visualization app of Protégé called OntoGraf: It should help the user to get an overview of the ontology with the help of a tree diagram.

Another way of representing the ontology are concept diagrams. These are supposed to improve visualizations such as those provided by OntoGraf. Concept diagrams can be seen by the visualization app “iCircles” (short for “Inductive Circles,” cf. [5, p. 55]), which can be embedded in Protégé. ICircles or concept diagrams (see Fig. 3) represent exactly the list that was previously visualized with tree diagrams in OntoGraf: *A* = world, *B* = representation, *C* = will, *D* = cognizan, *E* = reason, *F* = language, *G* = science, *H* = practical reasoning, *I* = logic, *J* = eristic. Thus, there is an isomorphism between circle diagrams (cd) and tree diagrams (td) which can also be expressed by the metaphors of containment (is_in) and subordination (is_under):

(cd) *B* & *C* is_in *A*; *D* & *E* is_in *B*; *H* & *F* & *G* is_in *E*, *I* & *J* is_in *F* and so on.
(td) *B* & *C* is_under *A*; *D* & *E* is_under *B*; *H* & *F* & *G* is_under *E*, *I* & *J* is_under *F* and so on.

Both types of diagrams indicate that there is a hierarchical structure of concepts that corresponds with the structure of objects in intuitive representations. And only on the basis of this structure, it can be decided whether judgments are true or false.

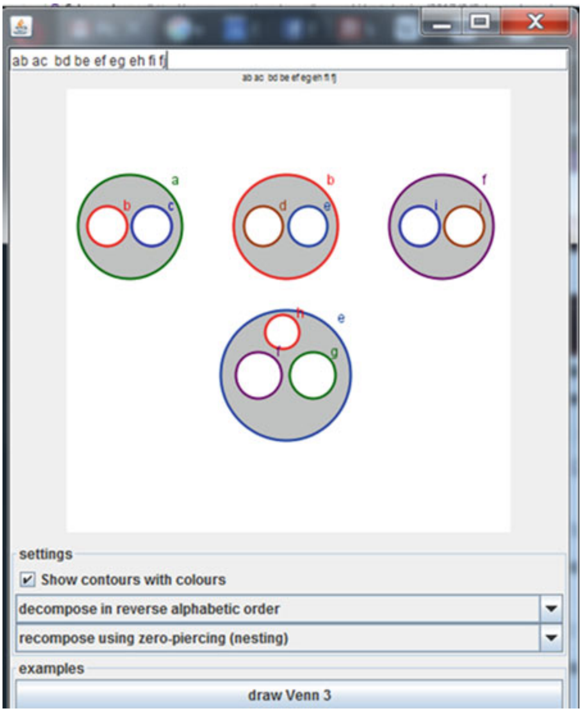

Fig. 3 iCircles

From the perspective of modern rationalists, concept diagrams thus fulfill the criteria of naive representationalism that Euler had in mind by claiming (E1) and (E2): Concept diagrams show the hierarchy of concepts that correlate with objects, and they show a process of abstraction that ranges, e.g., from I (logic) to A (world): For example, logic (I) correlates with our specific rational ability and is part of our language (F). The world (A) correlates with all mental (B) and material elements (C) and contains everything in itself (B–J), including logic (I). For rationalists this view is a naive form of philosophy.

If one asks about the origin of concept diagrams, ontologists often refer to Euler (cf. [7, 34]). However, there is a difference between Euler diagrams and concept diagrams, as stated above: Euler diagrams depict only judgments or inferences. They show the relation of concepts or classes within the context of a judgment, but not of isolated terms. Even if concept diagrams show relations of concepts and classes in a hierarchy, such as the relation given in (cd), these relations are only the basis for judging what true and false propositions are. Put simply, Euler-type diagrams signify the primacy of judgment and concept diagrams the primacy of the conceptual.

3 The Primacy Question in Logic

In Sect. 2 I have shown that although Euler takes a representationalist position, his diagrams satisfy the criteria of the rationalist position, namely the primacy of judgment or inference (Sect. 2.1). In contrast, for some years now there has been a representationalist approach that implicitly advocates a primacy of the conceptual, i.e., by using concept diagrams (Sect. 2.2).

In recent philosophical debates, the primacy question has been discussed without reference to diagrams. The fundamental question of philosophers interested in semantics is: how can we even talk about concepts? The prevailing response of philosophers to this question is a rationalist one, namely: only by using judgments one can talk about concepts. Since this answer represents a possibility of formulating the context principle and the use theory of meaning, it also reveals an argumentative structure: for rationalists, the context principle or the use theory of meaning are the reasons for asserting a primacy of judgments in logic and language philosophy.

I will discuss the primacy question in Sect. 3.1 without referring to logic diagrams. Rather, I will outline the history of holistic semantics often expressed by rationalists such as Robert Brandom, John MacFarlane, or Jim Mackenzie. According to them, Kant was the first philosopher to argue for the primacy of judgment, which was only then taken up again by Frege and Wittgenstein. Using some arguments provided by various researchers, Sect. 3.2 criticizes the Kant–Frege thesis of rationalism: Logicians and language philosophers have repeatedly shown that there were certainly approaches to rationalist semantics before Kant or also between Kant and Frege. This debate is not only relevant to the history of philosophy in order to place Schopenhauer's concept diagrams and his context principle (provided in Sect. 4) in the history of logic. Rather, the debate also shows that it may make sense to look at alternative historical approaches, as it is always a modern view with which we analyze philosophical texts of the last centuries.

3.1 The Kant–Frege Thesis

Whereas representationalists suggests that we structure the world primarily in concepts, rationalists draw our attention to the fact that we always have to use judgments in order to speak about concepts: A term has content if it can be used appropriately in a judgment. This position is held by rationalists such as Robert Brandom, John MacFarlane, or John Landy. I will illustrate this point of view with a few quotations. Brandom states:

> One of Kant's epoch-making insights, confirmed and secured for us also by Frege and Wittgenstein, is his recognition of the primacy of the propositional. The pre-Kantian tradition took it for granted that the proper order of semantic explanation begins with a doctrine of concepts or terms [. . .]." [2, p. 159]

Brandom's historiography is radical and very selective. It assumes the whole traditional (Aristotelian) logic before Kant, an atomistic and compositionalist approach. Furthermore, Brandom and Mackenzie [23, sect. 9] justify the primacy of judgment of Kant historically with the help of the context principle given by Frege. Brandom explains:

> One of [Kant's] cardinal innovations is the claim that the fundamental unit of awareness or cognition, the minimum graspable, is the judgment. For him, interpretations of something as classified or classifier make sense only as remarks about its role in judgment. In the Grundlagen Frege follows this Kantian line in insisting that "only in the context of a proposition [Satz] does a name have any meaning". [4, p. 79]

John MacFarlane also argues that Kant, among other things, needs the primacy of judgment and the categorical propositions, already explained in Sect. 2.1, in order to claim that a concept has content and can be applied to some possible objects of intuitive representation (cf. [22]). Concepts and classes are therefore the product of our logical activity and not their basis. A similar variant of this thesis is given by John Landy. For him, "concepts-qua-inferential-rules" organize Kant's picture theory of mental representation [17, sect. 2]. Only the inferential rules, in which concepts are embedded, allow a relationship between concepts and intuition.

Not only for traditional logic, but also for modern rationalists, the structure of philosophical logic consists of three parts: Concepts, judgments, and inferences. Brandom argues that, in traditional logic, everything starts from the bottom-up [2, pp. 80, 124, 159]: First of all, there are concepts, judgments are made up of concepts, and inferences are drawn from judgments. This structure is based on the traditional organization of the Aristotelian *Organon*: first the categories, then hermeneutics, and finally syllogisms. According to Brandom, the modern rationalist logic, however, begins either with inferences or with judgments. Judgments are derived from inferences and concepts are abstracted from judgments. This structure is thus top-down. Whereas traditional theory is atomistic because it starts with the smallest units, modern theory is holistic because it begins with whole and complex structures [2, p. 15]. We can also say that traditional theory is compositional because it is based on the composition of individual logical atoms, i.e., concepts; modern theory, on the other hand, is contextual because it always looks at contexts from which smaller units are derived.

With regard to the questions of how and when top-down contextualism has established itself, Brandom answers: Kant was the first and then Frege the next one:

> This insight into the fundamental character of judgment and so of judgeable contents is lost sight of by Kant's successors [. . .]. It is next taken up by Frege. [4, p. 80]

3.2 *Criticism of the Kant–Frege Thesis*

The question of whether Kant really wanted to establish a contextualist approach in his *Critique of Pure Reason* is difficult to decide. Numerous arguments for and against this were discussed in research. Especially the Kant–Frege thesis of Brandom and Mackenzie has been challenged. From a historical point of view, however, several researchers contradict the Kant–Frege thesis and cite authors before Kant or between Kant and Frege as evidence to the contrary. In the following, only a few researchers are listed who provide evidence for quotations regarding the use theory of meaning, the context principle, or the primacy of the propositional as arguments against the Kant–Frege thesis.

Michael Forster [9] maintains that Baruch de Spinoza, among others, has already emphasized the use theory of meaning:

> (MF) Verba ex solo usu certam habent significationem [. . .]. (Words get their exact meaning only from use [. . .].) [29, p. 146]

Theo M. Janssen [12] argues that Friedrich Schleiermacher in particular has already formulated a context principle:

> (TJ) the [. . .] meaning of a term is to be derived from the unity of the word-sphere and from the rules governing the presupposition of this unity. [30, M 50]

And Hans Sluga [33, p. 55] explained at the beginning of the 1980s that the primacy of the propositional can also be found in Hermann Lotze:

> (HS) [W]e can only say of concepts that they mean something, and they mean something because certain propositions are valid of them [. . .]. [21, p. 448]

In this respect, Brandom's historical notes are worthy of criticism. Valentin Pluder also showed in his contribution in the present volume that there was a debate on the primacy question in nineteenth century philosophy.

Of course, rationalists have objected to (MF), (TJ), and (HS) or they have simply tacitly adhered to Brandom's rationalist historiography (e.g., [23]). The main argument of rationalists is that one single quote makes not yet a whole theory. It would therefore be exaggerated to conclude from one counterexample found in (MF), (TJ), and (HS) that modern rationalism was anticipated.

At first glance, this counterargument of rationalists seems justified since there is no elaborated theory of contextualism in periods in question. At second glance, however, this counterargument is problematic. This can be illustrated by the context principle, for example. First of all, it must be noted that Frege and Wittgenstein have not elaborated a theory of contextualism: Wittgenstein formulates the context principle several times and in different ways, but the connection it has with a primacy of judgment or with the use theory has only become clear by interpretations of Wittgenstein's work. With Frege, the situation is even more problematic. Here the two contrary principles, namely the principle of compositionality and the context principle, are each described by interpreters as "Frege principle." How difficult it is to combine both approaches sensibly can be shown, for example, in the essay of Brandom entitled *Frege's Technical Concepts* [3, pp. 235–277].

I believe that these two points of criticism already show that it can make sense to deal with alternative historical approaches, especially when they offer us solutions to our current problems. At least that can be the case, as long as one considers that it is always a very modern view, with which one must look both at the texts of the early modern period and at the texts of early analytic philosophy. And this view is motivated by the fact that it is possible (with the help of alternative historical approaches) to develop a theory that is advantageous today.

4 Schopenhauer's Concept Diagrams and the Context Principle

Here, I come across a direct problem with the two theses hitherto mentioned: If Schopenhauer anticipates the representationalism of concept diagrams (as indicated in Sect. 2), then he cannot also anticipate the contextualism of modern rationalism (as indicated in Sect. 3) and vice versa. This really does appear to be a dilemma, but the problem can be solved by the fact that Schopenhauer allocates the primacy of the conceptual and the primacy of the propositional to different functions and tasks: When it comes to explaining where the semantics of concepts originates, Schopenhauer is a representationalist. But when he describes how we understand the semantics of terms, he is a rationalist. Hence, Schopenhauer tries to solve the primacy question by grasping both horns of the dilemma with different gloves. Instead of a naive representationalism or an overloaded rationalism, Schopenhauer opts for a way out of both extremes.

I will explain this solution in the following. Thereby, I only refer to Schopenhauer's *Berlin lectures*, which are published in the *Manuscripts Remains*. These

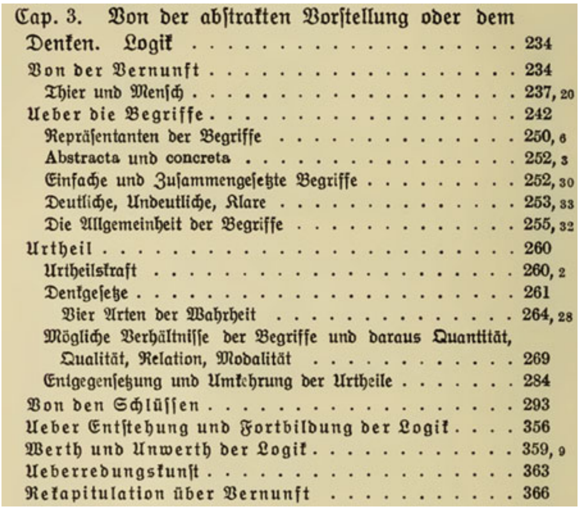

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Fig. 4 Table of contents of Schopenhauer's logic in [31]

lectures were elaborated in the 1820s, edited by Mockrauer for the first time in 1913, and re-printed in 1986. In what follows, I am referring to the Mockrauer edition of 1913 [31].

At first, I come back to the OntoGraf tree diagram in Fig. 2. The structure of the *Berlin lectures* is very similar to the structure of *The World as Will and Representation I*. This similarity in structure stems from the fact that Schopenhauer has taken *The World as Will and Presentation I* of 1819 as the basis for his lectures, but has developed academic topics such as logic much more strongly. Therefore Fig. 2 can also be used as an overview of the first book of his lectures. I will therefore only briefly repeat the structure, but this time bottom-up: Logic is contained in the chapter on language, language is included in the part on reason, reason is part of representation, and representation is a part of the world.

If we now ask how Schopenhauer's logic has been further structured in detail, we get an indication by looking into the table of contents (Fig. 4). This table indicates that Schopenhauer is working compositionally, atomistically, and bottom-up, starting with a primacy of the conceptual: He begins with concepts (this is the chapter "Ueber die Begriffe"), continues with judgments ("Urtheile"), and finally comes to inferences ("Von den Schlüssen"). The other paragraphs are mainly metalogical treatises (cf. Pluder and Beziau, in this volume).

The chapter I will mainly focus on is the chapter on concepts. Because of the bottom-up structure shown in Fig. 4 I will start with the representationalist approach in Sect. 4.1, which deals with an explanation of the origin, emergence, and development of concepts. However, since this is only one horn of the dilemma, this approach is supplemented in Sect. 4.2 by the rationalist use theory of language and the context principle, both of which deal with the understanding of concepts. In both sections we will see that Schopenhauer uses a very unique form of concept diagrams.

4.1 Representationalism and Explanation

In this section I will first deal with the question of what a concept is for Schopenhauer (4.1.1). After that I will present Schopenhauer's theory of concept formation and development and work on it in more detail (4.1.2). Finally, an interpretation of his representationalist theory will be offered (4.1.3).

4.1.1 Concepts and Spheres

In the chapter on concepts of his *Berlin lectures*, Schopenhauer deals among other things with the question of what concepts are. First of all, one must bear in mind that Schopenhauer distinguishes concepts from words. Concepts are meaningful words, or to put it another way: words are only the lexical characters of concepts. But what does the expression "concept" mean? How is it possible that words get a meaning and become concepts? First, Schopenhauer explains what a concept is. In doing so, he comes close to naive representationalism which we have already come to know as (E1) in Euler's words (see above, Sect. 2.1):

> [E]ach concept [...] has a sphere, a comprehension, i.e. several other, certain concepts, or at least many real objects, which therefore lie within its sphere, can be thought of with its help. The concept *conceives* [der Begriff *begreift*] several things: this is without doubt the origin of the name 'concept'. So the name is appropriate [...]. [31, p. 257]

The word "concept" comes etymologically from the Latin term "concipere." This means "to conceive," "to grasp," "to take up," "to contain," or "to comprehend" something. Also the German expression "Begriff" derives from the verb "begreifen," and, as Schopenhauer notes, the Greek expression "ὅρος" can also be understood metaphorically in the sense of "border," "termination," "definition," etc. As a result of this etymology, Schopenhauer argues that the expressions "concept," "Begriff," and "ὅρος," etc. are apt: Etymology indicates the property of all concepts, namely that a concept has a comprehension or a sphere. So a concept is something that grasps and restricts other concepts or a set of objects.

What it means to say that a concept has a sphere, a comprehension, or a boundary can be illustrated by an example given by Schopenhauer:

> We say, for example, pack animal contains all horses, camels, donkeys, and so on. [...] This is why such a general representation is called "concept", in contrast to the individual representation which is intuition [Anschauung]. [31, p. 257]

The example given by Schopenhauer aptly describes the function of the metaphor of the concept. Imagine we had the intuition of different horses, camels, donkeys, etc. Every intuition is an individual representation of something specific. In order to be able to talk about the set of these individual representations or objects in the way of abbreviated speech, we need a concept. If we cannot use concepts and classes, we would have to name (or to indicate) on every individual and every sub-class about which we want to make a statement. In order to avoid the enumeration of all individual representations or all sub-classes, we are summarizing them in one concept (cf. [31, pp. 252, 255]). This is done with the help of a mental "container schema" (cf. [16, p. 272]) called "sphere" or "circle." In our example, in which we presented various individuals from the donkey, camel, and horse classes, we include all within a sphere and then call the sphere itself "pack animal."

The concept is therefore a metaphor, because in order to understand what a concept does, one must use exactly what is not conceptual, i.e., intuition. This result in a strange, if not paradoxical situation: on the one hand, the concept is strictly separated from intuition, because intuition shows individuals and concrete representations, whereas concepts represent generality. In modern terms, intuition refers to elements, but concepts always to sets. This leads to different levels within the ontology: Some concepts are so concrete that they can almost be confused with intuition (cf. [31, p. 252]), other concepts are so abstract that one hardly knows what they cover and what not (cf. [31, p. 259]). But even if concepts are very concrete, they always remain separate from intuition. On the other hand, however, one needs intuitions in order to understand at all what concepts are (cf. [31, p. 251]) and what they can do, namely to grasp individuals or classes, to comprehend, to draw boundaries, etc. In short, the situation is paradoxical: We need intuitions to understand what concepts are and what they do, but also in order to understand that concepts are not intuitions.

4.1.2 Origin, Emergence, and Development of Concepts

But now the question arises, how does it come about that concepts work as they do. What is the origin of the conceptual and how have concepts evolved? Schopenhauer answers:

(S1) Because when concepts are formed, the reflection always proceeds in an abstract manner, i.e. from the properties of the visual objects, it only takes up certain of them and leaves others behind. But these other are just summarized again by another concept. Thus, from the same objects several concepts will be abstracted. [31, p. 257]

We can definitely see that here Schopenhauer is closely related to the abstraction theory that we already got to know as (E2) in Euler (see above, Sect. 2.1). Schopenhauer [31, pp. 257f.] uses two examples and two concept diagrams in order to explain exactly how we have to imagine this process of abstraction:

(S2) By ascending from an already formed concept (bird) to a wider one (animal), reason refrains from many properties and differences, merely in order to understand quite a lot by one concept (Fig. 5).

(S3) The concept 'green' originates from the intuition of a tree; furthermore, the concept 'flower-bearing'; the spheres of both are part of the concept 'tree' (Fig. 6).

(S2) refers to Fig. 5 which shows a circle diagram similar to Euler's diagram for affirmative universal propositions (cf. Fig. 1). Since Schopenhauer also refers to Euler and other authors with similar logic diagrams in the later course of his lectures

Fig. 5 "Bird" (Vogel) and "Animal" (Tier) in [31, p. 258]

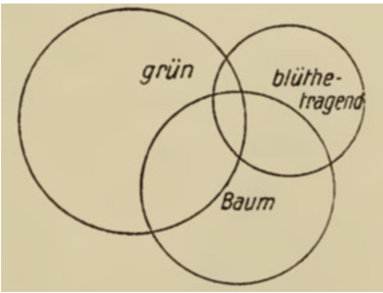


Fig. 6 "Green" (grün), "tree" (Baum), and "flower-bearing" (blüthetragend) in [31, p. 257]

on logic [31, p. 270], the diagram can also be described as an Euler or Euler-type diagram. In contrast to Euler-type diagrams, however, Schopenhauer does not use this diagram to represent propositions, e.g., a proposition with "bird" as subject and "animal" as predicate. Rather, he uses the two circles in Fig. 5 in order to explain how the concept "animal" originated from the concept "bird": The sphere of the concept "animal" was abstracted from the characteristics and differences of the concept "bird" and a new concept (i.e., "animal") emerged that contains more individuals than the first one.

(S3) refers to Fig. 6, which offers a diagram that can be interpreted almost as a Venn-type diagram (cf. [1, 25]). In a first glance, the different size of the circles seems to indicate a modern interpretation of Euler-type diagrams or to indicate a set diagram. But in my opinion Schopenhauer is not primarily concerned here with the proportions of the circles and also not primarily with the set of elements designated by the circle size. Rather, each circle displays a concept, and the overlap and non-overlap of several circles show, combinatorially, all possible sub-concepts or sub-classes that these concepts have. This is similar to Venn-type diagrams. Schopenhauer does not use these diagrams to represent propositions, however, but on the one hand to refer to objects and on the other to illustrate the development and emergence of concepts.

Let us first use the abbreviations G, T, and F for the three concepts "green," "being-a-tree," and "flower-bearing." Let us also use the diagram in Fig. 7, which is a Venn-type interpretation of Fig. 6. An object that is only green but cannot be characterized by the concept "flower-bearing" and "being-a-tree" is in the area of the left circle that has no intersection with any other circle. Thus, in Fig. 7, green is $G\overline{T}\,\overline{F}$. Something that is green and is a tree is $GT\overline{F}$. However, Schopenhauer does not want to use the diagram primarily to refer to objects of intuitive representations,

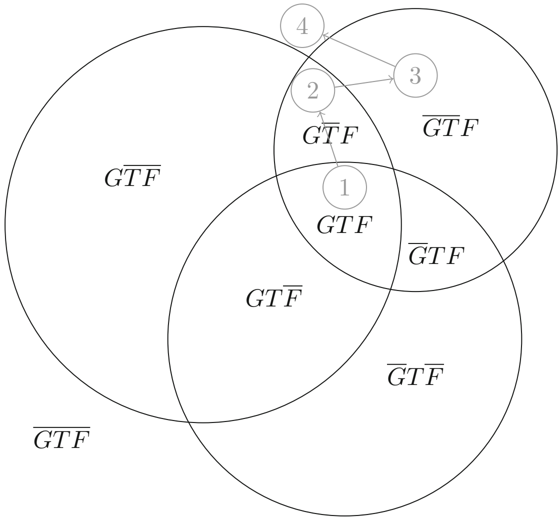


Fig. 7 Four steps of concept development

but to illustrate the development of concepts. To this end, we begin with the most concrete concept, which contains many properties and in which many concepts are combined. In the example of Fig. 6 or Fig. 7, it is the lens in the middle of the diagram where all circles intersect. The lens represents a concrete tree for which applies that it has the properties “green,” “flower-bearing,” and “being-a-tree,” i.e., GTF.

The origin and development of concepts described above in quote (S3) can now be presented in four steps as illustrated in Fig. 7: (1) The lens which represents a concrete object with at least three properties (i.e., GTF) is very close to intuition. At this junction to intuition the concept described by GTF is originated. Since this concept or sphere is composed of three conceptual properties, these properties can also be abstracted individually. (2) One can abstract “flower-bearing” from GTF and then arrive at area $GT\overline{F}$. (3) But if “green” is also abstracted from $GT\overline{F}$, the result would be $\overline{G}T\overline{F}$. (4) Finally, the negation of “being-a-tree” leads to $\overline{GTF}$, which is the area outside all circles. This area can also be identified by various words, but these no longer contain any known conceptual properties. These words denoting $\overline{GTF}$ are very abstract, e.g., such as “being,” “substance,” “the absolute,” etc. and it is no longer comprehensible what they exactly denote and represent and what they do not (cf. [31, p. 259] cf. also Xhignesse, in this volume).

Of course, the diagram given by Schopenhauer in Fig. 6 and my Venn-type interpretation in Fig. 7 are only one example for explaining the origin (1) and development (2–4) of concepts. For illustrating this exemplary route from (1) GTF to (4) $\overline{GTF}$ the four gray nodes in the Venn-type diagram of Fig. 7 are connected by arrows. All in all, one has to imagine the Venn-type method as much more complex, i.e., with many more concepts or circles, in order to really illustrate the emergence of concepts up to abstract concepts such as “being,” “substance,” and so on. Nevertheless, it should have become clear that and why Schopenhauer uses Euler- and Venn-type diagrams on the conceptual level. Furthermore, Schopenhauer offer a very special application of concept diagrams, for they not only show the ontological relationship of concepts but also the development of the whole ontology.

4.1.3 Interpretation

The question that now arises, however, is how this conceptual theory of Schopenhauer should be interpreted. I believe that there are two possible interpretations of this theory: either (1) subjective, singular, and ontogenetic or (2) intersubjective, collective, and phylogenetic.

(1) In the subjective, singular, or ontogenetic interpretation of Schopenhauer's conceptual theory, one would probably say that Schopenhauer is mainly influenced by Locke (cf. [6]). In Locke, we find two specific quotes that are usually provided to clarify a certain subjective semantics. These quotes from Locke read:

(L1) [W]ords, in their primary or immediate signification, stand for nothing but the ideas in the mind of him that uses them. [20, p. 347; III 2,2]

(L2) [I]deas become general by separating from them the circumstances of time and place and any other ideas that may tie them down to this or that particular existence. By means of such abstraction they are fitted to represent more than one individual. [20, p. 351; III 3,3]

(L1) illustrates the so-called main thesis of Locke's semantic theory, as Norman Kretzmann has pointed out [15]. This ideational theory says that words represent mental states, so words always have a subjective meaning at first. If these ideas refer to particular objects, as indicated in (L2), words represent objects and have been abstracted from the objectivity. The semantics of words emerges from the abstraction that everyone carries out himself. Semantics thus becomes an individual matter, a matter of the subject. We already became acquainted with a similar theory by Euler in Sect. 2.1.

According to the prevailing interpretation of Locke, (L1) and (L2) lead to nothing else than the private language argument of Wittgenstein. Put simply: if everyone in isolation abstracts concepts from the objects, the language that emerges would also be private. Schopenhauer's theory of concept diagrams can be precisely classified in this subjectivist, singular and ontogenetic interpretation, since a private language consists of words, of which only the speaker can know what they refer to (cf. [37, §243]). Or to put it another way: If in Schopenhauer every speaker carries out abstractions in isolation and autonomy, and if only one speaker knows which subclasses a term designates and which do not, then the theory of concept diagrams is subjectivistic, singular, and ontogenetic.

(2) However, there is also a second possibility of interpretation for which one can even cite textual evidence. First, Schopenhauer criticizes exactly the process of abstraction given by Locke [31, p. 253]. Furthermore, all quotations regarding the semantics of concepts, that can be found in the *Berlin lectures*, can also be

interpreted intersubjectively, collectively, or phylogenetically. I repeat again two quotes given above but with a different emphasis:

(S2) Because when concepts are formed, *the reflection* always proceeds in an abstract manner. . .

(S3) By ascending from an already formed concept (bird) to a wider one (animal), *reason* refrains from many properties. . .

It is not fully clear how to interpret these quotations. I suggest, however, that we should read it intersubjectively, collective and phylogenetically, as a theory of conceptual development that describes processes over many centuries and through many generations. This means not referring expressions such as "the reflection" and "reason" to a subject. Rather, we should read them as collective singulars. This means that it is not only one subject or a subject in isolation that makes this reflection, but that there are many speakers who repeatedly propose and discuss new concepts by abstracting individual relations of conceptual spheres. This is not a process that takes place within a lifetime or in the childhood of an individual speaker, but across all generations of the whole linguistic community. The process abstraction of one speaker thus becomes a proposal of abstraction within the community of speakers, and instead of the subjective, singular, and ontogenetic interpretation an intersubjective, collective, and phylogenetic is offered now. This interpretation is far from the private language argument and close to Wittgenstein's thesis that Allan Janik [11] and later John McDowell [24, p. 95] made known: "Commanding, questioning, recounting, chatting, are as much part of our natural history as walking, eating, drinking, playing" [37, §25].

The consequence of the second interpretation is that Schopenhauer escapes the naive representationalism that is attacked by rationalists. This way out of the dilemma between naive representationalism and an overloaded rationalism is made possible by limiting representationalism to the theory of the explanation of concepts and thus keeping the way open to limit rationalism to a theory of understanding.

4.2 Rationalism and Understanding

In Sect. 4.1 I have shown that Schopenhauer's logic and philosophy of language is underpinned by his intersubjective representationalism. Nevertheless, it will become clear that rationalism is indispensable for a theory of individual understanding. First, this argument is introduced by a theory of translation (see Sect. 4.2.1), which leads to the use theory of meaning and the context principle (Sect. 4.2.2). Finally, I will argue why we would be well advised to share such an interpretation of Schopenhauer.

4.2.1 Translating and Understanding Languages

In his chapter on concepts in the Berlin lectures, Schopenhauer not only explains how concepts have developed (as discussed in Sect. 4.1), but he also explains how we understand concepts. Schopenhauer does not illustrate this topic by using the example of childhood language development, which is known from the first paragraphs of Wittgenstein's *Philosophical Investigations*, for example, but rather by means of foreign language learning (which is more closely related to Willard Van Orman Quine's philosophy of meaning). In doing so, Schopenhauer makes a distinction between three types of translation for concepts. These three types show the possibilities that exist when translating from a source language into a target language. Interestingly, these three types are still used in modern translation studies, e.g., in Otto Kade (cf. [32, p. 52]). Schopenhauer explains each type of translation by using an example [31, p. 245]:

(1) **Total equivalence:** This type of translation claims that there are words in two languages that do not have different meanings. For example, 'tree' can be translated 1:1 with 'arbor' or 'δενδρον'.
(2) **Zero equivalence:** This type of translation claims that there can be no word in the target language that matches the word of the source language. Due to this 1:0 ratio, a loanword is introduced into the target language, such as for example: "Βαναυσος, Chaos, naiv".
(3) **Facultative equivalence:** This type of translation claims that there can be a word in a source language that does not correspond to exactly one word in the target language. Thus, there are several words in the target language that represent the concept of the original language, for example: "Frappant, auffallend, speciosum". We see this ratio in the diagram of Fig. 8.

Schopenhauer is now particularly interested in the third type, as it is the most common type of translation and it best illustrates the understanding and learning of concepts. But at first, in the more detailed study of total equivalence, Schopenhauer explains why lexical language learning or interlinear translation cannot work for all words and concepts:

> [I]n a lexicon, the word of one language is usually explained by several words of the other, none of which exactly corresponds to the word of the first language, but every word hits something apart, in all directions. But all together they denote the boundaries [of the circumference] between which the concept lies. [31, p. 245]

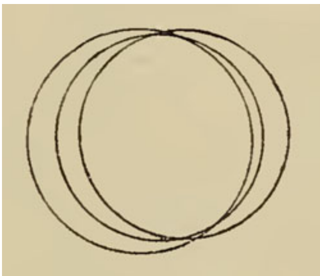


Fig. 8 "Frappant, auffallend, speciosum" in [31, p. 245]

Fig. 9 Honestum in [31, p. 246]

Total equivalence would be given if, for example, a lexem in a bilingual dictionary could be assigned to only one corresponding word of a target language, without there being any loss of meaning. But due to word-sense disambiguation and the problem of one word intersecting multiple concepts, total equivalence cannot be the only type used in the process of translation.

4.2.2 Schopenhauer's Use Theory of Meaning and His Context Principle

Due to the fact that total equivalence or lexical translation cannot be the only type of translation, Schopenhauer argues for a semantic theory of understanding which differs from representationalist theories (as described in Sect. 4.1). By referring to an example, he argues why a non-representationalist theory is needed:

> [T]ake the word *honestum*: its sphere is never hit concentrically by that of the word which any German word designates, such as *Tugendhaft*, *Ehrenvoll*, *anständig*, *ehrbar*, *geziemend* [i.e. virtuousness, honourable, decent, appropriate, glorious et al]: they all do not hit concentrically: but as shown [by Fig. 9]. [31, pp. 245f.]

Since total equivalence has to fail as the only translation option, Schopenhauer explains that we cannot learn languages 1:1 either (as representationalist theories may argue). Rather, instead of a total equivalence, there is usually a factual equivalence resulting from the phylogenetic abstraction theory discussed in Sect. 4.1. If one now refers the theory of translations to the question of how every subject learns languages, Schopenhauer comes to the conclusion that the principle of subjective language understanding cannot be learning by rote. Therefore a use theory of meaning is needed:

> That is why one learns not the true value of the words of a foreign language with the help of a lexicon, but only *ex usu* [by using], by reading in old languages and by speaking, staying in the country, by new languages: namely it is only from the various contexts in which the word is found that one abstracts its true meaning, finds the concept that the word designates. [31, p. 246]

This quotation shows that Schopenhauer limits the rationalistic semantics of the use theory of meaning and the context principle to a theory of understanding (languages). The quotation goes far beyond the text fragments we know from

Spinoza, Schleiermacher or Lotze, i.e., (MF), (TJ), and (HS) of Sect. 3.2. This becomes evident when one takes a closer look at the quote (for a detailed examination cf. [18]).

The first sentence expresses a use theory of meaning. This is already well-defined by the expression "ex usu" which refers to understanding languages in general. Schopenhauer explicitly states that the theory is about learning a language and the semantics of the individual words. The second part of the first sentence is a specification on how to learn exactly the meaning of a word. Here, Schopenhauer distinguishes between learning an old and a new language: The new language is learnt through active use on-site in interacting with native-speakers; old languages, on the other hand, can only be learnt by reading texts. So "by using" is concretized here by two possible activities: communicating with competent speakers and reading texts of competent writers.

The second sentence then begins with a justification of the use theory of meaning. It thus answers the question of why a use theory of meaning can claim validity at all. This justification is shown by the first word of the second sentence "namely" (nämlich). Here we find both the context principle and a primacy of the propositional (or the primacy of inference). The term "context" (Zusammenhang) is explicitly used here, and the fact that meaning can only be learned in connection with larger contexts indicates the universal validity of the context principle. The fact that the "true meaning" (wahre Bedeutung) is derived and abstracted from the larger context also indicates a holism. It remains unclear, however, what the context exactly is: Schopenhauer does not explicitly say whether the holistic context from which the meaning of the word is abstracted is a judgment or an inference. Nevertheless, it is obvious that the meaning of the word is taken from the context of a more complex structure, i.e., judgments or inferences.

4.2.3 Interpretation

The structure of the quotation is surprising: in the foreground is a use theory of meaning, which is underpinned by a context principle together with a holistic answer to the primacy question. The context principle and the primacy of judgment or inference are supported by the fact that there often is no total equivalence between words of two languages. Rather, there is a facultative equivalence that says that concepts are formed by the individual relationship of conceptual spheres and therefore the exact meaning of a word can only be understood by extracting or abstracting it from the context of a proposition or an inference. This subjective, singular, and ontogenetic abstraction is fundamentally different from the intersubjective, collective, and phylogenetic theory of abstraction given in Sect. 4.1. Here it is not explained how concepts came into being, but how the subject can understand them from the contextual use.

Critics have responded to my interpretation with various arguments (cf. e.g. [36]). The best of these arguments are (1) that the text passage on the use theory of meaning and the context principle is isolated in the entire work of Schopenhauer,

(2) that it does not conform to Wittgenstein's theory of early childhood language acquisition, (3) that Wittgenstein perhaps did not even know the text passage of Schopenhauer, and (4) that there are text passages to the contrary in Schopenhauer's complete works which are close to a representationalist theory of meaning. This may all be the case, and I am aware that my interpretation depends on decisions that do not need to be supported by other interpreters. But I believe that one has argumentative advantages in today's debate between rationalists and representationalists if one supports my interpretative decisions and develops from Schopenhauer a position that I would like to call "rational representationalism."

5 Rational Representationalism

In the previous Sect. 4 I developed two different perspectives on concept theory with the help of Schopenhauer's logic, which could be assigned to different functions: In Sect. 4.1, a representationalists perspective was adopted that dealt with the origin, emergence and development of concepts; in Sect. 4.2, arguments were made for a rationalist perspective, limited to the individual understanding of concepts.

In detail, Sect. 4.1. has shown that representationalism makes sense for an explanatory theory about the origin, emergence and development of concepts. As the structure of the *Berlin lectures* has already shown, Schopenhauer begins with the doctrine of concepts and builds up judgments and inferences (Fig. 4). This approach appears to be atomistic and compositional. Since words have always gained significance only in relation to representations, we must therefore speak of a representation-based theory of meaning. Unlike Locke's representationalism, however, Schopenhauer's theory does not have to be confronted with the private language argument. In contrast to Locke, Schopenhauer's representationalism is intersubjective, collective, and phylogenetic: concepts emerge through processes of abstraction from representations, but across many generations within the community of speakers.

Section 4.2 has shown that Schopenhauer's rationalism is made for a theory on the individual understanding of concepts: Speakers learn concepts by using judgments. For this reason, Schopenhauer also strengthens the primacy of the propositional and the context principle. Both are the reason for the use theory of meaning. In contrast to the phylogenetic process of Sect. 4.1, this theory is ontogenetic and therefore limited to the understanding of individual speakers.

From the point of view described in Sects. 1 and 2, both positions seem to be incompatible. Already the primacy question suggests that there may be only two alternative positions, tertium non datur: either a representationalist position in which concepts are primary, since they are the logical atoms from which compositional judgments and inferences are formed; or a rationalist position in which judgments or inferences are primary, since they provide the holistic context from which concepts are derived. With reference to Schopenhauer's logic, however, a third position opens up which unites the naive representationalism and the overloaded rationalism by assigning different roles to the two other: Rationalism and its semantic principles describe the understanding of concepts, representationalism and its intuitive and ontological basis explain the emergence of concepts. I would describe this third position as "rational representationalism."

What is striking here is that rational representationalism was developed with a constant reference to concept diagrams. For rational representationalism, concept diagrams or general logic diagrams play a decisive role: on the representationalist side, one can say that representations are illustrated by concepts (Sect. 4.1). On the rationalist side, one can say that concepts are again illustrated by representative diagrams (Sect. 4.2). In other words, individual representations are already operative in our conceptual ability, and vice versa. Intuitions and concepts are thus more closely intertwined than, for example, rationalists normally claim.

And rationalists usually have no trouble talking about something like the "concept of the concept." They make claims such as that

> for the concept of the concept CC it is valid (1) that CC itself is a concept C, (2) that all C is in CC, and (3) that (1) is true iff (2).

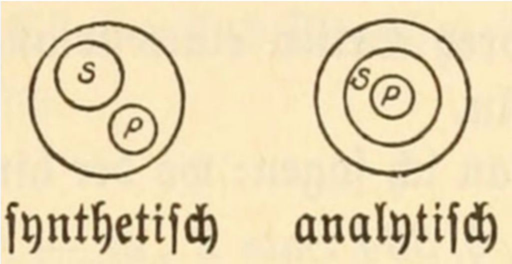
In making such claims, however, they ignore the fact that they use problematic metaphors that point out that the concept itself is not a concept, but a metaphor. Quine [27, p. 20] had already referred to the problematic metaphor of containment, which is also found in (2). Even substitutions of the problematic expression of (2) are again problematic metaphors themselves, e.g.,

> (2'): all C is_under CC.

The metaphors of containment, of substitution (cf. Sect. 2.2) as well as the metaphor of the concept itself point out that we need intuition to be able to explain what concepts are and do: We draw boundaries, circles, spheres, or the like and refer to them with words or sounds. These intuitive representations are already part of our rational ability, and we investigate them in the form of metaphors. However, the metaphor of containment only becomes a problem or a mysterious expression of rationalism if representationalism is narrowed down too much. The vicious circle—that we need concepts in order to agree on what happens to intuitions when concepts arise and are used—can, however, be broken up with the instruments of rational representationalism. The concept diagrams used by Schopenhauer are representations of intuitions. They show how concepts and intuitions relate to each other. Thus, they do exactly what the metaphors that seem mysterious to rationalists do.

Philosophers, who either only support rationalism or representationalism, overlook the connection between language and representations illustrated by concept diagrams. As an example, one can again take Quine's criticism of analytical judgments, since he claimed that "the notion of containment is left at a metaphorical level" [27, p. 20]. Schopenhauer and rational representationalist would contradict this and say that these expressions only left at a metaphorical level if they are not interpreted in a representationalist way. This way is achieved with the help of logic diagrams. Analytical judgments say that the subject is necessarily contained in the

Fig. 10 Analytic and synthetic judgments [13, p. 717]



predicate, and the metaphor of containment is explained by the relationship between subject and predicate in the diagram.

We can see in a quote from Kant's logic manuscripts that he also explains the metaphor of containment in analytical judgments by means of diagrams. He can therefore also be regarded as a precursor of rational representationalism, particularly since Schopenhauer developed the same strategy independently of him.

> The logical relation of all concepts is that one of them is contained under the *sphaera notionis* of the other: [. . .] The metaphysical relation consists of whether one [concept] is synthetically or analytically connected to the other: (Fig. 10). [13, p. 717; Refl. 3216]

In Fig. 10 we see that in analytic judgments, subject (S) and predicate (P) are only connected by the copula, which is represented by the outer circle. In synthetic judgments, the predicate is completely contained in the subject, so that the copula plays no ampliative role in the diagram. Schopenhauer could not know this quotation or this diagram from Kant's *Manuscripts Remains*. Nevertheless, he follows the same strategy to explain the analytic-synthetic distinction in his own lectures on logic. Based on his concept diagrams, Schopenhauer uses Euler-type diagrams in his theory of judgment in order to explain the difference between analytic and synthetic judgments (cf. [31, p. 270]): For Schopenhauer, analytic judgments have the form of universal-affirmative Euler-type diagrams, synthetic judgments the form of particular-affirmative ones (cf. Fig. 1): The analytic judgment "Gold is yellow" actually means "All gold is_in yellow" and is represented by the diagram for "Every A is B" in Fig. 1. But if one disagree with the classification that "Gold is yellow" is an analytic judgment (cf. [31, p. 123]), one probably refers "Gold is yellow" to the Euler diagram for affirmative-particular judgments, namely "Some A is B" and thus explains synthetic judgments. In both cases, the metaphor of containment loses its mysterious character—which it has among the pure concepts in language—with the help of the diagram. Similar to the metaphor, the diagram thus becomes a mediator between representation and rationality, between world and logic.

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A Comment on Lemanski's "Concept Diagrams and the Context Principle"

Gunnar Schumann

Abstract In this paper I make some critical comments on Jens Lemanski's article "Concept Diagrams and the Context Principle," mainly on his "rational representationalism." This is a view concerning the question whether it is either concepts, judgements, or inferences that may count as the primary element of a "logic." It suggests that concepts are considered to be primary for the explanation of linguistic *meaning*, whereas judgements are considered to be primary regarding our *understanding* of language. Criticism is put forward on the issues whether Schopenhauer proposed a "phylogenetic abstraction theory" (as Lemanski puts it) and a use theory of meaning. Also, a critique of the general idea of rational representationalism that Schopenhauerian concept diagrams can play the role of mediation between intuitive representation and rationality is developed.

Keywords Wittgenstein · Meaning · Word · Concept · Semantic theory · Representational theory of language · Use theory of meaning

Mathematics Subject Classification (2000) Primary 03B65, Secondary 03A05

In an article to this volume, Jens Lemanski argues for what he calls "rational representationalism" [5]. This is a view concerning the question whether it is either concepts, judgements, or inferences that may count as the primary element of a "logic." "Logic" here is to be taken in its historical meaning (especially prominent in German Idealism) as the philosophical analysis of the cognitive faculty of man, i.e. in its old meaning as epistemology, the task of which is the study of the structure of the representation of the world in the mind. This traditional debate began at the latest with Locke's theory of mental representation and concept formation. According to Locke, when we perceive an external object, it has an effect on our mind, such that a so-called *impression* is formed in our mind which is a mental representation of

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an individual thing. From several of these, the mind actively can create a *complex idea* in various ways [6, Bk. II, ch. II, VIII, XII]. One of them is *abstraction* by which general concepts of things are formed: Reason abstracts the particularizing properties (esp. those of time and location) from representations of individual things. Thus, by impressions of several individual red things (e.g., ripe tomatoes) and the workings of the mind, a subject attains the general concept (e.g., red) [6, Bk. III, ch. III]. Locke is also the first to turn to the question of how words attain their meanings. According to his famous ideational theory of meaning words stand for ideas in the mind of the speaker. In Locke's semantics, sentences are made up from words, these in turn refer to ideas in the speakers mind. General expressions refer to complex ideas, namely general ideas which are ultimately formed from impressions of individual things. Although Locke was not explicitly concerned with the problem of "primacy of logic" it is clear that his analysis of human cognition can be seen as a paradigm example of a "bottom up"-theory: The representation of the world in the human mind starts with ideas of individual things (*impressions*) of which general concepts are formed by reason, and sentences are made up of words that refer to complex or simple ideas.

According to Robert Brandom, it was a dedicated innovation by Immanuel Kant that reason can be understood as the faculty of judging and that all of reason's acts can be reduced to judgements. Hence, concepts gain their semantical content only by the fact that they can be used in propositions [1, ch. 2.2.1]. Thus, according to Brandom, Kant was the first to recognize the epistemological and semantic primacy of propositions over concepts. To speak of something as a general concept that subsumes several individual things under it only makes sense by referring to their role in judgements. Brandom presents Kant as the father of a "modern rationalist logic" according to which logic resp. the mental representation of the world begins with either judgements or inferences. Judgements would be gained from inferences and concepts would be derived from judgements. For Brandom, this form of contextualism is continued in Frege's context principle that states that concepts only have meaning in the context of the sentence they are embedded in [1, ch. 2.2.2]. As Brandom argues, this thesis about the primacy of the propositional can also be found in the statements from Wittgenstein's *Philosophical Investigations* that sentences are the only moves in a language game and that naming is just a preparation for the use of a word [1, ch. 2.2.2]. This would turn the traditional order in logic ("bottom up") into one of "top down." Traditional "logic" is compositional, whereas modern logic is contextual, for logical atoms are derived at from more complex units. (Lemanski calls the first view "representationalist," the second "rationalist" [5, pp. 47f.].

One of Lemanski's aims, if not the most important, is, if I understand aright, that concepts of logic like entailment, mutual exclusion, negation, etc. *stem from* (in a sense to be specified) intuition and thus ultimately from sensual experience of worldly objects. Ultimately we possess logical concepts because we can allegedly see logical relations in pictorial representations, foremost logical diagrams (cf. below). This idea is refreshing and interesting and certainly not to be done away with immediately. It should certainly not be forbidden by any philosophical theory of perception. From an Ordinary Language Philosophy standpoint (which I endorse) it is welcome to assign to the faculty of perception more tasks than have been traditionally assigned to it. It is a myth of a great deal of the twentieth century analytical metaphysics that we only can *infer* from sense data but not *perceive*, e.g. the sadness or the anger of a person, the whimsicalness of a smiling child, or that we could not *perceive* that it was the wave that knocked us over or that it was the car that scrunched the tin. Analogously, it makes good sense to speak of that one can see, e.g., a mathematical result or a logical relation by pointing towards a blackboard with written calculations or a diagrammatic illustration. But from this we must be careful not to jump to unjustified conclusions regarding our grasp of logical concepts.

In the following, I make some comments on the issue and Lemanski's line of thought.

(A) It should be made more clear what the question about the primacy in logic in fact amounts to. What is dependent on it? Is the question about the primacy not in fact a really artificial, philosophical one? Who cares about primacy and why is this an issue at all? It should also be made more clear in what regard the primacy question is asked. Is it the structure of our knowledge of the world, i.e. such that the primacy question is an epistemological one? Is our knowledge about the world a complex structure which in fact is composed from primordial epistemic units, as, e.g., Locke would have it? Or is it to be taken as a so-called ontological thesis about the structure of the world itself? The task then would be to find out what the ontological atoms of the world are and whether we find them by analyzing language logically into its semantical components, for there be an isomorphism between world and language. This resembles the program of Russell's Logical Atomism.

These foundation-compositionalist theories of meaning have been objected by (late) Wittgenstein in a very general and fundamental manner [3, ch. 5.2]. One of the strongest counter-arguments applicable both to Locke and Russell is that there is no one-to-one link between words and things which is allegedly established by ostensive definition. It does not suffice to point towards a ripe tomato and while doing so to say "red" in order to determine the meaning of the word "red." The meaning of "red" is not affixed to the word by that procedure for it is not yet clear which characteristic of the many simultaneous characteristics of that ripe tomato is relevant. The learner of the meaning of "red" also needs to know to what logical category the unknown word belongs (i.e., the *color* of the ripe tomato) but the category (to what is pointed to) already belongs to language. The teaching of the meaning of a word involves the teaching of *the way it is used* [8, §§ 28–31].

(B) Late Wittgenstein can hardly be taken to be a proponent of rationalism, as Brandom would have it. It is true that Wittgenstein repudiates the naive idea that all words gain their meaning by referring to objects. And it is also true that Wittgenstein holds that referring is no simple and mysterious relation between words and things. Rather, that words sometimes can be said to refer to things presupposes that they also can be put to use in other language games. But from this it does not follow that Wittgenstein holds that single words gain their meaning only by being embedded in whole sentences. The usage of a word, as Wittgenstein speaks of it, is not necessarily

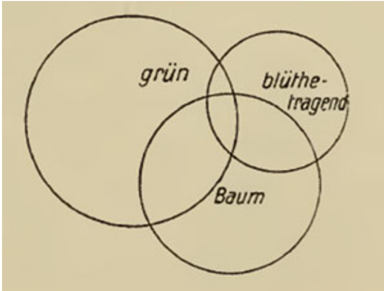

Fig. 1 "Green" (grün), "tree" (Baum), and "flower-bearing" (blüthetragend) in [7, p. 257]

the same as being embedded in a sentence or longer syntactic linguistic structure but he wants to be embedded in practical activities of men, in their form of life. Wittgenstein's claim that words gain their meaning by being embedded in a context of use is not synonymous to the claim that words gain their meaning only in the context of a whole sentence. Brandom thus distorts what Wittgenstein wants to say in the *Investigations* in order to present him as a predecessor of Brandom's own semantical theory, his so-called *Inferentialism*.

(C) Lemanski develops his "rational representationalism" on the basis of some considerations of Arthur Schopenhauer. "Rational representationalism" is the view that concepts are considered to be primary for the explanation of linguistic *meaning*, whereas judgements are considered to be primary regarding our *understanding* of language [5, p. 56].

According to Lemanski, Schopenhauerian concept diagrams play the role of mediation between intuitive representation and rationality. Schopenhauer explicates abstraction in a similar vein to Locke: Concepts are formed by abstraction from intuitions of observable objects, only some of the properties of the objects are kept and others are dropped. Schopenhauer adds that more than one concept can be abstracted from the same observed object. The less properties are kept, the more general a concept becomes. The more general term "animal" thus emerges from a more specific one "bird." In Schopenhauer's lectures on the issue, this is represented by conceptual diagrams with two concentrical circles with one having a small diameter ("bird") and the other having a larger one ("animal"). As Lemanski says, Schopenhauer's use of the diagrams is such that he does not simply want to illustrate conceptual relations between concepts of different degree of generality but he wants to illustrate how concepts are formed and acquired [5, p. 59f]. Lemanski calls this an "abstraction theory" [5, p. 65], but this can hardly be called a "theory," for it merely explicates the common meaning of "abstraction."

According to Lemanski, Schopenhauer gives another way of illustrating the origin of concepts: From one concrete particular object, given in intuition, several concepts can be abstracted, e.g. from the idea of an individual tree the three different concepts "green" ("grün"), "flower-bearing" ("blütetragend"), and "tree" ("Baum"). Schopenhauer again uses a diagram (Fig. 1).

Consider the intersection area of all three concepts. Lemanski says this lens "represents a concrete tree" [5, p. 61]. From the idea of a particular, or at least highly specific concept, all the other involved concepts are developed by abstracting one property after another from that highly specific idea. "Abstraction" is taken by Lemanski to mean "negation."

> The origin and development of concepts described above in quote (S3) can now be presented in four steps as illustrated in Fig. 2: (1) The lens which represents a concrete object with at least three properties (i.e., GTF) is very close to intuition. At this junction to intuition the concept described by GTF is originated. Since this concept or sphere is composed of three conceptual properties, these properties can also be abstracted individually. (2) One can abstract "flower-bearing" from GTF and then arrive at area GT¬F. (3) But if "green" is also abstracted from GT¬F, the result would be ¬GT¬F. (4) Finally, the negation of "being-a-tree" leads to ¬G¬T¬F, which is the area outside all circles. This area can also be identified by various words, but these no longer contain any known conceptual properties. These words denoting ¬G¬T¬F are very abstract, e.g. such as "being," "substance," "the absolute," etc. and it is no longer comprehensible what they exactly denote and represent and what they do not [5, pp. 61]

I think there are several problems with this "abstraction theory":

1. The intersection figure GTF does not represent an individual object. It is the result from the intersection of the representations of the extension of *concepts*, not *intuitions*. Concepts are linguistically represented by general expressions. (In modern logical phrasing: Concepts are linguistically represented by open sentences of the form "x is green/flower-bearing/a tree." The extension of a concept comprises all things which when inserted in an open sentence yield a true sentence.) Now, the intersection area represents the more specific concept of all things being green, flower-bearing trees—and that is still a concept, no

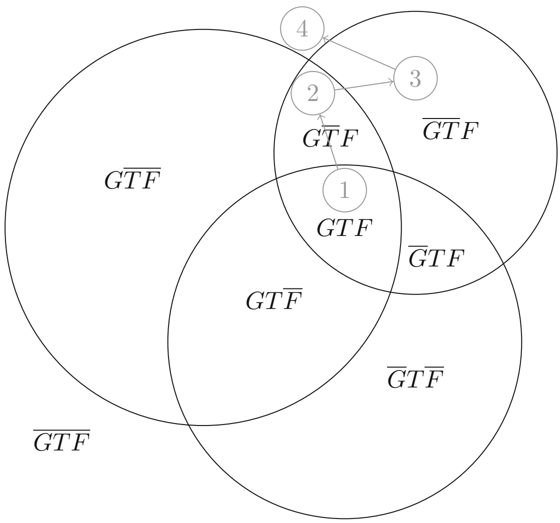

Fig. 2 Lemanski's "four steps of concept development", in [5, p. 61]

concrete individual[1] (—even if the extension of that concept would consist of only one element or even no element). Schopenhauer actually stresses the point that concepts cannot become intuitions (= ideas of particular things) and that both types of ideas have to be kept distinct. (Concepts do not originate continuously from intuitions by comparison between different intuitions but are a distinct faculty of reasoning ("Reflexion"), as he says. He also just says that from the intuition of a tree the concepts green and flower-bearing *arise* ("entsteht"), not that concrete specific concepts simply *are* intuitions.[2] Lemanski seems to acknowledge the point, for a moment later he speaks of the intersection GTF as representing a concept [5, p. 61].

2. In the passage before the quoted one above, Lemanski also claims that the concept of green is represented by the remaining area of the circle "green" minus the intersecting areas of circle "flower-bearing " and "tree."[3] But this is obviously self-contradicting, for the whole circle "green" is to represent the extension of the concept "green," not just the subtraction figure "G¬T¬F." (Green trees and green flower-bearers belong to the set of green things, too, and green flowering trees are still green trees.) It is also a misunderstanding of Lemanski to say that abstract concepts like "being" or "substance" denote the area outside of all three circles. For the abstract concepts "being" or "substance" comprise all three concepts and do not denote exclusively what is neither green nor flower-bearing nor a tree. Schopenhauer's main point in this section actually is to stress that there is a reverse proportionality between conceptual extension and intension and he uses this to criticize Schelling and his followers, for they had a "fetish" with very abstract concepts like "being," "substance," and "the absolute" (cf. [9]).
3. It is incorrect to think of an "abstraction process" as steps of consecutive negations. Schopenhauer's point is that from one intuition of an object several concepts can be abstracted, yes, but this feat is not done by consecutively *negating* the several properties of that intuited object. The negation of a concept is a logical operation which presupposes the possession of that very concept; by negation no new concept is gained but only the contradictory "formed." "To abstract" means to refrain from taking something into consideration, not to generate its contradictory—which would be to take it still into consideration and exercising a certain logical operation on it.

[1] The lens is the logical conjunction: "x is a tree and x is green and x is flower-bearing" which still is a general term, not a singular term.

[2] "Wenn ein einzelnes individuelles Ding vorgestellt wird; so ist diese Vorstellung immer eine Anschauung. Anschauungen sind immer einzelne Vorstellungen; Begriffe sind stets allgemein, d.h.es können mehrere Dinge durch sie gedacht werden." ("When a single individual thing is represented this idea is always an intuition. Intuitions are always individual ideas, concepts are always generic, i.e. several things can be thought by them.") [7, p. 255]. Therefore, Lemanski is not precise in formulating: "Every intuition is an individual representation of something specific" [5, p. 58]. It should rather read: "Every intuition is a representation of something individual."

[3] [5, p. 60]: "Thus, in Fig. 7, green is G¬T¬F. Something that is green and is a tree is GT¬F."

4. And what is illustrated by Fig. 2 cannot be called "steps of concept development." Schopenhauer does not put the diagrams in the passages quoted by Lemanski to such a use. By the diagram and his remarks he does not describe some sort of mental process that takes intuitions as its raw material and creates very abstract concepts. Even if one could find textual evidence for this claim, what is represented in these diagrams hardly can count as an description, let alone an explanation of "the origin and development of concepts" (p. 61)—whatever precisely is meant by that phrase. Nothing in Schopenhauer's lectures indicates that the diagrams represent something else than two different ways of abstracting from specific to more general concepts. In fact, Schopenhauer's point here is a piece of pretty straightforward logic: When someone has grasped the concept of a green, flower-bearing tree, then one has also grasped the three concepts green, flower-bearing and tree which have an intersecting area.

Lemanski then goes on to construe this "conceptual theory" of Schopenhauer, which sounds very Lockean so far (at least the first part), as actually opposed to Locke's position: Allegedly according to Schopenhauer, it is not "reflection" or "reason" that do the abstracting work in concept formation and it is also not a human being in the early days of her concept acquisition and language learning. Lemanski proposes to read "reflection" and "reason" in Schopenhauer's text as collective singulars. The "abstraction process" actually be a long-term and collective process of a whole linguistic community, continuously performed by several speakers over years and generations. Thus, Schopenhauer actually would be a proponent of a "phylogenetic abstraction theory."[4]

There are a lot of problems with this interpretation:

1. There is absolutely no textual evidence for that Schopenhauer actually took "reflection" and "reason" to be collective singulars referring to a speaker community. This would be such a fundamental shift of meaning of "reflection" and "reason" that this reading could hardly count as intelligible. Which philosopher in his right mind would use the words "reflection" and "reason" when in fact he wanted to talk about whole language communities—and then go on without indicating explicitly that "reflection" and "reason" would have to understood in such an extraordinary and artificial way? Well, Hegel used "objective spirit" as a word for hypostatized historical forms of morality, law and the state. But for Lemanski's interpretation there is no single clue in Schopenhauer's text.
2. It is not at all clear why Schopenhauer would propose such a theory. Lemanski's interpretation seems ad hoc for, if I see things right, it tries to save Schopenhauer's "theory of concept formation" from late Wittgenstein's private language argument. Lemanski takes the private language argument apply to Locke's theory of concept formation. But of course it does not. Wittgenstein's argumentation is

[4]It is unclear why Lemanski would call this a "*phylogenetic* theory", for he speaks of a process across several *generations* of speakers, not of one across several biological *species* on an evolutionary scale.

an objection against the semantical thesis, held by Locke and others, that words gain their meaning by reference to ideas in the minds of the speakers [4, ch. 5.]. As far as I know, Schopenhauer did not propose a theory of linguistic meaning at all. Thus, Schopenhauer would have to be taken to change the topic from the question how reason gains general concepts from intuitions to the question of how to account for the meanings of words and the threat of a private language objection (which was not familiar to any philosopher before late Wittgenstein) to something elaborate as an use theory of meaning *in nuce*—which seems nothing but absurd.

3. Again, why would Schopenhauer propose such a theory? Admittedly, the talk of "reason" or "reflection" abstracting general concepts from intuition is a metaphorical way of speaking. And it is best to not let oneself lead astray by the extensive use of this activity-talk of the inner working of the mind as in Locke, Hume, Kant, and the German Idealists. It is in fact *persons* and only that which is alike to them that can be properly said to abstract from specific ideas to more general ones. This should not be called a "process" for it facilitates the misunderstanding that abstracting is an activity or event in the mind of a person, whereas in fact it is an ability or faculty of rational beings. But this aside, how could this "abstraction process" be intelligibly thought of as a cross-generation process of a linguistic community? Is this about the idea that in earlier times speakers only could think and speak about individuals? (What would they have been stating of these individuals then—if not predicates? But predicates are nothing but general concepts—it is one of the prerequisites for a being to be called "rational" or "thinking" that it employs its faculty to use general concepts.) Would and could their language only be describable as entailing no general notions? Would and could we call this a language at all? And how would then, after some years or generations, later people have developed general concepts? Does the idea further imply that in earlier times, people only were able to talk and think about specific concepts such as "All birds in this forest" and "all children in this valley" but not about animals or humans in general? Were they unsure about whether "green" and "tree" were distinct concepts for they could not know for a period of time whether the extension of the concepts really are different (like, before the discovery of a copper beach)?

Then, Lemanski believes to find a use theory of meaning in Schopenhauer [5, pp. 65f]. Lemanski takes this from Schopenhauer's remarks about that the Latin word "honestum" is not completely synonymous with (and thus not directly translatable by) the German words "tugendhaft" ("virtuous"), "ehrenvoll" ("honorable"), "anständig" ("decent"), "ehrbar" ("reputable"), or "geziemend" ("seemly"). Schopenhauer again illustrates his point by a diagram consisting of several eccentric circles, representing the different extensions and their only partial overlappings. Lemanski quotes Schopenhauer:

> That is why one learns not the true value of the words of a foreign language with the help of a dictionary, but only *ex usu* [by using], by reading in old languages and by speaking, staying in the country, by new languages: namely it is only from the various contexts in

which the word is found that one abstracts its true meaning, finds the concept that the word designates [7, p. 263].

According to Lemanski, it is the first sentence of that quote that expresses a use theory of meaning in Schopenhauer, which also goes beyond what Baruch de Spinoza, Friedrich Schleiermacher and Hermann Lotze had to say about the relation of use and meaning. But I believe Schopenhauer cannot be made a predecessor of late Wittgenstein's use "theory" of meaning. Apart from the obvious problem that one single remark by Schopenhauer (which is unfortunately all the textual evidence Lemanski presents to us) does not amount to a *theory* of meaning, there is the problem that the first sentence of that quote is obviously about *learning a foreign language*, not an answer to the question how *words* in general (i.e., in every language) gain their *meaning*.[5] At best, Schopenhauer sketches a theory of *translation*, not one of *meaning*, not even a theory of primordial language acquisition. Schopenhauer clearly has in mind a subject that is already a competent speaker of at least one natural language who aims to learn a new one. For this he gives the good advice not to rely solely on dictionaries, but to derive the meaning of an unknown foreign word by attending to the context of its use. By this the learner will be immune to simply take one word of the foreign language as synonymous to one specific word of his native language, as the *honestum*-case is supposed to show.

Even the fact that Schopenhauer here speaks of "context" ("Zusammenhang") of which the learner will have to abstract the meaning of a word, is no hint for that Schopenhauer was a proponent of the "context principle,"[6] according to which a word gains its meaning only from being embedded in a sentence or inference. There is no textual evidence for this interpretation of Schopenhauer's talk of "context" (Where does Schopenhauer state something like the thesis that the primary unit of meaning would have to be taken as something more complex than a word?), and the "context" may as well be simply non-verbal. How could the fact that there is often no total semantical equivalence between two words of two different natural languages be evidence for the correctness of the context principle and the primacy of judgements or inferences to words? It is hard to believe that hardly anyone before Schopenhauer would have noticed this connection. Lemanski's interpretation of the

[5] Although Michał Dobrzański, in his contribution to this volume, points out that in the quotation Lemanski uses as evidence for his claim Schopenhauer speaks about the acquisition of a *foreign language*, he endorses Lemanski's argument that there is a use theory of meaning to be found in Schopenhauer here [2, pp. 17ff.]. Also, Dobrzański seems to take a use theory of meaning to be one in which the use of a word by a speaker is the only way of epistemically determining which meaning conceived as a private inner state of the speaker [2, pp. 14, 19]. Nothing could be farer from the truth. Wittgenstein argues that use determines meaning especially in contrast to the Lockean picture of meaning as the speaker's inner state. The ideational theory of meaning is a species of the representationalist theory of meaning against which Wittgenstein is fighting in his *Philosophical Investigations*. The meaning of words is neither fixed by their reference to objects in the "outside world" nor by reference to private objects in the mind of speakers. Use determines meaning, or, at least, every difference in meaning must be due to a difference of use. Use is not merely a heuristic device to find out inner states of speakers.

[6] This is as Lemanski takes it to be (cf. [5, p. 66]).

quote above as representing a "use theory of meaning, which is underpinned by a context principle together with a holistic answer to the primacy question" [5, p. 66] is unjustified.

Furthermore, there is the problem that Lemanski apparently equates learning the meaning of word of a foreign language with primordial learning the meaning of a word. It is the point of Wittgenstein's criticism of the Augustinian picture of language in his *Philosophical Investigations* that learning to become a competent speaker of one's first language is not equivalent to the process of learning a foreign language by a competent speaker of a natural language [8, § 32]. For, learning the meaning of a word implies learning the role the word plays in the language, what the word is used for in that language [8, § 30f]. (Ryleans would call this the *category* of the word: "green" is (mostly) a *color* word, tree is (mostly) a material and living object, "flowering" is (mostly) a word designating a state of plants.) To speak a language competently implies to know the role/category of a word, i.e. its use. And, learning a foreign language by looking in a dictionary or asking for the correct foreign word to signify something can only be done by someone who already knows the use of it. As Wittgenstein puts it: "Only someone who already knows how to do something with it can significantly ask a name" [8, § 31].

Lemanski concedes that there might be problems with his interpretation of Schopenhauer [5, p. 66f], but nonetheless he wishes to defend "rational representationalism" according to which concepts are primary regarding "the explanation of logic and language," whereas judgements are primary concerning "understanding." Concepts are generated or "emerge" "bottom up" from individual representations in the mind and the above discussed abstraction process, but they are understood "top down" from the supra-verbal contexts of their use [5, p. 67].

Again, there are several problems with that line of arguing:

1. Lemanski's claim involves that there is a distinction between *the explanation of how concepts come into being* and *the understanding* of concepts (and that allegedly Schopenhauer already drew it) [5, p. 66]. But what could that mean? Is there one way of general terms to gain their meaning and another one of how speakers understand them? What does *understanding a word* amount to if not *understanding the meaning of word*? As if there was a process, called "the coming into being" of a concept which is independent from the understanding of it. If so, could there be concepts that "are there" (wherever they would be) but are not (yet) understood by anyone?
2. The aiming point of Lemanski's remarks is that the logical diagrams Schopenhauer uses, or logical diagrams like these in general, fulfill an important function: they illustrate and make clear the talk of "entailment" regarding analytical statements. When an analytical statement is illustrated by two circles of different sizes, and the smaller being wholly inside the bigger one, one can allegedly simply *see* the logical relation the two concepts are in. Logical diagrams would thus become a mediator of world and rationality. Although Lemanski does not say very much on that issue, this is presumably so because diagrams are perceivable objects "in the world" from which one can recognize logical relations between concepts. Diagrams are thus mediators: Figuratively speaking, they

stand with one foot in the physical world (by being a material representation) with the other in the noumenal world (by expressing logical relations).

There are several problems with this idea:

(a) Diagrams such as Fig. 1 or 2 can indeed be taken as pictorial representations of logical relations. But this is so only because someone can understand them as that, provided they are embedded in an appropriate context. They do not convey logical relations (or any other meaning) by themselves, by being two or more concentrical or eccentrical circles. The picture of two concentrical circles of different size does not represent anything specific whatsoever *by itself*. A picture of something may count as an illustration of a certain thought only when the observer already knows of what the illustration is an illustration of. Concentrical circles of different size may be taken to represent an analytical relation between two concepts and certainly suggest this to people in the business of logical diagrams in certain contexts. But these circles may equally be taken as the pictorial representation of a sombrero hat seen from above, maybe with words inscribed on it. Understanding a word is not to have a picture available and the concept of entailment is not bestowed upon us by looking at a circle enclosed in another.

(b) Conceptual relations are not simply somewhere to be found "out there" or "in the world," not even in pictures. Conceptual relations are not a matter of observation nor perception, we do not get to know of them by experiencing certain things in the world. Conceptual relations like entailment, exclusion, or compatibility are not found in the world and abstracted from several instances of particular perceivable objects, not even diagrams. To be said to have the concepts of entailment, exclusion, or compatibility or any other logical concept a person must display a certain form of behavior that allows the ascription of these concepts to her. The most direct way to test this is if the person's verbal behavior suffices the criteria of rational speaking. (E.g., a candidate would have to be asked if she agrees if trees are living beings given that plants are living beings, a.s.o.) Relations between concepts are learned when we learn the kind of specific use we make of the concepts, namely if two or more concepts entail each other, are compatible or incompatible and these can only be taught simultaneous with the concepts of negation, equivalence, contradiction, possibility and impossibility, necessity and contingency, a.s.o. In fact, only if one has grasped these conceptual relations one can be said to have grasped the concept of a concept at all.

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The World as Will and I-Language: Schopenhauer's Philosophy as Precursor of Cognitive Sciences

Sascha Dümig

Abstract Academic discourse usually tends to regard Schopenhauer's reflections on language as precursory to Wittgenstein's usage-based language philosophy. As will be argued in this article, this interpretation is not only erroneous but, in fact, also an oppositional stance has to be taken. On basis of an in-depth analysis of the relevant passages in Schopenhauer's oeuvre, I will display the theory's correspondence to mentalism and cognitivism. To prove this point, I will primarily show that Chomsky's fundamental differentiation between E(xternalized) and I(nternalized) Language is inherent in Schopenhauer's language conception, in particular, in Schopenhauer discrimination between REDE (Speech) as object of outer experience on the one hand and the meaning of words which consists of abstract concepts on the other hand. Furthermore, it will be shown that Schopenhauer displays a concentration on and fosters a further examination of I-Language. This becomes particularly evident in his discussion of language acquisition and processing, and the ways in which the genuinely cognitively conceptualized language module is implemented in the human brain.

Keywords Concepts · Working memory · E-Language · I-Language

Mathematics Subject Classification (2020) Primary 91F20, Secondary 97C50, 03A05, 03B65, 00A99

1 Chomsky's Conception of E- and I-Language

One of the most fundamental conceptions in generative grammar is the distinction between E(xternalized)- and I(nternalized) Language [3]. "E-Language linguistics" work taxonomically, i.e. language is regarded and examined as some external object, a "Ding unter Dingen." E-Language is accordingly conceptualized independent of its (individual) speakers, there is no recourse to an internal, representational

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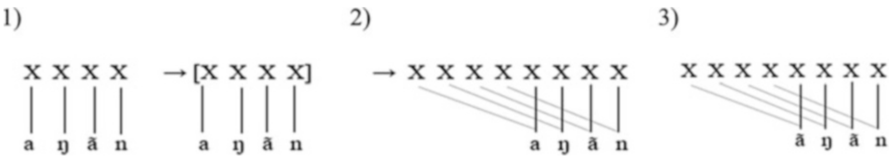
data procession. I-Language, on the other hand, is an internal, representational procession of data. Language is here conceptualized as a mental capacity that is creative in that it is capable of producing a potentially endless number of sentences. The legitimacy of this creative component in language acquisition, now, is based on the fact that the latter generally entails a restricted input: Every individual language learner has to work on the restricted amount of data, namely the data that are available in the target language.

> A speaker of a language has observed a certain limited set of utterances in his language. On the basis of this finite linguistic experience he can produce an indefinite number of new utterances which are immediately acceptable to other members of his speech community. He can also distinguish a certain set of 'grammatical' utterances, among utterances that he has never heard and might never produce. He thus projects his past linguistic experience to include certain new strings while excluding others. [2, p. 61]

The concept of an I-Language consequently projects that particular mental states can be realized in multiple physical forms, for otherwise language would be deprived of its inherent creative potential. [This can be exemplified by drawing on German teenage slang: How come that our youth produces a grammatical form such as dt. *gecheckt* > engl. *to check*"?]. In order to account for such multiple realizations, it is imperative to locate language as a mental knowledge system with complexly layered processing: The distinction between syntactical surface- and depth structure is by now almost classical. It accredits for semantic congruence despite syntactically different realizations. Despite this field, phonology also provides an amplitude of examples to illustrate internal cognitive working processes: In Malay, for instance, the phonological rule of reduplication is used to transform the word *aŋãn* (dream) into the word *ãŋãnãŋãn* (ambition). In comparing the phonological structures of the two words, it is striking that the initial [a] is nasalized in the reduplicated ãŋãnãŋãn while it is not in aŋãn. Malay, now, possesses a rule of nasalization according to which vowels that are subsequent to nasals are produced nasalized. Taking a look at the surface structure, i.e. the concrete physical realization, it becomes evident that the initial [a] in *ãŋãnãŋãn* is not preceded by a nasal. Following Frampton [7], the nasalization of initial vowels can only be understood once reduplication is comprehended as a process of copying (which includes representational levels in the process of copying). Primarily, the X-positions which encode the temporal duration of the segments are copied (cf. ex.1). These, however, are still associated with the basal segmental content. That is to say, before the structure is 'repaired', i.e. before every position is assigned a segment, phonological rules are applied. It is in this "mid-representation" which is never overtly produced, that a [n] is placed in front of the initial vowel (cf. ex.2). Here, on the mid-representational plane, the rule of nasalization is used: a nasalized form is copied into the initial x-position (cf. ex.3) ([7], p. 9).



As follows from this exemplary representation, an adequate explanation of linguistic data prerequisites symbol processing and, in conjunction with the latter, the possibility to back up mid-data and rule applications in working memory. Fodor and Pylyshyn [6] emphasize this difference between connectionist and the so-called classical architectures of computation:

> Connectionist theories acknowledge only causal connectedness as a primitive relation among nodes; when you know how activation and inhibition flow among them, you know everything there is to know about how the nodes in a network are related. By contrast, Classical theories acknowledge not only causal relations among the semantically evaluable objects that they posit, but also a range of structural relations, of which constituency is paradigmatic. [6, p. 12]
>
> [...] Classical theory is committed not only to there being a system of physically instantiated symbols, but also to the claim that the physical properties onto which the structure of the symbols is mapped are the very properties that cause the system to behave as it does. In other words the physical counterparts of the symbols, and their structural properties, cause the system's behavior. (ibid., p. 14)

Because connectionist networks operate on the basis of causal activation, they cannot be used to explain the most fundamental cognitive operations. A causal activation of a neuron by two further neurons is not to be equated with the logical operation of conjunction. The index "A and B" demands for object A and B to remain represented once a third neuron is activated. This can be achieved only by a means of a saved symbolic representation. Even if a causal realization, a realization with an indifferent activational energy, would be assumed, an established association between A and B and the third neuron which would have to be individually acquired would have to exist. In this case, a general rule would not exist; the rule of conjunction would have to be acquired for every combination of objects in the world. The same would hold true for every other logical operation. The absurdity of assuming such a lifelong process of learning seems quiet obvious. Nevertheless, this structure is still extensively used as explanatory model in neuroscientific discourse. Causal activation is here envisioned but as a transitional process from a particular cause to the corresponding effect (ad infinitum). The particular effect, however, is never semantically incorporated in the cause. Although recent neurosemantic approaches emphasize that there is an intrinsic connection between the sign and its use in language games (this, in reference to Wittgenstein), a causal connection is still maintained.

According to Fodor, approaches that only refer to external, observable units adhere to the proposition P: "For each mental predicate that can be employed in a psychological explanation, there must be at least one description of behavior to which it bears a logical connection." [5, p. 51] Mentalistic approaches negate proposition P due to the aforementioned reasons.

2 Schopenhauer's E- and I-Language

Schopenhauer assigns divergent classes of objects (for the subject) to E-Language and I-Language. Pertaining to the realm of objects of outer experience (external, observable objects), E-Language follows the principle of causality respectively the principle of sufficient reason of becoming:

> [. . .] before all things, that we should recognize, that this law refers solely and exclusively to changes of material states and to nothing else whatever; consequently, that it ought not to be brought in when these are not in question. The law of causality is the regulator of the changes undergone in Time by objects of our outer experience; but these objects are all material. Each change can only be brought about by another having preceded it, which is determined by a rule, and then the new change takes place as being necessarily induced by the preceding one. This necessity is the causal nexus. [9, p. 40]

Thus, the principle of sufficient reason of becoming is equivalent to the linguistic concept of E-Language. Schopenhauer here draws on the notion of REDE (Speech). He compares the latter with the transmission of a signal by a telegraph.[1] That which is visually or acoustically transmitted by a carrier medium, however, is an abstract code; it may not be equated with the particular external realization. As objects of outer experience, physically produced sound waves or written forms, in turn, only affect external objects, i.e. our sensory organs as well as our neuronal structure. Even if the so-called transducer converts distal physical stimuli into analogously structured neuronal events, the fundamental principle of cause and effect holds. From the perspective of the principle of sufficient reason of becoming, the body is accordingly but a collection of causes and effects: "It is given as an idea in intelligent perception, as an object among objects and subject to the laws of objects" [10, pp. 129–130].

The abstract code, i.e. the actual meaning of the Speech, is referred to as "concepts of reason" by Schopenhauer. As such, they fall under the principle of sufficient reason of knowing:

> The meaning of a speech is, as a rule, immediately grasped, accurately and distinctly taken in, without the imagination being brought into play. It is reason which speaks to reason, keeping within its own province. It communicates and receives abstract conceptions, ideas that cannot be presented in perceptions, which are framed once for all, and are relatively few in number, but which yet encompass, contain, and represent all the innumerable objects of the actual world. [10, p. 51]

Schopenhauer thus strictly discriminates between REDE (Speech), as object of outer experience, and the meaning of words, which consists of abstract concepts. The word itself is comprehended as the concept's sensual sign. This sign, however, is not to be confused with the objects of REDE (Speech) because of its distinct function in internal processing:

> [. . .] es dient den Begriff zu fixieren, d.h. das sonst ganz abgesonderte abstrakte Bewußtseyn in Verbindung zu erhalten mit dem sinnlichen, anschauenden und bloß thierischen Bewußtseyn [. . .]. [12, p. 243]
>
> ([. . .] it serves to fix the concept, to link the completely separated abstract consciousness to the sensory, intuitive and mere animalistic consciousness. (own translation))

[1] "Speech, as an object of outer experience, is obviously nothing more than a very complete telegraph, which communicates arbitrary signs with the greatest rapidity and the finest distinctions of difference." [10, p. 51].

At this point, it is necessary to briefly discuss the acquisition and processing of words and concepts respectively their interaction in the process of thinking and speaking, as some scientists do not take account of these distinctions and interactions as seen from a psycholinguistic perspective. As such, misinterpretations as the following come to the fore. As Lemanski [8] argues,

> Die Tatsache, dass Begriffe in der Schopenhauerschen Philosophie die Funktion haben, reale Gegenstände und ideale Tatsachen abzuspiegeln, bedeutet aber nicht, dass Schopenhauer auch glaubt, dass die Bedeutung der Begriffe allein darauf beruht, dass sie in der Philosophie für den Zweck der Repräsentation instrumentalisiert werden. Es ist somit möglich, dass Schopenhauer unabhängig von seiner Repräsentationstheorie auch eine semantische Gebrauchstheorie und einen Kontextualismus thematisiert oder sogar vertreten hat. [8, p. 185]
>
> (The fact that the role of concepts in Schopenhauer's Philosophy is to mirror real objects and ideal relations does not imply that Schopenhauer also believes that the meaning of concepts merely rests on their use for representational purposes (in philosophy). It is therefore possible that—despite of his representational theory—Schopenhauer discussed or even supported a semantic usage-based theory and a contextualism. (own translation))

In this examination it has, so far, been pointed out that, in Schopenhauer's oeuvre:

1. The objects of the external world are subsumed under abstract concepts, they are not merely mirrored. Words are the (physical) signs of these concepts. As such, they are—in Schopenhauer's terms—representations of representations of representations. As follows, the question that Lemanski posits, namely if Schopenhauer's language-theoretical reflections correspond to a representational theory of language (such as formulated by the early Wittgenstein) instead of a usage-based account thus becomes obsolete. In regard to concepts and words in Schopenhauer's work, it turn out to be a straw man argument.
2. The function of language is clearly defined by Schopenhauer. Significantly, it has no social function, i.e. no public character, but instead serves as tool of internal processing, namely as a means to fix mental concepts.

In the following, I will elaborate these two points in a more detailed analysis. As will be shown, the latter strengthens an argumentation against a usage-based interpretation.

3 Language Processing According to Schopenhauer and the Primacy of I-Language

In his considerations of the way in which words contribute to fix concepts, Schopenhauer displays unusually modern views:

> Die Zeichen der Begriffe, die Worte, sind ein so nothwendiges Hülfsmittel des Denkens, daß ohne sie keine willkürliche Vergegenwärtigung der Begriffe, folglich gar kein Denken möglich ist [...] Daher können wir ohne Worte oder Zeichen nicht einmal bis 20 zählen [...] Die Anschauung präsentiert sich bald die eine bald die andere Eigenschaft [...] und für unser sinnliches an Zeit und Succession gebundenes Bewußtseyn muß diese Gegenwart durch ein Wort bezeichnet werden. [12, pp. 243–244]

(The signs of concepts, words, are such a necessary tool of thinking that without them, no willful realization of concepts, hence no thinking, would be possible. [. . .] Thus, without words or signs we could not even count to 20. [. . .] Intuition constantly presents different properties [. . .] and to our sensory consciousness, bound to time and succession, this presence has to be designated by a word. (own translation))

This perspective on language as a tool ("Hülfsmittel") of thinking bears strong analogies to the multi-component model of working memory (2012, [1]). The latter assume that information can be kept "online," i.e. present, for mental processing by way of a phonological loop. It needs to be noted, however, that Schopenhauer does not suggest an identification of language and thought. Instead, he emphasizes the important role that language plays in consciously accessing one's memory:

So nothwendig auch zum Denken die Worte sind und so sehr auch der Begriff eines Zeichens bedarf; so beruht dennoch die Nothwendigkeit des Zeichens nicht darauf daß ohne dasselbe der Begriff überhaupt gar nicht gefaßt, gar nicht gedacht werden könnte (denn das kann er an und für sich, da oft uns ein Wort fehlt unsern Begriff auszudrücken), sondern darauf, daß die willkürliche, beliebige *Hervorrufung* des Begriffs nur durch das Zeichen möglich ist: das Zeichen dient nicht ihn zu denken, sondern ihn jederzeit zu vergegenwärtigen. [12, p. 247]

(Though words are necessary for thinking and a concept requires a sign, the necessity of the sign is not based on the fact that without it a concept would not be available, could not be thought (for this is possible as such, because quite often we lack the word to express a concept), but on the fact that the willful evocation of a concept is possible only by a sign: the sign does not make it thinkable, but serves to hold it present at any time. (own translation))

Words consequently function to evoke concepts ("Hervorrufung") and, at the same, to hold the latter present in working memory ("vergegenwärtigen"). What is more, the passages that pertain to memory are substantial for an examination of Schopenhauer's vision on second language acquisition (L2). Lemanski primarily refers to the following passage in order to suggest that a usage-based theory of semantics and the context principle is supported:

Darum lernt man nicht den wahren Werth der Wörter einer fremden Sprache durch das Lexikon, sondern erst ex usu, durch Lesen bei Alten Sprachen und durch Sprechen, Aufenthalt im Lande, bei neuen Sprachen: nämlich erst aus verschiednen Zusammenhang in dem man das Wort findet abstrahiert man sich dessen wahre Bedeutung, findet den Begriff aus, den das Wort bezeichnet. [8, pp. 186, 12, 246]

(For that reason no one learns the meaning of foreign words by studying the dictionary, but rather ex usu, by reading old languages and, concerning new languages, by talking and visiting the country: it is by drawing on the diverse contexts in which the word is met that one can accordingly abstract its meaning and, in such a way, find the concept the word designates. (own translation))

While the interpretation seems quiet comprehensive, Schopenhauer's statements referring to L2 acquisition need to be contextualized, for they cannot be analyzed without a consideration of his theory on first language acquisition (L1). As he notes,

die Bildung des Begriffs, sein Entstehn ist nicht, wie man früher meinte, das Vergleichen vieler anschaulicher Objekte und allmäliges Zusammenfassen ihrer Ähnlichkeiten: sondern der Begriff entsteht nicht allmälig, er entsteht mit einem Schlage indem man an die Stelle der anschaulichen Vorstellungen ein bloßes *Denken* setzt, eine ganz neue Thätigkeit des

> Geistes eintritt, die Reflexion, die Vernunft, und der Uebergang zu einer ganz anderen Klasse von Vorstellungen geschieht. [12, pp. 256–257]
>
> (Concept formation, its creation, is not—as was believed—based on the comparison of many perceived objects and the successive classification of their similarities: concept formation is not gradual, it happens all at once in such a way that abstract thinking replaces intuitive representations, a new operation of the mind emerges, reflection, reason, and the transition to a distinct class of representations takes place. (own translation))

Schopenhauer's explanation of the creation (Entstehung) of concepts (and meanings) is completely coherent with his general epistemological assumptions. As concepts of reason underly the principle of sufficient reason of knowing, they cannot be derived subsequently from representations. Schopenhauer also emphasizes that concepts form an entirely different field of representations (they are representations of representations). They are certainly premised based on representations, yet, underly completely different rules than the latter.

As a matter of fact, Schopenhauer thus anticipates what Goldfield and Reznick [4] come to label "naming insight" in language acquisition, for meanings are, following Schopenhauer, "[. . .] framed once for all, and are relatively few in number [. . .]" [10, p. 51].

As follows, Schopenhauer's explanations of L1- respectively L2-acquisition seem to represent a stark contradiction: On the one hand, concepts are figured to be "framed once for all"; on the other hand, however, concepts of a foreign language are to be discovered in its use.

It is at this point that the perspective of a processual working memory comes into play. For Schopenhauer, the ability to learn a foreign language is bound to the use of our memory:

> Everyone therefore who knows several languages, will do well to make a point of reading occasionally in each, that he may ensure to himself their possession. [9, p. 174]

The use of the verbal working memory is not to be equated with thought. In fact, an antagonistic relation between the two is established:

> People who have little capability for original thought do this all their lives (and moreover not only with intuitive representations, but with conceptions and words also); sometimes therefore they have remarkably good memories, when obtuseness and sluggishness of intellect do not act as impediments. [. . .] Still, on the whole, genius is seldom found with a very bad memory; because here a greater energy and mobility of the whole thinking faculty makes up for the want of constant practice. [9, p. 174]

The possibility of an antagonistic relation between working-memory based language processing and thought indicates that Schopenhauer envisions a modular model of mental processing.

Modern neuro-psychologists draw on lesion studies to prove the autonomous processing of individual mental modules. The fact that the loss of a particular function (visual perception of forms, for instance) does not have consequences on the performance of other functions (such as the visual perception of color) is a clear indication for separate modules with different functions. Schopenhauer displays an analogous argumentation:

> Injuries to the head, with loss of brain substance, affect the intellect as a rule very disadvantageously: they result in complete or partial imbecility or forgetfulness of language, Permanent or temporary, yet sometimes only of one language out of several which were known, also in the loss of other knowledge possessed [. . .] [11, p. 469]

Linguistic memory and concept-based thought are accordingly comprehended as distinct mental functions. The (memorized) representation of a word in the mental lexicon, the physical (sensual) sign of a concept, only relates but to a subset of conceptual content. Words from different languages are accordingly usually not completely concurrent in their meaning. Instead they form smaller or larger interfaces on the background plane of the concept:

> Sometimes a foreign language expresses a concept with a nuance which our own language does not give to it and with which we then exactly conceive it. Everyone, who is concerned with the precise expression of his own ideas, will then use the foreign word without paying any attention to the yelping of pedantic purists. [13, pp. 567–568]

The background plane is formed by the pure concept with which thought operates. It can accordingly not be matched by any word, as Schopenhauer emphasizes:

> In der Regel werden wir uns des Begriffs immer nur mit seinem Zeichen, dem Wort, zugleich bewußt: aber bisweilen auch ohne solches Zeichen, nämlich wenn wir ein Wort suchen, das unsere Sprache nicht hat: dann haben wir bloß den Begriff und suchen das Zeichen dazu, wobei uns ganz deutlich wird, wie der Begriff völlig verschieden ist vom Wort, seinem Zeichen, als auch von der anschaulichen Vorstellung [. . .]. Ein Beispiel zu geben hiervon hat einige Schwierigkeit: denn ich soll Ihnen als Beispiel einen Begriff mittheilen, für den es kein Wort gibt: das geht nicht: daher muss ich einen Begriff nehmen für den die eine Sprache ein Wort hat, die andre nicht [. . .]. [12, p. 244]
>
> (Usually we realize a concept in tune with its sign, the word: but sometimes we realize a concept without such a sign, namely in states when we are searching for a word our language does not provide: in this case, we only have got the concept and search for a fitting sign, while it become clear to us how completely distinct the concept is from the word, its sign, as well as from sensory representation [. . .] To give an example is quite difficult because I would have to provide a concept for which no word exists: this is impossible. Therefore I have to use a concept for which one language provides a word while another does not [. . .] (own translation))

According to Schopenhauer, we have to distinguish the linguistic concept and the meaning of a word from the pure concept of thought. The latter can only be approached by a comparative word analysis, i.e. by defining the spheres of a word's semantic content. The pure concepts predate the usage of words. They are thinkable but do not necessarily find an expression in a word of a target language (cf. [12, p. 244]). The amount of pure concepts always stays the same. The acquisition of new concepts only implies that aspects of a concept can be kept duratively online in working memory by signs. The *ex usu* in the above citation thus pertains to the production of a new relation between a phonological form and features of an existing conceptual representation. They do not imply a genuinely new formation. It is this relation respectively representational interface which is formed in usage and stored.

In his dissertation from 1813, Schopenhauer has already worked out a modern, processual conception of memory in which he also points out that representations in memory change in usage:

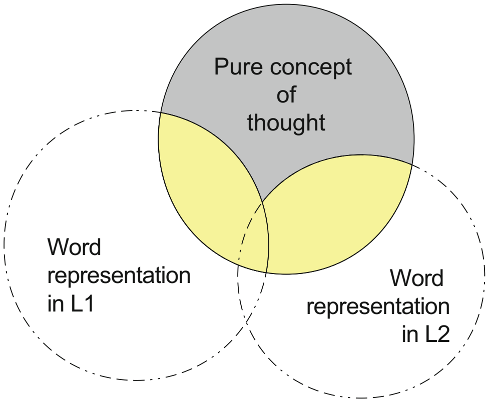


Fig. 1 The cognitive landscape of pure concepts and word representations

> I think the best would be that of a piece of drapery, which, after having been repeatedly folded in the same folds, at last falls into them, as it were, of its own accord. The body learns by practice to obey the will, and the faculty of representing does precisely the same. A remembrance is not by any means, as the usual view supposes, always the same representation which is, as it were, fetched over and over again from its store-house; a new one, on the contrary, arises each time, only practice makes this especially easy. [9, p. 173]

Memory retrieval and representations in memory can accordingly be comprehended as dynamic, as are the meanings of a word. Concepts, in turn, remain unaltered and stable ("[. . .] das Zeichen dient nicht ihn zu denken, sondern ihn jederzeit zu vergegenwärtigen." ("[. . .] the sign does not make it thinkable, but serves to hold it present at any time." (own translation) [12, p. 247]. In this way, the cognitive "landscape" can be envisioned as a doubly mapped out territory (cf. Fig. 1). On the basal plane we find the map of the pure concepts which are always unalterable and only accessible by thought. Upon this fundamental plane, then, linguistic representations form something akin to a system of spotlights: Words form brighter and weaker spotlights which shed more or less light on the concept plane. This system of spotlights is dependent on usage, though not in terms of the usage-based theory of the later Wittgenstein.

4 Results

In order to stress that Schopenhauer's elaborations or reflections on language cannot be interpreted in the latter way, a short summary of the essential points will be supplied.

- Externalized Language is distinguished from Internalized Language and, as such, Speech as E-Language from concepts as I-Language.
- Communication flows from reason to reason and is in itself independent from the usage of words in speech. Language is thus no social event.

- Concepts are formed once and for all. Words simply allow consciousness access to aspects of these pure concepts (as an aid during online processing). Therefore, word meanings shed light but on a subset of a concept. We can thus rather imagine a discovery procedure by means of usage than a genuine new formation of meanings.

In light of all arguments it should have become apparent that a detailed and correct reading of Schopenhauer's philosophy necessarily requires an interpretation of the latter as precursor of cognitive sciences. From my point of view, other or common readings result in historical distortions which took and still take place in regard to Schopenhauer's philosophy and the many seminal aspects it contains.

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Schopenhauer's Perceptive Invective

Michel-Antoine Xhignesse

Abstract Schopenhauer's invective is legendary among philosophers, and is unmatched in the historical canon. But these complaints are themselves worthy of careful consideration: they are rooted in Schopenhauer's philosophy of language, which itself reflects the structure of his metaphysics. This short chapter argues that Schopenhauer's vitriol rewards philosophical attention; not because it expresses his critical take on Fichte, Hegel, Herbart, Schelling, and Schleiermacher, but because it neatly illustrates his philosophy of language. Schopenhauer's epithets are not merely spiteful slurs; instead, they reflect deep-seated theoretical and methodological commitments to transparency of exposition.

Keywords Abstraction · Animals · Clearness · Concepts · Distinctness · Insults · Language

Mathematics Subject Classification (2020) Primary 03A05, Secondary 03B65

1 Introduction

Schopenhauer is perhaps best known today for his intemperate invective against his contemporaries, especially Fichte and Hegel, whom he subjected to legendary quantities of abuse. It seems fair to say that, so far as most readers are concerned, these venomous vituperations look rather like jealous ranting. And to some extent, the reaction is warranted: a careful catalogue of Schopenhauer's obloquies betrays a singular concern—an obsession, even—with the fame and fortune lavished upon those he most reviled, and the studied indifference with which his own work was first received. Franco Volpi, for instance, has recently published an alphabetized compendium of Schopenhauer's insults, in which he points to the

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irony of Schopenhauer's own sentiment that the *argumentum ad personam* is the last resort of a player who has otherwise lost the game, and has been bested (though perhaps also wronged) by a superior intellect [28, p. 5].

Volpi's catalogue presents Schopenhauer's insults as de-contextualized and unsourced objects of fun, which they certainly are; but taken in context, Schopenhauer's bilious rhetorical flourishes actually reward sustained critical attention. A closer examination of the insults he so clearly delighted in heaping upon Fichte, Hegel, Herbart, Schelling, and Schleiermacher (as well as the Danish Academy and the German public) reveals a deeper disagreement over the nature of concepts and the function of language, as well as with academic integrity and the norms and structure of intellectual discourse. In other words, Schopenhauer's insults hold a mirror up to his philosophy of language. It is also worth noting that although Schopenhauer's scolding may shock today's readers, it was actually quite common in nineteenth-century German academic circles to publicly excoriate one's rivals in order to underscore the importance of one's own views (if not quite to that extent) [1, pp. 74–75].

This is not to say that we can read a latent philosophy of language implicit in the text of his remonstrations; Schopenhauer's account of language is explicitly and independently presented in his main writings, although it is somewhat scattered and may seem insubstantial by today's standards. But it is important to remember that we have benefitted from three-quarters of a century of empirical linguistics and cognitive science—for a defence of Schopenhauer against the charge of etymological dilettantism, see [5, p. 152]; for a sketch of how Schopenhauer's ideas fit in with later linguistic and cognitive science frameworks, see also [5, p. 157–163], and [6, this volume].

Nor do I mean to suggest that Schopenhauer's remarks betray a failure of engagement with the ideas of his contemporaries. On the contrary, he frequently devotes lengthy passages to explaining the substance of his disagreement with the German Idealists. Indeed, it is worth noting that Schopenhauer was not given to slighting all those with whom he disagreed—quite the opposite, as his treatment of Berkeley, Hume, Kant, and Locke (among others) demonstrates. In fact, Julian Young has argued that Schopenhauer subscribes to the same concept-empiricism as the British empiricists [30, Chap. 2, §4, pp. 22–25], and David E. Cartwright has offered a careful analysis of Schopenhauer's intellectual debts to, admiration for, and disagreement with, John Locke [2]. My point, rather, is that we should read Schopenhauer's animadversions in light of his linguistic commitments, treating them as case studies offering a useful illustration of the principles animating his philosophy of language.

I shall begin, in the next section, by sketching the essentials of Schopenhauer's philosophy of language, focusing in particular on its basis in his analysis of concepts. It is here that we find Schopenhauer's explanation for the differences between animal and human cognition, as well as his explanation of the role abstraction plays in facilitating thought. From these premises, I will turn to a closer examination of Schopenhauer's charges against his contemporaries, which I argue centred on their misuse of our powers of abstraction and, consequently, their blatant disregard for the evidence given by intuitive perception.

2 Schopenhauer's Philosophy of Language

In order to discuss Schopenhauer's philosophy of language, one must first say something about his analysis of concepts. But my goal in this chapter is somewhat narrower in scope, since it concerns Schopenhauer's infamous remarks on Fichte, Hegel, Herbart, Schelling, and Schleiermacher, and what they can teach us about his philosophy of language. Accordingly, I do not have the space to offer a detailed exposition of Schopenhauer's analysis of concepts—an analysis which, in any case, has already been amply documented by Malter [13], Neeley [14], Dümig [5], and especially, Dobrzański [3, 4], this volume. Nevertheless, something must be said about concepts, since these hold the key to Schopenhauer's philosophy of language.

The first thing to note is that Schopenhauer relies on concepts to mark the boundary between the animal and the human realms. Humans and non-human animals alike all produce sounds, and according to Schopenhauer these sounds inevitably give voice to the stimulations and movements of individual expressions of the creature's Will [21, §298, p. 565]—they embody desires and hungers, needs and wants, and basic reactions. But human beings also enjoy the unique capacity to use these sounds to indicate objects in our environment; we are endowed with reason, which allows us to derive concepts (abstract representations) from our perceptions and to use sounds to designate these concepts [22, §3, p. 27].

This is not to say that we speak by means of aural signs communicating images of our intuited percepts, except insofar as we sometimes use a mental image as a representative of a concept, e.g. when we use the image of some particular dog (or stereotype of a breed of dog) to stand for the whole concept [21, §28, p. 102]. For more on Schopenhauer's denial that thinking requires mental images, see [7, this volume, esp. §4]. Such cases aside, Schopenhauer explicitly denies that concepts are mental images. The question of how it is that words become meaningful is discussed in [6, 12] (both in this volume), as well as in [4, 25]; here, I follow Dümig [6], who emphasizes Schopenhauer's remark that 'the meaning of the speech is immediately understood [. . .] Reason speaks to reason while remaining in its own province: it sends and receives abstract concepts, representations that cannot be intuited [. . .]' [23, §9, pp. 62–63; 17, §28]. The point, in other words, is that we just immediately grasp the meanings of words [25, p. 379].

Thinking, according to Schopenhauer, requires us to combine and separate concepts according to the rules codified by logic in the theory of judgements [19, §29, pp. 104–105]. Concepts are abstracted from what is given to us by the world in intuitive perception; they allow us to abstract from the individual instances given to us in intuitive perception and think about the world at the general level [27, p. 272]. Schopenhauer thus reverses the epistemic priorities set by the scholastics and rationalists such as Descartes, Leibniz, and Wolff: the representations of perception are intuitive and immediate, while concepts are merely abstract representations ([17, §26, p. 97]; Schopenhauer seems to have been largely in agreement with Locke on this point—see [2]). This means that only intuitions can ever be said to be 'clear', since they are unmediated perceptions. Concepts, on the other hand, are called 'distinct' only when they can be decomposed into their constituent attributes, down to the level of concrete, individual intuitions [27, pp. 270–272]. As Schopenhauer puts it, however, 'Concepts are confused if one does not quite know their sphere, that is, if one cannot specify the other conceptual spheres which intersect or fill them, or which surround them' [27, p. 272] [my translation]; for an excellent explanation of just what Schopenhauer means by the spheres' of concepts, see [12], this volume. In other words, a confused concept is one which is not grounded in a perceptual intuition. Abstraction does not proceed from the many to the one, but rather from the one (individuals given in intuition) to the many (ideas given by concepts) [27, pp. 273–274].

It is this facility of abstract representation that allows human beings to think and reason, to execute plans and decide upon courses of action in addition to acting from mere impulse. That said, Schopenhauer did not think that language is necessary for thought, since words do not exhaust the perceptual content behind concepts; words merely simplify the tasks of communication and reasoning [5 pp. 155–156; 4, this volume]. So, although thinking is the manipulation of concepts, concepts are not the ultimate ground of knowledge, since they are mere abstractions from intuited perception, representations of representations. Only intuitions can ground knowledge in this way [23, §9, p. 63; 27, pp. 270–272]. Notice, then, that abstract representation is something which we impose upon the world through our use of language. The world offers us only percepts; language mediates our experience of the world by imposing concepts upon the data given to us in perception [23, §9] (see also [25, p. 370]; whether this amounts to a representational theory of language is an open question; for an excellent discussion of this issue, see [4], this volume). Indeed, Schopenhauer goes so far as to task philosophy with describing, in the abstract, the essence of the world given to us by intuitive, concrete cognition [23, §15, pp. 108–110].

Curiously, although Schopenhauer clearly states that non-human animals are not capable of abstraction and thus have no language, he does think that they understand proper names [17, §26, p. 99]. Since Schopenhauer did not think that non-human animals could mobilize concepts, this might suggest that he did not think of proper names as abstractions—labels, on a causal or hybrid framework—but rather as directly associated with an entity's perceptual properties. And yet in his Berlin lectures, Schopenhauer makes it clear that concepts—among which he includes proper names—designate at the individual, or intuitive, level; they are abstractions [27, p. 293]. So how should we reconcile these two observations?

Presumably, Schopenhauer thought of names as abstractions because names are conferred by human beings, who use them to stand in for individuals. Indeed, he tells us that 'in the judgement "Socrates is a philosopher", several people could very well think of form, size, and other qualities of different people, which would nevertheless correspond to the concept of Socrates, because they never contain everything in the concept that is in the individual: the concept is always an abstractum, a thought, never a single individual thing' ([27, p. 293]; translated by Jens Lemanski). In other words, the name 'Socrates' does not stand for a *description* of the man and all of his properties, but rather designates a concept that abstracts from these and stands for his essence. Two people talking about Socrates may well associate different properties with the man, since the concept cannot hope to encompass them all; what matters is that their concepts each represent the essence of the man and, thus, converge on the right referent. So much for the human understanding of proper names; but what about non-human animals?

When Schopenhauer says that non-human animals can understand proper names, it seems most likely that he had his poodles—all of which shared the name 'Atma' [1, p. 136]—in mind, since dogs and other animals can easily be taught to respond to the sounds constituting their names. But what does Schopenhauer's use of 'Atma' tell Atma? Not very much, since, as we saw above, Schopenhauer did not think that animals could mobilize concepts, and because he characterized judgement as an operation exclusive to thinking, rather than to intuition [27, p. 293]. In other words, Schopenhauer's use of 'Atma' conveys no descriptive content whatsoever (e.g. '*Schopenhauer's white poodle*') to Atma; rather, so far as she is concerned, it refers directly to her. Atma knows that 'Atma' designates her, whatever she is, but she associates no judgement with her perception of the utterance. Atma knows enough to respond to the utterance of her name, but her conceptual reach goes no further. An animal can be taught to recognize and respond to uses of its name, but not to use it for communicative purposes of its own, since it lacks the capacity to mobilize concepts.

We should, of course, be wary of reading too much or too modern a theory of language into these remarks, especially given their apparent inconsistency. But if my explanation of Schopenhauer's remarks is correct, then it seems as though he subscribed to a theory of naming and reference akin to the Millian, even though Mill would only articulate his views on the subject 25 years after the publication of *The World as Will and Representation*, in *A System of Logic* (1843). To be sure, Schopenhauer's remarks on the subject are nothing like as sophisticated as Mill's, and they do not map on seamlessly; what they show, however, is that direct reference was in the air. At least, so long as we read Schopenhauer as tending more towards representationalism than towards a use-theory of meaning (see, e.g. [4], this volume, which mediates between the use- and picture-theories, but emphasizes Schopenhauer's representationalism; [12], by contrast, emphasizes the use-theory instead).

But let us return to Schopenhauer's account of concepts. By way of a helpful analogy, Schopenhauer says that concepts are related to their root percepts in much the same way as arithmetical formulae are related to the operations of thought which give them their content, or as logarithms are related to their number [17, §27, pp. 100–101]. Arithmetical formulae allow us to abstract away from particular cases to draw inferences and make generalizations grounded in logic. Consider Euler's formula:

Euler's formula (EF),

$$e^{ix} = \cos x + i \sin x$$

EF is obtained by abstracting from certain features of functions in complex analysis, including algebraic geometry and number theory, and it helps us to say a great many things in engineering, mathematics, and physics. Each of its constants and variables takes a particular content, given by the mathematician's domain of discourse; absent such specification, however, they remain free to roam across all possible interpretations.

It should be straightforwardly obvious, then, that the letters and symbols we use to express EF are largely meaningless without some guide to their interpretation. According to Schopenhauer, this is because they are only very loosely tied to perceptual intuitions, by means of subsidiary mathematical concepts which are themselves abstracted from other, more fundamental concepts. Eventually, when the yarn has been sufficiently unravelled, we will find some sort of perceptual intuition. In other words, the statements we express using EF are attenuated by the successive degrees of abstraction required to generate them, until they approximate pseudo-concepts or 'empty husks'.

In much the same way, Schopenhauer thought that the formulation of a concept pares a perceptual intuition down to its component parts, thereby allowing us to focus our attention on just some of its properties and relations at a time. The higher the level of abstraction, however, the less particular content remains in the concept [17, §26, pp. 98–99]. So, when we abstract from the name 'Yuni' to the kind 'unicorn', we strip our concept of its Yuni-the-unicorn-specific properties and boil it down to just its unicorn-specific properties; by the time we get from 'unicorn' to '*x*', the concept no longer has any named, horned-, or horse-content left to it at all, and can be manipulated in thought as easily as arithmetical formulae.

Concepts are thus necessarily severed from their intuitive content, and words are simply the arbitrary signs we use to fix concepts before us so that we can make use of them [17, p. 99]. But concepts do not refer to things in themselves, nor even to representations of things in themselves; they refer merely to the general representations we have created for ourselves by abstracting from the content of what is given to us intuitively in perception, which they then *represent* ([17, p. 99; 23, §9]; see also [4, this volume], which considers the question of just what Schopenhauer's concepts actually refer to in more detail). This helps to explain Schopenhauer's pessimistic take on dogmatic metaphysics, which argued that metaphysics was rooted in reason (and thus in abstraction) rather than experience (perceptual intuition) [8, pp. 430–431]. It also explains why Schopenhauer thought that perfect translations of most words are impossible, as well as why some languages lack words for concepts identified in others ([21, §299, pp. 567–568]; Schopenhauer's theory of translation plays a crucial role in introducing his account of individual understanding—see [12, this volume, esp. §4.2.1–4.2.2])—e.g. 'hygge' in Danish, Hawaiian 'pana po'o' or the Inuit 'iktsuarpok'. Different words pick out different concepts, which in turn are grounded in slightly different perceptual intuitions.

Consequently, 'even in mere prose the finest translation of all will relate to the original at best as the transposition of a given piece of music into another key relates to the original' [21, §299, p. 568]. Musical transposition shifts pitches up or down by a regular interval, and we usually transpose music when we want to play a piece on an instrument with a different range, because a musician or vocalist prefers a particular key, lacks the requisite range, or has not yet learned the original key. In other words, while the notes bear the same relationship to one another and the piece is therefore recognizably similar, it sounds quite different (since it is composed of different constituent parts; the sound will be higher or lower). To fill out the analogy, then, the idea is that translation is an exercise in approximation reflecting our inability to capture and communicate the givens of intuitive perception.

Schopenhauer goes on to argue that the health of a language is thus to be measured by the ratio of its words to its concepts, so that an oversupply of concepts without words is a sign of intellectual poverty—a state which he thought aptly characterized contemporary German [21, §300, p. 573]. As Dümig has cautioned, however, we should not thereby conclude that Schopenhauer subscribed to linguistic relativism, according to which linguistic categories determine cognitive categories [5, pp. 157–158]. For one thing, Schopenhauer explicitly rejects the identity of word and concept; for another, linguistic relativism gets the causal story backwards: words are derived from the need to communicate concepts, which are abstracted from intuitive perception. The perception precedes the development of cognitive categories, which in turn precede the deployment of linguistic signs.

It is also worth noticing that, according to Schopenhauer's story, we cannot communicate the content of our intuitions directly because our bodies offer no mechanism by which to do so. Sight, to use Schopenhauer's example, may well be more adept at or direct in discerning the world, but it does not come equipped with the ability to manipulate percepts so that they can easily be communicated to others ([21, §301, pp. 574–575]; c.f. [6, §1, this volume], which argues that for Schopenhauer language has no social function, and serves primarily as a tool of internal processing). Consequently, we must resort to using language, which trades in audible rather than visual symbols. The result is that the content we communicate suffers from a doubled imperfection: (1) because in perceiving the world we merely perceive the phenomenal realm and (2) because the tool we use to communicate this imperfect vision is itself only capable of transmitting a small part of what is intuited.

3 Learning from the Tirades

It would be impractical for me to reproduce all of Schopenhauer's delightfully barbed comments here. Those so interested, however, should start by consulting the following: *WWR1* [23]: xx–xxi, xxiv, §37–40, §147, §263*, §495–496, §508, and §517; *WWR2* [24]: p. 12–13, 34 fn. 6, 40–41, 65, 70, 84, 87, 192–193, 303, 316, 442–443, 464, 582, 590, and 616; *Two Problems* [22]: Preface 1 (esp. ¶xvii–xxvi, ¶xxix–xxx), Preface 2 ¶xli; and therein, pp. 99–100, ¶85–86 of *FW*, and p. 149 ¶147 of *OBM*; *PP1*: 6, 21–28, 23 fn. 2, 70, 94–96, 132, 135, 141–142, 141 fn. 2, 144–146, 148, 153, 156–158, 161–163, 166–176, 178–180, 182, 191–192, 196, 375 fn. 18, and 396; *PP2*: §9 pp. 8–9, §10 p. 11, §11 p. 12, §21 p. 19, §28 p. 38, §42 p. 59, §74 pp. 104–105, §76 p. 109, §77 p. 112, §106 p. 196, §141 p. 279, §219 p. 431, §239 p. 456, 458, §241 p. 468, 470, §250 p. 483, §255 p. 486, §255 p. 486, §283 pp.

516–518, 524, 541–542, §297 pp. 561–562, and §377 p. 641–642; *FR* [17]: §V pp. vi–vii, §VII pp. 11–12., §VIII p. 15 & 17, §14 p. 22, §20 pp. 39–40, §21 pp. 83–84, §26 p. 99, §34 pp. 112–113, 117–118, 124; *On Vision* [18]: §III p. vi*; and *Will in Nature* [19]: xxi–ii, 6–7, and 141.

Worse, readers might find such a reproduction tedious and repetitive, since these all articulate the same basic complaints—and because Schopenhauer's meticulous editorial process saw him repeat his best turns of phrase across later editions of his works. Yet since my goal is to explain the root criticism that informs Schopenhauer's contumely remarks, I would be remiss if I did not identify at least a few representative instances of his tongue-lashings:

1. A tendency of minds to operate with such abstract and too widely comprehended concepts has shown itself at almost all times. Ultimately it may be due to a certain indolence of the intellect, which finds it too onerous to be always controlling thought through perception. Gradually such unduly wide concepts are then used like algebraical symbols, and cast about here and there like them. In this way philosophizing degenerates into a mere combining, a kind of lengthy reckoning, which (like all reckoning and calculating) employs and requires only the lower faculties. In fact, there ultimately results from this a mere *display of words,* the most monstrous example of which is afforded us by mind-destroying Hegelism, where it is carried to the extent of pure nonsense. But scholasticism also often degenerated into word-juggling ([24, p. 40]; Schopenhauer says much the same about Schellingians in his Berlin Lectures [27, p. 276]. Pluder [15, §1 and §2.2 (this volume)] offers an excellent explanation of Schopenhauer's derision for Scholasticism).
2. [...] a very peculiar device is often employed whose invention is traceable to Messrs. Fichte and Schelling. I refer to the artful trick of writing abstrusely, that is to say, unintelligibly; here the real subtlety is so to arrange the gibberish that the reader must think he is in the wrong if he does not understand it, whereas the writer knows perfectly well that it is he who is at fault, since he simply has nothing to communicate that is really intelligible, that is to say, has been clearly thought out [20, p. 162].
3. They therefore summarily forsook the only correct path found in the end by those wise men [viz., Bacon, Kant, and Locke], and philosophized at random with all kinds of raked-up concepts, unconcerned as to their origin and true content, so that Hegel's pretended wisdom finally resulted in concepts which had no origin at all, but were rather themselves the origin and source of things [24, p. 41].
4. [...] good style depends mainly on whether a writer really has something to say; it is simply this small matter that most of our present-day authors lack and is responsible for their bad style. But in particular, the generic characteristic of the philosophical works of the nineteenth century is that of writing without really having something to say; it is common to them all and can therefore be just as well studied in Salat as in Hegel, in Herbart as in Schleiermacher. Then according to the homeopathic method, the weak minimum of an idea is diluted with a fifty-page torrent of words and now with boundless confidence in the truly German

> patience of the reader the author calmly continues the twaddle on page after page. The mind that is condemned to such reading hopes in vain for real, solid, and substantial ideas; it pants and thirsts for any ideas as does a traveler for water in the Arabian desert and must remain parched [20, p. 163].

Word-juggling and philosophizing at random; ideas diluted to a weak minimum and spewed in a torrent of empty words; abstruse gibberish arranged to puff up the writer and denigrate his interlocutors; these are Schopenhauer's complaints. Yet there is much more substance to Schopenhauer's diatribes than to the grousing of a grumpy old man raging against the dying of the *licht*.

The problem, as Schopenhauer sees it, is that Hegel and his ilk are preoccupied with manipulating the 'empty husks' of concepts, turning them over and over again in order to derive ever-new and fanciful results bearing no essential connection to the world of perceptual intuition [24, p. 84]. G. Steven Neeley characterizes these as 'pseudo-concepts' and 'nonsensical utterances' [14, p. 49]. In other words, Schopenhauer's complaint was that the German Idealists misused abstractions, without regard for either (1) the fact that the more abstract the concept, the less perceptual content it possesses and, thus, the less epistemically sound its grounding and (2) the laws of thought which govern the manipulation of abstract symbols (viz., identity, non-contradiction, excluded middle, and sufficient reason; for an explanation of just what Schopenhauer and his contemporaries considered 'laws of thought', see [15, this volume, §2.1]). Allow me to explain.

When Schopenhauer accuses his contemporaries of employing a homeopathic method that dilutes ideas to a weak minimum, he is expressing a concern *about the level of abstraction* employed when they contrast one concept with its opposite—e.g. of finite and infinite, being and non-being, or unity, plurality, and multiplicity [24, p. 84; 17, §26, p. 99]—and from this process draw inferences about the nature of the noumenal. The abstract ideas resulting from this process of comparison are so detached from the perceptual kernel underlying them that they cannot be used to communicate, to impart knowledge about the author's perceptual intuition. Unaccompanied by perception, these concepts can yield only the most general knowledge of the thing represented, if that. They are an invitation for us to explain words with other words—that is to say, to use one imperfect communicative mechanism to communicate another, with ever-diminishing returns [24, pp. 71–72]. In other words, a-perceptual concepts are as empty of particular content as logic's constants and variables. No concept can hope to communicate its associated perception directly since, mediated as it is by the use of words, it amounts to an incomplete abstraction from perception. The most abstract concepts (such as those of logic and mathematics) are necessarily the least closely tied to perception, and the less grounded in perception a concept is, the more meaningless the words signifying it become. That said, Schopenhauer took exception to Euclidean geometry because he thought that its reliance on diagrams forestalled the possibility of gaining any real insight into the laws of spatial relationships underpinning those diagrams. See, for example, [22, §15; 7, this volume, esp. §3].

Consider, for example, the uses to which we put predicates and variables in first-order logic: we use *P(x)* to denote any statement *P* concerning the variable object *x*. We combine variables with constants to get terms, and we combine terms with connectives, delimiters (i.e. brackets), and quantifiers to build the well-formed formulae that allow us to draw valid inferences regardless of the particular content of each abstract variable, constant, or predicate. From Schopenhauer's perspective, we start with propositions, which express linguistic content, and abstract away from them by introducing variables, constants, etc., in order to focus our attention on the logical relationships underpinning our use of words. That said, Schopenhauer was quite dismissive of logic, and he followed his contemporaries in characterizing it as largely useless, since the rules of logic were supposed to govern thinking at the pre-reflective level. For an excellent explanation of Schopenhauer's and his contemporaries' attitude towards logic, see [15], this volume, esp. §1. But Schopenhauer made one significant exception: the rules of logic are useful when it comes to exposing deliberate attempts to deceive in an argument. And this is exactly what Schopenhauer accuses his contemporaries of attempting.

The concepts deployed by the German Idealists, Schopenhauer thinks, are no more than logical constants or variables, capable of being filled by any particular content whatsoever. Worse still, according to Schopenhauer, the Idealists do not offer us a guide to their interpretation, but leave us to interpret their pseudo-concepts as we please. This, then, is the crux of the problem, for the allegation here is that the use of these abstractions is not even underpinned by a communicative intention (as is the case in logic and mathematics).

This is why Schopenhauer thinks that, even when speaking in their native tongue, 'those of limited ability' (including, presumably, the 'thick-' and 'shallow-skulled' followers of 'Hegelry'—[20, p. 166]

> always merely make use of hackneyed phrases (*phrases banales*, *abgenutzte Redensarten*); and even these are put together with so little skill that we see how imperfectly aware they are of their meaning and how little their whole thinking goes beyond the mere words, so that it is not very much more than parrot chatter. For the opposite reason, originality of idiom and individual fitness of every expression used by a man are an infallible symptom of outstanding intellect [21, §299, pp. 569–570].

For more on Schopenhauer's association of genius with transparency of exposition and the ability to compare concepts with perceptions, see [26]. Rather than use language to communicate their intuitive perceptions of the noumenal, Hegel and his 'mercenary followers' [20, p. 96] instead offer us a string of symbols and instruct us to make of them what we will. We are given *words*—'empty, hollow, disgusting verbiage' [17, §20, pp. 127–128]—rather than *ideas*; not so much a *thought* as a parrot's *squawk*, perhaps paired with a *squeak* and contrasted to a *screech*. Nor is this cacophony governed by the laws of thought, since these are gleefully jettisoned in service of the infamous 'dialectic' which plays havoc with identity, non-contradiction, and excluded middle, if not also sufficient reason; c.f. [16]. Preface 1, esp. p. 16–19, xx–xxiii, where Schopenhauer takes issue with Hegel's apparent misunderstanding of syllogistic logic and the notion of a contradiction. Indeed, how could it be so-governed when its component sounds are utterly devoid of representational content in the first place? The result, Schopenhauer says, is that 'their voice found an echo which even now reverberates and spreads in the numb skulls of a thousand stooges' [17, §20, pp. 127–128].

4 Conclusion

I have tried, throughout this chapter, to remain neutral about the merit of Schopenhauer's accusations against his contemporaries. Determining their worth requires much more, and more careful, scholarship than I have had the space to undertake here. And, indeed, this is work that many have already tackled—see, e.g., [9–11, 29, 31]. Instead, I have simply tried to show that Schopenhauer's blistering invective gains its content—and its bite!—from his analyses of concepts and of language more broadly. I have argued that the precise content of the vitriol Schopenhauer infamously directs at his contemporaries is worthy of philosophical attention—not because it expresses his critical take on Fichte, Hegel, Herbart, Schelling, and Schleiermacher (which it does), but because it neatly illustrates his philosophy of language, and his analysis of concepts. In particular, it draws our attention to his emphasis on the epistemic value of perceptual intuition. As long as we stick to what is given to us in intuition, Schopenhauer thought, then we cannot err: intuition is sufficient unto itself [23, §8, p. 58]. Abstract reasoning helps us to communicate our concepts to others [23, §6, p. 43], but it also introduces new sources of doubt and error, since it takes us further away from the givens of intuitive perception. The Idealists' cardinal sin was just to abuse our faculty of abstraction, piling ever more concepts atop one another without pausing to anchor them in intuitive perception. The result, I have argued, is that Schopenhauer's epithets are not merely spiteful slurs. Instead, they reflect a deep-seated *theoretical*, as well as methodological, commitment to transparency of exposition.

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Part II
Logic

Schopenhauer's Eulerian Diagrams

Amirouche Moktefi

Abstract Philosopher Arthur Schopenhauer included some logic diagrams in his major work: *The World as Will and Representation*, published in 1818. Few years later, he made a thorough use of diagrams in his Berlin Lectures that have not been published until 1913. These works are seldom mentioned in logic diagrams literature. This paper surveys and assesses Schopenhauer's diagrams and the extent to which they conform to the scholarship of his time. It is shown that Schopenhauer adopted a scheme that is largely inspired from Leonhard Euler's circles but includes some interesting innovations that were unknown to Euler. Two curiosities are particularly inspected: an inventory of logical relations and a diagram on the routes to good and evil.

Keywords Schopenhauer · Euler diagram · Gergonne relations · Syllogistic · Eristic

Mathematics Subject Classification (2020) Primary 03A05, Secondary 01A55

1 Introduction

It is little known that Philosopher Arthur Schopenhauer made use of logic diagrams in his main work *The World as Will and Representation* (1818) [42] and the subsequent Berlin Lectures [41]. His name does not appear in the surveys on the history of the subject [1, 11, 18, 40], with the notable exception of [8]. The aim of this paper is to survey and assess the Eulerian diagrams used by Schopenhauer and the extent to which they conform to what we would expect from a logician of the early nineteenth-century. We open with a brief exposition of Euler's circle diagrams and their linear counterpart. Then, we consider Schopenhauer's familiarity

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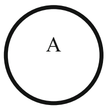
with such diagrams and their principles of representation. The following sections discuss two important curiosities that are found in his major work: an inventory of the relations of concepts with the aid of diagrams and a surprising complex diagram involving a high number of concepts. We conclude with an appreciation of the role of diagrams in logic as advocated by Schopenhauer.

2 Eulerian Diagrams

The use of diagrams in logic is ancient [14, 21]. Some figures such as the square of opposition and the tree of porphyry were particularly common. However, in this chapter, we are specifically interested in diagrams that show the composition of propositions and that were used to solve simple logic problems such as syllogisms. Such diagrams, hereafter Euler(ian) diagrams, are named after the mathematician Leonhard Euler who popularized them in the second volume of his *Letters to a German Princess* (1867) [15]. Euler's diagrams are first published in Letter *CII*, dated 14 February 1761. They are introduced to represent logic propositions:

> These four species of propositions may likewise be represented by figures, so as to exhibit their nature to the eye. This must be a great assistance towards comprehending more distinctly wherein the accuracy of a chain of reasoning consists.
>
> As a general notion contains an infinite number of individual objects, we may consider it as a space in which they are all contained. Thus, for the notion of *man* we form a space [...]



> in which we conceive all men to be comprehended [16, p. 339]

Hence, a space (here a circle) stands for the extension of the term it represents. It is then easy to represent propositions by making the topological relations of those spaces stand for the logical relations of the terms that form those propositions. For instance, if we are told that 'Every Animal is Beautiful', it suffices to draw two circles that would stand for the extensions of 'Animal' and 'Beautiful', respectively, in such a way as to have the circle 'Animal' strictly included in the circle 'Beautiful', as shown in Fig. 1. Similarly, if we wish to represent the proposition 'No Animal is Beautiful', we simply draw two disjoint circles standing for 'Animal' and 'Beautiful', respectively, as shown in Fig. 2.

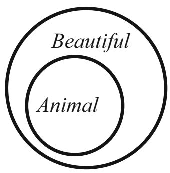


Fig. 1 Every Animal is Beautiful

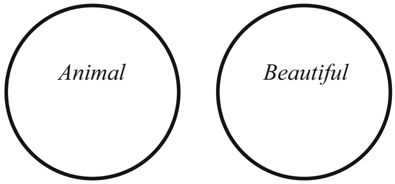


Fig. 2 No Animal is Beautiful

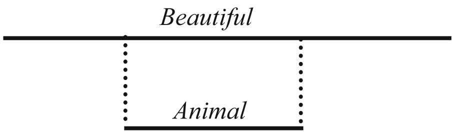


Fig. 3 Every Animal is Beautiful

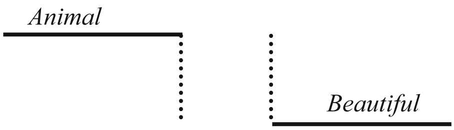


Fig. 4 No Animal is Beautiful

Although Euler favoured circles, other shapes could have been similarly used as long as the figure encloses the extension of the term that is represented. In particular, several logicians in Euler's time preferred to use line segments for a similar purpose [3]. One simply represents terms with line segments (instead of circles). For instance, to represent the above proposition 'Every Animal is Beautiful', we simply represent each term 'Animal' and 'Beautiful' with a line segment. Then, we indicate their logical relation by drawing the segment 'Animal' strictly under the segment 'Beautiful', as shown in Fig. 3. The vertical dotted lines precisely indicate that the smaller segment is to be understood as part of the larger one. Likewise, if we were to represent the other proposition 'No Animal is Beautiful', it suffices to represent two disjoint segments 'Animal' and 'Beautiful', as shown in Fig. 4. Hence, linear diagrams proceed in the same manner as Euler's circles: terms are represented with a geometrical figure, and then logical relations between those terms are represented by the geometrical relations between their corresponding figures. In this sense, linear diagrams may be said to be Eulerian diagrams as well [33, pp. 608–609].

Now that we managed to represent propositions with the help of these figures, it is possible to use them to solve some logic problems, in particular syllogisms that students commonly face in formal logic. It is reminded that a syllogism is formed of a set of three propositions: two premises and a conclusion. A traditional syllogism contains three terms, notably a middle term which is found in both premises, while the other two terms are found in the conclusion. A syllogistic form is said to be valid if whenever its premises are true, its conclusion must be true. It is easy to use Eulerian diagrams to check the validity of a given syllogistic form. It suffices to represent the premises jointly in a single diagram and to verify if the latter conveys the conclusion as well. For instance, let us consider the following syllogistic form:

- Every Animal is beautiful
- Every Cat is an Animal
- Therefore, Every Cat is Beautiful

Fig. 5 A syllogism with circle diagrams

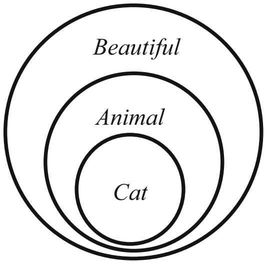
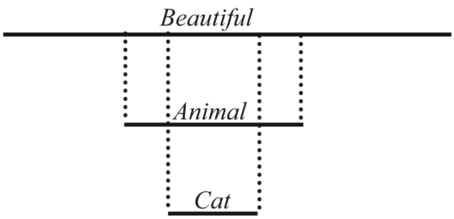


Fig. 6 A syllogism with linear diagrams



The representation of the premises with (circular) Euler diagrams is obtained by drawing three circles 'Cat', 'Animal' and 'Beautiful' in such a way as to depict the inclusion of the first in the second and the second in the third, as shown in Fig. 5. The same information can be represented with linear diagrams, as shown in Fig. 6. It is now easy to observe on both diagrams that the conclusion of the syllogistic form holds. For instance, this is evidenced in the circular Euler diagram by the inclusion of the circle 'Cat' in the circle 'Beautiful'. Similarly, the linear diagram shows the segment 'Cat' under the segment 'Beautiful'. One may thus conclude that 'Every Cat is Beautiful' and that the syllogistic form above is indeed valid.

These diagrammatic schemes provide simple and intuitive methods to represent propositions and solve elementary logic problems such as syllogisms. They convey the information contained in the premises and allow an easy access to the conclusion through observation [47]. It is hence easy to understand their success in Schopenhauer's time. A close look shows however that the appeal to these schemes in traditional logic faces some difficulties. Indeed, these diagrams offer an imperfect correspondence with the propositional forms that were commonly used by logicians. For instance, if we consider a universal proposition of the form 'Every A is B', it is unclear whether A coincides with B or is strictly included in it. Hence, there actually are two possible schemes that correspond to that proposition. The situation is even more complex with particular propositions (starting with 'Some') which troubled Euler and most of subsequent logicians until the end of the nineteenth-century [39]. Because they represent actual relations between terms, Eulerian diagrams poorly handle our uncertainty as to the scope of a given proposition. In such cases, the logician is often forced to manipulate several diagrams exhibiting each a possible configuration. This difficulty was overcome in 1880 by John Venn who introduced his own diagrams as an improvement over Euler's circles [49].

3 Spheres and Concepts

When Schopenhauer enters the scene at the beginning of the nineteenth-century, Eulerian diagrams already had a long history. Indeed, it is not rare to meet with such diagrams in logic texts prior to him, and it might even be said that they enjoyed some popularity among his contemporaries and immediate predecessors, specifically in the German-Speaking world [26]. Both circular and linear diagrams were already known to Gottfried Leibniz [9]. Subsequently, Johann Heinrich Lambert, Gottfried Ploucquet and naturally Euler are among the main representatives of this diagram tradition in the second part of the eighteenth-century [25]. Among Schopenhauer's immediate predecessors, Immanuel Kant is an important figure that made use of Eulerian diagrams and it is likely that such scheme spread after him through his critics and followers [30]. Schopenhauer certainly knew of earlier works for he briefly refers to several among them:

> The presentation of these spheres by figures in space is an exceedingly happy idea. Gottfried Ploucquet, who had it first, used squares for the purpose. Lambert, after him, made use of simple lines placed one under the other. Euler first carried out the idea completely with circles [42, p. 42].

It is known that Schopenhauer attended and was influenced by Gottlob Ernst Schulze's lessons in Göttingen. Schopenhauer's logic notes from 1811 include few logic diagrams, both circular and linear, and an explicit reference to Euler [10, pp. 232, 237]. Although no such diagrams are found in Schulze's *Grundsätze der Allgemeinen Logik* (1810) which was used for the lessons [43], it is likely that they were used during the lectures. Later on, Schopenhauer included some logic diagrams in *The World as Will and Representation* (1819), his major work [42]. In the following years, he made a thorough use of diagrams in his Berlin Lectures that were first published in 1913 (and reprinted in 1986 [41]; see [27]). This lesser-known work suggests both familiarity and consideration for diagrams and their use in logic.

Schopenhauer's conception of logic diagrams resembles to a large extent that of Euler. He encloses the sphere (i.e. extension) of a concept within a circle. Then, the relations of the circles express the relations of those concepts:

> From what has been said it follows that every concept, just because it is abstract representation, not representation of perception, and therefore not a completely definite representation, has what is called a range, an extension, or a sphere, even in the case where only a single real object corresponding to it exists. We usually find that the sphere of any concept has something in common with the spheres of others, that is to say, partly the same thing is thought in it which is thought in those others, and conversely in those others again partly the same thing is thought which is thought in the first concept; although if they are really different concepts, each, or at any rate one of the two, contains something the other does not. In this relation every subject stands to its predicate. To recognize this relation means to *judge* [42, p. 42].

Schopenhauer clearly adopted an extensional interpretation of Euler diagrams. This fact might look trivial to modern readers who are accustomed to this dominant interpretation in elementary logic and set theory, but it must be reminded that an intensional interpretation had its supporters in earlier periods of history as well. Leibniz, for instance, is known for his work on logic calculi that would admit both interpretations [2]. It is reminded that while the extension of a concept refers to the set of individuals it denotes, the intension (or intent) rather refers to the qualities that are shared by those individuals. To produce intensional Eulerian diagrams, we need to make circles stand for the intension of the concept, and hence, to enclose the qualities that it connotes. If we were to represent the proposition ‘Every Cat is an Animal’ intensionally, we need to draw a circle ‘Animal’ inside a circle ‘Cat’. Indeed, being a cat requires more qualities than merely those required to be an animal in the first place. We see how the intension of the concept ‘Cat’ includes that of the concept ‘Animal’, while, inversely, the extension of the concept ‘Cat’ is actually included in that of the concept ‘Animal’. Although counterintuitive, we see that it is possible to represent our proposition ‘Every Cat is an Animal’ with intensional Euler diagrams. The situation is more complicated for other forms of propositions and it may be said that logicians made little success in handling them. It appears that the topological relations of Eulerian diagrams match better with the extensional interpretation [48].

Yet, intensional interpretations of logic were still widespread in Schopenhauer’s time and it is merely by the end of the nineteenth-century that extensional interpretations dominated the scene. Clarence I. Lewis argued in 1918 that the successes of George Boole’s symbolic logic and his followers can be explained by their appeal to extension unlike earlier attempts, especially in the German-Speaking world, which favoured intension [29, pp. 35–37]. However, one might object that many intensional logicians also adopted extensional interpretations when they attempted to construct logical calculi. Leibniz is a good example but others can be found even among Boole’s British followers, such as William Stanley Jevons [44]. As discussed earlier, Euler clearly intended his circles to represent the extension of the concepts they stand for, and so did Schopenhauer.

Although several clues show that Schopenhauer knowingly worked out his diagrams in the manner of Euler, it should not be inferred that his knowledge of logic diagrams sprang solely from Euler’s work. Indeed such diagrams were popular in Schopenhauer’s time and he acknowledged that this ‘schematism of concepts [. . .] has been fairly well explained in several textbooks’ [42, p. 44]. Also, several features of Schopenhauer’s diagrams are not found in Euler’s work. We already alluded to linear diagrams which are absent from Euler’s *Letters* but were certainly known to Schopenhauer [41, p. 287]. Another interesting feature that pertains to circular diagrams is the appeal to discontinuous lines to represent uncertainty. As indicated earlier, there is an imperfect correspondence between traditional forms of logic propositions and Euler’s diagrams. For instance, if one holds the proposition ‘Every *A* is *B*’, it may be that *A* is strictly included in *B* or that *A* rather coincides with *B*. Without further specification, both configurations are legitimate. Hence, if one wishes to rigorously represent that proposition, it is needed to offer two diagrams: one in which circle *A* is shown to be strictly included in circle *B* and another in which circles *A* and *B* coincide. If one uses linear diagrams, she will similarly need a diagram where a segment *A* is shown to be strictly under a segment *B* and another

Fig. 7 Every *A* is *B*

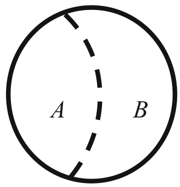


Fig. 8 Every *A* is *B*

diagram where segments *A* and *B* coincide. This is not a happy solution since it decreases the visual aid offered by the diagrams.

To overcome this difficulty, several logicians introduced a simple convention to combine those two legitimate configurations. The idea is to express uncertainty with dotted lines. For instance, in Fig. 7, it is not known if the dotted line exists. If it does exist, then the space *A* is shown to be strictly included in the space *B* (which is the entire circle). If the dotted line does not exist, then spaces *A* and *B* will simply coincide. This trick makes it possible to have both configurations of the above proposition represented within a single diagram. A similar solution can be adopted for linear diagrams as shown by Fig. 8.

This convention, which is already found in Leibniz's logic writings [9, pp. 30–31], was later used by Johann H. Lambert (1764) [23] and apparently spread among subsequent authors. It is, for instance, found in Johann Maass's 'remarkable diagrams' published in his *Grundriss der Logik* in 1793 ([31]; see [4]). It was certainly known to many early nineteenth-century logicians and it is, hence, unsurprising to find it in Schopenhauer's Lectures [41, p. 325]. Yet, it attests that Schopenhauer's acquaintance with logic diagrams did not solely derive from Euler.

4 Relations of Concepts

Schopenhauer uses the relations between circles to express the relations between concepts, since each circle stands for the sphere of a concept:

> For logic, however, it is a very fortunate circumstance that all the relations of concepts can be made plain in perception, even according to their possibility, i.e., *a priori*, through such figures [42, p. 42].

Schopenhauer proceeds to listing five relations of concepts to which '[a]ll combinations of concepts may be referred to' ([42, p. 44]; see also [12] in this volume), as shown in Fig. 9:

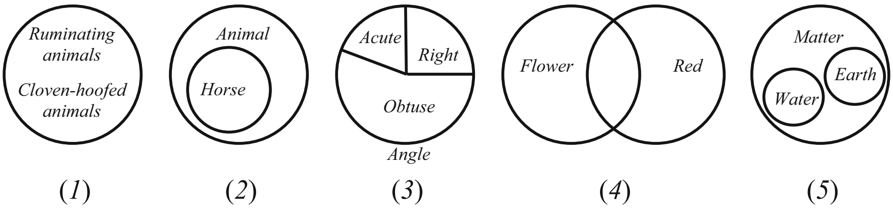


Fig. 9 Relations of concepts

1. The first relation exhibits two concepts whose spheres ‘are equal in all respects’ [42, p. 42]. He then provides several examples, such as the two ‘convertible’ concepts: ‘ruminating animals’ and ‘cloven-hoofed animals’. Although he does not provide a visual representation of it, Schopenhauer explains that such ‘concepts, then, are represented by a single circle that indicates either the one or the other’. [42, p. 43].
2. In the second relation, the ‘sphere of one concept wholly includes that of another’ [42, 43]. Schopenhauer illustrates this case with a figure that exhibits the sphere of the concept ‘Horse’ entirely included in that of the concept ‘Animal’.
3. In the third relation, a ‘sphere includes two or several which exclude one another, and at the same time fill the sphere’ [42, p. 43]. Schopenhauer provides the example of the sphere ‘Angle’ which includes the spheres ‘Acute’, ‘Right’ and ‘Obtuse’. Evidently, the latter three spheres were meant to be ‘Acute angle’, ‘Right angle’ and ‘Obtuse angle’, respectively.
4. In the fourth relation, we are told that two ‘spheres include each a part of the other’ [42, p. 43]. For instance, the spheres of ‘Flower’ and ‘Red’ intersect but none includes the other since there are flowers that not red and there are red objects that are not flowers.
5. Finally, in the fifth relation described by Schopenhauer, ‘Two spheres lie within a third, yet do not fil it [. . .] This last case applies to all concepts whose spheres have nothing immediately in common, for a third one, although often very wide, will include both’ [42, p. 44]. We are given the example of the spheres ‘Earth’ and ‘Water’ which are separate, yet both are included within the sphere ‘Matter’.

This inventory is an interesting direction in working with diagrams, especially when it serves for the construction of higher structures by combining different relations. There is however an oddity that makes this catalogue of relations difficult to apprehend. Indeed, only cases (1), (2) and (4) express *simple* relations (between two spheres). The two remaining cases (3) and (5) involve relations with more than two spheres: case (5) exhibits three spheres, while case (3) may involve further spheres. This is uncharacteristic since other complex relations that are not listed by Schopenhauer can well be constructed. Also, one might argue that listing such complex relations is superfluous in the sense that they can be reduced to a combination of simpler relations that are already in the inventory.

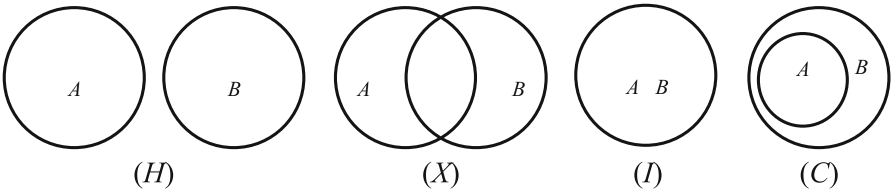


Fig. 10 Gergonne relations

In order to make sense of these relations, it is worthwhile comparing Schopenhauer's inventory with that of one of his contemporaries, Joseph D. Gergonne, who undertook a similar task in 1817:

> Let us presently examine the diverse circumstances in which two ideas, when compared to each other, can be relatively to their extension. This question evidently goes back to inquiring about the diverse sorts of circumstances in which two closed figures whatever, two circles for instance, drawn on a same plane, can stand one to the other; the extension of each circle representing here that of each idea. ([19]; translated by the author)

Although Gergonne does not actually draw such circles, he subsequently referred to the task of drawing such 'systems of circles' worked out 'in the manner of Euler', thus attesting his acquaintance with Euler's diagrams [19, p. 200]. Gergonne explicitly restricted his inquiry to simple propositions and hence investigated only relations of two extensions to each other. Let those extensions be *A* and *B*. Gergonne listed and named four relations (*H*), (*X*), (*I*) and (*C*), shown in Fig. 10 and defined as follows:

- (*H*) *A* and *B* are strictly disjoint.
- (*X*) *A* and *B* partly overlap.
- *(I)* *A* and *B* coincide.
- (*C*) *A* is strictly included in *B*.

Gergonne additionally considered the case where '*A* strictly includes *B*' and listed it as a separate relation when *A* and *B* are viewed as the terms of a proposition [17, 20]. Yet, that proposition can be expressed as '*B* is strictly included in *A*', and hence this case can be viewed as a mere variation (reversal) of the existing relation (*C*).

Gergonne identified four *simple* relations while only three were found in Schopenhauer's inventory. A quick look shows that Schopenhauer's *simple* relations (1), (2) and (4) correspond to Gergonne's relations (*I*), (*C*) and (*X*), respectively. However, we observe that Gergonne's simple relation (*H*) is not distinctly listed in Schopenhauer's inventory. Instead, one finds two complex relations (3) and (5). It is challenging at first to explain this discrepancy. One reason is that Schopenhauer chose to illustrate his relation (3) with a diagram that relates three spheres to a fourth. This makes it difficult to see how this relation stands to the others. The puzzle diminishes if, instead, we were offered an illustration involving a number of spheres that is similar to that which is found in relation (5), as shown in Fig. 11.

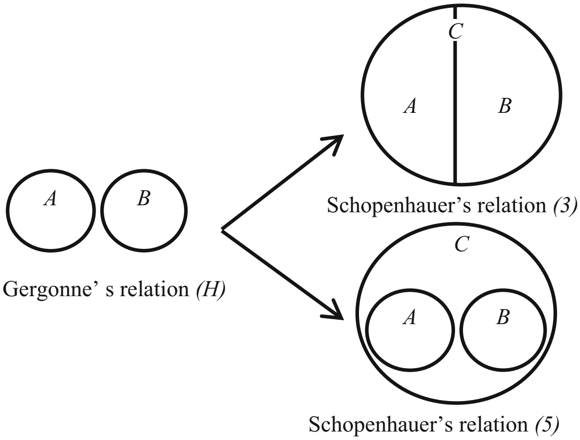


Fig. 11 From Gergonne to Schopenhauer

It becomes easier to see that Schopenhauer's relations (3) and (5) both depict Spheres (for instance, *A* and *B*) that are strictly disjoint and, hence, that stand to each other in the Gergonne relation (*H*). Interestingly, Schopenhauer considers next how these Spheres *A* and *B* stand to a third (*C*) that includes both of them. If the outer region is empty, we are in the situation described by the relation (3), otherwise we obtain the relation (5). It is unclear why Schopenhauer generalized to a higher number of spheres the relation (3) alone. It is also perplexing why he did not similarly consider how spheres in relations (1), (2) and (4) may stand to another sphere that includes them.

Yet, despite its lack of systematicity, Schopenhauer's inquiry is interesting because it addresses the problem, seldom considered in his time, of the outer region (left open) of an Euler diagram. This issue will resurface in later debates on the diagrammatic representation of the Universe of discourse and the treatment of negative terms [5]. Against Venn, other logicians such as Lewis Carroll chose to restrict the Universe and hence to devote a limited space to the outer region [7, 32]. We saw how Schopenhauer extended a simple Gergonne relation and identified two distinct complex relations depending on how spheres stand to a higher sphere that includes them. A generalization of this process was carried out by John Neville Keynes who identified seven relations between terms *S*, *P*, and their complementary terms *not-S* and *not-P*:

> In Euler's diagrams, as ordinarily given, there is no explicit recognition of *not-S* and *not-P*; but it is of course understood that whatever part of the universe lies outside *S* is *not-S*, and similarly for *P*, and it may be thought that no further account of negative terms need be taken. Further consideration, however, will shew that this is not the case; and, assuming that *S*, *not-S*, *P*, *not-P* all represent existing classes, we shall find that *seven*, not five, determinate class relations between them are possible [22, p. 170].

Unsurprisingly, Schopenhauer's two complex relations (3) and (5) are listed among Keynes' relations and numbered (vii) and (vi), respectively [22, p. 171]. It may be said that Keynes independently generalized the application of this technique that is merely initiated by Schopenhauer.

5 Complex Diagrams

So far, we have discussed only applications of diagrams to simple logical problems such as syllogisms. However, logicians also handled more complex problems, known as sorites, involving a higher number of terms. Suppose we had to face a problem involving four terms and were given the following propositions as premises:

(1) Every *A* is *B*
(2) Every *C* is *D*
(3) No *B* is *D*

To deduce the relation of terms *A* and *C* is easy with an Euler diagram depicting the relations expressed in the premises between the terms *A*, *B*, *C* and *D*, as shown in Fig. 12.

One immediately observes that circles *A* and *C* are disjoint and hence may conclude that:

- No *A* is *C*

However, this conclusion can also be obtained simply by handling our premises in pairs in order to produce a series of syllogisms. For instance, one may take propositions (1) and (3) to be the premises of a syllogism whose conclusion is:

(4) No *A* is *D*

Then, the new proposition (4) is associated with Proposition (2) which has not been considered yet. They form together the premises of a syllogism whose conclusion is:

- No *A* is *C*

Hence, we obtained the same conclusion that was observed in Fig. 12. It is thus unnecessary, but not necessarily unhelpful, to construct diagrams for more than three terms (which is the number demanded by syllogisms). No such complex diagrams are found in Euler's *Letters* where it is said that the syllogism is 'the only method of discovering unknown truths. Every truth must always be the conclusion of a syllogism, whose premises are indubitably true' [16, p. 350]. It is merely later, with the development of symbolic logic that the need for more complex diagrams was felt, especially after the introduction of Venn diagrams [37, 38].

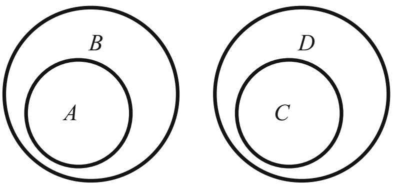


Fig. 12 Euler diagram for 4 terms

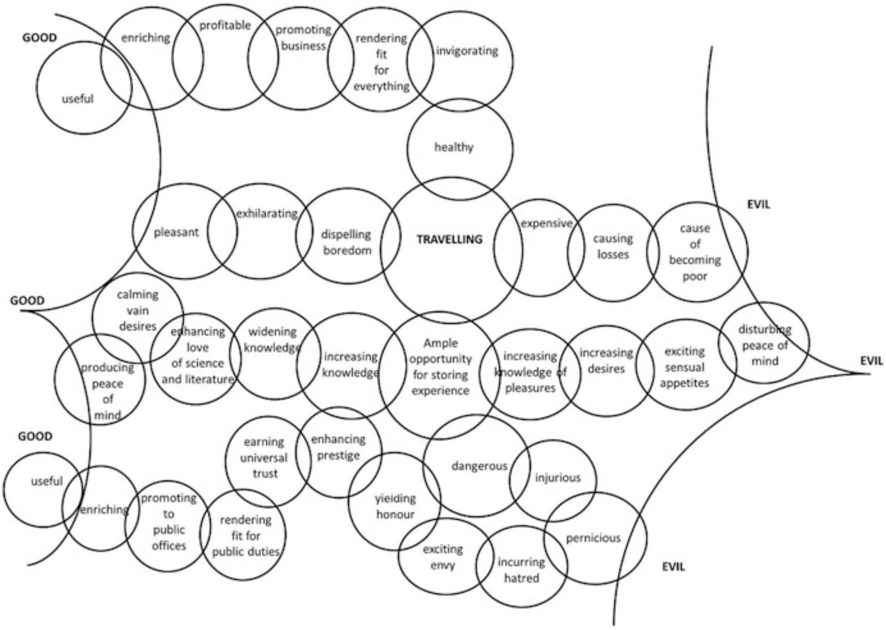


Fig. 13 Schopenhauer's diagram of Good and Evil

Prior to Venn, it was rather rare to meet with such complex diagrams. Sorites were rather reduced to series of syllogisms (which can be handled with simple diagrams). Schopenhauer was familiar with this technique for, after explaining how syllogisms can be treated with the help of diagrams, he observed that if 'many spheres are brought together in this way, there arise long chains of syllogisms' [42, p. 44]. Yet, his Berlin Lectures show many complex diagrams involving more than three spheres [41, pp. 382–383]. In *The World as Will and Representation* ([42, opposite to p. 49]; see also [41, p. 384]), we find a remarkably complex diagram depicting about 35 spheres within a large area situated between the spheres of 'Good' and 'Evil', as shown in Fig. 13.

This complex diagram might look at first as a remarkably rare instance of a pre-Venn diagram exhibiting the relations of a high number of concepts. However, a look at its description by Schopenhauer suggests a different, but still remarkably rare, interpretation [28]. Let us imagine a speaker who attempts to convince others that travelling is evil. For the purpose, she needs to stand on the sphere that stands for 'travelling' (at the centre of the diagram) and to follow the route that leads to the big area that stands for 'Evil' (on the right of the diagram). This is achieved by 'jumping' from one sphere to another by exploiting their intersections, as shown in Fig. 14. For instance, such a speaker may argue that travelling is expensive, and hence causes losses which, in turn, cause becoming poor. The latter is, then, said to lead to evil. Hence, the overall argument aims at persuading that travelling is evil.

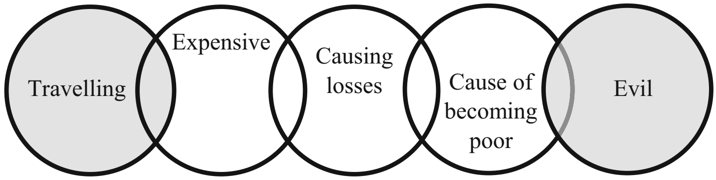


Fig. 14 Travelling is evil

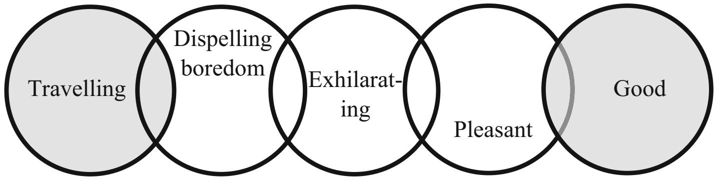


Fig. 15 Travelling is good

Yet, the diagram itself paradoxically depicts the sphere of travelling strictly outside the sphere of Evil. It is thus to be understood that the diagram is not expected to represent the actual relations of the concepts that it contains. It merely shows the 'route' that our speaker has to follow to move from a sphere to another which is not adjacent to it. Naturally, using the same procedure, a different speaker could have followed an entirely different route to argue that travelling is actually good (Fig. 15).

Neither speaker is consistent with the actual one-to-one relations of spheres displayed by the chains of diagrams along the route. Each unduly generalizes what is known of part of a sphere to the whole sphere itself. For instance, we are told by the first speaker that travelling is expensive, while the diagram shows that only some travelling is expensive. Similarly, the second speaker would state that travelling dispels boredom, while only some travelling is seen on the diagram to be dispelling boredom. Both speakers (fallaciously) operate as such for the purpose of persuading us of their claim. Schopenhauer describes this method as follows:

> The sphere of a concept is almost invariably shared by others, each of which contains a part of the province of the first sphere, while itself including something more besides. Of these latter concept-spheres we allow only that sphere to be elucidated under which we wish to subsume the first concept, leaving the rest unobserved, or keeping them concealed. On this trick all the arts of persuasion, all the more subtle sophisms, really depend [42, p. 49].

It follows that the diagram of good and evil should not be interpreted as a depiction of the relations between the concepts it contains. The diagram rather depicts routes of persuasion where it is possible to move from a concept to another through a sequence of concepts. For instance, if one moves along a sequence *A*–*B*–*C*, the diagram shows the relation between *A* and B, then between *B* and *C*, but it does not intend to depict the relation between *A* and *C*. It merely shows the chain of moves that are to be made to move from *A* to *C*. This is an interesting feature that is seldom noticed in diagram studies even though other examples of diagrams depicting sequences (a task that often comes to depicting a sequence of diagrams) are sometimes described in literature. Some good examples are found in medieval mathematical manuscripts where the diagrams include an expression of their process of drawing [45]. Recent formal diagrammatic systems also convey the conception of diagrams as objects that are treated in sequences in accordance with strict rules of inference [36, 46].

6 Conclusion

Schopenhauer made a thorough use of Eulerian diagrams in his Lectures. As such, he considered them as useful pedagogical tools to facilitate logical reasoning. However, it would be misleading to confine his interest in diagrams to the educational province. Indeed, a survey of his diagrams shows that he was acquainted with the main diagrammatic innovations of his time and has contributed to the field, notably through the design of his uncharacteristic diagrams for eristic. Also, Schopenhauer promoted a high status for diagrams in logic, not only for the discovery of new truths, but also for their justification since syllogistic rules can be reduced to diagrammatic manipulations:

> This schematism of concepts [. . .] can be used as the basis of the theory of judgements, as also of the whole syllogistic theory, and in this way the discussion of both becomes very easy and simple. For all the rules of this theory can be seen from it according to their origin, and can be deduced and explained [42, p. 44].

Further evidence of Schopenhauer's high consideration for diagrams is given by Jens Lemanski in [24]. However, this enthusiasm has to be moderated for two main reasons.

First, Schopenhauer had a limited esteem for the practical utility of logic itself. This view contrasts with that of some subsequent logicians, such as Carroll who thought of logic as a public good and worked on its social promotion [34]. Although Schopenhauer's scepticism does not undermine the status of diagrams within the logical realm, it weakens the plea for diagrams in practical setting.

Second, Schopenhauer noted how fortunate it was that relations of concepts can be represented with the aid of the Eulerian scheme but failed to explain why it is so:

> On what this exact analogy between the relations of concepts and those of figures in space ultimately rests, I am unable to say [42, p. 42].

Schopenhauer's trust in diagrams thus seems to partly spring from faith rather than evidence in their aptitude to convey logical reasoning. This puzzle is naturally not restricted to diagrams. Later on, Boole and his followers faced a similar challenge when the former introduced his logical notation which exhibited 'analogies' between the algebra of logic and 'quantitative' algebra [6, 13]. Part of the mystery vanishes when it is reminded that these languages (both algebraic and diagrammatic) have been precisely designed to fulfil specific functions and may prove unsatisfactory in other settings [35]. Hence, the above analogy may be said to rest to a large extent on the appropriate choice of logical and geometrical objects [33].

It remains that Schopenhauer's enthusiasm attests that he highly regarded these Eulerian diagrams and worked on their promotion. This effort went largely unnoticed and has hardly been acknowledged by historians of logic diagrams so far. It is hoped and expected that the present volume will contribute to making Schopenhauer's diagrammatic work better known and to encourage further exploration.

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Schopenhauer's Logic in Its Historical Context

Valentin Pluder

Abstract Schopenhauer never wrote a whole book on logic, but there are nonetheless several passages in his works where he reflects extensively on the topic. His approach to logic is dominated by two beliefs that were very common in the period: firstly, that there had been hardly any developments in the field of logic since Aristotle and, secondly, that everybody intuitively and unwittingly follows the rules of logic without first needing to be taught. Although Schopenhauer argues that there had been no crucial developments in logic since the days of Aristotle, he does give a short list of the enhancements and additions that logic had undergone in the intervening period. However, Schopenhauer does not prove himself to be a historian of logic. Rather, he positions himself within the context of the contemporary debate on logic. As a result, he places a clear emphasis on, firstly, the principle of sufficient reason of knowing and, secondly, the separation of concepts and representations of perception. This paper works through Schopenhauer's own list of the main developments in the history of logic and offers critical commentary on it. It concludes by examining some of the issues that do not appear on Schopenhauer's list.

Keywords Schopenhauer · Logic · Nineteenth century

Mathematics Subject Classification (2020) Primary 03A05, Secondary 01A55

1 Introduction

Schopenhauer never wrote a whole book on logic, but several passages in which he reflects extensively on logic can nonetheless be found in his works. To name the most important of these passages in chronological order: the remarks on the principle of sufficient reason of knowing in *The Fourfold Root of the Principle of Sufficient Reason* (1813) [19, pp. 114–126], §9 in the first volume of *The World*

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as Will and Representation [WWR] (1819) [20, pp. 39–50], the long passages in the manuscripts of his 1820s lectures [21, pp. 234–366] and finally the passages on logic in general and on syllogistics in the supplements to the first book of the WWR (1844) [22, pp. 102–117].

Schopenhauer's texts on logic span a period of at least 31 years—and that is without considering the variations in later editions. Bearing in mind the changes in Schopenhauer's thought, it is unsurprising that the texts do not fit together perfectly, but are instead somewhat heterogeneous. Thus, it is not possible to assemble all the pieces into a single, coherent system of logic.

The paper starts by examining Schopenhauer's views on logic in general: specifically, he holds that not only is logic an already-perfected science, but also that everyone intuitively follows the rules of logic. The second section then presents the most important *Zusätze und Verbesserungen* (*additions and enhancements*) that Schopenhauer believes logic had undergone in the time since Aristotle. Finally, the third section looks at some of the aspects Schopenhauer did not deem worthy of being mentioned among the *Zusätze und Verbesserungen*: in particular, the *Port-Royal Logic* [1] and Kant's contributions to the logic of the nineteenth century. The purpose of these reflections on Schopenhauer's writings on the history of logic is not to present him as a historian of logic, which he certainly is not. Rather, his brief and occasional remarks on the history of logic reveal the aspects of logic that he considers noteworthy, something that is reflective not only of his perspective on logic but also of how logic was commonly understood in his contemporary context.

2 Logic as an Already-Perfected Science and Intuitive Way of Thinking

If Schopenhauer's writings on logic are considered as a whole, his position looks highly heterogeneous with numerous discontinuities. However, it is possible to give a general outline of the kind of logic Schopenhauer has in mind throughout the different periods of his thought. As is clear simply from the section headings of his lecture on logic (the manuscripts of which were published in 1913, edited by Paul Deussen [21]), Schopenhauer's general idea of logic—like most works on logic in the nineteenth century—roughly follows the traditional structure of Aristotle's *Organon*, which was established not by Aristotle himself but by editors in the first century BC, including Andronicus of Rhodes. Accordingly, Schopenhauer deals first with concepts (*Begriffe*), corresponding to Aristotle's *Categories* (*Categoriae*), and then with judgements (*Urteile*), corresponding to Aristotle's *On Interpretation* (*De interpretatione*), before turning to arguments or inferences (*Schlüsse*), corresponding to Aristotle's *Prior Analytics* (*Analytica priora*). In the nineteenth century, as well as in earlier periods, these three parts were normally followed by a fourth part on scientific methods, corresponding to the Aristotelian *Posterior Analytics* (*Analytica posteriora*) and parts of the *Topics* (*Topica*). Unlike in Schopenhauer's lectures, the contents of the *Sophistical Refutations* (*De sophisticis elenchis*) were usually not given a separate section. At the time when Schopenhauer wrote his texts on logic, what was generally known as 'logic' covered all these different subjects, and sometimes even additional ones such as metalogical questions. A theory of deductive reasoning can be found—mainly in the form of syllogisms—in the third part of the *Organon*. Deductive reasoning in a strict sense is therefore only one element among others, and it was far from being the chief topic of scholarly interest at this time.

It is not only for traditional reasons that the concept or term is located at the beginning of Schopenhauer's logic. Rather, the term is the basis on which everything else is built. Although there are exceptions, the idea of starting logic with the proposition rather than with the term did not become commonly known in the German-speaking context until the works of Adolf Trendelenburg (see below). In line with this tradition, Schopenhauer clearly states that the structure of his logic is based on concepts or terms: 'Logic presupposes the existence of terms and now teaches how one has to operate correctly with them' [21, p. 259].[1]

One might be tempted to apply the sequence 'concept, judgement, inference' to both volumes of *The World as Will and Representation*, with the intention of assembling a 'complete' logic. However, §9 of the first volume—which is dedicated entirely to logic—deals mainly with concepts and judgements, and does not elaborate on inferences. Only the second volume appears to complete the sequence, as its tenth chapter is exclusively dedicated to syllogisms. But this appearance is deceptive. The two volumes were written at an interval of 25 years—or 24 years according to Schopenhauer himself—and cannot be read as if they were one; the conceptual changes are too extensive.

The parallel structure and content of the *Organon* and Schopenhauer's texts on logic is not surprising in the light of Schopenhauer stating that Aristotle had described logic to an 'extent of perfection' [21, p. 357] that left barely anything to add to bring it to the state it had attained by Schopenhauer's time, when logic was 'rightly regarded as an exclusive, self-subsisting, self-contained, finished, and perfectly safe branch of knowledge, to be scientifically treated by itself alone and independently of everything else' [20, §9, p. 46]. This was a very common view in the German-speaking world until the end of the nineteenth century. It was famously pre-formulated by Kant, who remarked on the fact that 'since the time of Aristotle it [logic] has not had to go a single step backwards [. . .]. What is further remarkable about logic is that until now it has also been unable to take a single step forward, and therefore seems to all appearance to be finished and complete' [15, p. Bviii/p. 106].

The belief that logic is a science that had already been brought close to perfection in the ancient world is paired with another belief that was equally common in the nineteenth century: namely, that the rules of logic are grasped intuitively. The idea is that the mind thinks logically as it is. Thus, it is not possible for the mind to work

[1] 'Die Logik setzt das Vorhandensein der Begriffe voraus und lehrt nun wie man regelrecht damit zu operiren habe' [21, p. 259].

against the rules of logic, because these rules prescribe the laws of thinking. For this reason, reflection on the performance of thinking—by means of an inductive method—necessarily reveals the universally valid rules of logic. Logic 'is the universal knowledge of the reason's method of procedure, expressed in the form of rules. Such knowledge is reached by self-observation of the faculty of reason, and abstraction from all content. But that method of procedure is necessary and essential to reason; hence reason will not in any case depart from it' [20, §9, p. 45]. That renders the study of logic quite useless in practice, because everybody follows the rules of logic anyway. There is only one exception: in an argument, making reference to logic allows deliberate attempts to deceive to be unveiled. If invalid conclusions are drawn intentionally, they can be referred to using the relevant technical terms [20, §9, p. 47]. In every other practical respect, logic is useless. Hence, Schopenhauer speaks in derogatory fashion of the elaborate logic of scholasticism. He concludes: 'To seek to make practical use of logic would therefore mean to seek to derive with unspeakable trouble from universal rules what is immediately known to us with the greatest certainty in the particular case. It is just as if a man were to consult mechanics with regard to his movements, or physiology with regard to his digestion' [20, §9, p. 45]. The latter, wittily expressed view was so common in the early nineteenth century that Schopenhauer is even prepared to agree on this point with his arch nemesis Hegel [12, p. 8].

The two assumptions, namely that logic has already been perfected as a science and that logic is just a reflection of the way the mind works anyway, sparked a rich debate about logic in the German-speaking world in the nineteenth century. This debate focused not on the traditional topics of concept, judgement and inference, but rather on metalogical issues such as the ultimate foundation of logic, its unity, its relation to content, etc. Subjects that came to the fore in the English-speaking world at the same time, such as the quantification of the predicate, the arithmetisation of logic and the distinction between term logic and propositional logic, were simply of no interest for most German philosophers, including Schopenhauer. (There are of course exceptions such as Bolzano and Drobisch [9], not to mention authors from the eighteenth century.)

3 The History of Logic After Aristotle

Schopenhauer does not say much about the history of logic. That comes as no surprise given that he thinks the development of logic as a science was (almost) completed 2000 years ago. Indeed, Schopenhauer and his fellow logicians must have thought that there was hardly a 'history' of logic at all. Regarding logic before Aristotle, Schopenhauer points out both: 'the awkward and tedious way in which logical truths are brought out in many of Plato's dialogues' and 'what Sextus Empiricus tells us of the controversies of the Megarics concerning the easiest and simplest logical laws, and the laborious way in which they made such laws plain and intelligible' [20, §9, p. 48]. Schopenhauer does not comment on the apparent discrepancy between the claims, on the one hand, that it was so tedious to bring out the first logical rules and, on the other, that these rules are supposed to be the ahistorical and as it were 'natural' ground of all thinking across all times and places. This may be an inconsistency, but it does not have to be. The difficulties may relate not to thinking and arguing according to the rules of logic, but to expressing these rules in a general and unambiguous form.

In Schopenhauer's view, this was already achieved by Aristotle and since his time only a few 'additions and enhancements' had been added. In his lectures, Schopenhauer explicitly names six of these in a non-chronological order which has been retained here: (1) The universal laws of thought at the beginning of logic; (2) the scholastic mnemonics; (3) hypothetical and disjunctive inferences; (4) the separation between concepts and representations of perception; (5) the fourth figure; he concludes (6) with criticisms of certain aspects of Aristotelian logic [20, p. 357].

3.1 The Laws of Thought at the Beginning of Logic

The laws of thought are ranked first in Schopenhauer's list of additions and enhancements. Specifically, he speaks of 'the positioning of the universal laws of thought as starting point' [20, p. 357]. Schopenhauer recognises four laws of thought, all of which he regards as 'metalogical' propositions: the law of identity, the law of contradiction, the law of excluded middle and the law of sufficient reason. Apart from the latter, these laws are of course neither additions nor enhancements, as they were already well established at the time of Aristotle.

In the nineteenth century, the first three laws were often understood as actually being one law or as derived from one basic law. A typical example of this is the analytical logic of August Twesten [30], which stands in the tradition of Kant's logic, is at least akin to Schopenhauer's logic and attempts to unite all the different aspects that logic involved in this period. Laws of thought, concepts, judgements, inferences and methods ought, according to Twesten, to be understood as parts of one system of logic and not as an assortment of unrelated phenomena [30, §§29–30, p. 13]. Based on this system, Twesten believes that the law of identity and the law of contradiction are two expressions of one basic law, and that the law of excluded middle is another derivative of the basic law [30, §§25–26, p. 11]. The only law he cannot accommodate within his system is the 'new' law, namely Leibniz's law of sufficient reason, and so this law is excluded from his analytical logic [30, §27, p. 12].

Schopenhauer shares the understanding of the first three laws as essentially one, even if he ties them back to the law of excluded middle. But he differs from Twesten in retaining the law or principle of sufficient reason as an irreducible part of logic: 'It seems to me that the doctrine of the *laws of thought* could be simplified by our setting up only two of them, namely the law of the excluded middle, and that of sufficient reason or ground' [22, p. 103]. In *The Fourfold Root* Schopenhauer elevates the fourth law or, more precisely, the principle of sufficient reason of knowing (i.e. the ground of knowledge) to the foundation of his whole doctrine of inferences: ‘The whole syllogistic science, in fact, is nothing but the sum-total of the rules for applying the principle of sufficient reason to the mutual relations of judgments’ [19, §30, p. 125]. Thus, on the one hand, Schopenhauer states that there are two irreducible laws, but, on the other hand, talks about only one. This heterogeneity might be due to the mingling of the principles of term logic and propositional logic. In respect of the semantic content of the ‘principle of sufficient reason of knowing’ [19, §29, p. 123], Schopenhauer explains that ‘if a *judgment* is to express *knowledge* of any kind, it must have a sufficient reason: in virtue of which quality it then receives the predicate *true*. Thus *truth* is the reference of a judgment to something different from itself’ [19, §29, p. 124; cf. V263].

The first three laws of thought specify which judgements are thinkable at all. The fourth law, i.e. the principle of sufficient reason of knowing, is the ground of possibility for assessing whether a judgement is true or not. According to Schopenhauer, there are four different kinds of grounds of knowledge that judgements may refer to and, accordingly, four different kinds of truth: logical, empirical, transcendental and metalogical [19, §§30–33, pp. 124–129]. A judgement is logically true if it has its ground in another judgement (cf. [21, p. 264]. The second judgement on which the truth of the first is founded may be founded on another judgement and so on. This line of argument comes to an end when it hits the laws of thought themselves, for these laws are judgements themselves. Finally, the reason for the laws of thought to be true is that thinking is only possible in accordance with them. They ‘are founded on the formal conditions of all thinking, which are contained in the Reason; and in this case its truth is of a kind which seems to me best defined as *metalogical* truth’ [19, §33, p. 127]. That means the laws of thought must be followed intuitively in any case. However, to become aware of them we must reflect on the way we think. We ‘then find out, that it is just as impossible to think in opposition to them [the laws of thought], as it is to move the members of our body in a contrary direction to their joints’ [19, §33, p. 123].

Schopenhauer regards the law or principle of sufficient reason as one of the enhancements that logic had undergone in the time since Aristotle. Additionally, he claims that these metalogically true judgements are to be situated at the beginning of every logic. In the nineteenth century, it was not unusual to begin a book on logic with the four (or sometimes only three) laws of thought. One example of this is the section on logic in Joseph Beck’s *Grundriß der empirischen Psychologie und Logik* (*Fundamentals of Empirical Psychology and Logic*) [2], which was very popular in the nineteenth century and even into the twentieth century: between 1841 and 1928, 21 editions were published, with a series of different editors after Beck’s death in 1883 (cf. [7, p. XXV]). Beck’s logic is not especially remarkable in itself, but it provides a good sense of what was commonly meant by ‘logic’ in the nineteenth century. Beck’s logic, like Schopenhauer’s, starts with the four laws of thought and then, in line with the structure of the *Organon*, moves on to the doctrines of concept, judgement and inference, before concluding with a reflection on the methods of science. But while most logicians did not alter the sequence of concept, judgement, inference and method, the position of the laws of thought varied within works on logic. Given that they are judgements, it made sense to place the laws after the sections on judgements (cf. Drobisch [10, §35]). Interestingly, Schopenhauer himself favours the latter option in his lecture [21, p. 261]. Hence, it does not seem to be very important for Schopenhauer where exactly the four laws of thought are presented.

3.2 The Scholastic Mnemonics

The second point on Schopenhauer's list of additions and enhancements that logic had undergone since Aristotle is: 'The invention of the naming of quantity and quality by using letters and as a consequence the naming of the types of inferential figures by using words whose consonants indicate the rules of reduction to the first figure through inversion' [20, p. 357].[2] Whereas Schopenhauer quite clearly regards the principle or law of sufficient reason as an enhancement, he probably only considers the 'barbaric words' [21, p. 358] which name the 24 valid types of syllogisms, to be an addition. At most, this addition might be of historical interest, but it is of even less practical use than logic in general: 'In the actual exposition of logic, these matters are still presented as one displays old and no longer used weapons in an armoury' [21, p. 358].[3] Knowledge of the scholastic mnemonics dwindled in the course of the nineteenth century. For example, Hermann Ulrici, who was far from being an irrelevant logician in his time, states that Barbara, Cesare, Datisi, etc. are: 'meaningless words in which only the vowels are significant' [31, p. 189].[4] In 1860, Ulrici (unlike the early Schopenhauer) was no longer aware of the sophisticated functions served by the consonants in the barbaric words. For example, that they indicate the rules and means to reduce imperfect modes to the perfect modes of the first syllogistic figure.

In this context, it is surprising that Schopenhauer, on the one hand, deems the traditional but (in his eyes) useless scholastic mnemonics worthy of mention in his brief list of additions and enhancements that logic had undergone since Aristotle but, on the other, does seemingly not include the illustration of logic by means of diagrams on his list at all, even though he not only praises highly the works of Ploucquet, Lambert and above all Euler [20, §9, p. 42] but also uses diagrams himself throughout his texts on logic. This would be understandable if Schopenhauer had listed only additions and enhancements that concern the content of logic, such as the principle of sufficient reason, while regarding diagrams merely

[2] 'Die Erfindung der Bezeichnung der Quantität und Qualität durch Buchstaben, und demnach der modi der Schluß-Figuren durch Wörter, deren Consonanten die Regeln der Zurückführung auf die erste Figur durch Umkehrung angeben' [21, p. 357].

[3] 'Im eigentlichen Vortrag der Logik führt man diese Sachen noch vor, wie man in einer Rüstkammer alte aus dem Gebrauch gekommene Waffen zeigt' [21, p. 358].

[4] 'sinnlose Wörter, in denen nur die Vocale von Bedeutung sind' [27, p. 189].

as formal or technical means of illustration. But in fact the mnemonics are listed as techniques. Thus, even as a technique diagrams should be part of the list as well.

3.3 Hypothetical and Disjunctive Inferences

The third point on Schopenhauer's list is: 'The consideration of hypothetical and disjunctive inferences, while Aristotle confined himself to categorical inferences' [21, p. 357].[5] In fact, Aristotle himself does not discuss hypothetical inferences, but they were addressed during ancient times by the Peripatetic and Stoic schools (cf. [4]). Later, they can be found in the *Port-Royal Logic* [1, p. 287], and in the nineteenth century they were part of the standard repertoire of all different kinds of logic (Kant's, Hegel's, Beck's, etc.). Schopenhauer discusses them in his lecture [21, pp. 333–339], where he points out that 'the disjunctive and hypothetical inferences are of a distinctly different nature to the categorical ones' [21, p. 339].[6] According to Schopenhauer, categorical inferences are directly based on concepts, while disjunctive and hypothetical inferences are based on the relations of judgements. One might get the impression that *modus ponens* and *modus tollens* do not quite fit in the framework of Schopenhauer's early logic, which at the time of his lectures was largely a term logic. This had changed completely by the time Schopenhauer published his supplements to the first book of the WWR in 1844. At the same time that George Boole was working in Britain on a propositional calculus with algebraic structures [5], Schopenhauer discarded major parts of his earlier logic by granting primacy to judgements. This indicates that at the latest from the 1840s German logicians too began to doubt whether logic had in fact been perfected long ago.

3.4 The Separation of Concepts and Representations of Perception

Schopenhauer credits the fourth point on his list to himself: one of the enhancements that logic had undergone during its history was 'my sharp separation of concepts and representations of perception, i.e. things' [21].[7] This statement might be a little bit surprising, as this distinction is usually attributed to Kant. But Schopenhauer

[5] 'Die Betrachtung der hypothetischen und disjunktiven Schlüsse, während Aristoteles sich auf die kategorischen beschränkte' [21, p. 357].

[6] 'Sie sehn daß die disjunktiven und hypothetischen Schlüsse merklich andrer Natur sind als die kategorischen' [21, p. 339].

[7] 'Meine scharfe Sondierung der Begriffe von den anschaulichen Vorstellungen, d. h. den Dingen' [21, p. 357].

replies to this objection: 'Unfortunately this [i.e. that Kant made this distinction] was not the case, although the reproach for this has not yet become known, and is therefore perhaps unexpected' [20, p. 437]. According to Schopenhauer, the Kantian thing as an 'object of experience' 'is not the representation of perception, nor is it the abstract concept; it is different from both, and yet is both at the same time, and is an utter absurdity and impossibility' [20, p. 437]. Schopenhauer opposes to Kant's assumption that even perceptions are always conceptually formed through the categories. Instead, Schopenhauer's philosophy distinguishes clearly between the 'mere *sensation* in the sense-organs' [20, p. 438] and representations. The latter are produced by the intellect, which converts sensations through both understanding—which follows the law of causality—and the forms of perception into representations of perception. Concepts do not appear at all prior to this point. However, Schopenhauer begins §9 of the WWR with the sentence 'The concepts form a peculiar class, existing only in the mind of man, and differing entirely from the representations of perception so far considered' [20, §9, p. 39]. I shall briefly elaborate (a) on the concepts being 'only in the mind of man' (*allein im Geiste des Menschen vorhanden*) and (b) on the peculiarity (*Eigentümlichkeit*) of these concepts.

(a) Concepts are the fundamental building blocks of Schopenhauer's logic. These concepts are, in his view, 'only in the mind', and hence logic exists only in the mind of man. Against the background of Schopenhauer's philosophy, that is no surprise. But for the nineteenth century, this position was not an obvious one to take. Especially in the mid-nineteenth century, more and more logicians attempted to align themselves more closely with the successful natural sciences. As a consequence, realism became fashionable amongst logicians. This view holds that the structures of reality match the structures of reasoning not merely because reality originates in the human mind, but because there is a reality outside the mind which has a logical structure. This ontological belief is combined with the epistemological view that the logic of thinking represents the logic of a reality which is held to be independent of the mind (cf. [13]). As well as these realisms or even materialisms of concepts, there were also positions more or less distantly related to Hegel's *Science of Logic* [12]. These approaches merge logic and metaphysics by, on the one hand, considering the structure of reality to be analogous to the structure of the mind, but, on the other, holding that reality is not contained within the human mind; rather, the human mind is an aspect of a reality that, as a kind of overarching mind, houses the human mind within itself.

(b) Regarding the peculiarity of concepts as a class that is entirely different from representations of perception, Schopenhauer points out that concepts cannot be experienced through the senses but can only be understood discursively, and that they are not located in time or space but only within thoughts. However, at the same time concepts are not independent of representations of perception. In fact, they are connected because concepts are generated through a process of abstraction that starts from perception. This means that, in a certain sense, the concepts reflect the representations of perception. This reflection 'is necessarily the copy or repetition of the originally presented world of perception, though a copy of quite a special kind in a completely heterogeneous material. Concepts, therefore, can quite appropriately be called representations of representations' [20, §9, p. 40] or, more precisely, abstract representations of more concrete representations. Being abstract means to be universal, so all concepts have at least in principle 'a range, an extension, or a sphere' [20, §9, p. 42]. According to Schopenhauer, that is why concepts can be presented by spatial figures or diagrams, which 'is an exceedingly happy idea' [20, p. 42]. This might indicate that by '*anschauliche Vorstellung*' Schopenhauer does not mean 'representation of perception' but rather 'visual representation'. In that case, the mystery of the absent diagrams in Schopenhauer's list of enhancements would be solved: they were actually never absent but were included the whole time under the fourth point on the list. This thought is supported by Schopenhauer himself when he claims to be the first to have completely replaced the Aristotelian proofs with diagrams [21, p. 272].[8]

Since all concepts originate from perception, they remain in the final analysis bound to perception. This understanding illustrates again the cardinal role that the principle of sufficient reason as ground of knowledge plays in logic and especially in syllogistics: 'the abstract representation has its whole nature simply and solely in its relation to another representation that is its ground of knowledge. Now this of course can again be a concept or an abstract representation [. . .]. However, this does not go on *ad infinitum*, but the series of grounds of knowledge must end at last with a concept which has its ground in knowledge of perception' [22, pp. 40–41]. The truth in question here is of course not the logical or metalogical truth, like that mentioned above, but the empirical truth.

3.5 The Fourth Figure

The fifth point on Schopenhauer's list, the 'fourth figure', is another addition. Originally, Aristotelian syllogistics only recognised three figures. The supplementation of a fourth figure is usually ascribed to the Greek physician Galenus (c. 129–215). But Schopenhauer—like Theophrastus before him—does not regard it as an enrichment. He thinks that 'it is clear that this figure is merely the *first* wilfully [!] turned upside down, and by no means the expression of an actual process of thought natural to our faculty of reason' [22, p. 115]. Hence, this figure is obsolete. Although Schopenhauer is not the only thinker who regards the fourth figure as non-natural (cf. e.g. Twesten [29, p. 104] and Kant [14, p. 17]) and even if it is often regarded as

[8] 'Die Aristotelischen Beweise hat man schon längst aus der Logik weggelassen; aber man hat ihnen die Verdeutlichung durch anschauliche Schemata noch nicht so durchgängig substituirt, wie ich es thun werde.' [21, p. 272].

redundant (cf. e.g. Trendelenburg [26, pp. 235–237]) it did not disappear from logic (cf. e.g. Victorin [32, pp. 108, 201–208]) but remained a source of disagreement among the logicians of the nineteenth century (Stammler [23, p. 29]). A detailed discussion of the fourth figure can be found in the paper written by Hubert Martin Schüler and Jens Lemanski in this volume.

3.6 Criticisms of Aristotelian Logic

A short recapitulation reveals that Schopenhauer considers only two of the five points on his list to be enhancements: namely, the laws of thought, including the principle of sufficient reason, and Schopenhauer's own sharp separation between concepts and representations of perception. Meanwhile, he regards the scholastic mnemonics as obsolete, does not consider hypothetical and disjunctive inferences in depth and believes the fourth figure is simply unnecessary. He also deems some parts of Aristotelian logic to be unnecessary. In his lecture, he mentions explicitly only 'inferences whose modality is problematic', namely inferences that include judgements that are not actual or necessary but only possible. Modal logics and all other kinds of non-classical logic—in the sense of logics that transgress the principle of bivalence or the principle of extensionality or both—did not attract much interest in nineteenth-century German-speaking philosophy. Aristotle's remarks on such logics in the *Organon* were 'long since ignored. With good reason' [21, p. 339].[9]

But Schopenhauer is mistaken when he claims that in his time the original *Organon* was read 'very rarely [. . .], because it is a sparsely rewarding and very difficult read that takes a lot of time' [21, p. 357].[10] Contrary to this claim, Immanuel Bekker published the first volume of the collected works of Aristotle in 1831, 10 years after Schopenhauer's lecture. Bekker followed the Hellenistic tradition of opening with the *Organon*, which he subdivided (again traditionally) into concept, judgement, inference, etc. Even more interesting in relation to the reception of Aristotle in nineteenth-century logic is the rearrangement of the *Organon*'s traditional order by Trendelenburg, inspired by Otto Friedrich Gruppe [11, p. 38]. Mainly for philological reasons, in his *Elementa logices Aristotelicae* (1837) [24] Trendelenburg has Aristotelian logic start not with concepts but with judgements. As a schoolbook that was republished over and over again and eventually even several times in German [27], Trendelenburg's *Elementa* can be regarded as very influential. It seems natural that this new reading of the original Aristotelian texts paved the way for the rejection of the traditional order of logic (concept, judgement,

[9] 'Man hat diesen Theil seiner [Aristoteles] Logik schon längst unbenutzt gelassen. Mit Recht' [21, p. 339].

[10] '[Es ist] höchst selten gelesen [...], da es ein wenig lohnendes und sehr schwieriges Studium ist, was sehr viel Zeit erfordert' [21, p. 357].

inference) in favour of a logic that starts with judgements and therefore follows the context principle, as Frege [16] later did.[11]

4 Left Off Schopenhauer's List: The Port-Royal Logic and Kant's Logic

It is not just the aspects Schopenhauer mentions explicitly that are of interest, but also those he does not. They include (a) the *Port-Royal Logic* and (b) Kant's logic.

(a) The history of logic can roughly be divided into three periods: first, ancient and medieval logic starting from Aristotle (fourth century BC); second, early modern logic starting from the *Port-Royal Logic* (1632) [1] and third, modern logic starting from Frege's *Begriffsschrift* (1879) [28]. Even given that Schopenhauer was a critic of Cartesian logic [21, p. 254], it is, in view of the work's importance, remarkable that he makes absolutely no mention of the *Port-Royal Logic* or any of its innovative ideas; although the work was not widely discussed in the early nineteenth century, it was far from unknown (cf. e.g. Degerando [8, pp. 377–381]).

The *Port-Royal Logic*, on the one hand, stands in the tradition established by the *Organon*: its structure follows the sequence of 'concept, judgement, inference, method'. On the other hand, however, it also introduced some innovations that went on to have a profound influence on the subsequent period. They included, in particular, situating the topic of perception at the beginning of logic, thereby bringing epistemological and, above all, psychological explanations into logic, as well as a theory of signs. Schopenhauer does not appear to regard the last points as particularly important, given that he does not deem them worthy of inclusion in his list of significant events in the history of logic, although they are mentioned elsewhere. Overall, Schopenhauer's texts on logic are very rich in content despite their brevity.

(b) A clue as to why the integration of psychology into logic is not mentioned by Schopenhauer can again be found in Kant. Following the remark that logic has neither taken a single step backwards nor a single step forwards since Aristotle, Kant states: 'For if some moderns [i.e. modern logics, like the *Port-Royal Logic*] have thought to enlarge it [Aristotle's logic] by interpolating *psychological* chapters about our different cognitive powers [...], or *metaphysical* chapters about the origin of cognition or the different kinds of certainty in accordance with the diversity of objects [...], or *anthropological* chapters about our prejudice [...], then this proceeds only from their ignorance of the peculiar nature of this science [i.e. logic]' [15, p. Bviii/p. 106]. In a sense, Kant presents here his own list of unnecessary additions to logic. Kant's own idea of logic is that of a formal logic, and indeed it is Kant who first coined this term (cf. [15, p. B170/p. 267]). But what does formal

[11] Some thinkers believe the context principle first appeared earlier: Brandom, for example, argues that it can already be found in the work of Kant [6, 13].

logic mean according to Kant? Firstly, a formal logic has to abstract away from all content (empirical or transcendental). Metaphysics as an ontology therefore has no place in a formal logic. Secondly, a formal logic is purely a priori, which means it has no empirical principles either. Thus, psychology and anthropology have no place in logic [15, pp. B74–76/pp. 193–195].

Schopenhauer shares this concept of logic as formal logic. That does not mean, of course, that he does not talk about psychology and metaphysics. But it does mean that these issues are not to be found within logic. By endorsing the Kantian concept of a formal logic, Schopenhauer situates himself in what was a very broad current in at least the early part of the nineteenth century. Within this current, there was an undercurrent that equated logic with analytics [15, p. B170/p. 267]. Twesten [30], Beneke [3] and Schopenhauer were part of this undercurrent: 'I [Schopenhauer] think, however, that logic has only a theoretical interest in coming to know the essence of the lawful process of reason, that it should therefore merely be analytic' [21, p. 359].[12]

An answer to the question of how Schopenhauer's logic is embedded in its temporal context, of course, also has to make reference to the counterposition, or at least supposed counterposition: namely, what were known as transcendental logics, another term coined by Kant. But in fact, Kant's transcendental logic and his formal logic are not in conflict. According to Kant, transcendental logic is only an application of formal logic to objects which are known a priori (cf. [15, p. B82/pp. 197–198]). That means transcendental logic is the logic of the conditions of possible objects. Transcendental logic thus has content, in contrast to formal logic. It is a popular narrative to describe the development of logic after Kant in the nineteenth-century Germany as starting from a schism among the supporters of transcendental logic (such as Hegel [12], Ritter [18] and Prantl [17]) and the supporters of formal logic (such as Schopenhauer). Whether this is consistent in detail, I shall not discuss here. But the fact is that logic in the nineteenth century was by no means exclusively pure or formal, but could also be defined in terms of content. Towards the middle of the nineteenth century, a further current became increasingly prevalent: motivated by a desire not to practise any metaphysics, logicians aligned themselves more with the natural sciences. That led to a logic that bordered on being a theory of science. Examples include Trendelenburg's *Logische Untersuchungen* (*Logical Investigations*) [25].

In summary, Schopenhauer's logic is a fairly typical example of a formal logic in Kant's tradition at the beginning of the nineteenth century, if only the surface is considered. However, this does not say anything about how he treats the individual elements in detail in his logic. Beneath the surface there are some very interesting reflections on topics such as diagrams.

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[12] 'Ich [Schopenhauer] halte indessen dafür daß die Logik bloß ein theoretisches Interesse hat, um das Wesen, das Gesetzmäßige Verfahren der Vernunft kennen zu lernen: daß sie also bloß Analytik seyn soll' [21, p. 359].

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Arthur Schopenhauer on Naturalness in Logic

Hubert Martin Schüler and Jens Lemanski

Abstract The question of naturalness in logic is widely discussed in today's research literature. On the one hand, naturalness in the systems of natural deduction is intensively discussed on the basis of Aristotelian syllogistics. On the other hand, research on "natural logic" is concerned with the implicitly existing logical laws of natural language, and is therefore also interested in the naturalness of syllogistics. In both research areas, the question arises what naturalness exactly means, in logic as well as in language. We show, however, that this question is not entirely new: In his *Berlin Lectures* of the 1820s, Arthur Schopenhauer already discussed in depths what is natural and unnatural in logic. In particular, he anticipates two criteria for the naturalness of deduction that meet current trends in research: (1) Naturalness is what corresponds to the actual practice of argumentation in everyday language or scientific proof; (2) Naturalness of deduction is particularly evident in the form of Euler-type diagrams.

Keywords Natural deduction · Natural logic · Natural language · Syllogistics · Euler diagrams

Mathematics Subject Classification (2000) Primary 03B65; Secondary 00A00

1 Introduction

Naturalness is a widely discussed topic in logic today, especially in relation to (1) systems of natural deduction and in the field of (2) natural logic. (1) The first systems of natural deduction were invented in the 1920s and 1930s by Gerhard Gentzen and Stanisław Jaśkowski. Gentzen intended to set up a calculus "which comes as

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close as possible to actual reasoning" [9, p. 68]. Also Jaśkowski expected that his system "will be more suited to the purposes of formalizing practical proofs" [11, p. 32]. Both, Gentzen and Jaśkowski initially oriented towards natural reasoning in mathematical practices. Both had noticed that axiomatic calculi in the tradition of Gottlob Frege, Bertrand Russell, and David Hilbert did not do justice to the actual reasoning of mathematicians.

This is an essential criterion for distinguishing between axiomatic and natural systems. (For more criteria see [21, chap. 2].) Another criterion is mentioned by Danielle Macbeth:

> In an axiomatic-system, a list of axioms is provided (perhaps with a long explicitly stated rule or rules of inference) on the basis of which to deduce theorems. Axioms are judgements furnishing premises for inferences. In a natural deduction system one is provided not with axiom but instead with a variety of rules of inference governing the sorts of inferential moves from premises to conclusions that are legitimate in the system. In natural deduction, one must furnish the premises oneself; the rules only tell you how to go on. [19, p. 75]

The difference between axiomatic and natural deduction also concerned the interpretation of Aristotelian syllogistics. The axiomatic interpretation of Aristotle was mainly represented by Jan Łukasiewicz and Günter Patzig (cf. [17, 25]). From the 1970s, however, the interpretation of John Corcoran and Timothy Smiley prevailed in research. This interpretation of the Aristotelian *Organon* comes closer to a system of natural deduction as provided by Gentzen. For Corcoran "Aristotle's syllogistic is an underlying logic which includes a natural deductive system and [...] is not an axiomatic theory [...]" [6, p. 85]. According to John N. Martin, Corcoran argues against two main theses of Łukasiewicz "that syllogisms should be construed as conditional sentences in an object-language [...] and that Aristotle's reduction of the valid syllogisms to the 'perfect syllogisms' should be viewed as axiomatic theory in which Barbara and Celarent serve as axioms" [20, p. 1]. In recent years, Neil Tennant interprets syllogistics in a Gentzen–Prawitz system and "aim[s] to show that fresh logical insights are afforded by a proof-theoretically more systematic account of all four figures" [28, p. 120].

(2) Whereas systems of natural deductions are oriented towards the mathematical practice of proof, the so-called natural logic is more focused on the linguistic structure of everyday argumentation. In 1970, George Lakoff [13, p. 254] defined natural logic as "the empirical study of the nature of human language and thought." And Johan van Benthem interprets natural logic as a "system of reasoning based directly on linguistic form, rather than logical artefacts." [3, p. 109]. Therefore, natural logic is based on the conviction that natural language implies logical laws that do not have to be artificially formalized, but can be made explicit in a regimented language (cf. [12]). After all, there are always disadvantages both in artificial formalization and in natural regimentation of everyday language: Whereas formalization always requires interpretation, regimentation is difficult to calculate with.

Since the 2000s natural logic took on a Janus-faced character: on the one hand, it looked backward, because it revisited linguistic form in the same way as pre-Fregean philosophy had done, and on the other hand, it looked forward, because its methods and principles are increasingly being used for neural networks and artificial intelligence (cf., e.g., [5, 24]). However, similar to the research on natural deduction systems, the question of what naturalness exactly means also arises in the area of natural logic. At present, various ancient, medieval, or early modern logicians are being discussed in research as precursors of natural logic [2]. Although these logicians are considered to be the source of ideas for today's research, the naturalness of logic is not explicitly addressed as a topic in their texts.

Surprisingly, however, already in the early nineteenth century a discussion about the naturalness of deduction in syllogistics can be found: In his *Berlin Lectures* of the 1820s, Arthur Schopenhauer distinguished between a natural and an unnatural kind of deduction within traditional syllogistics. But what are the criteria for natural and unnatural reasoning in syllogistics?

Schopenhauer discusses this question on almost 40 printed pages of his *Berlin Lectures* [26, pp. 293–331] including many different arguments. Since there has not been research on these text passages so far, we can here only take a first step. Thus we would like to concentrate here mainly on two criteria for naturalness given by Schopenhauer, for which we think that they are particularly interesting from today's point of view: (1) For Schopenhauer, the naturalness of deduction depends on the practice of actual reasoning and proving; (2) the naturalness of deduction is promoted by Euler-type diagrams. (1) is a mental-linguistic criterion, (2) a diagrammatic one.

In Sect. 2, we will first discuss how Schopenhauer distinguishes between unnaturalness and naturalness in logic. We will first examine the concept of unnaturalness (2.1) and then outline which inferences are natural in logical reasoning (2.2–2.4). Section 3 will show that the unnaturalness of reasoning results from the completeness of natural deduction in syllogistic. One could thus say that Sect. 2 presents a positive, whereas Sect. 3 is a negative approach towards the question of naturalness in logic. Finally, in Sect. 4, we will argue that Schopenhauer's criteria for naturalness and unnaturalness often include arguments that can be still relevant to today's research discussion: First, Schopenhauer argues that certain inferences are natural because they correspond to our current use in scientific and everyday reasoning. Second, he argues that diagrams play a specific role for the naturalness of deduction.

2 Unnatural and Natural Deduction in Syllogistics

In this section, we will first refer to a relevant quote from the section *On Inferences* in Schopenhauer's *Berlin Lectures*, which makes his criticism of unnatural deduction in syllogistic explicit. Here, Schopenhauer labels all inferences of the fourth syllogistic figure—the so-called Galenic figure—as unnatural. We will then show in each of the following sections (Sects. 2.2–2.4) why Schopenhauer considers the first three figures to be natural. This will reveal a criterion of completeness, which we will examine later in Sect. 3.

2.1 Unnaturalness of $\mathcal{F}_{IV}$

In syllogistics we usually distinguish four figures for up to 24 valid types of inferences (modi). Depending on the position of the three terms (termini major = M, minor = m, and medius = μ) in the premises, the 24 modi are classified into one of the four figures ($\mathcal{F}$).

$$\mathcal{F}_I \begin{bmatrix} \mu & M \\ m & \mu \\ m & M \end{bmatrix} \mathcal{F}_{II} \begin{bmatrix} M & \mu \\ m & \mu \\ m & M \end{bmatrix} \mathcal{F}_{III} \begin{bmatrix} \mu & M \\ \mu & m \\ m & M \end{bmatrix} \mathcal{F}_{IV} \begin{bmatrix} M & \mu \\ \mu & m \\ m & M \end{bmatrix}$$

In Aristotle we find only $\mathcal{F}_{I-III}$; $\mathcal{F}_{IV}$ is often attributed to the Greek physician Galen of Pergamon (~129–216) (cf. [18]). This is one of the reasons why Schopenhauer claims that there are basically not four, but rather only three figures since $\mathcal{F}_{IV}$ only turns $\mathcal{F}_I$ upside down. Since only the position of μ is relevant for the assignment of $\mathcal{F}$, one can argue that $\mathcal{F}_I = \mathcal{F}_{IV}$ or $\binom{\mu M}{m\mu} = \binom{M\mu}{\mu m}$, in which μ is in a diagonal position in both cases. For Schopenhauer, the change from $\mathcal{F}_I$ to $\mathcal{F}_{IV}$ results in grammatical and diagrammatical problems that do not meet the requirements of natural logic and deduction. A first relevant quote of Schopenhauer on this topic is given in $Q1$:

> ($Q1$) Aristotle has only the first three: the fourth is (according to a legend of the Arabs) invented by Galen. It's just the reversed first. Actually, there is no unique relation between concepts: it is quite unnatural and really only the very first figure turned upside down: therefore Aristotle intentionally did not set it up. [26, p. 305]
> Aristoteles hat nur die drei ersten: die vierte soll (nach einer Sage der Araber) von Galen erfunden seyn. Sie ist bloß die umgekehrte erste. Ihr liegt eigentlich kein besondres Verhältniß der Begriffe zum Grunde: sie ist ganz unnatürlich und wirklich nur die ganz auf den Kopf gestellte 1ste Figur: daher Aristoteles sie absichtlich nicht aufstellte.

$Q1$ does not explicitly answer the question of what naturalness of deductive inferences mean and why $\mathcal{F}_{I-III}$ can be considered as natural; however, $Q1$ clearly states that $\mathcal{F}_{IV}$ and associated inferences can be characterized as unnatural. What is given in $Q1$ only as an unjustified claim will be discussed in Sect. 3 in more detail. But before we come to the arguments of Schopenhauer which deal with the naturalness of $\mathcal{F}_{I-III}$, let us examine what role Aristotle played for Schopenhauer ($Q1.1$) what "unique relation between concepts" in $Q1$ mean ($Q1.2$).

($Q1.1$) Schopenhauer claimed that Aristotle did not introduced $\mathcal{F}_{IV}$ because he recognized it as unnatural. This is not an untypical argument raised by Schopenhauer since it can already be found in the *Port-Royal Logic* (III 4) in a similar way. From the perspective of today's research, however, it is not uncontroversial. Theodor Ebert and Günther Patzig have pointed out that there are of course only three figures in ancient syllogistics, but Aristotle treats the modes of the fourth figure "implicitly as additional (indirect) modes of the first figure" [7, p. 13]:

> The Aristotelian indirect modes of the first figure and the modes of our fourth figure differ only in the arrangement of the premises, a difference that becomes recognizable only in the definition of a standard formulation, and which is irrelevant to the question of its validity.

> That is why all the judgements about an alleged inferiority of the fourth figure, as they were given by Aristotle's lack of this figure by former logicians and logic historians, lie on the one hand on an inaccurate reading of the Aristotelian texts, and on the other on a misunderstanding of the underlying logical facts. [7, p. 14]

Of course, Schopenhauer does not explicitly equate unnaturalness with inferiority, but the Schopenhauer/Port-Royal argument that suggests that Aristotle did not attach the fourth figure intentionally is nonetheless problematic. Most modern interpretators of Aristotle agree that there is no text passage of the *Organon* from which one can read without a hitch that Aristotle intentionally considered $\mathcal{F}_{IV}$ to be unnatural. However, before Schopenhauer there were already several authors who characterized $\mathcal{F}_{IV}$ as unnatural (cf. [18, sect. 2], [27]); and, as far as we know, nobody discusses the topic of naturalness and unnaturalness in as detailed a way as Schopenhauer does. Furthermore, one fact seems to be certain for Schopenhauer and all his precursors: Unnaturalness does not necessarily have anything to do with invalidity.

($Q1.2$) In $Q1$, the lack of unique relation is an important characteristic for the unnaturalness of $\mathcal{F}_{IV}$. Thus we have to clarify the question of what Schopenhauer means with "unique relation between concepts" in $Q1$. An essential feature of Schopenhauer's lectures on logic is that he represents the relation of concepts with the help of Euler-type diagrams (cf. [23]). Even though he designates the respective diagrams by scholastic mnemonics, he mainly uses a regimented language based on rules to explain the inferences. These rules are listed in [26, pp. 324–325], but cannot discussed here in detail. In the following quote ($Q2$), Schopenhauer names the point of view that and why the regimented inferences should best be represented by Euler-type diagrams:

> ($Q2$) Namely, between the possible relations that concepts can have to each other and the positions in which circles can be put together is a very precise and absolutely consistent analogy. ([26, p. 269])
> Nämlich zwischen den möglichen Verhältnissen, die Begriffe zu einander haben können, und den Lagen in welch[en] man Kreise zusammenstellen kann ist eine ganz genaue und schlechthin durchgängige Analogie.

The relationship between concepts and Euler diagrams can therefore basically be understood as an analogy: In the same way in which two concepts behave towards each other in logic, two circles can behave towards each other in geometry. Euler diagrams thus graphically depict the conceptual relations expressed in linguistic inferences. From the analogy-thesis of $Q2$ and the reference to the peculiarity of the conceptual relations of $Q1$, it can be concluded that there are specific Euler-type diagrams for $\mathcal{F}_{I-III}$, but not for $\mathcal{F}_{IV}$. Thus one can also conclude that the naturalness of $\mathcal{F}_{I-III}$ can be demonstrated with the help of geometric forms and that unnaturalness is given by the fact that no unique or autonomous Euler-type diagram of $\mathcal{F}_{IV}$ can be constructed.

Since the central objective of his chapters *On inferences* is to show the naturalness of the first three and the unnaturalness of the fourth figure, Schopenhauer gives his audience the following instruction ($Q3$):

($Q3$) Draw them [sc. the diagrams] down in order to be able to follow my remarks about them: it is precisely in our reflections on the various combinations of concepts underlying the various syllogistic figures that you will receive deep insight into the essence of concepts in general, into the mechanism of thought, and thus into the nature of our reason itself. [26, p. 306]
Zeichnen Sie solche [sc. die Diagramme] auf, um nachher meinen Bemerkungen darüber folgen zu können: eben an unsern Betrachtungen über die verschied[nen] Kombinationen der Begriffe welche den verschied[nen] syllogistischen Figuren zum Grunde liegen, werden Sie tiefre Einsicht erhalten in das Wesen der Begriffe überhaupt, in den Mechanismus des Denkens und somit in die Natur unsrer Vernunft selbst.

This instruction ("Draw them down...") which was originally meant for students discloses not only a didactic procedure, but it also opens up a kind of characterization concerning the question of how logic works. Those who draw Euler diagrams

(3.1) will achieve theoretical knowledge of *conceptual relations* between spheres of concepts because concepts resp. circles either partially overlap or are completely included or excluded,
(3.2) will achieve a deeper understanding of the *essence of concepts* because concepts behave to each other like geometrical circles behave to each other,
(3.3) will achieve knowledge of the *mechanism of thought* because by applying the Eulerian rules of construction one does consciously what reason otherwise automatically does (cf. [26, p. 306]),
(3.4) will gain "*insight*" [26, p. 306]—and we think this is meant literally—*into the nature of reason in general*. In other words: At the moment we see a diagram, we simultaneously see the validity of reasoning via an analogy.

In $Q2$ we have found out that Schopenhauer considers $\mathcal{F}_{IV}$ to be unnatural: It does not have a unique conceptual relationship that can be depicted with the help of logic diagrams. As seen in $Q3$, this unique relationship illustrated by Euler diagrams is important since the diagrams give a deep insight into the essence of natural language. From this consideration one could conclude that $\mathcal{F}_{I-III}$ are natural in so far as each provides a unique relationship of concepts resp. circles. We will now focus on the quotes in which Schopenhauer describes the naturalness of the first three figures. In Sect. 3, we will focus again on Schopenhauer's arguments concerning the unnaturalness of $\mathcal{F}_{IV}$.

2.2 Naturalness of $\mathcal{F}_I$

But let us now come to the arguments for the naturalness of the inferences in $\mathcal{F}_{I-III}$. Schopenhauer characterizes $\mathcal{F}_I$ as "the simplest and most natural" form of reasoning [26, p. 301]. The following quote ($Q4$) repeats this judgment in even more detail:

($Q4$) [T]he 1st figure is the most perfect, because every thought can finally take its form: very natural: just subsumption of one concept under a wider one and this again under a wider one is the simplest and most essential operation of reason: it is the mere retrospective view of a wider abstraction to the narrower [...]. [26, pp. 302–303]

> Sodann beweist dies [...] daß die 1ste Figur die vollkommenste wäre, indem jeder Gedanke zuletzt ihre Gestalt annehmen kann: sehr natürlich: grade Subsumtion eines Begriffs unter einen weitern und dieses wieder unter einen weitern ist die einfachste und wesentlichste Vernunftoperation: es ist der bloße Rückblick von einer weitren Abstraktion, auf die enger[n] [...].

In $Q4$ we already find two arguments for the naturalness of $\mathcal{F}_I$, of which the following is to be significantly emphasized: The first argument is a mental-linguistic one ($Q4.1$), the second argument a diagrammatic one ($Q4.2$).

($Q4.1$) We characterize logical thinking as natural, if a thought can be expressed in a unique syllogistic form. In order to express a thought, an intentional state, a propositional attitude, an inferential context, etc., one needs a certain linguistic form. This linguistic form can correspond to one of the three syllogistic figures, i.e., $\mathcal{F}_{I-III}$. The choice of $\mathcal{F}$ depends on which thought one intends to articulate in an inferential context. If every thought can be expressed in only one $\mathcal{F}$, this naturalness would be perfect. Following Kant, Schopenhauer claims that in $\mathcal{F}_I$ this is the case.

The cognitive activity that one can express with the help of $\mathcal{F}_I$ is the decision or resolution ("Entscheidung," [26, p. 326]). $\mathcal{F}_I$, containing four modes, always expresses a resolution about the relationship between M and m (in the conclusion). In each mode the conclusion shows one of the four possible categorical propositions in syllogistics (A, E, I, O). Therefore, any possible resolution between m and M (within the conclusion) can be represented by one of the four modes:

Barbar*A*	`All`	mM
Celar*E*nt	`No`	mM
Dari*I*	`Some`	mM
Feri*O*	`Some...not`	mM

This high expressivity of $\mathcal{F}_I$ is one reason why Schopenhauer says that this one is the "most perfect" and also a "very natural" figure above all others [26, p. 302].

($Q4.2$) The diagrammatic criterion is difficult to deal with in brief, since it depends strongly on the logical interpretation of the diagrams and this in turn depends exceedingly on the diagrammatic interpretations of the editors. (In Schopenhauer's manuscripts, the diagrams are often drawn very inaccurately.) However, Schopenhauer emphasizes especially the role of μ in the respective figure as a diagrammatic criterion. In $\mathcal{F}_I$, μ is the "mediator" ("Vermittler") between M and m: what does (not) belong to μ, also does (not) belong to m. Due to this function, Schopenhauer also speaks metaphorically of μ as a "Handhabe" ("manipulator," [26, p. 309]): μ grasps one concept and passes it on to the other. Schopenhauer alludes here to the metaphor of the concept (cf. [14]).

For example, in the modus Barbara (Fig. 1), the middle-sized concept (i.e. μ) grasps the narrower concept (i.e., m) and subsumes it under the widest one (i.e. M)

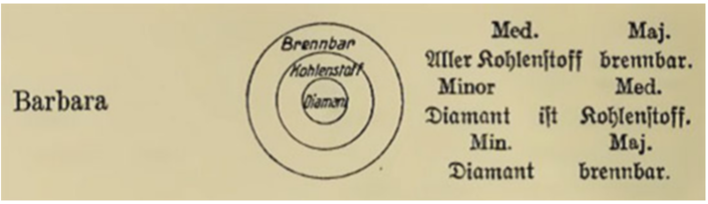


Fig. 1 Barbara [26, p. 304]

(cf. [26, pp. 297, 301, 321]). This function also explains the origin of the names "medius," "major," and "minor."

`All`	carbon$^{\mu}$		flammable^{M}
	diamond^{m}	`is`	carbon$^{\mu}$
	diamond^{m}		flammable^{M}

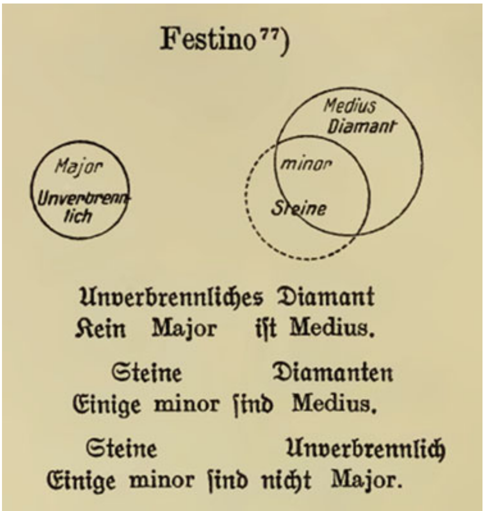
As we can see in this example (Fig. 1) naturalness is given, because of the specific function of μ. The conclusion ("All diamonds are flammable.") consists in its diagrammatic representation of a wider sphere of M ("flammable") which contains an entirely narrower sphere of m ("diamonds") with the help of μ which mediates between M and m. The expression "the mere retrospective view" in $Q4$ means nothing else than the fact that the conclusion goes back to the first premise. In other words, the resolution about the relationship between M and m in the conclusion depends on the relationship between μ and M in the first premise. Due to these and other functions of μ not discussed here, $\mathcal{F}_I$ in the Euler-type diagrams proves to be natural.

2.3 Naturalness of $\mathcal{F}_{II}$

For Schopenhauer, the inferences resp. modi of the $\mathcal{F}_{II}$ also belong to a system of natural deduction and natural logic. This can be seen in the following quote ($Q5$):

> ($Q5$) The 2nd figure is therefore the natural form of our thought process, when we want to separate concepts, to distinguish things, and to establish characteristic features of their difference. So we use the second figure mainly when we want to avoid misunderstandings and confusion of concepts. In all such cases it is the natural form of thought, not the first.
> Die 2te Figur ist daher die natürliche Form unsers Gedankenganges, wann wir Begriffe trennen, Dinge unterscheiden wollen, und hiezu karakteristische Merkmale ihres Unterschiedes aufstellen. Wir gebrauchen also die 2te Figur hauptsächlich wenn wir Verwechselung und Konfusion der Begriffe verhüten wollen. In allen solchen Fällen ist sie die natürliche Form des Gedankens, nicht die erste. [26, p. 310]

Fig. 2 Festino [26, p. 308]



Even if we get more references to the mental-linguistic criterion ($Q5.1$) and less references to a diagrammatic one ($Q5.2$), we can, with reference to the context of $Q5$, indicate some essential criteria for the naturalness of $\mathcal{F}_{II}$:

($Q5.1$**)** Negative propositions can be represented in $\mathcal{F}_I$ by Celar$E$nt and Feri$O$. But if one does not want to emphasize one's own resolution, but rather intend to express a difference between the concepts of the conclusion, i.e., m and M, the linguistic form of all inferences within $\mathcal{F}_{II}$ is better suited than the structure of $\mathcal{F}_I$. With the help of $\mathcal{F}_{II}$ a speaker can make clear to his audience that his intention is to avoid a misunderstanding or to prevent a confusion of concepts. Regarding such intentions, $\mathcal{F}_{II}$ is more natural than $\mathcal{F}_I$.

($Q5.2$**)** A peculiarity of Schopenhauer's logic is that he only gives one single diagram for each mode. As Schopenhauer should have known from Euler [8, l. CIV] (cf. also [10] or [4, p. 45 et seq.]), in many cases only one diagram is not sufficient in order to decide whether a mode is valid or not. However, Schopenhauer is more concerned with the question of naturalness than with the question of the validity of modes. This can be seen, for example, by Festino (Fig. 2) and Baroco (Fig. 3).

`No`	incombustible^M	`is a`	diamond$^\mu$
`Some`	stones^m	`are`	diamonds$^\mu$
`Some`	stones^m	`are not`	incombustible^M

`All`	meerschaum bowls^M	`are`	turning brown$^\mu$
`Some`	pipe bowls^m	`will not`	turn brown$^\mu$
`Some`	pipe bowls^m	`are not`	meerschaum bowls^M

Fig. 3 Baroco [26, p. 308]

These modes require several different diagrams since the conclusion in both is particular and negative (Festin$O$, Baroc$O$). However, Schopenhauer only gives the Euler-type diagrams in which the O-prop. is interpreted as a shortened variant of the E-prop. (universal negative). It is precisely through this interpretation that he emphasizes the mental-linguistic criterion of $\mathcal{F}_{II}$.

In all diagrammatic modes of $\mathcal{F}_{II}$, μ functions as a "Scheidewand" ("septum," [26, pp. 308–309]) between M and m. In relation to μ, m does the opposite of what M, in relation to μ, does. With the help of the diagrams, Schopenhauer tries to show that μ "grasps" one of the two other concepts (either M or m) and thereby separates the other remaining one from it. The naturalness of the mental-linguistic criterion, namely separation and differentiation, is thus shown in the separation of the Euler-type circles for M and m. (Whether this thesis is coherent or not, cannot be discussed here.)

2.4 Naturalness of $\mathcal{F}_{III}$

Schopenhauer discusses many arguments for the naturalness of $\mathcal{F}_{III}$. The following quote ($Q6$) offers a relevant text passage:

> ($Q6$) The 3rd figure is thus the natural form of thinking when noting an exception, only it is rarely pronounced in extenso, but almost always contracted: the medius is briefly given as the example of the exception, which is the argument of the proposition. ([26, p. 316])
> Die 3te Figur ist also die natürliche Form des Denkens beim Anmerken einer Ausnahme: nur wird sie selten in extenso ausgesprochen, sondern fast immer kontrahirt: der medius wird kurz angegeben als das Beispiel der Ausnahme, welches das Argument des Satzes ist.

Also in $Q6$ we again see two criteria for naturalness, whereby the mental-linguistic criterion ($Q6.1$) is emphasized more clearly than the diagrammatic criterion ($Q6.2$):

($Q6.1$) If it is the intention of a speaker to represent an anomaly or exception, she naturally applies $\mathcal{F}_{III}$, even if she is often unaware of the theoretical function. Schopenhauer explains that we use contracted forms of $\mathcal{F}_{III}$ "countless times" ("unzählige Mal," [26, p. 312]) in everyday reasoning and argumentation. The structure of the contracted form is not very complicated and can be summarized by a simple formula:

$$\text{Some } m \text{ are (not) } M, \text{ such as } \mu.$$

This formula already shows a conspicuous characteristic of $\mathcal{F}_{III}$: The conclusion in $\mathcal{F}_{III}$ is always a particular proposition (I-prop. or O-prop.: "Some m are (not) M") and is justified by adding the exception to M ("such as μ"). Furthermore, another mental component comes into play: Schopenhauer repeatedly speaks of the fact that $\mathcal{F}_{III}$ expresses something unexpected ("unerwartet"). The contracted form only works to the extent that we can already assume the acquaintance of a certain rule (given in the second premise). But if there is an exception (positive or negative) to this well-known rule, the exception is expressed by adding μ to the conclusion. This is illustrated by the enthymeme "Some water dwellers are not fish, such as dolphins." However, the explicit form of this enthymeme is given in Fig. 4:

Fig. 4 Felapton [26, p. 313]

`No`	dolphin$^{\mu}$	`is a`	fish^{M}
`All`	dolphins$^{\mu}$	`are`	water dweller^{m}
`Some`	water dweller^{m}	`[are] not`	fish^{M}

(Q6.2) Similar to (Q4.2) and (Q5.2), the diagrammatic criterion for naturalness of $\mathcal{F}_{III}$ depends on μ. In $\mathcal{F}_{III}$, μ is called an "Anzeiger" ("indicator," [26, p. 316]). In the case of a particular positive conclusion (I-prop.), μ is the indicator of an unexpected difference between the normally homogeneous concepts m and M. In the case of a particular negative conclusion (O-prop.), μ is the indicator of an unexpected congruence between the normally heterogeneous concepts m and M. In Felapt$O$n, Bocard$O$, and Feris$O$n, μ indicates an unexpected difference; in Darapt$I$, Disam$I$s, and Datis$I$, μ indicates an unexpected congruence. This is well illustrated by the diagram of the example for contracted forms in Q6.1 (cf. Fig. 4): "water dweller" and "fish" are very homogeneous terms, but μ makes the difference between both obvious. Due to the well-known rule (second premise), "All dolphins are water dwellers," it is expected that the circle for dolphins also at least intersects the circle for fish. However, the conclusion makes explicit that this expectation does not always have to come true. In the diagram the circles for dolphins and fish are completely separated, but both are completely within the circle for water dwellers.

3 Natural Completeness and Unnatural Redundancy

In Sects. 2.2–2.4 we have given a mental-linguistic as well as a diagrammatic criterion in order to clarify Schopenhauer's thesis that only $\mathcal{F}_{I-III}$ can be part of a system of natural deduction and natural logic. A justification to the claim that $\mathcal{F}_{IV}$ is unnatural, however, was only insufficiently addressed in Sect. 2.1. There, Schopenhauer explained that $\mathcal{F}_{IV}$ turns $\mathcal{F}_{I}$ upside down. But this fact alone is no argument for declaring $\mathcal{F}_{IV}$ as being unnatural.

In this section, we would like to further strengthen Schopenhauer's argument that $\mathcal{F}_{IV}$ is unnatural. He himself already envisages at least three central backings for his argument in only one short text passage [26, p. 329]. But since we have already selectively emphasized criteria in Sect. 2, we concentrate here again only on some mental-linguistic (3.1) and some diagrammatic aspects (3.2). In both criteria, Schopenhauer's argumentative strategy is as follows: whereas $\mathcal{F}_{I-III}$ already form a complete system of natural deduction and logic, $\mathcal{F}_{IV}$ is mentally linguistic as well as diagrammatically redundant.

3.1 Mental-Linguistic Completeness and Redundancy

One criterion that Schopenhauer repeatedly used as an argument for the naturalness of $\mathcal{F}_{I-III}$ concerned the relationship between intentional states and their linguistic expressions: If I want to express a decision or resolution, I should use $\mathcal{F}_I$. But if it is my intention to make a difference clear, $\mathcal{F}_{II}$ is usually considered. If, however, an unexpected exception is to be expressed, it is clever to formulate this in the form of $\mathcal{F}_{III}$. Schopenhauer leaves open whether the choice of $\mathcal{F}$ is conscious or unconscious.

However, Schopenhauer claims that with $\mathcal{F}_{I-III}$ the expressivity of natural logic is exhausted. $\mathcal{F}_{IV}$ is redundant for several reasons: (1) On the one hand, it is unnatural because its syntax is not in accordance with our natural feeling for language. (2) On the other hand, $\mathcal{F}_{IV}$ does not expand our expressiveness, since no thought or intentional state is expressed more clearly in it than in one of the other $\mathcal{F}$. Both cases are confirmed for Schopenhauer by the fact that no examples for $\mathcal{F}_{IV}$ can be found in ordinary or scientific language. All typical examples appear to be artificially constructed.

Let us first look at what Schopenhauer says in the following quote ($Q7$) about the unnatural syntax in $\mathcal{F}_{IV}$:

> ($Q7$) Therefore, this figure [sc. $\mathcal{F}_{IV}$] is always unnatural, and one will never think in it. Most naturally it still appears in Fesapo: but apparently the upper sentence of the same is a Conversio: One will never originally think "No Christian is a Bashkir": but rather "No Bashkir is a Christian": for one always takes the narrower term to the subject, the further to the predicate.
>
> Daher ist diese [sc. die vierte] Schlußart immer unnatürlich, und nie wird man in ihr denken. Am natürlichsten erscheint sie noch in Fesapo: aber doch ist offenbar der Ober-Satz desselben eine Conversio: Man wird nie ursprünglich denken "Kein Christ ist ein Baschkire": sondern "Kein Baschkir ist ein Christ": denn man nimmt immer den engern Begriff zum Subjekt, den weitern zum Prädikat. [26, p. 323]

In $Q7$, Schopenhauer alludes to the following modus Fesapo given in Fig. 5:

`No`	Christian^M	`is a`	Bashkir$^\mu$
`All`	Bashkirs$^\mu$	`are`	Russians^m
`Some`	Russians^m	`are not`	Christians^M

For Schopenhauer, it would be more natural if the intended resolution would have been expressed in a mode of $\mathcal{F}_I$. But in contrast to a $\mathcal{F}_I$ inference the first premise of Fesapo was changed by a conversio simplex and the second one by a conversio per accidens. Schopenhauer claimed that the unnaturalness is already apparent in the first premise: One would usually not think or express oneself in this way, since it is more natural to place the wider concept in the subject position and the narrower concept in the predicate position, and not vice versa (as in Fesapo).

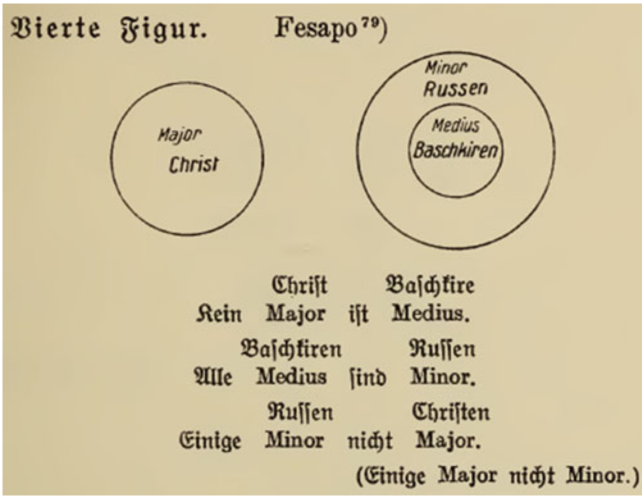


Fig. 5 Fesapo [26, p. 319]

Let us now come to the argument that $\mathcal{F}_{IV}$ does not expand our logical expressiveness. This argument is reflected in the following quote ($Q8$):

> (Q8) If one meanwhile also wants to give a rational basic thought to the 4th figure, it would be the following one: the 1st figure always has the purpose to *decide* a case by a general rule: therefore it subsumes the case under the rule: the 4th, which is its straight conversion, also has the opposite purpose: it wants to *confirm* a rule by a case, the case should be the proof of the rule[.]
> Will man inzwischen auch der 4ten Figur einen vernünftigen Grundgedanken unterlegen; so wäre es dieser: die 1ste Figur hat immer den Zweck einen Fall durch eine allgemeine Regel zu *entscheiden*: daher sie den Fall der Regel subsumirt: die 4te, welche ihre grade Umkehrung ist, hat auch den umgekehrten Zweck: sie will nämlich eine Regel durch einen Fall *bestätigen,* der Fall soll der Beleg der Regel seyn[.] ([26, p. 323]; emphasized by H.M.S. & J.L)

$Q8$ points out that Schopenhauer also regards $\mathcal{F}_{IV}$ as redundant insofar as it cannot express a thought that could not already be expressed by $\mathcal{F}_{I-III}$. The only intention that can be expressed by using $\mathcal{F}_{IV}$, namely confirmation, is already contained in $\mathcal{F}_I$ as one aspect of the resolution: In the conclusion, $\mathcal{F}_I$ decides whether the relationship between M and m is positive or negative. A certain case can thus be confirmed or rejected by a rule. However, some inferences of $\mathcal{F}_{IV}$ (which are more complicated than Fesapo, e.g., Calemes) reverse this relationship and can only confirm the rule. Consequently, its expressivity is already covered by $\mathcal{F}_I$, which also has a much more natural syntax. In summary, it can be said that $\mathcal{F}_{IV}$ is mentally linguistically superfluous and thus unnatural, since it does no more than $\mathcal{F}_{I-III}$.

3.2 Diagrammatic Completeness and Redundancy

According to Schopenhauer, the completeness of $\mathcal{F}_{I-III}$ and thus the redundancy of $\mathcal{F}_{IV}$ can also be shown diagrammatically. The following ($Q9$) is a relevant quote for this thesis and deals mainly with the diagrammatic possibilities of μ:

> ($Q9$) Thus, three times we find the possibility [sc. of μ] consistently exhausted. 1) Three ways in which the medius can be the reason of the judgment in the conclusion: every possible inference corresponds to one of it. 2) Three possible positions of the medius in the premises [. . .]. 3) In the spheres: The medius is either the widest, or the narrowest, or the middle sphere. [26, pp. 329–330]
> Wir finden also die Möglichkeit [sc. von μ] drei Mal übereinstimmend erschöpft. 1) Drei Arten wie der Medius Grund des Urtheils der conclusio seyn kann: in jedem möglichen Schluß ist ers auf eine dieser drei Arten. 2) Drei mögliche Stellungen des Medius in de[n] Prämissen [. . .]. 3) In den Sphären: der Medius ist entweder die weiteste, oder die engste, oder die mittlere Sphäre.

In his chapter *On inferences* Schopenhauer treats all three points of $Q9$ in detail. We can only briefly discuss the three points in the following:

($Q9.1$**)** We have already shown in Sects. 2.2–2.4 that μ has its own diagrammatic function, which Schopenhauer describes metaphorically:

1. In $\mathcal{F}_I$, μ functions as the "Handhabe" (manipulator)
2. In $\mathcal{F}_{II}$, μ functions as a "Scheidewand" (septum)
3. In $\mathcal{F}_{III}$, μ functions as an "Anzeiger" (indicator)

As the argument of the whole inference, μ is always a kind of reason (cf. [26, pp. 297, 325]). According to Schopenhauer, the diagrammatic completeness of $\mathcal{F}_{I-III}$ can be found in the fact that there are exactly three and no more than three reasons. Depending on how μ interacts with the other terms in the Euler-type diagram, it becomes one of the following reasons:

1. "Entscheidungsgrund," the reason for resolution [26, p. 326];
2. "Unterscheidungsgrund," the reason for distinction [26, p. 327];
3. "Ausscheidungsgrund," the reason for inclusion and difference [26, p. 327].

($Q9.2$**)** The easiest way to interpret the second point is to illustrate the position of μ in $\mathcal{F}_{I-IV}$ (cf. above, Sect. 2.1): Schopenhauer argues that μ can have only three possible positions in the premises, namely either

($P1$) diagonally (sometimes as subject, sometimes as predicate),
($P2$) or completely right (always as predicate),
($P3$) or completely left (always as subject).

If one now look at the position of μ, one can see that $\mathcal{F}_{I-III}$ corresponds to P1–3, but $\mathcal{F}_{IV}$ repeats P1:

$$(P1) = \begin{bmatrix} \mu & M \\ m & \mu \\ m & M \end{bmatrix} (P2) = \begin{bmatrix} M & \mu \\ m & \mu \\ m & M \end{bmatrix} (P3) = \begin{bmatrix} \mu & M \\ \mu & m \\ m & M \end{bmatrix} (P1) = \begin{bmatrix} M & \mu \\ \mu & m \\ m & M \end{bmatrix}$$

In $\mathcal{F}_I$, μ is always diagonal $(P1)$, $\mathcal{F}_{II}$ completely right $(P2)$, and $\mathcal{F}_{III}$ completely left $(P3)$. $\mathcal{F}_{IV}$ is thus diagrammatically a repetition of the diagonal position $(P1)$, already fulfilled by $\mathcal{F}_I$, and thus redundant.

($Q9.3$**)** The third point is the most interesting one, but it is difficult to explain if Schopenhauer's Euler-type diagrams are used as a diagrammatic criterion. It is easier to explain, however, if one uses $P1 - P3$ already discussed in $Q9.2$. Additionally, one has to assume the rule that the predicate is always wider than the subject, as discussed in $Q7$. In regards to these rules, Schopenhauer argues for diagrammatic completeness and redundancy:

$(D1)$ If $(P1)$, as given in $\mathcal{F}_I$, μ represents a medium-sized sphere.
$(D2)$ If $(P2)$, as given in $\mathcal{F}_{II}$, μ represents the widest sphere.
$(D3)$ If $(P3)$, as given in $\mathcal{F}_{III}$, μ represents the narrowest sphere.

This, however, exhausts all possibilities of the relationship between μ and the other concepts m and M [26, pp. 327–328]. Moreover, in $\mathcal{F}_{IV}$ as well as in $\mathcal{F}_I$, μ represents a medium-sized sphere, since in both μ is sometimes in the position of the subject and sometimes in the position of the predicate (i.e., $(P1)$). Thus $\mathcal{F}_{IV}$ is only a repetition of $\mathcal{F}_I$.

4 Conclusion and Outlook

In this section we would like to summarize the central criteria and arguments from Sects. 2 and 3 and then examine some aspects of Schopenhauer's approach from today's perspective.

4.1 Summary

We have seen in Sects. 2 and 3 that Schopenhauer indicates a mental-linguistic criterion in order to separate the natural from the unnatural inferences. Inferences are natural (1) if the position of the concepts in the judgment and (2) the position of the judgments in the inference meet our ordinary linguistic usage and thus (3) express a specific mental intent. Inferences are unnatural if they have an unusual syntax and if they only allow us to express something that can be better expressed by other inferences.

Regarding the classification into four figures in syllogistics, Schopenhauer can say that all valid inferences of the first three figures are natural, whereas the inferences of the fourth figure are unnatural. With the first of the three figures all forms of resolutions can be expressed, with the second figure confusions and misunderstandings can be clarified, and with the third figure exceptions and paradoxes are proved. However, the fourth figure has an unnatural syntax and no autonomous or unique expressivity.

For Schopenhauer there is a complete analogy between the logical relations of the concepts in the judgment and the spatial relation of circles in geometry. For this reason, all differences regarding inferences can also be represented with the help of Euler-type diagrams. If one looks only at the medius (at the argument of the inference), one can see that it always clarifies the relationship between the major and the minor term. In the first figure it decides whether the termini major and minor completely or partially overlap or exclude each other. In the second figure it always separates the major term from the minor one and in the third figure it shows exceptions between the termini major and minor, which are usually expected to be in correspondence.

4.2 Schopenhauer's Logic in the Context of Current Debates

We are of the opinion that Schopenhauer puts forward many arguments which are still worth discussing today or which are still being discussed in current research. Since we have only focused on two criteria in Sects. 2 and 3, we will concentrate here only on (1) the mental-linguistic criterion that might be of interest to natural logic and (2) the diagrammatic criterion which is in line with current trends in systems of natural deductions.

1. Schopenhauer's criterion for naturalness and unnaturalness in syllogistics does not depend on the method of mathematical proof, but on a mental-linguistic criterion. It therefore does not have much to do with the approach that became known through Gentzen and Jaśkowski (cf. Sect. 1) for which reasoning and proving were mainly limited to the activity of the mathematician. For Schopenhauer, logic is not an *ancilla mathematicae*. The naturalness of logic is not a criterion that can only be oriented towards the scientific practice of mathematical proof. This does not mean, of course, that his logic based on criteria of naturalness ignores or even excludes mathematical proofs. On the contrary, Schopenhauer explains several times how proofs fit into his logical approach. However, the decisive criterion for him concerns the question of whether his logic can represent the naturalness of both scientific and everyday thinking and reasoning or not. His mental-linguistic criterion says that naturalness is given if we actually think this way ("countless times," *Q*6.1) and that unnaturalness is given if we would not think in this or that certain way. Similar to current trends in "natural logic" (cf. [1, 12]), Schopenhauer's criteria for naturalness lead to a regimented fragment of natural language and argues against the formalization of Aristotelian scholasticism. However, Schopenhauer sees the basis of a natural logic not in a regimented

language, but in the description of the corresponding intuition with the help of Euler diagrams.
2. As we have seen in Sect. 1 of Danielle Macbeth's quote, a criterion for the difference between axiomatic and natural systems concerns the role of axioms, assumptions, and rules. Schopenhauer, however, proves the validity of inferences by their correspondence to the Aristotelian rules. We have not discussed this in detail here, since Schopenhauer separates the question of validity from the question of naturalness (cf. above Sect. 2.1). The naturalness of inferences is not based on their validity, but on their actual application in everyday argumentation and can be illustrated with the help of the construction rules of Euler diagrams. In these diagrams Schopenhauer sees not only reasoning aids, but the actual foundation of logic (cf. [15]). Furthermore, Schopenhauer sees another advantage of the diagrams in the fact they can automatically make implicit information explicit (see above $Q3.3$ and $Q3.4$). With these arguments Schopenhauer supports current developments in the information sciences in which various systems of natural deduction are presented more and more by using Venn- and Euler-type diagrams, e.g., in [16, 22], etc.

In this paper, we have only presented a few ideas on naturalness in Schopenhauer and given some hints as to how they might be of relevance for current research on natural logic and natural deduction. Detailed investigations on Schopenhauer's logic, but also on his predecessors, seem to be necessary in order to reach a verdict on the subject of naturalness in pre-Fregian logic. So far, it seems to be unique that Schopenhauer combines a mental-linguistic criterion with a diagrammatic one in dealing with the question of the naturalness of logic.

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Schopenhauer and the Equational Form of Predication

Anna-Sophie Heinemann

Abstract Given the common narrative of the history of the nineteenth century logic, it may seem surprising that in some passages of his logic lectures, Schopenhauer invokes an equation sign to express relations of predication as in "A = B". The present paper proposes an assessment of Schopenhauer's use of the equation sign. Departing from an analysis of Schopenhauer's account of concepts and judgments, it offers a survey of logic textbooks which Schopenhauer was acquainted with. The preliminary conclusion will be that for some of Schopenhauer's sources, the equational notation is justifiable as they do suggest certain revisions of logic which point towards the possibility of quasi-"algebraic" models. Schopenhauer's own use of the equation sign, however, fails to come up to the conceptual prerequisites that would allow for an "algebraic" approach. In particular, Schopenhauer does not seem to be aware of the possibility to invoke an equational notation to express implication in the sense of stating equivalences between propositions and their transformations.

Keywords Mathematization of logic · Laws of thought · Syllogistic predication · Interpretation of the copula · Negative terms · Conversion and contraposition · Identity and equivalence

Mathematics Subject Classification (2020) Primary 03A05, Secondary 01A55, 00A22

1 Introduction

It is well-known that around the middle of the nineteenth century, some formative steps were taken towards what came to be named the "algebra of logic." Most of its early promotors are to be found among the British authors of that time. Most

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prominently, we remember George Boole, who tried to cast logical problems into the form of quasi-algebraic equations, invoking notational means borrowed from mathematics, most prominently the equation sign. Even today, Boole's attempts are still alive to the extent that Boole's project is recognizable in today's "equational logics." Given this narrative, it may seem surprising to find that even before Boole's times, there were continental authors whose choice of symbols appears to be comparable to Boole's in spirit. Such cases can be found for the use for plus-signs and minus-signs in various interpretations. The equation sign, however, is usually employed to express either predication, as in "A = B" for "A is B", or a sense of implication by stating an equivalence between several propositional expressions, as in "'All As are Bs' = 'Some Bs are As'".

Interestingly, the equation sign can also be found in Schopenhauer's logic of the *Berlin Lectures*. The present paper will try to give some elucidation and an assessment of Schopenhauer's use of the equation sign. The first step will be to consider equational forms throughout his logic. The second step will be to look at some relevant context within Schopenhauer's logic, namely his account of judgments as being made up of concepts, and his views on the quantity and quality of judgments. The third step consists in a survey of logic textbooks which Schopenhauer was acquainted with, aiming at a reconstruction of where he might have taken his equational notation from. Finally, there remains the question whether Schopenhauer's choice of notation matches the conceptual prerequisites of his logic. The preliminary conclusion will be that it does not. Rather, the equational notation mirrors certain revisions of logic which point towards "algebraic" or algorithmic models, and which are suggested in some of his sources, but which Schopenhauer himself fails to come up to.

2 Schopenhauer's Use of Equality Signs in Logic

Like many of the textbooks of his time, Schopenhauer's logic lectures[1] are composed of a section on concepts, followed by a section on judgments, and a section on inference. Schopenhauer's account of judgments contains a subject matter which more commonly comes under the caption of "immediate inferences," such as inferences by subordination and opposition. A special case of such inferences are those transformations of judgments which are effected by "conversion" and by "contraposition." Schopenhauer concludes his section on judgments with a summary of their admissible forms, in the following arrangement [19, p. 293].

[1]Schopenhauer's logic lectures are contained under Chap. 3 in [19, pp. 234–363].

(a)	*Convertiren simpliciter, lassen sich*		
(1)	Allgemein verneinende	Kein A = B	Kein B = A
(2)	Partikulär bejahende	Einige A = B	Einige B = A
(b)	*Convertiren per accidens*		
(1)	Allgemein bejahende	Alle A = B	Einige B = A
(c)	*Contraponiren simpliciter*		
(1)	Allgemein bejahende	Alle A = B	Kein Nicht B = A
(2)	Partikulär verneinende	Einige A = nicht B	Einige Nicht B = A
(d)	*Contraponiren per accidens*		
(1)	Allgemein verneinende	Kein A = B	Einiges Nicht B = A

Generally speaking, Schopenhauer's account of transformation by conversion and contraposition is not very exceptional. To convert a proposition (*Satz*) means to turn the predicate into the subject and the subject into the predicate (*das Prädikat zum Subjekt und das Subjekt zum Prädikat machen*; [19, p. 289]). In order to preserve their meaning on such an interchange, some kinds of judgments will require a change of quantity and quality (ibid.). The conversion is simple (ibid.) in case neither quantity nor quality are changed. If there is a change in quantity only, the conversion is per accidens (ibid.). If there is a change in quality, a contraposition takes place (ibid.). If in this case, the quantity remains unchanged, the contraposition is, again, simple (ibid.). If, however, both quality and quantity are changed, the contraposition is, again, per accidens (ibid.).

For the admissible transformations, Schopenhauer relies on an extension of the scholastic codification of a set of rules: Universal affirmatives are converted per accidens [19, p. 290] or by simple contraposition (ibid.). Universal negatives are converted *simpliciter* or by contraposition per accidens [19, p. 291]. Particular affirmatives, too, may be converted *simpliciter* (ibid.), except if the predicate happens to be wholly included in the subject. In the latter case, particular affirmatives allow for conversion per accidens (ibid.). However, there is no possibility of contraposition for particular affirmatives (ibid.). Particular negatives, on the other hand, cannot at all be converted, except for by simple contraposition (ibid.). Accordingly, Schopenhauer would have (i) "All A are B" as an "equivalent" (loosely speaking) for "Some B are A," or "No non-B is an A;" (ii) "No A is a B" as an equivalent for "No B is an A," or "Some non-B are A;" (iii) "Some A are B" as an equivalent for "Some B are A;"[2] and, finally, (iv) "Some A are not B" as an equivalent for "Some non-B are A."[3] These are just the forms collected in Schopenhauer's table, (i) being listed

[2]According to Schopenhauer, this may turn out "All B are A" in special cases. One such case would be "Some trees are firs," but "All firs are trees" ([19, p. 291]; hence "Some trees are firs" in a sense turns out a converted universal).

[3]Schopenhauer exemplifies these rules as follows. Universal affirmatives are exemplified by "All rocks are solids," turning into "Some solids are rocks" on conversion, and into "No non-solid is a rock" on contraposition [19, p. 291]. As an universal negative, Schopenhauer gives "No rock is an animal," resulting in "No animal is a rock," or, by contraposition, in "Some non-animals are

under (b)(1) and (c)(1); (ii) under (a)(1) and (d)(1); (iii) under (a)(2), and (iv) under (c)(1).

Thus, the contents of Schopenhauer's table of conversions are not very surprising. What is surprising is the way the contents are presented. Their arrangement results from a classification according to the distinction of kinds of conversion.[4] But what is particularly noteworthy is Schopenhauer's choice of the equality sign "=" to indicate predication, but precisely not to indicate the "equivalence" between the members of the pairs of judgments in each line.

By casting judgments into an "equational" form, Schopenhauer seems to anticipate early proponents of a "mathematization" of logic such as Moritz Wilhelm Drobisch,[5] who in 1836 put it this way:

> To express affirmative judgments by equations seems to be the most appropriate way of signifying that in a certain respect, they always represent an identity of the subject and the predicate, which comes to light most distinctly on conversion."
>
> (Die bejahenden Urtheile durch Gleichungen auszudrücken scheint am zweckmäßigsten, um zu bezeichnen, daß sie in gewisser Hinsicht immer eine Identität des Subjects and Prädicats darstellen, wie aus der Umkehrung am deutlichsten hervorgeht; [1, p. 132].[6])

This quote from Drobisch is representative for the conception of logic which first allowed for a "mathematical" approach. With particular regard to the use of equality signs, one may ascribe to it at least two necessary prerequisites, namely (1) a reduction of forms to the acceptance of affirmative judgments only, and (2) an implicit quantification of concept terms, which allows one to compare their quantities in order to test whether they can be said to "equal" one another. This is because an "equivalence," or, as Drobisch put it, an "identity," just cannot be negative. It is about shared, i.e., "equivalent," or "identical" parts of terms, most naturally taken to be parts of their extensions. But of course, still there remain judgments which express denials. The most convenient approach to deal with them would be to admit negative terms. Another way, however, would be to depart from certain operations of "addition" and "subtraction" in logic: the negative counterpart of a concept term would then be expressible by subtracting the concept term itself

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rocks" (ibid.). The case of particular affirmatives is illustrated by "Some birds a predators"; so "Some predators are birds" (ibid.). For particular negatives, Schopenhauer gives "Some animals are not endothermic"; so "Some non-endothermics are animals" [19, p. 292].

[4] In a footnote, Schopenhauer refers to this listing as a "table," namely one which is of a "peculiar symmetry" (*sonderbare Symmetrie*; [19, p. 293]). This "peculiar symmetry" consists in "the lower part of the table reading just like the upper one" (ibid.). Schopenhauer himself gives no explanation for his comment, and it seems far from clear what he refers to. Moreover, no clue is to be found in any of the textbooks discussed in Sect. 3 of the present article. Therefore, the interpretation of Schopenhauer's sense of symmetry must remain arcane as it is for the present purposes.

[5] For reasons of space, the present article is confined to discuss contributions from the German logic scene at around Schopenhauer's time. Therefore, references to some very important British developments of the early nineteenth century will be omitted.

[6] All translations of German into English are by the author.

from a given larger term extension (cf. ibid.).[7] Schopenhauer, however, gives no clue of what he means by his equational forms of judgments, nor of why he thinks that he is justified in replacing the copula "is" by an equality sign.

It should be noted that up to his table of transformations by conversion and contraposition, Schopenhauer employed an equality sign only four times in his logic lectures. One out of these four instances is within the expression of the arithmetical equation "$3 \times 7 = 21$", cited as an example for what Schopenhauer named "metaphysical truth," i.e., a truth which is independent of experience [19, p. 267]. The remaining three instances of Schopenhauer's use of the equality sign are to be found among his discussion of the so-called laws of thought (*Denk-Gesetzen*; [19, p. 261]), and as in the table given above, they seem to be meant to chiefly mirror predication.

The first instance of an equality sign in Schopenhauer's lectures is to be found within his version of the Principle of Identity, which reads: "a concept is selfsame" (*der Begriff ist sich selbst gleich*; [19, p. 262]), "no matter if I think of it as a whole [. . .] or I dissolve it into all the concepts it contains as predicates" (*ich mag ihn nun [. . .] denken im Ganzen [. . .] oder auflösen in seine sämmtlichen Pr[ä]dikate*; ibid.). In other words: "a concept is equal to the sum of its predicates" (*der Begriff ist gleich der Summe seiner Prädikate*; ibid.). Hence Schopenhauer put the Principle of Identity as "A = A" (ibid.).

As the second law of thought, Schopenhauer lists the Principle of (Non-)Contradiction: "The predicate must not annul the subject, neither as a whole nor even partially" (*Das Prädikat darf das Subjekt nicht aufheben, weder ganz noch zum Theil*; ibid.). In other words: "it [the predicate] must not contradict it [the subject], i.e., what is affirmed in the subject must not be denied in the predicate nor vice versa, neither directly nor indirectly" (*d.h. es darf ihm nicht widersprechen, d.h. was im Subjekt bejaht ist darf im Prädikat nicht verneint seyn und umgekehrt, weder mittelbar noch unmittelbar*; ibid.). Interestingly, Schopenhauer abbreviated this principle by a somewhat curious formula, namely: "$A = -A = 0$" (ibid.).

Schopenhauer's third law of thought is the Principle of the Excluded Middle: "Any and every subject either has any one predicate or does not have it; it is to be affirmed or denied of it (*Jedem Subjekt kommt jedes Prädikat entweder zu oder nicht; ist entweder von ihm zu bejah*[*n*] *oder zu verneinen*; [19, p. 263]), i.e., "non datur tertium" (ibid.). Schopenhauer's short-hand formula reads "A aut = b, aut = non b" (ibid.).

Schopenhauer's list of "laws of thought" is then completed by his interpretation of the Principle of Sufficient Reason, which he takes to be the principle of sufficient reason for cognition (ibid.), i.e., for taking cognizance of a judgment being true (ibid.). as warranted by something external to itself [19, p. 263f.]. Hence this

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[7]This is Drobisch's option of choice. However, it should be noted that Drobisch does not employ equality signs in negative judgments. Therefore, Drobisch does not face the problem of expressing negatives by "identities" between counterparts of terms.

principle does not relate to predication, taken as a judgment's inner structure. Therefore, Schopenhauer does not make use of any arithmetical signs to state it.

3 Schopenhauer's Account of Quantity and Quality of Judgments

The next interpretive step to Schopenhauer's use of the equality sign should be to consider some relevant parts of his logic as to their conceptual groundwork. Is there anything peculiar about Schopenhauer's views on logic that warrants the use of equality signs to indicate predication? The present section will relate to Schopenhauer's account of judgments as composed of concepts, and his views on quantity and quality of judgments.

Indeed, Schopenhauer's opinion was that "to judge" means "to discern the proportions"[8] between "given concepts" (*Die Verhältnisse gegebener Begriffe zu einander erkennen, heißt urtheilen*; [19, p. 260]; cf. *Jedes Urtheil ist also die Erkenntniß des Verhältnisse*[*s*] *zwischen Begriffen*; [19, p. 261]). To discern proportions between concepts means to discover "their linkage, or lack thereof, respectively" ([*die Erkenntniß*] *ihrer Verbindung oder auch Nicht-Verbindung*; ibid.). But to discover their linkage means to recognize that one concept is thought "within another concept either wholly or partially" (*d.h. die Erkenntniß daß in einem Begriff ein andrer entweder ganz oder zum Theil mitgedacht ist*; ibid.). or that "there is no linkage of this kind at all, to the effect of a negative judgment" (*oder aber umgekehrt daß er gar nicht mit ihm verbunden ist; dann ist das Urtheil negativ*; ibid.).

According to Schopenhauer, it is making and stating such cognitions what is meant by "thinking proper" (*eigent[lich] Denken*; ibid.). Hence in order to think at all, "one starts off with one concept, of which it is to be discovered that it is contained in a second concept, wholly or partially" (*Man geht stets von einem Begriff aus, den man als ganz oder zum Theil im andern enthalten erkennt*; ibid.). These two concepts are what for the purposes of logic are called "subject" and "predicate." The first concept, i.e., the one to start from is the subject; the other one, i.e., the one in which the subject is contained, the predicate (ibid.). However, Schopenhauer pointed out that the second concept, i.e., the predicate, "is just as well contained in the first, i.e., the subject, either wholly or partially" (*allemal ist aber auch der zweite ganz oder zum Theil im ersten enthalten*; ibid.). Therefore, the second can become the subject, and the first the predicate (ibid.). The proportions may differ since of two concepts A and B, "A may be in B wholly, while B is in A only partially" (ibid.). However, a transposition by conversion (*Umkehrung*) should in any case be possible (ibid.).

[8] The German has "Verhältniß," which admits of a broader set of interpretations, such as "relation," but also "ratio."

Now, one might expect Schopenhauer to have thought of the proportion between concepts which are said to be positively related as an overlap between sections of the concepts' "spheres," i.e. their extensions,[9] and hence as a partial identity. Regarding the category of quantity, to determine such an overlap would then imply an explicit consideration of the sizes of the respective portions of both terms. Secondly, as to the category of quality, the most suitable option to express negative judgments by way of determining overlaps of concept spheres would be to confine the form of judgments to that of affirmative categoricals while admitting negative concept terms. However, while Schopenhauer did concentrate on categoricals as to the form of judgments [19, p. 278], a closer look at his account of judgment reveals that as to quality and quantity, neither of the aforementioned suppositions seems to be the case.

From Schopenhauer's account of judgments, it seems fairly clear that he should be committed to conceive of the copula as an indicator of the linkage of concepts by proportion. Indeed, at first sight he did so in defining the copula as "the word which indicates the proportion of the concepts" (*Das Wort, welches das Verhältniß der Begriffe andeutet*; [19, p. 261]). However, Schopenhauer modified this determination in calling it "a trope" (*uneigentlich*; ibid.). The reason is that while the copula is described as a connector, it may also serve the separation of concepts. In this latter case, its expression is not "is" but rather "is not" (ibid.). Hence Schopenhauer seems claim that there are in fact two copulae which differ in quality. Similarly, while discussing the quality of judgments, Schopenhauer noted that quality is either about the union (*Vereinigung*) or about the separation (*Trennung*) of concepts, or rather, their extensions (*Begriffssphären*; [19, p. 274]). But according to Schopenhauer, a judgment's quality is not to be expressed by any of the words which designate a judgment's concepts [19, p. 275]. Rather, it is expressed by "is" or "is not," which indicate two copulae, or, classically speaking: two modifications of the copula: *Affirmatio aut Negatio afficit copulam* (ibid.).

While Schopenhauer holds that a judgment's copula carries the expression of its quality, he is also clear that a judgment's quantity is expressed by the subject term (*Der Ausdruck der Qualität in der Copula, der Ausdruck der Quantität im Subjekt*; [19, p. 278]). A difference in quantity depends on whether the subject term is to be taken by the whole of its extension or by part of it only (*ob das Subjekt in seinem ganzen Umfange genommen werden soll, oder nur ein Theil desselben*; [19, p. 276]). Hence it remains unclear how a copula which can be either affirmative or negative should conform with a single equation sign, and how the idea of an equivalence between the two sides of it should be justified if the quantity of the subject term only is considered.

9 Schopenhauer is explicit that the notion of a concept's "sphere" is to be interpreted extensionally [19, p. 271].

4 Possible Sources for Schopenhauer's Equational Symbolism

As shown in the previous sections, Schopenhauer himself offers no consistent account of his employing equation signs for predication. Thus, there remains the question if he adopted such means from other works on logic. The following section will serve to make out some possible sources.

In his logic lectures, Schopenhauer seems to relate to three authors from the century before him, notably the philosopher-logician Gottfried Ploucquet (1716–1790), the mathematician Johann Heinrich Lambert (1728–1777), and the physicist-mathematician Leonhard Euler (1701–1783). These references concern questions of diagrammatical representation in logic: While treating of intersecting or nested circles of different sizes to represent proportions between concept "spheres," Schopenhauer adds a marginal note referring to Lambert's employing lines of different lengths to serve the same purpose [19, p. 270], Ploucquet's use of squares and Euler's introducing circles [19, pp. 269–270].[10]

In the introductory part of his lectures, Schopenhauer also makes notice of a more contemporary writer on logic: Johann Friedrich Herbart (1776–1841). Schopenhauer refers to the lengthy logical appendix to Herbart's *Hauptpuncte der Metaphysik* [4, pp. 101–130; 19, p. 248]. Presumably he also knew Herbart's *Lehrbuch zur Einleitung in die Philosophie* [5, §§34–71], which contains some extended sections on logic.

Moreover, some of Schopenhauer's early manuscripts[11] prove that he was at least acquainted with some more logical literature of his time. He refers to Herrmann Samuel Reimarus's (1694–1768) slightly earlier *Vernunftlehre*, published in 1756 [18, 20, p. 52]. There is one reference to Kant's *Logik* [8, 20, p. 53]. Schopenhauer also relates to Johann Gebhard Ehrenreich Maass (1766–1823), who had authored an influential *Grundriß der Logik*, first published in 1793 [13, 20, p. 52], and to Ludwig Heinrich von Jakob's (1759–1827) *Grundriss der allgemeinen Logik* from 1788 ([9]; ibid.). He also refers to Johann Christoph Hoffbauer's (1766–1827) *Anfangsgründe der Logik* ([6]; ibid.).[12] Furthermore, he mentioned Ernst Platner's (1744–1818) *Philosophische Aphorismen* [15, 20, p. 53].[13] There is also a reference to Johann Gottfried Karl Christian Kiesewetter's (1766–1819) *Grundriß der Logik* ([10]; ibid.). Last not least, Schopenhauer refers to Salomon Maimon's (1751–1800) *Versuch einer neuen Logik* [14, 20, p. 52], and to his own former teacher Gottlob Ernst Schulze's (1761–1833) *Grundsätze der allgemeinen Logik* [20, 21, p. 51].[14]

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[10] Schopenhauer's references are [12, 17, pp. 157–204], and [2, vol. 2, p. 106].

[11] Schopenhauer's early manuscripts are contained in [20].

[12] Presumably Schopenhauer also knew Hoffbauer's *Analytik der Urtheile und Schlüsse* [7].

[13] It is probable that Schopenhauer was also acquainted with Platner's *Lehrbuch der Logik und Metaphysik* [16].

[14] All of the listed authors share the opinion that (general) logic is a "formal" science, treating of nothing but the forms, i.e., the necessary conditions and hence "laws" of thought—but abstracting

Now, while Lambert and Ploucquet were of course quite concerned about a reasonable symbolic notation, neither of them employed equality signs. Equality signs are neither to be found in von Jakob, Hoffbauer, Maass, Platner nor in Kiesewetter. However, the remaining authors did give some specimens of a quasi-algebraical notation, some more reasonable than others. These cases will be considered in the following sections of the present paper.

4.1 Reimarus, Kant, and Fichte

By the late eighteenth century, it was not quite uncommon to employ an equality sign to express identity in the sense of a concept's self-sameness. An earlier case of equational expression for predication is to be found in Reimarus's *Vernunftlehre*. Reimarus invoked an equality sign in "a = b", as provable by reference to both being "equal" (*gleich*) to a third term c [18, p. 470]. However, in 1794, Johann Gottlieb Fichte famously expressed the Principle of Identity as "A = A" in his *Grundlage der gesammten Wissenschaftslehre*, noting that "this is the meaning of the logical copula" (*denn dies ist die Bedeutung der logischen Copula*; [3, p. 5]). Fichte explicitly contended that "A = A" is the foundational principle of logic (*Grundsaz der Logik*; [3, p. 14]). He also noted that the proposition (*der Saz*) "−A nicht = A" is equally accepted as axiomatic as "−A = −A" is but another way of putting "A = A" [3, p. 18].

Even in the preface to Kant's logic lectures, Kant's student and his lectures' editor Gottlob Benjamin Jäsche invoked an equation sign to express the Principle of Identity by "A = A" within his critique of Fichte [8, p. XVII].[15] Moreover, Jäsche similarly used a minus sign to express negation when positing that besides "A = A", there is an "−A = −A" [8, p. XVII]. But while in Fichte, "−A = −A" looks like another form of the Principle of Identity for negative terms, Jäsche declared that this formula indicates the Principle of Non-Contradiction (ibid.). Thus, Jäsche's version of the Principle of Contradiction is but a way of positing a positive identity for apparently negative terms, which of course is not a contradiction.

In any case it is possible that Schopenhauer took his equational expressions and his minus sign in "A = −A = 0" from Fichte or from Kant's logic as edited by Jäsche.

from all content whatsoever. (As such a view had been prominently put forward by Kant, they are classified as 'Kantian' in Friedrich Ueberweg's *System der Logik*, cf. [23, pp. 51–52].) Thus, they embrace the premise that the exposition and the justification of the laws of thought are independent of both psychological and ontological or metaphysical considerations. A similar starting point was also shared by Herbart and his followers such as Drobisch, quoted in the introductory section of the present paper.

15 As to an explicit justification for the use of the equality sign in "A = A" departing from a critique of Fichte and Schelling, more material is to be found in Wilhelm Traugott Krug's *Denklehre oder Logik* [11, pp. 43–60].

4.2 Herbart

Some more specific applications of the equality sign are to be found in Herbart. In Herbart's appendix to his *Hauptpuncte der Metaphysik*, there is an equality sign to identify the minor premise by its subject term, as in the derivation of "S P" from "M P" being the major premise and "S P" being the minor, i.e., the minor "= S" [4, p. 122].[16]

In his *Lehrbuch*, which Schopenhauer does not explicitly refer to, Herbart gave another instance of equality signs by casting the premises of syllogisms into an equational form: Given the major "A B" and the minor "M N," it is possible to distinguish the following "equations" (*Gleichungen*): "1) A = N. 2) B = N. 3) A = M. 4) B = M." [5, p. 59]. Herbart also invoked the equality sign to state that a "series" (*Reihe*) of terms gives another concept, as in "A, B, C, D = p" [5, p. 31]. Furthermore, he employs "A = A" to signify a (semantic) tautology (rather than an ontologically grounded identity), such as "What is evaporated evaporates" (*das Verdunstende verdunstet*; [5, p. 47]). This case of the equality sign is within a more global argument of Herbart's, namely that the predicate is somewhat restricted, depending on what subject it is applied to. Hence as applied in "Water evaporates," "to evaporate" would be taken to mean evaporation as applicable to water, i.e., depending on water being the subject, only a portion of the set of characteristic marks determining the predicate is considered (ibid.). Only in "What is evaporated evaporates," the whole set will be relevant.

While Schopenhauer's use of the equation sign in logic does not quite come up to Herbart's, still he might have adopted Herbart's talk of judgments as "equations."

4.3 Maimon

Herbart's talk of judgments as "equations" seems to have been conceptually anticipated in Maimon's *Versuch einer neuen Logik*. In this work, Maimon explicitly stated that the affirmation of a relation of inclusion between subject and predicate (*die Bejahung, deren Bedeutung ist, daß das Prädikat im Subjekte enthalten ist*) should be indicated by "=" as a sign of equality (*Gleichheit*; [14, pp. 68–69]).[17] Moreover, Maimon held that the affirmation of a relation of agreement "within" an object (*Uebereinstimmung im Objekte*) should be signified by the algebraic "plus" symbol, "+", and the affirmation of a relation of negation within an object by a

[16]"Setzt alsdann der Untersatz [...] einen bestimmten Fall, in welchem das Subject (das *antecedens*) Statt finde, oder das Prädicat (das *consequens*) nicht Statt habe: so gleicht die Conclusion, welchem diesem bestimmten Falle (= S) das andre Glied des Obersatzes zueignet oder abspricht, ganz den gewöhnlichen Schlüssen."

[17]It is noteworthy that according to this passage, what is affirmed is the inclusion of the predicate in the subject, not of the subject in the predicate.

"minus" sign, "−", while "infinity" (*Unendlichkeit*) should be signified by "0" [14, p. 69]. It should be remembered that an "infinite judgment" was one which states the denial of a concept term by its contradictory opposite. Accordingly, Maimon wanted an affirmative judgment to be expressed by "a + b", while a negative judgment should be expressed by " a − b", and an "infinite" judgment by "a 0 b" (ibid.). As in the case of "a 0 b", the conjoined concepts cannot at all determine each other, so their relation does not alter any one of them. Therefore, Maimon thought that their conjunction equals 0: it is "= 0" (ibid.).

As to the meaning of "agreement" between concept terms, Maimon noted that there is a three-fold interpretation of being either "mutually or unilaterally identical" (*wechselseitig oder einseitig identisch*) or jointly determining an object (*zur Bestimmung eines Objekts übereinstimmen*; [14, p. 71]. Mutually identical judgments are such that they give "a = a," "ax = ax," "an = an;" hence they are co-extensive (*von gleichem Umfange*; ibid.). An unilaterally identical judgment, however, is of the form "ax + a," which means that it contains "a" among its determinations, to the effect that as to their consequences, "ax" is equivalent (*einerlei*) to "a," while being of smaller extension (*von kleinerem Umfange*; ibid.). In other words, in this case, "ax = a." Furthermore, there is an equivalence of "a + b = ab," which means that some "a," namely such that are conjoined with "b," are "ab" (ibid.). [The same goes for "a + ab" and "b + ab" (ibid.)].

Moreover, Maimon noted that due to the applicability of algebraic rules, an universal negative judgment should be regarded as equivalent (*gleichgeltend*) with an universal affirmative one, with an opposing predicate [14, p. 72]. Symbolically, this relation is mirrored by "two times minus giving plus" (*da minus minus plus giebt*; ibid.). Hence "ax − (−a)" would be equivalent (*gleich*) to "ax + a" (ibid.). Obviously, Maimon seems to have claimed that there is in fact no distinction of quality in categoricals, at least inasmuch as negative judgments can be translated into affirmative ones and vice versa. Another aspect to the same consequence is that judgments can be translated into one another according to their quantity, as in, e.g., "Some a are b" meaning "Not no a is b" [14, p. 68].

Hence it is imaginable that Maimon is one source of inspiration for Schopenhauer to develop some grasp of the applicability of symbols of algebraic operations to logic. However, of course Schopenhauer's "A = −A = 0" does not quite come up to Maimon's exposition in either the sense of "−" or the sense of "0". Moreover, Maimon clearly employed the equality sign "=" to signify an "equivalence" of different forms of judgment. Schopenhauer, however, fails to do precisely this. Rather, he employs the equality sign to express predication.

4.4 Schulze

Some more clues to Schopenhauer's equational notation are to be found in his teacher Schulze. In his *Grundsätze der allgemeinen Logik*, Schulze made an effort to cast the Principle of (Non-)Contradiction into the form of a pseudo-mathematical equation. As a symbolic paraphrase of "Contradictories are unthinkable," he gave "A = non A = 0" [22, p. 32],[18] which obviously corresponds to Schopenhauer's "A = −A = 0" [19, p. 262].

Now, it is plausible to think of Schopenhauer adapting elements from his teacher's logic into the preparatory manuscript of his own lecture. However, he does not give one word of explanation for his choice of symbols. In Schulze himself, the short-hand "A = non A = 0" is commented on only once, and indirectly. The comment is to be found in an addition to the section on the principles or "laws" of thought. This addition has it that the Principle of Identity should be taken to mean "Everything is what it is" (*Jedes Ding ist das, was es ist*; [22, p. 44])—which, like in Schopenhauer [19, p. 262], should be written as "A = A" [22, p. 44]. Accordingly, its complementary Principle, namely that of (Non-)Contradiction, should be understood as "Nothing is what it is not"; however, at this passage Schulze did not even bother to insert the somewhat clumsy "A = non A = 0" once more (ibid.)

There is only one more instance of Schulze's employing equality signs to express predication. It occurs in Schulze's exposition of a syllogism with two premises and one conclusion. These are represented as "A = B", "C = A", "Ergo C = B" [22, p. 117]. Obviously, Schulze's choice of short-hand for judgments corresponds to that employed by Schopenhauer in his table of conversions. Unfortunately, Schulze gave no more elucidations of his notation than did Schopenhauer, and its scattered use seems similarly nonsystematic. But nonetheless, Schulze's approach to logic allows for an implicit vindication, which seems to be lacking in Schopenhauer.

Schulze's implicit vindication for his casting judgments into an equational form relates to his conceptions of their quantity and quality, as follows. Treating of categoricals, Schulze distinguished between the subject and the predicate of judgments. While he described the subject as a judgment's "fundamental term" (*Grundbegriff*; [22, p. 74]), he considered the predicate as its "appending term" (*Beilegungsbegriff*; [22, p. 75]). The subject is what can be determined (*das Bestimmbare*; [22, p. 74]), but the predicate is the determination (*die Bestimmung*; [22, p. 75]). As related in a judgment, the subject and the predicate enter a certain proportion (*Verhältniß*; ibid.), which itself determines the judgment's form (*Form*; ibid.) Its verbal expression is the "conjunction term" (*Bindewort*), or copula (ibid.).

Departing from such—quite traditional—premises, Schulze noted that the predicate, i.e., the predicated mark of an object to be represented in thought, can be applied to the subject's whole extension ([auf] *den ganzen Umfang des Grundbegriffes*), or to part of it only (*oder nur auf einen Theil davon*; [22, p. 78]). Depending on which of these is the case, the judgment is of universal or particular extension (*Umfang*), i.e., quantity (*Größe, quantitas*; [22, p. 79]). However, Schulze criticized the tradition of attaching quantity to judgments according as the subject term only is considered. Hence while treating of subordination of judgments, he noted that

[18]Schulze's textbook is quoted in the fourth edition from 1822, which of course Schopenhauer could not yet have at his hands while preparing his logic lectures.

> "up to the present time, logicians have without sufficient reasons considered relations of subordination between judgments only as their subject terms are subordinated to one another" (*Die Logiker haben bisher, allein ohne hinreichenden Grund, nur diejenigen Urtheile als im Verhältnisse der Unterordnung zu einander stehend aufgestellt, welche in Ansehung des Grundbegriffes einander untergeordnet sind*; [22, p. 89]).

Departing from this remark of Schulze's, it may be conjectured that he was aware that only if there was a quantitative determination of the predicate, there would be a possibility to compare predicate terms of different judgments as to their extensions' sizes. Moreover, there would be a possibility to compare not only to which portion of the subject the predicate is applied, but also which portion of the predicate is applied. Thus, Schulze's remarks insinuate a quantitative relation of size between the subject's and the predicate's extension, respectively. Therefore, it is imaginable that he (more or less) consistently regarded this relation as one of equating portions of both extensions; hence his possible association of judgments with quasi-arithmetical equations.

Schulze also noted that judgments are normally said to differ in texture (*Beschaffenheit*), or quality (*qualitas*), according as they are classed as affirmative or negative [22, p. 80]. However, Schulze refrained from attaching quality to the copula. In Schulze, it is not the copula which has a double character of either attaching a predicate to a subject or separating them. Rather, Schulze held that the common talk of affirmative and negative judgments concerns one and the same operation of thought (*Handlung des Verstandes*; [22, p. 81]), namely the one of including one concept's extension into another concept's extension: In affirmative judgments, the subject's extension is included into that of a positive term (*wird die Vorstellung, welche dem Urtheile zu Grunde liegt, in den Umfang eines bejahenden Begriffes gehörig gedacht*; [22, p. 80].) In negative judgments, the subject's extension is included into that of a negative term (*wird die Vorstellung, welche dem Urtheile zu Grunde liegt, in den Umfang eines verneinenden Begriffes gehörig gedacht*; [22, p. 81].)[19] But in both cases, one concept is (positively) related to a notion as a mark (*der Begriff als ein Merkmahl mit der Vorstellung verbunden*; ibid.). This is why Schulze explicitly opposed the view that there could be such a thing as a negative copula, which serves a separation of concepts. Rather, a negative copula would be a "conjunction term" (*Bindewort*) to effect a disjunction—which, as Schulze claimed, would be a "logical absurdity" (*eine durch negation affizirte copula wäre eine solche, die nicht verbände, also ein logisches Unding*; ibid.).[20] Rather, it is possible to conceive of negative predicates, i.e., of predicates by which something is denied, or excluded from the subject, as well as of judgments containing such predicates [22, pp. 80–81]. Thus, Schulze seems to have thought of what is expressed by the copula as an unchangeably positive relation of mapping (portions of) concepts' extensions

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[19]Extensionally negative terms or concepts are introduced even in the introductory parts of Schulze's textbook [22, p. 28]. Treating of inferences, Schulze also spoke of negative marks of concepts [22, p. 118], which seems to be an intensional equivalent.

[20]Similarly, Krug admits of no negative copula but negative concepts since "a negative copula, i.e., a copula which does not copulate, is a contradiction in itself" [11, p. 206].

onto each other. In the case of the so-called negative judgments, such a mapping would take place onto a privative term, i.e., the opposite of what is spoken of as separated. Again, it is imaginable that Schulze thought of equating portions of both extensions, which may be somehow remind of quasi-arithmetical equations.

5 A Tentative Assessment

Now, how exactly does Schopenhauer's account of judgments and their relations of opposition connect to his table of conversions and contrapositions? In fact, Schopenhauer's table of conversions seems to revoke his account of both quality and quantity. It contains negative terms instead of negative copulae, and his use of equality signs suggests that some more or less definite parts of the subject and predicate terms are positively equaled, even if the determination of the predicate term is omitted.

If Schopenhauer intended to model the proportion between concepts as an overlap between some sections of their "spheres," it is unclear how he should integrate a negative copula into this model. Again, if Schopenhauer meant to assimilate the relation between the subject and predicate of judgments to a partial identity, it is unclear how he could have their quantity depend on the subject term only. Moreover, one might doubt whether an equality sign is the right choice of symbol to express such an overlap or partial identity at all.

It is imaginable that Fichte's *Wissenschaftslehre* or Kant's logic lectures as edited by Jäsche were one source for Schopenhauer's attempt at expressing negation by a minus sign. Schopenhauer's use of the equation sign in logic might relate to Herbart's talk of judgments as "equations." Maybe Schopenhauer also took some inspiration from Maimon, who developed a more consistent way of using "+", "−", "=", and "0". Finally, Schopenhauer's "$A = -A = 0$" seems to come from his teacher Schulze. However, Schopenhauer's applications of such signs remain unjustified. The reason is that unlike his teacher, Schopenhauer proposed a negative copula to the exclusion of negative terms, and to a quantitative determination of judgments by consideration of the subject only—at least explicitly. However, what Schopenhauer *does* in some places opposes to what he *says* about quality and quantity of judgments, and this seems to be the case with his employing equational forms in his table of conversions.

One may conclude that on the whole, Schopenhauer seems to be going in different directions at the same time. On the one hand, what he says explicitly comes close to a typical textbook account of quality and quantity of judgments. But on the other hand, there are some passages where Schopenhauer seems to be pointing towards some revisions of this account, concerning the conceptions of the copula and of the quantities of terms. If Schopenhauer had set out on this track, this would account for his attempt to cast predicative judgments into an equational form, especially as in his table of conversions. However, it seems doubtful that Schopenhauer did so consciously and consistently.

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From Euler Diagrams in Schopenhauer to Aristotelian Diagrams in Logical Geometry

Lorenz Demey

Abstract In this paper I explore the connection between Schopenhauer's Euler diagrams and the Aristotelian diagrams that are studied in contemporary logical geometry. One can define the Aristotelian relations in a very general fashion (relative to arbitrary Boolean algebras), which allows us to define not only Aristotelian diagrams for *statements*, but also for *sets*. I show that, once this generalization has been made, each of Schopenhauer's concrete Euler diagrams can be transformed into a well-defined Aristotelian diagram. More importantly, I also argue that Schopenhauer had several more general, systematic insights about Euler diagrams, which anticipate general insights and theorems about Aristotelian diagrams in logical geometry. Typical examples include the correspondence between *n-partitions* and α*-structures* (a particular class of Aristotelian diagrams), and the fact that many families of Aristotelian diagrams have distinct *Boolean subtypes*. Because of his various concrete Euler diagrams and, especially, his more systematic observations about them, Schopenhauer can rightly be considered a distant forerunner of contemporary logical geometry.

Keywords Euler diagram · Aristotelian diagram · Schopenhauer · Logical geometry · Square of opposition · α-structure · Bitstring

Mathematics Subject Classification (2000) Primary 03A05; Secondary 03G05, 68T30, 01A55

1 Introduction

Arthur Schopenhauer (1788–1860) is best known for his deeply pessimistic outlook on the world, a world that he took to be cruel and filled with violence. By emphasizing the fundamental irrationality and absurdity of the universe, he not only anticipated much of twentieth-century continental philosophy, but also influenced

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major literary figures such as Samuel Beckett and Jorge Luis Borges [35]. Much less known is that Schopenhauer was also concerned with what is perhaps the purest of all rational undertakings, viz. *logic*. There is a relatively brief discussion of logic in the first of the four books of his magnum opus, *The World as Will and Representation* [29], and a much more extensive treatment in a set of lecture notes [28], which were meant for his university lectures in Berlin in the 1820s [21, pp. 113–115].

Unsurprisingly, Schopenhauer holds logic in rather low esteem in everyday use, since he deems the discipline to be utterly useless for the purposes of daily life:

> We no more need logic to avoid false reasoning than we need its rules to help us reason correctly; and even the most learned logician completely puts it aside when actually thinking. [29, p. 68]

Nevertheless, he does recognize the special epistemological status of logic, and therefore concludes that it is worth studying after all:

> Even though it has no practical use, logic must nevertheless be preserved because of its philosophical interest as a special branch of knowledge concerning the organization and action of reason [. . .] it is a self-contained, self-subsistent, internally complete and perfect discipline that achieves absolute certainty. [29, pp. 69–70]

This double perspective on the discipline of logic (practically useless, but philosophically valuable) is also manifested in his remarks on the origin and history of logic [29, pp. 70–72].

In his discussion on logic, Schopenhauer makes extensive use of the so-called *Euler diagrams* to visually represent the relationship between concepts. He also briefly touches upon the historical development of these diagrams[1]:

> every concept has what may be termed an extension [*Umfang*] or sphere, even in cases where only a single real object corresponds to it [. . .] The idea of presenting these spheres by means of spatial figures is very felicitous. It occurred first to Gottfried Ploucquet, who used squares to do it; Lambert, who came after him, used plain lines positioned under each other; but it was Euler who perfected the idea by using circles. [29, p. 65]; also cf. [28, pp. 286–287]

Schopenhauer takes Euler diagrams to be absolutely central to logic. They not only facilitate the teaching of this discipline, but also capture its very essence, by allowing one to derive all the logical inference rules (also cf. [21, pp. 115–118]):

> This schematism of concepts, which is already explained quite well in many textbooks, can be used to ground the doctrine of judgement as well as the whole of syllogistic logic and makes it very easy and uncomplicated to teach them both. [. . .] The essence of thought proper, i.e., of judgement and inference, can be presented by combining conceptual spheres according to the spatial schema described above, and all the rules of judgement and inference can be derived from this schema by construction. [29, pp. 68–70]

In this paper, I will explore the connection between Schopenhauer's Euler diagrams and another type of logical diagrams, viz. *Aristotelian diagrams*. Such

[1] More detailed (and historically accurate) accounts of the history of Euler-type diagrams can be found in [2, 3, 21–23].

diagrams visualize the Aristotelian relations of contradiction, contrariety, etc. that hold between a given number of propositions. The oldest and most well-known example of an Aristotelian diagram is the so-called square of opposition for the categorical statements from syllogistics, but there are also many other (larger, more complex) Aristotelian diagrams, often developed in very different logical systems than traditional syllogistics. In contemporary logic, it has become clear that these diagrams can be fruitfully studied as objects of independent mathematical and philosophical interest, which has led to the burgeoning subfield of *logical geometry* [6, 8, 11, 16, 32, 33].

As far as I know, no square of opposition (or any other Aristotelian diagram) drawn by Schopenhauer has been handed down to us.[2] Nevertheless, I will show in this paper that the *particular* Euler diagrams used by Schopenhauer can be 'translated' or 'transformed' into particular Aristotelian diagrams. More importantly, I will also argue that Schopenhauer formulated some more *general* insights about entire 'classes' or 'series' of Euler diagrams, which again have direct analogues in the realm of Aristotelian diagrams. Because of these systematic observations regarding entire classes of logic diagrams, Schopenhauer can rightly be considered a distant forerunner of contemporary logical geometry.

Before continuing, I should briefly say something about the scope of this paper. My argumentation will primarily be based on the relatively small number of Euler diagrams that appear in *The World as Will and Representation* [29]. There is also a plethora of Euler diagrams in Schopenhauer's university lecture notes [28], but for reasons of space, those diagrams will not be the main focus of this paper. I will only draw upon material from the lecture notes insofar as it allows me to further illustrate or reinforce a key claim in my overall argumentation. Finally, I will not say anything in this paper about the very large and complex 'Bonum/Malum diagram' [29, p. 74]. Schopenhauer himself already indicates that this diagram does not belong to logic proper, but rather to 'the *art of persuasion* [*Überredungskunst*]' [29, p. 72]. A more detailed discussion of this particular diagram can be found in [24].

The paper is organized as follows. Section 2 provides some necessary background on logical geometry, focusing on a very general way of defining the Aristotelian relations, and a particular class of Aristotelian diagrams, viz. the so-called α-structures. Section 3 shows how two of Schopenhauer's simplest Euler diagrams can be transformed into well-defined Aristotelian diagrams, viz. a classical and a degenerate square of opposition. Section 4 deals with Schopenhauer's infinite series of Euler diagrams for partitions and shows how it can be transformed into an infinite series of strong α-structures. Section 5 is concerned with a final class of

[2] In his university lecture notes, Schopenhauer did explicitly discuss the opposition relations that hold between the categorical statements [28, pp. 305ff.]. Furthermore, in the immediate textual surroundings of this discussion (e.g., on p. 297 and p. 304), we frequently find the reference '*Illustr.*', which indicates that an actual diagram is missing from the manuscript, probably because Schopenhauer drew it from memory during his actual lectures. It can reasonably be assumed that at least one of these occurrences of '*Illustr.*' refers to a square of opposition. Thanks to Jens Lemanski for some interesting discussion about this.

Euler diagrams and shows how each of them can be transformed into both a strong and a weak α-structure. Section 6 briefly summarizes the results that have been obtained in this paper.

2 Some Background on Logical Geometry

This section provides some necessary background on logical geometry. In Sect. 2.1, I define the Aristotelian relations with respect to an arbitrary Boolean algebra. This will later enable us to transform Schopenhauer's Euler diagrams into well-defined Aristotelian diagrams. Next, in Sect. 2.2, I introduce a particular class of Aristotelian diagrams, viz. the so-called α-structures, and discuss some of their key properties. These α-structures will turn out to play a crucial role in Sects. 4 and 5.

2.1 Defining the Aristotelian Relations in a Boolean Algebra

The Aristotelian relations can be characterized with various degrees of abstractness and generality [10, 12]. For the purposes of this paper, it will be useful to consider a very general definition, in the mathematical setting of Boolean algebra [17]; afterwards, I will show how the more well-known characterizations of the Aristotelian relations can be obtained as special cases of this definition.

Definition 2.1 Let $\mathbb{B} = \langle B, \wedge, \vee, \neg, \top, \bot \rangle$ be an arbitrary Boolean algebra. Two elements $x, y \in B$ are said to be

$\mathbb{B}$-*contradictory*	iff	$x \wedge y = \bot$	and	$x \vee y = \top$,
$\mathbb{B}$-*contrary*	iff	$x \wedge y = \bot$	and	$x \vee y \neq \top$,
$\mathbb{B}$-*subcontrary*	iff	$x \wedge y \neq \bot$	and	$x \vee y = \top$,
in $\mathbb{B}$-*subalternation*	iff	$\neg x \vee y = \top$	and	$x \vee \neg y \neq \top$.

Note that by De Morgan's laws, the condition $x \vee y = \top$ is equivalent to $\neg x \wedge \neg y = \bot$, while the conditions $\neg x \vee y = \top$ and $x \vee \neg y \neq \top$ are equivalent to resp. $x \wedge \neg y = \bot$ and $\neg x \wedge y \neq \bot$. This means that the relations of contradiction, contrariety and subcontrariety are all defined in terms of the 'symmetrical' elements $x \wedge y$ and $\neg x \wedge \neg y$, whereas the relation of subalternation is defined in terms of the 'asymmetrical' elements $\neg x \wedge y$ and $x \wedge \neg y$. This conceptual split in the definitions of the Aristotelian relations is explored in much more detail in [31].

Definition 2.1 provides a characterization of the Aristotelian relations in an arbitrary Boolean algebra. However, it is also important to have a clear grasp of what it means for two elements *not* to stand in any Aristotelian relation whatsoever. This corresponds to the notion of *unconnectedness* [15, 31], which is defined in terms of four conditions:

Definition 2.2 Let $\mathbb{B} = \langle B, \wedge, \vee, \neg, \top, \bot \rangle$ again be an arbitrary Boolean algebra. Two elements $x, y \in B$ are said to be $\mathbb{B}$-*unconnected* iff (i) $x \wedge y \neq \bot$ and (ii) $x \wedge \neg y \neq \bot$ and (iii) $\neg x \wedge y \neq \bot$ and (iv) $\neg x \wedge \neg y \neq \bot$.

The first condition of Definition 2.2 implies that x and y are neither $\mathbb{B}$-contradictory nor $\mathbb{B}$-contrary, while the fourth condition implies that x and y are neither $\mathbb{B}$-contradictory nor $\mathbb{B}$-subcontrary. The second condition implies that there is no $\mathbb{B}$-subalternation from x to y, and similarly, the third condition implies that there is no $\mathbb{B}$-subalternation from y to x. Together, these four conditions thus imply that the elements $x, y \in B$ do not stand in any Aristotelian relation in the Boolean algebra $\mathbb{B}$.

We can move from these very general definitions to more well-known characterizations of the Aristotelian relations and unconnectedness, by plugging in a concrete Boolean algebra for $\mathbb{B}$. I will now discuss two key examples: (1) letting $\mathbb{B}$ be a Boolean algebra of statements, and (2) letting $\mathbb{B}$ be a Boolean algebra of sets.[3]

First of all, consider the case where $\mathbb{B}$ is a Boolean algebra of *statements*. (This can typically be achieved by taking $\mathbb{B}$ to be the Lindenbaum–Tarski algebra of some suitable logical system S.[4]) The top and bottom elements of such a Boolean algebra are resp. the tautological and self-contradictory statements, while the algebraic operations of meet, join and complement correspond to the logical operations of resp. conjunction, disjunction and negation. By applying Definition 2.1, we find that two statements P and Q are contrary in this Boolean algebra iff $P \wedge Q = \bot$ and $P \vee Q \neq \top$, i.e., iff the conjunction of P and Q is self-contradictory, while the disjunction of P and Q is not tautological. The first part means exactly that P and Q cannot be true together, while the second part means that P and Q can be false together. We have thus obtained the 'familiar' definition of contrariety for statements (in terms of being able to be true/false together) as a special case of Definition 2.1. The familiar definitions of contradiction, subcontrariety, subalternation and unconnectedness for statements can be obtained from Definitions 2.1 and 2.2 in a completely analogous fashion.

Secondly, consider the case where $\mathbb{B}$ is a Boolean algebra of *sets*. (Because of the Stone representation theorem, every Boolean algebra is isomorphic to a Boolean algebra of this kind [17].) The top and bottom elements of such a Boolean algebra are resp. some domain of discourse D and the empty set $\emptyset$, while the algebraic operations of meet, join and complement correspond to the set-theoretical operations of resp. intersection, union and complementation (with respect to D). By

[3]The close relationship between these two examples was already noted by Keynes, who wrote: 'These seven possible relations between *propositions* (taken in pairs) will be found to be precisely analogous to the seven possible relations between *classes* (taken in pairs)' [19, p. 119, my emphases]. Note that Keynes talks about *seven* relations, because in addition to the four Aristotelian relations and unconnectedness, he is considering two others. However, this difference is irrelevant for our current purposes.

[4]'Suitable' here means that the logical system S has all the connectives and axioms that are needed to guarantee that its Lindenbaum–Tarski algebra effectively is a *Boolean* algebra. This is mainly a technical caveat, and it is further irrelevant for the purposes of this paper.

applying Definition 2.1, we find that two sets X and Y are contrary in this Boolean algebra iff $X \cap Y = \emptyset$ and $X \cup Y \neq D$, i.e., iff the intersection of X and Y is empty, while the union of X and Y does not exhaust the domain of discourse D. The definitions of contradiction, subcontrariety, subalternation and unconnectedness for sets can be obtained from Definitions 2.1 and 2.2 in a completely analogous fashion. For future reference, all these definitions are listed below:

Definition 2.3 Let $\mathbb{B} = \langle B, \cap, \cup, \backslash, D, \emptyset \rangle$ be a Boolean algebra of sets. Two sets $X, Y \in B$ are said to be

$\mathbb{B}$-*contradictory*	iff	$X \cap Y = \emptyset$	and	$X \cup Y = D$,
$\mathbb{B}$-*contrary*	iff	$X \cap Y = \emptyset$	and	$X \cup Y \neq D$,
$\mathbb{B}$-*subcontrary*	iff	$X \cap Y \neq \emptyset$	and	$X \cup Y = D$,
in $\mathbb{B}$-*subalternation*	iff	$(D\backslash X) \cup Y = D$	and	$X \cup (D\backslash Y) \neq D$,
$\mathbb{B}$-*unconnected*	iff	$X \cap Y \neq \emptyset$	and	$X \cap (D\backslash Y) \neq \emptyset$
	and	$(D\backslash X) \cap Y \neq \emptyset$	and	$(D\backslash X) \cap (D\backslash Y) \neq \emptyset$.

Note that the conditions $(D\backslash X) \cup Y = D$ and $X \cup (D\backslash Y) \neq D$ are equivalent to resp. $X \subseteq Y$ and $X \not\supseteq Y$. In a Boolean algebra of sets, subalternation thus corresponds to the *proper subset*-relation: there is a subalternation from X to Y iff $X \subset Y$.

Because of the generality of Definitions 2.1 and 2.2, we are now able to deal with Aristotelian relations not only in the case of *statements*, but also in the case of *sets*. This insight will be absolutely crucial when we transform Schopenhauer's Euler diagrams into Aristotelian diagrams, because those Euler diagrams also represent relations between *sets*. More specifically, Schopenhauer's Euler diagrams represent relations between spheres/extensions of concepts (cf. the quotation provided in Sect. 1), and these extensions are indeed sets. For example, the extension of the concept Horse is the set of all concrete horses that exist in the world.

2.2 The α-Structures and Their Properties

Now that the Aristotelian relations and unconnectedness have been defined relative to arbitrary Boolean algebras, we can likewise define the notion of an Aristotelian diagram:

Definition 2.4 Let $\mathbb{B} = \langle B, \wedge, \vee, \neg, \top, \bot \rangle$ be an arbitrary Boolean algebra and consider a fragment $\mathcal{F} \subseteq B\backslash\{\top, \bot\}$. Suppose that $\mathcal{F}$ is closed under $\mathbb{B}$-complementation, i.e., if $x \in \mathcal{F}$, then $\neg x \in \mathcal{F}$. An *Aristotelian diagram for* $\mathcal{F}$ *in* $\mathbb{B}$ is a diagram that visualizes an edge-labeled graph $\mathcal{G}$. The vertices of $\mathcal{G}$ are the elements of $\mathcal{F}$, and the edges of $\mathcal{G}$ are labeled by the Aristotelian relations holding

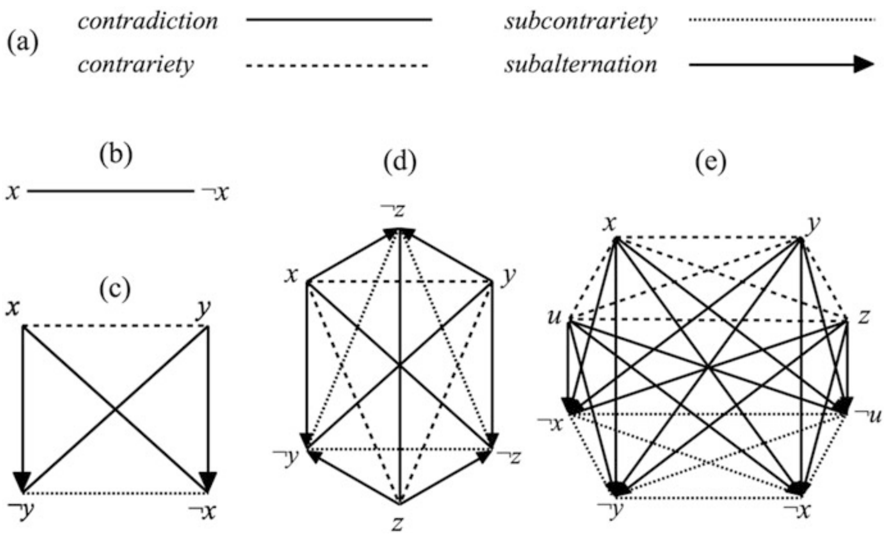

Fig. 1 (**a**) Code for visually representing the Aristotelian relations; examples of (**b**) PCD, (**c**) classical square of opposition, (**d**) JSB hexagon, (**e**) Moretti octagon

in $\mathbb{B}$ between those elements, i.e., if $x, y \in \mathcal{F}$ stand in some Aristotelian relation in $\mathbb{B}$, then this is visualized according to the code in Fig. 1a.

Note that by definition, the set $\mathcal{F}$ of elements appearing in an Aristotelian diagram is closed under complementation and only contains non-trivial elements (i.e., neither $\top$ nor $\bot$). There are various historical and systematic reasons for these restrictions [31, Subsection 2.1], which need not concern us here. Later on in this paper, we will focus on a particular subclass of Aristotelian diagrams, viz. the so-called α-structures or α-diagrams (the term 'α-structure' is due to Moretti [25])[5]:

Definition 2.5 Let $\mathbb{B}$ again be a Boolean algebra, and let $n \geq 1$ be a natural number. An *α_n-structure in* $\mathbb{B}$ is an edge-labeled graph $\mathcal{G}$. The vertices of $\mathcal{G}$ form a fragment $\{x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n\} \subseteq B \backslash \{\top, \bot\}$, where all distinct x_i, x_j are pairwise $\mathbb{B}$-contrary, i.e., such that x_i and x_j are $\mathbb{B}$-contrary for all $1 \leq i \neq j \leq n$. The edges of $\mathcal{G}$ are labeled by the Aristotelian relations holding in $\mathbb{B}$ between those elements An *α_n-diagram in* $\mathbb{B}$ is an Aristotelian diagram that visualizes such an α_n-structure in $\mathbb{B}$.

The condition of pairwise $\mathbb{B}$-contrariety between all distinct x_i, x_j immediately implies that there are several other Aristotelian relations in an α_n-structure as well.

[5] Strictly speaking, the term 'α-structure' refers to the (abstract) underlying graph structure, while the term 'α-diagram' refers to the (concrete) diagrammatic visualization. However, for the purposes of this paper, this distinction will not matter much, so we will usually not distinguish between these two terms, and follow Moretti [25] in simply talking about 'α-structures'.

In particular, it follows that $\neg x_i$ and $\neg x_j$ are $\mathbb{B}$-subcontrary and that there are $\mathbb{B}$-subalternations from x_i to $\neg x_j$, for all $1 \leq i \neq j \leq n$. Furthermore, as in any Aristotelian diagram, it holds that x_i and $\neg x_i$ are $\mathbb{B}$-contradictory, for all $1 \leq i \leq n$.

Although there certainly exist Aristotelian diagrams that do *not* belong to the class of α-structures, this particular class does contain some of the most well-known examples of Aristotelian diagrams. For example, consider the four smallest members of this class, i.e., the α_n-structures for $n \in \{1, 2, 3, 4\}$:

- The α_1-structure is simply a *pair of contradictory elements* (PCD).[6] An example is shown in Fig. 1b. Because Aristotelian diagrams are supposed to be closed under complementation, this is the smallest possible Aristotelian diagram. PCDs do not frequently appear in the literature, but they have considerable theoretical importance, since they can be thought of as the fundamental 'building blocks' for all other, larger Aristotelian diagrams [13, 14].
- The α_2-structure is a *classical square of opposition*. An example is shown in Fig. 1c. Without a doubt, this is the oldest, most well-known and most frequently-used type of Aristotelian diagram.
- The α_3-structure is a so-called *Jacoby–Sesmat–Blanché hexagon* (JSB hexagon), which is named after Jacoby [18], Sesmat [30] and Blanché [5]. An example is shown in Fig. 1d. After the classical square of opposition, this is probably the most well-known and most frequently used type of Aristotelian diagram. Within a JSB hexagon, one can discern a triangle of contraries that interlocks with a triangle of subcontraries.
- The α_4-structure is a so-called *Moretti octagon*, which is named after Moretti [25] (who drew it as a cube, rather than an octagon). An example is shown in Fig. 1e. Within a Moretti octagon, one can discern a trapezoid of contraries and a trapezoid of subcontraries. (If this diagram is drawn as a cube instead of an octagon, then these two trapezoids correspond to a tetrahedron of contraries that interlocks with a tetrahedron of subcontraries. Such a tetrahedron of contraries was already known (and drawn) by Charles S. Peirce; cf. [1, p. 60] and [20, p. 569].)

One of the main theoretical insights of logical geometry is that a given family of Aristotelian diagrams can have multiple *Boolean subtypes*, i.e., it is perfectly possible for two Aristotelian diagrams to exhibit exactly the same configuration of Aristotelian relations among their respective sets of elements, and yet have completely different Boolean properties [9, 15]. The first concrete example of this phenomenon was pointed out by Pellissier [26] and concerns the JSB hexagons. One can show that there are two Boolean subtypes of JSB hexagons, with completely different Boolean properties: in a *strong* JSB hexagon, the join of the 3 contrary

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[6] Note that an α_1-structure does not contain any *distinct* x_i, x_j, and hence, no contrarieties either. Since the pairwise contrarieties among distinct x_i, x_j constitute the characteristic feature of the α-structures (cf. Definition 2.5), the α_1-structure is clearly seen to be a 'limiting' (or 'degenerate') case of the α-structures in general.

elements equals $\top$, whereas in a *weak* JSB hexagon, the join of the 3 contrary elements is not equal to $\top$. These kinds of (differences in) Boolean properties are nowadays usually characterized in terms of *bitstring length*, i.e., the smallest number of bits with which a given diagram can be encoded. For example, a strong JSB hexagon can be encoded by bitstrings of length 3 (its 3 contrary elements are then encoded as 100, 010 and 001, so that $100 \vee 010 \vee 001 = 111$), whereas a weak JSB hexagon requires bitstrings of length 4 (its 3 contrary elements are then encoded as 1000, 0100 and 0010, so that $1000 \vee 0100 \vee 0010 = 1110 \neq 1111$) [15, 34].[7]

We are now in a position to systematically examine the Boolean subtypes of the various α-structures. Theorem 2.6 below summarizes the situation. For reasons of space, this theorem will not be proved in this paper, but it is based on a straightforward application of bitstring analysis. Note that the important cutoff happens at $n = 3$. This is not a coincidence: because of their *binary* nature, the Aristotelian relations cannot capture the full Boolean complexity that may arise in larger sets [7].[8]

Theorem 2.6

1. *The family of* α_1*-structures is Boolean homogeneous, i.e., it has just a single Boolean subtype, which requires bitstrings of length 2.*
2. *The family of* α_2*-structures is Boolean homogeneous, i.e., it has just a single Boolean subtype, which requires bitstrings of length 3.*
3. *For* $n \geq 3$*, the family of* α_n*-structures has two Boolean subtypes: (i) a* strong *subtype, which requires bitstrings of length* n*, and (ii) a* weak *subtype, which requires bitstrings of length* $n + 1$.

The cases $n = 1$ and $n = 2$ of this theorem mean that the family of PCDs and the family of classical squares of opposition are both Boolean homogeneous, which is well-known in logical geometry. The case $n = 3$ means that the family of JSB hexagons has two Boolean subtypes, viz. the strong JSB hexagons (which require bitstrings of length 3) and the weak JSB hexagons (which require bitstrings of length 4). We have already seen that this was first pointed out by Pellissier [26]. For a final example, note that the case $n = 4$ means that the family of Moretti octagons has two Boolean subtypes, viz. the strong Moretti octagons (which require bitstrings of length 4) and the weak Moretti octagons (which require bitstrings of length 5). A concrete example of a strong Moretti octagon can be found in Moretti [25] (drawn as a cube), while a weak Moretti octagon can be found in Reichenbach [27] (again drawn as a cube).

In a Boolean algebra $\mathbb{B} = \langle B, \wedge, \vee, \neg, \top, \bot \rangle$, a finite set $\Pi = \{x_1, \ldots, x_n\} \subseteq B \backslash \{\top, \bot\}$ (with $n \geq 2$) is said to be an *n-partition of* $\mathbb{B}$ iff (i) $x_i \wedge x_j = \bot$ for all

[7] There also exist Aristotelian families that have more than two Boolean subtypes. For example, the family of Buridan octagons has *three* Boolean subtypes: one that requires bitstrings of length 4, one that requires bitstrings of length 5 and one that requires bitstrings of length 6 [9, 15].

[8] For an easy illustration from classical propositional logic, note that the 3-element set $\{p \vee q, \neg p, \neg q\}$ is inconsistent, while each of its 2-element subsets is consistent.

distinct $x_i, x_j \in \Pi$ and (ii) $\bigvee \Pi = \top$. There is a clear correspondence between partitions and (strong) α-structures. This is made fully precise in Theorem 2.7 below. Note that there is again a cutoff at $n = 3$, and that α_2-structures (i.e., classical squares of opposition) do *not* correspond to any partitions.

Theorem 2.7

1. *Each 2-partition* $\{x, \neg x\}$ *gives rise to an* α_1*-structure with elements* $\{x, \neg x\}$*, and a contradiction holding between* x *and* $\neg x$.
2. *For* $n \geq 3$*, each 3-partition* $\{x_1, \ldots, x_n\}$ *gives rise to a strong* α_n*-structure with elements* $\{x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n\}$*, with contradictions between* x_i *and* $\neg x_i$*, and contrarieties between* x_i *and* x_j*, for all* $1 \leq i \neq j \leq n$.

The case $n = 2$ of this theorem means that each 2-partition corresponds to a PCD. The case $n = 3$ means that each 3-partition corresponds to a strong JSB hexagon, which has the 3 elements of the partition on its triangle of contraries. The case $n = 4$ means that each 4-partition corresponds to a strong Moretti octagon, which has the 4 elements of the partition on its trapezoid of contraries.

3 Two Basic Examples

After this brief overview of logical geometry, we are now ready to turn to Schopenhauer. In this section I will show how two of Schopenhauer's simplest Euler diagrams can be transformed into well-defined Aristotelian diagrams. This will be a valuable exercise in itself, but it will also serve as a useful preparation for the more involved transformations of Euler diagrams into Aristotelian diagrams that will be discussed in Sects. 4 and 5.

3.1 From an Euler Diagram to a Classical Square of Opposition

Schopenhauer begins his discussion of Euler diagrams by mentioning the most basic case, viz. that of two concepts that completely coincide with each other. (The example he gives involves the concepts of Ruminantia and Bisulca, i.e., ruminants and animals with cloven hoofs.) He does not explicitly provide an Euler diagram for this situation, stating that 'Such cases may be presented using a single circle that signifies the one as much as the other' [29, p. 66].

Next, Schopenhauer turns to the case where 'The sphere of one concept completely encloses the sphere of another' [29, p. 66]. The example he gives involves the concepts of Horse and Animal; the accompanying Euler diagram is shown in Fig. 2a. This diagram clearly shows that (the extension/sphere of) Horse is a *proper*

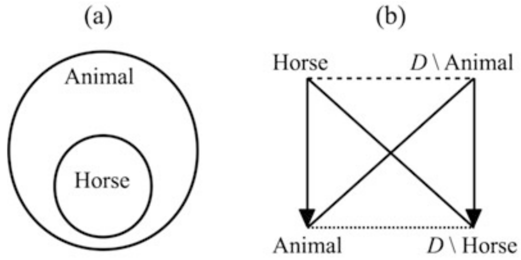

Fig. 2 (**a**) Schopenhauer's Euler diagram; (**b**) the corresponding Aristotelian diagram: a classical square of opposition

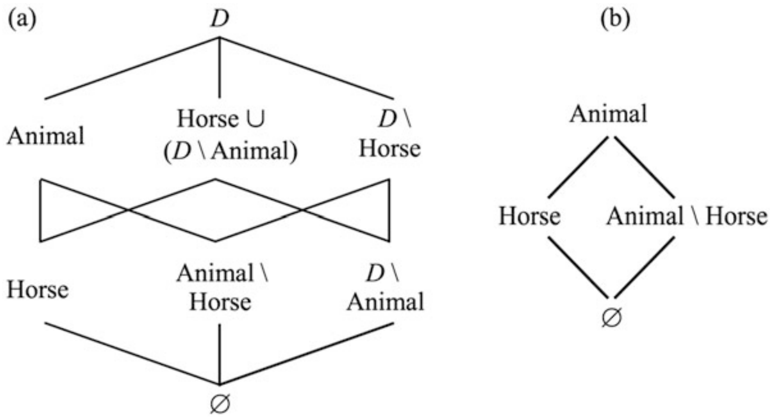

Fig. 3 Hasse diagrams for (**a**) the Boolean algebra $\mathbb{B}_3$ and (**b**) the Boolean algebra $\mathbb{B}_2$

subset of (the extension/sphere of) Animal.[9] We have already seen in Sect. 2.1 that this proper subset-relation essentially amounts to a *subalternation* from Horse to Animal, in some underlying Boolean algebra of sets. The smallest Boolean algebra of sets that non-trivially[10] contains Horse and Animal has $2^3 = 8$ elements. This Boolean algebra, $\mathbb{B}_3$, has top element D (for Domain), bottom element $\emptyset$ and three atomic elements, viz. Horse, Animal\Horse and D\Animal. The Hasse diagram for $\mathbb{B}_3$ is shown in Fig. 3a; note that $\mathbb{B}_3$ is isomorphic to the powerset algebra $\wp(\{1, 2, 3\})$.

[9]One might object that Schopenhauer's words ('completely enclose') commit him to Horse being a subset of Animal, but not necessarily a *proper* subset. This objection is misguided, for the following two reasons. First of all, the Euler diagram used by Schopenhauer contains a significant amount of space for the part Animal\Horse, which is a clear visual suggestion that Horse is a *proper* subset of Animal. Second, if the inclusion were non-proper, then it would be possible that Horse and Animal are exactly the same concept, so that they should have been represented by just a single circle instead (cf. the first case discussed by Schopenhauer).

[10]I.e., in such a way that neither Horse nor Animal end up being the top or bottom element of the Boolean algebra. See Remark 3.3 for a more detailed discussion.

We can now determine the Aristotelian relations holding between some of the sets of this Boolean algebra $\mathbb{B}_3$ (recall Definition 2.3). We have already seen that Horse $\subset$ Animal, which means exactly that there is a $\mathbb{B}_3$-subalternation from Horse to Animal. Furthermore, since Horse$\cap(D\backslash$Horse$) = \emptyset$ and Horse$\cup(D\backslash$Horse$) = D$, it follows that Horse and $D\backslash$Horse are $\mathbb{B}_3$-contradictories; similarly, Animal and $D\backslash$Animal are also $\mathbb{B}_3$-contradictories. Furthermore, since Horse $\cap$ $(D\backslash$Animal$) = \emptyset$ and Horse $\cup$ $(D\backslash$Animal$) \neq D$, it follows that Horse and $D\backslash$Animal are $\mathbb{B}_3$-contraries. Completely analogously, one can show that Animal and $D\backslash$Horse are $\mathbb{B}_3$-subcontraries, and that there is a $\mathbb{B}_3$-subalternation from $D\backslash$Animal to $D\backslash$Horse. All these Aristotelian relations can be summarized by means of a classical square of opposition, as shown in Fig. 2b.

We have thus succeeded in transforming Schopenhauer's original Euler diagram (Fig. 2a) into an Aristotelian diagram, viz. a classical square of opposition (Fig. 2b). It bears emphasizing that the Aristotelian relations that are visualized by this square are all mathematically well-defined: they hold relative to the underlying Boolean algebra $\mathbb{B}_3$ (cf. Fig. 3a) and are thus specific instantiations of the general characterization of the Aristotelian relations provided by Definition 2.1. To finish this subsection, I will now make three remarks, in increasing order of importance.

Remark 3.1 The transformation process that has just been described is by no means an injection, i.e., it is perfectly possible for two distinct Euler diagrams to be transformed into one and the same Aristotelian diagram. For example, we have just seen how the Euler diagram for the proper inclusion of Horse in Animal is transformed into a classical square of opposition. It is easy to see how another Euler diagram, which visualizes the proper inclusion of $D\backslash$Animal in $D\backslash$Horse, would be transformed into exactly the same classical square. Alternatively, one can view the original Euler diagram in Fig. 2a as a visual representation of *both* proper inclusion relations—albeit, perhaps, with different degrees of visual perspicuity. More generally, from this alternative perspective, the single Euler diagram in Fig. 2a at once visualizes *six* relations among Horse, Animal, $D\backslash$Horse and $D\backslash$Animal, all six of which are also visualized by the classical square of opposition in Fig. 2b.

Remark 3.2 Let us reiterate once more that the elements visualized by the square in Fig. 2b are not *statements*, but *sets* (more specifically: extensions of concepts). After all, the original Euler diagram in Fig. 2a also visualizes a relation (viz. proper inclusion) between two *sets*. Note, however, that this Euler diagram can also be seen as a visual representation of the categorical A-statement 'all horses are animals'.[11] One can then consider the corresponding I-, E- and O-statements, i.e., 'some horses are animals', 'no horses are animals' and 'some horses are not animals'. Together, these four categorical statements yield another classical square of opposition. However, this second square is very different from the one shown in Fig. 2b: the square in Fig. 2b is a diagram for *sets* and Aristotelian relations between sets (e.g., having empty or non-empty intersection), whereas the second square just

[11] Schopenhauer himself also took this view [28, p. 290].

described would be a diagram for *statements* and Aristotelian relations between statements (e.g., being able or not being able to be true together).

Remark 3.3 Recall that $\mathbb{B}_3$ is the smallest Boolean algebra of sets that *non-trivially* contains Horse and Animal. This non-triviality condition means that neither Horse nor Animal are identical to D or $\emptyset$, i.e., to the top or bottom element of $\mathbb{B}_3$ (cf. the Hasse diagram in Fig. 3a). Without this non-triviality condition, there is a *smaller* Boolean algebra of sets that contains Horse and Animal. This smaller Boolean algebra, $\mathbb{B}_2$, has $2^2 = 4$ elements; its top element is Animal and its bottom element is $\emptyset$; its two remaining elements are Horse and Animal\Horse. The Hasse diagram for $\mathbb{B}_2$ is shown in Fig. 3b; note that $\mathbb{B}_2$ is isomorphic to the powerset algebra $\wp(\{1, 2\})$. The Boolean algebra of sets $\mathbb{B}_2$ contains Horse and Animal, but Animal ends up being identical to the top element of $\mathbb{B}_2$. Consequently, next to the 'expected' results, $\mathbb{B}_2$ also yields some very counter-intuitive results; for example, there is a $\mathbb{B}_2$-subalternation from Horse to Animal (because Horse $\subset$ Animal), but additionally, there is also a $\mathbb{B}_2$-subcontrariety between Horse and Animal (because Horse $\cap$ Animal $\neq \emptyset$ and Horse $\cup$ Animal $=$ Animal). Furthermore, $\mathbb{B}_2$ does not contain enough elements to construct a square of opposition (recall that by definition, Aristotelian diagrams cannot contain a Boolean algebra's top or bottom elements). These issues illustrate the importance of respecting the non-triviality condition when transforming a given Euler diagram into an Aristotelian diagram.

3.2 *From an Euler Diagram to a Degenerate Square of Opposition*

Another of Schopenhauer's Euler diagrams illustrates the case where 'Two spheres each include a part of the other'. [29, p. 67]. The example he gives involves the concepts of Flower and Red; the accompanying Euler diagram is shown in Fig. 4a. The smallest Boolean algebra of sets that non-trivially (recall Remark 3.3) contains Flower and Red has $2^4 = 16$ elements. This Boolean algebra, $\mathbb{B}_4$, has top element D (for Domain), bottom element $\emptyset$ and four atomic elements, viz. Flower $\cap$ Red, Flower $\cap$ (D\Red), (D\Flower) $\cap$ Red and (D\Flower) $\cap$ (D\Red). The Hasse diagram for $\mathbb{B}_4$ is shown in Fig. 5; note that $\mathbb{B}_4$ isomorphic to the powerset algebra $\wp(\{1, 2, 3, 4\})$.

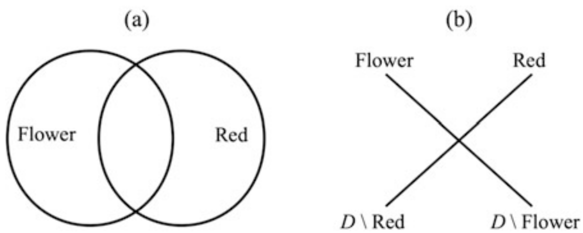

Fig. 4 (**a**) Schopenhauer's Euler diagram; (**b**) the corresponding Aristotelian diagram: a degenerate square

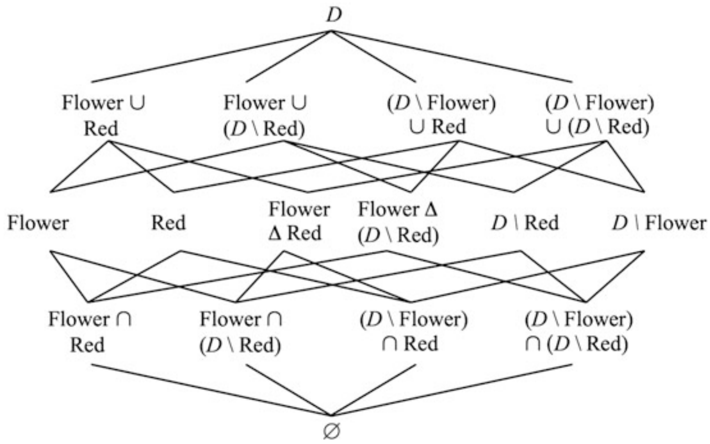

Fig. 5 Hasse diagram for the Boolean algebra $\mathbb{B}_4$. Note that Δ denotes the *symmetrical difference* between two sets, i.e., $X \Delta Y := (X \cap (D \backslash Y)) \cup ((D \backslash X) \cap Y)$

We can now determine the Aristotelian relations holding between some of the sets of this Boolean algebra $\mathbb{B}_4$ (again, recall Definition 2.3). First of all, since Flower $\cap$ ($D\backslash$Flower) $= \emptyset$ and Flower $\cup$ ($D\backslash$Flower) $= D$, it follows that Flower and $D\backslash$Flower are $\mathbb{B}_4$-contradictories; similarly, Red and $D\backslash$Red are also $\mathbb{B}_4$-contradictories. Furthermore, since (i) Flower $\cap$ Red $\neq \emptyset$, (ii) Flower $\cap$ ($D\backslash$Red) $\neq \emptyset$, (iii) ($D\backslash$Flower) $\cap$ Red $\neq \emptyset$ and (iv) ($D\backslash$Flower) $\cap$ ($D\backslash$Red) $\neq \emptyset$, it follows that Flower and Red are $\mathbb{B}_4$-unconnected, i.e., these two sets do not stand in any Aristotelian relation whatsoever in $\mathbb{B}_4$.[12] Completely analogously, one can show that Flower and $D\backslash$Red are $\mathbb{B}_4$-unconnected, $D\backslash$Flower and Red are $\mathbb{B}_4$-unconnected, and $D\backslash$Flower and $D\backslash$Red are $\mathbb{B}_4$-unconnected. This can all be summarized by means of a 'degenerate' square of opposition (or 'X of opposition' [4, pp. 11–12]), as shown in Fig. 4b.

We have thus succeeded in transforming Schopenhauer's original Euler diagram (Fig. 4a) into an Aristotelian diagram, viz. a degenerate square of opposition (Fig. 4b). Once again, it bears emphasizing that the Aristotelian relations (or rather: lack thereof, in the four cases of unconnectedness) that are visualized by this

[12]The Euler diagram in Fig. 4a can be seen as a visual representation of the statement 'some flowers are red', as acknowledged by Schopenhauer [28, p. 294]. This means exactly that Flower $\cap$ Red $\neq \emptyset$; cf. condition (i) above. However, Schopenhauer explicitly indicates that this same Euler diagram *also* represents the statements 'some flowers are not red' and 'some red things are not flowers' [28, p. 294], i.e., Flower $\cap$ ($D\backslash$Red) $\neq \emptyset$ and ($D\backslash$Flower) $\cap$ Red $\neq \emptyset$; cf. conditions (ii) and (iii) above. As far as I know, Schopenhauer never explicitly discussed the interpretation 'some non-red things are not flowers' of this same Euler diagram, which corresponds to condition (iv) above. Finally, note that by attaching equal importance to the four regions Flower $\cap$ Red, Flower $\cap$ ($D\backslash$Red), ($D\backslash$Flower) $\cap$ Red and ($D\backslash$Flower) $\cap$ ($D\backslash$Red), we are essentially reinterpreting the diagram in Fig. 4a as a *Venn diagram*.

square are all mathematically well-defined: they hold relative to the underlying Boolean algebra $\mathbb{B}_4$ (cf. Fig. 5) and are thus specific instantiations of the general characterizations of the Aristotelian relations and unconnectedness provided by Definitions 2.1 and 2.2.

Remarks 3.1–3.3 from the previous subsection continue to apply in the present situation. For example, we once again observe the *non-injective* nature of the transformation process from Euler diagrams to Aristotelian diagrams. In particular, note that the Euler diagram in Fig. 4a, which visualizes that Flower $\cap$ Red $\neq \emptyset$, has been transformed into the degenerate square of opposition in Fig. 4b, but that another Euler diagram, for example, one which visualizes that $(D\backslash\text{Flower}) \cap (D\backslash\text{Red}) \neq \emptyset$, would be transformed into exactly the same degenerate square. Alternatively, one can again view the single Euler diagram in Fig. 4a as a visual representation of *six* relations (or rather: lack thereof, in the four cases of unconnectedness) among Flower, Red, $D\backslash$Flower and $D\backslash$Red, all six of which are also visualized by the degenerate square in Fig. 4b. Also note that the Euler diagram in Fig. 4a can be viewed as a visual representation of the categorical I-statement 'some flowers are red' (cf. Footnote 12). One can then consider the corresponding A-, E- and O-statements, i.e., 'all flowers are red', 'no flowers are red' and 'some flowers are not red'. Together, these four categorical statements yield a classical square of opposition. The difference between these two squares is now even clearer than in Sect. 3.1: the diagram in Fig. 4b is a *degenerate* square of opposition for *sets*, whereas the second diagram just described would be a *classical* square for *statements*.

To conclude this section, let us summarize the results that have been obtained. It is well-known in contemporary logical geometry that there are exactly two types of Aristotelian squares, viz. classical squares and degenerate squares. In Sect. 3.1 we have seen that one of Schopenhauer's Euler diagrams can be transformed into a classical square, and in Sect. 3.2 we have seen that another one of his diagrams can be transformed into a degenerate square. In other words, Schopenhauer's stock of Euler diagrams is sufficiently rich so as to contain analogues of each of the two types of Aristotelian squares that are nowadays studied in logical geometry.

4 Partitions, Euler Diagrams and Aristotelian Diagrams

In this section we will deal with yet another Euler diagram that was used by Schopenhauer—or rather, an entire *class* of Euler diagrams.[13] These diagrams illustrate the case where 'A sphere includes two or more further spheres, which are mutually exclusive and at the same time exhaust the first sphere' [29, p. 66]. The

[13] Strictly speaking, it might be better to call the diagrams used by Schopenhauer that are discussed in this section (and the next one) 'Eulerian diagrams' or 'Euler-type diagrams', rather than simply 'Euler diagrams', because they do not have direct counterparts in Euler's original writings.

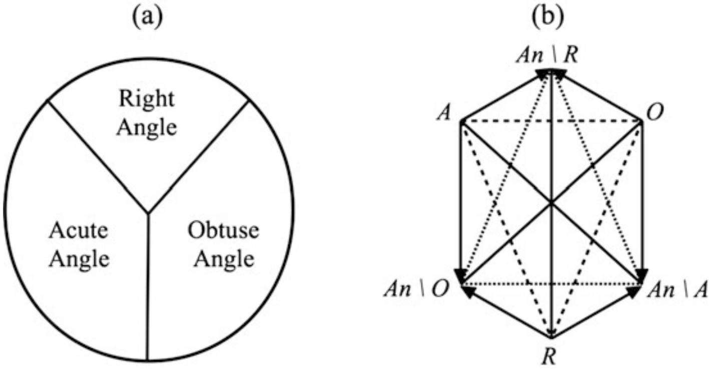

Fig. 6 (**a**) Schopenhauer's Euler diagram; (**b**) the corresponding Aristotelian diagram: a strong JSB hexagon

example he gives involves the concepts of Acute Angle, Right Angle and Obtuse Angle; the accompanying Euler diagram is shown in Fig. 6a. The smallest Boolean algebra of sets that non-trivially contains Acute Angle, Right Angle and Obtuse Angle has $2^3 = 8$ elements. This Boolean algebra, which we will again label $\mathbb{B}_3$, has top element An (for Angle),[14] bottom element $\emptyset$ and three atomic elements, viz. A (for Acute Angle), R (for Right Angle) and O (for Obtuse Angle).

We can now determine the Aristotelian relations holding between some of the sets of this Boolean algebra $\mathbb{B}_3$ (recall Definition 2.3). Since Acute Angle, Right Angle and Obtuse Angle are mutually exclusive, we have $A \cap R = \emptyset$, $A \cap O = \emptyset$ and $R \cap O = \emptyset$. Furthermore, since neither of these three concepts' extensions are empty, it follows that resp. $R \cup O \neq An$, $A \cup O \neq An$ and $A \cup R \neq An$. We thus find that there are pairwise $\mathbb{B}_3$-contrarieties between each of A, R and O. These three sets thus constitute a triangle of contraries in $\mathbb{B}_3$. Completely analogously, one can show that, for all distinct $X, Y \in \{A, R, O\}$, there are $\mathbb{B}_3$-contradictions between X and $An \backslash X$, as well as $\mathbb{B}_3$-subcontrarieties between $An \backslash X$ and $An \backslash Y$ and $\mathbb{B}_3$-subalternations from X to $An \backslash Y$. All these Aristotelian relations can be summarized by means of a JSB hexagon, as shown in Fig. 6b.

Schopenhauer's remark that Acute Angle, Right Angle and Obtuse Angle are *mutually exclusive* thus essentially means that his Euler diagram for these three concepts can be transformed into a well-defined Aristotelian diagram, viz. a *JSB hexagon*. However, Schopenhauer also notes that these three spheres are jointly exhaustive, i.e., $A \cup R \cup O = An$. This remark goes beyond the *Aristotelian* rela-

[14]Since Acute Angle, Right Angle and Obtuse Angle are meant to 'exhaust the first sphere', this first sphere has to be interpreted as a 'restricted domain of discourse', i.e., Angle. Of course, there also exist objects that are *not* angles (e.g., flowers and horses), but if these were also taken into account, then Acute Angle, Right Angle and Obtuse Angle would *not* exhaust the domain of discourse.

tions[15] and provides additional *Boolean* information. This additional information determines the Boolean subtype of the Aristotelian diagram under consideration: the hexagon in Fig. 6b is a *strong* JSB hexagon.

The Euler diagram in Fig. 6a has thus been transformed into the strong JSB hexagon in Fig. 6b. This should not come as a surprise. After all, Schopenhauer's remarks that Acute Angle, Right Angle and Obtuse Angle are mutually exclusive and jointly exhaustive mean that the set $\Pi := \{A, R, O\}$ is a *3-partition* of $\mathbb{B}_3$. By Theorem 2.7 (item 2), this 3-partition gives rise to a strong α_3-structure, i.e., a strong JSB hexagon, which has the three elements of Π on its triangle of contraries. This JSB hexagon is exactly the diagram shown in Fig. 6b.

Although the example involving Acute/Right/Obtuse Angle is the only one that is explicitly given by Schopenhauer in [29], his theoretical discussion is much more general than this. Recall his description: 'A sphere includes *two or more* further spheres, which are mutually exclusive and at the same time exhaust the first sphere' [29, p. 66, my emphasis]. The number of (mutually exclusive and jointly exhaustive) spheres that are included in the first sphere is thus left unspecified. The concrete Acute/Right/Obtuse Angle example is based on *three* spheres, but Schopenhauer could equally easily have given examples based on *two* spheres, *four* spheres, etc.

In his university lecture notes [28, p. 296], Schopenhauer explicitly provides an example of a sphere that includes *two* further spheres which are mutually exclusive and jointly exhaustive. His example involves the concepts of Organic and Inorganic (both included in Body [*Körper*]); the accompanying Euler diagram is shown in Fig. 7a. The smallest Boolean algebra that non-trivially contains Organic and Inorganic has 4 elements. This Boolean algebra, which we will again label $\mathbb{B}_2$, has top element Body, bottom element $\emptyset$ and two atomic elements, viz. Organic and Anorganic. The Hasse diagram for $\mathbb{B}_2$ is shown in Fig. 7b. Since Organic $\cap$ Anorgnic $= \emptyset$ and Organic $\cup$ Anorganic $=$ Body, it follows that Organic and Anorganic are

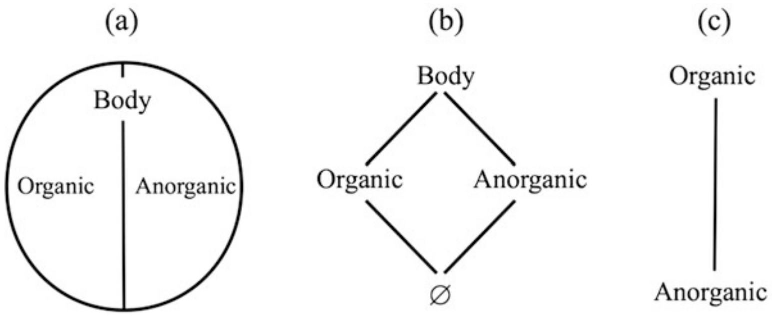

Fig. 7 (**a**) Schopenhauer's Euler diagram; (**b**) Hasse diagram for the Boolean algebra $\mathbb{B}_2$; (**c**) the corresponding Aristotelian diagram: a PCD

[15] Again, note that the Aristotelian relations are *binary* in nature, and thus cannot capture the information that $A \cup R \cup O = An$.

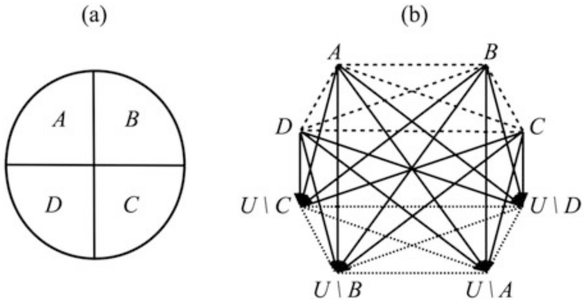

Fig. 8 (**a**) Schopenhauer-inspired Euler diagram; (**b**) the corresponding Aristotelian diagram: a strong Moretti octagon

$\mathbb{B}_2$-contradictory. Schopenhauer's Euler diagram in Fig. 7a can thus be transformed into the (very simple) Aristotelian diagram in Fig. 7c. This Aristotelian diagram is a PCD. Again, this should not come as a surprise. After all, Schopenhauer's remarks that Organic and Anorganic are mutually exclusive and jointly exhaustive mean that the set {Organic, Anorganic} is a *2-partition* of $\mathbb{B}_2$. By Theorem 2.7 (item 1), this 2-partition gives rise to an α_1-structure, i.e., a PCD. This PCD is exactly the diagram shown in Fig. 7c.

As far as I know, Schopenhauer nowhere discussed an example of a sphere that includes *four* further spheres which are mutually exclusive and jointly exhaustive. However, one can easily construct such an example, cf. the Euler diagram shown in Fig. 8a. The smallest Boolean algebra of sets that contains A, B, C and D will again be called $\mathbb{B}_4$. This Boolean algebra has 16 elements in total, including top element U (for Universe), bottom element $\emptyset$ and four atomic elements, viz. A, B, C and D. One can easily determine the Aristotelian relations that hold among A, B, C, D and their complements in $\mathbb{B}_4$. This yields a Moretti octagon, as shown in Fig. 8b. Furthermore, since A, B, C and D are jointly exhaustive (i.e., $A \cup B \cup C \cup D = U$), the diagram in Fig. 8b is a *strong* Moretti octagon. Once again, this should not come as a surprise. After all, the fact that A, B, C and D are mutually exclusive and jointly exhaustive means that the set $\Pi := \{A, B, C, D\}$ is a *4-partition* of $\mathbb{B}_4$. By Theorem 2.7 (item 2), this 4-partition gives rise to a strong α_4-structure, i.e., a strong Moretti octagon, which has the four elements of Π on its trapezoid of contraries. This strong Moretti octagon is exactly the diagram shown in Fig. 8b.

I will again finish this section by summarizing the results that have been obtained. It is well-known in contemporary logical geometry that there is a precise correspondence between partitions and α-structures (cf. Theorem 2.7). We have seen that Schopenhauer explicitly discussed Euler diagrams where a given sphere contains two or more spheres that are mutually exclusive and jointly exhaustive—i.e., that constitute a partition of the first sphere. His concrete example of a 2-partition [28, p. 296] can be transformed into a PCD, i.e., an α_1-structure, which is in line with item 1 of Theorem 2.7. His concrete example of a 3-partition [29, p. 66] can be transformed into a strong JSB hexagon, i.e., a strong α_3-structure, which is in line with item 2 of Theorem 2.7. For the sake of illustration, I have also discussed an example of a 4-partition, showing that it can be transformed into a strong Moretti octagon, i.e., a strong α_4-structure, which is again in line with item 2 of Theorem 2.7. Furthermore, since Schopenhauer left the number of mutually exclusive and jointly exhaustive spheres that are included in the first sphere unspecified, he was implicitly considering an *infinite series* of Euler diagrams (for each $n \geq 2$, there is a distinct Euler diagram corresponding to an n-partition). Each Euler diagram in this series can be transformed into a distinct Aristotelian diagram, viz. an α-structure (from $n = 3$ onwards, this α_n-structure is a strong one). In other words, by leaving the number of cells in the partition unspecified, Schopenhauer essentially anticipated the entire *infinite series* of α-structures from contemporary logical geometry.

5 Boolean Subtypes of Aristotelian Diagrams

In this section we consider a final class of Euler diagrams discussed by Schopenhauer. These diagrams illustrate the case where 'Two spheres lie inside a third, but do not exhaust it' [29, p. 67]. The example he gives involves the concepts of Water and Earth, both of which lie inside Matter. The accompanying Euler diagram is shown in Fig. 9a. The smallest Boolean algebra of sets that non-trivially contains Water and Earth has $2^3 = 8$ elements. This Boolean algebra, which we will again label $\mathbb{B}_3$, has top element M (for Matter), bottom element $\emptyset$ and three atomic elements, viz. W (for Water), E (for Earth) and $M \backslash (W \cup E)$.[16]

We can now determine the Aristotelian relations holding between some of the sets of this Boolean algebra $\mathbb{B}_3$ (recall Definition 2.3). Since Water and Earth are mutually exclusive, we have $W \cap E = \emptyset$.[17] Furthermore, since Water and Earth do not exhaust Matter, we have $W \cup E \neq M$. This means that W and E are $\mathbb{B}_3$-contrary. Completely analogously, one can show that there are $\mathbb{B}_3$-contradictions between W and $M \backslash W$ and between E and $M \backslash E$, as well as $\mathbb{B}_3$-subalternations from W to $M \backslash E$ and from E to $M \backslash W$, and finally, a $\mathbb{B}_3$-subcontrariety between $M \backslash E$ and $M \backslash W$. All these Aristotelian relations can be summarized by means of a classical square of opposition, as shown in Fig. 9b.

The Euler diagram in Fig. 9a has thus been transformed into a well-defined Aristotelian diagram, viz. the classical square of opposition in Fig. 9b. However, if we consider not only W and E (and their complements), but also the third atomic element of $\mathbb{B}_3$, i.e., $M \backslash (W \cup E)$ (and its complement), then this very same Euler diagram can also be transformed into another Aristotelian diagram. One can easily show that $M \backslash (W \cup E)$ is $\mathbb{B}_3$-contrary to W as well as to E. These

[16] Since Water and Earth do not exhaust matter, it follows that $M \backslash (W \cup E) \neq \emptyset$.

[17] Schopenhauer does not explicitly say that the two spheres that lie inside the third one have to be mutually exclusive, but his Euler diagram does display them as such; cf. Fig. 9a. Furthermore, immediately after giving the example, he does state that 'The last case applies to all concepts whose spheres *do not have anything directly in common*' [29, p. 67, my emphasis].

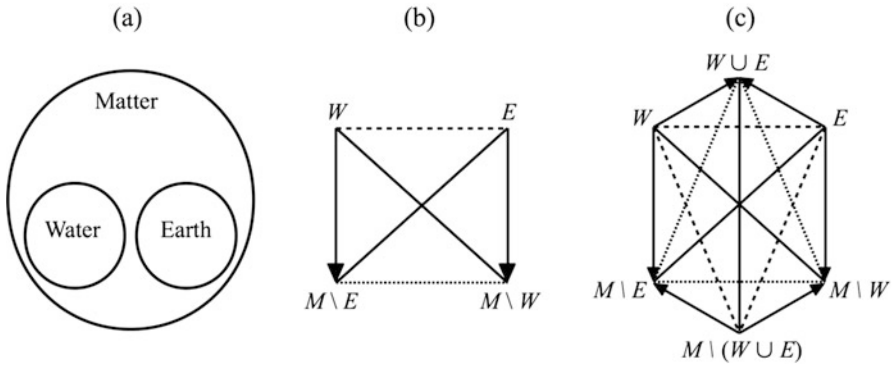

Fig. 9 (**a**) Schopenhauer's Euler diagram; (**b**) first corresponding Aristotelian diagram: a classical square of opposition; (**c**) second corresponding Aristotelian diagram: a strong JSB hexagon

three sets thus constitute a triangle of contraries in $\mathbb{B}_3$. By also taking the three other non-trivial elements of $\mathbb{B}_3$ into account, we can construct a JSB hexagon, as shown in Fig. 9c. Although Water and Earth by themselves do not exhaust Matter (i.e., $W \cup E \neq M$), adding this third 'remainder' concept $M\backslash(W \cup E)$ does exhaust Matter (i.e., $W \cup E \cup M\backslash(W \cup E) = M$). This means that the diagram in Fig. 9c is a *strong* JSB hexagon. This should not come as a surprise. After all, Schopenhauer's remarks that Water and Earth are mutually exclusive, together with the definition of $M\backslash(W \cup E)$ as a 'remainder' concept, imply that the set $\Pi := \{W, E, M\backslash(W \cup E)\}$ is a *3-partition* of $\mathbb{B}_3$. By Theorem 2.7 (item 2), this 3-partition gives rise to a strong α_3-structure, i.e., a strong JSB hexagon, which has the three elements of Π on its triangle of contraries. This JSB hexagon is exactly the diagram shown in Fig. 9c.

The Euler diagram in Fig. 9a can thus be transformed into a classical square, i.e., an α_2-structure, but also into a strong JSB hexagon, i.e., a strong α_3-structure. An analogous situation arises when we move to Euler diagrams with higher numbers of spheres. Schopenhauer himself only considered and illustrated the case where '*Two* [disjoint] spheres lie inside a third, but do not exhaust it' [29, p. 67, my emphasis], and unlike the case discussed in Sect. 4, he did not generalize this to higher numbers of spheres. I will now discuss two such generalizations, which are completely in line with Schopenhauer's thinking.

The first generalization can be described as a case where '*three* disjoint spheres lie inside a fourth, but do not exhaust it'. A concrete example involves the concepts of Water, Earth and Air, all of which lie inside Matter; cf. the Euler diagram in Fig. 10a. The smallest Boolean algebra of sets that non-trivially contains Water, Earth and Air has $2^4 = 16$ elements. This Boolean algebra, which we will again label $\mathbb{B}_4$, has top element M (for Matter), bottom element $\emptyset$ and four atomic elements, viz. W (for Water), E (for Earth), A (for Air) and $M\backslash(W \cup E \cup A)$. One can easily determine the Aristotelian relations that hold among Water, Earth and Air and their complements in $\mathbb{B}_4$. This yields a JSB hexagon, as shown in Fig. 10b. Since Water, Earth and Air do not exhaust Matter, this diagram is a *weak*

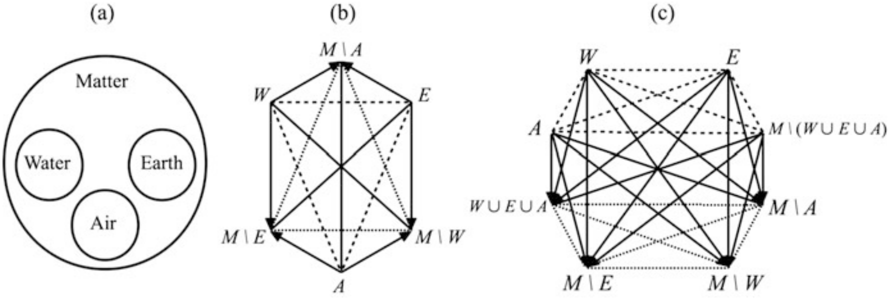

Fig. 10 (**a**) Schopenhauer-inspired Euler diagram; (**b**) first corresponding Aristotelian diagram: a weak JSB hexagon; (**c**) second corresponding Aristotelian diagram: a strong Moretti octagon

JSB hexagon, i.e., a weak α_3-structure. If we consider not only W, E and A (and their complements), but also the fourth atomic element of $\mathbb{B}_4$, i.e., $M\backslash(W \cup E \cup A)$ (and its complement), then the Euler diagram in Fig. 10a can also be transformed into another Aristotelian diagram. One can easily show that $M\backslash(W \cup E \cup A)$ is $\mathbb{B}_4$-contrary to W, to E and to A. These four sets thus constitute a trapezoid of contraries in $\mathbb{B}_4$. By also taking the four complements of W, E, A and $M\backslash(W \cup E \cup A)$ into account, we can construct a Moretti octagon, as shown in Fig. 10c. Although Water, Earth and Air by themselves do not exhaust Matter (i.e., $W \cup E \cup A \neq M$), adding this fourth 'remainder' concept $M\backslash(W \cup E \cup A)$ does exhaust Matter (i.e., $W \cup E \cup A \cup M\backslash(W \cup E \cup A) = M$). This means that the diagram in Fig. 10c is a *strong* Moretti octagon. Once again, this should not come as a surprise. After all, the fact that Water, Earth are Air are mutually exclusive, together with the definition of $M\backslash(W \cup E \cup A)$ as a 'remainder' concept, implies that the set $\Pi := \{W, E, A, M\backslash(W \cup E \cup A)\}$ is a *4-partition* of $\mathbb{B}_4$. By Theorem 2.7 (item 2), this 4-partition gives rise to a strong α_4-structure, i.e., a strong Moretti octagon, which has the four elements of Π on its trapezoid of contraries. This Moretti octagon is exactly the diagram shown in Fig. 10c.

I now briefly turn to a second generalization, which can be described as a case where '*four* disjoint spheres lie inside a fifth, but do not exhaust it'. A concrete example involves the concepts of Water, Earth, Air and Fire, all of which lie inside Matter; cf. the Euler diagram in Fig. 11a. The smallest Boolean algebra of sets that non-trivially contains Water, Earth, Air and Fire has $2^5 = 32$ elements. This Boolean algebra, $\mathbb{B}_5$, has top element M (for Matter), bottom element $\emptyset$ and five atomic elements, viz. W (for Water), E (for Earth), A (for Air), F (for Fire) and $M\backslash(W \cup E \cup A \cup F)$. One can easily show that Water, Earth, Air and Fire yield a Moretti octagon in $\mathbb{B}_5$, as shown in Fig. 11b. Since Water, Earth, Air and Fire do not exhaust Matter, this diagram is a *weak* Moretti octagon, i.e., a weak α_4-structure. If we consider not only W, E, A and F (and their complements), but also the fifth atomic element of $\mathbb{B}_5$, i.e., $M\backslash(W \cup E \cup A \cup F)$ (and its complement), then the Euler diagram in Fig. 11a can also be transformed into another Aristotelian diagram, viz. an α_5-structure (which is not shown here, for reasons of space). Although Water, Earth,

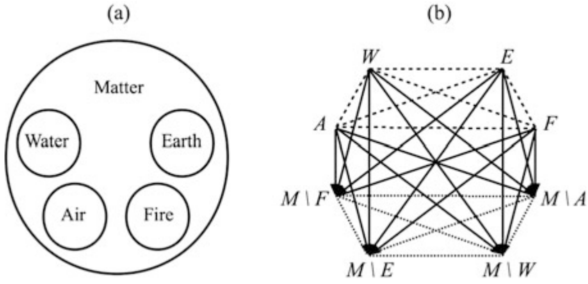

Fig. 11 (**a**) Schopenhauer-inspired Euler diagram; (**b**) one corresponding Aristotelian diagram: a weak Moretti octagon (There also exists another one, which is not shown here.)

Air and Fire by themselves do not exhaust Matter (i.e., $W \cup E \cup A \cup F \neq M$), adding this fifth 'remainder' concept $M\backslash(W \cup E \cup A \cup F)$ does exhaust Matter (i.e., $W \cup E \cup A \cup F \cup M\backslash(W \cup E \cup A \cup F) = M$). This means that the α_5-structure is a *strong* α_5-structure, which is again perfectly in line with item 2 of Theorem 2.7, since $\{W, E, A, F, M\backslash(W \cup E \cup A \cup F)\}$ is a *5-partition* of $\mathbb{B}_5$.

By now, the pattern that emerges should be very clear:

- The Euler diagram in Fig. 9a can be transformed into an α_2-structure (Fig. 9b), but also into a strong α_3-structure (Fig. 9c).[18]
- The Euler diagram in Fig. 10a can be transformed into a weak α_3-structure (Fig. 10b), but also into a strong α_4-structure (Fig. 10c).
- The Euler diagram in Fig. 11a can be transformed into a weak α_4-structure (Fig. 11b), but also into a strong α_5-structure (not shown).

We can now combine these results with those of Sect. 4. In that section, we have seen that whenever $n \geq 3$ mutually disjoint spheres lie inside a given sphere and also exhaust that sphere, they give rise to a *strong α_n-structure*. In this section, we have seen that whenever $n \geq 3$ mutually disjoint spheres lie inside a given sphere but do not exhaust it, they give rise to a *weak α_n-structure*.[19] By explicitly distinguishing between cases where $n \geq 3$ mutually disjoint spheres are jointly exhaustive and cases where they are not jointly exhaustive, Schopenhauer thus clearly anticipated the distinction between strong and weak α_n-structures (cf. Theorem 2.6), and hence, more generally, the insight that families of Aristotelian diagrams can have *distinct Boolean subtypes*.

6 Conclusion

In this paper I have explored the connection between Schopenhauer's Euler diagrams and the Aristotelian diagrams that are studied in contemporary logical

[18] Since the family of classical squares is Boolean homogeneous, it makes little sense to talk about a 'weak' α_2-structure (recall Theorem 2.6, item 2).

[19] Recall that for $n < 3$, the family of α_n-structures is Boolean homogeneous; cf. items 1 and 2 of Theorem 2.6.

geometry. One can define the Aristotelian relations in a very general fashion (relative to arbitrary Boolean algebras), which allows us to define not only Aristotelian diagrams for *statements*, but also for *sets*. I have shown that, once this generalization has been made, each of Schopenhauer's concrete Euler diagrams can be transformed into a well-defined Aristotelian diagram. More importantly, I have also argued that Schopenhauer had several more general, systematic insights about Euler diagrams, which anticipate general insights and theorems about Aristotelian diagrams in contemporary logical geometry.

For example, it is well-known in logical geometry today that there are exactly two types of Aristotelian squares, viz. classical squares and degenerate squares. Schopenhauer had Euler diagrams that can be transformed into each of these two types of Aristotelian squares (cf. Sect. 3). Furthermore, logical geometry shows that there is a clear correspondence between n-partitions and (strong) α_n-structures. Schopenhauer anticipated this correspondence, by considering Euler diagrams for n-partitions, each of which can be transformed into the corresponding (strong) α_n-structure; he also discussed this correspondence in its full generality, i.e., by considering n-partitions for arbitrary n (cf. Sect. 4). Finally, logical geometry emphasizes that many families of Aristotelian diagrams have distinct Boolean subtypes. In particular, for $n \geq 3$, the family of α_n-structures has two Boolean subtypes (strong and weak). By explicitly distinguishing between cases where a number of (mutually disjoint) spheres are jointly exhaustive and cases where they are not jointly exhaustive, Schopenhauer also displayed a remarkable sensitivity to the subtle interplay between Aristotelian and Boolean considerations (cf. Sect. 5).

In sum: because of his various concrete Euler diagrams and, especially, his more systematic observations about them, Schopenhauer can rightly be considered a distant forerunner of contemporary logical geometry, which studies Aristotelian diagrams as objects of independent mathematical and philosophical interest.

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Metalogic, Schopenhauer and Universal Logic

Jean-Yves Beziau

Abstract Schopenhauer used the word “metalogical” since his first work, *On the Fourfold Root of the Principle of Sufficient Reason* (1813), being the first to give it a precise meaning and a proper place within a philosophical system. One century later the word “Metalogic” started to be used and promoted in modern logic by the Russian logician Nicolai Vasiliev and the Polish School (Łukasiewicz, Tarski, Wajsberg). The aim of this paper is to examine the relations between the different uses of this word and doing that to try to have a better understanding of what Metalogic is and also logic *tout court*.

In a first section we examine and clarify the meaning of Metalogic in modern logic, comparing Metalogic to Metamathematics and Universal Logic. We make in particular a distinction between two trends in Metalogic that can be crystallized through *metatheorem vs. meta-axiom*.

In a second section we present Schopenhauer’s use of the word, which is essentially through the notion of *metalogical truths*. We describe their locations within Schopenhauer’s framework, standing side by side with other kinds of truths (metaphysical truths, logical truths, empirical truths), constituting altogether the Principle of Sufficient Reason (PSR) of Knowledge, one of the four roots of the PSR. We explain why Schopenhauer thinks that mathematical truths do not need to have a logical ground and present his view according to which metalogical truths are fundamental laws of thought that cannot be changed. We discuss the feminine nature he attributes to them and establish a parallel with Aristotle’s vision of logic.

In a third section we examine how modern logic arose from a double challenge of the fundamental laws of logic: their reformulation and relocation, their relativization and rejection. We emphasize that this dynamic evolution was performed on the basis of some semiotical and conceptual changes at the heart of logic and Metalogic.

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La Fête de la Raison dans Notre-Dame de Paris le 10 novembre 1793. Painting by Charles-Louis Müller. Collection Musée de la Ville de Poitiers and Société des Antiquaires de l'Ouest. ©Musées de Poitiers/Christian Vignaud

> Reason is of a feminine nature: it can give only after it has received. On its own, it possesses nothing but the empty forms of its own operation. Completely pure rational cognition gives us in fact only four things, the very metalogical truths.—Arthur Schopenhauer

1 Indiosyncralogical Schopenhauer

Arthur Schopenhauer (1788–1860) has been very popular during the second half of the nineteenth century and beginning of the twentieth century, in particular among artists: Richard Wagner, Guy de Maupassant, Thomas Mann. Here is how he is nowadays presented by Mary Troxell in the *Internet Encyclopedia of Philosophy*: "Arthur Schopenhauer has been dubbed the artist's philosopher on account of the inspiration his aesthetics has provided to artists of all stripes. He is also known

Fig. 1 Schopenhauer's spherical thoughts

as the philosopher of pessimism, as he articulated a worldview that challenges the value of existence. His elegant and muscular prose earns him a reputation as one of the greatest German stylists" [117]. This is a good summary of the general picture people have about Schopenhauer, in particular with no connection to logic.

Few people know that Schopenhauer had some interesting ideas about logic, at best they have heard about his essay on Eristical Dialectic generally known as *The Art of Persuasion*, but this has more to do with sophistry than logic itself. Schopenhauer was interested in particular in spheric representation of concepts (Fig. 1), in the line of Leonhard Euler (1707–1783), and showed how we can go in this way from Good to Evil, from Paradise to Hell, and back, in a not so expensive way (See [73], a good starting point to explore Schopenhauer's circus of conceptual circles).

In the present paper we are dealing with a fundamental notion, Metalogic, examining what Schopenhauer said about that and comparing it with Metalogic as conceived in modern logic. Our objective is, on the one hand, to give a more open approach to the philosophical discussion about central concepts of modern logic, often reduced to contemporary problems without a general historical perspective (this is the case of the most famous books on philosophy of logic of the last decades, the one by Susan Haack published in 1978 [56]), and, on the other hand, to give a better vision of Schopenhauer, who had interesting views on many topics: the theory of colors (he was a friend of Goethe), biology (he knew the work of Lamarck), language (he knew many languages and translated Baltasar Gracián from Spanish to German), religion (he was the first Western philosopher to be interested in Oriental philosophy, both Hinduism and Buddhism), ... and also logic!

For this reason the present paper has been written in a way so that it can be of interest both for aficionados of Schopenhauer knowing few things about logic and logic lovers knowing quite nothing about Schopenhauer. We have given precise references both for the sake of rigor and as further readings for those wanting to know more.

Establishing a bridge between Schopenhauer and contemporary mathematical logic may look strange, not to say extravagant. But this is not so absurd if we consider that Schopenhauer developed some ideas about logic and also mathematics (connected to ideas of Ludwig Wittgenstein, 1889–1951, and Luitzen Egbertus Brouwer, 1881–1966) and that he is not so far away in time from modern logic (he was 66 year old when George Boole, at the age of 39, published the *Laws of Thought* [30]). And anyway it is good to try to have a general perspective, establishing connections between ideas of different times and origins, without being afraid to eventually fall into the sin of anachronism. We are not promoting sin (Chronos bless us), but, as they say in Germany "no risk, no fun", and relating different things from different times is anyway tautologically anachronical.

We could have written a paper only about Metalogic according to Schopenhauer, but this would have been at best a good popular paper for lazy people having no time to read Schopenhauer. We cannot explain Schopenhauer better than himself. He was a philosopher who at the same time had his own vigorous original style and the capacity to write things rigorously, clearly, and succinctly, showing that we can seriously write serious things without being boring (we will try to do the same here).

He wrote: "To use many words to communicate thoughts is everywhere the unmistakable sign of mediocrity. To gather much thought into few words stamps the man of genius" (PPA, V2, Ch23). And describing the general attitude of the philosopher he states that: "The real philosopher always looks for limpidity and precision, he will invariably try to resemble not a turbid, impetuous torrent, but instead a Swiss lake which by its calmness preserves transparency despite its great depth, a great depth revealing itself precisely through its great transparency" (4RP, §3).

This metaphor of the Swiss lake (Fig. 2) is interesting because Switzerland is a country not only with lakes but with high mountains and to be at the top of the mountain having a panoramic view is also the perspective of Schopenhauer who can be qualified as a "panoramic philosopher." This quality is described as follows by Friedrich Nietzsche (1844–1900) in his essay *Schopenhauer as Educator* (1874 [84]):

> His greatness is that he can stand opposite the picture of life, and interpret it to us as a whole: while all the clever people cannot escape the error of thinking one comes nearer to the interpretation by a laborious analysis of the colours and material of the picture; with the

Fig. 2 Swiss lake and mountains: panoramic transparency. Ouchy and on the top left Montreux where was organized the 1st world congress on universal logic in 2005. ©Courtesy of Lausanne Tourism Office, Switzerland

> confession, probably, that the texture of the canvas is very complicated, and the chemical composition of the colours undiscoverable. Schopenhauer knew that one must guess the painter in order to understand the picture. But now the whole learned fraternity is engaged on understanding the colours and canvas, and not the picture: and only he who has kept the universal panorama of life and being firmly before his eyes, will use the individual sciences without harm to himself; for, without this general view as a norm, they are threads that lead nowhere and only confuse still more the maze of our existence. Here we see the greatness of Schopenhauer, that he follows up every idea, as Hamlet follows the Ghost, without allowing himself to turn aside for a learned digression, or be drawn away by the scholastic abstractions of a rabid dialectic.

Synopticity (in German: *Übersichtlichkeit*) was also promoted by Wittgenstein, much influenced by Schopenhauer, having read his books in his youth and once again after having written the *Tractatus* (see [82] and [80]).

In the perspective of this panoramic approach and to give an "avant-goût" of the content of our paper, we present the following chronological list of works we will talk about (not an exhaustive list, but a representative one):

- 1787, Immanuel Kant (1724–1804), 2nd edition of the *Critique of Pure Reason* [67]
- 1813, Arthur Schopenhauer (1788–1860), *On the Fourfold Root of the Principle of Sufficient Reason* [99]
- 1854, George Boole (1815–1864), *The Laws of Thought* [30]
- 1880, Charles Sanders Peirce (1839–1914), "A Boolian Algebra with One Constant" [87]
- 1881, John Venn (1834–1923), *Symbolic Logic* [121]
- 1913, Nicolai Vasiliev (1880–1940), "Logic and metalogic" [120]

- 1921, Emil Post (1897–1954), "Introduction to a General Theory of Elementary Propositions" [88]
- 1922, David Hilbert (1862–1943), "Neubegründung der Mathematik: Erste Mitteilung" [61]
- 1930, Jan Łukasiewicz (1878–1956) and Alfred Tarski (1901–1983), "Introduction into the sentential calculus" [16, 77]
- 1937, Mordechaj Wajsberg (1902–194?), "Metalogische Beiträge. I" [124]
- 1952, Stephen Kleene (1909–1994), *Introduction to Metamathematics* [68]

But we will not follow in our paper a chronological order, neither forward, nor backward. Our itinerary is as follows: we start first with the conception of Metalogic in modern logic at the beginning of the twentieth century, we then present Schopenhauer's views on the Metalogical, and in a third part we treat the question of Metalogic through a panoramic analysis of the development of modern logic.

Our travel in time is to highlight the present to go ahead, not to relive the past or/and to spend happy vacations in the nineteenth century in Frankfurt am Main with Arthur Schopenhauer.

Nevertheless, according to the structure of our travel, Schopenhauer is at the center of our attention. It is the main course of our philosophical menu and at the very middle of this menu we have the Sect. 3.2. entitled *Schopenhauer's Theory of the Metalogical* which is the main dish. The whole menu should pamper both gourmets and gluttons.

2 Metamathematics, Metalogic, and Universal Logic

The word "Metalogic" is a neologism combining the prefix "meta" with the substantive "logic". To understand the meaning of this combination in modern logic, we will start by examining another neologism, with the same prefix: "Metamathematics". This is a good point of departure because, on the one hand, the meaning of this neologism is quite clear and, on the other hand, the neologism "Metalogic" has been used in modern logic in particular under the influence of "Metamathematics".

2.1 *Origin and Nature of Metamathematics*

David Hilbert (1862–1943) made "Metamathematics" famous. He did not create the word but he was the first to give a precise meaning to it and to use it in a systematic way.[1] Before Hilbert the word was used in discussions about non-

[1] The same can be said about a central terminology and a central character of modern logic: "truth-value" and Gottlob Frege (1848–1925), see [20].

Euclidean geometry as a kind of synonymous to "Metageometry" which also was used, and Hilbert knew about that (for details see [127]).

But Hilbert started to use the word in a new way, as synonymous to another expression he promoted: "Proof Theory" (in German: *Beweistheorie*). The reason why is that for him the object study of Metamathematics are mathematical proofs, which are themselves the core of mathematics. A belief shared by many mathematicians. Nicolas Bourbaki (1935–1968) starts his famous multi-volume treatise, the Bible of modern mathematics [31], with the sentence "Depuis les Grecs qui dit mathématique dit démonstration" (literally: "Since the Greeks who says mathematic says demonstration"; inexact published translation: "Ever since the time of the Greeks, mathematics has involved proof"). The Greek prefix "meta" has different meanings but the way Hilbert is using it is *above*, in the intuitive sense that we study an object by being outside of it, upside being a good position, like when having a panoramic view at the top of a mountain.

Hilbert had the idea that Metamathematics was in some sense superior to mathematics, because it is the understanding of what mathematics is:

> The axioms and provable theorems, i.e. the formulae that arise in this interplay, are the images of the thoughts that make up the usual procedure of traditional mathematics; but they are not themselves the truths in an absolute sense. Rather, the *absolute truths* are the insights that my proof theory furnishes into the provability and consistency of these formal systems. [62]

The emphasis on "absolute truth" is ours. "Truth" in modern logic is often contrasted to "Proof", an opposition related to the contrast between *Model Theory* and *Proof Theory*.[2] But the use of "truth" Hilbert is doing here is not in the perspective of Model Theory (which did not exist at that time) but in the sense of a more fundamental and philosophical level, which is indeed the perspective of his Metamathematics.

Following the influence of Hilbert and his school, logic in the first half of the twentieth century has been at some point identified with Metamathematics. Hilbert started to use word "Metamathematik" in the following two papers:

- "Neubegründung der Mathematik: Erste Mitteilung" (1922) [61],
- "Die logischen Grundlagen der Mathematik" (1923) [62].

As we can see, none of them has this word in the title. And there are no papers and books by Hilbert with this word in the title.[3] But Stephen Cole Kleene (1909–1994) published in 1952 a book entitled *Introduction to Metamathematics* [68],

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[2]This is also expressed as an opposition between semantics and syntax. "Proof Theory" as coined by Hilbert concentrates on mathematical proofs from a syntactical point of view, according to which mathematics, Hilbert says, "becomes a stock of formulae" [62]. "Model Theory" was coined by Tarski [114] and deals with the interpretation of the syntax, the *models* of the theories.

[3]The two books by Hilbert on logic are entitled *Grundzüge der theoretischen Logik* (1928), co-authored with Wilhelm Ackermann, 1896–1962 [63] (in English: *Principles of Mathematical Logic*) and *Grundlagen der Mathematik* (in English: *Foundations of Mathematics*), (Volume 1, 1934 - Volume 2, 1939), co-authored with Paul Bernays, 1888–1977 [64].

which was frequently re-edited. This is an important textbook of modern logic which influenced a whole generation, as emphasized by Michael Besson in the foreword of the re-printed 2009 edition: "Stephen Kleene was one of the greatest logicians of the twentieth century, and had an enormous influence on the subject. The book in your hands is the textbook that spread that influence far beyond his own students, to an entire generation of logicians." [70]

In this book there is no Model Theory. This is one of the main reasons why nowadays such a book is not considered as a serious introductory book to logic. Kleene himself explains/justifies the contents of this book in a paper entitled "The writing of *Introduction to Metamathematics*" [71].[4]

Kleene's famous book was developed under Hilbert's perspective of logic, often called "Hilbert's program", that we will not present here in details (to know more about it see [133]). But we will make some remarks explaining how "Metamathematics" can be understood in a way different than Hilbert's one and clarifying the relation between logic and Metamathematics in view of our discussion on Metalogic. We present a list of four points followed by comments.

(a) Metamathematics does not reduce to Proof Theory.
(b) The study of mathematical reasoning does not reduce to Proof Theory.
(c) Proof Theory does not reduce to Hilbert's formalist approach.
(d) Logic does not reduce to the study of mathematical reasoning.

(a) If we consider Metamathematics as a science having as object of study mathematics, there is no reason to reduce it to Proof Theory unless we reduce mathematics to proofs. There is also a whole conceptual and semiotical aspect. Boole in fact changed mathematics by considering operations on objects other than numbers or of geometrical nature. And Philosophy of Mathematics can be considered as part of Metamathematics.

(b) Moreover mathematical reasoning can be studied by other means than Proof Theory, in particular Model Theory, and also in psychological and cognitive perspectives.

(c) Hilbert is famous for having promoted Proof Theory using reduced means in particular regarding the question of finiteness. But it is possible to develop Proof Theory without such restrictions. Even in the school of Hilbert, Gerhard Gentzen (1909–1945) was not afraid to use infinistic methods, in particular for his famous proof of consistency of arithmetic using induction up to ϵ_0 [53].[5] The Polish School is famous not to have endorsed Hilbert's finitism and to have allowed the use of any mathematical tools as, for example, the axiom of choice. This has been

[4]Kleene later on (1967) published another textbook entitled *Mathematical Logic* [69] full of model theory, giving up Metamathematics both syntactically and semantically. In particular he uses in this book the expression "Model Theory" for propositional logic, which is up to now unfortunately not so common.

[5]Gentzen also worked on "natural deduction", developing formal systems supposed to catch in a more natural way reasoning. For a discussion about that see [103], which makes a connection with Schopenhauer who was already concerned by this point.

crystallized by the punny title of the book of Rasiowa and Sikorski: *The mathematics of metamathematics* (1963) [89].

(d) Logic can be defined as the study of all kinds of reasoning. We have to be careful since the word "logic" is also used to talk about reasoning itself (about this confusion see our paper "Logic is not logic" [17]). Although mathematical reasoning is important, this is not the only form of reasoning. Aristotle when developing the science of reasoning, in particular through a particular system, syllogistic, was considering reasoning about anything. This was also the case of the Stoics and later on of Boole himself, whose book title is *The Laws of Thought*, not *The Laws of Mathematical Reasoning*, nor *The Laws of Mathematical Thought*. But at some point in the development of modern logic people were focusing on mathematical reasoning: Peano, Frege, Whitehead, Russell, Hilbert…

2.2 From Metamathematics to Metalogic

As we have seen in the previous section, the word "Metamathematics" was promoted by Hilbert, identifying Metamathematics with Proof Theory. But the word "Metalogic" was not used by Hilbert, as explained by Haskell Curry (1900–1982), who studied in Göttingen and was one of the last students of Hilbert (see [40]): "Anyone who looks at all seriously at formalistic work of modern mathematical logic can hardly avoid noticing a great variety of words beginning with the prefix 'meta-'. One meets 'metalanguage,' 'metasystem,' 'metatheorem,' 'metalogic,' 'metacalculus,' 'metasemiosis,' and, in German, '*Metaaussagenkalkül*'. All these terms are described as in principle due to Hilbert. Actually the only one of them which Hilbert himself used is 'metamathematics'; the rest were invented by his followers on the basis of some analogy." ([43, pp. 86–87])

At the beginning of the twentieth century, "Metalogic" was used, independently and with different meanings by:

- the Russian logician Nicolai Alexandrovich Vasiliev (1880–1940)
- the Lvov-Warsaw Polish School (1915–1944)[6]

Vasiliev used this word before the people of the Lvov-Warsaw School and before Hilbert started to use the word "Metamathematics". In particular he published in 1913 a paper in Russian entitled "Logic and Metalogic." This paper was translated

[6]To fix the ideas we have symbolically put here as dates of birth and death of this school, respectively, the coming of Łukasiewicz to Warsaw University and his departure from this university. Of course one can argue that this school started before 1915 and did not stop in 1944, that it is still alive, see the recent book *The Lvov-Warsaw School, Past and Present* [50].

Fig. 3 Vasiliev's views on metalogic

EARTHLY LOGIC	ARISTOTELIAN, EMPIRICAL, THE LOGIC OF EARTH
IMAGINARY LOGIC	ANY LOGICAL SYSTEMS, APPLYING TO IMAGINARY WORLD OR OTHER PLANETS
METALOGIC	THE LOGIC OF THE FORM OF OUR THOUGHT, ABSTRACT AND NON-EMPIRICAL
THREE ASPECTS OF LOGIC - VASILIEV	

in English only 80 years later, in 1993 [120].[7] The view of Vasiliev on metalogic can be summarized by the table on (Fig. 3)[8].

Here is the detailed explanation of Vasiliev of the nature of metalogic and its relation with other aspects of logic:

> I would call a logic without any empirical elements metalogic. The name "metalogic" is better suited for this discipline, as it indicates a formal analogy to metaphysics. Metaphysics is the knowledge of being regardless of the conditions of experience. Metalogic is the knowledge of thought regardless of the conditions of experience. Metaphysics is the science of pure being. It constitutes an abstraction from the world of phenomena, and it is the knowledge of that which is common to all empirical things. Metalogic is a discipline of pure thought. It is an abstraction from everything in thought that is empirical. There may be many worlds, but the essence of being is one. Such is the basic premiss of metaphysics. There may be Many logics, but they all have something in common which is only One, viz. metalogic. Metalogic, then, is the discipline of the formal aspect of thought regardless of its content. Therefore, the only formal logic is metalogic. [119]

We see that Vasiliev is using here the prefix "meta" by analogy with "Metaphysics". The word "Metaphysics" was originally used as the title of a book by Aristotle, based on the sense of the Greek prefix "meta" meanings "after" (different from the other sense we already talked about, "above"). This book was ordered in the corpus of Aristotle's work by commentators just after one entitled *Physics*, and since they were not able to find a proper name expressing the rather mysterious content of this book they decided just to name it *Metaphysics*. The meaning of the prefix "meta" Vasiliev is using is not an "afterward" syntactic meaning, it is related to the accidental semantics of the word "Metaphysics", which however essentially makes sense if we interpret "meta" as "above", Metaphysics being above experience.

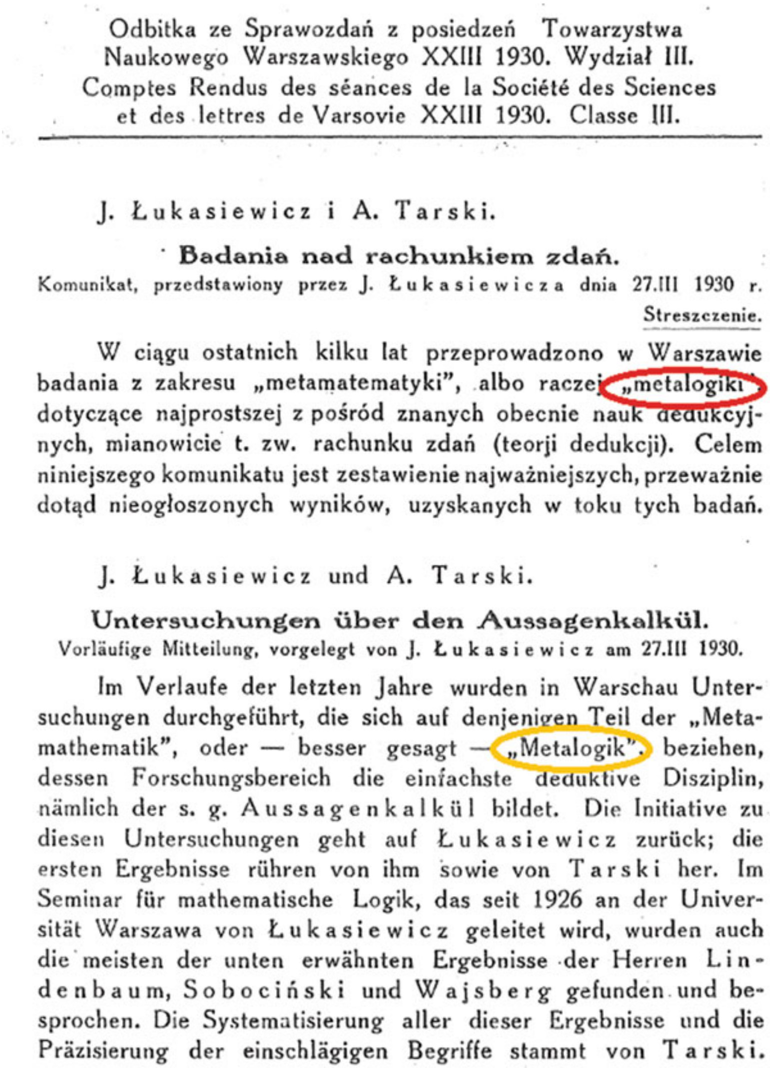
According to Vasiliev's quotation it is quite clear that he did not use the word "Metalogic" under the influence of the pre-Hilbertian use of "Metamathematics" and the correlated use of "Metageometry". But there is a common background: Vasiliev's work was developed by analogy with the school of Non-Euclidean

[7]See also two papers by Vasiliev of the same period: [118] and [119]. For a general presentation of Vasiliev and his work, see [2, 3].

[8]This figure is extracted from our previous paper "Is Modern Logic Non-Aristotelian?" [22] related to a lecture presented at a conference in honor of Vasiliev, October 24–25, 2012 at Lomonosov Moscow State University. And it was published in a book with other papers presented at this conference.

geometry (like Lobachevsky he was connected to Kazan): he created the expressions "Non-Aristotelian logic" and "Imaginary logic" (see [120]).

Let us see now how "Metalogic" was used in the Lvov-Warsaw School, about 15 years later, at the end of the 1920s, after Hilbert had started to use the word "Metamathematics". At the beginning of the paper by Łukasiewicz and Tarski "Introduction into the sentential calculus", originally published in 1930 in German with a Polish summary (see Fig. 4), it is written:



Odbitka ze Sprawozdań z posiedzeń Towarzystwa Naukowego Warszawskiego XXIII 1930. Wydział III.
Comptes Rendus des séances de la Société des Sciences et des lettres de Varsovie XXIII 1930. Classe III.

J. Łukasiewicz i A. Tarski.

Badania nad rachunkiem zdań.

Komunikat, przedstawiony przez J. Łukasiewicza dnia 27.III 1930 r.

Streszczenie.

W ciągu ostatnich kilku lat przeprowadzono w Warszawie badania z zakresu „metamatematyki", albo raczej „metalogiki", dotyczące najprostszej z pośród znanych obecnie nauk dedukcyjnych, mianowicie t. zw. rachunku zdań (teorji dedukcji). Celem niniejszego komunikatu jest zestawienie najważniejszych, przeważnie dotąd nieogłoszonych wyników, uzyskanych w toku tych badań.

J. Łukasiewicz und A. Tarski.

Untersuchungen über den Aussagenkalkül.

Vorläufige Mitteilung, vorgelegt von J. Łukasiewicz am 27.III 1930.

Im Verlaufe der letzten Jahre wurden in Warschau Untersuchungen durchgeführt, die sich auf denjenigen Teil der „Metamathematik", oder — besser gesagt — „Metalogik", beziehen, dessen Forschungsbereich die einfachste deduktive Disziplin, nämlich der s. g. Aussagenkalkül bildet. Die Initiative zu diesen Untersuchungen geht auf Łukasiewicz zurück; die ersten Ergebnisse rühren von ihm sowie von Tarski her. Im Seminar für mathematische Logik, das seit 1926 an der Universität Warszawa von Łukasiewicz geleitet wird, wurden auch die meisten der unten erwähnten Ergebnisse der Herren Lindenbaum, Sobociński und Wajsberg gefunden und besprochen. Die Systematisierung aller dieser Ergebnisse und die Präzisierung der einschlägigen Begriffe stammt von Tarski.

Fig. 4 Łukasiewicz and Tarski on Metalogic. ©Courtesy of Warsaw Scientific Society, Poland

> In the course of the years 1920–30 investigations were carried out in Warsaw belonging to that part of metamathematics—or better metalogic—which has as its field of study the simplest deductive discipline, namely the sentential calculus.

Why "Metalogic" is better than "Metamathematics"? To answer this question we have to examine what is the meaning of "Metalogic" in the Polish School. It is not that simple because there are two meanings which are entangled.

The first meaning is the most common one and is directly related with other meta-words, in particular "metatheorem". There is a logic system, for example, sentential logic, that must be differentiated from the study of this system, which is *Metasentential Logic*. This expression is explicitly used by Mordechaj Wajsberg (1902–194?) in a paper using this very word in the title: "Beiträge zum Metaaussagenkalkül" (1935) (in English: "Contributions to metasentential logic") [124]. This is part of Metalogic, as well as the study of other logical systems.

If we consider Classical Sentential Logic presented as a so-called Hilbert system, there are some axioms and rules, from which some theorems are derived. For example, $p \rightarrow p$ is such a theorem. This is a theorem of the system. But there are also theorems about this system, for example, the decidability of this system. Such a theorem is called a "metatheorem".

In general we do not make the difference between a system or a theory and the study of this system or theory. Because the two come together. A theory about the physical world, like the theory of relativity is called a "physical theory", not a "metaphysical theory", although this could make sense if we consider that it is about the physical world. And the study of the theory of relativity is also not called a metaphysical theory, in particular because it is not clearly distinguished from the theory itself.

Now let us consider the theory of natural numbers. It is standardly called "number theory" and its objects of study are the natural numbers. And there is something which is called *Peano Arithmetic*, bearing the name of the famous Italian logician Giuseppe Peano (1858–1932). This theory reduces to a small group of axioms, like "every number has a successor", which is formally expressed in First-Order Logic (the main logical system of modern logic) as $\forall x \exists y \; s(x) = y$. The intended meaning of "s" is successor, but this meaning has to be specified by the formal apparatus. Due to the basic framework of First-Order Logic it is a unary function. We have then another axiom stating that this function is injective, expressing the fact that a number cannot have two successors, shaping the concept of successor into immediate successor, etc.

Peano Arithmetic is a theory about the theory of numbers, describing reasoning about these numbers. This is mathematical reasoning. For this reason, Peano Arithmetic can be considered as part of Metamathematics. Now what about the study of Peano Arithmetic? Is it Metametamathematics? Generally it is simply considered as part of Metamathematics, because no clear distinction is made between Peano Arithmetic and the study of it. Let us consider a theorem such as Euclid's theorem, according to which there are infinitely prime numbers. The formalization of this theorem in Peano Arithmetic can be considered as part of Metamathematics, but this formalized theorem is not considered as a metatheorem. Here are some "real" metatheorems about arithmetic:

- The two Gödel's incompleteness theorems, 1932 [54]
- Existence of non-standard models of Peano Arithemtic, Skolem, 1934 [105]
- Relative consistency of arithmetics, Gentzen, 1936 [53]

We can say that these metatheorems are part of Metamathematics because they are results about a system describing mathematical reasoning. We cannot say the same about results about the incompleteness of a physical theory (cf. the work by Newton da Costa and Francisco Doria [39]) or the fact that classical physics can be axiomatized in first-order logic with only universal quantifiers (cf. the work by Rolando Chuaqui and Patrick Suppes [35]). A name, promoted by the very Polish School, that has been used for this kind of research, is "Methodology of Deductive Sciences".

If we consider the decidability of Classical Sentential Logic, we can say that it is part of the Methodology of Deductive Sciences, but can we say that it is part of Metamathematics? Is it a result about a system describing mathematical reasoning? Sentential Logic is a general system describing all kinds of reasoning, close to Stoic Logic (as shown by Łukasiewicz [76]). So what we have is a theorem about a logical system. That is why it makes more sense to consider it as part of Metalogic, rather than as part of Metamathematics. Many results of the Polish School are about Sentential Logic, classical or not, in particular the results of Wajsberg [123], who used *Metalogical Contributions* as the title of a work published in two papers (in 1937 [125] and in 1939 [126]). And if we have systems supposed to describe reasoning about physics or biology, like, respectively, those of Paulette Février (1915–2013) [49] and Joseph Henry Woodger (1894–1981) [132], both good friends of Tarksi, it is better to call the study of these systems "Metalogic" rather than "Metamathematics".

Emil Post, who proved the main and most important metatheorems about Classical Propositional Logic (functional completeness, completeness, decidability, and maximality), simply used the expression "General Theory of Elementary Propositions" (cf. the title of his 1921 paper [88]).[9]

Tarski used "metamathematics" in several papers and, most important, as the title of his book gathering his pre-WW2 papers: *Logic, Semantics, Metamathematics—Papers from 1923 to 1938* [115]. But this was probably due to the strong influence of the Hilbert School which lasted during many years. Tarski's book was published in 1956, only a few years after Kleene's book, which was dominating the market, as they say in the country of Walt Disney. However it is worth pointing out that there is only one occurrence of the word "Metalogic" in [115] (the one we have mentioned above) together with 4 occurrences of the adjective "metalogical".

Later on for the Polish edition of Tarski's work, Jan Zygmunt decided to use "Metalogika" as the subtitle of a book entitled *Logico-Philosophical Papers—Vol 2*, volume sharing many papers with [115] (cf. Fig. 5).

[9] "Sentential Logic" and "Propositional Logic" are both used. The first expression is used by people who want to emphasize, not to say to force, a syntactic or/and linguistic interpretation.

Fig. 5 Polish edition of Tarski's work, Volume 2. ©Courtesy of Polish Scientific Publishers—PWN

> We subtitle this volume *Metalogic* in an effort to sum up in one succinct word what we feel is most distinctive about the range of issues these works address. Tarski himself uses the terms 'metalogic' and 'metalogical' in various senses and contexts. He sometimes speaks of the metalogical conception of truth (and other notions). At other times he speaks of metalogic as a subject of study, or a field of research. He considers sentential calculi and their theories as properly belonging to metalogic. He also puts *Principia Mathematica* in this camp, together with set theory in all its multifarious versions. Tarski uses 'metamathematics' more often than 'metalogic' and 'metamathematical' more often than 'metalogical' [134].[10]

But Zygmunt notes that Tarski used the word Metalogic on a review published in 1938 in the *Journal of Symbolic Logic*, saying that this work, which is Mostowksi's doctoral thesis:

> contains a succession of very valuable and interesting results from the domain of metalogic. As the subject of his research the author has chosen the system of *Principia Mathematica*, based on a simplified theory of types, and enlarged it by adding the axiom of infinity ... However, all the results obtained are, according to the author, applicable also to other kindred formal systems, in particular to the formalized system of Zermelo. [113]

If we prove theorems about propositional logic using mathematics, we can say that we are doing mathematics, mathematics of logic, or mathematical logic (this last expression is ambiguous because it can also be interpreted as "the logic of mathematics"). But what kind of mathematics is it exactly, is it a special mathematical theory? Most of the time Metalogic is developed in an informal way, like standard mathematics. But one may work on that, develop a theory about that,

[10] Translation from Polish courtesy of Robert Purdy—checked and revised by Zygmunt.

this is what people have done in Poland, in particular Alfred Tarski with the theory of consequence operator. Then we go to another sense of Metalogic that will be the main treat of our next section.

2.3 From Metalogic to Universal Logic

In the previous section, we have seen two different and independent uses of the word "Metalogic", one due to Vasiliev, in between the geometrical use of "Metamathematics" and Hilbert's use, another one due to the Polish School, following, and inspired by, Hilbert's Metamathematics. In this section we will see another aspect of the notion of Metalogic emerging from the Polish School and that can be related to Vasiliev's notion. An aspect that also will be a useful bridge between the modern understanding of Metalogic and Schopenhauer's notion of metalogical truth.

After Hilbert's Metamathematics, we have many meta-words, as pointed out by Curry. But the word "meta-axiom" is not common at all. Difficult to find some occurrences of it, if any. On the other hand, it would make sense to call Tarski's axioms for the consequence operator "meta-axioms", if we consider that these axioms are axiomatizing the so-called axiomatic systems (see [15]).

"Axiomatic systems" in logic is an expression used to qualify proof-theoretical systems in which there are lots of axioms and few rules. These systems are also often called "Hilbert systems" because Hilbert used to use them, but he was not the first and the only one. Such systems were also promoted by Whitehead and Russell in *Pincipia Mathematica* [129]. Let us emphasize that a Hilbert system with many axioms and only one rule can equivalently be presented with many rules and few axioms (see, e.g., [90]). And this is not the essential feature of these systems by contrast with other proof-theoretical systems, such as Gentzen's sequent systems. The distinction between Hilbert and Gentzen's systems has to do indeed with Metalogic, since we can say that Gentzen's systems incorporate some metalogical principles at the logical level.[11]

Tarski's axioms for the consequence operator characterize some properties of the notion of logical consequence generated by such "Hilbert axiomatic systems", properties that are nowadays standardly called "reflexivity", "monotonicity", and "transitvity". Curiously these properties are the same as the ones of a consequence relation model-theoretically (or semantically) defined by Tarski in his 1936s paper on logical consequence [110]. In the two cases we can call these properties "meta-properties", considering they are at the level of the infrastructure, part of a general theory of all existing or possible logical systems. They can be presented as follows:

[11] For a presentation of the different kinds of proof-theoretical systems, the relation between them and their metalogical features, see [12].

- $T \vdash a$, when $a \in T$ (Reflexivity)
- If $T \vdash a$ and $T \subseteq U$, then $U \vdash a$ (Monotonicity)
- If $T \vdash a$ and $U, a \vdash b$, then $T, U \vdash b$ (Transitivity).

where "'a" and "b" denote abstract objects intended to represent propositions, "T" and "U" denote sets of such objects, called *theories*, and "$\vdash$" a binary relation between them, called *consequence relation*.

These properties can be interpreted as proof-theoretical meta-axioms or model-theoretical meta-axioms for a consequence relation. Proof-theoretically, "$T \vdash a$" means that there is proof leading from T to a, model-theoretically, that if the propositions of which T is made of are all true according to a given interpretation, then a is also accordingly true. We have used the symbol "$\vdash$", but Tarski was not using it.[12] This symbol has been used in a different way by Frege, Whitehead and Russell. Sometimes the symbol "$\vdash$" is used for proof-theoretical consequence by contrast to the symbol "$\models$" used by Tarski in the 1950s for the notion of model-theoretic consequence. The way we are using "$\vdash$" here is above this difference.

Such properties have been compared, and sometimes identified, with the *structural rules* of Gentzen's sequent systems directly inspired by what Paul Hertz, student of Hilbert, called *Satzsysteme* [59]. Hertz's rules look more like Tarski's axioms, belonging to the meta-level. And the common feature between Tarski and Hertz's approaches is that there is only one binary relation acting on some abstracts objects, no connectives or other logical operators (quantifiers, modalities, etc.) being specified. Tarski later on applied his theory of consequence operator to the study of specific logical systems by mixing the two in particular in his joint paper with Łukasiewicz [77]. Hertz did not do that himself, this was however done by Gentzen who started his research activities by further developing Hertz's ideas. But first of all Gentzen developed a work about Hertz's framework staying at Hertz's general abstract level, proving a general completeness result [51], that can be viewed as establishing the link between the two Tarski's frameworks (but this was done independently of Tarski's work that Gentzen did not know). The next step for Gentzen was to apply Hertz's framework to the study of some particular systems, classical and intuitionistic systems. To do that he incorporated Hertz's rules as rules of his sequent systems, calling them "structural rules" [52].

There are two major ambiguities about the understanding of these structural rules. The first is that although structural rules are clearly distinguished from logical rules about logical operators, the two are at the same level (the same happens with Tarski's theory of consequence operator). The second is that structural rules may be confused with the properties of the consequence relations generated by these rules. Gentzen himself did not make a clear distinction between the two, because he

[12]Moreover Tarski was not considering a relation but a function, an "operator" acting on theories. It seems that for developing his theory he was influenced by the topological work of Kuratowski, with whom he collaborated at some point. The three properties presented here are not the same as, but are equivalent to, the ones of Tarski's consequence operator which look like those of a topological space, see [107].

did not explicitly consider the consequence relations generated by the rules of his sequent systems.

Serious misunderstandings may arise, a typical one is about the cut rule. The cut rule in a sequent system is a rule of the system. Gentzen with his famous *cut-elimination theorem* showed that the sequent system without this rule generates the same system as the system with the cut rule. These two systems generate therefore the same logics, having the same metalogical properties, in particular transitivity holds, transitivity being a property at the meta-level analogous to the cut rule. Gentzen's cut-elimination theorem can be properly understood only by making a distinction between the two things, using two different names "cut" and "transitivity" and two different corresponding symbols.

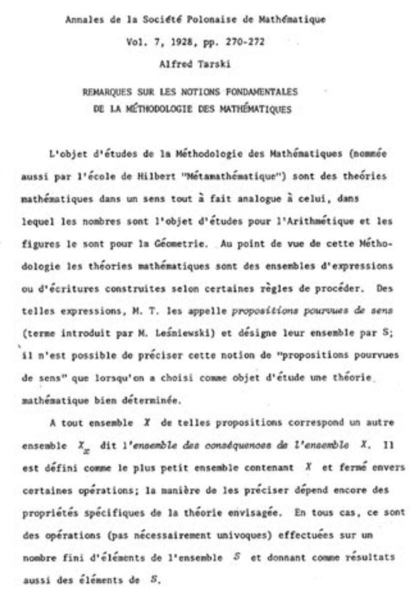
Gentzen's cut rule is taken from Hertz who did not use this terminology, but called it the "Syllogismus" [60], because for him that was a formulation of the Barbara syllogism, the canonical example of syllogism. Cut is indeed the basic mechanism of syllogistic, every syllogism is a cut: the middle term disappears, is "cut". Surprisingly Gentzen showed that it is possible to exclude this mechanism from reasoning. He did that by presenting a system in which there is no cut in any logical rules. The cut mechanism is expressed and isolated in only one rule, the cut rule. Gentzen then showed that the cut rule is redundant, that it is possible to get the same results without using it. This is the consequence of his cut-elimination theorem, which is in fact a true metatheorem, one of the most impressive metatheorems of modern logic, both by the inner quality of its proof (using double recursion at the time when recursion theory was just starting), its philosophical value (seriously challenging Aristotle's syllogistic), its numerous consequences (e.g., decidability), and applications (e.g., relative consistency of arithmetic).

The first publication about the consequence operator is a 1928 abstract in French entitled "Remarques sur les notions fondamentales de la méthodologie des mathématiques" [107] (Fig. 6). Tarski presents the theory of consequence operator as part of Methodology of Mathematics, Metamathematics, or Methodology of Deductive Sciences (see [108, 109]). This last expression, which is the more general, is used in the title of the last paper where he stresses the following:

> For the purpose of investigating each deductive discipline a special metadiscipline should be constructed. The present studies, however, are of a more general character: their aim is to make precise the meaning of a series of important metamathematical concepts which are common to the special metadisciplines, and to establish the fundamental properties of these concepts. [109]

It is clear therefore that Tarski is conscious that he is opening another dimension. One could qualify this as "Metametalogic". But this would be a bit monstrous.

In a project we have started to develop since the beginning of the 1990s [26], we have decided to use the expression "Universal Logic". The choice of this terminology was in particular motivated by the analogy with "Universal Algebra". Universal Algebra is a general theory of algebraic structures. The expression was coined by James Jospeh Sylvester (1814–1897), then used by Alfred North Whitehead (1861–1947), but its actual meaning is due to Garrett Birkhoff (1911–1996).



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Alfred Tarski

REMARQUES SUR LES NOTIONS FONDAMENTALES
DE LA MÉTHODOLOGIE DES MATHÉMATIQUES

L'objet d'études de la Méthodologie des Mathématiques (nommée aussi par l'école de Hilbert "Métamathématique") sont des théories mathématiques dans un sens tout à fait analogue à celui, dans lequel les nombres sont l'objet d'études pour l'Arithmétique et les figures le sont pour la Géometrie. Au point de vue de cette Méthodologie les théories mathématiques sont des ensembles d'expressions ou d'écritures construites selon certaines règles de procéder. Des telles expressions, M. T. les appelle *propositions pourvues de sens* (terme introduit par M. Leśniewski) et désigne leur ensemble par S; il n'est possible de préciser cette notion de "propositions pourvues de sens" que lorsqu'on a choisi comme objet d'étude une théorie mathématique bien déterminée.

A tout ensemble X de telles propositions correspond un autre ensemble X_x dit l'*ensemble des conséquences de l'ensemble* X. Il est défini comme le plus petit ensemble contenant X et fermé envers certaines opérations; la manière de les préciser dépend encore des propriétés spécifiques de la théorie envisagée. En tous cas, ce sont des opérations (pas nécessairement univoques) effectuées sur un nombre fini d'éléments de l'ensemble S et donnant comme résultats aussi des éléments de S.

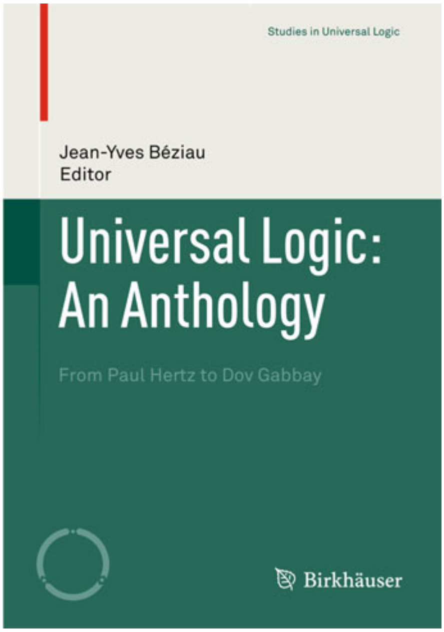


Fig. 6 Tarski's paper included in the Anthology of Universal Logic. ©Courtesy of Warsaw Scientific Society, Poland and Birkäuser, Switzerland

Universal Algebra is a conceptual general framework for developing the study of any algebraic system. It could have been called meta-algebra, since the object of study of Universal Algebra are algebras.

But this proposal did not show up[13] and in some sense the meta way of speaking although it can be clear and meaningful, like in the case of "metatheorem", is not so nice and sometimes confusing due to the most famous meta-word, "Metaphysics", which was accidentally chosen as we have already pointed out. The meaning of the word "Metaphysics" is not clear for two related reasons: the topics dealt with in Aristotle's book are difficult and abstract, the word itself does not express and/or explain what these topics are. Let us emphasize that in this book Aristotle is dealing with the principle of contradiction and we can say that Metalogic is part of Metaphysics in the sense of Aristotle if we consider Metalogic not just as a collection of technical results about logical systems but any examination and discussion surrounding, motivating, justifying, founding these systems.

The fact that "Metaphysics" is quite confusing is the first reason not to use the word "Metalogic". The second reason is that "Metamathematics" is also quite confusing due to the fact that it is much attached to Hilbert who used it in a very particular sense. There are therefore two reasons not to use the word "Metalogic",

[13] Abraham Robinson (1918–1974) however talked about "The metamathematics of algebra" [91].

both related to the "meta" prefix. And then we have a third reason directly connected with this prefix: the idea is to reject the very use of a prefix.

In "Universal Logic", "universal" is not a prefix. It opens a new dimension, a new perspective, which is not only superposition. Semantically speaking the word "universal" is very powerful because it means at the same time unity and generality. However there is an ambiguity with "Universal Logic" since it can be interpreted and/or understood as a universal system of logic, which is an opposite view. This latter view corresponds to the spirit of *Principia Mathematica* or the work of Stanislaw Leśniewski (1886–1939) who was the PhD advisor of Tarski. But despite this inherent ambiguity we were not afraid to choose the expression "Universal Logic", in particular because the meaning of "Universal Algebra" as promoted by Birkhoff is quite clear, established, and well-known, at least among mathematicians.

Moreover Universal Logic is directly inspired by Birkhoff's approach according to which there are no axioms. In both cases we are in the Realm of *Axiomatic Emptiness* (see [18]). We have the following parallel: in Universal Algebra, an *Abstract Algebra* is a set with a family of operators obeying no axioms, $A = \langle \mathbb{A};\, f_i \rangle$, in Universal Logic, an *Abstract Logic* is a set with a consequence relation obeying no axioms, $L = \langle \mathbb{L}; \vdash \rangle$.

It was a way to make a clear demarcation from Tarski's approach to Metalogic which is based on (meta)axioms, and also from Vasiliev who, though he rejected the law contradiction from the sphere of his Metalogic, still considered that it consists of some basic fundamental principles.

In 2012 we published a book entitled *Universal Logic: an Anthology* [29] (Fig. 7) including 15 items, chronologically classified, each one presented and commented by a specialist. Among them the papers by Hertz and Tarski we talked about and Part 6 is about Curry. It includes the two first chapters of his 1952 book *Leçons de logique algébrique* [42], translated in English and commented by his former student Jonathan Seldin. With this book Curry introduced for the first time the expression "Algebraic Logic". Ten years later he published a book which is a kind of extended version of the 1952 one with the title *Foundations of Mathematical Logic* [43]. This expression can also be seen as an alternative way to speak about Metalogic. Curry at the same time that he presents some technical tools, develops a lot of philosophical discussions which can properly be considered as part of Metalogic. The extracts of his French book presented in the Anthology of Universal Logic are in fact rather philosophical. Beside this Part 6 the only philosophical paper in this anthology is the one by Louis Rougier, "The Relativity of Logic" [95], that we will talk about in the third part of our paper after having presented Schopenhauer's ideas.

The aim of the Universal Logic project [26] is not only to develop Metalogic in the wide sense of a general theory of logical systems and structures but also to discuss and develop philosophical ideas related to such kind of theory and the basic concepts of logic. An expression such as "philosophy of logic" is not so good, as other expressions of the type "philosophy of . . . ", because it gives the impression of an afterward, as if philosophy would be comments on an already manufactured product. For this reason better not to use it if we think that philosophy is part of the production, a fundamental element of the conception, which can even be considered as the first stage.[14] There is also the expression "philosophical logic", but its meaning is even more confusing (see [106]) than the one of "mathematical logic" (that can be interpreted in two different non-equivalent ways: *logic of mathematics* and *mathematics of logic*). We are therefore glad to welcome philosophical aspects of logic, including some of historical flavor, under the umbrella "Universal Logic" and that is why it makes sense to have the volume in which the present paper is included in the book series *Studies in Universal Logic*.

3 Schopenhauer's Theory of the Metalogical

As indicated by the title of his dissertation of 1813, *On the Fourfold Root of the Principle of Sufficient Reason* (in German: *Über die vierfache Wurzel des Satzes vom zureichenden Grunde*), Schopenhauer distinguishes four roots of the principle of sufficient reason (hereafter PSR). Here are they:

- PSR of becoming
- PSR of knowing
- PSR of being
- PSR of acting

We present them because it is important to have the general picture but we will not enter here in details for each of them. Our main interest is for the PRS of knowing where the metalogical is located. However it is important to say a few words about the PSR *tout court*, for people who have never heard about it, and also to present and discuss a bit the PRS of being, to have a proper understanding of Schopenhauer's vision of logic, considering its relation with mathematics. This is what we will do in the first section of this second part of our paper.

There are two versions of the essay of Schopenhauer, the original version of 1813 and a revised version in 1847. Further ideas about metalogical truth and logic can be found elsewhere in Schopenhauer's work. Here is the list including abbreviations we will use:

- *On the Fourfold Root of the Principle of Sufficient Reason* (4RP), 1813 and 1847
- *The World as Will and Representation* (WWR), 1818, 1844, and 1859 [100].
- *Parerga and Paraliponema* (PPA), 1851 [101].
- *Handwritten Manuscripts* (HWM), 1913 [102].

And here is a list of the parts of these works especially relevant for our discussion:

- 4RP, Chapt V, §29. PSR of knowing
- 4RP, Chapt V, §30. Logical Truth

[14] Compare with what S.Haack says in the section *Logic, philosophy of logic, metalogic* of her 1978 book.

- 4RP, Chapt V, §32. Transcendental Truth
- 4RP, Chapt V, §33. Metalogical Truth
- 4RP, Chapt V, §34. Reason
- WWR, Vol.1, 1st Book, §9.
- WWR, Vol.1, 1st Book, §10.
- WWR, Vol.1, 1st Book, §15.
- WWR, Vol.1, Appendix: Criticism of the Kantian Philosophy
- WWR, Vol.2, Chpt IX. On Logic in General
- WWR, Vol.2, Chpt X. On the Science of Syllogisms
- WWR, Vol.2, Chpt XIII. On the Methods of Mathematics
- PPA, Vol.2, Chpt II. Logic and Dialectic
- HWM, Berlin lectures, 1820s, §Metalogical truth
- HWM, Eristical Dialectic, 1830s

3.1 The Tricky and Crutchy Euclid

The PRS is not an invention of Schopenhauer, his original contribution is to have distinguished four roots of the PRS. For many people the PRS is strongly connected or due to Gottfried Wilhelm Leibniz (1646–1716). But Schopenhauer in Chapter 2 of 4RP entitled *General survey of the most important views hitherto held concerning the principle of sufficient reason* of about 20 pages has only half a page about Leibniz (§9 of 4RP). Schopenhauer wrote the following: “Leibniz first put forth the principle of reason formally as a fundamental principle of all knowledge and science. He proclaimed it very pompously in many passages in his works, thereby even putting on airs about it, and portraying himself as if he were the first one to discover it; however, he knew nothing further to say about it, except that anything and everything must always have a sufficient reason why it is so and not otherwise, which must have been quite well-known to the world before him.”[15]

Schopenhauer quotes the French formulation of Leibniz of the PRS: “En vertu du principe de la raison suffisante, nous considérons qu’aucun fait ne saurait se trouver vrai ou existant, aucune énonciation véritable, sans qu’il y ait une raison suffisante, pourquoi il en soit ainsi et pas autrement.” Leibniz also uses the Latin formulation *Nihil est sine ratione*, to which Martin Heidegger (1889–1976) gives much importance in his book *The Principle of Reason* (in German: *Der Satz vom Grund* [57]), a book in which Schopenhauer is strangely never mentioned. Schopenhauer does not focus on a specific linguistic formulation of the PRS. Although he considers that Leibniz was the first to put the PRS in the first place,

[15] Maybe Schopenhauer is too harsh with the philosopher known for claiming that we are living in the best of all possible worlds, by contrast to Schopenhauer’s idea, according to which we maybe are in the worst of all possible worlds. For a more neutral assessment of Leibniz on the Principle of Reason see [85].

Schopenhauer traces back the PRS up to Plato, quoting *Philebus* and *Timaeus* where Plato claims that everything which occurs, occurs with a cause, and then criticizes Aristotle and more generally the classical philosophers: “We see that the Ancients still did not attain a clear distinction between the requirement (der Forderung) for a knowledge ground for founding a judgment (eines Erkenntnissgrundes zur Begründung eines Urtheils) and that of a cause for the occurrence of a real event (einer Ursache zum Eintritt eines realen Vorganges)” (last paragraph of §6 of 4RP).

Schopenhauer is indeed the first to make a clear distinction between what we can call the epistemological and ontological versions of the PRS but he does not stop at the level of this simplistic dichotomy. He goes further on with a fourfold distinction. The ontological version is duplicated in two: the PRS of becoming concerning material phenomena (the law of causality) and the PRS of acting concerning human action. And so is duplicated the epistemological version: the PRS of knowing concerning knowledge in general and the PRS of being concerning *a priori* knowledge. The very name “PRS of being” is quite ambiguous and one may rather see it as an ontological version of the PRS. But Schopenhauer is a follower of Kant on the question of the pure *a priori* intuitions. The PRS of being is ruling these intuitions, which according to Kantian philosophy are not reality, but conditions of apprehension of reality.

As explained in a paper I wrote many years ago [8], Schopenhauer is very critical to the use of logic in mathematics, because as a follower of Kant he believes in the grounding of mathematics in the pure *a priori* intuitions of space and time, on which geometry and arithmetics are, according to the Kantian perspective, respectively based (cf. §37, §38, and §39 of 4RP). Schopenhauer goes a step further than Kant by strongly insisting that mathematical truth therefore does not need logic. In particular he scapegoats Euclid:

> The principle of non-contradiction compels us to admit that everything Euclid demonstrates is true: but we do not find out why it is so. We have almost the same uncomfortable sensation people feel after a conjuring trick, and in fact most of Euclid’s proofs are strikingly similar to tricks. The truth almost always emerges through a back door, the accidental result of some peripheral fact. An apagogic proof often closes every door in turn, leaving open only one, through which we are forced simply because it is the only way to go ... by our lights the Euclidean method can only appear as a brilliant piece of perversity (eine sehr glänzende Verkehrtheit) (WWR, §15).

Both Brouwer (cf. [47, 72]) and Wittgenstein (cf. Chapter 14 of [80]) have been strongly influenced by Schopenhauer’s views of mathematics. But let us emphasize that for Schopenhauer logic is not leading us in the wrong direction, at the end we arrive at the same location. The point is that its “method” is an intricate path. Schopenhauer describes this with the following nice river metaphor:

> Euclid’s logical way of treating mathematics is a useless precaution, a crutch for sound legs ... it is like a night traveler who, mistaking a clear and solid path for water, takes care not to tread on it and instead walks along the bumpy ground beside it, happy all the while to keep to the edge of the supposed water (WWR, §15).

Schopenhauer says that Euclid uses “intuitive evidentness to support only what he absolutely had to (the axioms), supporting everything else with inference ... In

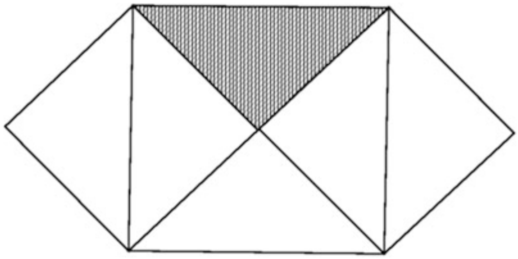


Fig. 7 Pictorial proof of the Pythagorean theorem

mathematics, according to Euclid's treatment, the axioms are the only indemonstrable first principles, and all demonstrations are in gradation strictly subordinated to them." But according to Schopenhauer the theorems can also be supported by evidence and they do not need to be derived from the axioms: "every proposition again begins a new spatial construction. In itself, this is independent of the previous constructions, and can actually be known from itself, quite independently of them, in the pure intuition of space, in which even the most complicated construction is just as directly evident as the axiom is" (WWR, §15).

The idea of Schopenhauer is that mathematical reasoning, whether in geometry or about numbers, does not need to be based on logic and that it is better to have mathematical proofs directly based on what is really supporting their truth, the pure *a priori* intuitions of space and time. He concludes Chapter 6 of 4PRS, devoted to the PRS of being, by saying: "I cannot refrain from again providing a figure which has already been given in other places, the mere appearance of which, without further discussion provides twenty times the conviction of the truth of the Pythagorean theorem than Euclid's mousetrap proof" and by providing the diagram reproduced in (Fig. 7).

A criticism that can be addressed to Schopenhauer is that visual reasoning, on the one hand, does not necessarily reduce to intuition of space, on the other hand, does not only apply to space, it can be applied to anything. Reasoning involving colors, for example, can be developed (see [24], and for a general perspective see the multi-volume book *Proofs without Words* by Roger Nelsen [83]). This is not against Schopenhauer's examples of visual proofs, but it seriously challenges the space-to-space basis of his neo-Kantian philosophy of mathematics.

3.2 Metalogical Truths: Where They Are and What They Are

The PRS of knowing is about truth. Schopenhauer presents it as follows: "truth is the relation (Beziehung) of a judgment to something out of it, its sufficient reason" (4RP §39). There are four types of truth according to the kind of reason on which a judgment is based. The reason may be:

- a judgment (*formal or logical truth*)
- a sensible representation (*empirical truth*)

- a pure intuition (*transcendental or metaphysical truth*)
- the formal conditions of thought (*metalogical truth*).

Schopenhauer defines metalogical truth as follows: “a judgment may be founded on the formal conditions of all thinking, which are contained in the Reason; and in this case its truth is of a kind which seems to me best defined as metalogical truth” (4RP §33). In this essay Schopenhauer gives some formulations of these metalogical truths that we will present later on. Here we just list them with the names he gives to them in WWR (at the beginning of §10):

- identity
- non-contradiction
- the excluded middle
- PRS of knowing.

In (Fig. 8) is presented a general picture describing the place of metalogical truths within the framework of the PRS.

As we can see Schopenhauer does not use “Metalogic” as a substantive, but as an adjective applied to truth. Metalogical is a quality of truth. We do not find the word “Metalogik” in his writings and he does not consider that there is a field of study corresponding to that.

Nevertheless we can talk about Schopenhauer’s “Theory of the Metalogical.” By that we mean his views on metalogical truths, where they are and what they are and the general philosophy explaining/justifying that. We have worked up to now on their position within the general Schopenhauer’s PRS framework. If we want to have a better understanding of what they are we have to go upstream and downstream.

- Upstream: what are the basis of the metalogical truths, how we know them, why they are four, and why they are these four?
- Downstream: what does arise from these four metalogical truths, in which sense are we using them, what are their relations with reasoning?

The reason why there are exactly four metalogical truths seems a bit artificial. As well shown by Fig. 9, Schopenhauer has a general systematic 4-scheme. His main book *The World as Will and Representation* also has 4 parts (which do not correspond to the 4 roots of the PRS). We can say that Schopenhauer is often following a kind of 4-ideology, not to say 4-mysticism (see [97]), by contrast to Georg Wilhelm Friedrich Hegel (1770–1831), or Peirce later on, praising the 3. The first three metalogical truths are not due to Schopenhauer, we will come back to their formulations and meanings in the third part of our paper. To add the PRS of knowing as a fourth metalogical truth is a bit weird: the PRS of knowing is the fourth part of itself. This is somewhat circular, like a dog biting its tail. Anyway it allows to square everything . . . Let us emphasize that Schopenhauer considers metalogical truths as judgments, and that therefore the PRS of knowing is a judgment about judgments, a “metajudgment”. Schopenhauer is not using this word but this would be a reason to “metafy” it.

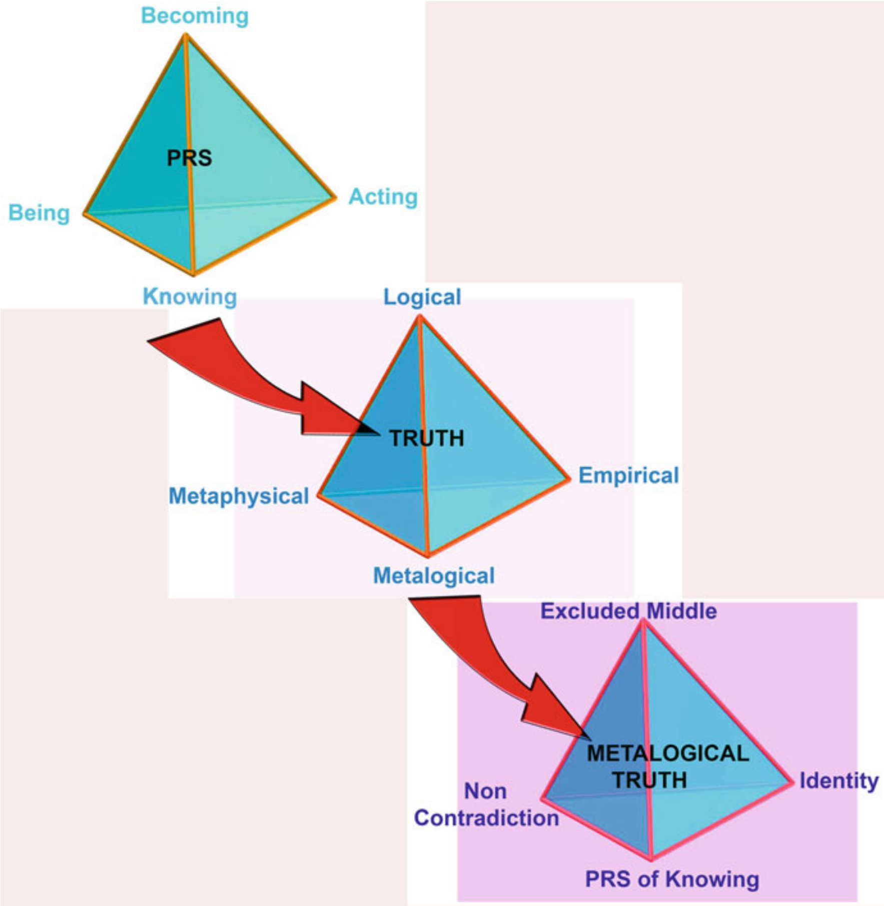


Fig. 8 The location of metalogical truth

Later on, in the supplements of WWR, Schopenhauer proposed a reduction of these four metalogical truths to only two: on the one hand, the PRS of knowing, on the other hand, the condensation of the three other ones in only one that he called "the law of excluded middle", but that would be better called "law of dichotomy," to avoid the confusion with the previous formulation he gave of the law of excluded middle and because it better fits with what it really is (WWR, V2, §9).

Schopenhauer says that metalogical truths "were discovered long ago by induction" and that:

> it is by means of a kind of reflection which I am inclined to call Reason's self-examination, that we know that these judgments express the conditions of all thinking, and therefore have these conditions for their reason. For, by the fruitlessness of its endeavors to think in opposition to these laws, our Reason acknowledges them to be the conditions of all possible thinking: we then find out, that it is just as impossible to think in opposition to them, as it is

> to move the members of our body in a contrary direction to their joints. If it were possible for the subject to know itself, these laws would be known to us *immediately*, and we should not need to try experiments with them on objects, i.e. representations" (4PR §33).

Since the justification of these metalogical truths is an important feature, let us also quote another formulation by Schopenhauer, similar but slightly different, that can be found in his *Handwritten Manuscript*, corresponding to his lectures in Berlin in the 1820s:

> The reason for these judgments is the consciousness of reason that only according to these rules one can think. However, reason does not come to the realization of this directly, but only through a self-examination, through a reflection on what can be thought (not experienced) at all. In this way it recognizes that it tries in vain to think against those laws; e.g. it cannot think that a circle is triangular, or a piece of wood of being iron: thus it recognizes those laws as the conditions of the possibility of all thinking. It is thus the same as we learn about the movements possible to the body (just as we learn about the properties of every other object) with the help of experiments. If the subject could recognize itself (what is, however, impossible), we would recognize those laws directly and not only with the help of experiments on objects, i.e. representations (HDW, 1913, p. 268).[16]

What is not clear and not detailed by Schopenhauer is how these four metalogical truths precisely emerged. The situation of the PRS of knowing seems a bit different from the three other metalogical truths. Considering the principle of non-contradiction, it rather seems that this principle was formulated by induction, not in the sense presented by Schopenhauer, i.e., that we cannot reason in a different way, but because everything in nature was seen as based on dichotomy (cf. the Pythagorean table of opposite). Then this natural phenomenon was transformed into an artificial device, classical negation, which became the main tool to develop reasoning. But we can indeed experiment without much problem other tools (see [28]).

Considering the downstream aspect, we can compare Schopenhauer with Aristotle. There are three different aspects of Aristotle's logic which are quite independent (in parentheses, their location in Aristotle's corpus):

- syllogistic, which is a system with rules describing and/or prescribing how to rightly reason (*Prior and Posterior Analytics*)
- criticism and description of false ways of reasoning (*Sophistical Refutations*)
- presentation and defense of the principle of non-contradiction (*Metaphysics*).

The relation between syllogistic and the principle of non-contradiction is clear neither with Aristotle, nor with Schopenhauer. In both cases the justification of the principle of non-contradiction is not based on any syllogistic argument, and, on the other hand, syllogistic also does not seem to depend on this principle. It has even been argued that the rejection of the principle of non-contradiction is compatible with Aristotle's syllogistic (see [55]). However in the theory of the square of opposition the notion of contradiction is used to classify and organize

[16]English translation courtesy of Jens Lemanski. No English translation of this *Handwritten Manuscript* has yet been published.

the four kinds of propositions that are used in syllogistic. This would be a reason to consider the principle of non-contradiction as a metalogical principle in Aristotelian logic. But if we consider the square of opposition as part of the metatheory of syllogistic, it encompasses also two other notions of opposition (contrariety and subcontrariety) as well as subalternation.

Schopenhauer did not present a new logical system. He supports syllogistic and try to improve it, in particular by use of diagrams, further developing the works of Gottfried Ploucquet (1716–1790), Jean-Henri Lambert (1728–1777), and Euler, that he knew (WWR1, §9). He points out and praises the fundamental character of syllogism (corresponding to the cut phenomenon) that he describes as creative. He does that with two metaphors, one chemical, the other electrical:

> From *one* proposition there cannot result more than what is already to be found therein, that is to say, more than it itself states for the exhaustive comprehension of this meaning. But from *two* propositions, if they are syllogistically connected to premisses, more can result than is to be found in each of them taken separately; just as a body that is a chemical compound displays properties that do not belong to any of its constituent elements considered separately. On this rests the value of syllogisms (PPA V2 §24).

> The voltaic pile may be regarded as a sensible image of the syllogism. Its point of indifference, at the centre, represents the middle, which holds together the two premisses, and by virtue of which they have the power of yielding a conclusion. The two different conceptions, on the other hand, which are really what is to be compared, are represented by the two opposite poles of the pile. Only because these are brought together by means of their two conducting wires, which represent the copulas of the two judgments, is the spark emitted upon their contact—the new light of the conclusion (WWR §10).

But Schopenhauer claims: “we no more need logic to avoid false reasoning than we need its rules to help us reason correctly; and even the most learned logician completely puts it aside when actually thinking” (WWR §9). He makes a comparison with the two other corners of the basic triangle of the pyramid of philosophy made of truth, goodness, and beauty, saying:

> We do not have to burden our memory with all the rules, since logic can only be of theoretical interest and never of practical use for philosophy. It may be said that logic is to rational thought as the figured bass is to music, or, more loosely, as ethics is to virtue or aesthetics to art; but it should be borne in mind that no one has ever become an artist by studying aesthetics or achieved nobility of character by studying ethics, that people composed music both beautifully and correctly long before Rameau and that we do not need to have mastered the system of figured bass to recognize dissonance. In just the same way, we do not need to know logic to avoid being deceived by sophisms (WWR1 §9).

Nevertheless Schopenhauer also has described 38 stratagems (in German: Kunstgriffe) which can be compared with the 13th fallacies described by Aristotle in his famous *Sophistical Refutations* (Fig. 9). It is part of an essay written in the 1830s which was not concluded during Schopenhauer’s lifetime. It was published only after his death, sometimes presented in an ambiguous way with invented controversial titles or subtitles, such as *The Art of Being Right—38th ways to win when you are defect*. In §26 of Volume 2 of PPA, entitled *On Logic and Dialectic*, Schopenhauer talks about this essay emphasizing the distinction between the form and matter of the sophisms:

Fig. 9 Schopenhauer won the sophistic game against Aristotle

> The tricks, dodges, and chicanery, to which they resort in order to be right in the end, are so numerous and manifold and recur so regularly that some years ago I made them the subject of my own reflection and directed my attention to their purely formal element after I had perceived that, however varied the subjects of discussion and the person taking part therein, the same identical tricks and dodges always came back and were very easily to recognize. This led me at this time to the idea of clearly separating the merely formal part of these tricks and dodges from the material and of displaying it, so to speak, as a neat anatomical specimen. I therefore collected all the dishonest tricks so frequently occurring in arguments and clearly presented each of them in its characteristic setting, illustrated by examples and given a name of its own. Finally, I added the means to be used against them, as a kind of guard against their thrusts; and from this was developed a formal *Eristical Dialectic* (PPA, V2, §26).

Due to this comment the essay was posthumously baptized in German *Eristische Dialektik* and in English *Controversial Dialectic* or *Eristical Dialectic* (for a recent study on this essay, see [86]).

But what prevails is Schopenhauer's critical view of the weak, not to say null, utility of the practical aspect of logic, as a tool for reasoning rightly and recognizing wrong reasoning. This leads him to make the following consideration:

> The teaching of logic should not take the form so much of a science oriented towards practice, and should not merely set down unembellished rules for the correct conversion of judgments and inferences etc.; instead it should be directed towards making known the essence of reason and concepts, and towards a detailed consideration of the principle of sufficient reason of knowing. After all, logic is merely a paraphrase of this principle, and indeed only for cases in which the ground for a judgment's truth is neither empirical nor metaphysical but rather logical or metalogical. In addition to the principle of sufficient reason of knowing, we must introduce three more fundamental laws of thought or judgements of metalogical truth that are just as closely related; the whole technique of reason emerges little by little from these (WWR1 §9).

If we reduce a course of logic to Schopenhauer's Theory of the Metalogical it would be a quite short course, because he does not say much about the metalogical truths. After having located the metalogical truths in his system, as one of the four kinds of truths, itself part of the fourth root of the PRS, and stated that they are 4, Schopenhauer does not go much further. As he himself writes: "I attribute metalogical truth to these laws because they come purely from reason and are not to be explained any further" (WWR Appendix on Kant). No comments!

On the other hand, in the above quote Schopenhauer suggests to include in the teaching of logic, the knowledge of the *essence of reason and concepts* that he puts side by side with the PRS of knowing and other metalogical truths. It seems reasonable to indeed consider that Schopenhauer's Theory of the Metalogical does not reduce to the metalogical truths but includes his ideas on reason and concepts which are directly connected with them. This is what we will explore in the next section.

3.3 The Femininity and Triviality of Metalogical Truths

The essence of reason is not necessarily something easy to catch. Especially if we consider that reason is the essence of human beings, a basic idea promoted by the Ancient Greeks that Schopenhauer fully embraces despite his fondness for dogs, music, and the Buddha. He considers this idea as truly universal:

> The unanimous view of every age and people is that these various and far-reaching manifestations all spring from a common principle, from a special mental power that distinguishes humans from animals and that is called *Vernunft*, *Logos*, *Ratio*. (WWR1 §8)

He starts by saying that reason is easily recognized and qualified by everybody:

> Everyone also knows very well how to recognize the manifestations of this faculty, and can tell what is rational and what is irrational; everyone can tell where reason emerges in contrast to the other human capacities and characteristics.

and that even philosophers agree about it:

> The philosophers of all ages also generally agree with this common knowledge of reason, and in addition emphasize several of its especially important manifestations: mastery of affects and passions, the ability to make inferences and to lay down universal principles, even those that can be ascertained prior to any experience, etc.

but Schopenhauer adds:

> However, all their explanations of the true essence of reason are wavering, vaguely delineated, long-winded, and lack both unity and focus (WWR1 §8).

Let us see if the poodle philosopher himself has a clear and distinct explanation of the essence of reason.

Schopenhauer writes: "Reason is of a feminine nature: it can give only after it has received. On its own, it possesses nothing but the empty forms of its own operation. Completely pure rational cognition gives us in fact only four things, the

Fig. 10 Chaumette symbolizing Reason at Notre-Dame in 1793

very metalogical truths" (WWR, §10). From this we can infer that metalogical truths are the feminine structure of our thought.

We can illustrate Schopenhauer's metaphor with a 1878 painting by Charles-Louis Müller (1815–1892), entitled "La fête de la Raison dans Notre-Dame de Paris le 10 novembre 1793" (Fig. 10).

We have chosen this painting because Reason is herein represented, not to say advertised, by a woman, but also because this celebration of reason took place in the most famous church of Paris (Notre-Dame) and was supposed to replace the erroneous religious cult. During this ceremony, the girl, named Chaumette Momoro, made the following declamation:

> Vous l'avez vu, citoyens législateurs, le fanatisme a lâché prise et a abandonné la place qu'il occupait à la Raison, à la justice, à la vérité; ses yeux louches n'ont pu soutenir l'éclat de la lumière, il s'est enfui. Nous nous sommes emparés des temples qu'il nous abandonnait, nous les avons régénérés. Aujourd'hui tout le peuple de Paris s'est transporté sous les voûtes gothiques, frappées si longtemps de la voix de l'erreur, et qui, pour la première fois, ont retenté du cri de la vérité ([1, p. 301]).

Schopenhauer, who claimed that "a man cannot serve two masters, so it is either reason or the scriptures" (PPA, V2, Ch15), could have been a follower of Chaumette (in 1793 however he was only 5 years old). The choice of this painting is also to emphasize the problematic rationalism of Schopenhauer, grounded on emptiness.

The feministic view of reason promoted by Schopenhauer is compatible with Aristotle's views on logic. It fits well with the Stagirite's hylemorphism, which at the logical level corresponds to the distinction between the form of a reasoning and its matter. The form of reasoning is described by Aristotle with his syllogistic figures and the matter is the possible interpretations of the subject and the predicate which can be anything fitting within this dual categorization (Socrates and other individuals being excluded).[17] As we have emphasized in previous papers [16, 22] this formal character of logic is one aspect of Aristotelian logic that is still predominating in modern logic. What has changed is the form of the form not the essential formal nature of logic. Nobody indeed seems shocked by the use of the expression "formal logic" in modern logic, expression due to Kant who famously claimed that Aristotle's logic will never change (Preface of the 2nd edition of the *Critique of Pure Reason*). And in fact its hylemorphic character has survived.

In case of Schopenhauer we can also see a difference on the form with Aristotle. He considers that the form is characterized by metalogical truths, not syllogisms. According to him metalogic truths are "formal conditions of thought". But, on the other hand, Schopenhauer says:

> The essence of thought proper, i.e. of judgment and inference, can be presented by combining conceptual spheres according to the spatial schema described above, and all the rules of judgment and inference can be derived from this schema by construction. The only practical use that can be made of logic is to prove that an opponent in debate is using intentional sophistries (not making genuine logical mistakes) by pointing out their technical names (WWR1 §9).

The articulation between this combinatorial essence of thought and the feminine metalogical one is not explicitly explained by Schopenhauer. We can see that all these essences are formal. We can argue that the combinatorial essence is produced or justified by the metalogical one (our paper "Opposition and order" goes in that direction [23]). Schopenhauer was not able to properly explain the phenomenon as he himself recognizes: "I am unable to say what the ultimate basis is for this exact analogy between the relations of concepts and those of spatial figures. But it is in any event a fortunate circumstance for logic that the very possibility of all conceptual relationships can, in the following way, be presented intuitively and *a priori* by means of such figures ... All combinations of concepts may be reduced to these cases" (WWR1 §9). This is here another example of use of spatial devices to reason about something else than geometrical space. And this is especially important, because it is an application of spatial devices to describe and explain reasoning.

The square of opposition is another geometrical figure which is very famous on the history of logic. This square is at a deep metalogical level if we consider that it gives an explanation of what are the different categorical propositions by classifying them, using in particular the notion of contradiction to do that, and that moreover it explains what contradiction is by, on the one hand, distinguishing it from other oppositions, and, on the other hand, showing how it works, giving an example of application. By contrast, spheres of concepts are devices to describe and/or practice reasoning, although they also have a theoretical systematic aspect emphasized by

[17] About the distinction between form and matter in syllogistic, see [32–34].

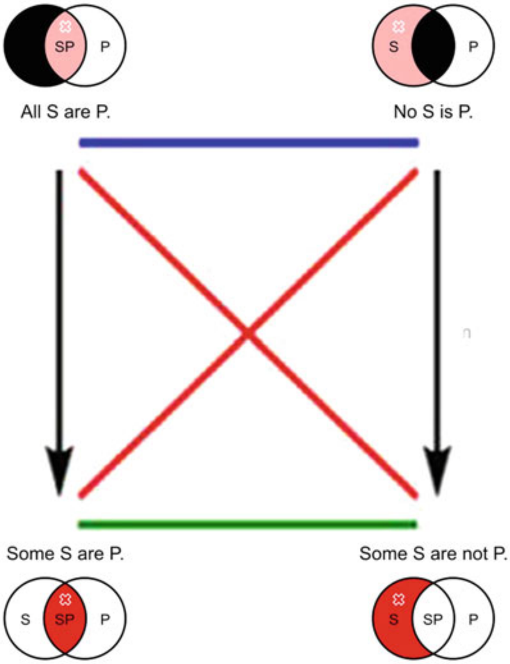


Fig. 11 Circling the square of opposition

Schopenhauer, explaining all the possibilities, providing the general picture. The two kinds of figures, square and circle, can be mixed together in the diagram presented in Fig. 12, an improvement of the one by Tilman Plesk, which is presently the main top illustration on the entry about the square on Wikipedia. We have replaced the inside square by a colored square, where red represents contradiction, blue contrariety, green subcontrariety, and the black arrows, subalternation, using hence colors additionally to spatial representation, putting in activity another one of our senses (about the square and colors see [21] and [66], and for alternatives of Fig. 11, see [4]).

Schopenhauer also pretends to explain the relation between reason, conceptualization, and understanding, but his explanation is rather strange and complicated. He says: "Reason has only *one* function: the formation of concepts, and all the phenomena mentioned above can be very easily and in fact trivially explained on the basis on this simple function: it is what distinguishes the life of humans from that of animals; and everything that has been, at any time or place, described as rational or irrational points to the application or non-application of this function" (WWR1, §8 p.62).

Concepts are abstracts but they are not abstraction of reality, they are more like reflects of the reality of phenomena. They are representations of representations. The basic representations are intuitive representations ruled by the law of causality (PRS of becoming) which is connected to understanding (cf. WWR1 §8 and 4RP §26 and

Fig. 12 The moon: the kingdom of logic with concepts of reason and Boole's Crater

§27). Schopenhauer reinforces his feminal metaphor by contrasting the femininity of reason to the masculinity of understanding: "…its nature is feminine; it only conceives, but does not generate. It is not by mere chance that the Reason is feminine in all Latin, as well as Teutonic, languages; whereas the Understanding is in variably masculine" (4RS, pp. 136–137d). This metaphor which is duplicated in a Sun-Moon metaphor, the intuitive representation being equated to the Sun, and the concepts to the Moon (Fig. 12): "As if from the direct light of the sun into the borrowed reflection of the moon, we now pass from immediate, intuitive representation (which presents only itself and is its own warrant) into reflection, the abstract, discursive concepts of reason (which derive their entire content only from and in relation to this intuitive cognition) (WWR1, §8).

What is difficult to understand in the philosophy of Schopenhauer is not only the articulation of the PRS of becoming (corresponding to understanding) with the PRS of knowing (corresponding to reasoning) but also the articulation between these two and the PRS of being (corresponding to mathematics). For example, on the basis on these three principles how Schopenhauer would explain how work a physical theory making use of mathematics like Newton's theory of gravitation or Einstein's theory of relativity based on Non-Euclidean geometry? A central point of Schopenhauer's theory on which he himself insists is that there are not four different separated PRS but one PRS with four roots as indicated by the very title of his essay. That is nice but it is not completely clear how everything is articulated especially considering some claims of Schopenhauer which look a bit paradoxical, at least as they are phrased, such as "understanding, considered in itself, is unreasonable" (WWR1, §6), or his "phenomenal" claim according to which science has nothing to do with the inner essence of the world (WWR1, §7).

4 Formulation, Axiomatization, Interaction, Reflection

Schopenhauer had the idea that the fundamental basis of reasoning that he claimed was fairly described by what he characterized as the four metalogic truths was predetermined and fixed and therefore would never change. We will examine in the third part of this paper if this idea makes sense in the light of the development and evolution of modern logic, in particular by making the distinction between reasoning, its formulation and description. For conducting this analysis we will take advantage of the clarification about modern Metalogic made in the first part of our paper.

4.1 Reformulations, Semiotical Changes, and Mathematical Interaction

According to *Encyclopaedia Britannica*, the *laws of thought* are "traditionally, the three fundamental laws of logic: (1) the law of contradiction, (2) the law of excluded middle (or third), and (3) the principle of identity." [46] Schopenhauer also calls them laws of thought, but he additionally characterizes them as "metalogical truths". And the original contribution of Schopenhauer is also to have considered, as we have seen, a fourth law, the PRS of knowing, according to which: "truth is the ratio of a judgment to something out of it". He uses the same name for these laws but did not formulate them in the same way as in *Encyclopaedia Britannica*. Here are his formulations:

- (identity) A subject is equal to the sum total of its predicates, or $a = a$.
- (contradiction) No predicate can be attributed and denied to a subject at the same time, or $a = -a = o$.
- (excluded middle) One of two opposite, contradictory predicates, must belong to every subject.

This is rather unsatisfactory. From a modern point of view this is rather weird, not to say false. But let us note that in *Encyclopaedia Britannica* these laws are also formulated in a unsatisfactory way: (1) and (2) are expressed in the language of modern propositional logic and (3) of first-order logic. Considering that the entry is about the *traditional* laws of thought, this is an anachronism. And we have a disparity, because (1) and (2) are expressed in propositional logic and (3) in first-order logic, moreover (1) and (2) are called "laws" and (3) a "principle".

We will not here develop a critical analysis of Schopenhauer's formulations because, on the one hand, this would require a better understanding based on a historical and philological research, comparing formulations of these laws by other authors of the period (in the line of the work of Anna-Sophie Heinemann included in the present volume [58]), and, on the other hand, this is not so important for our present discussion.

The first important point is that these formulations are not proper original inventions of Schopenhauer, it is at best a reformulation of something which was in the air at his time. Schopenhauer indeed does not emphasize or claim any personal contribution. Before presenting the above formulations, he writes: “There are only four metalogically true judgments of this sort, which were discovered long ago by induction, and called the laws of all thinking; although entire uniformity of opinion as to their expression and even as to their number has not yet been arrived at, whereas all agree perfectly as to what they are on the whole meant to indicate” (PRS 33). Writing this he does not even consider that the fourth metalogical truth is due to himself, as it is indeed the case, as well as the idea to put it at the same level as the three other metalogical truths.

The second important point is that Schopenhauer was not interested to question or furthermore investigate these formulations. His position can be understood on the basis that according to him, although we do not have a direct access to them (we know them only by self-reflection), they are obvious, they do not need further explanation, things cannot be otherwise, it is like the way we use our foots for walking. Let us go on and ahead with this walking metaphor:

- (WALK-1) We are walking in a certain way, we see how it is by practicing it.
- (WALK-2) It is not possible to better walk, to walk in a different manner.
- (WALK-3) It is not useful to further describe how we are walking.

Schopenhauer’s position is in accordance with these three points. This also the case up to a certain point of the positions of René Descartes (1596–1650) and Blaise Pascal (1623–1662). But both French philosophers think that syllogistic is not only useless but also misleading—Schopenhauer is not so critical—and they both present new methodologies, that we have summarized in two tables in [17]. Descartes’s methodology is very general and quite far from any logical principles or systems (although exhaustion can be viewed as an extended version of the principle of excluded middle). Pascal’s methodology is the promotion of the axiomatic method, it has strongly inspired Tarski and has some connections with our present discussion on Metalogic, we will deal with this in the next section. Schopenhauer’s position is much more conservative. Nevertheless he has two original contributions we have talked about in the second part of our paper: to reduce the three traditional laws of thought to only one, to consider five possible basic positions between spheres of concepts. His ideas are interesting but not presented in a very satisfactory way and moreover he does not develop them much.

Kant had the idea that Aristotle’s science of logic was perfect. Maybe he gave more value to it than Descartes, Pascal, and Schopenhauer. Kant famously claimed in the preface of the second edition of the *Critique of Pure Reason* that the science of logic is firmly and definitively established, therefore a dead science. Schopenhauer’s position is in the same vein as the one of Kant, despite some light differences.

Ironically, few years after Kant’s morbid declaration (1787), was born a man known as George Boole, who developed a line of work, when Schopenhauer was still alive, that revolutionized the science of logic and from which modern logic arose. It is worth to stress however that Boole was not a revolutionary by birth, nature, or behavior (this was more the case of his wife, a forerunner of homeopathy, who by experimenting it on her husband caused his premature death, throwing cold water on him, after he went back home strongly wetted by a heavy Irish rain). So how to explain that Boole changed the history of logic?

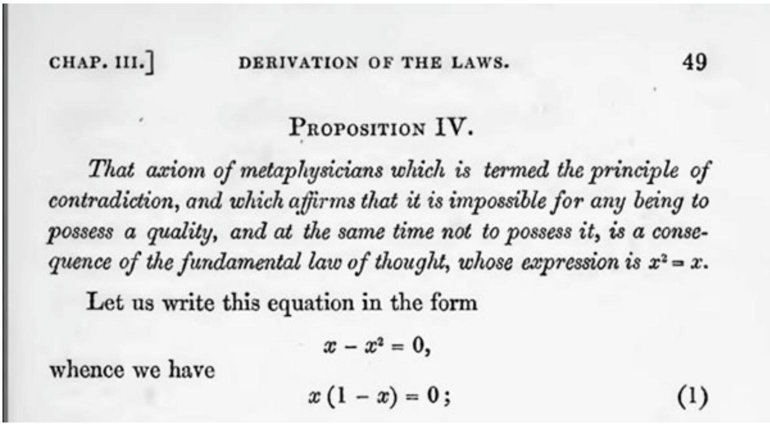
We can understand that through an important distinction. Modern logic has challenged the traditional laws of thought in two different ways, not to be confused:

- The traditional laws of thought are not necessarily fundamental laws.
- The traditional laws of thought do no not always hold.

We will discuss the first point here and the second in the next section. The first point means that a law, like the law of non-contradiction, can be derived from some more fundamental laws (but still is valid). This point was made by Boole. This is one of the reasons he can be considered as the father of modern logic. And this is related to one of the characteristics of the new methodology he promoted, i.e., to use mathematical tools and symbolisms to develop logic, which is also typical of modern logic.

Boole considered that the fundamental law of thought is $x^2 = x$. He claimed that in his famous 1954 book the *Laws of Thought*, a claim supported by a "proof" that it is possible to derive from it the law of contradiction using the symbolism he is promoting (cf. Fig. 13). We have examined this point in details in a recent paper entitled "Is the principle of non-contradiction a consequence of $x^2 = x$?" [27].

Few years later (1880) Peirce showed that it is possible to derive/define all the 16 connectives of classical propositional logic with only one. He did that in a semantical way, using a method similar to what is now presented as the bivalent semantics of propositional logic. Later on Jean Nicod (1893–1924) showed that it is possible to axiomatize classical propositional logic with this sole connective. Wajsberg gave another version of such axiomatization and Henry Sheffer (1882–



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PROPOSITION IV.

That axiom of metaphysicians which is termed the principle of contradiction, and which affirms that it is impossible for any being to possess a quality, and at the same time not to possess it, is a consequence of the fundamental law of thought, whose expression is $x^2 = x$.

Let us write this equation in the form

$$x - x^2 = 0,$$

whence we have

$$x(1 - x) = 0; \quad (1)$$

Fig. 13 Boole's symbolic proof that the principle of contradiction is derivable from $x^2 = x$, the fundamental law of thought for him

1964) independently rediscovered this connective. Whithead, Russell, Wittgenstein [130] knew this result and thought it was important, and in fact it is (for details about his connective see [81]). Not only the principle of contradiction and excluded middle can be derived from this connective, but also all other principles governing classical connectives, in particular the one corresponding to what Boole considered as the fundamental law of thought. Peirce therefore went a step ahead of Boole.

But what is important is that in the cases both of Boole and Peirce there is a crucial semiotical change in the very formulation of logic (and this is also the case later on with a third famous father of modern logic, Frege). Boole was much influenced for that by the British School of *Symbolic Algebra*. He did perform the semiotical change of considering operations on signs rather than on their values due to this school but made a fundamental new step by considering algebraic operations on signs having values other than numbers or quantities (see [44, 45, 104, 128]). Peirce's contributions to semiotic are well-known. He indeed is considered as the father of semiotics and had the idea that logic is part of semiotics.

Many people have the tendency to reduce the development and emergence of modern logic to a phenomenon of *formalization* of logic. But as we have pointed out in [16] the expression "formal logic" is highly ambiguous having 5 different meanings. So it is better to say that the changes who led to modern logic were due to new *formulations*, based on semiotical changes. And to point out that these semiotical changes cannot be characterized or reduced to a "mathematization of logic". Modern logic was inspired by mathematics but also changed mathematics, what we have is a real interaction.

Results like those of Boole and Peirce can be considered as metatheorems, part of Metalogic, but all the semiotical aspect of their work leading to a new conception of logic also can be considered as part of Metalogic. Schopenhauer's Theory of the Metalogical is far from all this but at the same time it is interesting to see that he is lightly touching this dimension, on the one hand, by promoting spherical diagrams, which are in the spirit of Euler and partly resemble the so-called "Venn diagrams", developed by John Venn, (to whom is attributed the expression "symbolic logic" [121] and who is considered as an important figure of modern logic), on the other hand, by using some mathematical symbols to express the law of identity and contradiction. He is however not doing that for the two other metalogical truths: the law of excluded middle and the PRS of knowing. Although the first three metalogical truths discussed by Schopenhauer have been reformulated and relocated (they are not necessarily at fundamental first positions) in modern logic, they still are there at the logical or metalogical level. On the other hand, the PRS of knowing, an original idea of Schopenhauer, has completely disappeared. Heinrich Scholz (1884–1956), a good friend of Łukasiewicz, wrote a book on the history of logic [98] where he claimed that this is because it cannot be formalized. Against this view Newton da Costa presented a formalization of it using modal propositional logic with quantification on propositions (see [7, 9]). This idea has not yet been systematically developed. Doing so could lead to an interesting new logical theory inspired by Schopenhauer's Theory of the Metalogical.

The criticism and rejection of the traditional laws of thought were not initiated by Boole, Peirce, or Frege. From this point of view we can say that they were not fundamentally against Schopenhauer's metalogic truths. On the other hand, if we consider that semiotics is a fundamental part of Metalogic, we can say that their metalogical views are quite different from those of Schopenhauer.

4.2 The Modern Axiomatic Methodology

In the previous section we have seen how the law of non-contradiction was relativized by Boole and Peirce, being derivable from other laws, this being done by the reformulation of the basic logical framework. In this section we will see another point: the rejection of the traditional laws of logic, in particular the very law of non-contradiction. One of the first to perform this rejection was the Russian logician Nicolai Vasiliev. We already talked about him in Sect. 2.2. pointing out he was using the word "Metalogic". His relativization has to be understood through his conception of Metalogic but also through his promotion of the axiomatic method. When we are talking about the axiomatic method, we are talking about the new axiomatic method, which was in particular promoted from Kazan, the city of Vasiliev's family (his father was a friend of Nikolai Lobatchevski). Vasiliev considered that the principle of non-contradiction can be treated as the parallel postulate:

> Non-Euclidean geometry is a geometry without the 5th postulate, [that is] without the so-called axiom of parallels. Non-Aristotelian logic is a logic without the law of contradiction. It is worth mentioning here that it was precisely non-Euclidean geometry that has served us as a model for the construction of non-Aristotelian logic ([119, p. 128]).

According to the modern axiomatic methodology, a very important tool of the modern world (Fig. 14), axioms are relative in two complementary and non-exclusive senses:

(A) An axiom can be replaced by another one.
(B) An axiom is not considered as an absolute truth.

(A) is quite specific of modern axiomatic and was emphasized by Tarski (see [111] and chapter 6 of [112]). It is related to, but not only, the formulation of axioms with different primitive terms, primitive terms that are also therefore not absolutely primitive. A theory can be axiomatized in different ways. A Boolean algebra can, for example, be seen as an idempotent ring or as a distributive complemented lattice. And a given axiom can have many different equivalent formulations, a typical example is the one of the axiom of choice.

(B) is not completely new. Plato had the idea that mathematics was based on hypotheses rather than on absolute truths. The search for truth was for him the task of philosophy, therefore a higher science (cf. Book 6 of *Republic*).

What is the most important is that modern axiomatic was applied to logic, and this led to the relativization of logical axioms not only in the sense of (A) but also

Fig. 14 The axiomatic method: a winning strategy

of (B). There are different logical systems starting with different axioms leading to different theorems. The study of the different logical systems is part of Metalogic in the Polish sense, as we have explained in Sect. 2.2. To play with axioms of logic was a favorite game in the 1920s. In Poland they liked the idea to reduce everything to one axiom. An other idea was to develop independent axiomatizations, in the sense that in an axiomatic system one axiom cannot be derived from the other ones. Paul Bernays in his PhD ([5, 6]) showed that the system of axioms for propositional logic in the *Principia Mathematica* was no independent and provided an independent one, showing its independence using three-valued matrices. Independence is a typical metalogical concept or/and result. The use of many-valued matrices to prove such a metatheorem is part of Metalogic, in a more essential way that the use of such a device to develop a logical system (a logical matrix, or set of logical matrices, can be seen itself as a logical system).

If we consider an axiomatic theory, let us say Peano Arithmetic (PA), and a theorem of this theory, let us say the infinity of prime numbers (IP), according to Schopenhauer's terminology, this is a *logical truth*, because its reason consists in other judgments, ultimately PA axioms. Although this fits with the spirit of modern logic, the language used is not the same. In modern logic it is said that a theorem of PA logically follows from the axioms of PA, but not that it is a logical truth.

In modern logic the expression "logical truth" is attributed to truths which are not depending on non-logical axioms, such as the axioms of PA. We say that a proposition is a *logical truth* if it is true in virtue of logic itself. Alternatively it is synonymously said that such a proposition is *logically valid* or that it is a *tautology*.

The infinity of prime numbers is not a logical truth in the sense of modern logic, but the formula PA' $\rightarrow$ IP is.[18] This fact can be used to argue that all results of mathematics are nothing else than tautologies or formal truths. This is a position defended in particular by philosophers who know very few about mathematics, who do not know what is the thrill of proving a theorem by being directly in touch with beautiful mathematical objects, not to say creatures. Working mathematicians who are living for and from such thrills do not support a tautological view of mathematics (although there are some exceptions such as Saunders MacLane, see [79]). They would certainly be more sympathetic to Schopenhauer's philosophy of mathematics. In fact despite the fact that Arithmetic has finally been axiomatized after several thousands of year (by contrast to geometry which was axiomatized right at the start), generally mathematicians working in number theory are not interested to show how their theorems can be derived from PA.

In an axiomatic system for a mathematical theory like Peano Arithmetic, the rules are supposed to be orchestrated (specified and described) by logic. Now if we have an axiomatic system for logic, is logic itself orchestrating the rules and how? Is this Metalogic?

If we consider a tautology such as $p \vee \neg p$, can we say that it is a metalogical truth in the sense of Schopenhauer? One could claim that, arguing it is a modern formulation of what Schopenhauer calls the law of excluded middle, one of his four metalogical truths. We can consider a Hilbert proof-theoretical system in which the formula $p \vee \neg p$ is an axiom. From this axiom and another axiom expressing the commutativity of disjunction, it is possible to prove that $\neg p \vee p$ is a theorem (not a metatheorem). We can go the other way round: start with $\neg p \vee p$ as an axiom and derive $p \vee \neg p$ as a theorem. This illustrated the point (A). In this example there is nothing dramatic because the two formulas are quite the same and both can be called "excluded middle". The situation is different if we derive $p \vee \neg p$ from $((\neg p \rightarrow q) \wedge (\neg p \rightarrow \neg q)) \rightarrow p$, a formula corresponding to the strong version of the reduction to the absurd. It is indeed possible to do so not using other axioms or rules for negation but only principles ruling implication, disjunction, and conjunction. It would not make really sense to call this formula, then an axiom, the excluded middle. From it, it is also possible to deduce in fact various formulas that can be interpreted as formulations of the law of non-contradiction. As we have seen, Schopenhauer had a similar proposal, deriving the law of excluded middle from a more fundamental law from which he says the law of non-contradiction can also be derived. For Schopenhauer however this does not change the essential value of the law of excluded middle, it is still a metalogical truth.

From the viewpoint of modern logic, even if we agree that $p \vee \neg p$ is a formula having a real axiomatic value, not just a formula lost in the infinite jungle of all formulas, it would be a bit strange to call it a metalogical truth. The reason why is that it is awkward to apply the metaterminology to axioms of a logical system, because

[18] PA' is here the conjunction of the propositions of a finite subtheory of PA, from which IP is a consequence.

they are part of the system in the same way as the theorems. The metaterminology is reserved for things about the system. For example, dedicability, which is considered as a metatheorem. That is the reason why it does not make sense to talk about meta-axiom, unless we develop a full metatheory such as Tarski's consequence operator which can be seen as axiomatizing the properties of a consequence relation generated by a Hilbert system.

This is an important difference between Schopenhauer's Theory of Metalogic and the modern one, because Schopenhauer is not going that high as a theory of consequence. The second important difference is that in modern logic we have on the (B) side the rejection of the law of excluded middle, an easy game, which is in fact facilitated by the axiomatic method applied to logic but also to its metatheory. First of all we can construct a Hilbert proof-theoretical system in which $p \vee \neg p$ is neither an axiom, nor a theorem, the canonical example being Heyting's system of intuitionistic logic. In this system in which $p \vee \neg p$ is not an axiom, to prove that it is also not a theorem is a metatheorem. This metatheorem can be proved in various ways in particular using logical matrices.

The intuitionistic system of logic can also be generated by a Gentzenian sequent system. What is surprising is that the standard system presented by Gentzen for intuitionistic logic has the same logical rules as the system for classical logic, in particular the same rules for negation. The difference is at the level of the structure of the system, not at the level of the structural rules, but at the level of external determinations: the sequents being not the same as the classical ones, having only one formula on the right. As we have described the situation in a previous paper (see the table *The Architecture of Sequent Systems* in [13]), in sequent systems the structural principles can be divided into internal ones (structural rules) and external ones. It would not make sense to both call them metalogical principles because there are at two different levels, and the meta prefix contains the idea of differentiation of levels. But for course all this corresponds to the field of Metalogic that we indeed prefer to call *Universal Logic* as emphasized in Sect. 2.3. This change of terminology is also important to stress that Metalogic does not reduce to an axiomatic game, that the foundation of logic, if any, is much more conceptual and semiotical.

4.3 Multi-Level Analysis and Productive Self-Reflection

For Schopenhauer metalogical truths are not immediately and directly perceptible, nevertheless they are obvious. Louis Rougier (1889–1982), promoter of the Vienna Circle, in an interesting book with a beautiful poetic title *Les Paralogismes du Rationalisme* (*Paralogisms of Rationalism*) [94] published in 1920 criticized rationalism based on some obvious truths like "the whole is bigger than the part," one of his favorite targets being Leibniz. These considerations and seeing himself later on the development of modern logic with many non-classical systems led him to a spectacular, not to say dramatic, claim (cf. Fig. 15): "Avec la découverte du

Fig. 15 Shall we burn Reason with her four metalogical truths?

caractère conventionnel et relatif de la Logique, l'esprit humain a brûlé sa dernière idole" (in English: With the discovery of the conventional and relative character of logic, human spirit has burnt his last idol) [96]. But, as we have said in a previous paper [19], at the end Rougier defends a wishy-washy scientism, contrasting with his smashy declaration.

More interesting was the behavior of one of the main leaders of the Lvov-Warsaw School, namely Jan Łukasiewicz. In the prehistory of this school (1910), he wrote a book in which he precisely analyzes and criticizes Aristotle's arguments supporting the principle of non-contradiction [74]. Such an approach was not motivated by an ideology according to which contradiction is the basis of everything but by a rational inquiry. In an appendix of this same book, Łukasiewicz presents the ideas of Ernst Schröder (1841–1902) about logic. According to Jan Woleński [131], this is the first presentation of "formal logic" in the circle that will become one of the most important schools of modern logic. Later on Łukasiewicz was led to construct a formal system of logic not rejecting the principle of non-contradiction (this was done in Poland much later—1948—by Staniłsaw Jaśkowski (1906–1965) [65], for reasons having nothing to do with Łukasiewicz's book), but rejecting the principle of excluded middle [75]. And this was not done in a non-Aristotelian perspective, on the contrary, this was done in view of supporting Aristotle's views on future contingents. Łukasiewicz's logic is both a three-valued logic and a modal logic.

As we have seen in Sect. 2.2., Metalogic *stricto sensu* is the study of some logical systems. But we can consider that the philosophical analysis of basic laws of logic is part of Metalogic *lato sensu*, as well as the creation of new logical systems generated by this analysis, systems developed by a methodology which itself is part of Metalogic *stricto sensu*, whether it is the use of logical matrices, sequent systems, or possible worlds.

Łukasiewicz's three-valued logic is not against Aristotle, but it goes a step further by, on the one hand, giving a better *understanding* of contingency and the possibility to go beyond the truth-falsity dichotomy, on the other hand, providing *techniques* with some useful applications.

Now let us examine the law of non-contradiction, also considered by Schopenhauer as a metalogical truth. If we consider the theory of the square of opposition, we could say that Aristotle was not absolutely defending this law, since in the square, among the three notions of opposition, there is subcontrariety, according to which two propositions can be true together and opposed. But that would be an anachronical and false view, because Aristotle explicitly says that he does not consider subcontrariety as an opposition (cf. *Prior Analytics*, 63b21-30). The square of opposition with 3 oppositions was firmly established only later on, in particular by Apuleius and Boethius (see [37]). Nevertheless Aristotle introduced the basic distinction which led to the square, the distinction between contrariety and contradiction. The introduction of contrariety next to contradiction according to which two propositions can be false together can be seen at the same time as a rejection of the excluded middle and as a relativization of the notion of opposition, not reducible anymore to contradiction. This indeed can even be interpreted as a relativization of the notion of contradiction and the related principle of non-contradiction.

As we have pointed out in a previous paper [14], it is possible to establish a correspondence between the 3 notions of opposition of the square of opposition, contradiction, contrariety, and subcontrariety and the 3 kinds of negation, respectively, classical, paracomplete, and paraconsistent negations. This does not mean that all aspects of negation are already inside the square, but the square is a general picture.

The understanding of the law of non-contradiction can and has been developed in different ways in modern logic. There are various formulations both syntactical and semantical. And what is very interesting is the study of the relation between this law and other properties of negation. It is possible to put all the properties of classical negation into one axiom, the strong version of the reduction to the absurd, from which everything can spring, not only the law of non-contradiction and excluded middle, but also the *exfalso sequitur quod libet*, all versions of contraposition, etc. This is nice, but what also is nice is the complete deconstruction of this very single axiom in many pieces and the relations between these different pieces (see [10]). To do that we do not have to take a position, to believe or not that the law of non-contradiction is absolutely true. It is indeed better to carry on these metalogical investigations in a neutral and objective way.

And it is better to consider that these investigations are part of "Universal Logic" rather than "Metalogic". First because the properties of negation are at different levels: a logical level, like a property of negation expressed by a formula such as $\neg(p \wedge \neg p)$, or at the metalogical level, like the replacement theorem, according which, for example, $\neg(p \wedge q)$ is logically equivalent to $\neg(q \wedge p)$. It is a bit confusing to call Metalogic the study of the relation between logical and metalogical properties. To call it "Metametalogic" would, on the one hand, not be very nice and, on the other hand, would not solve the problem, because we may need to go to a further meta-level. The second reason to choose "Universal Logic" is that one of the central ideas beyond Universal Logic is a systematic comparative study of all logical systems, the examination of the different properties of negation being a natural part to this study. And a third reason is that Universal Logic does not reduce to mathematical or/and formal studies of the properties of logical systems (and/or logical operators such as negation), there is also a philosophical dimension. For example, in the case of negation the idea is to simultaneously study the technical properties of negation, their interpretations, and meanings. We can then really see if the law of non-contradiction makes sense or not and if it is possible to reason in a coherent way without it or with only part of it (see [28]).

A logical system in which there is a negation not obeying the full law of non-contradiction is called a "Paraconsistent Logic" and such a negation is called a "Paraconsistent Negation". Newton da Costa (1929–20??), who chose this terminology [38], started to work on this topic motivated by Russell's paradox, according to which the *principle of abstraction* leads to a contradiction. The principle of abstraction states that every property determines a set. It can be seen as an axiom or a fundamental law of thought. This was not done by Schopenhauer or other "traditional" logicians, who did not think of it as a principle but rather as a mechanism of conceptualization. The obviousness of this principle of abstraction can be seen as higher as the one of the principle of non-contradiction. One may then want to reject the principle of non-contradiction if this allows to save the principle of abstraction. Unfortunately this does not work in an easy and simple way due to Curry's paradox [41], which is a version of Russell's paradox using only some basic properties of implication.

In modern logic, logical systems without negation have been studied, the most well-known being *positive propositional logic*. In some sense we can say that these systems reject the law of non-contradiction, since they are not even involving negation. And we can also say that in the case of a metalogical system like Tarski's original theory of consequence in which at the first stage no connectives at all are involved.

Someone may say: that is very fine! but which logic are you using to do all that? Certainly all these investigations cannot be packed in one big logical system. They are not carried on in one system. We can consider various systems reflecting them and reflect about these systems, *ad infinitum*...

We can agree with Schopenhauer that we do not immediately know/perceive the laws of reason, and we can even go further saying that even when our reason is put in action they do not fully unveiled. Someone may think that we are more pessimistic than the king of pessimism. In some sense it is true, but we can see also a beauty in that: the unveiling is possible and pleasant, and this is an infinite pleasure, because it never ends, there is no final understanding.

Would it be interesting to face the very essence of reason depicted as the metalogical truths formulated by Schopenhauer or in another way? If we have seen it, so what? Or: what then can we do?

Someone may look at her face in a mirror but that would be a mistake to think that she then knows who she is. Self-knowledge is not that easy. This face in the mirror is just one of her aspect. And it is also not by seeing her whole body naked in the mirror that she will reach complete full-fledged self-knowledge of herself.

We do not only have to unveil, we also need to act or, better, to interact. In the case of logic, we have to reason about reasoning.[19] By doing so we have a better understanding of what reasoning is and we further develop reasoning, getting higher.

Acknowledgements and Memories

I started to be interested by Schopenhauer when I was 20 years old, in particular by reading two books by Clément Rosset (1939–2018): *Schopenhauer, philosophe de l'absurde* [92] and *L'Esthétique de Schopenhauer* [93]. I started then to read most of the works of Schopenhauer and up to now this is the philosopher I have read the most and who I think is one of the greatest philosophers of all time. I have been interested in all the aspects of his philosophy (religion, sexuality, language, etc.) which is indeed about everything, like a true philosophy must be.

After defending a Master Thesis on Plato's cave under the supervision of Sarah Kofman in 1988 at the University of Paris 1 (Panthéon-Sorbonne), I was seriously thinking of doing a PhD on Schopenhauer with Clément Rosset. This did not happen because, on the one hand, Rosset was professor in Nice and I was in Paris (about one thousand kilometers by road) and, on the other hand, because I started to concentrate more and more on logic.

But the following years when doing at the same time a PhD in philosophical logic and a PhD in mathematical logic I wrote four papers on Schopenhauer: one on suicide [11], dedicated to Sarah Kofman who committed suicide on the date of Nietzsche's 150th birthday, October 14, 1994, one on Schopenhauer's criticism of the use of logic in mathematics [8], two on the principle of sufficient reason, related to the proposal of formalization of this principle in quantified propositional modal logic by one of my advisors: Newton da Costa ([7] and [9]). This second one is an extended abstract of a talk presented at the *38th Conference of History of Logic*, November 17–18, 1992, Kraków, Poland.

In 1992–1993 I spent 1 year and a half in Poland (for details see [25]) and I remember that when there I read the recently published biography of Wittgenstein by Ray Monk [82] and was pleased to learn that Schopenhauer was the philosopher that Wittgenstein had read the most, extensively reading it when a teenager and re-

[19] Roy Cook says: "Metalogic can be captured, loosely, by the slogan *reasoning about reasoning*", we agree with him but we do not reduce reasoning about reasoning to metalogic as he describes it, i.e., the "mathematical study of formal systems that are intended to capture correct reasoning." [36, p. 188].

reading it after having written *Tractatus*, before the start of his second period where he developed ideas on philosophy of mathematics influenced by Schopenhauer.

After all these years I am glad to be back to Schopenhauer and I would like to thank Jens Lemanski who invited me to take part to the event he organized at the University of Hagen, December, 7–8, 2017, *Mathematics, Logic and Language in Schopenhauer*, and to contribute to this volume. Moreover Jens made many comments on a first draft of this paper, useful for its improvement.

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Part III
Mathematics

Schopenhauer and the Mathematical Intuition as the Foundation of Geometry

Marco Segala

Abstract Schopenhauer did not write extensively on mathematics, but he discussed the subject in almost all of his works. His thesis about the superiority of intuition in establishing the truth of geometrical theorems became a battle against the traditional demonstrative procedure in geometry. Commentators have generally provided internal readings of Schopenhauer's texts on mathematics but have neglected their context.

This paper examines Schopenhauer's philosophy of mathematics by discussing its relationship with both his views on the acquisition of knowledge and his familiarity with the contemporary British discussions of mathematics. An overview of his ideas on the primacy of intuition in both mathematics and its teaching is the basis of this inquiry into the connection of those ideas with both his conception of the role of mathematics in natural philosophy and his encounter with the 1830s British texts on mathematics, which he quoted in the second volume of *The World as Will and Representation*. By making him aware that Euclidean geometry required a thorough scrutiny of its foundations—notwithstanding its undisputed reputation—these texts contributed to the hitherto unappreciated modifications in his mathematical considerations.

Schopenhauer participated in an early phase of the debate on the foundations of geometry by taking a fresh look at intuition: not only as an alternative to demonstration, but also as the ground of truth and certainty in the Euclidean system.

Keywords Schopenhauer · British philosophy · Intuition · Philosophy of mathematics · Foundations of geometry

Mathematics Subject Classification (2020) Primary 03-03, Secondary 01A55, 03A99, 00A35

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1 Introduction

Schopenhauer did not write extensively on mathematics, but he discussed the subject in almost all of his works, from the 1813 Dissertation *On the Fourfold Root of the Principle of Sufficient Reason* until *Parerga and Paralipomena* (1851). His thesis about the superiority of intuition in establishing the truth of geometrical theorems is not unique in the history of philosophy but is certainly noticeable. It battled against the traditional demonstrative procedure in geometry—"as akin to someone cutting off his legs so that he can go on crutches" [WI, § 15, p. 95/83]—that leads "to the obvious detriment of the science" [WI, § 15, p. 95/83] and breaks the unity of mathematics: "arithmetic and algebra are not taken up with the kind of proofs that fill geometry; rather, their whole content simply amounts to an abbreviated way of counting" [WI, §15, p. 101/90].

Interpreters and scholars have generally provided internal readings of Schopenhauer's texts,[1] with the exception of François Rostand—who traces similar views on the importance of intuition in mathematics back to Descartes, Locke, Pascal, Malebranche, Leibniz and recalls Kant [43]. But the philosopher's provocative stance has also called forth resolute response, as reminded by Jens Lemanski: criticism—especially by mathematicians—around 1900 [32, pp. 330–331] and appreciation—in the second half of the twentieth century—of the view of intuition in geometry either as an alternative approach to demonstration or as an essential pedagogical instrument [32, pp. 331–333].

Yet, what still lacks in the analysis of Schopenhauer's views on mathematics is an attention to their context—with respect to both Schopenhauer's conception of philosophical knowledge and his familiarity with contemporary discussions of mathematics. A severe judgement like Cajori's ("Schopenhauer attacked mainly the logic of mathematics as found in Euclid. As a critique of the logic as used by Euclid the attack is childish and has no value for us")[2] is based on an inadequate appreciation of Schopenhauer's inquiry into the role of mathematics in philosophical and scientific knowledge—an inquiry that derived from a thoughtful assessment of contemporary discussions, and not just from the internal exigencies of his philosophy. Generally, commentators do not delve into the role that Abraham Gotthelf Kästner's approach to the "Parallelenproblem"—and Kant's reception of it—played in Schopenhauer's criticism of Euclid[3]; they do not assess Schopenhauer's divergence from Herbart and Fries in qualifying the importance of mathematics in metaphysics and philosophy of nature[4]; they neglect the importance of the 1830s British debate on mathematics, despite Schopenhauer's reference to it. It is not even mentioned that Schopenhauer was no stranger to mathematics' new

[1] See [4, 38, 41], [2, pp. 60–63].

[2] [13, p. 371]. See also p. 368: "his criticism is focused directly upon questions of logic, of mode of argumentation and of sufficiency of proof".

[3] See [39, pp. 141–153]. On Kant's philosophy of mathematics and its context, see [19, 29, 34].

[4] On the affinities and the important differences between Herbart and Fries, see [8].

course of the nineteenth century, with the affirmation of non-geometrical analysis—and yet his library included a book of Ernst Gottfried Fischer (cf. [17]), his professor of physics at the University of Berlin in the winter semester 1812–1813, introducing a logico-philosophical interpretation of analysis that sustained and encouraged the Lagrangian approach.[5]

This paper analyses Schopenhauer's philosophy of mathematics by discussing its relationship with both his views on the acquisition of knowledge and his reading of mathematical-related publications—focusing on the intellectual context provided by the British discussions. After an overview of his ideas on the primacy of intuition in mathematics—based on the Dissertation and *The World as Will and Representation*—the second section explores how those ideas were connected to his conception of the role of mathematics in natural philosophy. The third section deals with the 1830s British texts dedicated to mathematics that Schopenhauer read and quoted in the second volume of *The World as Will and Representation*. It appears that those publications contributed to hitherto unappreciated modifications in Schopenhauer's mathematical consideration: on the one hand, he expressed negative judgements on mathematical formalism and mathematical-physics that are not present in the works preceding *Parerga and Paralipomena*; on the other hand, his treatment of intuition in mathematics developed in a new form. As argued in section four, he appreciated that the authors of his British readings were debating on the very foundations of the Euclidean geometry—and not only on the "Parallelenproblem"—and he developed the notion that intuition could have been the answer to their questions.

It is generally maintained that Schopenhauer's philosophical theses did not change, if not marginally, after their first version in the system of 1819. This paper takes care to underscore mathematical-related ideas and contents that changed over time—as shown in his publications but even referring to the manuscripts when it is relevant. It aims to demonstrate that Schopenhauer participated in an early phase of the debate on the foundations of geometry by taking a fresh look at intuition: not only as an alternative to demonstration—something that was clearly unpopular among the mathematicians—but also as the ground of truth and certainty in the Euclidean system.

2 Intuition in Mathematics

Schopenhauer's philosophy of mathematics was mainly focused on geometry. His criticism started from the psychological observation that the logical method of proof in geometry provides "the conviction that the demonstrated proposition is true, but in no way does one see why what the proposition asserts is as it is" [Diss, § 40,

[5]See [HNV, p. 285]. About Fischer's non-peripherical role in the analytic movement in Germany, see [44, p. 562].

p. 135]. He blamed "the Euclidean method" for this separation of the *what* from the *why* that lets us "know only the former, not the latter" [WI, § 15, p. 98/86]. The consequence was that Euclid's system provided a conceptual knowledge, like that of medical theories: "a mere empirical and non-scientific knowledge" [VorI, p. 457].

A significant consequence was the challenge to the distinction between axioms and theorems in Euclid's *Elements*. It was probably connected to the question of the axiomatic nature of the fifth postulate: admitting that "the axioms themselves are no more immediately evident than any other geometrical theorems; they are simply less complicated because they have less content" [WI, § 15, p. 100/89] was a simple solution of the "Parallelenproblem". He sustained that "every theorem introduces a new spatial construction that is in itself independent of its predecessors" and can be demonstrated "through pure spatial intuition, in which even the most involved construction is actually as immediately evident as an axiom" [WI, § 14, p. 88/75]. On the contrary, the Euclidean method required that theorems "are proven logically, that is, by presupposing the axioms and then by means of consistency with the assumptions made in a theorem or with a prior theorem, or by means of the inconsistency of the negation of a theorem with the assumptions, with the axioms, with prior theorems or even with itself" [WI, § 15, p. 100/89].

Another important dissent concerned the use of the *reductio ad absurdum* in demonstrations, for it is the principle of non-contradiction that obliges to accept a conclusion—not the content of the theorem: "the truth almost always emerges through a back door, the accidental result of some peripheral fact. An apagogic proof often closes every door in turn, leaving open only one, through which we are forced simply because it is the only way to go" [§ 15, p. 96/84]. Thirty years later, in the 1847 edition of *On the Fourfold Root of the Principle of Sufficient Reason* [G, § 39, p. 139], his denunciation of the demonstrative method as blind and forced was expressed by the metaphorical designation of Euclid's proof of the Pythagorean theorem as a "mousetrap"—an image with a certain appeal.

Schopenhauer first published his views in his 1813 Dissertation *On the Fourfold Root of the Principle of Sufficient Reason*. He established his theses on the premise that mathematics—traditionally articulated in arithmetic and geometry—pertains to space and time as intuited a priori, "just as the infinite extension and infinite divisibility of space and time are objects only of pure intuition and are foreign to empirical intuition" [Diss, § 36, p. 130]. Succession and position define the relations within, respectively, portions of time and space; they "are intelligible to us simply and solely by means of pure, a priori intuition"—never by concepts. The law governing those relations is the *principle of sufficient reason of being* and the geometrical example of "the connection between the sides and angles of a triangle" shows that it "is completely different both from that between cause and effect and from that between cognitive ground and consequence" [Diss, § 37, p. 131].

According to these notions, he defined arithmetic and geometry. As it conveys the "nexus of the parts of time", the former "is the basis of all counting" and "teaches absolutely nothing but methodical abbreviations of counting" [Diss, §

39, p. 133]. The latter is intuitive, non-conceptual "insight" into "the nexus of the positions of the parts of space"; this brings to the famous notion that "every geometrical proposition would have to be reduced to this intuition, and the proof would merely consist in clearly bringing out the nexus whose intuition is at issue" [Diss, § 40, p. 133]. The long § 40 of the 1813 Dissertation develops these ideas by analysing the intuitive nature of Euclid's 12 axioms as distinguished from the demonstrative character of the theorems. Demonstrations compel to accept the truth of theorems, but "thus, the logical truth, not the transcendental truth of the theorem, is demonstrated" [Diss, § 40, p. 135]. The former "produces mere conviction (*convictio*), not insight (*cognitio*)" and "leaves behind an unpleasant feeling" [Diss, § 40, p. 135], while "the ground of being of a geometric proposition recognized through intuition gives satisfaction" [Diss, § 40, p. 136].

To substantiate his point, he offered alternative, intuitive demonstrations of Euclid's 6th and 16th propositions and concluded his exploration with a caveat: "through all of this I have in no way proposed a new method of mathematical demonstration, no more than my proof will take the place of Euclid's" [Diss, § 40, p. 138]. More modestly he intended to underline how the lack of insight and satisfaction in demonstrative geometry might contribute to disliking mathematics.

In the first edition of *The World as Will and Representation* (1819) Schopenhauer revisited the discrepancy between immediate, intuitive truth and "truth that is grounded in proof" [WI, § 14, p. 89/77] and refined his notions on mathematics within a wider discourse. He emphasised the epistemic value of "feeling" geometrical truths by drawings [WI, § 11] and explained that intuition, as an immediate apprehension of truth, is more convincing than reasoning. Once again, however, a caveat clarifies that even if not immediately connected to truth, nonetheless abstraction and demonstration are necessary for precise communication and reliable application of knowledge: "in pure intuition we are perfectly acquainted with the essence and lawlike nature of a parabola, a hyperbola or a spiral. [. . .] Differential calculus does not really extend our cognition of curves in any way. [. . .] But it does change the kind of cognition we have: it converts intuitive cognition into an abstract cognition that is so rich in consequences for practical application" [WI, § 12, p. 78/63]. Arithmetic can really benefit from conceptualisation because numbers "can be expressed in abstract concepts that correspond exactly to them" [WI, § 12, p. 79/64]. It is not the case of geometry, where abstract cognition cannot precisely express spatial relations: it is easier to *see* "how the cosine decreases as the sine increases" [WI, § 12, p. 79/64] than to explain it conceptually. Schopenhauer's thesis is that geometry must "be translated" into numbers "if it is to be communicable, precisely determined, and applicable in practice" [WI, § 12, p. 79/64]. But such a translation is unnatural: the three dimensions of space must be expressed by numbers, which conceptualise the single dimension of time. Schopenhauer commented: "how the single dimension of time must suffer, as it were, to reproduce the three dimensions of space" [WI, § 12, p. 79/65]. The conclusion derived from these premises is that "a Euclidean proof, or an arithmetic solution to a spatial problem" [WI, § 12, p. 80/66] cannot acquiesce the mind looking for *real* comprehension.[6]

In 1819 Schopenhauer was able to elaborate a radical philosophy of geometry where theorems are nothing more than complex axioms, the *reductio ad absurdum* should be banned, and the logical demonstration is judged as useless or, worse, detrimental. He stated not only that "every truth discovered through inferences and communicated through proofs could also, somehow, have been recognized directly, without inferences or proofs" [WI, § 14, p. 91/78] but also that mathematics could gain from such a radical change in perspective: "abandoning the prejudice that a proven truth is at all preferable to one that we have intuitive cognition of" can lead to "an improved method in mathematics" [WI, § 15, p. 99/87]. Geometry

> never relies on the stilted march of a logical proof, since such a proof always misses the point and is usually soon forgotten without affecting anyone's conviction; we could even dispense with proof entirely and geometry would remain just as evident because it is quite independent of such proof, which only ever demonstrates something that we were already completely convinced of beforehand by a different kind of cognition. So logical proof is like a cowardly soldier who inflicts another wound on the corpse of an enemy already killed by someone else, but then boasts of finishing him off [WI, § 15, p. 102/90–91].

To strengthen his point, Schopenhauer introduced a visual demonstration of Pythagoras' theorem [WI, § 15, p. 98/87] and recalled that an implicit confirmation of his theses could be found in Kant's doctrine of space and time in the Transcendental Aesthetic of *Critique of pure reason*. According to his reading, Kant

> did not finish his train of thought, since he did not reject the whole Euclidean method of demonstration, even after saying [. . .] that all geometric knowledge is immediately evident in intuition. It is quite remarkable that even one of his opponents, and in fact the most astute of them all, G. E. Schulze (Critique of Theoretical Philosophy, II, 241), drew the conclusion that Kant's doctrine would give rise to an entirely different treatment of geometry than the usual one. He meant this to be an apagogic proof against Kant, but in fact he unwittingly began a war against the Euclidean method [WI, Appendix, pp. 465–6/519].

In the following years, Schopenhauer reiterated some aspects of his views on the primacy of intuition. "On the method of mathematics", chapter 13 of the second volume of *The World as Will and Representation* (1844), made explicit the criticality of the "Parallelenproblem" while emphasising the necessity of reform in the standard model of demonstration. The chapter on mathematics in the second edition of *On the Fourfold Root of the Principle of Sufficient Reason* (1847) explicitly referred to § 15 of the *The World as Will and Representation*, reproduced the visual demonstration of Pythagoras' theorem, and introduced the notion of "mousetrap" [G, § 39].

It is worth noting that in the second edition of the *The World as Will and Representation* (1844) he added this sentence to § 14: "all ultimate, i.e. original *evidentness* is *intuitive*: as the word already indicates" [1844, § 14, p. 78; 1859,

[6]Such a stance implied a negative judgement of both analytic geometry and mathematical analysis.

§ 14, p. 91/78]. Another addition is in the third edition (1859): "it is only on this sort of a geometrical basis (i.e. by means of *a priori* intuition) [. . .] that significant progress can be made with inferences" [1859, § 14, p. 92/79]. He clearly intended to strengthen the intuitive approach to mathematics by a more incisive praise of immediateness.

There are analogous remarks in *Parerga and Paralipomena* (1851). It is recalled that mathematics is not analytical: "the synthetical nature of geometrical propositions can be demonstrated by the fact that they contain no tautology. This is not so obvious in the case of arithmetic, but yet it is so" [PII, On logic and dialectic, § 23, p. 22/20]. And a passage from the chapter on the history of philosophy summarises: "mathematics is based on *intuitive perceptions* on which its proofs are supported; yet because such perceptions are not empirical but *a priori,* its theories are apodictic. [. . .] Accordingly, philosophy is now a science from mere *concepts,* whereas mathematics is a science from the *construction* (intuitive presentation) of its concepts". [PI, *Fragments for the history of philosophy*, § 13, pp. 79/74–75].

3 Role and Purpose of Mathematics

It is debatable whether Schopenhauer's belligerent attitude toward Euclid was really aimed at rewriting the traditional *corpus* of the geometry. Some passages in *The World as Will and Representation*, likewise the concluding remarks in § 39 of the 1813 Dissertation, suggest a concern for pedagogy in mathematics instead. He complained that Euclid's model deprives

> students of any insight into the laws of space, indeed, it gets them quite out of the habit of investigating the ground and inner nexus of things, and teaches them instead to let themselves to be satisfied with the historical knowledge *that* it is so. The exercise of acumen that wins Euclid's method such incessant praise amounts to no more than this: schoolchildren practise making inferences (i.e. applying the principle of non-contradiction), but more particularly they strain their memories remembering all the data whose mutual agreements have to be compared [WI, §15, p. 101/89].

For this reason, he admitted that "for teaching mathematics, I altogether prefer the analytical method to Euclid's synthetic method, even though it runs into very serious—if not insuperable—problems in the case of complicated mathematical truths" [WI, § 15, p. 99/87]. He also specified: "the most decisive step in this direction has been taken by Herr *Kosack*, a teacher of physics and mathematics at the Nordhausen Gymnasium, who has added a thoroughgoing attempt to treat geometry according to my principles to the schedule for school examination on the 6th of April 1852" [WI, § 15, p. 99/87]. Such a reference was not disinterested: as a matter of fact, Carl Rudolph Kosack mentioned Kant and Schopenhauer as his sources of the idea that demonstration in geometry requires eminently intuition (cf. [30, p. 10]; see [32]).

It would be limiting, however, to insist on the primacy of intuition and the pedagogical issue as the only relevant claims of Schopenhauer's philosophy of mathematics. He held a more complex view of mathematics and its role in the construction of knowledge that is not easily noticed in his texts—even because it was partially expunged from the pages of the *The World as Will and Representation* after its first edition. It had to do with the excess of abstraction and formalism not only in demonstrations but even in mathematical content. It timidly emerged in the manuscripts of 1813 and in the first edition of *The World as Will and Representation*, where he referred to Abel Bürja, Ferdinand Schweins and Bernhard Friedrich Thibaut, who had been his mathematics professor at the University of Göttingen (1809–1811).

Abel Bürja was the author of two treatises on autodidacticism in arithmetic and geometry [9, 10] which Schopenhauer borrowed from the Weimar Library in summer 1809, just before leaving for Göttingen, and in summer 1813, while writing the Dissertation.[7] Bürja's observations on the explanation of geometrical theorems were later mentioned in an 1813 manuscript regarding Kant's third *Critique* [HNI, pp. 63–64/83–84: § 95]. We cannot establish whether Bürja was a source of Schopenhauer's views on mathematics, but it is worth noting that after reading his books Schopenhauer chose Thibaut's mathematical course at Göttingen, whose manual of mathematics mentioned intuition as grounding geometrical notions [50, pp. 187–188, 310–311]. It was likely through Thibaut that Schopenhauer heard about Schweins, who had studied and taken his doctoral degree at Göttingen in 1807. Before moving to Heidelberg in 1810, where he became full professor in 1816, Schweins had taught mathematics at Darmstadt, where in 1810 he published the book later mentioned by Schopenhauer [45].

In 1817 Thibaut's manual was briefly discussed in the manuscripts [HNI, p. 447/602: § 655]. The reference was enriched by comments on Schweins in the first edition of *The World as Will and Representation*:

> Professor Thibaut in Göttingen has performed a great service in his Outline of Pure Mathematics [Grundriß der reinen Mathematik], although I would like a much more decisive and thorough substitution of the evidentness of intuition in place of logical proof. Professor Schweins in Heidelberg (Mathematics for primary scientific instruction [Mathematik für den ersten wissenschaftlichen Unterricht] 1810) has also declared himself against the Euclidean treatment of mathematics and attempted to move away from it. Only I find that his improvement reaches only as far as the presentation and not the method of treating mathematics itself, which still remains wholly Euclidean. He has certainly adopted a more coherent, more pragmatic approach rather than the fragmentary approach of Euclid, and that is definitely praiseworthy; but then he has abandoned Euclid's strict form without in the least moving away from his method as such, that is, logical proof in places where immediate evidentness would have been available [W1, pp. 571–72/109–110].

Schopenhauer praised the "pragmatic" approaches of those books to mathematics, but it is evident that he was not satisfied by their notions and methods; this is probably the reason why they were expunged from the 1844 and 1859 editions of the *The World as Will and Representation*. He looked for clarity and visibility of

[7] About the loans, see [HNV, p. 284]. In summer 1813 he also borrowed the 1800 German edition of Euclid's *Elementa* (see [DSW, vol. 16, p. 108]).

the truth, like in intuition, but also for concreteness against formalism, because the validity of theorems must not "reveals itself accidentally [*per accidens*]" [W1, p. 572/109].

His ideal of mathematics was related to his philosophy of science. He sought a philosophy of nature as a synthesis of the natural sciences and metaphysics—whose grounding, by the way, was in intuition. The former would make available empirical content and exhibit the effectiveness of metaphysics of will in providing knowledge of the world [46]. He praised factual and verifiable content as the solid foundation of scientific knowledge and the main source of progress. Instead, mathematics was abstraction, and even if he had accepted Kant's view of mathematical truths as synthetic, nevertheless he did not consider them as contributing to the advancement of learning. In an unpublished manuscript written in 1832, he clearly expressed the view that logic and mathematics "do not teach anything more than what we already *apriori* know" [Pandectae, p. 39].[8] The project of reinstalling intuition in mathematical demonstration was the way to preserve the connection between mathematics and knowledge.

On the contrary, the pernicious logical demonstrative procedure in mathematics had contaminated philosophy and contributed to widening the gap between metaphysics and reality. His criticism of Spinoza's *more geometrico* [WI, p. 102/91 footnote] is a clear example of his low esteem of the benefits of mathematics to philosophy. Something similar, even if inverted, could be observed in Schelling's philosophical procedure of "construction": here philosophy aimed to ground the mathematical demonstration.[9] In one way or the other, mathematics had widened its detachment from reality.

On the front of the sciences, things were not better. Abstraction and logical demonstrations had become values of the mathematised sciences. Schopenhauer's penchant for Goethe was probably related to his polemics against Newton and the mathematical description of the world. Melanchthon's famous acclamation of arithmetic and geometry as "the wings of human minds" [36, p. 288]—which had contributed to the boosting of the scientific revolution in the Reformed lands—never persuaded Schopenhauer. Notwithstanding Melanchthon's explicit reference to Plato, Schopenhauer was deeply convinced that the concrete truth about the world cannot derive from the abstractions of mathematics.

As a consequence, Schopenhauer was generally reluctant to consider mathematics as philosophically and epistemically relevant. When assessing scientific knowledge, Schopenhauer valued the role that empirical truth plays in establishing a sound theory. Precision and certainty of mathematics (and logic), on the contrary, do not provide content and knowledge. He certainly recognised the profound impression of Euclid's model of explanation on metaphysics and natural philosophy in the modern era, and his criticism was both a response to the undue honour paid

[8]"[. . .] sie uns eben nichts weiter lehren, als was wir schon vorher (a priori) wußten".

[9]On the relationship between construction, demonstration, and the project of transcendental philosophy in Schelling, see [55, pp. 188–193]. See also [7, 25].

to the traditional deductive procedure and a reminder of the privileged access to knowledge provided by intuition. Besides, he was aware that history had indelibly marked the fate of mathematics and a reversal would be implausible. He was not pursuing a quixotic dream, rather he reflected upon mathematics as a concrete form of knowledge, something intrinsically useful in everyday life, schools, the sciences, and even philosophy.

4 A British Debate

An explicit expression of those ideas appeared in print at the end of chapter 13 of the second volume of *The World as Will and Representation*, by referring to "the sense in which Plato recommended geometry to philosophers [. . .] as a preliminary exercise, by which the mind of the pupils became accustomed to dealing with incorporeal objects, after this mind had hitherto in practical life had to do only with corporeal things" [WII, 13, p. 131/144]. Schopenhauer pointed out that an interesting perspective had emerged in Britain, in the review of a book of William Whewell by the Scottish philosopher William Hamilton [21].[10] Described as "an investigation of the influence of mathematics on our mental powers and of its use for scientific and literary education in general", Hamilton's review was interpreted by Schopenhauer as assessing that "the value of mathematics is only indirect, and is found to be in the application to ends that are attainable only through it; it is by no means necessary; in fact, it is a positive hindrance to the general formation and development of the mind. [. . .] The only immediate use left to mathematics is that it can accustom fickle and unstable minds to fix their attention" [WII, 13, pp. 131/144–5]. Hamilton's "fine" essay was later mentioned again in the chapter "On learning and the learned" of *Parerga and Paralipomena*, where Schopenhauer acknowledged the peculiarity of the "aptitude for mathematics", which "does not by any means run parallel to the other mental faculties, and in fact has nothing in common with them" [PII, §256, p. 489/409].

If we want to understand Schopenhauer's convinced reference to Hamilton, we should consider the context that stimulated both Whewell's intervention about mathematical education in relationships with higher learning and Hamilton's response to it. The starting point was the so-called 'analytic revolution', around 1800, when Lagrange's seminal work and its dissemination by Lacroix showed the superiority of analysis over synthetic-geometric mathematics to pursue generality.[11] In a few decades, mathematicians would acknowledge that geometry had become inadequate to scientific investigation. Notwithstanding the "peculiar excellence" of the "method of synthesis", "the very circumstances, which cause its perspicuity and

[10] Schopenhauer added a reference to [22], the German translation of Hamilton's review.

[11] On that seminal moment, see [20, Chap. 2].

evidence, render it unfit for the deduction of truths that are remote and intricate".[12] As a consequence, not only mathematics underwent an inevitable transformation: it became clear that mathematical education required substantial reformation, too.[13] Above all, it was questioned whether learning mathematics should still be part of a general education because skills and the talent required to be a proficient mathematical analyst were peculiar and rare.

4.1 Whewell on the Study of Mathematics

Whewell reflected upon these events from the extraordinary point of view of tutor and professor at the University of Cambridge from 1818. According to him, "the object of a liberal education is to develop the whole mental system of man, and thus to bring it into consistency with itself; to make his speculative inferences coincide with his practical convictions; to enable him to render a reason for the belief that is in him".[14] The analytic revolution, as recalled by Harvey Becher, "challenged the entire Cambridge educational system, for mathematics formed the core of the liberal education that was Cambridge's *raison d'être*" [3, p. 3]. Synthetic-geometric mathematics functioned as trainer of logical and open minds, necessary to ground culture and the intellectual abilities of an elite which would pursue professional and clerical careers. Instead, pure analysis's vocation was abstraction and formalism, which dismissed geometry and its intuitive foundation: "there exist certain modes of treating the study of mathematics, and certain views concerning its foundations, which must diminish its benefits as a mental discipline and a preparation for all other branches of philosophical speculation" [52, p. 168].

In the 1830s Whewell had already developed critical views against a privileged role of analysis in Cambridge education. He considered analysis as having limited or even pernicious effects on the mind: "analysis too often merely gives us results which exercise no intellectual faculty, nor convey any satisfactory knowledge" [52, p. vi]. To ground his stance, Whewell embarked on a series of inquiries in the area of pedagogy: *Thoughts on the Study of Mathematics as a part of a Liberal Education* (1835), *On the Principles of English University Education* (1837), *Of a Liberal Education in General, and with Particular Reference to the Leading Studies of the University of Cambridge* (1845). The last one offered harsh criticism like the following: "the destructive effect of mere analysis upon the mind"; "so far as the analytical method has superseded the geometrical, I am obliged to say [. . .], the result has been very unfortunate"; analysis is "of little value as a discipline

12These were the words of a British reviewer of Lacroix's *Traité du calcul différentiel et du calcul integral* (1797–1798) in *Monthly Review* (see [1, p. 492]).

13At this time the challenge of non-Euclidean geometries was not present yet: Euclidean geometry was still the cornerstone of the English liberal education. See [42].

14Whewell, *Thoughts on the Study of Mathematics as a part of a Liberal Education*, in [52, p. 139].

of the reason for general purposes. [. . . It] belongs to a class of intellectual habits which it is the business of a good education to counteract, correct, and eradicate, not confirm, aggravate, and extend".[15] The good education could be found in the old curriculum of Euclid and Newton, whose *Principia* contained "beautiful examples of mathematical combination and invention, following the course of the ancient geometry". A person educated according to the traditional programmes "had commonly acquired a command of certain mathematical methods, and a love of mathematics, which he retained through life" [51, pp. 35, 185]. It is worth noting that Whewell was quite candid about the aim of mathematical education: "the use of mathematical study [. . .] is not to produce a school of eminent mathematicians, but to contribute to a Liberal Education of the highest kind" [51, p. 77].

The primacy of "liberal education" and the protection of the youngsters' minds from the aridity of formalism was at first defended in the brief pamphlet (less than 50 pages) *Thoughts on the Study of Mathematics as a part of a Liberal Education* (1835). After maintaining the educational superiority of the study of mathematics ("teaching of reasoning by practice") over the study of logic (teaching of reasoning "by rule"),[16] Whewell asserted that mathematics can train minds to deal "with other kinds of truth" and "on any particular subject" [*Thoughts*, p. 141, 142] only if conventional or empirical views of its first principles are banished and excessive formalism and generalisation are avoided [*Thoughts*, p. 142]. Otherwise, "we not only sow the seeds of endless obscurity and perplexity [. . .], but we also weaken his [the student's] reasoning habits and disturb his perception of speculative truths; and thus make our mathematical discipline produce, not a wholesome and invigorating, but a deleterious and perverting effect upon the mind" [*Thoughts*, p. 156]. He was adamant that "the foundation of all geometrical truth resides in our general conception of space" and that the teaching of differential calculus according to the new course of analysis was misleading [*Thoughts*, pp. 149–153]. In order to learn at best geometry and calculus, then, the sources were still Euclid and Newton's *Principia*, notwithstanding all of modern mathematics.

In conclusion, to be part of a liberal education, mathematics must be rigorous, not abstract, and grounded in the notions of geometrical space and arithmetic number: "I believe that the mathematical study to which men are led by our present requisitions has an effect, and a very beneficial effect, on their minds: but I conceive that the benefit of this effect would be greatly increased, if the mathematics thus communicated were such as to dissipate the impression, that mathematical reasoning is applicable only to such abstractions as space and number" [*Thoughts*, p. 174].

[15] [51: dedicatory letter to Airy; p. 204; p. 45].

[16] Whewell, *Thoughts on the Study of Mathematics as a part of a Liberal Education*, in [52, p. 141]: mathematics, then, is to be considered "as a means of forming logical habits better than logic itself".

4.2 Hamilton's Review

One year after Whewell's *Thoughts on the Study of Mathematics*, *The Edinburgh Review* published a long review by William Hamilton—in fact as long as Whewell's pamphlet. Together with Dugald Stewart, Hamilton (1788–1856) was the most influent interpreter of Thomas Reid's common sense realism and pillar of the Scottish philosophical movement—at least until John Stuart Mill would demolish his philosophy in the memorable *Examination of Sir William Hamilton's Philosophy* (1865).[17] He visited Germany in 1817 and 1820 and contributed to the diffusion of Kantian and post-Kantian philosophy in Britain. His fame in the second quarter of the century was certainly related to his extensive knowledge of Continental philosophy; besides, he was a brilliant philosopher, a talented logician,[18] and a respected reviewer in influent journals like *The Edinburgh Review*. To be reviewed by Hamilton could be crucial for the success of a book—as acknowledged by Mill, who expected his forthcoming *System of Logic* (1843) would be reviewed by the "hostile, but intelligent" Scottish philosopher.[19]

Tackling Whewell over the subject of mathematics as a means of liberal education brought Hamilton to discussing the nature of mathematical principles, the notion of liberal education itself, and the comparison between mathematical and philosophical knowledge—while expressing opinions, critiques and strong dissent that would stimulate Whewell's reaction.[20] His conclusions were that the primacy of mathematics at Cambridge was "indirectly discouraging the other branches of liberal education", tended "positively to incapacitate and to deform the mind", was worthless "for the conduct of the business, or for the enjoyment of the leisure" [21, pp. 453–454], and was not serving the cause of mathematics, as no Cambridge mathematician had ever gained recognition in the field [21, p. 410].

Hamilton's analysis started from a different view about what a "liberal education" should be: "we speak not now of *professional*, but of *liberal* education; not of that, which makes a mind an instrument for the improvement of science, but of this, which makes science an instrument for the improvement of the mind" [21, p. 411]. Such a perspective, it is evident, would not admit the curricular primacy of

[17] On Hamilton (1788–1856) and his fame at the time of Schopenhauer's reference, see [35, pp. 113–114, 120–133].

[18] His decennial (1846–1856) controversy with Augustus De Morgan about the priority in theorising the quantification of the predicate was also famous. See [18, 31, 40].

[19] "If you do not review the book it will probably fall into the hands either as you suggest, of Sir W. Hamilton, or of Brewster. The first would be hostile, but intelligent, the second, I believe, favourable, but shallow": John Stuart Mill to John Austin, July 7, 1842, in [37, p. 528].

[20] On January 23rd 1836, Whewell wrote a letter to *The Edinburgh Review* (vol. XLIII, n. 127, 1836, pp. 270–272; then reprinted in [52, pp. 186–189]) making clear that his pamphlet was about "what kind of mathematics is most beneficial as a part of a liberal education" and not "a vindication of mathematical study" as Hamilton had suggested—"having thus made me work at a task of his own devising" [52, pp. 186–187]. Such a casual missive was nevertheless followed by the more committed works of 1837 and 1845.

mathematics. But it was its "utility as an intellectual exercise" that he essentially contested: instead of "its importance as a logical exercise", the "evidence" speaks "of its contracted and partial cultivation of the faculties"; besides, "the most competent judges" and "the authorities" of the philosophical tradition have generally sustained "that the tendency of a too exclusive study of these sciences, is, absolutely, to disqualify the mind for observation and common reasoning" and, even more precisely, that "none of our intellectual studies tend to cultivate a smaller number of the faculties, in a more partial manner, than mathematics" [21, pp. 411, 412, 419].

Amongst those authorities Hamilton quoted Aristotle and the notion of virtuous man as educated through a varieties of disciplines, German pedagogic books, Goethe ("the cultivation afforded by the mathematics is, in the highest degree, one-sided and contracted"), Voltaire ("j'ai toujours remarqué que la geometrie laisse l'esprit ou elle le trouve"), Franklin and even first-rank mathematicians like D'Alembert and Descartes.[21] Other authors were recalled to support the view that geometry—as based on imagination and senses—does not reinforce understanding or the capacity of generalisation: Mersenne, Digby, Coleridge, Kant, Duhamel, Pestalozzi and Warburton ("the routine of demonstration [is] the easiest exercise of reason, where much less of the vigour than of the attention of mind is required to excel" [21, pp. 425–429]. He also reproduced long passages from mathematicians like Pascal, Berkeley, s'Gravesande, D'Alembert and from other illustrious intellectual and philosophers in order to support his argument about narrowness and proneness to error of the mathematical mind [21, pp. 434–441]. The conclusion that mathematicians "are disposed to one or other of two opposite extremes—credulity and skepticism" gave Hamilton the opportunity, on the one hand, to express his Reidian anti-metaphysical stance and condemn as bad philosophers (because too credulous) the mathematicians "Pythagoras, Plato, Cardan, Descartes, Mallebranche, and Leibnitz"[22]; on the other hand, to denounce mathematicians' inclination towards atheism, negation of moral freedom and denial of the soul.[23]

Hamilton's criticism of Whewell's arguments was strong. Firstly, he demolished Whewell's idea that mathematics is more apt than logic to ground a liberal education. Hamilton reproached Whewell of having overlooked the distinctions between *theoretical* and *practical* logic and between *practical* logic "as specially applied to Necessary Matter=Mathematical reasoning" and "as specially applied

[21] [21, pp. 417–421]. See [21, p. 421] for the quotations. Here Hamilton was amply using his first-hand knowledge of German philosophy and literature.

[22] [21, p. 443]: "Conversant, in their mathematics, only about the relations of ideal objects, and exclusively accustomed to the passive recognition of absolute certainty, they seem in their metaphysics almost to have lost the capacity of real observation, and of critically appreciating comparative degrees of probability. In their systems, accordingly, hypothesis is seen to take the place of fact; and reason, from the mistress, is degraded to the handmaid, of imagination."

[23] [21, pp. 445–450]. On this subject, Hamilton quoted Patristic authors, philosophers like Berkeley, Kant, Fries and added a long passage (without any reference) from Jacobi's 1815 Preface to *David Hume on Faith, or Idealism and Realism, a Dialogue* (1787), in [26, pp. 51–55]. It is worth noting that the same passage from Fries was in a footnote of Jacobi's text [26, pp. 52–53].

to Contingent Matter=Philosophy and General reasoning". It is the latter, stated Hamilton, that helps to "cultivate the reasoning faculty for its employment on contingent matter". On the contrary, Whewell ignored practical logic and erroneously concluded for the primacy of mathematics [21, p. 413].

Secondly, he attacked Whewell on the nature of mathematical first principles. According to Hamilton, Whewell was addressing a question of philosophy of mathematics without actually referring to philosophical notions or authors. On this subject, Hamilton the philosopher showed pertinence and precision—and made it evident that Whewell had offered an interpretation of the foundations of mathematics that misinterpreted Kant's views [21, pp. 414–417].

Thirdly, he defended the superiority of philosophical education over the mathematical by considering their different objects, ends and "modes of considering their objects". While mathematics "take no account of things",[24] "philosophy, on the other hand, is mainly occupied with realities; it is the science of a real existence, not merely of a conceived existence" [21, p. 422]. As to the ends, they tend to two different kinds of knowledge: in mathematics the whole science is contained in the principles—which "afford at once the conditions of the construction of the science, and of our knowledge of that construction (*principia essendi et cognoscendi*)"—and "it is only the evolution of a potential knowledge into an actual, and its procedure is thus merely explicative". Philosophy is quite different: "its principles are merely the rules for our conduct in the quest, the proof, and the arrangement of knowledge: it is a transition from absolute ignorance to science, and its procedure is therefore ampliative" [21, p. 423]. But even more relevant is the difference in the modes of considering their objects: mathematical science "contemplates the general in the particular", while philosophy "views the particular in the general"]; mathematics is perfectly expressed by its own language, while philosophy struggles with common linguistic expressions of concepts which do not mirror its notions[25]; in mathematics deductions are "apodictic or demonstrative", while in philosophy "such demonstrative certainty is rarely to be attained" [21, p. 424]. All this considered, "it will easily be seen how an excessive study of the mathematical sciences not only does not prepare, but absolutely incapacitates the mind, for those intellectual energies which philosophy and life require. We are thus disqualified for observation either internal or external—for abstraction and generalization—and for common reasoning; and disposed to the alternative of blind *credulity* or irrational *scepticism*" [21, p. 424].

All in all, Hamilton denied any positive effect of studying mathematics while pursuing a liberal education: mathematical demonstration is counterproductive "as a practice of reasoning in general"; it "educates to no sagacity" and "allows no room for any sophistry of thought". Against Whewell's convinced view that mathematics establishes "logical habits better than logic itself", Hamilton rebutted that the very

[24] Hamilton used "mathematics" as a plural noun.

[25] [21, p. 424]. Hamilton speaks of "the absolute equivalence of mathematical thought and mathematical expression".

perfection of mathematical reasoning makes it useless: the "art of reasoning *right* is assuredly not to be taught by a process in which there is no reasoning *wrong*" [21, pp. 426–427]. He also explained why mathematics appears extremely easy to the inclined and acutely painful for many other students: the simplicity and monotony of demonstrations require an unbearable attention from minds "endowed with the most varied and vigorous capacities". Paradoxically, "to minds of any talent, mathematics are only difficult because they are too easy", because in "mathematics dullness is thus elevated into talent, and talent degraded into incapacity" [21, p. 430].

The only way to benefit from the study of mathematics, Hamilton concluded, was under restricted conditions: "if pursued in moderation and efficiently counteracted, [it] may be beneficial in the correction of a certain vice, and in the formation of its corresponding virtue. The vice is the habit of mental distraction; the virtue the habit of continuous attention" [21, p. 450]. Such a benefit, however, would not redeem mathematics from the disadvantage of narrowness—while the mind needs "an extensive, a comprehensive, or an intensive application of thought"—and in any case it cannot train students without inclination to attention: "after all, we are afraid that D'Alembert is right; mathematics may distort, but can never rectify the mind" [21, pp. 452–453].

5 Intuition and the Foundations of Geometry

Generally, historians have assessed Hamilton's attack on Whewell by looking at the battle of the latter against the former's common sense philosophy and in particular at the dispute about the philosophy of mathematics—the nature of axioms and definitions in geometry and the general interpretation of mathematical truth.[26] Schopenhauer did not miss this foundational controversy—that offered a new perspective over the "Parallelenproblem"—and was inspired by Hamilton's treatment of the role of mathematics in education and the production of knowledge. We can discern what he appreciated in Hamilton's observations, comments and sources.

Firstly, the vast and informed quotations from authors (many of them from German sources) who supported his argument certainly impressed Schopenhauer: he, too, was used to this kind of justification in his writings—in the manuscripts even more than in publications. It is worth noting that in the 1859 edition of *The World as Will and Representation* he added the same quotation from Baillet's *Life of Descartes* which Hamilton had translated in his review.[27] Secondly, the Scottish philosopher expressed knowledge of and admiration for Kant's views on space, time

[26] [11, 12, 15, 16, 47], [48, pp. 86–89].

[27] [WII, p. 132/145], [21, p. 421].

and mathematics—something that certainly captivated Schopenhauer.[28] Besides, even if cursorily the review mentioned intuition as essential to the understanding of mathematics: "the principles of mathematics are self-evident; [. . .] every step in mathematical demonstration is intuitive" [21, p. 428]. Thirdly, Hamilton insisted that mathematical truth and knowledge had nothing in common with the same notions in natural philosophy: "the truth of mathematics is the harmony of thought and thought; the truth of philosophy is the harmony of thought and existence" [21, p. 423]. His observation that in philosophy "demonstrative certainty is rarely to be attained" and is not comparable to the apodictic truth of mathematics [21, p. 424] was similar to Schopenhauer's remark on Spinoza [WI, p. 102/91 footnote], who had mixed the two of them—a remark that was introduced in the second edition of *The World as Will and Representation*, i.e. after having read Hamilton.

Schopenhauer agreed with Hamilton about the distance between mathematical formalism and natural philosophy. It is often sustained, claimed Hamilton, that the mathematics is "the passport to other important branches of knowledge. In this respect mathematical sciences (pure and applied) stand alone: to the other branches of knowledge they conduce —to none directly, and if indirectly to any, the advantage they afford is small, contingent, and dispensable" [21, p. 453]. Schopenhauer's distrust of mathematics as contributing to knowledge was evidently strengthened by Hamilton. In 1851, he attacked vehemently arithmetic as a tool for arid calculations:

> that the lowest of all mental activities is arithmetic is proved by the fact that it is the only one that can be performed even by a machine. In England at the present time, calculating machines are frequently used for the sake of convenience. Now all analysis *finitorum et infinitorum* ultimately amounts to repeated reckoning. It is on these lines that we should gauge the 'mathematical profundity', about which Lichtenberg is very amusing when he says: 'The so-called professional mathematicians, supported by the childish immaturity of the rest of mankind, have earned a reputation for profundity of thought that bears a strong resemblance to that for godliness which the theologians claim for themselves' [PII, Psychological remarks, § 356, p. 610/493].

Such a diminishing appreciation helped to build a case against Newton—"the great mathematician" who enjoyed "ludicrous veneration" [PII, On philosophy and natural science, § 80, p. 126/99]—and his theory of colours: "Goethe had the true objective insight into the nature of things, a view that is given up entirely to this. Newton was a mere mathematician, always anxious to measure and calculate and taking as the basis of this purpose a theory that was pieced together from the superficially understood phenomenon" (PII, On the theory of colours, §107, p. 197–

[28] [21, p. 423]: "without entering on the metaphysical nature of Space and Time, as the basis of concrete and discrete quantities, of geometry and arithmetic, it is sufficient to say that Space and Time, as the necessary conditions of thought, are, severally, to us absolutely one; and each of their modifications, though apprehended as singular in the act of consciousness, is, at the same time, recognised as virtually, and in effect, universal. Mathematical science, therefore, whose conceptions (as number, figure, motion) are exclusively modifications of these fundamental forms, separately or in combination, does not establish their universality on any a posteriori process of abstraction and generalization; but at once contemplates the general in the particular."

8/211].[29] Similar harsh criticism was levelled at Laplace and the French Newtonian physics around 1800: "le calcul! le calcul! This is their battle-cry. But I say: ou le calcul commence, l'intelligence des phénomènes cesse".[30]

Schopenhauer's reference to Hamilton should not be overlooked. He was the most important author in orienting Schopenhauer towards an utter devaluation of mathematics as a source of actual knowledge. Before reading that iconoclastic text, Schopenhauer's was simply considering the vindication of intuition against demonstration and the reasons of pedagogy—but he had never questioned that mathematics was essential in culture and education. The attacks against mathematical-physics were a substantial leap from the previous position but it should be misleading viewing them as motivated by incomprehension of advanced mathematics. Hamilton had offered several arguments to sustain the idea that mathematics was not only useless to education and knowledge but even harmful. He had provided examples of how good mathematician had turned into bad (natural) philosophers. In his long quotation from Jacobi there was the same kind of criticism of the mathematical-physics tradition as barren and empty later exploited by Schopenhauer: "he [the mathematical-physicist] no longer marvels at the object, infinite as it always is, but at the human intellect alone, which, in a Copernicus, Kepler, Gassendi, Newton, and Laplace, was able to transcend the object, by science to conclude the miracle, to reave the heaven of its divinities, and to disenchant the universe" [21, p. 449].[31]

Hamilton's review also engaged Schopenhauer's attention to the question of the foundations of mathematics—and specifically of geometry—as debated in Britain. In particular, he acknowledged that the generalised perplexities about the fifth postulate could be related to deeper questions. He could fully appreciate them in 1838, when *The Edinburgh Review* published another review on Whewell: Thomas Flower Ellis analysing the *Mechanical Euclid* [14, 53].[32] Two years younger than Whewell, Ellis (1796–1861) had graduated from Trinity College, Cambridge, in the 1810s and was acquainted with the philosopher.[33] Yet his review was not sympathetic: it discussed the question of the foundations of geometrical certainty and Ellis confronted Whewell's position on the absolute necessity of mathematical truths from Dugald Stewart's point of view, who had maintained Euclid's axioms and theorems "to consist, in truth, of definitions and of propositions requiring proof" [14, p. 87].[34] While Whewell asserted that "deductive proofs consist of many steps, in each of which we apply known general propositions in particular cases;— 'all triangles have their angles equal to two right angles, therefore this triangle has; therefore, &c.'" [53, p. 182], Ellis countered:

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[29]The quotation comes from Deussen's posthumous edition of *Parerga* [DSW, vols. 3–4].

[30]Originally in [Sen, p. 32], the quotation was included in [F, p. 90] by Frauenstädt.

[31]The original text in [26, p. 52].

[32]On Whewell's work, see [27].

[33]On Ellis, see [33].

[34]Ellis referred to Stewart's *Elements of the Philosophy of the Human Mind* [49, p. 43, p. 40, p. 527].

> the reception of one truth does not precede the reception of the other in the order of reasoning. These axioms are, in truth, practical laws of thought; they are a part of the machinery by which the reason works, not of the material from which it obtains its results. Again, it is not possible for human ingenuity to deduce a single geometrical inference from these axioms. [. . .] The science therefore does require the definitions, but does not require the axioms [14, p. 88].

Overtly relying on Stewart, who had been an object of Whewell's criticism, Ellis defended the notion that geometry could be founded only on self-evident truths: "the definition requires the possibility of the thing as defined. The possibility should therefore be presented as a self-evident proposition, that is, as an axiom" [14, p. 92]. According to him, the difference between Stewart and Whewell "appears to be on the question, merely, whether what we have here called the second class of axioms be truly axioms. Mr Stewart thinks that they consist of definitions and propositions requiring proof; while Mr Whewell considers them to be truly axioms" [14, p. 94].[35] If an axiom were not self-evident and required a definition to be understood, it was not an actual axiom, but rather a theorem whose truth benefitted of the definition's self-evidence. He rhetorically asked: is it correct to consider

> as an axiom which merely supplies the incompleteness of the definition? Is that properly called an axiom, which adds to the properties given in the definition, or explains the meaning of the words? Is that properly called a definition, which conveys an incomplete or indefinite (or 'vague') conception, till an explanation be added, or an addition supplied, by an axiom? [14, p. 96].

If an axiom is an addition, Ellis concluded, and is neither a definition nor a self-evident proposition, "we protest against founding any argument, respecting mathematical reasoning, on a part of the system which is acknowledged to be a violation of the principles of such reasoning" [14, p. 97].

Schopenhauer appreciated Ellis's contribution, which had distinctly expressed criticism of the diffuse praise of Euclid's geometry. He immediately registered some passages from the review in his manuscripts and later he elaborated them in chapter 13 of the second volume of *The World as Will and Representation*.[36] Ellis had developed arguments that supported Schopenhauer's own denunciation of the Euclidean system—with its "futile attempts to demonstrate the *directly* certain as merely *indirectly* certain" [WII, 13, p. 130/144]—and his unconventional view of the equivalence between postulates and theorems.[37] Both of the themes were recapitulated in the first paragraph of chapter 13, but without considering the

[35] Ellis's "second class of axioms" corresponds to Euclid's five postulates. At the time, they called "axioms" all the fundamental propositions of Euclidean geometry, the seven axioms and the five postulates.

[36] [HNIV(1), pp. 289–290/254–255: Spicilegia, § 36 (1838)], [WII, 13, pp. 130–131/144].

[37] As an example, Ellis had insisted that Euclid's fifth postulate required a demonstration and had advanced a general consideration that appealed to Schopenhauer: "the proposition has, universally we believe, been allowed to require demonstration, and to be improperly termed an axiom. It is surely not correct to assert that a chain of truths owes its peculiar certainty to its resting upon that which itself requires, and has not received, a demonstration. Euclid's twelfth axiom is, indeed,

reference to Ellis it seems that Schopenhauer was simply reasserting the primacy of intuition over demonstration in geometry. Instead, such an insistence reveals that Schopenhauer had acquired full awareness of the debate on the foundations of geometry—and of its importance and width in Britain[38]—and thus reshaped the notion of intuition as foundational. He took the similarity of his views to those of Ellis (and Stewart), and specifically their thesis that geometry's only ground was in definitions——and consequently there was not any difference between axioms and theorems—as the occasion to reconsider demonstration and *reduction ad absurdum* as symptoms of a more serious problem: the lack of foundations. Even the notion of intuition as an alternative to demonstration—like he had presented it in the 1810s—did not confront the real question. The point at issue was neither the "Parallelenproblem" nor substituting the (overly complicated) demonstration of theorems with the immediate vision of their truth; instead it was the promise of certain and definitive truth of geometry itself. Rather than for the (better) procedure, the quest was for the foundation.

Hamilton's and Ellis's reviews of Whewell had shown Schopenhauer a lively and heated debate that gave new meaning to his own view of intuition in mathematics—as the foundation of geometry. Whereas in the 1810s he had developed his philosophy of mathematics as Kant's follower and as a consequence of his praise for the fundamental role of intuition in metaphysics,[39] in the 1830s he explored the possibility of intuition as the ultimate foundation of geometry. The British debate had demonstrated that the traditional interpretation of the Euclidean geometry brought to both inconclusive discussions about the demonstration of the fifth postulate and the denigration of mathematics as a part of education. Schopenhauer realised that the relevant philosophical questions concerned the foundations of mathematics and that the entire mathematical structure was less firm than believed. His harsh judgement of the calculus's abstraction derived from a more radical view encouraged by the British discussion: if intuition was the foundation, abstraction became an actual perversion.

Overlooking Schopenhauer's reading of *The Edinburgh Review* has persuaded commentators that he was far from the mainstream of mathematics; thus, some may have considered his views as vitiated by a substantial incomprehension of the new course of mathematics in the nineteenth century.[40] Certainly, he never exhibited an aptitude for the exact sciences, but this is not the point. Intuition as both an

merely an indication of the point at which geometry fails to perform that which it undertakes to perform" [14, p. 91].

[38]Whewell continued the discussion with Ellis in the second book of [54].

[39]In an annotation of spring 1820 he celebrated "the joy of *conceiving directly* and intuitively, correctly and sharply, the *universal and essential aspect of the world*" [HNIII, p. 23/19: Reisebuch, § 61].

[40]Cajori bluntly commented: "Schopenhauer discloses no acquaintance with such modern mathematical concepts as that of a function, of a variable, of coordinate representation, and the use of graphic methods. With him Euclid and mathematics are largely synonymous. Because of this one-sided and limited vision we can hardly look upon Schopenhauer as a competent judge of the

alternative to logical demonstration and a pedagogical aid was Schopenhauer's starting point; but it evolved, enriched by the encounter with the British debate. At the time geometry was still solidly Euclidean and it represented the model for any pursuit of truth and certainty—within and without mathematics.[41] Doubts were typically related to the fifth postulate, but until the acceptance of the non-Euclidean geometries—thanks to Eugenio Beltrami [5], Hermann von Helmholtz [23, 24] and Felix Klein [28]—there was not a real interest in discussing the foundations of geometry. The British debate stimulated Schopenhauer to reconsider his approach based on intuition as an answer to the questions raised by Stewart, Whewell, Hamilton and Ellis on the foundations of truth in mathematics and geometry—especially after calculus had abandoned the geometrical model, and abstraction and conceptualisation had gained centrality in analysis. Schopenhauer was certainly not equipped to discuss analysis, but he lucidly saw that if logical deduction had become the only guarantee of mathematics, the synthetical character of mathematics would have been lost. In 1844 and 1851, condemnation of both abstraction in analysis and deduction in geometry appeared as consequences of Schopenhauer's development of his treatment of intuition—after reflecting on the more stringent question of the foundations of mathematics in general and geometry in particular.

A few years later, unfortunately, the non-Euclidean geometries revolutionised the philosophy of geometry: that precocious British debate on the Euclidean system inexorably aged and Schopenhauer's participation was easily forgotten.

Abbreviations

Arthur Schopenhauer's Works

Sämtliche Werke, hrsg. von Arthur Hübscher, dritte Aufl., 7 Bände, Wiesbaden, Brockhaus, 1972

Diss *Ueber die vierfache Wurzel des Satzes vom zureichenden Grunde. Eine philosophische Abhandlung*, Rudolstadt, in Commission der Hof-Buch und Kunsthandlung, 1813, in Band VII, pp. 1–94 (tr.: *On the Fourfold Root of the Principle of Sufficient Reason and Other Writings*, ed. by D. Cartwright, E. Erdmann, C. Janaway, Cambridge, Cambridge UP, 2015)

G *Ueber die vierfache Wurzel des Satzes vom zureichenden Grunde. Eine Philosophische Abhandlung*, zweite Auflage, Frankfurt a.M., Hermann'sche Buchhandlung, Suchsland, 1847, in Band I (tr.: *On the Fourfold Root of the Principle of Sufficient Reason and Other Writings*, ed. by D. Cartwright, E. Erdmann, C. Janaway, Cambridge, Cambridge UP, 2015)

educational value of modern mathematics" [13, p. 367]. Similar accusations around 1900, when Schopenhauer was even called "enemy" of mathematics, are mentioned by [32, pp. 330–331].

[41] The only exception, Bolzano's *Beyträge zu einer begründeteren Darstellung der Mathematik*, [6], which proposed a logico-mathematical foundation of geometry. Bolzano admitted that his first source and guide to such an approach had been Kästner's *Anfangsgründe der Arithmetik* (1758). Like most of Bolzano's works, the *Beyträge* were ignored until the second half of the nineteenth century.

W1 *Die Welt als Wille und Vorstellung*, Leipzig, Brockhaus, 1819 (tr.: *The World as Will and Representation*, ed. by C. Janaway, J. Norman, A. Welchman, Cambridge, Cambridge UP, 2014)

WI *Die Welt als Wille und Vorstellung*, 1. Band, *Vier Bücher nebst einem Anhange, der die Kritik der Kantischen Philosophie enthält*, dritte Auflage, Leipzig, Brockhaus, 1859, in Band II (tr.: *The World as Will and Representation*, ed. by C. Janaway, J. Norman, A. Welchman, Cambridge, Cambridge UP, 2014)

WII *Die Welt als Wille und Vorstellung*, 2. Band, *welcher die Ergänzungen zu den vier Büchern des ersten Bandes enthält*, dritte Auflage, Leipzig, Brockhaus, 1859, in Band III (tr.: *The World as Will and Representation*, vol. 2, ed. by E.F.J. Payne, New York, Dover, 1966)

PI *Parerga und Paralipomena: kleine philosophische Schriften*, 1. Band, Berlin, Hayn, 1851, in Band V, (tr.: *Parerga and Paralipomena*, vol. 1, ed. by E.F.J. Payne, Oxford, Oxford UP, 1974)

PII *Parerga und Paralipomena: kleine philosophische Schriften*, 2. Band, Berlin, Hayn, 1851, in Band VI, (tr.: *Parerga and Paralipomena*, vol. 2, ed. by E.F.J. Payne, Oxford, Oxford UP, 1974)

F *Ueber das Sehen und die Farben*, hrsg. von Julius Frauenstädt, Leipzig, F.A. Brockhaus, 1870

DSW *Arthur Schopenhauers sämtliche Werke*, hrsg. von Paul Deussen, München, Piper, 1911–1942

Arthur Schopenhauer's Manuscripts

Der handschriftliche Nachlaß in fünf Bänden, hrsg. von Arthur Hübscher, München, Deutscher Taschenbuch Verlag, 1985 (tr.: *Manuscript Remains*, 4 vols., ed. by E.F.J. Payne, Oxford, Berg, 1988)

HNI *Frühe Manuskripte (1804–1818)*, in Band I (tr.: vol. 1)

HNII *Kritische Auseinandersetzungen (1809–1818)*, in Band II (tr.:vol. 2)

HNIII *Berliner Manuskripte (1818–1830)*, in Band III (tr.:vol. 3)

HNIV(1) *Die Manuskriptbücher der Jahre l830–l852*, in Band IV.1 (tr.:vol. 4)

HNIV(2) *Letzte Manuskripte. Gracians Handorakel*, in Band IV.2 (tr.:vol. 4)

HNV *Randschriften zu Büchern*, in Band V

VorI *Philosophische Vorlesungen*, I: *Theorie des gesammten Vorstellens, Denkens und Erkennens*, in Schopenhauer 1986

Pandectae *Pandectae: Philosophische Notizen aus dem Nachlass*, hrsg. von Ernst Ziegler, München, Beck, 2016

Sen *Senilia. Gedanken im Alter*, hrsg. von Franco Volpi und Ernst Ziegler, München, Beck, 2010

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Schopenhauer on Intuition and Proof in Mathematics

Jason M. Costanzo

Abstract Although Arthur Schopenhauer adopts the substance of Kantian idealism within his work, his own theory of cognition nonetheless involves a departure from Kant's original views. He both rejects Kant's deduction of the pure concepts of understanding and likewise formulates what amounts to a "radical" divide between intuitive and abstract knowledge. Within this essay, the larger consequences of these changes are explored as they apply to his account of the nature and distinction between intuition and proof within mathematics, and in particular, within geometry. As will be shown, Schopenhauer tends to be largely dismissive of proof procedures within geometry that rely heavily upon the use of formal reasoning and logic on the grounds that such procedures serve only to rationally confirm what is already known on an intuitive basis insofar as mathematics, following Kant, is understood to be ultimately rooted in a priori cognition. In contrast, Schopenhauer proposes an intuitive model for proof on the basis of his own account of the principle of sufficient reason, where the "ground of being" of a theorem is directly exposed, thereby rendering any additional proof procedure redundant. Although offering an interesting perspective in regards to the nature of intuitive evidence within mathematics, Schopenhauer's proposal nonetheless leads to a number of difficulties in regards to our knowledge of mathematics that in the end renders such a proposal problematic. These problems are explored and the essay concludes with an examination of the historical context out of which Schopenhauer's views on this issue might be read and interpreted.

Keywords Schopenhauer · Kant · Intuition · Proof · Mathematics · Geometry · Abstraction · Principle of sufficient reason · Truth · Judgment

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1 Preliminary Remarks

A self-pronounced student of Kant, Arthur Schopenhauer was also a sharp critic, and in many ways his philosophy can be seen as both an interpretation and (as he saw it) a correction to Kant's original version of transcendental idealism. This is perhaps nowhere more evident than in the critical appendix to his magnum opus *The World as Will and Representation* (WWR), where he there attributes a number of errors to Kant within the *Critique of Pure Reason* (CPR) and other writings. Foremost among these errors is his view that Kant overlooks a detailed examination of experience and so fails to provide a robust theory of the nature and origin of empirical perception itself.[1]

In consequence, Schopenhauer concludes that Kant's great work is riddled with a: "terrible confusion of intuitive and abstract knowledge" [15, W1, p. 434]. Now irrespective of whether Schopenhauer's assessment of Kant is accurate, in the attempt to resolve this perceived confusion, Schopenhauer adopts a theory of cognition that in the end leads to a number of unpalatable results, as I see it, in relation to his account of our knowledge of mathematics.

In this section and the next, I begin by discussing Schopenhauer's corrective, as it were, to this "terrible confusion" in the form of the theory of cognition that he provides, a theory that ultimately involves a "radical" divide, as I call it, between intuition and abstraction. Following this, I indicate the consequences that this theory has upon the relationship between intuition and proof in mathematics, and in particular, within geometry.

2 Schopenhauer's Theory of Cognition

At first glance, Schopenhauer's criticism of Kant might appear puzzling. For was it not Kant who first distinguished between these two sources of knowledge within the CPR? Of course, Schopenhauer was quite aware of this fact so that his criticism ought *not* to be read as the suggestion that Kant failed to distinguish between intuition and abstraction, but rather as the suggestion that he failed to *adequately* distinguish them.[2] In support of this argument, Schopenhauer cites

[1]Thus Schopenhauer states that: "An essential difference between Kant's method and that which I follow is to be found in the fact that he starts from indirect, reflected knowledge, whereas I start from direct and intuitive knowledge" [15, W1, pp. 452–453]. NB: I employ W1 and W2 both here and throughout for the first and second volumes to Schopenhauer's *The World as Will and Representation*, respectively.

[2]Paul Guyer reads this similarly: "He cannot be doing both, so one can only assume that he is blaming Kant not for having failed to make any distinction between intuition and concept at all, but for somehow having made a false or inadequate distinction between them" [5, p. 114].

the *Transcendental Logic*, where Kant highlights both the relationship and the distinction between sensibility (*Sensibilität*) and understanding (*Verstand*):

> Our cognition arises from two fundamental sources in the mind, the first of which is the reception of representations (the receptivity of impressions), the second the faculty for cognizing an object by means of these representations (spontaneity of concepts); through the former an object is given to us, through the latter it is thought in relation to that representation (as a mere determination of the mind). [10, A50/B74]

As Schopenhauer sees it, such a description is false. In the first place, it is absurd to speak of an object separately and in relation solely to the faculty of sensibility, for according to him an object is only ever represented following the *combined* operation of sensibility and understanding. From this perspective, in relation to the receptivity of impressions we have not an object proper but only "a mere *sensation* in the sense-organ" [15, W1, p. 438]. In the second place, understanding is not a faculty of thinking. To the contrary, like sensibility, it is quite intuitive in nature, having nothing whatsoever to do with concepts. Indeed, it is precisely here that Schopenhauer's account sharply diverges from that of Kant in regards to the nature and relationship between intuitive and abstract knowledge. As Schopenhauer sees it, following the representation of an object within perception, nothing further is added on the basis of the faculty of understanding and indeed: "no concepts and no thinking are needed in addition; therefore the animal also has these representations" [15, W1, p. 439].

This in turn helps to explain Schopenhauer's above remarks that Kant has confused these two sources of knowledge. Whereas Kant intermingles abstraction and intuition through the faculty of understanding for the reason that this faculty is involved in both the production of pure concepts *and* the representation of the objects of perception, Schopenhauer radically distinguishes the two. For Schopenhauer, the object as *perceived* is fundamentally differentiated from the object as *conceived*. Although Schopenhauer retains an idealist position insofar as he holds that the perceived world is the representation of the subject's cognition, his epistemology is nevertheless empiricist in flavor insofar as reflection is seen as a second-order (derived) act that follows upon first-order sensory (albeit, intuitive) experience.

From this perspective, we might appreciate the significance of Schopenhauer's remarks within the introductory chapters (§3) to his WWR to the effect that: "The main difference among all our representation is that between the intuitive and the abstract" [15, W1, p. 6]. In the first place, there are intuitive representations, which include both the objects and the underlying cognitive forms of perception. Whereas sensibility provides the forms of space and time, understanding has for its "sole function" the union of space and time through the form of causality [15, W1, p. 11]. Whereas Kant associated pure concepts (or categories) with understanding, Schopenhauer denies such an association. Again, whereas Kant identifies a conceptual role for understanding, Schopenhauer instead insists, as he notes within his earlier doctoral dissertation on *The Fourfold Root of the Principle of Sufficient Reason* (PSR) that: "This intellectual operation does not . . . take place discursively or reflectively *in abstracto*, by means of conceptions and words; it is, on the contrary, an intuitive and quite direct process" [13, p. 61].

It is only in relation to abstract representations that all manner of discursive and reflective thought takes place. Such an operation likewise belongs solely to the faculty of reason (*Vernunft*), which as understanding has only "one function," that is, "the formation of the concept"[3] [15, W1, p. 39]. As objects and indeed representations, they nonetheless differ, "entirely from the representations of perception" [15, W1, p. 39]. They are *abstract* as opposed to *intuitive*.

Effectively, we find that Schopenhauer has formed a "radical" divide between thinking (abstraction /reason) and perception (intuition/understanding). We might furthermore oppose this to Kant's own account as involving a "moderate" divide. With Kant, understanding is associated with the production of pure concepts or categories that play a role in the eventual cognition of objects. With Schopenhauer, any and every such role is denied to understanding. Perception is radically distinguished from conception, and alternatively, intuitive from abstract representations insofar as the two are seen as fundamentally distinct operations. Although abstract representations might refer back to intuitive representations, Schopenhauer cannot *in the same way and sense* assert as Kant has famously done that: "Intuition and concepts therefore constitute the elements of our cognition, so that neither concepts without intuition corresponding to them in some way nor intuition without concepts can yield a cognition" [10, A50/B74]. To the contrary, for Schopenhauer, it is quite possible to intuit without conceiving at all. For this reason Schopenhauer is able to assert that other animals (as noted above) enjoy a similar faculty of understanding *without* conception, the latter of which exists: "only in the mind of man" [15, W1, p. 39].

3 The Principle of Sufficient Reason

Having radically divided intuition from abstraction, it is incumbent upon Schopenhauer to establish the way in which the two now relate. This is in fact what he sets about to do in his earlier doctoral dissertation. Within this work, Schopenhauer there notes (PSR §28) that all abstract reflection takes place either through the use of words or images or some combination between the two. In first case (the use of words alone), thought is divided into two kinds. First, there is the purely logical use of reasoning, as in logic. Second, there is the formation of judgments in the attempt to mediate the divide between the singular or particular (given in perception) and the universal (as concept). This latter act likewise takes place in either one of two ways. For thought can either seek for the universal that governs (the conception or

[3] As Christopher Janaway notes in his *Self and World in Schopenhauer's Philosophy*: "Schopenhauer reserves the title *reason* for the capacity to operate with concepts, yet retains for the understanding a (concept-free) role in empirical intuition" [9, p. 51].

rule for) the singular or else for the singular that might be subsumed beneath the universal and governing case. In either case, we find that the *faculty of judgment* is precisely that which bridges the gap between thought and perception and so serves as, "the mediator between intuitive and abstract knowledge" [13, p. 121]. Still more, "The true kernel of all knowledge is that reflection which works with the help of intuitive representations" [13, p. 122]. In other words, it is in relation to the faculty of judgment that true and genuine knowledge is obtained. At face value, such a statement might seem hardly controversial. Looked at broadly, however, it serves to express a larger epistemological program identifiable throughout Schopenhauer's work, a program that tends toward the negative evaluation of both logic and abstract thought. I will take up this point later following the discussion of mathematics.

Inasmuch as the judgments that we form involve the mediation between two things that stand opposed to one another, that is, the universal in thought and the singular in experience, to that extent there arises the possibility for both truth and error. Thus Schopenhauer notes that: "if a judgment is to express knowledge of any kind, it must have a sufficient reason: in virtue of which it then receives the predicate *true*" [13, p. 124]. From this perspective, Schopenhauer introduces the principle of sufficient reason (*principium rationis sufficientis*) which holds that: "Forever fact F, a reason or justification must be given why F is the case"[4] [11].

In fact, Schopenhauer identifies not one but rather four distinct roots (*Wurzeln*), as he calls them, of sufficient reason. As there are two fundamental forms of knowledge (intuitive and abstract), these various roots may likewise be further subdivided into intuitive and abstract sufficient grounds. First, with respect to intuitive knowledge, a principle of sufficient reason may be demanded for: the recognition of causal relations (1_ground of becoming), motivations (2_ground of action), and mathematical intuitions (3_ground of being). Second, with respect to abstract knowledge, the principle of sufficient reason governs both the form of thinking itself along with any rational justifications that follow from the logical processes of reasoning (4_ground of knowing).

It is in terms of the relationship between the sufficient ground of being and knowing that Schopenhauer's views of mathematics evolve. On the basis of logic, thought can proceed from one judgment to another in such a way that the truth (or falsity) of the conclusion is dependent upon the truth (or falsity) of the premises. This being the case, it follows that justifications for inferred conclusions will be "founded" upon prior justifications in relation to premises. Of course, there is nothing surprising in such a result, certainly in relation to mathematics itself. The problem, however, is that Schopenhauer has asserted what amounts to a radical divide between intuition and abstraction so that in the case of mathematics, the justification of second-order abstract truths will be founded upon the justification of first-order intuitive truths. Although this may not at first glance appear problematic,

[4]Schopenhauer uses Christian Wolff's formulation: "Nihil est sine ratione cur potius sit quam non sit. Nothing is without a reason for its being" [13, p. 5].

in relation to the kinds of proof procedures commonly employed by mathematicians, a number of difficulties result in terms of his account of them.

To understand this, it is necessary to further discuss Schopenhauer's account of the nature of truth itself. Following discussion of the faculties of Reason and Judgment, Schopenhauer turns his attention within the PSR (§§ 29–33) to the discussion of truth, where a variety of distinctions are identified. He first divides truths into those that are *material*, as founded upon "intuitive representations," from those that are *formal*, as founded upon either logic or the "formal conditions of all thinking" [4, pp. 126–127]. Both classes of truths are further subdivided into two types. Material truths are subdivided into *empirical* and *transcendental* truths, the former being founded upon an empirical representation (space, time, and causality), the latter being founded upon a formally intuitive representation (either space or time), as in the case of mathematics.[5] For example, whereas the statement that "Socrates is a man" would be an empirical truth, the assertion that "A straight line is the shortest distance between two points" would be a transcendental truth.

Formal truths are further subdivided into *metalogical* and *logical* truths. The former are founded upon the laws of thinking themselves, and the latter are hierarchically ordered truths that have as their sufficient ground a prior judgment as the direct result of the formal processes of reasoning. Examples of metalogical truths would be the principle of identity, non-contradiction, excluded middle and sufficient reason. Alternatively, an example of a logical truth would be "A triangle is a space enclosed within three lines" for the reason that, as Schopenhauer holds, the judgment is founded upon the principle of identity [13, p. 125].

This last example of logical truth is of utmost significance to our discussion. Logical truths and their corresponding judgments may be founded not only upon the formal truths of reason, but they may also be founded upon material truths, which is precisely the case, as Schopenhauer sees it, for the proof procedures employed by mathematicians. To his discussion of such procedures, I now turn.

4 Geometrical Proof Procedures

Within the PSR, Schopenhauer notes that for geometry it is only in: "dealing with axioms that we appeal to intuition. All the other theorems are demonstrated" [13, p. 159]. Here Schopenhauer's account patterns classical Euclidean geometry where from the assumption of indemonstrable, self-evident axioms, the geometer proceeds to the demonstration of second-order propositions or theorems that are logically certain, but not self-evident. For Schopenhauer, the self-evidence of the axioms (as transcendental truths) is rooted in the principle of sufficient reason, in particular, the

[5] I omit discussion of statements founded upon the principle of sufficient reason of action (in relation to motives and the will) for the reason that such examples have no direct bearing upon the argument here.

ground of being as tied to our intuition into space, as one of the two fundamental forms of human sensibility. On the other hand, the logical certainty of the theorems (as logical truths) result from an entirely different source, which is to say, the ground of knowing, as tied to reason and judgment. Of this latter procedure, Schopenhauer notes that: "The logical truth of the theorem is thus shown, but not its transcendental truth . . . as it lies in the reason of *being* and not in the reason of *knowing*" [13, p. 159].

What we see is that geometrical axioms and corresponding theorems ultimately rest upon what amounts to entirely distinct sufficient grounds (being vs. knowing) as well as kinds of truth (material or transcendental vs. formal or logical). This being the case, the whole of Euclidean geometry, as Schopenhauer conceives it, may itself be divided on a similar basis. For there are the truths of the axioms, being materially transcendental (hence, intuitive) in nature. Alternatively, there are the truths of the theorems, being formally logical (hence, abstract) in nature. Although the latter truths are in some sense tied to the former, for in the proofs of the theorems the axioms are supplied as assumed and supporting truths, the interesting consequence is that the two kinds of truths have in the end very little to do with one another. Schopenhauer likewise sees logical truths as quite redundant in a twofold way.

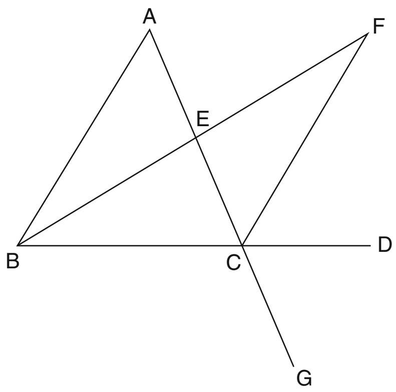
First, logical truths are redundant from the point of view of our knowledge of geometry. As geometry is founded upon an a priori intuition into the form of space, to that extent the logical certainty derived from proofs themselves is never in and of itself sufficient to explain *why* any particular geometrical truth is the case, but only *that* it is the case. Thus Schopenhauer asserts that: "proof by indicating the reason of knowledge only effects conviction (*convictio*), not knowledge (*cognitio*)" [13, p. 159]. In effect, logical truths result in a kind of "epistemic vacuity" (or emptiness), as I will later discuss in more detail, from the perspective our knowledge and further leave us, as he notes, with a "disagreeable feeling"[6] [13, p. 159].

In the second place, logical truths are redundant from the point of view of geometry itself as a mathematical theory. For although proofs can have "confirmatory" utility—as Schopenhauer sees it—we simply do not need them. In support of this view, Schopenhauer offers a number of examples within the PSR that serve to intuitively complement the logical proofs for propositions given within Euclid's *Elements*. (PSR § 39) As these examples are of some significance to this discussion, in what follows I reproduce Schopenhauer's discussion of Proposition 16. First, of this proposition, Euclid states the following:

[6]Schopenhauer is in particular opposed to proof procedures that make use of *reductio*, as Dale Jacquette notes: "Schopenhauer criticizes Euclid's style of mathematical proof for its lack of intuitive insight into what he refers to as the *ground of being* (*ratio essendi*) of mathematical theorems. He maintains that *reductio* reasoning in mathematics offers only conviction (*convictio*) based on reasoning (*Vernunft*) without understanding (*Verstand*). The latter epistemic state he believes can only result from a perceptual grasp of the basis for a mathematical truth, which *reductio* thinking never affords. Proof by contradiction, according to Schopenhauer, at most convinces us *that* a proposition is true without offering any satisfactory insight into *why* it is true" [8, p. 247].

Proposition 16. *In any triangle, if one of the sides by produced, the exterior angle is greater than either of the interior and opposite angles*.

Let ABC be a triangle, and let one side of it BC be produced to D; I say that the exterior angle ACD is greater than either of the interior and opposite angles CBA, BAC.

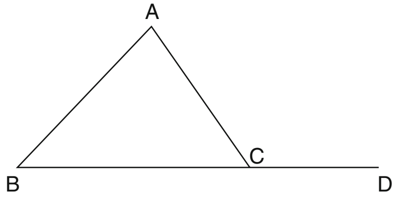


Let AC be bisected at E, and let BE be joined and produced in a straight line to F; let EF by made equal to BE, let FC be joined, and let AC be drawn through to G. Then, since AE is equal to EC, and BE to EF, the two sides AE, EB are equal to the two sides CE, EF, respectively; and the angle AEB is equal to the angle FEC, for they are vertical angles. Therefore, the base AB is equal to the base FC, and the triangle ABE is equal to the triangle CFE, and the remaining angles are equal to the remaining angles, respectively, namely those which the equal sides subtend; therefore, the angle BAE is equal to the angle ECF. But the angle ECD is greater than the angle ECF; therefore, the angle ACD is greater than the angle BAE. Similarly, also, if BC is bisected, the angle BCG, that is, the angle ACD, can be proved greater than the angle ABC as well. Therefore, etc. Q.E.D. [7, pp. 279–280]

The primary point to here identify is that Euclid proceeds *first*, through the construction of lines and triangles, and *second*, confirms the proposition logically, and thus indirectly on the basis of the ground of knowing. From this perspective, the theorem has been shown to be true. But have we obtained any insight into the nature of it? This Schopenhauer denies. To the contrary, this proof, and similar kinds of proof offer: "no insight as to *why* that which it asserts is what it is" for the reason that "we have not found its reason of Being" [13, p. 159]. Effectively, such and similar kinds of proof leave us with what Schopenhauer has called a "disagreeable feeling."

For Schopenhauer, inasmuch as mathematics is fundamentally intuitive in nature, to that extent our *knowledge* of mathematics must be founded upon an immediate insight into the ground of being so that logical proofs, at a fundamental level, become for the most part redundant. In response to Euclid's proof, Schopenhauer thus offers his own *intuitive* version and proof procedure for Proposition 16:

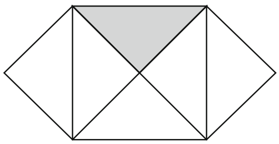
For the angle BAC to be even equal to let alone greater than, the angle ACD, the line BA toward CA would have to lie in the same direction as BD (for this is precisely what is meant by equality of the angles), i.e., it must be parallel with BD; that is to say, BA and BD must meet (reason of being), and must thus do the contrary of that which would be required for the angle BAC to be of the same size as the angle ACD.



For the angle ABC to be even equal to, let alone greater than, the angle ACD, line BA must lie in the same direction toward BD as AC (for this is what is meant by equality of angles), i.e., it must be parallel with AC, that is to say, BA and AC must never meet; but in order to form a triangle BA and AC must meet and must thus do the contrary of that which would be required for the angle ABC to be of the same size as ACD. [13, p. 163]

Again, the proposition has been "proved," but in this case intuitively. The difference, however, is that Schopenhauer's version, as he contends, shows us *why* it is that for any triangle, in producing any of its sides, the exterior angle should be larger than the interior and opposite angles. Rather than appealing to reason, that is, to logical confirmation for his proof (ground of knowing), Schopenhauer instead appeals to our inner intuitive understanding (ground of being) of spatial structure. On the basis of such knowledge, it becomes evident that line BA toward CA can never lie in the same direction as BD, for then the triangle would collapse; and yet this is precisely what the opposite angle BAC would require if it were to be equal to or larger than ACD. He then repeats this same procedure for the other interior angle ABC.

As a final example, I reproduce Schopenhauer's example below.[7] The figure is employed by Schopenhauer as evidence in support of Euclid's proof of Proposition 47 (the Pythagorean Theorem) that holds that for any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the two sides:



By way of explanation, as each side of the larger square forms the hypotenuse of one of the right triangles that divide it, the square of the hypotenuse of each of these right triangles is thus equal to the area of the larger square itself. So too as each

[7] Schopenhauer makes use this example twice within his work—within the WWR (W1, §15) and the PSR (§39).

side of each of the right triangles constitutes an area that is equal to exactly *half* of the larger square, to that extent it follows that the sum of each of the sides is equal to the whole. Of course, the given explanation is unnecessary. And this is precisely Schopenhauer's point. For when once the figure is seen, that is, intuited—insight into the (transcendental) truth of the theorem follows.

Based upon the above examples, the redundancy of logical proofs for geometry is found to be twofold. First, it follows from the distinction between logical and transcendental truth, since our *knowledge* of mathematics is founded upon the latter. Second, and more significantly, this former division is itself founded upon the broader distinction that Schopenhauer has already drawn between intuition (material truth / ground of being /understanding) and abstraction (formal truth/ground of knowing/reason). So Schopenhauer concludes that not only can evidence, as instanced above, be given for the two propositions 16 and 47, but he further contends that similar kinds of intuitive insight: "might be brought to evidence in every theorem"[8] [13, p. 161]. Within the WWR, he announces an even stronger claim: "We demand the reduction of every logical proof to one of perception (*anschauliche*)" [15, W1, p. 69]. This latter demand likewise follows from Schopenhauer's larger epistemological program, which is in general inimical to forms of inquiry that tend to be highly abstract and logical in nature, to be shortly discussed.

For now, were one so inclined to accept Schopenhauer's account of geometry, his demand for the exposure of the ground of being for geometrical theorems might appear to offer some benefit. For we might compare the above "visual argument," as Roger Nelsen characterizes such examples in his *Proofs Without Words* [12, p. vi], with the procedure adopted by Euclid throughout the *Elements*. Whereas the visual argument once intuited yields immediate insight into the truth of the Pythagorean theorem, Euclid's proof of that same theorem demands a procedure consisting of more than ten logically interconnected assertions for the purpose of confirming that very same theorem.[9] So too, whereas the visual argument is given in-itself and without any other presuppositions, Euclid's proof demands a host of axioms and prior theorems. As a final point, as no assumptions must be made in order to ascertain the truth exposed in the above visual argument, it follows that no (formal) knowledge of geometry is required in order to comprehend even the most complex geometrical theorem, insofar as the ground of being has been intuitively exposed for it. On the other hand, following Euclid's method, a grasp of even the basic truths of geometry would require the quite tedious work of plodding through an increasing number of prior assumptions.

[8]Following Gordon Brittan [2], we might interpret Schopenhauer as espousing an "evidentialist" approach to mathematics. Based upon his espousal of Kant's views, one might likewise and with some confidence assume that Schopenhauer also read Kant in this way.

[9]For a more detailed discussion of this subject in relation to Euclid's demonstration of the Pythagorean theorem, I refer the reader to my article on the topic [3].

5 Some Problems

Although Schopenhauer's approach to geometry may doubtless have value, especially in regards to its application to education, there are nonetheless problems that ought to be identified and discussed. In what follows, I identify *three* such problems. It is to be noted that although the discussion here centers around Schopenhauer, some of these difficulties doubtless find their source in Kant's own account of mathematical intuition, though discussion of the matter extends beyond the confines of this essay.

Problem 1 Misleading Intuitions

In the first place, there is the problem that intuition, for all its immediacy and self-evidence, may nonetheless mislead with respect to our knowledge of geometrical truth. In this, Euclid's fifth or parallel postulate serves as a representative example. Indeed, no other postulate has had a more turbulent history, nor has any other geometrical problem yielded more surprising results. The heart of the difficulty is that from the perspective of normal intuition, parallel lines ought never to meet. But "normal" intuition here refers to normal Euclidean intuition and the problem, as geometers would later come to recognize, is that our intuitions into space need not necessarily be governed by such Euclidean notions of space. Inevitably, doubt regarding the self-evidence of the parallel postulate led to various attempts to prove (or disprove) it, and although these attempts were ultimately unsuccessful, such efforts eventually led to the development of alternative and quite consistent forms of non-Euclidean geometry where the parallel postulate failed to hold. As Howard Eves notes in his *Foundations and Fundamental Concepts of Mathematics*: "it is now known that the parallel postulate cannot be deduced as a theorem from the other assumptions of Euclidean geometry but is independent of those other assumptions" [4, p. 61].

It is noteworthy that Schopenhauer failed to grasp the significance of the doubt underlying Euclid's parallel postulate even during his own time. Indeed, he saw attempts to prove (or disprove) it as futile for the reason that the postulate, according to his view, is intuitively self-evident, as he notes in the second book to his WWR: "no such proof can be produced," he says, for the reason that, "there is nothing more immediate" [15, W2, p. 130]. In other words, it is not normal intuition that is to be censured but rather mathematicians for censuring normal intuition.

But the problem is that although our intuitions may guide insight and discovery, without the certainty of logical demonstrations, such intuitions may also serve as false guides. Left to intuition alone, mathematics must stand on quite precarious grounds. On this point, Schopenhauer could of course respond that his particular intuitive approach to mathematics is not necessarily intended to replace but rather to complement the logical approach standardly employed by mathematicians.[10]

[10]Schopenhauer thus remarks in the PSR that: "I do not mean to suggest the introduction of a new method of mathematical demonstration, nor the substitution of my own proof for that of Euclid . . .

Despite this, the example of the parallel postulate along with modern developments in the foundations of mathematics where logic has been shown to serve as a fundamental basis of our knowledge of mathematics (including geometry) leaves, as I see it, little "space" for justification of Schopenhauer's views. Pedagogically speaking, his views do perhaps find utility, but apart from this it seems that mathematicians do well to stick to the standard fare of logical proof and demonstration.

Problem 2 The Particularity of Intuitive Evidence

A second problem to be identified is the particularity of intuitive evidence. Consider once again the above visual argument for the Pythagorean theorem. We might inquire into precisely what it is that is intuited in this figure. The answer of course seems obvious. We intuit a square and embedded right triangles that point us to the theorem of Pythagoras. But how do we know that this applies *for all cases*? In response, Schopenhauer might perhaps appeal to Kant for an answer, who notes:

> [T]o construct a concept means to exhibit *a priori* the intuition corresponding to it. For the construction of a concept, therefore, non-empirical intuition is required, which consequently, as intuition, is an individual object, but that must nevertheless, as the construction of a concept (of a general representation), express in the representation universal validity for all possible intuitions that belong under the same concept. [10, A713/B741]

Kant's account of the process may be summarized as follows. Within thought, a mathematical concept (e.g., "right triangle") is constructed. For such a construction, two things are required. First, it is necessary to intuitively produce the figure either within imagination (purely) or else to draw it (empirically), e.g., on the board. Now the resultant figure is itself singular or particular, being an object that is either imagined or seen. At the same time, in the construction of the concept, the figure serves as a *representative* of intuition, so that it can be said to hold *for all cases*.

Second, the manifold of intuitions, of which the examples serves as but a representative, must be schematized. This is in fact the far more significant point in relation to Schopenhauer's own views. For with Kant, it is not enough that an intuition be produced for the construction of a concept. What is further required is that the particular figure or image be schematized in relation to pure concepts. In the CPR, Kant notes that:

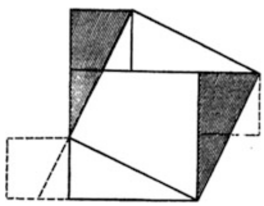
> No image of a triangle would ever be adequate to the concept of it. For it would not attain the generality of the concept, which makes this valid for all triangles, right or acute, etc., but would always be limited to one part of this sphere. The schema of the triangle can never

I merely wished to show what the reason of being is, and wherein lies the difference between it and the reason of knowing" [13, pp. 163–164]. At the same time and elsewhere he seems to suggest the opposite: "To improve the method of mathematics, it is specially necessary to give up the prejudice that demonstrated truth has any advantage over truth known through perception or intuition, or that logical truth, resting on the principle of contradiction, has any advantage over metaphysical truth, which is immediately evident, and to which also belongs the pure intuition of space" [15, W1, p. 73].

> exist anywhere except in thought, and signifies a rule of the synthesis of the imagination in regard to pure shapes in space.[11] [10, A141/B180]

Returning to Schopenhauer, we encounter a difficulty. The difficulty is that the radical division that he has formulated between creates a problem in regards to the representative functionality of singular intuitions. In the first place, Schopenhauer has entirely stripped intuitive cognition of its conceptual content. For he *insists* that nothing abstract or discursive be included within the content of perception. Within the context of Schopenhauer's thought there is, therefore, no room for either pure concepts or else for the schematism of pure concepts. But then it seems the particular intuition that is produced within the construction of a mathematical concept *cannot be synthesized*, so that the generality that can be said to hold for intuition in Kant's case cannot be said to hold in the case of Schopenhauer. Instead, we are left with the question of how intuitions can serve as both representatives while simultaneously having no *a priori* contact with the conceptual constructions that are thereafter produced. As Schopenhauer suggests, (mathematical) concepts are the forms of second-order knowledge. They are universal abstractions of singular or particular intuitions. They are (second-order abstract) representations of (first-order intuitive) representations.

As a final point, it is worth noting that later on in life, Schopenhauer appears to have recognized these difficulties.[12] This is seen in his correspondence with Johann August Becker, an attorney in Mainz and enthusiast of Schopenhauer's works. Commenting on Schopenhauer's suggestion (WWR I, §15) that a visual argument can be given not only in the case of an isosceles right triangle (as in the above example) but: "Even in the case when unequal sides contain the right angle," [15, W1, p. 73], Becker produces the following figure in the postscript to his letter (25 May 1852) to Schopenhauer [1, p. 69]:



What is evident is that the above figure offers an instance of the Pythagorean relation for the case of a non-isosceles right triangle. What is further evident, however, is that only a singular instance has been illustrated, but not all possible instances. Recognizing this, Schopenhauer responds:

[11] Regarding Kant's explanation here, Lisa Shabel notes that: "A mental act of mathematical construction must accord with a rule of synthesis prescribed by a pure concept of understanding" [16, p. 113].

[12] Special thanks to Dr. Jens Lemanski (Fern Universität in Hagen) for pointing out this correspondence.

> Your figure accomplishes the matter well, but is as a number of devilishly glued together pieces of furniture, which break with use, and as one piece is fixed, the other slides out of hand. One will be quite confused in the process. This stands in contrast to my simple figure, which really satisfies the heart. Overall, this is due to the equality of the sides. Without this, the matter is terribly difficult, if not impossible (to show). I have often tried.[13] [1, p. 70]

Problem 3 Epistemic Vacuity

There is then a third and final problem, which is that for Schopenhauer logical proofs become epistemologically vacuous. Here the problem is that intuition and proof have been far too strongly differentiated within the context of Schopenhauer's philosophy of mathematics. As already shown, Schopenhauer downplays the logical role of proofs so that in the end they have little to no epistemic value. But this inevitably renders proofs vacuous from the point of view of our knowledge of geometry. In other words, proofs can help to confirm what is already known on an intuitive basis, but they do little, epistemologically speaking, in terms of contributing to our knowledge of truth.

It should be noted that "epistemic vacuity" does not imply that proofs fail to provide knowledge of the truth (or validity) of a theorem. It is that proofs demonstrate only that, but not why, it is true. But from this perspective, the epistemic certainty obtained on the basis of a proof appears *negative* or *privative* in nature. In other words, from the point of view of our *knowledge* of some theorem, a proof reveals only *that* it is not-false, but not *why* it is true. For a positive (epistemologically grounded) knowledge of truth, the theorem must be related back to its ground of being in an intuition into space, following which the privative confirmation "that" it is, is replaced by the positive understanding "why" it is. Thus Schopenhauer notes in the PSR that: "When once the reason of being is found, we base our conviction of the truth of the theorem upon the reason alone, and no longer upon the reason of knowing given us by the demonstration" [13, p. 160]. Effectively, logical proofs can be dispensed with. But then such proofs at bottom lack epistemic value.

6 On Disinclination for Mathematics

Now the heart of the difficulties associated with Schopenhauer's account of intuition and proof in mathematics is rooted, I contend, in the radical distinction that he has drawn between intuition and abstraction. For unlike Kant who distinguishes these two sources of knowledge while nonetheless retaining a link on the basis of the pure concepts or categories, Schopenhauer has effectively jettisoned all manner of universality from the cognitive nature of perception. Although his criticism of Kant's account in this respect is in some sense warranted, it nonetheless creates a "rift," as it were, between these two sources that, as I see it, is unsuccessfully

[13] My translation.

mediated by his later account of the faculty of judgment and the principle of sufficient reason. Still more, Schopenhauer argues that the faculty of understanding is equally possessed by both humans and animals. In consequence, apart from a few abstract trimmings, humanity can make no boast of its insights into mathematical truths—such insight being the equal possession of both human and animal. Then again, this appears to be Schopenhauer's view of the matter.

A few points might be summed up in support of this claim. In the first place, Schopenhauer sees abstract thought itself as second-order cogitation. He characterizes concepts in Humean terms as being nothing more than a: "copy or repetition of the originally presented world of perception" [15, W1, p. 40] and again as, "representations drawn from representations" [13, p. 115]. Of logic, he notes that it "can never be of practical use," being a mere, "knowing in the abstract what everyone knows in the concrete" [15, W1, p. 45]. In regards to mathematics, he is particularly critical. Although offering certitude, for reason of its *a priori* nature, mathematics can teach us "only what we already knew beforehand" for the reason that both it as well as logic are sciences that are spun "entirely out of ourselves" [15, W2, p. 121]. Thus humanity is no better off than the animal in its knowledge of mathematics. Finally, in remarks characteristic of Schopenhauer, the genuine individual of knowledge, which is to say the *genius*, will feel a "disinclination for mathematics" so that mathematics (as well as logic and similar forms of abstract thought) will be "repugnant to genius" [15, W1, p. 189]. Of course, Schopenhauer's views here echo his larger epistemological program where art and the (intuitive) contemplation of the Platonic ideas serve as the very height of knowledge. But not only. For Schopenhauer's views echo both his own personal distaste (or "disinclination") for mathematics as well as popular notions about mathematics expressed during his own time.

Two historical examples, noted by Schopenhauer himself, may be given in support of this. First, Schopenhauer notes that his own approach mirrors developments among mathematicians and teachers of mathematics seen in Germany during his own time, in particular:

> The most positive work in this direction has been done by Herr Kosack, instructor of mathematics and physics at the Nordhausen Gymnasium, who added to the programme for the school examination of 6 April 1852 a detailed attempt to deal with geometry in accordance with my main principles. [15, W1, p. 73]

Second, in at least two separate places Schopenhauer cites in support of his views a review article written by W. Hamilton and published in the *Edinburgh Review* (1836)[14] [6]. The review examines an earlier essay (later appended to a book) written by a Rev. William Whewell entitled *Thoughts on the Study of Mathematics as a part of a Liberal Education* [17]. Among the many claims made by Whewell within this essay, the most important for this discussion is his view that the object of liberal education is to develop the "whole mental system of man" and in particular

[14]This article is cited by Schopenhauer in both the WWR [15, W2, p. 131] as well as his *Parerga and Paralipomena* [14, p. 489n5].

the "reasoning power" as this will: "enable persons to proceed with certainty and facility from fundamental principles to their consequences" [17, p. 139]. Logic and mathematics are thus recommended as the most effective means toward attaining this end. From this perspective, Hamilton's review serves as a sustained attack upon Whewell's ideas and indeed many of the points that he makes echo Schopenhauer's own views of the value of both logic and mathematics.

A few examples will suffice to highlight the connection. Hamilton notes that the study of mathematics is often recommended for its cultivation of the powers of reasoning and argumentation. But as most human reasoning deals with contingent matters: "the inutility (of mathematical reasoning) is perhaps the greatest" [6, p. 426]. It is of little use in "detecting and avoiding" fallacies [6, p. 427]. As every step in its logical procedure follows from the previous without any constructive criticism, to that extent mathematics: "calls forth an absolute minimum of thought" and so exercises the faculty of reason at "its most limited development" [6, p. 428]. Finally, echoing Schopenhauer's views on genius, Hamilton notes that: "to minds of any talent, mathematics are *only difficult because they are too easy*" and so become "more peculiarly intolerable by minds endowed with the most varied and vigorous capacities" [6, p. 430].

Although logical proof procedures find utility to the extent that they serve to confirm and to add conviction (*convictio*) to that which is already known (*cognitio*) on an intuitive level, undo emphasis upon such procedures inevitably leads to what amounts to the stunted-growth of inner human potential. For this reason, the genius will be inherently *disinclined* to do mathematics. The disinclination does not result, however, from the fact that the genius *cannot do* mathematics. Rather, the disinclination follows from the fact that any excessive emphasis upon reasoning (especially in regards to education) might otherwise deter the development of genius, a development that naturally demands the extension of its intellectual capacities beyond the limitations of second-order forms of knowing.

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Schopenhauer on Diagrammatic Proof

Michael J. Bevan

Abstract The present paper discusses the treatment of diagrammatic proof in Schopenhauer's philosophy of mathematics. 'Picture proofs' have been the subject of some scattered contemporary debate, and my aim here is to see whether Schopenhauer's treatment might prove fruitful in the context of recent discussion. In particular I argue that Schopenhauer's remarks on diagrammatic proof, though few and far between, might be able to provide conceptual tools adequate to meet some of the broader challenges facing the legitimacy of such proof. In § 1 the notion of a picture proof is introduced and two general objections to its legitimacy are formulated. In § 2 I set out what I take to be the substance of Schopenhauer's advocacy of picture proofs and in § 3 I formulate replies to these challenges based on the Schopenhauerian distinction between a proposition's ground of knowledge (*Erkenntnißgrund*) and its ground of being (*Seynsgrund*).

Keywords Schopenhauer · Diagram · Picture · Proof

Mathematics Subject Classification (2020) Primary 03A05, Secondary 00A66

1 Two Challenges to The Legitimacy of Picture Proofs

A quite ubiquitous way of characterising proof in mathematics is as a certain species of argumentation—for example, as sound, deductive argumentation which is non-circular, commits no fallacies, etc. Some might emphasise additional criteria:

I would like to thank Severin Schroeder for encouraging my interest in the present topic and for offering very helpful comments on an earlier draft of this paper. I also owe thanks to Jens Lemanski and the other organisers and attendees of the conference on *Logik, Mathematik und Sprache bei Schopenhauer* at the Hagen FernUniversität—for which this paper was originally written and delivered as a talk—for their hospitality. Their comments and suggestions, along with those of two anonymous referees, have improved the content immeasurably.

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intuitionists will add that a proof must be constructive, relevance logicians will say that a proof must have a conclusion relevant to its premises, and so on. The general assumption that demonstration is a species of argumentation can be detected as far back as Aristotle, for whom the study of proof taken up in the *Prior Analytics* [1] consists of the study of categorical premises and their syllogistic consequences.[1] Yet if one takes up the assumption that mathematical demonstration in particular must be argumentative, the diagram below (Fig. 1) presents a problem.

This hexagon is the very one supplied by Schopenhauer in the *World as Will and Representation* ([10], Vol. 1, § 15) and in the *Fourfold Root* ([11], § 39). What is most notable about it is that it seems sufficient, by itself and unaccompanied by words, to establish a general geometrical proposition about right triangles—namely, that those with two equal sides satisfy the Pythagorean theorem.

'Picture proofs' have been the subject of scattered debate in contemporary philosophy of mathematics. They cast the aforementioned argumentative assumption about mathematical proof into doubt, for pictures are not arguments—in fact they seem to be a radically different sort of thing: arguments can be stated and defended, deemed cogent, persuasive, circular or fallacious, and in what sense can any of these descriptions be ascribed to pictures? What can it mean to assert or deny that a picture is question-begging, or otherwise circular? Moreover all arguments must have premises and yet, evidently, if we were to ask someone who offered us a picture proof what the *premises* of their picture were we would be making a category error.[2] If pictures are not arguments, then we have two options in proceeding. Either we reject the assumption that all demonstration is argumentation, or we somehow account for our receiving general mathematical knowledge from such pictures in such a way as to steer just clear of calling them *proofs*. One way to cash out the latter option is to say that, rather than being a proof, a diagram instead *represents* or in some way *encodes* an argument, and that this argument is what establishes the proposition.[3] Now there might appear to be an element

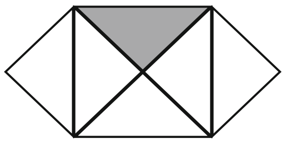

Fig. 1 Schopenhauer's Pythagorean Hexagon

[1]The historical emphasis on linguistic-argumentative proof and the corresponding marginalisation of diagrammatic proof is a theme explored in detail by Greaves [4].

[2]I should note that Norton [9], contrary to the preceding considerations, idiosyncratically *does* hold both thought experiments and picture proofs to be arguments. Under such a position, the legitimacy of picture proofs is rendered entirely unproblematic, and the present discussion is entirely uninteresting.

[3]This offloading of epistemic work, as it were, onto represented or encoded arguments can be seen implicitly even in very sympathetic treatments of diagrammatic proof. To my mind, the most conspicuous example is the recent tradition of diagrammatic proof theory (cf. Shin [12]; Mumma [7]; Shin et al. [13, § 2ff.]), wherein proof-theoretic techniques are applied to precisely defined

of stubbornness in this response—why *not* allow diagrams to be proofs in their own right? Why should we treat the assumption that proofs must be arguments as anything more than a historically entrenched prejudice? But in fact, there are some grounds for caution—challenges to the legitimacy of diagrammatic proof as such. Two particularly pressing challenges are what I will term the *objection from particularity* and the *objection from misleading pictures*.[4]

The objection from particularity is the descendant of an old family of reservations against diagrammatic reasoning, one ancestor of which we find addressed by Proclus in his commentary on Euclid [6, p. 162]. The problem is this: where there is a geometrical diagram, there is a diagram of a particular geometric figure. Hence in reasoning with a diagram, one is reasoning and making judgement about a single figure. How then can a geometer ever be justified in arriving at general results by the use of diagrams? Now when diagrams are used in tandem with arguments, there is a natural way of answering this which Proclus adopts: provided that the accompanying argument only makes reference to the relevant features of the figure, the result will apply to all figures which share these features and thus one may come, via the proof, to knowledge of a general proposition. Such a strategy is not open to the advocate of picture proofs however, for a picture proof is by hypothesis an unaccompanied diagram. There is no argument accompanying Schopenhauer's hexagon for example, and so we cannot talk about it 'making use of', in the sense of referring to, this or that feature of the right triangle pictured.

The second objection is based on the thought that some pictures can be misleading—they can suggest the truth of a proposition which is false. The existence of such pictures gives rise to a problem of epistemic luck: if there is no inherent difference between misleading pictures and picture proofs, then even in cases where one happens to gain a true belief through a diagram, one will merely have been lucky that they were not actually looking at a misleading one. To contrast this with arguments: an argument is misleading (in the sense of seeming sound but being really unsound) only if it is either invalid or has a false premise. Thus, prior to knowing the truth-value of the conclusion, it is possible in principle to see whether or not an argument might be misleading by checking the truth of its premises, or its form. But if the only way to know that a picture is not misleading is to have an independent proof which establishes the proposition in question, then how can a picture by itself ever serve to provide mathematical knowledge?

systems of drawing and manipulating diagrams. Such, in effect, treat diagrams as another sort of mathematical notation, and thus their mode of proof as largely discursive (rather than purely intuitive).

[4]These names are not widespread, but the objections are. Each is highlighted in various ways by, for example, Shin (cf. [12, p. 3ff.]), Brown (cf. [2, p. 161ff.]), Norman (cf. [8, p. 144]), Starikova (cf. [14, p. 85]), Mumma (cf. [7, pp. 255–262]).

2 Schopenhauer's Advocacy

Despite the fact that Schopenhauer provides a paradigm case of diagrammatic proof in his Pythagorean hexagon (Fig. 1) his advocacy of such has been, to the best of my knowledge, entirely neglected in the contemporary literature on the topic.[5] My hope is to rectify this by consulting his remarks on the matter, and then to formulate responses based on these to the challenges just outlined.[6]

Because Schopenhauer's advocacy of diagrammatic proof has been so neglected, I think it is prudent that I should first say a little to establish its existence beyond doubt, before summarising what I take to be its main thrust. For perhaps one might think it a leap to call Schopenhauer an advocate of diagrammatic *proof*—after all, he never explicitly refers to his own example as a proof (*Beweis*). He comes close in a number of places, seemingly within a hair when he says that "[t]he mere sight of it without any words conveys twenty times more conviction than does Euclid's mousetrap proof" [11, p. 205], and apparent near-misses like this might lead one to suspect that he withholds the term on purpose. Moreover one might feel as if Schopenhauer has some reason to withhold the term of 'proof': consider the following passage taken from the middle of the same section of the *Fourfold Root*. Commenting on Euclid's proof that in any triangle, sides subtending equal angles are equal, he says that

> [w]hen we have the ground of being, our conviction of the truth of the proposition is based solely thereon, and certainly no longer on that of [the ground of] knowledge which is given by demonstration. [11, p. 201]

As it relates to geometry, Schopenhauer's distinction between the ground of knowledge (*Erkenntnißgrund*) and ground of being (*Seynsgrund*) of a proposition applies in the following way. For Schopenhauer, 'demonstrations', with particular reference to those of Euclid, give one insight into the ground of knowledge, but not that of being: they give knowledge *that* the theorems are true, compel one to assume their truth on pain of contradiction, but rarely do they grant insight into *why* the theorems are true (cf. [11, pp. 200–202]). This being so, perhaps Schopenhauer avoids calling unaided diagrams 'proofs' so as to avoid suggesting that they supply knowledge in any way analogous to deductive demonstrations. That is to say, unlike the 'mousetrap proof', Schopenhauer's hexagon gives one knowledge via direct

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[5]In the process of review, it was brought to my attention by Dr Lemanski that towards the end of the twentieth century, Schopenhauer's remarks on mathematics yet enjoyed something of a new advocacy amongst a number of German and Swiss mathematicians (cf. [5, pp. 333–334]). Regrettably, this advocacy seems not to have had any detectible interaction with the wider literature on picture proofs and the like.

[6]These remarks are contained in Vol. 1, §15 and Vol. 2, Ch. 13 of the *World as Will and Representation* and, in particular, §39 of the *Fourfold Root*.

intuition into the ground of being of the theorem,[7] and thus supplies knowledge in a radically different way to that in which proof does.

This would be the best case I can think to make in favour of denying Schopenhauer's advocacy of diagrammatic proof, but we can reply to it adequately with two points. Firstly, one may well recognise Schopenhauer's withholding of the term 'proof' when it comes to pictures, and yet doubt that it reflects anything further than a stylistic consideration. Certainly one can see a stylistic reason Schopenhauer might have had in withholding the term 'proof' from pictures since, as was admitted, this helps to avoid the suggestion that pictures and deductive demonstrations supply knowledge in anything like the same way. But this being so, the absence of the term in certain passages cannot then establish deeper philosophical motivation; style would be explanation enough. Indeed, for our second point, we may contradict the thought that Schopenhauer had *philosophical* motivation to withhold the term 'proof' by citing the following passage.

> The whole of geometry also rests on the nexus of the position of the parts in space. It would thus be an insight into that nexus; but, as I have said, as such an insight is not possible through mere concepts, but only through intuition, every geometrical proposition would have to be reduced to this, *and the proof* (Beweis) *would consist merely in our clearly bringing out the nexus whose intuition is required; more we could not do.* ([11, p. 198]; my italics)

Here is a use of the term not in reference to Euclidean demonstrations, but rather to the act of evoking intuition into the ground of being, as occurs in the case of his hexagon. Given this, it is quite clear that he takes his diagram as proving its proposition in the requisitely strong sense.

Establishing this much has allowed me to introduce the crucial distinction between the ground of being and ground of knowledge of a proposition. With this to hand, the core of Schopenhauer's stance on diagrammatic proof can, I think, be stated succinctly as follows. A diagrammatic proof, like Schopenhauer's hexagon, proves a proposition to be the case by displaying the ground of being of this proposition, such that by contemplating the picture, we are able to intuit this ground and come to immediate knowledge both *that* the proposition is true and *why* it is true. Insofar as a picture is able to do this, it is in fact superior to those purely argumentative proofs which present only the grounds of knowledge of their conclusion—that is, the latter do not grant understanding as to *why* the conclusion holds. Such understanding can only be received through intuition of the ground of being, which diagrams are particularly well suited to supply.

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[7]—Albeit, the theorem as restricted to right triangles with two equal sides. It seems as if Schopenhauer took his diagram to establish the general Pythagorean theorem; whether or not it does is irrelevant to our discussion.

3 Applying Schopenhauer's Remarks

We turn back now to the two objections directed at diagrammatic proof that were set out previously. Recall that these were the following.

Particularity If a diagram only represents a particular geometrical figure (a particular triangle or rectangle, etc.), then it seems no conclusions of a general nature can be drawn justifiably by its use—particularly in the case of picture proofs, which are *unaccompanied* diagrams.

Misleading Pictures Some pictures suggest the truth of a proposition that is false. If it is not possible for one to tell merely by looking at a picture whether or not it is misleading, then it is always a matter of luck whether or not one gets a true belief from such pictures. This being so, one must never be able to gain knowledge of a mathematical proposition from a picture alone.

3.1 Particularity and Generality

Starting with the objection from particularity, our task is to offer an account of how general mathematical knowledge can be drawn from an unaccompanied picture. To make the matter concrete, we consider Schopenhauer's hexagon (Fig. 1) as an example, which is supposed to prove the restricted form of the Pythagorean theorem—that is, as restricted to triangles with two equal sides. Call the grey-shaded triangle in the picture T. I distinguish the following propositions:

(P_T) The square of T's hypotenuse is equal to the sum of the squares of T's other sides.

(P) Any right triangle with two equal sides is such that the square of the hypotenuse is equal to the sum of the squares of the other sides

P_T is just the instance of P in the case of T. As I hope to show now, granted that Schopenhauer's hexagon establishes P_T, then on the back of the remarks set out in §2, we can also show it to establish P. The assumption that the hexagon can establish the particular proposition P_T is not so problematic, since the possibility of diagrams establishing *particular* geometrical propositions is not what the objection from particularity calls into question. I also take the assumption to be plausible in itself.

Our reply runs as follows. Supposing that the diagram establishes P_T, on Schopenhauer's account we say that a diagram displays the ground of being of P_T—that is, it shows us *in virtue of what* it is the case that the square of T's hypotenuse is equal to the sum of the squares of T's other two sides. Now if one allows that it shows this to be so *in virtue of* T's being a right triangle with two sides equal, this then serves as the link between the particular and general propositions: since T's being a right triangle with two sides equal *makes it* such that the square of its hypotenuse is equal to the sum of the squares of the other two sides, we can legitimately infer that any other right triangle with two sides equal will also be so. These two features are shown in this instance to stand in a relationship of grounding, and so we see that one is a sufficient condition for the other. Put schematically for the sake of clarity, my thought is that Schopenhauer has the resources to say that a general mathematical proposition of the form ‘all Fs are Gs’ may be proven by a picture if this picture indeed shows that a figure x is G merely in virtue of being F.

This reply functions in a similar way to Proclus’ mentioned previously. Both emphasise that in proofs which work with (for us: are identical to) a diagram, only certain general features of the figure depicted should be considered salient. This notion of salience is, however, cashed out in different ways. For Proclus, it means that the mathematician, in giving an argumentative proof, only “make[s] use of” [6, p. 162] the relevant general features of the figure in formulating his premises. For us, it means that the picture shows that the particular figure has some property in virtue of possessing the relevant general features. In the case of Schopenhauer’s hexagon, the proof shows T’s satisfaction of the Pythagorean theorem as grounded in its being a right triangle with two sides equal.

3.2 Misleading Pictures

Passing to the objection from misleading pictures, I adapt an example from Brown [2], pp. 178–179, cf. [3] to make things concrete. On the Euclidean plane, draw four circles centred at the points $(\pm 1, \pm 1)$ and a fifth at the origin just large enough to touch the other four (Fig 2a). Note that the centre circle is contained in the box $\{(x, y)| -2 \leq x, y \leq +2\}$. Again for three-dimensional Euclidean space: draw eight spheres centred at $(\pm 1, \pm 1, \pm 1)$, and a ninth sphere at the origin touching the other four (Fig. 2b). Note that the centre sphere is entirely contained within the enclosing box $\{(x, y, z)| -2 \leq x, y, z \leq +2\}$.

By drawing these diagrams, it seems as if we now see why the results hold in the two- and three-dimensional cases, and so we may consequently accept the following generalisation.

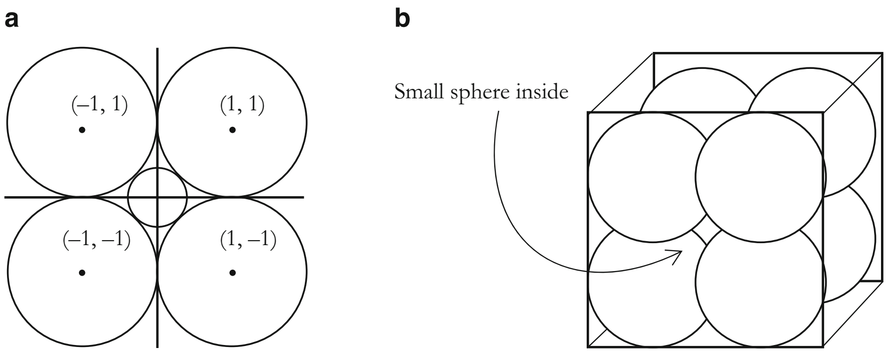

Fig. 2 Brown’s Examples

> For every natural number n: suppose that in n-dimensional Euclidean space we have 2^n $(n - 1)$-spheres each of radius 1 centred at $(\pm 1, \pm 1, \ldots, \pm 1)$, and an additional $(n - 1)$-sphere centred at the origin which just touches the other spheres. Then the $(n - 1)$-sphere at the origin is contained within $\{(x_1, \ldots, x_n) | -2 \leq x_1, \ldots, x_n \leq +2\}$.

But this proposition fails first at $n = 10$ (Ibid., 178). This is a clear case in which pictures mislead us. We can now set out the general objection in detail. Provisionally, say that a candidate picture proof of the proposition *p* is misleading when *p* is false. Now, if our only way of telling apart genuine picture proofs from other pictures is a kind of general feeling (i.e., that this or that picture just seems to show that *p*), and if these feelings are unreliable, then we have no way of knowing whether or not a picture is misleading prior to establishing the truth or falsity of the proposition in question by other means. But if we cannot independently know whether a given picture is misleading without already knowing the truth-value of the proposition it is supposed to prove, then it does not seem as if the picture itself can establish mathematical knowledge.

If the premise of this objection is correct—if, without knowledge of the truth-value of the supposed theorem, our only way of distinguishing misleading pictures from genuine picture proofs is a kind of gut-feeling—then it is clear that the objection is devastating. For we see here, and know from experience, that in mathematics such feelings are often mistaken, and cannot be taken as evidence. Therefore our task must be to find a way to deny this premise, and identify some other way by which one might identify genuine picture proofs independently of prior knowledge of the truth-value of the would-be theorem.

I take it that a Schopenhauerian can reply to this as follows—though I am more tentative as regards success than against the previous objection. In the case of misleading cases such as Brown's, I agree with the objector that the truth of the general proposition is merely suggested by the pictures. Perhaps, as here, it is suggested in such a way as to make the generalisation very plausible, but I say that the suggestion of plausibility is *all* that occurs. In the case of a genuine picture proof, I say to the contrary that an entirely different event takes place, and that this difference must be detectable by introspection. That is, rather than being suggested or made plausible to the subject, I say that the truth of the proposition in such cases is instead *seen* or *grasped*. Unlike in Brown's example, one is not merely given good evidence of the proposition's truth on which to base a justified induction to the general case ('induction', that is, as in the empirical sense). With picture proofs, as in Schopenhauer's case for instance, one instead comes to *see* immediately the truth of the proposition in question, in that they come to observe why it is the case. In doing so, that the (potentially quite complex) proposition is true becomes as immediate a realisation as that two and two are four, or that a ball cannot be red and green all over. This introspective difference can be illustrated by the empirical fact that, when we are told by a mathematician that Brown's generalisation fails at $n = 10$, we may likely do little more than raise an eyebrow. For even though the diagrams make it plausible that Euclidean spaces of any dimension are such that the centre sphere does not escape the box, we can hardly be said to *see* that *being a Euclidean space* is what makes it true in the space that the centre ball does not escape the box—not only because it cannot be this alone which makes it so, else the generalisation would in fact hold, but also because it is not clear that we intuit the general property of *being an n-dimensional Euclidean space* at all,[8] let alone intuit that this property makes anything to be thus and so. Whereas on intuiting its ground of being as displayed in Fig. 1, if one were to tell us that the restricted Pythagorean theorem were false, we would be as certain of their error as if they had told us that 7 and 5 did not make 12.

Unlike the attitude one has to a proposition which is merely very plausible, the attitudes of grasping-that and seeing-that are factive and imply certitude. That is, one cannot grasp that *p*—see that it is the case that *p*—without it actually *being* the case that *p* and without one being certain that *p*. I therefore say that the inherent difference between picture proofs and misleading pictures—that is, that the former but not the latter display the ground of being of their respective propositions—reflects a detectable difference in the attitudes of, on the one hand, merely being persuaded of some proposition, and on the other, grasping its truth. This being so, it now appears far more difficult to claim that in a genuine case of picture proof, one's belief in the theorem is merely fortunately in accordance with facts. When one *grasps* the truth of a proposition, one does not gain a merely fortunately true belief, but one must instead gain knowledge. Such can only occur when a picture is in fact a proof.

4 Conclusion

I had set out to show that, despite their scarcity and neglect, Schopenhauer's remarks on diagrammatic proof are very fruitfully applicable to the contemporary debate over the legitimacy of picture proofs—in particular, that they provide us with conceptual tools adequate to address the most pressing and general concerns about such proofs. I considered that Schopenhauer's conception of a proposition's ground-of-being as intuited via a picture could be drawn out and adapted so as first to address how pictures might establish general geometrical propositions and, second, to defend picture proofs against sceptical worries stemming from the existence of misleading pictures. I hope that this helps towards a corrective to the neglect which Schopenhauer's remarks on the matter have suffered, and that it might thereby open new avenues of discussion.

[8]That is, as opposed to the specific properties of *being a three-dimensional Euclidean space* or of *being a two-dimensional Euclidean space*, of which it is far more plausible that we have intuitions. If there *were* a general intuitive grasping of the property of being an n-dimensional Euclidean space, I suspect the often counter-intuitive results of higher-dimensional geometry (Brown's $n = 10$ case being just one example) would be far less so.

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From Necessary Truths to Feelings: The Foundations of Mathematics in Leibniz and Schopenhauer

Laura Follesa

Abstract I take into account Schopenhauer's study of Leibniz's work, as it emerges not only from his explicit critique in his dissertation *On the Fourfold Root of the Principle of Sufficient Reason* (1813) and in *The World as Will and Idea* (3rd edition 1859), but also from his annotations of Leibniz's writings. Schopenhauer owned many books of Leibniz in his private library and they are full of intriguing annotations. Many of these annotations concern the discussion on logic and mathematical truths and so they are particularly relevant for the study of Schopenhauer's philosophy of mathematics. After a comparison between Leibniz and Schopenhauer's definition of necessary and innate truths, I put alongside what the two authors stated about system and fundamental axioms. Two questions arise from Leibniz's interpretation of Euclid's axioms: the role of 'images' in knowledge and the notion of 'confused' knowledge. These two questions are worth of attention, as they allow to focus on Schopenhauer's theory of 'feeling' mathematical knowledge, as I show in the last section of this paper. To Schopenhauer, knowledge works with intuitive representations, intuition, perception, and, for this reason, feeling is the basis of all conceptions. Schopenhauer's provided a new point of view regarding feeling and intuitive knowledge that involves a special meaning for his philosophy of mathematics.

Keywords Leibniz · Innate and necessary truths · Intuition · Euclid's axioms · Feeling

Mathematics Subject Classification (2020) Primary 03A05, Secondary 01A99

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1 A Denied Source

In his dissertation *On the Fourfold Root of the Principle of Sufficient Reason* (1813), Schopenhauer depicted Leibniz, in comparison to Kant, as 'a poor rushlight'. Kant—he continued—was 'a mastermind, to whom mankind is indebted for the discovery of never-to-be-forgotten truths' [24, p. 206]. Schopenhauer expressly stated that one of Kant's 'chief merits' was 'to have delivered us from Leibniz and his subtleties: from pre-established harmonies, monads and *identitatis indiscernibilium*. Kant has made philosophy serious and I am keeping it so' [24, p. 206]. Despite these assertions, Schopenhauer appointed Leibniz as an indispensable source for the development of Kantian philosophy [24, p. 105]. In his own copy of Leibniz's *Nouveaux Essais sur l'entendement humain* (1703), Schopenhauer annotated in Latin ('*prolusio philosophiae kantianae*') his conviction that Leibniz's interrogative regarding the dependence of truths from experience remained at the origin of Kant's reflection [18, p. 195], [16, p. 49].[1]

I examine Schopenhauer's reception of Leibniz, especially in his handwritten annotations, and its meaning for Schopenhauer's reflection on logic and mathematics. After a short exposition of Leibniz's critique of Plato's doctrine of reminiscence, that attracted Schopenhauer's attention—as this latter's annotations clearly show—for its meaning concerning mathematical truths, I juxtapose Leibniz's and Schopenhauer's judgments of Euclid's system and the nature of this latter's axioms. Two questions emerge from Leibniz's works that are worth of attention for our understanding of Schopenhauer's view of mathematical knowledge: the role of 'images' and the notion of 'confused' knowledge. After a brief consideration of these two aspects of Leibniz's philosophy in the third section, I ultimately investigate Schopenhauer's concepts of 'feeling' and intuitive knowledge, as they involve a special meaning for his philosophy of mathematics. In this way, I argue that Schopenhauer's reading of Leibniz work represented a significant passage in the development of his original thesis of 'feeling' mathematical truths.

In *On the Fourfold Root*, Schopenhauer declared that Leibnitz was the first author who formally defined the principle of sufficient reason and its importance for our knowledge and science. He noticed, at the same time, that this principle was not Leibniz's invention, but was already known before him [24, p. 20]. Notwithstanding his endeavour to diminish the importance of Leibniz in a sort of history of the principle of sufficient reason, Schopenhauer could not avoid a confrontation with Leibniz, one of the first philosophers to have formulated it. Schopenhauer's critical attitude towards Leibniz was carried on in his later works, such as *The World as Will and Idea*, where it became even sharper. Leibniz was depicted by him as someone who just repeated 'with full approval' what Locke had already affirmed in his *Essay Concerning Human Understanding*, 'only with more confusion and indistinctness'

[1] The annotated book was collected as a part of the Schopenhauer-Nachlass at the Archivzentrum—Universitätsbibliothek of Goethe University in Frankfurt am Main, signature UBA Ffm, Na 50, Schop 603/283.

[26, p. 40]. According to Schopenhauer, Leibniz belonged to that kind of 'men who live more in words than in deeds, who have seen more on paper and in books than in actual life, and who in their greatest degeneracy become pedants and lovers of the mere letter' [26, p. 111]. A great error of Leibniz, Wolff, and their followers was, according to Schopenhauer, the fact that they explained knowledge of perception 'as merely confused abstract knowledge', instead of recognising its relation between 'abstract knowledge' and 'what is perceived' [26, p. 111].

These strong words against Leibniz and his school explain the lack of attention that the relationship between Leibniz and Schopenhauer has attracted from scholars.[2] No specific study has been provided to date on this relationship as concerns the discussion of human knowledge and the reflection on logic and on mathematical truth, while very few are the studies concerning Schopenhauer's philosophy of mathematics and logic (cf. [8, 21, 29]). It has been noted that Schopenhauer was not a mathematician as, for instance, Leibniz himself was; he nevertheless developed 'a unique, intrinsically interesting, and in some ways remarkably defensible position' [8, p. 43]. Moreover, Schopenhauer's negative attitude toward Leibniz, which emerges in both his published works and letters, and manuscripts, seems quite surprising if we consider how many of Leibniz's writings he collected in his private library: Latin, French, and German collections of works (*Opera philosophica*, *Oeuvres philosophiques*, *Kleine philosophische Schriften*), Leibniz's biography, and his correspondence with other scholars of his time. Many of these books, especially *Opera philosophica* (1839–1840), exhibit plenty of handwritten underlining, marks, and comments, and what is most intriguing is that they are related to mathematical and logic topics.[3] Schopenhauer's private library additionally contains some books belonging to authors of the so-called Leibnitian school, such as Christian Wolff and Moses Mendelssohn, testifying that Schopenhauer analysed these sources thoroughly.[4] His study of Leibniz proceeded—as he himself admitted—by means of comparison between different versions of the same work and, therefore, not merely through intermediate sources (e.g. by his reading of Kant).[5] Despite this, he never hesitated to display his opposition to Leibniz and to criticise his theories with vehemence.

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[2]Studies on the relationship between Leibniz and Schopenhauer often focused on their metaphysical view (the juxtaposition of Schopenhauer's 'pessimism' with Leibniz metaphysics is a typical topic, cf. [9], [28, pp. 225–238]), and concern also religious, moral, or even psychological matter (cf. [1, 2, 22], [7, pp. 215–223], [5], [23, pp. 679–686]).

[3]Leibniz 1765, sign: UBA Ffm, Na 50, Schop 603/28 [17]; Leibniz 1839–1840, sign. UBA Ffm, Na 50, Schop 603/283 [2]; Leibniz 1720, sign. UBA Ffm, Na 50, Schop 581 [19]; Leibniz 1772, sign. UBA Ffm, Na 50, Schop 603/240 [13]; Leibniz 1740, sign. UBA Ffm, Na 50, Schop 603/239 [15]; Leibniz 1745, sign. UBA Ffm, Na 50, Schop 551 [12]; Leibniz 1734–1735, sign. UBA Ffm, Na 50, Schop 550 [14]; Lamprecht 1740, sign. UBA Ffm, Na 50, Schop 547 Nr. 1 [11].

[4]It is now possible to find them online, because many of these books have been digitalised at the Archive in Hessen Website (Archinsys) <https://arcinsys.hessen.de/arcinsys/start> (last cons. 10/5/2018).

[5]In some of Schopenhauer's handwritten annotations, we find Leibniz defined as "miserable" (e.g. Schop 603/283 [2, pp. 554, 601, 624]) and other similar epithets.

2 'Necessary Truths' and Reminiscence

From the detailed examination of Schopenhauer's handwritten comments, underlining, and marks, it undoubtedly emerges that his study of Leibniz's was related to his meditation on the theoretical foundations of knowledge, especially of logic and mathematics, and the possibility of 'necessary truth'. One of the main works of Leibniz that Schopenhauer may have read many times and on different occasions is the *Nouveaux essais sur l'entendement humain* (*New Essays on Human Understanding*, 1707), that he owned both in French and in Latin version [17, pp. 1–496], [18, I, pp. 194–498]. Here, Leibniz defined 'pure mathematics' as the sole discipline that provides 'necessary truths'[17, p. 34], [16, p. 50]. According to Leibniz, these truths must have principles 'whose proof does not depend on instances nor, consequently, on the testimony of the senses, even though without the senses it would never occur to us to think of them' [16, p. 50]. These principles are, in fact, the principles of logic, of reason. They are independent from the senses, or, in other words, 'innate', as Leibniz explained in the *New Essays*. In this work he maintained that arithmetic and geometry should be considered as innate, as they are implicit in us. Our task is to carefully and methodically 'find them within ourselves', without any reference to empirical truths [16, p. 77].[6]

Leibniz did not deny that all our actual knowledge requires the contribution of sensitivity. However, the senses are not sufficient to him to provide all knowledge, because they just provide instances, that is, 'particular or singular truths' [16, p. 49].[7] In opposition to Aristotle and Locke's notion of 'tabula rasa' and the idea that knowledge can only originate from empirical experience, Leibniz traced back the necessary truths, as Plato did, to their belonging to a 'higher' metaphysical world, and founded our knowledge of them on the argument of 'innatism'. He himself established a connection with the Platonic theory of reminiscence and, in particular, with the ideas expressed by Plato in his dialogue *Meno*, 82 b concerning mathematical truths. Plato demonstrated, according to Leibniz, that one could construct mathematics and geometry 'even with one's eyes closed, without learning from sight or even from touch any of the needed truths' [16, p. 77].[8]

Necessity and innatism, to Leibniz, did not require the existence of the soul in a 'previous' world before its arrival on earth. So, if innate ideas exist, they should be eternally in the soul and cannot be derived from outside. In Leibniz's monadology

[6]Schopenhauer underlined the same sentences in two different versions of the same writing, both of which he owned in his library, namely, in Schop 603/283 [2, p. 208] and in Schop 603/284 [17, p. 33].

[7]Cf. Schopenhauer's copy of this work: Schop 603/283 [2, p. 195].

[8]See the same passage in Schop 603/283 [2, p. 196] and in Schop 603/284 [17, pp. 62–63], where Schopenhauer made his annotations. See also the writing *Reflexions sur l'essai de Mr. Locke*: "Je ne suis nullement pour la *Tabula rasa d'Aristote*; et il y a quelque chose de solide dans ce que *Platon* appeloit la *réminiscence*. Il y a même quelque chose de plus, car nous n'avons pas seulement une *réminiscence* de toutes nos pensées passés, mais encore un *pressentiment* de toutes nos pensées" [2, p. 137].

(theory of monads), each soul is a simple, active substance whose activity only consists of inner representations; no communication, no interaction, no exchange among monads is possible. Schopenhauer was well acquainted with these problems, and he also annotated many passages concerning metempsychosis and recollection in Leibniz's work, in addition to his meditation on necessary truth. At the same time, he confirmed the distance between Plato and Leibniz in his *The World*, assuming that the spirit of Plato 'certainly did not rest' on Leibniz [26, p. 224].[9]

Schopenhauer approached these topics partly through a Kantian perspective: The so-called metaphysical truths to which, with Kant, we assign the position of 'first principles of natural science' are the 'form of all knowledge' and are known directly, that is, a priori [26, p. 88]. In his dissertation *On the Fourfold Root*, he quite rarely made use of the word 'innate' and, when he did, just with this meaning: 'All that is innate in the whole of our cognitive faculty, all that is therefore a priori and independent of experience, is strictly limited to the *formal* part of knowledge: that is, to the consciousness of the peculiar functions of the intellect and of the only why in which they can possibly act' [24, p. 135]. Our knowledge depends on what we acquire from outside, while its necessity depends on the form by which we 'necessarily' acquire it. In this Kantian perspective, Schopenhauer affirmed that the metaphysical foundation of all truths 'cannot lie in abstract principles', but only in the 'immediate consciousness of the forms of the idea' [26, p. 88]. In this way, Schopenhauer also specified that 'the most universal forms of the phenomenon space and time', upon which the procedure of mathematics is based, are 'modes of the principle of sufficient reason', or, in other words, modes of the logical method [26, p. 244]. As we know something by 'applying' these forms to the empirical experience, every phenomenon becomes, to Schopenhauer 'absolutely necessary' and 'determined in that chain of causes and effects which admits of no interruption' [26, p. 370].

It is possible here to notice the difference between the Leibnitian definition of necessary truth and Schopenhauer's ideas. For a better understanding of the latter, with respect to Leibniz's thought, especially as concerns his philosophy of mathematics, a brief summary of the different kinds of truth in Leibniz is required. What he called 'necessary truths' do not belong to the empirical world (the world of 'truths of facts'), but to the world of logic; necessary truths are innate 'truths of reason' [18, p. 138, pp. 147–148]. Leibniz distinguished two sorts of the primary truths, which we know by 'intuition': the 'truths of reason' and the 'truths of facts'. Only the first kind of truth is necessary, while the second is contingent [16, p. 361]. Leibniz maintained that the primary truths of reason are necessary and generally known as 'identities' [16, p. 361].

[9] Schopenhauer underlined some passages about metempsychosis in his copy of the *New Essays* (cf. Schop 603/283 [2, pp. 125–126]. Schopenhauer also owned a book of Mendelssohn's philosophical writings, sign. UBA Ffm Bestand Na 50 Nr. Schop 603/89 [20].

3 Necessary Truths in Euclid's System

Schopenhauer's examination of Leibniz's writings is meaningful for studying the development of his philosophy of mathematics. In particular, Schopenhauer focused his attention on the concept of 'identity' and its relation with 'necessary truths', as well as on Leibniz's remarks on the Euclid system. In Leibniz's *Reflexions sur l'essai de l'entendement humain de Mr. Locke* (*Reflections on Mr. Locke's Essay on Human Understanding,* 1696), Schopenhauer found a definition of the 'axiom of identity' as equivalent to the 'axiom of non-contradiction' [18, p. 136]. According to Leibniz, Euclid used few axioms as the fundamental basis of his system and considered them neither primary truths nor improvable [18, p. 137].[10] He just needed to start from these few, unproved propositions in order to create a mathematical system where rigorous logical concatenation dominated, and he—Leibniz stated—let to other people the hard task to demonstrate these first principles. After Euclid, mathematicians and thinkers such as Apollonius and Proclus tried to demonstrate some of his axioms (see also [16, pp. 108, 371, 407, 428]). We see later in this paper that this advocacy of Euclid's method does not reflect Schopenhauer's point of view, as the latter affirms the importance of intuitive knowledge above merely abstract logic reasoning (cf. [8, pp. 44–50]). The difference between empirical and rational mathematics is what—according to Leibniz—distinguishes the mathematical science of the Greeks from the mathematics of other cultures. While the Greeks 'reasoned with the greatest possible accuracy' and 'bequeathed to mankind models of the art of demonstration', the geometry of Babylonians and Egyptians 'went a little beyond the empiric level' and, for this reason, nothing remains of it [16, p. 371]. Experience can only confirm a syllogism, but does not provide—according to Leibniz—any necessary validity.

Schopenhauer did not agree with this statement, and felt the urge to define the Euclidean method of mathematics as 'perverted' ([26, p. 90]; on the mathematical method cf. [18, p. 167]). Schopenhauer went further and declared that Euclid's method determined an 'obvious detriment of the science' [26, p. 91]. The few axioms (or, precisely, the postulates) on which the Euclidean system is based are just

[10]Both Leibniz and Schopenhauer referred to Euclid's 'primitive propositions' using only the word 'axioms' (*axiomes*, *Axiome*), exactly in the same way we modern do it. But in the first book of Euclid's *Elements* we find a clear distinction between the five axioms, or 'common notions' (*χοιναὶ ἔννοιαι*, which mainly concerned an equivalence in size), and the five 'postulates' (*αἰτήματα*, which means 'requests' and include the fifth postulate of parallel lines), although there have been some uncertainty among scholars concerning the authenticity of some axioms and postulates (cf. [4]). Leibniz was—at least—aware of this distinction, but he did not use this kind of terminological differentiation. Indeed, he just limited the use of the word 'postulate' (in French *demande*) to express the Aristotelian meaning of it: 'These principles were postulates rather than axioms (with postulates understood not in Euclid's way but in Aristotle's, namely as assumptions which we are willing to agree on while awaiting an opportunity to prove them)' [3, p. 420]. Moreover, when he wanted to discuss Euclid's recourse to undemonstrated propositions, such as 'two straight lines do not have any parts in common' and two straight lines 'do not enclose a space', he again consciously choose the term 'axioms' instead of 'postulates' [3, p. 452].

the result of an agreement ('*accordé*', to Leibniz), or they are, using Schopenhauer's words, arbitrary. This is the case of the definition of parallel lines or of the postulate that two straight lines can meet only once: They do not stand on demonstrations of logic, but depend on our empirical perception and imagination (on parallel lines, see [16, p. 296]). Leibniz explained that our imagination does not allow us to depict two straight lines meeting more than once because it draws on sense-experience. Moreover, he added that imagination cannot provide the right foundation to science, as it is not able to establish connections between distinct ideas [16, p. 451]. Euclid's decision to found his systems on postulates was a kind of mistake, since they are—according to Leibniz—strictly connected with images derived from the senses. Leibniz stated that images deriving from the sense produce confused ideas and a kind of knowledge that we are not able to demonstrate [16, p. 451]. Nevertheless, Leibniz claimed that Euclid had no distinct idea of a straight line, as he offered a provisory and unclear definition of it, which is even useless in his demonstrations; Euclid, Leibniz continued, was just 'obliged' to use two axioms (but here he means: postulates) in his demonstrations: (1) that two straight lines do not have any parts in common and (2) that they do not enclose a space [16, p. 451].

This problem clearly arises as regards the postulate of parallel lines (defined by Euclid as 'straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction', cf. [3, p. 7]), which is—according to Leibniz—also derived from our 'limited' experience. Only very distinct and clear knowledge could provide us the true definition of 'parallel lines'.[11] In this sense, Leibniz maintained that mathematical knowledge still needed some improvement, in order to eliminate the need to assume 'confused truths' or to refer to images delivered by the senses. In any case, Euclid's method was, in Leibniz's époque, the 'main model for deductive reasoning, and the touchstone of logical analysis and epistemology in general' [3, p. 3].

Schopenhauer developed a completely opposite interpretation. To him, it is true that Euclid set up certain 'arbitrarily chosen propositions', but gave 'a logical ground of knowledge of them, through a laborious logical demonstration, based upon the principle of contradiction' [26, p. 91]. Euclid's method and the force of the principle of contradiction, Schopenhauer continued, compel us to admit that 'what Euclid demonstrates is true, but we do not comprehend *why* it is so' [26, p. 92]. This fact was, in Schopenhauer's opinion, a kind of 'juggling trick' [26, p. 92]. What Leibniz defined as an unavoidable part of Euclid's method (that is: deriving logical truths from undemonstrated postulates) was described by Schopenhauer as a 'trap', or as a 'very brilliant piece of perversity' that takes 'prisoner the assent of the astonished learner' [26, p. 92]. Schopenhauer insisted that even a thorough study of the Euclidean system may not provide 'a real insight into the laws of space-relations', but only knowledge 'by heart' of certain results which follow from those laws [26, p. 92].

[11] Leibniz never published his studies on the parallel postulate, and his posthumous manuscripts were published only about two centuries later [25, p. 5].

Schopenhauer rooted the origins of the (perverted) Euclidean method in the rationalist endeavour to overtake the uncertainty of empirical knowledge. He stated that Euclid constructed the science of mathematics 'compelled by necessity' and founded his primitive propositions 'upon evidence of perception ($\varphi\alpha\iota\nu\acute{o}\mu\varepsilon\nu o\nu$)', while basing all the rest 'upon reasoning ($\nu o o\acute{\upsilon}\mu\varepsilon\nu o\nu$)' [26, p. 93]. Schopenhauer then explained that some authors in the past, such as Proclus (in his comment on Euclid) and Kepler (in his *Harmonia mundi*), seem to have known this distinction between pure and empirical intuition, or perception. Yet, only after Kant's discovery of the a priori intuition of space and time—maintained Schopenhauer—do we understand that Euclid's logical method is a 'useless precaution' in mathematics, and 'a crutch for sound legs, that it is like a wanderer who during the night mistakes a bright, firm road for water, and carefully avoiding it, toils over the broken ground beside it, content to keep from point to point along the edge of the supposed water' [26, p. 94]. In this way, Schopenhauer could conclude that the necessary element in the perception of a figure does not come from the figure itself, nor from an abstract concept under which we think it: the necessity derives 'from the form of all knowledge of which we are conscious a priori' [26, p. 94].

An improvement of the mathematical method in fact goes in a direction that is opposite to that suggested by Leibniz (who wished to provide a demonstration for the parallel postulate). According to Schopenhauer, indeed, it requires one 'to overcome the prejudice that demonstrated truth has any superiority over what is known through perception', or to recognise the importance, near logical truths and the principle of contradiction, to what metaphysical truth, which is immediately evident, and includes the pure intuition of space [26, p. 96]. Now, Schopenhauer strongly affirmed that Euclid's axioms 'have no more immediate evidence than any other geometrical problem, but only more simplicity on account of their smaller content' [26, pp. 97–98].

As to confirm his position, Schopenhauer added that Euclidean method of proof (based on axioms and postulates) was applied only to geometry and not to arithmetic, because arithmetical truth only rests on perception (of time), that is, 'simply in counting' [26, p. 99]. We therefore cannot represent numbers with 'a sensuous schema like the geometrical figure'. For this reason, Schopenhauer explained that 'the suspicion that perception is merely empirical, and possibly illusive, disappeared in arithmetic' [26, p. 99]. Counting is, according to him, an a priori activity through which alone everything in mathematics is ultimately demonstrated [26, p. 99].

4 Confused Ideas Between Images and Feeling

The analysis of Schopenhauer's reception of Leibniz work and especially the comparison between Leibniz's and Schopenhauer's interpretation of Euclid's axioms and postulates acquire a special meaning for the actual debate on Schopenhauer's hermeneutics, especially concerning the notion of 'feeling' —*Gefühl*—(see [27, pp.

356–361]). In this perspective, I will focus on two issues, that is, the function of 'images' and the notion of 'confused' knowledge, that emerge from the examination in the previous sections. In *New Essays*, Leibniz defined four different degrees of ideas: 'Obscure ideas' originate when memory keeps them without preserving their 'original exactness' or 'first freshness' so that they are 'faded or tarnished by time' [16, p. 254]. An idea is 'clear' when 'it enables us to recognize the thing and distinguish it from other things', although we hardly have, according to Leibniz, perfectly clear ideas of sensible things [16, p. 255]. Besides the 'clear ideas', Leibniz enumerated the 'distinct' ideas, by means of which we are able to 'distinguish in the object the marks which make it known, thus yielding an analysis or definition'. [16, p. 255]. Finally, ideas that are not distinct are, to Leibniz, called 'confused' and depend on the fact that our nature is so imperfect that we have perforce confused ideas [16, p. 256]. Leibniz defined confused ideas using examples. We may have a confused idea of a heap of stones, as we do not know exactly (or distinctly) its 'properties' (including the number of stones). Likewise, if we look at a thousand-sided figure, without knowing the exact number of its sides, we can have only a confused idea of it. The sole thing we need, added Leibniz, is 'distinct properties' that permit to bring order into confusion [16, pp. 257–258].

The definition of 'confused ideas' forced Leibniz into a distinction between 'idea' (or, better, distinct ideas) and 'image'. According to him, only by means of ideas do we obtain exact information about a regular geometrical form, and this is possible even if we do not see the figure. On the other hand, sight and imagination do not provide these distinguished properties, and they just create a 'confused' idea of it. The more the senses and the imagination are subtle, the more we are able to distinguish, for instance, one figure from another very similar to it. Leibniz précises that knowledge of figures does not depend upon the imagination, more than knowledge of numbers does and that, nevertheless imagination 'may be a help' [16, p. 261]. He placed the '*clear image'* or 'precise feeling' of a regular figure as at the origins of 'confuse ideas'; so, they are confused just because they derive from the senses and not the properties of that figure, as a 'distinct idea' can. Euclid's axioms are, very peculiarly, such a kind of 'confused' ideas.

Schopenhauer analysed the issues deriving from the separation between distinct ideas (that is, in abstract thought), on the one side, and clear but confused ideas (depending on intuition), on the other [25, pp. 270–272]. However, he noticed what Leibniz had affirmed about the fact that abstract thought somehow requires the aid of imagination, and that 'distinct ideas' are often associated with 'confused ideas' [19, p. 409].[12] In *On the Fourfold Root* he distinguished the representation from the 'mental image' and affirmed that only representation—here, he was referring to Kant—'shows reality' [24, p. 104]. By means of this distinction, he argued, for example, against Aristotle, that thinking 'without pictures of the

[12] 'Les plus abstraites pensées ont besoin de quelque imagination; et quand on considère ce que c'est que les pensées confuses, qui ne manquent jamais d'accompagner les plus distinctes que nous puissions avoir, on reconnoît qu'elles enveloppent toûjours l'infini' [19, p. 409].

imagination' is possible. A long tradition, starting with Aristotle, affirmed that we need imagination for our thinking, and this idea had a strong influence—according to Schopenhauer—on the history of philosophy between the fifteenth and sixteenth centuries [24, pp. 122–123]. In opposition to this idea, Schopenhauer suggested that 'the true kernel of all knowledge' is not imagination, but properly 'reflection', a notion that he probably found in Johann Gottfried Herder's work on the origin of language (cf. the notion of *Besonnenheit* in [6]; on the role of Herder's notion of 'reflection' in Schopenhauer's thought see also [10, p. 121]). 'The reflection works with the help of intuitive representations' and is 'the basis of all conceptions' [24, p. 122]. According to this perspective, Schopenhauer affirmed that every primary notion or philosophical theorem must proceed from an intuitive view as its root which 'imparts life and spirit to the whole analysis' [24, p. 123]. Schopenhauer did not refer to 'mental images' as a mere product of imagination, but to 'intuition', to perception, that is, to an immediate 'view' that is unavoidable to acquire some new knowledge, in opposition to abstract reasoning [24, p. 123]. In *The World*, Schopenhauer remembered that in Euclid's geometry one must begin with figures before proceeding to rigorous demonstration [26, p. 67]. Two other authors, Friedrich Schleiermacher and Wilhelm Gottlieb Tennemann, also affirmed, respectively, a 'logical and mathematical feeling' and the concept of 'feeling' in mathematics. Schopenhauer then specified his view, explaining that the concept of 'feeling' must be regarded 'from the right point of view' to avoid any kind of 'misunderstanding and controversy' [26, p. 67]. In particular, he continued, we class under the concept of 'feeling' 'every modification of consciousness' which is different from an 'abstract concept' [26, p. 67]. And, among the 'feelings', Schopenhauer also listed 'the apprehension of space relation presented a priori in perception', as well as 'the knowledge of the pure understanding' [26, pp. 66–67]. According to Schopenhauer's position, feelings are 'all truth, of which we are first conscious only intuitively', and that we formulate in abstract concept only later [26, pp. 66–67]. In this sense, it is possible to say that we 'feel' a truth.

Going back again to Leibniz's *Reflexions sur l'essai de Mr. Locke*, Schopenhauer underlined some passages on it which regard the notion of feeling [18, p. 137]. These passages, together with other passages I have previously quoted, acquire a special meaning in relation to Schopenhauer's view and allow us to recognise Leibniz's work as a source for Schopenhauer. Starting from these kinds of sources and from his interpretation of Kant's philosophy, Schopenhauer developed his own theory of knowledge, where he assigned, on the one hand, great importance to the possibility of 'pure' knowledge (like a rationalist) and, on the other hand, also to 'feeling', with the meaning I have shown. He provided a new definition of 'intuitive knowledge', which includes perception and intellectual intuition, and this idea involves a special meaning for his philosophy of mathematics. Many questions are left open from this analysis and, in particular, the fact that Schopenhauer's particular ideas on mathematics, developed from both Leibnitian and Kantian's philosophies, occupy a special place in the history of Western thought, as he did not belong to an empiricist or to a rationalist tradition.

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