Completeness

Completeness is a property of a topological vector space as a 'uniform space'. We do not explicitly use uniform spaces but mention that the linear structure allows to define neighbourhoods of 'uniform size' for all $x \in E$ by taking the translates $x + U$ for $U \in \mathcal{U}_0(E)$. This allows to introduce the notion of Cauchy filters, and completeness requires Cauchy filters to be convergent.

After some discussion on completeness and quasi-completeness, we come to Grothendieck's description of the completion of a locally convex space, Corollary 9.16, as the main result of this chapter.

Let E be a topological vector space, $A \subseteq E$. A filter $\mathcal{F}$ in A is called a **Cauchy filter** if for every $U \in \mathcal{U}_0(E)$ there exists $B \in \mathcal{F}$ such that $B - B \subseteq U$.

The set $A \subseteq E$ is called **complete** if every Cauchy filter in A is convergent to an element of A, and A is called **sequentially complete** if every Cauchy sequence in A is convergent to an element of A. A sequence (x_n) in E is called a **Cauchy sequence** if the elementary filter generated by the sequence is a Cauchy filter, i.e., if for each neighbourhood of zero U there exists $n_0 \in \mathbb{N}$ such that $x_n - x_m \in U$ for all $m, n \geqslant n_0$.

The space E is called **quasi-complete** if every closed bounded subset of E is complete.

Remarks 9.1 (a) If $\mathcal{F}$ is a filter in A, $\mathcal{F}$ convergent to $x \in A$, then $\mathcal{F}$ is a Cauchy filter. (Let U be a neighbourhood of zero. Then there exists a neighbourhood of zero V such that $V - V \subseteq U$. Then $(x + V) \cap A \in \mathcal{F}$, by hypothesis, and one obtains $\big((x + V) \cap A\big) - \big((x + V) \cap A\big) \subseteq V - V \subseteq U$.)

(b) Let $\mathcal{F}$ be a Cauchy filter in A, and let $x \in A$ be a cluster point of $\mathcal{F}$. Then $\mathcal{F} \to x$. (Let U be a neighbourhood of zero, V a neighbourhood of zero with $V + V \subseteq U$, $B \in \mathcal{F}$ with $B - B \subseteq V$ (in particular, $B \subseteq b + V$ for all $b \in B$). Then $B \cap (x + V) \neq \varnothing$, and therefore $B \subseteq B \cap (x + V) + V \subseteq x + V + V \subseteq x + U$. This shows that $\mathcal{F} \to x$.)

(c) If E is Hausdorff and A is complete, then A is closed. (For $x \in \overline{A}$ there exists a filter $\mathcal{F}$ in A with $\mathcal{F} \to x$. Then $\mathcal{F}$ is a Cauchy filter, which is convergent in A. Then $x \in A$, as the limit is unique.)

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(d) If A is complete and $B \subseteq A$ is relatively closed in A, then B is complete. (If $\mathcal{F}$ is a Cauchy filter in B, then $\mathcal{F}$ is a Cauchy filter base in A, which is convergent in A. Since B is closed in A and $B \in \mathcal{F}$, every limit of $\mathcal{F}$ in A belongs to B.)

(e) If E is a topological vector space possessing a countable neighbourhood base of zero $(U_n)_{n\in\mathbb{N}}$, then E is complete if and only if E is sequentially complete. (For the necessity let (x_n) be a Cauchy sequence, i.e., the collection $\big\{\{x_k\,;\ k \geqslant n\}\,;\ n \in \mathbb{N}\big\}$ is a Cauchy filter base, and a limit of this filter base is also a limit of the sequence. For the sufficiency let $\mathcal{F}$ be a Cauchy filter. Then there exists a decreasing sequence $(B_n)_n$ in $\mathcal{F}$, $B_n - B_n \subseteq U_n$ $(n \in \mathbb{N})$. For $n \in \mathbb{N}$ choose $x_n \in B_n$. Then (x_n) is a Cauchy sequence, which by hypothesis converges, $x_n \to x$. It is easy to see that then x is a cluster point of $\mathcal{F}$, and therefore $\mathcal{F} \to x$, because $\mathcal{F}$ is a Cauchy filter.)

(f) Let E be a metrisable locally convex space, and let d be a translation invariant metric on E inducing the topology. Then E is complete if and only if the metric space (E, d) is complete (i.e., E is a Fréchet space). This follows immediately from (e) above and the property that $(B_d(0, 1/n))_{n\in\mathbb{N}}$ is a countable neighbourhood base of zero.

(g) Let E, F be topological vector spaces, $u\colon E \to F$ linear and continuous, and let $\mathcal{F}$ be a Cauchy filter in E. Then $\mathrm{fil}(u(\mathcal{F}))$ is a Cauchy filter in F. (If V is a neighbourhood of zero in F, then $u^{-1}(V)$ is a neighbourhood of zero in E. Therefore, there exists $A \in \mathcal{F}$ such that $A - A \subseteq u^{-1}(V)$, and this implies that $u(A) - u(A) \subseteq u(u^{-1}(V)) \subseteq V$.) △

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Theorem 9.2
Let E be a Hausdorff topological vector space. Then there exist a complete Hausdorff topological vector space $\tilde{E}$ such that E is isomorphic to a dense subspace of $\tilde{E}$. The space $\tilde{E}$ is unique up to isomorphism and is called the ***completion*** *of E.*

We will not prove the existence, but rather refer to [Hor66, Chap. 2, § 9, Theorem 1] or [Sch71, Chap. I, § 1.5] for a proof. For locally convex space s we will give a proof later in this chapter. However, we will prove the uniqueness, and for this property we need the following preparations. The first of these is a fundamental fact from topology.

Proposition 9.3 *Let X and Y be topological spaces, Y Hausdorff and regular. Let $X_0 \subseteq X$ be a dense subset, $u_0\colon X_0 \to Y$ continuous, and suppose that for each $x \in X \setminus X_0$ the limit $u(x) := \lim_{y\to x,\, y\in X_0} u_0(y)$ exists. On X_0 define $u := u_0$. Then u is the unique continuous extension of u_0 to X.*

Recall that **regular** means that every point $y \in Y$ has a neighbourhood base consisting of closed sets. The existence of $\lim_{y\to x,\, y\in X_0} u_0(y)$ means that the image filter base $u_0(\mathcal{U}_x \cap X_0)$ is convergent, where $\mathcal{U}_x$ is the neighbourhood filter of x, and $\mathcal{U}_x \cap X_0 = \big\{U \cap X_0\,;\ U \in \mathcal{U}_x\big\}$. The limit is unique because Y is Hausdorff.

Proof of Proposition 9.3
Concerning the uniqueness, assume that u and $\tilde{u}$ are continuous extensions of u_0. Then the set $\{x \in X;\ u(x) = \tilde{u}(x)\}$ is closed (because the diagonal of $Y \times Y$ is closed) and contains X_0, hence is equal to X.

To show the continuity of u, let $x \in X$, and let V be a closed neighbourhood of $u(x)$. By hypothesis, there exists an open neighbourhood U of x such that $u_0(U \cap X_0) \subseteq V$. Then U is a neighbourhood of each of its points z; hence, $u(z) = \lim_{y \to z,\, y \in U \cap X_0} u_0(y) \in \overline{u_0(U \cap X_0)} \subseteq \overline{V} = V$. This shows that $u(U) \subseteq V$ and proves the continuity of u at x. □

Proposition 9.4 *Let E and F be topological vector spaces, $E_0 \subseteq E$ a dense subspace, F Hausdorff and complete, and let $u_0 \colon E_0 \to F$ be a continuous linear mapping. Then there exists a unique continuous extension $u \colon E \to F$ of u_0, and u is linear.*

Proof
Note that F is regular, because the closed neighbourhoods of zero in F form a neighbourhood base of zero. Let $\mathcal{U}_0$ be the neighbourhood filter of zero in E, and let $x \in E \setminus E_0$. Then

$$\mathcal{F}_x := (x + \mathcal{U}_0) \cap E_0 = \{(x + U) \cap E_0;\ U \in \mathcal{U}_0\}$$

is a filter in E_0 converging to x, hence a Cauchy filter. This implies that $u_0(\mathcal{F}_x)$ is a Cauchy filter base in F, hence convergent. Now Proposition 9.3 yields the existence and uniqueness of the continuous extension u of u_0.

In order to show the linearity of u we let $\lambda \in \mathbb{K}$ and note that the set

$$\{(x, y) \in E \times E;\ u(\lambda x + y) = \lambda u(x) + u(y)\}$$

is a closed subset of $E \times E$ and contains the dense subset $E_0 \times E_0$, hence is equal to $E \times E$. □

Proof of the uniqueness in Theorem 9.2
Assume that $\tilde{E}$ and $\hat{E}$ are completions, with embeddings $\tilde{j}_0 \colon E \hookrightarrow \tilde{E}$, $\hat{j}_0 \colon E \hookrightarrow \hat{E}$. Interpreting, for the moment, E as a subspace of $\tilde{E}$, we conclude from Proposition 9.4 that $\hat{j}_0$ extends uniquely to $\hat{j} \colon \tilde{E} \to \hat{E}$. Similarly, $\tilde{j}_0$ extends to $\tilde{j} \colon \hat{E} \to \tilde{E}$. As $\tilde{j} \circ \hat{j}$ is continuous, and is the identity on E, it follows that $\tilde{j} \circ \hat{j}$ is the identity on $\tilde{E}$; hence $\hat{j} \colon \tilde{E} \to \hat{E}$ is an isomorphism. □

The next part of the chapter serves to collect miscellaneous properties concerning completeness.

Proposition 9.5
(a) *Let $(E_\iota)_{\iota \in I}$ be a family of topological vector spaces, and assume that E_ι is (quasi-)complete for all $\iota \in I$. Then $E := \prod_{\iota \in I} E_\iota$ is (quasi-)complete.*
(b) *Let I be a set. Then $\mathbb{K}^I$ is complete.*
(c) *Let E be a vector space. Then $(E^*, \sigma(E^*, E))$ is complete.*

Proof

(a) for 'complete': Let $\mathcal{F}$ be a Cauchy filter in E. Then $\mathrm{pr}_\iota(\mathcal{F})$ is a Cauchy filter base in E_ι, convergent to x_ι ($\iota \in I$). Then $\mathcal{F} \to (x_\iota)_{\iota\in I} \in E$, by Proposition 4.6. The proof for 'quasi-complete' is analogous; observe that, for a bounded set $B \subseteq E$ the images $\mathrm{pr}_\iota(B)$ are bounded ($\iota \in I$).

(b) is a direct consequence of (a).

(c) Recall that E^* is a closed subset of $\mathbb{K}^E$ (Lemma 4.8) and that $\sigma(E^*, E)$ is the restriction of the product topology to E^*. □

Besides being of interest in its own right, the following result serves to prepare the presentation of examples of quasi-complete spaces which are not complete.

Lemma 9.6 *Let E be a barrelled locally convex space. Then $(E', \sigma(E', E))$ is quasi-complete.*

Proof

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Let $B \subseteq E'$ be $\sigma(E', E)$-bounded and closed. Then B is equicontinuous (Theorem 6.14), i.e., there exists $U \in \mathcal{U}_0(E)$ such that $B \subseteq U^\circ$. By the Alaoglu–Bourbaki theorem, U° is $\sigma(E', E)$-compact, and therefore complete. (If $\mathcal{F}$ is a Cauchy filter in U°, $\hat{\mathcal{F}}$ a finer ultrafilter, then $\hat{\mathcal{F}}$ is convergent, $\hat{\mathcal{F}} \to x$; therefore x is a cluster point of $\mathcal{F}$, $\mathcal{F} \to x$.) This implies that B is complete. □

Examples 9.7

(a) Let E be a Hausdorff locally convex space, and assume that there exists a linear subspace which is not closed. Then the dual pair $\langle E, E'\rangle$ is separating in E, and passing to the dual pair $\langle E, E^*\rangle$, we note that Corollary 2.10 implies that E' is $\sigma(E^*, E)$-dense in E^*. It is not difficult to show that under the above hypotheses $E' \neq E^*$, and therefore $(E', \sigma(E', E))$ is not complete.

(b) Let E be an infinite-dimensional Banach space. Then $(E', \sigma(E', E))$ is quasi-complete, by Lemma 9.6, but part (a) shows that $(E', \sigma(E', E))$ is not complete. Indeed, it follows from Baire's theorem that countably infinite-dimensional subspaces of E are not closed. △

The following result presents an interesting and surprising interplay concerning completeness in different topologies. It will be important and applied repeatedly in Chapter 14.

Theorem 9.8

Let E be a vector space, let $\sigma \subseteq \tau$ be two linear topologies on E, and assume that τ has a neighbourhood base of zero $\mathcal{U}$ consisting of σ-closed sets.

(a) *Let $\mathcal{F}$ be a τ-Cauchy filter, $x \in E$, $\mathcal{F} \xrightarrow{\sigma} x$. Then $\mathcal{F} \xrightarrow{\tau} x$.*

(b) *Let $A \subseteq E$ be σ-complete. Then A is also τ-complete.*

Proof

(a) Let $U \in \mathcal{U}$. There exists $B \in \mathcal{F}$ such that $B - B \subseteq U$. For $y, z \in B$ one therefore has $y - z \in U$, and as U is σ-closed one obtains $y - x \in U$. This implies that $B \subseteq x + U$, and therefore $\mathcal{F} \xrightarrow{\tau} x$.

(b) This is clear from (a), because every τ-Cauchy filter is a σ-Cauchy filter. □

The analogous result also holds for the 'sequential setup', with 'closed' replaced by 'sequentially closed', 'Cauchy filter' by 'Cauchy sequence', and 'complete' by 'sequentially complete'.

Example 9.9

Let $1 \leqslant p \leqslant \infty$. On ℓ_p let τ be the norm topology, and let σ be the restriction of the product topology on $\mathbb{K}^{\mathbb{N}}$.

The closed unit ball B_{ℓ_p} is easily seen to be sequentially σ-closed and sequentially σ-complete. Therefore the sequential version of Theorem 9.8 is applicable, and part (b) yields that B_{ℓ_p} (and therefore ℓ_p) is complete.

This (seemingly complicated) proof of the completeness of ℓ_p is nothing but an abstract version of the usual proof of the completeness of ℓ_p. △

The next aim is to prove the following result.

Theorem 9.10

Let E be a quasi-complete locally convex space. Then every $\sigma(E', E)$-bounded subset of E' is $\beta(E', E)$-bounded, i.e., $\mathcal{B}_\beta = \mathcal{B}_\sigma$, in the terminology of the end of Chapter 6.

Before we start with the preparations for the proof we mention a consequence of this result.

Corollary 9.11 *Let E be a quasi-complete quasi-barrelled locally convex space. Then E is barrelled.*

Proof

We will use the terminology of the end of Chapter 6. The fact that E is quasi-barrelled is equivalent to $\mathcal{E} = \mathcal{B}_\beta$ (Theorem 6.8), whereas the quasi-completeness implies that $\mathcal{B}_\beta = \mathcal{B}_\sigma$ (Theorem 9.10). Putting this together we conclude that $\mathcal{E} = \mathcal{B}_\sigma$ which is equivalent to E being barrelled (Theorem 6.14). □

Let (E, τ) be a locally convex space, and let $B \subseteq E$ be absolutely convex, bounded and closed. Define

$$E_B := \bigcap_{n\in\mathbb{N}} nB = \operatorname{lin} B,$$

with semi-norm p_B. Then $(E_B, p_B) \hookrightarrow (E, \tau)$ is continuous (because B is bounded).

If p_B is a norm and (E_B, p_B) is a Banach space, then B is called a **Banach disc**. Note that p_B is a norm if E is Hausdorff.

Lemma 9.12 *Let E be a locally convex space, and let $B \subseteq E$ be absolutely convex, bounded, closed and sequentially complete.*
(a) *Then (E_B, p_B) is complete. In particular, if p_B is a norm, then B is a Banach disc.*
(b) *Let $D \subseteq E$ be a barrel. Then D absorbs B.*

Proof
(a) follows from the 'sequential version' of Theorem 9.8, applied to E_B, with $\sigma_{E_B} := \tau \cap E_B$, $\tau_{E_B} := \tau_{p_B}$. The conclusion is that the ball $B = \{x \in E_B;\ p_B(x) \leqslant 1\}$ is p_B-complete.

(b) (E_B, p_B) is semi-normed and complete, therefore a Baire space (see Appendix B), hence barrelled (Theorem 6.9). The set $D \cap E_B$ is a barrel in (E_B, p_B), therefore a neighbourhood of zero, and therefore absorbs B. □

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Proof of Theorem 9.10
Let $B \subseteq E'$ be $\sigma(E', E)$-bounded. Then B° is a barrel. If $A \subseteq E$ is bounded, then $A^{\circ\circ} = \overline{\operatorname{aco}}\, A$ is closed and bounded, and therefore complete, by hypothesis. Then Lemma 9.12(b) implies that B° absorbs $A^{\circ\circ}$, and therefore $B \subseteq B^{\circ\circ}$ is absorbed by $(A^{\circ\circ})^\circ = A^\circ$. This shows that B is $\beta(E', E)$-bounded. □

With the following theorem we start the proof of the existence of the completion of a locally convex space; in fact, this theorem is the main ingredient of the proof and also provides a description of the completion.

Theorem 9.13 (Grothendieck)
Let E be a Hausdorff locally convex space. Let $\mathcal{M}$ be a directed covering of E, consisting of bounded, closed, absolutely convex sets. Let

$$F := \{u \in E^*;\ u|_A \text{ continuous } (A \in \mathcal{M})\}.$$

Then $\mathcal{M}$ can be used to define a polar topology on F in the dual pair $\langle E, F\rangle$, and $(F, \tau_{\mathcal{M}})$ is a completion of $(E', \tau_{\mathcal{M}})$.

For the proof we need several preparations.

Lemma 9.14 *Let E be a Hausdorff locally convex space, and let $A \subseteq E$ be absolutely convex and closed. Let $u \in E^*$, $u|_A$ continuous at 0, and let $\varepsilon > 0$. Then there exists $x' \in E'$ such that $|u(x) - \langle x, x'\rangle| \leqslant \varepsilon$ $(x \in A)$.*

Proof
It is clearly sufficient to show this for $\varepsilon = 1$.

The continuity of $u|_A$ at 0 implies that there exists an absolutely convex closed neighbourhood of zero $U \subseteq E$ such that $|u(x)| \leqslant 1$ $(x \in A \cap U)$. The polar $U^\bullet$ (taken in the dual pair $\langle E, E^*\rangle$) is a subset of E', $\sigma(E^*, E)$-compact (by the Alaoglu–Bourbaki theorem) and absolutely convex. Therefore Lemma 7.3(b) implies that $A^\bullet + U^\bullet$ is $\sigma(E^*, E)$-closed. Evidently, $A^\bullet + U^\bullet$ is also absolutely convex, and therefore $A^\bullet + U^\bullet = (A^\bullet + U^\bullet)^{\bullet\bullet}$, by the bipolar theorem. Now,

$$u \in (A \cap U)^\bullet = (A^{\bullet\bullet} \cap U^{\bullet\bullet})^\bullet = (A^\bullet \cup U^\bullet)^{\bullet\bullet} \subseteq (A^\bullet + U^\bullet)^{\bullet\bullet} = A^\bullet + U^\bullet.$$

(In the second equality we have used Remark 3.3(c).)

This shows that there exist $w \in A^\bullet$, $x' \in U^\bullet \subseteq E'$ such that $u = w + x'$, and this implies $|u(x) - x'(x)| = |w(x)| \leqslant 1$ for all $x \in A$. □

Lemma 9.15 *Let X be a topological space, $\mathcal{S} \subseteq \mathcal{P}(X)$. Then the space*

$$C_{\mathrm{b}}(X, \mathcal{S}) := \big\{f \colon X \to \mathbb{K};\ f|_A \text{ bounded and continuous } (A \in \mathcal{S})\big\},$$

with the semi-norms p_A,

$$p_A(f) := \sup_{x\in A} |f(x)| \quad (f \in C_{\mathrm{b}}(X, \mathcal{S}),\ A \in \mathcal{S})$$

is complete.

Proof

Without loss of generality we may assume that $\bigcup \mathcal{S} = X$.

For $A \in \mathcal{S}$ the space $C_{\mathrm{b}}(A)$ (bounded continuous functions with sup-norm) is complete. Let $\mathcal{F}$ be a Cauchy filter in $C_{\mathrm{b}}(X, \mathcal{S})$. Then for $A \in \mathcal{S}$ the image filter $\mathcal{F}_A$ in $C_{\mathrm{b}}(A)$ under the mapping $f \mapsto f|_A$ is a Cauchy filter, therefore convergent. This implies that there exists $g \in C_{\mathrm{b}}(X, \mathcal{S})$ such that $\mathcal{F} \to g$. (Observe that for $A, B \in \mathcal{S}$ with $A \cap B \neq \varnothing$ the limits g_A, g_B of $\mathcal{F}_A, \mathcal{F}_B$ coincide on $A \cap B$. Also, recall Proposition 4.6(b).) □

Proof of Theorem 9.13

We work in the dual pair $\langle E, F\rangle$.

First we show that $\mathcal{M} \subseteq \mathcal{B}_\sigma(E, F)$ (which makes it clear that $\mathcal{M}$ defines a polar topology on F). Let $A \in \mathcal{M}$, $u \in F$. There exists $U \in \mathcal{U}_0(E)$ such that $|u(x)| \leqslant 1$ $(x \in A \cap U)$. Also, $\lambda A \subseteq U$ for suitable $\lambda \in (0, 1]$ (because A is bounded). For $x \in A$ it follows that $\lambda x \in A \cap U$, $|u(x)| \leqslant \frac{1}{\lambda}$. Therefore A is $\sigma(E, F)$-bounded.

From Lemma 9.14 one concludes that E' is dense in F. (Recall that $\mathcal{M}$ is directed. This implies that $\mathcal{U} := \big\{\varepsilon B_{q_A};\ A \in \mathcal{M},\ \varepsilon > 0\big\}$ is a neighbourhood base of zero for $\tau_{\mathcal{M}}$.)

Finally, $(F, \tau_{\mathcal{M}})$ is complete: $C_{\mathrm{b}}(E, \mathcal{M})$ is complete, by Lemma 9.15, and $\mathrm{id} \colon C_{\mathrm{b}}(E, \mathcal{M}) \hookrightarrow \mathbb{K}^E$ is continuous (with the product topology on $\mathbb{K}^E$). Moreover E^* is a closed subspace of $\mathbb{K}^E$ (Lemma 4.8). This shows that $F = C_{\mathrm{b}}(E, \mathcal{M}) \cap E^*$ is closed in $C_{\mathrm{b}}(E, \mathcal{M})$, hence complete. □

Corollary 9.16 (Grothendieck) *Let E be a Hausdorff locally convex space, and recall the notation $\mathcal{E} = \{B \subseteq E';\ B \text{ equicontinuous}\}$. Then*

$$\tilde{E} := \{u \in E'^*;\ u|_B\ \sigma(E', E)\text{-continuous}\ (B \in \mathcal{E})\},$$

with the polar topology $\tau_{\mathcal{E}}$, is a completion of E. In particular, E is complete if and only if $E = \tilde{E}$.

Proof

This is obtained by applying Theorem 9.13 to $(E', \sigma(E', E))$ and $\mathcal{M} := \{U^\circ;\ U \in \mathcal{U}_0(E)\}$. Then $(E', \sigma(E', E))' = E$, and $\tau_{\mathcal{M}} = \tau_{\mathcal{E}}$ is the original topology on E. □

Remark 9.17 If one is just interested in the existence of a completion of a Hausdorff locally convex space E, one can proceed by a reduced method as follows. We only sketch this procedure and refer to [MeVo97, Proposition 22.21] for more details.

With a neighbourhood base of zero $\mathcal{U}$ in E one equips

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$$E'^\times := \{u \in E'^*;\ u|_{U^\circ} \text{ bounded } (U \in \mathcal{U})\}$$

with the semi-norms q_{U°,

$$q_{U^\circ}(u) := \sup\{|\langle u, y\rangle|;\ y \in U^\circ\} \qquad (u \in E'^\times,\ U \in \mathcal{U}).$$

Then $E \subseteq E'^\times$ isomorphically, in the natural way, and $E'^\times$ is complete; hence a completion of E is obtained as $\tilde{E} := \overline{E}^{E'^\times}$. △

Corollary 9.18 (Banach) *Let E be a Banach space, and let $u \in E'^*$ be $\sigma(E', E)$-continuous on $B_{E'}$ (the closed unit ball of E'). Then u belongs to E.*

Proof

By hypothesis, u is $\sigma(E', E)$-continuous on all equicontinuous sets $B \subseteq E'$. Applying Corollary 9.16 and using that E is complete one obtains $u \in E$. □

We conclude this chapter with a result on the completeness of dual spaces.

Theorem 9.19

Let E be a bornological locally convex space. Then $(E', \beta(E', E))$ is complete.

We need preparations for the proof.

Lemma 9.20 *Let E be a topological vector space. Then a set $B \subseteq E$ is bounded if and only if, for every sequence $(x_n)_{n\in\mathbb{N}}$ in B and every null sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{K}$, the sequence $(\lambda_n x_n)_{n\in\mathbb{N}}$ is a null sequence.*

Proof
For the necessity, let (x_n) and (λ_n) be as assumed above, and let $U \in \mathcal{U}_0$. There exist $\varepsilon > 0$ such that $\lambda B \subseteq U$ for $|\lambda| \leqslant \varepsilon$, $n_0 \in \mathbb{N}$ such that $|\lambda_n| \leqslant \varepsilon$ $(n \geqslant n_0)$. Then $\lambda_n x_n \in U$ $(n \geqslant n_0)$.

For the sufficiency, assume that B is not bounded. Then there exists $U \in \mathcal{U}_0$ such that $B \not\subseteq nU$ $(n \in \mathbb{N})$. With $x_n \in B \setminus nU$ one obtains $\frac{1}{n}x_n \notin U$ $(n \in \mathbb{N})$, $\frac{1}{n}x_n \not\to 0$. □

Lemma 9.21 *Let E, F be locally convex spaces, $u\colon E \to F$ linear, $B \subseteq E$ bounded and absolutely convex, $u|_B$ continuous at 0. Then $u(B)$ is bounded.*

Proof
Let (x_n) be a sequence in B, (λ_n) a null sequence in $\mathbb{K}$. Then $\lambda_n x_n \in B$ for large n, $\lambda_n x_n \to 0$ by Lemma 9.20, and by hypothesis $\lambda_n u(x_n) = u(\lambda_n x_n) \to 0$ $(n \to \infty)$. Therefore Lemma 9.20 implies that $u(B)$ is bounded. □

Remark 9.22 In Lemma 9.21 (as well as in Lemma 9.14) a linear mapping u was used whose restriction to an absolutely convex set is continuous at 0. It can be shown that the continuity at 0 is equivalent to the continuity on the whole absolutely convex set; cf. [Hor66, Chap. 3, § 11, Lemma 1]. △

Proof of Theorem 9.19
We apply Theorem 9.13 with

$$\mathcal{M} := \{A \subseteq E;\ A \text{ bounded, closed, absolutely convex}\};$$

then $\tau_{\mathcal{M}} = \beta(E', E)$.

Let $u \in E^*$, $u|_A$ continuous for all $A \in \mathcal{M}$. By Lemma 9.21, $u(A)$ is bounded for all $A \in \mathcal{M}$, and therefore Proposition 6.18 implies that u is continuous, i.e., $u \in E'$. Now Theorem 9.13 implies that $(E', \beta(E', E))$ is complete. □

Remark 9.23 As metrisable locally convex spaces are bornological, Theorem 9.19 implies that the duals of the following spaces are complete: $C_0^\infty(\Omega)$, $\mathcal{E}(\Omega)$, for open $\Omega \subseteq \mathbb{R}^n$, $\mathcal{S}(\mathbb{R}^n)$, and $C(X)$, for σ-compact Hausdorff locally compact spaces X. △

Notes The material of this chapter, up to Lemma 9.12, is rather standard; Proposition 9.3 is from [Bou07c, Chap. I, § 8.5, Théorème 1, p. I.57]. Theorem 9.8 is an interesting result which can be used to prove completeness of a set if completeness is known in a finer topology; its counterpart for uniform spaces can be found in [Bou07c, Chap. II, § 3.3, Proposition 7 and Corollaire]. Theorem 9.13 and Corollary 9.16 are due to Grothendieck [Gro50]. Following Horváth [Hor66, Chap. 3, § 11, Corollary 4], the author attributes Corollary 9.18 to Banach, although he did not find a direct reference to this result in Banach's publications. However, we will show in Remark 12.3 that it is an immediate consequence of another result of Banach's.