Reflexivity

We start by discussing semi-reflexivity and Montel spaces and present a number of examples of function spaces. At the end we present duality properties for reflexive spaces and Montel spaces.

We recall from Chapter 3 that a locally convex space E is called semi-reflexive if it is Hausdorff and the canonical embedding $\kappa\colon E \hookrightarrow E''$ is surjective. E is called reflexive if additionally κ is continuous, where the image space is equipped with the strong topology.

From Theorems 6.7 and 6.8 we know that (E, τ) is reflexive if and only if E is semi-reflexive and quasi-barrelled, or equivalently (because always $\tau \subseteq \beta(E, E')$, by Proposition 6.4) if and only if E is semi-reflexive, and $\tau = \beta(E, E')$, or equivalently (by Theorem 6.14), if and only if E is semi-reflexive and barrelled.

This is the reason why in the following we will mainly discuss semi-reflexivity.

Theorem 8.1
Let E be a Hausdorff locally convex space. Then E is semi-reflexive if and only if every bounded set in E is weakly relatively compact.

Proof
For the necessity we note that semi-reflexivity implies that $\beta(E', E) = \mu(E', E)$. Therefore, if $A \subseteq E$ is bounded, then A° is a $\mu(E', E)$-neighbourhood of zero, and there exists a $\sigma(E, E')$-compact barrel $C \subseteq E$ such that $A^\circ \supseteq C^\circ$. Then $A \subseteq A^{\circ\circ} \subseteq C^{\circ\circ} = C$.

For the sufficieny we note that the condition implies that $\beta(E', E) = \mu(E', E)$, which in turn implies that $(E', \beta(E', E))' = (E', \mu(E', E))' = E$. □

Remark 8.2 Note that the condition in Theorem 8.1 is a generalisation of the known criterion for the reflexivity of Banach spaces. △

J. Voigt, *A Course on Topological Vector Spaces*, Compact Textbooks in Mathematics,
https://doi.org/10.1007/978-3-030-32945-7_8

A **semi-Montel space** is a Hausdorff locally convex space in which every bounded set is relatively compact. (This terminology reminds of Montel's theorem from complex analysis; see Example 8.4(d).) A **Montel space** is a quasi-barrelled semi-Montel space.

Corollary 8.3 *If E is a semi-Montel space, then E is semi-reflexive. If E is a Montel space, then E is reflexive.*

Proof

This is obvious from Theorem 8.1. □

For use in the following example (b) we mention the notation $C_0(\Omega)$, for the space of continuous functions 'vanishing at ∞', on a Hausdorff locally compact space Ω:

$$C_0(\Omega) := \big\{f \in C(\Omega);\ \forall\, \varepsilon > 0\ \exists\, K \subseteq \Omega \text{ compact}\colon |f(x)| < \varepsilon\ (x \in \Omega \setminus K)\big\}.$$

8

For a function $f \in C(\Omega)$, the **support** is defined by $\operatorname{spt} f := \overline{\big\{x \in \Omega;\ f(x) \neq 0\big\}}$.

Examples 8.4

(a) The space s of rapidly decreasing sequences is a **Fréchet–Montel space**, i.e., a Fréchet space which also is semi-Montel (hence Montel, because Fréchet spaces are barrelled). Indeed, if $(x^k)_{k\in\mathbb{N}}$ is a bounded sequence in s, then one can choose a subsequence converging in each coordinate, and it is easy to show that this subsequence is convergent in s. Hence s is reflexive.

(b) Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Then

$$C_0^\infty(\Omega) := \big\{f \in C^\infty(\Omega);\ \partial^\alpha f \in C_0(\Omega)\ (\alpha \in \mathbb{N}_0^n)\big\},$$

with norms

$$p_m(f) := \max\big\{\|\partial^\alpha f\|_\infty;\ |\alpha| \leqslant m\big\} \quad (m \in \mathbb{N}_0,\ f \in C_0^\infty(\Omega)),$$

is a Fréchet–Montel space, therefore reflexive.

Indeed, $C_0^\infty(\Omega)$ is a Fréchet space. Also, every bounded set is relatively compact because of the Arzelà–Ascoli theorem, and therefore the space is semi-Montel.

A partial description of the dual is given as follows. If $\eta \in C_0^\infty(\Omega)'$, then there exist $m \in \mathbb{N}_0$ and $c \geqslant 0$ such that $|\eta(f)| \leqslant c p_m(f)$ $(f \in C_0^\infty(\Omega))$. The mapping

$$\Phi\colon (C_0^\infty(\Omega), p_m) \to C_0(\Omega)^{\{\alpha;\ |\alpha|\leqslant m\}}, \quad f \mapsto \big(\partial^\alpha f\big)_{|\alpha|\leqslant m},$$

is linear and isometric, and therefore the Hahn–Banach theorem implies that there exists $\hat\eta \in \big(C_0(\Omega)^{\{\alpha;\ |\alpha|\leqslant m\}}\big)'$ such that $\hat\eta \circ \Phi = \eta$. The Riesz–Markov theorem (see [Rud87, Theorem 2.14]) implies that there exists a family $\big(\mu_\alpha\big)_{|\alpha|\leqslant m}$ of finite Borel measures on Ω such that

$$\hat{\eta}(g) = \sum_{|\alpha|\leqslant m} \int g_\alpha \, d\mu_\alpha \quad \big(g = \big(g_\alpha\big)_{|\alpha|\leqslant m} \in C_0(\Omega)^{\{\alpha;\ |\alpha|\leqslant m\}}\big).$$

For $f \in C_0^\infty(\Omega)$ this means that

$$\eta(f) = \sum_{|\alpha|\leqslant m} \int \partial^\alpha f \, d\mu_\alpha = \Big(\sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} \partial^\alpha \mu_\alpha\Big) f,$$

where the derivatives of the measures should be interpreted in the sense of distributions. (Strictly speaking, the last formula would only be valid for $f \in C_c^\infty(\Omega)$, but the distributions can be extended by continuity to $f \in C_0^\infty(\Omega)$.)

(c) Let $\Omega \subseteq \mathbb{R}^n$ be open. Then $\mathcal{E}(\Omega) := C^\infty(\Omega)$, with semi-norms

$$p_{K,m}(f) := \max\big\{\|\partial^\alpha f\|_K\,;\ |\alpha| \leqslant m\big\} \quad (K \subseteq \Omega \text{ compact},\ m \in \mathbb{N}_0,\ f \in \mathcal{E}(\Omega))$$

(where $\|\cdot\|_K$ denotes the sup-norm on K) is a Fréchet–Montel space, in particular reflexive.

Let $\big(\Omega_k\big)_{k\in\mathbb{N}}$ be a **standard exhaustion** of Ω, i.e., Ω_k is open, relatively compact in Ω_{k+1} $(k \in \mathbb{N})$, and $\bigcup_{k\in\mathbb{N}} \Omega_k = \Omega$. Define $K_k := \overline{\Omega_k}$ $(k \in \mathbb{N})$. Then any compact subset of Ω is contained in some K_k; hence the topology of $\mathcal{E}(\Omega)$ is generated by the set $\big\{p_{K_k,m}\,;\ k \in \mathbb{N},\ m \in \mathbb{N}_0\big\}$; therefore $\mathcal{E}(\Omega)$ is metrisable, and also it is complete. (Note that, even though we use the standard exhaustion for the proof of the above properties, the topology does not depend on the choice of the exhaustion.)

Next we sketch why $\mathcal{E}(\Omega)$ is semi-Montel. As an intermediate step let $k \in \mathbb{N}_0$, and let (f_j) be a sequence in $\mathcal{E}(\Omega)$, $\sup_j \big\{\|\partial_l f_j\|_{K_{k+1}}\,;\ 1 \leqslant l \leqslant n\big\} < \infty$. Then the sequence (f_j) is bounded on K_{k+1} and equicontinuous on K_k, and by the Arzelà–Ascoli theorem there exists a $\|\cdot\|_{K_k}$-Cauchy subsequence. Now let (f_j) be a bounded sequence in $\mathcal{E}(\Omega)$. This means that $\sup_j p_{K,m}(f_j) < \infty$ for all compact $K \subseteq \Omega$, $m \in \mathbb{N}_0$. Applying the previous remark and a suitable diagonal procedure one obtains a subsequence which is a $p_{K,m}$-Cauchy sequence for all compact $K \subseteq \Omega$, $m \in \mathbb{N}_0$, i.e., a Cauchy sequence, and therefore convergent in $\mathcal{E}(\Omega)$.

(d) Let $\Omega \subseteq \mathbb{C}$ be open, $\mathcal{H}(\Omega) := \big\{f\colon \Omega \to \mathbb{C}\,;\ f \text{ holomorphic}\big\}$, with semi-norms

$$p_K(f) := \|f\|_K \qquad (f \in \mathcal{H}(\Omega),\ K \subseteq \Omega \text{ compact}).$$

Then $\mathcal{H}(\Omega)$ is a Fréchet–Montel space, therefore reflexive.

The Montel property of $\mathcal{H}(\Omega)$ is just **Montel's theorem**, and for completeness we recall its proof. Let $H \subseteq \mathcal{H}(\Omega)$ be a bounded set. Let (Ω_n) be a standard exhaustion of Ω, and for $n \in \mathbb{N}$ let $K_n := \overline{\Omega_n}$. For all $n \in \mathbb{N}$ one has

$$C_n := \sup\big\{\|f\|_{K_n}\,;\ f \in H\big\} < \infty,$$

and there exists $r_n > 0$ such that $K_n + B_{\mathbb{C}}[0, r_n] \subseteq K_{n+1}$. Then Cauchy's integral formula for the derivative,

$$f'(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B(z,r)} \frac{f(\zeta)}{(\zeta - z)^2} \, \mathrm{d}\zeta,$$

implies that $|f'(z)| \leqslant C_{n+1}(r_n/2)^{-2}$ for all $z \in K_n + B_{\mathbb{C}}(0, r_n/2)$, $f \in H$, and this estimate shows that $H_{K_n} := \{f|_{K_n};\ f \in H\}$ is equicontinuous. From the Arzelà–Ascoli theorem we conclude that H_{K_n} is a relatively compact subset of $C(K_n)$.

Now, starting with a sequence (f_k) in H we can choose a subsequence $(f_{k_j})_{j\in\mathbb{N}}$ such that $\big(f_{k_j}|_{K_n}\big)_{j\in\mathbb{N}}$ converges in $C(K_n)$ for all $n \in \mathbb{N}$, i.e., $(f_{k_j})_{j\in\mathbb{N}}$ is convergent in $C(\Omega)$. This shows that H is relatively sequentially compact in the metric space $\mathcal{H}(\Omega)$, hence relatively compact.

(e) Let $\Omega \subseteq \mathbb{R}^n$ be open,

8

$$H(\Omega) := \{f \in C^2(\Omega);\ f \text{ harmonic}\},$$

with semi-norms

$$p_K(f) := \|f\|_K \quad (K \subseteq \Omega \text{ compact},\ f \in H(\Omega)).$$

We recall that **harmonic** means that $\Delta f = \sum_{j=1}^n \partial_j^2 f = 0$. We will explain that then $H(\Omega)$ is a Fréchet–Montel space.

(i) Let $P := \sum_{|\alpha|\leqslant m} a_\alpha \partial^\alpha$ be a partial differential operator with constant coefficients. Then it is easy to see that the space

$$E_P(\Omega) := \{f \in \mathcal{E}(\Omega);\ Pf = 0\}$$

is a closed subspace of $\mathcal{E}(\Omega)$, therefore a Fréchet–Montel space; see Theorem 8.8(b) below. In the following we will sketch why $H(\Omega) = E_\Delta(\Omega)$.

(ii) We recall that harmonic functions f have the mean value property, i.e., if $x \in \Omega$, $r > 0$ are such that $B[x, r] \subseteq \Omega$, then

$$f(x) = \frac{1}{\sigma_{n-1}} \int_{S_{n-1}} f(x + r\xi) \, dS(\xi).$$

We refer to [Eva98, Section 2.2.2, Theorem 2] (or any other textbook on partial differential equations) for this property.

(iii) Let $(\Omega_k)_{k\in\mathbb{N}}$ be a standard exhaustion of Ω, $K_k := \overline{\Omega_k}$, $d_k := \operatorname{dist}(K_k, \Omega \setminus \Omega_{k+1})$, and let $\rho_k \in C_c^\infty(\mathbb{R}^n)$, $\rho_k \geqslant 0$, $\operatorname{spt} \rho_k \subseteq B(0, d_k)$, $\int \rho_k(x)\, dx = 1$, $\rho_k(x) = \rho_k(y)$ if $|x| = |y|$ $(k \in \mathbb{N})$. Then, for $f \in H(\Omega)$, the convolution $\rho_k * f$,

$$\rho_k * f(x) := \int_{\Omega_{k+1}} \rho_k(x - y) f(y) \, dy,$$

is defined for $x \in \Omega_k$, and in fact is equal to $f(x)$, because of the mean value property of f. Differentiating under the integral sign, one concludes that f is infinitely differentiable on Ω_k, and that

$$\partial^\alpha f(x) = \int_{\Omega_{k+1}} \partial^\alpha \rho_k(x-y) f(y)\,dy \quad (x \in \Omega_k,\ \alpha \in \mathbb{N}_0^n).$$

(iv) From (iii) it follows that, for $k \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^n$ there exists a constant $c_{k,\alpha}$ such that

$$\|\partial^\alpha f\|_{K_k} \leqslant c_{k,\alpha} \|f\|_{K_{k+1}} \quad (f \in H(\Omega)).$$

This shows that the topology on $H(\Omega)$ defined above is the topology induced by $\mathcal{E}(\Omega)$. Therefore the assertion follows from (i).

(f) The **Schwartz space** $\mathcal{S}(\mathbb{R}^n)$, also called the space of **rapidly decreasing functions**, is defined by

$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n);\ x \mapsto (1+|x|^2)^m \partial^\alpha f(x) \text{ bounded } (m \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^n) \right\},$$

with norms

$$p_{m,k}(f) := \max\left\{ (1+|x|^2)^m |\partial^\alpha f(x)|;\ x \in \mathbb{R}^n,\ |\alpha| \leqslant k \right\} (m, k \in \mathbb{N}_0,\ f \in \mathcal{S}(\mathbb{R}^n)).$$

It is standard to show that $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space. Next we show that $\mathcal{S}(\mathbb{R}^n)$ is a Montel space.

Let $m \in \mathbb{N}_0$, (f_k) a sequence with $M := \sup_k p_{m+1,m+1}(f_k) < \infty$. We show that then there exists a $p_{m,m}$-Cauchy subsequence. Let $\varepsilon > 0$; choose $R > 0$ such that $\frac{M}{1+R^2} < \varepsilon$. Then

$$\sup_k \left\{ (1+|x|^2)^m |\partial^\alpha f_k(x)|;\ |x| \geqslant R,\ |\alpha| \leqslant m \right\} \leqslant \varepsilon.$$

For $|\alpha| \leqslant m$ the set $\left\{ \partial^\alpha f_k;\ k \in \mathbb{N} \right\}$ is $\|\cdot\|_\infty$-bounded and equicontinuous on $B[0, R]$, and therefore, by the Arzelà–Ascoli theorem, there exists a subsequence $(f_{k_j})_j$ such that $(\partial^\alpha f_{k_j})_j$ is $\|\cdot\|_\infty$-convergent on $B[0, R]$, for all $|\alpha| \leqslant m$. Repeating this argument for smaller and smaller ε and choosing suitable subsequences, we obtain a $p_{m,m}$-Cauchy subsequence.

If (f_k) is a bounded sequence in $\mathcal{S}(\mathbb{R}^n)$, then the previous procedure can be carried out for arbitrary $m \in \mathbb{N}$, yielding a Cauchy sequence in $\mathcal{S}(\mathbb{R}^n)$.

We mention the remarkable fact that $\mathcal{S}(\mathbb{R})$ is isomorphic to the space s; see [MeVo97, Example 29.5(2)]. An analogous result for $\mathcal{S}(\mathbb{R}^n)$ is presented in [ReSi80, Theorem V.13]. △

After these examples we come back to some further theory.

Theorem 8.5
Let E be a reflexive Hausdorff locally convex space. Then the space $(E', \beta(E', E))$ is reflexive.

Proof
Let τ be the topology of E. By hypothesis and Theorem 6.8, $(E, \tau) = (E'', \beta(E'', E'))$. Therefore $E''' = (E'', \beta(E'', E'))' = (E, \tau)' = E'$, with $\beta(E''', E'') = \beta(E', E)$. □

Theorem 8.6
Let E be a Montel space. Then $(E', \beta(E', E))$ is a Montel space.

For the proof we need a preparation. Let E be a topological vector space. We define the topology τ_c on E' to be the **topology of compact convergence**, i.e., the polar topology $\tau_{\mathcal{M}_c}$ corresponding to the collection $\mathcal{M}_c$ of compact subsets of E.

The fact proved next is, in principle, a property of a uniformly equicontinuous set of functions on a *uniform space*; topological vector spaces are special uniform spaces. In fact, part of the proof is just a generalised version of the proof of the following standard property: If B is an equicontinuous set of functions on a compact metric space A, and $f \in B$, $\varepsilon > 0$, then there exists a finite set $F \subseteq A$ such that

$$\Big\{g \in B;\ \sup_{x \in F} |g(x) - f(x)| \leqslant \varepsilon/3\Big\} \subseteq \Big\{g \in B;\ \|g - f\|_\infty \leqslant \varepsilon\Big\}.$$

Proposition 8.7 *Let E be a topological vector space, and let $B \subseteq E'$ be equicontinuous. Then $\tau_c \cap B = \sigma(E', E) \cap B$.*

Proof
The inclusion '$\supseteq$' follows from $\tau_c \supseteq \sigma(E', E)$. For '$\subseteq$' it is sufficient to show: For $y_0 \in B$ and compact A, there exists a finite set $F \subseteq E$ such that

$$\Big\{y \in B;\ \sup_{x \in F} |\langle x, y - y_0\rangle| \leqslant 1/3\Big\} \subseteq \Big\{y \in B;\ \sup_{x \in A} |\langle x, y - y_0\rangle| \leqslant 1\Big\}.$$

(This property expresses that each τ_c-neighbourhood in B of y_0 contains a suitable $\sigma(E', E)$-neighbourhood in B of y_0.) As B is equicontinuous, there exists a balanced $U \in \mathcal{U}_0$ such that

$$\sup_{x \in U,\, y \in B} |\langle x, y\rangle| \leqslant 1/3.$$

Due to the compactness of A, there exists a finite set $F \subseteq A$ such that $A \subseteq F + U$. Now let $y \in B$ be such that $\sup_{\tilde{x} \in F} |\langle \tilde{x}, y - y_0 \rangle| \leqslant 1/3$. For $x \in A$ there exists $\tilde{x} \in F$ such that $x - \tilde{x} \in U$, which implies that

$$|\langle x, y - y_0 \rangle| \leqslant |\langle x - \tilde{x}, y \rangle| + |\langle \tilde{x}, y - y_0 \rangle| + |\langle \tilde{x} - x, y_0 \rangle| \leqslant 1;$$

hence $\sup_{x \in A} |\langle x, y - y_0 \rangle| \leqslant 1$. □

Proof of Theorem 8.6
The space $(E', \beta(E', E))$ is reflexive, by Corollary 8.3, therefore barrelled. Let $B \subseteq E'$ be $\beta(E', E)$-bounded, convex and closed. Theorem 6.8 implies that B is equicontinuous, therefore $\sigma(E', E)$-compact (by the Alaoglu–Bourbaki theorem). Now Proposition 8.7 implies that B is τ_c-compact. Since E is a Montel space, $\tau_c \cap E' = \beta(E', E)$, and therefore B is $\beta(E', E)$-compact. □

Theorem 8.8
Let E be a locally convex space, $F \subseteq E$ a closed subspace. Then:
(a) *If E is semi-reflexive, then F is semi-reflexive.*
(b) *If E is a semi-Montel space, then F is a semi-Montel space.*

Proof
(a) is a consequence of Theorem 8.1, because $\sigma(F, F') = \sigma(E, E') \cap F$ (recall Corollary 2.16).
(b) is obvious. □

Remark 8.9 The analogue of Theorem 8.8 with 'reflexive' instead of 'semi-reflexive' or 'Montel' instead of 'semi-Montel' does not hold. There even exists a Montel space with a non-reflexive closed subspace. We refer to [Sch71, Chap. IV, Exercises 19, 20] for an example. △

Notes The author was not able to trace the origins of (semi-)reflexivity and the (semi-)Montel property in locally convex spaces. The examples are standard in analysis. The isomorphy of $\mathcal{S}(\mathbb{R})$ and s, mentioned in Example 8.4(f) is due to Simon [Sim71, Theorem 1]. Theorem 8.6 can be found in [Köt66, VI, § 27.2], [Sch71, Chap. IV, § 5.9].