Weakly Compact Sets in $L_1(\mu)$

In view of the discussion of properties of weakly compact sets in the last chapters, it seems appropriate to present examples of weakly compact sets in a non-reflexive space. Besides the characterisation of weak compactness of subsets of $L_1(\mu)$, we will also show that $L_1(\mu)$ is weakly sequentially complete.

In all of this chapter $(\Omega, \mathcal{A}, \mu)$ will be a measure space.

A set $H \subseteq L_1(\mu)$ is called **equi-integrable** if H is bounded and for any sequence (A_n) in $\mathcal{A}$ with $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$, one has $\sup_{f\in H} \int_{A_n} |f| \,\mathrm{d}\mu \to 0$ as $n \to \infty$.

The main objective of this chapter is to prove the Dunford–Pettis theorem, which asserts that weak relative compactness for a subset of $L_1(\mu)$ is equivalent to equi-integrability; see Theorem 15.4.

We warn the reader that the notion of equi-integrability (also sometimes called 'uniform integrability') in some references is defined without the requirement of boundedness, and quite generally, there are various definitions of equi-integrability around, not all equivalent.

For functions $f, g\colon \Omega \to \mathbb{R}$ we will use the notation $[f > g] := \big\{x \in \Omega;\ f(x) > g(x)\big\}$, and similarly for $[f > 0]$, etc. In order to obtain another formulation of equi-integrability where in the condition the terms $\int_{A_n} |f| \,\mathrm{d}\mu$ are replaced by $\big|\int_{A_n} f \,\mathrm{d}\mu\big|$, we make the following observation. For $f \in L_1(\mu)$, $A \in \mathcal{A}$ there exists $B \in \mathcal{A}$, $B \subseteq A$ such that $\int_A |f| \,\mathrm{d}\mu \leqslant 4\big|\int_B f \,\mathrm{d}\mu\big|$. To show this we first observe that $\int_A |f| \,\mathrm{d}\mu \leqslant \int_A |\mathrm{Re}\, f| \,\mathrm{d}\mu + \int_A |\mathrm{Im}\, f| \,\mathrm{d}\mu$, and without loss of generality we can assume that $\int_A |\mathrm{Im}\, f| \,\mathrm{d}\mu \leqslant \int_A |\mathrm{Re}\, f| \,\mathrm{d}\mu$. Let $A_\pm := [\pm \mathrm{Re}\, f > 0]$; also without loss of generality we may assume that $-\int_{A_-} \mathrm{Re}\, f \,\mathrm{d}\mu \leqslant \int_{A_+} \mathrm{Re}\, f \,\mathrm{d}\mu$. Then with $B := A_+$ one obtains

$$\int_A |f| \,\mathrm{d}\mu \leqslant 2\int_A |\mathrm{Re}\, f| \,\mathrm{d}\mu \leqslant 4\int_B \mathrm{Re}\, f \,\mathrm{d}\mu \leqslant 4\Big|\int_B f \,\mathrm{d}\mu\Big|.$$

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In what follows we will use the abbreviation $A_n \downarrow \varnothing$ for a decreasing sequence (A_n) of sets satisfying $\bigcap_n A_n = \varnothing$.

Lemma 15.1 *A set $H \subseteq L_1(\mu)$ is equi-integrable if and only if it is bounded and for all sequences (A_n) in $\mathcal{A}$ with $A_n \downarrow \varnothing$ one has $\sup_{f\in H} \left|\int_{A_n} f \,\mathrm{d}\mu\right| \to 0$ as $n \to \infty$.*

Proof

It is trivial that the equi-integrability of H implies the condition. The converse implication will be proved by contraposition. Thus, assume that H is not equi-integrable. It is not difficult to show that then there exist $\varepsilon > 0$, a sequence (B_n) in $\mathcal{A}$, $B_n \downarrow \varnothing$, and a sequence (f_n) in H such that $\int_{B_n} |f_n| \,\mathrm{d}\mu \geqslant \frac{9}{8}\varepsilon$ and $\int_{B_{n+1}} |f_n| \,\mathrm{d}\mu \leqslant \frac{\varepsilon}{8}$ for all $n \in \mathbb{N}$. Note that this implies that $\int_{B_n \setminus B_{n+1}} |f_n| \,\mathrm{d}\mu \geqslant \varepsilon$ for all $n \in \mathbb{N}$. Then, by the observation preceding the lemma, for each $n \in \mathbb{N}$ there exists a set $C_n \in \mathcal{A}$, $C_n \subseteq B_n \setminus B_{n+1}$ such that $\left|\int_{C_n} f_n \,\mathrm{d}\mu\right| \geqslant \frac{1}{4}\int_{B_n\setminus B_{n+1}} |f_n| \,\mathrm{d}\mu \geqslant \frac{\varepsilon}{4}$. Defining $A_n := \bigcup_{k=n}^{\infty} C_k$ $(n \in \mathbb{N})$ we obtain $A_n \downarrow \varnothing$ and

$$\begin{aligned}\left|\int_{A_n} f_n \,\mathrm{d}\mu\right| &\geqslant \left|\int_{A_n\setminus A_{n+1}} f_n \,\mathrm{d}\mu\right| - \int_{A_{n+1}} |f_n| \,\mathrm{d}\mu \\ &\geqslant \left|\int_{C_n} f_n \,\mathrm{d}\mu\right| - \int_{B_{n+1}} |f_n| \,\mathrm{d}\mu \geqslant \frac{\varepsilon}{4} - \frac{\varepsilon}{8} = \frac{\varepsilon}{8};\end{aligned}$$

hence, $\sup_{f\in H} \left|\int_{A_n} f \,\mathrm{d}\mu\right| \geqslant \frac{\varepsilon}{8}$ for all $n \in \mathbb{N}$. □

In the proof that equi-integrability implies weak relative compactness we will use the following weak compactness criterion for sets in Banach spaces.

Lemma 15.2 (Grothendieck) *Let E be a Banach space, and let $A \subseteq E$. Assume that for all $\varepsilon > 0$ there exists a weakly compact set $A_\varepsilon \subseteq E$ such that $A \subseteq A_\varepsilon + \varepsilon B_E$. Then A is weakly relatively compact.*

15

Proof

Obviously A is bounded, and therefore $\overline{A}^{\sigma(E'',E')}$ is $\sigma(E'', E')$-compact. It is sufficient to show that $\overline{A}^{\sigma(E'',E')} \subseteq E$. For $\varepsilon > 0$ one has

$$\overline{A}^{\sigma(E'',E')} \subseteq \overline{A_\varepsilon + \varepsilon B_E}^{\sigma(E'',E')} \subseteq A_\varepsilon + \varepsilon B_{E''},$$

where for the last inclusion we have used that $A_\varepsilon + \varepsilon B_{E''}$ is $\sigma(E'', E')$-compact. This implies that

$$\overline{A}^{\sigma(E'',E')} \subseteq \bigcap_{\varepsilon>0} (A_\varepsilon + \varepsilon B_{E''}).$$

Given $x \in \overline{A}^{\sigma(E'',E')}$, one obtains sequences (x_n) in E, (y_n) in E'', $x_n \in A_{1/n}$, $\|y_n\| \leqslant 1/n$, $x_n + y_n = x$ $(n \in \mathbb{N})$. From $y_n \to 0$ one concludes that $x_n \to x$ $(n \to \infty)$, hence $x \in E$. □

In order to apply this criterion we have to deduce the required ε-approximation from the equi-integrability. This will be provided by the following lemma.

Lemma 15.3 *Let $H \subseteq L_1(\mu)$ be equi-integrable. Then:*
(a) *For any $\varepsilon > 0$ there exists $\delta > 0$ such that $B \in \mathcal{A}$, $\mu(B) < \delta$ implies that $\int_B |f| \, d\mu \leqslant \varepsilon$ for all $f \in H$.*
(b) *For any $\varepsilon > 0$ there exists $B \in \mathcal{A}$ with $\mu(B) < \infty$ such that $\int_{\Omega \setminus B} |f| \, d\mu \leqslant \varepsilon$ for all $f \in H$.*
(c) *For any $\varepsilon > 0$ there exist $B \in \mathcal{A}$ with $\mu(B) < \infty$ and $n \in \mathbb{N}$ such that $\sup_{f \in H} \int (|f| - n\mathbf{1}_B)^+ \, d\mu \leqslant \varepsilon$.*

Proof
(a) Assume that the assertion does not hold. Then there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ one can find a set $B_n \in \mathcal{A}$ with $\mu(B_n) \leqslant 2^{-n}$ and $f_n \in H$ such that $\int_{B_n} d\mu \geqslant \varepsilon$. Then $B_0 := \bigcap_{n \in \mathbb{N}} \bigcup_{k=n}^{\infty} B_k$ is a μ-null set, and setting $A_n := \big(\bigcup_{k=n}^{\infty} B_k\big) \setminus B_0$ one obtains a sequence (A_n) in $\mathcal{A}$ such that $A_n \downarrow \varnothing$ and $\int_{A_n} |f_n| \, d\mu \geqslant \int_{B_n} |f_n| \, d\mu \geqslant \varepsilon$ for all $n \in \mathbb{N}$, which contradicts the equi-integrability of H.

(b) Assume that the assertion does not hold. Then there exists $\varepsilon > 0$ such that for all $B \in \mathcal{A}$ with $\mu(B) < \infty$ one can find $f \in H$ such that $\int_{\Omega \setminus B} |f| \, d\mu > \varepsilon$. This implies that there exist a disjoint sequence (B_n) in $\mathcal{A}$ with $\mu(B_n) < \infty$ for all $n \in \mathbb{N}$ and a sequence (f_n) in H such that $\int_{B_n} |f_n| \, d\mu \geqslant \varepsilon$ for all $n \in \mathbb{N}$. Setting $A_n := \bigcup_{k \geqslant n} B_k$ we obtain a sequence (A_n) in $\mathcal{A}$, $A_n \downarrow \varnothing$, with $\int_{A_n} |f_n| \, d\mu \geqslant \varepsilon$ for all $n \in \mathbb{N}$, which contradicts the equi-integrability of H.

(c) Let $\varepsilon > 0$. Because of part (b) above, there exists $B \in \mathcal{A}$ with $\mu(B) < \infty$ such that

$$\int_{\Omega \setminus B} |f| \, d\mu \leqslant \varepsilon/2 \quad (f \in H). \tag{15.1}$$

Define $c := \sup_{f \in H} \|f\|$. By part (a), there exists $\delta > 0$ such that $\int_A |f| \, d\mu < \varepsilon/2$ for all $f \in H$ and all $A \in \mathcal{A}$ with $\mu(A) < \delta$. For $n \in \mathbb{N}$, $f \in H$ we obtain

$$c \geqslant \int_{[|f| > n} |f| \, d\mu \geqslant n\mu([|f| > n]).$$

For $n > c/\delta$ we conclude that $\mu(|f| > n]) \leqslant c/n < \delta$; hence

$$\|(|f| - n)^+\| = \int_{[|f| > n]} (|f| - n) \, d\mu < \varepsilon/2 \quad (f \in H, \ n > c/\delta). \tag{15.2}$$

Combining (15.1) and (15.2) we obtain the assertion. □

We mention in passing that in fact a set $H \subseteq L_1(\mu)$ is equi-integrable if and only if H is bounded and the properties asserted in (a) and (b) of Lemma 15.3 are satisfied. Another noteworthy consequence of part (b) is that an equi-integrable set H always 'lives on a σ-finite subset of Ω': There exists a σ-finite subset $B \in \mathcal{A}$ such that $f|_{\Omega \setminus B} = 0$ for all $f \in H$.

For one of the equivalences in the main result of this chapter we introduce the following notation. For a disjoint sequence (B_n) in $\mathcal{A}$ we define the mapping

$$L_{(B_n)}\colon L_1(\mu) \to \ell_1,\ f \mapsto \Big(\int_{B_n} f\,\mathrm{d}\mu\Big)_{n\in\mathbb{N}}.$$

Obviously $L_{(B_n)}$ is a continuous linear operator, even a contraction.

Theorem 15.4 (Dunford–Pettis)
For $H \subseteq L_1(\mu)$ the following properties are equivalent:
(i) *H is equi-integrable;*
(ii) *H is weakly relatively compact;*
(iii) *for each disjoint sequence (B_n) in $\mathcal{A}$ the operator $L_{(B_n)}$ maps H to a relatively compact subset of ℓ_1.*

Proof
(i) $\Rightarrow$ (ii). Let $\varepsilon > 0$. We choose B and n as asserted in Lemma 15.3(c). In $L_2(B, \mu_B)$, where μ_B denotes the restriction of the measure μ to $\mathcal{A} \cap B$, the set $\big\{f \in L_2(\mu_B);\ |f| \leqslant n\big\}$ is bounded, convex and closed, hence weakly compact (because $L_2(\mu_B)$ is reflexive). The embedding $L_2(\mu_B) \hookrightarrow L_1(\mu_B)$ is continuous, hence, by Lemma 6.3, continuous with respect to the weak topologies, and as a consequence the set $\big\{f \in L_1(\mu);\ |f| \leqslant n\mathbf{1}_B\big\}$ is weakly compact in $L_1(\mu)$. The inequality in Lemma 15.3(c) shows that $H \subseteq \big\{f \in L_1(\mu);\ |f| \leqslant n\mathbf{1}_B\big\} + B_{L_1(\mu)}(0, \varepsilon)$. Now Lemma 15.2 implies that H is weakly relatively compact.

(ii) $\Rightarrow$ (iii). As $L_{(B_n)}\colon L_1(\mu) \to \ell_1$ is a continuous operator, this operator is also continuous with respect to the weak topologies; hence $L_{(B_n)}(H)$ is a weakly relatively compact subset of ℓ_1, and Corollary 5.10 implies that $L_{(B_n)}(H)$ is relatively compact.

(iii) $\Rightarrow$ (i). Let (A_n) be a sequence in $\mathcal{A}$, $A_n \downarrow \varnothing$. We define $B_n := A_n \setminus A_{n+1}$ $(n \in \mathbb{N})$. Then (B_n) is a disjoint sequence in $\mathcal{A}$.

Clearly, $L_{(B_n)}(H)$ is bounded. Recall from Example 5.6(i) that the relative compactness of $L_{(B_n)}(H)$ is equivalent to $\sup_{f\in H} \sum_{k=n}^{\infty} \big|\int_{B_k} f\,\mathrm{d}\mu\big| \to 0$ as $n \to \infty$. Observe that

$$\Big|\int_{A_n} f\,\mathrm{d}\mu\Big| \leqslant \sum_{k=n}^{\infty}\Big|\int_{B_k} f\,\mathrm{d}\mu\Big| \qquad (f \in H,\ n \in \mathbb{N}).$$

Hence, Lemma 15.1 implies that H is equi-integrable. □

As the second important result of the present chapter we show that $L_1(\mu)$ is weakly sequentially complete. For ℓ_1, this property had already been shown in Theorem 5.8; see also Remark 5.9.

15

Theorem 15.5
Let (f_n) be a Cauchy sequence in $L_1(\mu)$ with respect to the weak topology. Then (f_n) is weakly convergent.

Proof
Let (B_k) be a disjoint sequence in $\mathcal{A}$. Then it is immediate that $(L_{(B_k)} f_n)_n$ is a weak Cauchy sequence in ℓ_1; recall Lemma 6.3. Theorem 5.8 implies that $(L_{(B_k)} f_n)_n$ is convergent, and therefore the range of the sequence is relatively compact in ℓ_1.

Now Theorem 15.4 shows that the set $\{f_n;\ n \in \mathbb{N}\}$ is weakly relatively compact. This implies that the sequence (f_n) possesses a weak cluster point. Being a weak Cauchy sequence, it is convergent in the weak topology, by Remark 9.1(b). □

We conclude this chapter by some additional comments.

Remarks 15.6 (a) It is not difficult to show that the equi-integrability of a set $H \subseteq L_1(\mu)$ is equivalent to the condition that for each $\varepsilon > 0$ there exists $g \in L_1(\mu)_+$ such that $\sup_{f\in H} \int_{[|f|>g]} |f|\, d\mu < \varepsilon$.
Concerning the necessity of this condition, the function g can be found in the form $g = c\mathbf{1}_B$ for suitable $c > 0$ and $B \in \mathcal{A}$ with $\mu(B) < \infty$; see Lemma 15.3(c). The sufficiency is rather immediate.

(b) Theorem 15.4 implies: If $H \subseteq L_1(\mu)$ is weakly relatively compact, then the set

$$\{f \in L_1(\mu);\ \text{there exists } g \in H \text{ such that } |f| \leqslant |g|\}$$

is weakly relatively compact. In particular, for every $g \in L_1(\mu)_+$ the order interval

$$[-g, g] := \{f \in L_1(\mu);\ -g \leqslant f \leqslant g\}$$

is weakly compact.

Similarly: If $g \in L_1(\mathbb{R})_+$, then the set $\{g(\cdot - y);\ 0 \leqslant y \leqslant 1\}$ is compact (because the mapping $y \mapsto g(\cdot - y)$ is continuous); hence,

$$\{f \in L_1(\mu);\ |f| \leqslant g(\cdot - y) \text{ for some } y \in [0, 1]\}$$

is weakly relatively compact. △

Notes Theorem 15.4 is due to Dunford and Pettis [DuPe40, Theorem 3.2.1]. Lemma 15.2 is attributed to Grothendieck in [Die84, XIII, Lemma 2]. With the aid of this lemma the proof that equi-integrability implies weak relative compactness, in Theorem 15.4, is rather natural. The author was at a loss for finding a short 'measure theory-free' proof of the reverse implication, in the literature. The device to use the operators $L_{(B_n)}$ in Theorem 15.4 is present in the original paper [DuPe40], for 'decompositions' of the measure space. The weak sequential completeness of $L_1(\mu)$, Theorem 15.5, is due to Dunford and Pettis as well [DuPe40, p. 377].

Our definition of equi-integrability can be found implicitly in [DuSc58, Theorem IV.8.9, Corollary IV.8.10 and their proofs]. The characterisation of equi-integrability mentioned in Remark 15.6(a) is taken as the definition in [Bau90, § 21] and appears in [Bog07, Theorem 4.7.20] as one of the equivalences of weak relative compactness of a set.