The Banach–Dieudonné and Krein–Šmulian Theorems

In this and the following two chapters we discuss some surprising properties concerning the weak topology of Banach spaces. (However, the discussion will not be restricted to Banach spaces!)

For the first result stated below we will give an interesting and motivating application in the subsequent example. The proof of this result and the more genreral Krein–Šmulian theorem requires the consideration of several additional topologies on locally convex spaces.

Theorem 12.1 (Banach)
Let E be a Banach space, $F \subseteq E'$ a subspace. Then F is $\sigma(E', E)$-closed if and only if $F \cap B_{E'}$ is $\sigma(E', E)$-closed.

Example 12.2
Let E be a complex Banach space, $\Omega \subseteq \mathbb{C}$ open, $f : \Omega \to E$. A 'traditional' result is then Dunford's theorem: f is holomorphic if and only if $x' \circ f$ is holomorphic for all $x' \in E'$ ([Dun38, Theorem 76], [Yos80, Section V.3]). ('Holomorphic' is defined as complex differentiable, and the $\mathbb{C}$-valued theory of functions of one complex variable carries over to E-valued functions, with the result that E-valued holomorphic functions are analytic.) It is relatively standard that the hypothesis in Dunford's theorem can be weakened to the requirement that $x' \circ f$ is holomorphic for all $x' \in F$, where F is an almost norming subspace

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of E'. Using Theorem 12.1 one can show that even this condition can be replaced by a weaker requirement:

Let $f\colon \Omega \to E$ be locally bounded, and assume that the set

$$F := \{x' \in E';\ x' \circ f \text{ holomorphic}\}$$

is separating in E. Then f is holomorphic.

We start the proof by noting that F is a subspace of E', and that the hypothesis implies that F is $\sigma(E', E)$-dense in E'; see Corollary 2.10. Now we show that the closed unit ball of F,

$$B_F = \{x' \in F;\ \|x'\| \leqslant 1\} = B_{E'} \cap F,$$

is $\sigma(E', E)$-closed. We introduce the mapping $\varphi\colon E' \to \mathbb{C}^\Omega$, $x' \mapsto x' \circ f$ and note that φ is continuous with respect to $\sigma(E', E)$ and the product topology on $\mathbb{C}^\Omega$. By Montel's theorem – see Example 8.4(d) –, the set

$$H := \{g\colon \Omega \to \mathbb{C} \text{ holomorphic};\ |g(z)| \leqslant \|f(z)\|\ (z \in \Omega)\}$$

is a compact subset of $C(\Omega)$ (provided with the topology of compact convergence); therefore H is closed in $\mathbb{C}^\Omega$. Then the equality

$$B_F = B_{E'} \cap \varphi^{-1}(H)$$

12

shows that B_F is $\sigma(E', E)$-closed.

Now we conclude from Theorem 12.1 that F is $\sigma(E', E)$-closed, and therefore $F = E'$. Then the assertion follows from Dunford's theorem.

The result quoted above is due to Grosse-Erdmann ([GrE92]). The above elegant proof is a variant of the proof given by Arendt and Nikolski ([ArNi00, Theorem 3.1]); see also [ABHN11, Theorem A.7]. △

For another application of Theorem 12.1, resulting in a generalisation of Pettis' theorem on measurability of Banach space-valued functions we refer to [ABHN11, Corollary 1.3.3].

Remark 12.3 Corollary 9.18 can be derived from Theorem 12.1. Indeed, if $u \in E'^*$ is $\sigma(E', E)$-continuous on $B_{E'}$, then $u^{-1}(0) \cap B_{E'}$ is $\sigma(E', E)$-closed; hence $u^{-1}(0)$ is a $\sigma(E', E)$-closed subspace of E', and u is $\sigma(E', E)$-continuous, i.e., $u \in E$. △

The proof of Theorem 12.1 will be given at the end of this chapter; the remainder of the chapter is devoted to preparations for the proof of a more general version.

For a locally convex space E we define a topology τ_{f} on E' by

$$\tau_{\mathrm{f}} := \{A \subseteq E';\ A \cap B \in \sigma(E', E) \cap B \text{ for all equicontinuous sets } B \subseteq E'\};$$

it is not difficult te check that τ_f is indeed a topology on E'. Expressed differently, we equip the sets $B \in \mathcal{E}$ (the collection of equicontinuous subsets of E') with the trace $\sigma(E', E) \cap B$ of the weak topology and equip E' with the finest topology on E' for which all injections $j_B \colon B \hookrightarrow E'$ are continuous. If $\mathcal{E}_0$ is a cobase of $\mathcal{E}$, for instance $\mathcal{E}_0 = \{U^\circ;\ U \in \mathcal{U}\}$ where $\mathcal{U}$ is a neighbourhood base of zero in E, then τ_f is also the finest topology for which all j_B, for $B \in \mathcal{E}_0$, are continuous. (Concerning terminology: A **cobase** of a collection $\mathcal{A}$ of sets is a subcollection $\mathcal{A}'$ of $\mathcal{A}$ such that for all $A \in \mathcal{A}$ there exists $A' \in \mathcal{A}'$ such that $A \subseteq A'$.)

Clearly, a set $A \subseteq E'$ is τ_f-closed if and only if $A \cap B$ is $\sigma(E', E) \cap B$-closed for all B belonging to a cobase $\mathcal{E}_0$ of $\mathcal{E}$.

Proposition 12.4 *Let E be a locally convex space, and let τ_f be the topology on E' defined above. Then $\tau_f \supseteq \tau_c$ (topology of compact convergence, see Chapter 8). The topology τ_f is Hausdorff, translation invariant, and every τ_f-neighbourhood of zero is absorbing and contains a balanced τ_f-neighbourhood of zero.*

Proof

It was shown in Proposition 8.7 that $\tau_c \cap B = \sigma(E', E) \cap B$ for all equicontinuous sets $B \subseteq E'$. As τ_f is the finest topology coinciding with $\sigma(E', E)$ on the equicontinuous sets, it follows that $\tau_f \supseteq \tau_c$.

The topology τ_f is Hausdorff because $\tau_f \supseteq \sigma(E', E)$, and τ_f is translation invariant because the collection of equicontinuous sets and the topology $\sigma(E', E)$ are translation invariant.

Let V be a τ_f-neighbourhood of zero, $x' \in E'$, $B \subseteq E'$ equicontinuous, balanced and containing x'. Then there exists a balanced $\sigma(E', E)$-neighbourhood of zero W such that $W \cap B \subseteq V \cap B$. There exists $\alpha \in (0, 1)$ such that $\lambda x' \in W$ for $|\lambda| \leqslant \alpha$, and therefore

$$\lambda x' \in W \cap B \subseteq V \cap B \subseteq V \quad (|\lambda| \leqslant \alpha).$$

This shows that V is absorbing.

Let U be a τ_f-neighbourhood of zero, and let

$$V := \bigcup \{A \subseteq U;\ A \text{ balanced}\}$$

be its 'balanced core' (the largest balanced subset of U). Let $B \subseteq E'$ be equicontinuous and balanced. There exists a balanced $W \in \mathcal{U}_0(\sigma(E', E))$ such that $W \cap B \subseteq U \cap B \subseteq U$. Since $W \cap B$ is balanced, one concludes that $W \cap B \subseteq V$, and this implies that $W \cap B \subseteq V \cap B$. This shows that V is a τ_f-neighbourhood of zero. □

Remark 12.5 Why can one only show 'balanced' in Proposition 12.4(b)? The reason in the proof is that there does not exist an 'absolutely convex core' of sets. In fact, the reason is deeper, because it is known that in general τ_f is not a linear (let alone a locally convex) topology ([Kōm64, § 2]).

The index 'f' in τ_f is historical and probably just stands for 'finest'. △

Theorem 12.6 (Banach–Dieudonné)
Let E be a metrisable locally convex space. Let

$$\mathcal{M}_{\rm ns} := \bigl\{\{x_n;\ n \in \mathbb{N}\} \cup \{0\};\ (x_n) \text{ null sequence in } E\bigr\}.$$

Then $\tau_{\rm f} = \tau_{\rm c} = \tau_{\rm ns} := \tau_{\mathcal{M}_{\rm ns}}$.

Proof
(i) '$\tau_{\rm f} \supseteq \tau_{\rm c} \supseteq \tau_{\rm ns}$'. The first inclusion is part of Proposition 12.4; the second inclusion holds because every $A \in \mathcal{M}_{\rm ns}$ is compact.

(ii) '$\tau_{\rm ns} \supseteq \tau_{\rm f}$'. Let U be an open $\tau_{\rm f}$-neighbourhood of zero. It suffices to show that there exists $A \in \mathcal{M}_{\rm ns}$ such that $A^\circ \subseteq U$.
Because E is metrisable, there exists a decreasing neighbourhood base of zero $(V_n)_{n\in\mathbb{N}_0}$ in E, $V_0 = E$, and all V_n absolutely convex and closed. In part (iii) of the proof we will show:

$$\begin{aligned}&\text{For each } n \in \mathbb{N}_0 \text{ there exists a finite set } B_n \subseteq V_n \text{ such that } A_n^\circ \cap V_n^\circ \subseteq U,\\ &\text{where } A_n := \textstyle\bigcup_{k=0}^{n-1} B_k\ (n \in \mathbb{N}_0).\end{aligned} \tag{$*$}$$

Assuming this, we set $A := \bigl(\bigcup_{k=0}^\infty B_k\bigr) \cup \{0\}$. Then obviously $A \in \mathcal{M}_{\rm ns}$. Also $A^\circ \subseteq A_n^\circ$, and therefore $A^\circ \cap V_n^\circ \subseteq U$ $(n \in \mathbb{N})$. From $\bigcap_{n\in\mathbb{N}} V_n = \{0\}$ one obtains $\bigcup_{n\in\mathbb{N}} V_n^\circ = E'$, and therefore $A^\circ \subseteq U$.

(iii) We prove ($*$) by induction. For $n = 0$, the assertion holds with $B_0 = \varnothing$. Assume that B_k has been obtained for $k = 0, \dots, n-1$. We have to find a finite set $B_n \subseteq V_n$ such that $(A_n \cup B_n)^\circ \cap V_{n+1}^\circ \subseteq U$.
Set $C := V_{n+1}^\circ \setminus U$. The polar V_{n+1}° is compact for $\sigma(E', E)$, by the Alaoglu–Bourbaki theorem. Because V_{n+1}° is equicontinuous, the topologies $\tau_{\rm f}$ and $\sigma(E', E)$ agree on V_{n+1}°; therefore, V_{n+1}° is also compact for $\tau_{\rm f}$, and as a consequence the closed subset C is compact for $\tau_{\rm f}$. Since $A_n^\circ \cap V_n^\circ \subseteq U$ and $U \cap C = \varnothing$, we know that $A_n^\circ \cap V_n^\circ \cap C = \varnothing$. For all $x \in V_n$ the set $\{x\}^\circ \cap A_n^\circ \cap C$ is a closed subset of C, and

$$\bigcap_{x\in V_n} \bigl(\{x\}^\circ \cap A_n^\circ \cap C\bigr) = \Bigl(\bigcap_{x\in V_n} \{x\}^\circ\Bigr) \cap A_n^\circ \cap C = V_n^\circ \cap A_n^\circ \cap C = \varnothing.$$

Now the compactness of C implies that the family $\bigl(\{x\}^\circ \cap A_n^\circ \cap C\bigr)_{x\in V_n}$ cannot have the finite intersection property. This means that there exists a finite subset $B_n \subseteq V_n$ such that $\varnothing = B_n^\circ \cap A_n^\circ \cap C = (A_n \cup B_n)^\circ \cap \bigl(V_{n+1}^\circ \setminus U\bigr)$, hence $(A_n \cup B_n)^\circ \cap V_{n+1}^\circ \subseteq U$. □

Remark 12.7 The usual way to formulate Theorem 12.6 is to use the topology τ_{pc}, the topology of uniform convergence on the precompact sets of E, instead of τ_{c}. An inspection of the proof of Proposition 8.7 immediately yields that it also shows that $\tau_{\text{pc}} \cap B = \sigma(E', E) \cap B$ for all equicontinuous sets B. This implies that in Theorem 12.6 one also obtains $\tau_{\text{f}} = \tau_{\text{pc}} = \tau_{\text{ns}}$ (which is the traditional assertion in the Banach–Dieudonné theorem). △

Let E be a locally convex space,

$$\mathcal{M}_{\text{cc}} := \{A \subseteq E;\ A \text{ convex and compact}\}.$$

Then $\tau_{\text{cc}} := \tau_{\mathcal{M}_{\text{cc}}}$, the **topology of compact convex convergence**, is a polar topology on E'. Observe that, in view of Lemma 4.9, the set

$$\mathcal{M}'_{\text{cc}} := \{A \subseteq E;\ A \text{ absolutely convex and compact}\}$$

is a cobase of $\mathcal{M}_{\text{cc}}$, hence $\tau_{\mathcal{M}'_{\text{cc}}} = \tau_{\mathcal{M}_{\text{cc}}}$. Note that $\sigma(E', E) \subseteq \tau_{\text{cc}} \subseteq \mu(E', E)$; therefore $(E', \tau_{\text{cc}})' = b_1(E)$ $(= E$ if E is Hausdorff).

If E is Hausdorff and quasi-complete, then $\tau_{\text{c}} = \tau_{\text{cc}}$ is compatible with the dual pair $\langle E, E' \rangle$.

Theorem 12.8 (Krein–Šmulian)
Let E be a Fréchet space, and let $\mathcal{U}$ be a neighbourhood base of zero in E. Then a convex set $A \subseteq E'$ is $\sigma(E', E)$-closed if and only if $A \cap U^\circ$ is $\sigma(E', E)$-closed for every $U \in \mathcal{U}$.

Proof
The necessity is trivial.

For the sufficiency, we recall that A is τ_{f}-closed, which by Theorem 12.6 implies that A is τ_{c}-closed. By the above preliminary remark, $\tau_{\text{c}} = \tau_{\text{cc}}$ is compatible with the dual pair $\langle E, E' \rangle$, and therefore the convex set A is $\sigma(E', E)$-closed as well. □

Proof of Theorem 12.1
This follows immediately from Theorem 12.8. □

Notes Theorem 12.1 is contained in [Ban32, Chap. VIII, § 3, Lemme 3]. In order to understand this it should be mentioned that the subspaces of E' whose intersection with the closed unit ball is $\sigma(E', E)$-closed occur in [Ban32] as ‘transfiniment fermé’, whereas $\sigma(E', E)$-closed subspaces are ‘régulièrement fermé’. A translation into more modern terminology was given by Bourbaki [Bou38], and a new proof was given by Dieudonné [Die42, Théorème 23]. (Interestingly enough, the proof by contraposition in [Ban32, Chap. VIII, § 3, Lemme 2] seems to have persisted in the literature, where usually in step (iii) of the proof of Theorem 12.6, the existence of a finite set B_n is shown by contraposition.) The new methods introduced by Dieudonné then served to extend Theorem 12.8 – proved in [KrŠm40, Theorem 5] only for the case of Banach spaces – to more general settings. For this and a variety of related results obtained by these methods we refer to Köthe [Köt66, § 21.10] and Schaefer [Sch71, Chap. IV, § 6.4].