Precompact – Compact – Complete

This chapter is a short survey on the technical properties mentioned in the title, for subsets of topological vector spaces and locally convex spaces.

Let E be a topological vector space, $A \subseteq E$. The set A is called **precompact** if for all $U \in \mathcal{U}_0$ there exists a finite set $F \subseteq E$ such that $A \subseteq F + U$.

Remarks 11.1 (a) If A is precompact, then $\overline{A}$ is precompact.
(b) If A is compact, then A is precompact.
(c) Subsets, scalar multiples and finite unions of precompact sets are precompact.
(d) If A is precompact, then A is bounded.
(e) The notion 'precompact' can be defined in the more general framework of uniform spaces. △

Theorem 11.2
Let E be a topological vector space, $A \subseteq E$. Then the following properties are equivalent:
(i) *A is precompact;*
(ii) *every filter in A possesses a finer Cauchy filter;*
(iii) *every ultrafilter in A is a Cauchy filter.*

Proof
The equivalence '(ii) $\Leftrightarrow$ (iii)' is clear, because every filter possesses a finer ultrafilter.

(i) $\Rightarrow$ (iii). Let $\mathcal{F}$ be an ultrafilter in A, and let $U \in \mathcal{U}_0$. Then there exists a finite set $F \subseteq E$ such that $A \subseteq F + U$. Using Remark 4.3(b) one concludes that one of the sets $(x + U) \cap A$ $(x \in F)$ belongs to $\mathcal{F}$. This implies that $\mathcal{F}$ is a Cauchy filter.

J. Voigt, *A Course on Topological Vector Spaces*, Compact Textbooks in Mathematics,
https://doi.org/10.1007/978-3-030-32945-7_11

(ii) $\Rightarrow$ (i). Assume that A is not precompact. Then there exists $U \in \mathcal{U}_0$ such that $A \setminus (F + U) \neq \varnothing$ for all finite sets $F \subseteq E$. Then the collection

$$\{A \setminus (F + U);\ F \subseteq E \text{ finite}\}$$

is a filter base, with a finer Cauchy filter $\mathcal{F}$. There exists a set $B \in \mathcal{F}$ with $B - B \subseteq U$. For $x \in B$ one deduces that $B \subseteq x + U$, $(x + U) \cap A \in \mathcal{F}$. But also $A \setminus (x + U) \in \mathcal{F}$ by construction. Therefore

$$\varnothing = (A \cap (x + U)) \cap (A \setminus (x + U)) \in \mathcal{F},$$

a contradiction. □

Theorem 11.3
Let E be topological vector space, $A \subseteq E$. Then A is compact if and only if A is precompact and complete.

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Proof
For the necessity the precompactness is clear. In order to prove the completeness, let $\mathcal{F}$ be a Cauchy filter. Then Proposition 4.5 implies that $\mathcal{F}$ possesses a cluster point $x \in A$. Then $\mathcal{F} \to x$, because $\mathcal{F}$ is a Cauchy filter.

For the sufficiency let $\mathcal{F}$ be an ultrafilter in A. Then $\mathcal{F}$ is a Cauchy filter, by Theorem 11.2, and therefore converges. Now Proposition 4.5 implies that A is compact. □

Theorem 11.4
Let E be a topological vector space, $A \subseteq E$ precompact.
(a) *Then* bal A *is precompact.*
(b) *If E is locally convex, then* aco A *is precompact.*

Proof
(a) Let $U \in \mathcal{U}_0$ be balanced. There exists a finite set $F \subseteq E$ such that $A \subseteq F + U$. Then

$$\operatorname{bal} A = \bigcup_{|\lambda| \leqslant 1} \lambda A \subseteq \bigcup_{|\lambda| \leqslant 1} (\lambda F + \lambda U) \subseteq \operatorname{bal} F + U.$$

The set bal F is compact; therefore, there exists a finite set $B \subseteq E$ such that bal $F \subseteq B + U$, and so

$$\operatorname{bal} A \subseteq \operatorname{bal} F + U \subseteq B + (U + U).$$

(b) Let $U \in \mathcal{U}_0$ be absolutely convex. Then there exists a finite set $F \subseteq E$ such that $A \subseteq F + U$. The set

$$\operatorname{aco} F = \Big\{ \sum_{y\in F} \lambda_y y;\ (\lambda_y) \in \mathbb{K}^F,\ \sum_{y\in F} |\lambda_y| \leqslant 1 \Big\}$$

is compact, because it is the continuous image of the compact set

$$\Big\{ (\lambda_y)_{y\in F} \in \mathbb{K}^F;\ \sum_{y\in F} |\lambda_y| \leqslant 1 \Big\}.$$

Therefore there exists a finite set $B \subseteq E$ such that $\operatorname{aco} F \subseteq B + U$.

Let $x \in \operatorname{aco} A$. Then $x = \sum_{j=1}^m \mu_j x_j$, with $x_1, \dots, x_m \in A$, $\sum_{j=1}^m |\mu_j| \leqslant 1$. Then $x_j = y_j + z_j$ with suitable $y_j \in F$, $z_j \in U$ $(j = 1, \dots, m)$; therefore

$$x = \sum_{j=1}^m \mu_j x_j = \sum_{j=1}^m \mu_j y_j + \sum_{j=1}^m \mu_j z_j \in B + U + U.$$

Hence $\operatorname{aco} A \subseteq B + (U + U)$. □

Corollary 11.5 *Let E be a quasi-complete topological vector space, $A \subseteq E$ compact.*
(a) *Then $\overline{\operatorname{bal}}\, A$ is compact.*
(b) *If E is locally convex, then also $\overline{\operatorname{aco}}\, A$ is compact.*

Proof
This is immediate from Remark 11.1(a) and Theorems 11.3 and 11.4. □

Notes Theorems 11.3 and 11.4 are analogous to what is standard in metric spaces. The remaining facts contain useful information and preparations for later results. For the closed convex hull of a compact set in a Banach space, Corollary 11.5 is due to Mazur [Maz30].