1 Initial Topology, Topological Vector Spaces, Weak Topology

The main objective of this chapter is to present the definition of topological vector spaces and to derive some fundamental properties. We will also introduce dual pairs of vector spaces and the weak topology. We start the chapter by briefly recalling concepts of topology and continuity, thereby also fixing notation.

Let X be a set, $\tau \subseteq \mathcal{P}(X)$ (the power set of X). Then τ is called a **topology**, and (X, τ) is called a **topological space**, if

for any $\mathcal{S} \subseteq \tau$ one has $\bigcup \mathcal{S} \in \tau$,
for any finite $\mathcal{F} \subseteq \tau$ one has $\bigcap \mathcal{F} \in \tau$.

(This definition is with the understanding that $\bigcup \varnothing = \varnothing$, $\bigcap \varnothing = X$, with the consequence that always $\varnothing, X \in \tau$.) Concerning notation, we could also write

$$\bigcup \mathcal{S} = \bigcup_{U \in \mathcal{S}} U, \quad \bigcap \mathcal{F} = \bigcap_{A \in \mathcal{F}} A.$$

If $\mathcal{S} = (U_\iota)_{\iota \in I}$ or $\mathcal{F} = (A_n)_{n \in N}$ are families of sets, with N finite, then one can also write

$$\bigcup \{U_\iota ;\ \iota \in I\} = \bigcup_{\iota \in I} U_\iota, \quad \bigcap \{A_n ;\ n \in N\} = \bigcap_{n \in N} A_n.$$

The sets $U \in \tau$ are called **open**, whereas a set $A \subseteq X$ is called **closed** if $X \setminus A$ is open. For a set $B \subseteq X$ we define

$\mathring{B}$ $(= \operatorname{int} B) := \bigcup \{U ;\ U \in \tau,\ U \subseteq B\}$, the **interior** of B (an open set),
$\overline{B}$ $(= \operatorname{cl} B) := \bigcap \{A ;\ A \supseteq B,\ A \text{ closed}\}$, the **closure** of B (a closed set).

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For $x \in X$, a set $U \subseteq X$ is called a **neighbourhood** of x if $x \in \mathring{U}$, and the collection

$$\mathcal{U}_x := \{U \subseteq X;\ U \text{ neighbourhood of } x\}$$

is called the **neighbourhood filter** of x. (Note that $U \cap V \in \mathcal{U}_x$ if $U, V \in \mathcal{U}_x$.) A **neighbourhood base** $\mathcal{B}$ of x is a collection $\mathcal{B} \subseteq \mathcal{U}_x$ with the property that the neighbourhood filter coincides with the collection of supersets of sets in $\mathcal{B}$. (Note that neighbourhoods need not be open sets.)

A topological space (X, τ) is called **Hausdorff** if for any $x, y \in X$, $x \neq y$, there exist neighbourhoods U of x, V of y such that $U \cap V = \varnothing$.

If (X, d) is a **semi-metric space**, i.e., X is a set and the **semi-metric** $d\colon X \times X \to [0, \infty)$ is symmetric and satisfies $d(x, x) = 0$ $(x \in X)$ as well as the **triangle inequality**

$$d(x, y) \leqslant d(x, z) + d(z, y) \qquad (x, y, z \in X),$$

then d induces a topology τ_d on X: A set $U \subseteq X$ is defined to be open if for all $x \in U$ there exists $r > 0$ such that $B(x, r) \subseteq U$, where

$$B(x,r) = B_X(x,r) = B_d(x,r) := \big\{y \in X;\ d(y,x) < r\big\}$$

is the **open ball** with centre x and radius r. The corresponding **closed ball** will be denoted by

$$B[x,r] = B_X[x,r] = B_d[x,r] := \big\{y \in X;\ d(y,x) \leqslant r\big\}.$$

(We mention that our definition of 'semi-metric' often runs under the name 'pseudo-metric'; we found our notation more convenient, as it is parallel to 'semi-norm', mentioned later.) The topology τ_d is Hausdorff if and only if d is a **metric**, i.e., additionally to the previous properties one has that $d(x, y) = 0$ implies $x = y$.

A topological space (X, τ) is called **(semi-)metrisable** if there exists a (semi-)metric on X such that $\tau = \tau_d$.

If $\tau \supseteq \sigma$ are topologies on a set X, then τ is said to be **finer** (or **stronger**) than σ, and σ is said to be **coarser** (or **weaker**) than τ. The **trivial topology** $\{\varnothing, X\}$ is the coarsest topology on X, and the **discrete topology** $\mathcal{P}(X)$, i.e., the collection of all subsets of X, is the finest topology on X.

Let (X, τ), (Y, σ) be topological spaces, $f\colon X \to Y$, $x \in X$. Then f is **continuous at** x if $f^{-1}(V)$ is a neighbourhood of x, for all neighbourhoods V of $f(x)$. The mapping f is called **continuous**, if f is continuous at every $x \in X$, and this is equivalent to the property that $f^{-1}(V) \in \tau$ for all $V \in \sigma$. The mapping f is a **homeomorphism**, if f is continuous and bijective, and the inverse $f^{-1}\colon Y \to X$ is also continuous.

Remark 1.1 Let X be a set, $\Gamma \subseteq \mathcal{P}(\mathcal{P}(X))$ a set of topologies. Then it is easy to see that $\bigcap \Gamma$ is a topology on X. In order to spell this out more explicitly, we note that

$$\bigcap \Gamma = \bigcap_{\tau \in \Gamma} \tau = \{A \subseteq X;\ A \in \tau \text{ for all } \tau \in \Gamma\}.$$

(In this case, because of the subscript '$\tau \in \Gamma$', $\bigcap \tau$ does *not* mean $\bigcap_{U \in \tau} U$.) △

Let X be a set, $\mathcal{S} \subseteq \mathcal{P}(X)$. Then

$$\operatorname{top} \mathcal{S} := \bigcap \{\tau;\ \tau \text{ topology on } X,\ \tau \supseteq \mathcal{S}\}$$

is the coarsest topology containing $\mathcal{S}$, called the **topology generated by $\mathcal{S}$**, and $\mathcal{S}$ is called a **subbase** of $\operatorname{top} \mathcal{S}$.

If τ is a topology, $\mathcal{B} \subseteq \tau$, and for all $U \in \tau$ one has that

$$U = \bigcup \{V \in \mathcal{B};\ V \subseteq U\},$$

then $\mathcal{B}$ is called a **base** for τ. If $\mathcal{S}$ is a subbase of τ, then it is not difficult to show that

$$\mathcal{B} := \left\{\bigcap \mathcal{F};\ \mathcal{F} \subseteq \mathcal{S},\ \mathcal{F} \text{ finite}\right\} \tag{1.1}$$

is a base of τ.

Let X be a set. Let I be an index set (i.e., a set whose elements we use as indices), and for $\iota \in I$ let (X_ι, τ_ι) be a topological space and $f_\iota \colon X \to X_\iota$ a mapping. The topology

$$\operatorname{top}\{f_\iota^{-1}(U_\iota);\ U_\iota \in \tau_\iota,\ \iota \in I\} \tag{1.2}$$

is the coarsest topology on X for which all mappings f_ι are continuous; it is called the **initial topology** with respect to the family $(f_\iota;\ \iota \in I)$. A base of the initial topology is given by

$$\left\{\bigcap_{\iota \in F} f_\iota^{-1}(U_\iota);\ F \subseteq I \text{ finite},\ U_\iota \in \tau_\iota\ (\iota \in F)\right\}; \tag{1.3}$$

this is a consequence of (1.1) and (1.2).

The **product topology** on $\prod_{\iota \in I} X_\iota$ is the initial topology with respect to the family $(\mathrm{pr}_\iota;\ \iota \in I)$ of the canonical projections. A base of the product topology is given by

$$\left\{\prod_{\iota \in F} U_\iota \times \prod_{\iota \in I \setminus F} X_\iota;\ F \subseteq I \text{ finite},\ U_\iota \in \tau_\iota\ (\iota \in F)\right\}.$$

The following theorem is an important key result on initial topologies, which will be used repeatedly in this treatise.

Theorem 1.2
Let (Y,σ), (X,τ), (X_ι,τ_ι) $(\iota\in I)$ be topological spaces, $g\colon Y\to X$, $f_\iota\colon X\to X_\iota$ $(\iota\in I)$, τ the initial topology with respect to $(f_\iota;\ \iota\in I)$. Let $y\in Y$. Then:
(a) *g is continuous at y if and only if $f_\iota\circ g$ is continuous at y $(\iota\in I)$.*
(b) *g is continuous if and only if $f_\iota\circ g$ is continuous $(\iota\in I)$.*
(c) *The initial topology on Y with respect to g is the same as the initial topology with respect to $(f_\iota\circ g;\ \iota\in I)$.*

Proof
(a) The necessity is clear. In order to show the sufficiency, let U be a neighbourhood of $g(y)$. There exist a finite set $F\subseteq I$ and $U_\iota\in\tau_\iota$ $(\iota\in F)$ such that $\bigcap_{\iota\in F} f_\iota^{-1}(U_\iota)\subseteq U$ is a neighbourhood of $g(y)$. (Recall that these sets constitute a base of the initial topology.) Therefore, the set

$$g^{-1}\Big(\bigcap_{\iota\in F} f_\iota^{-1}(U_\iota)\Big)=\bigcap_{\iota\in F} g^{-1}\big(f_\iota^{-1}(U_\iota)\big)=\bigcap_{\iota\in F}(f_\iota\circ g)^{-1}(U_\iota)$$

is a neighbourhood of y, and is a subset of $g^{-1}(U)$.

(b) is a consequence of (a).

(c) is an immediate consequence of (b). □

Next we define topological vector spaces and derive some basic properties.

Let E be a vector space over the field $\mathbb{K}$ (where $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$), and let τ be a topology on E. Then τ is called a **linear topology**, and (E,τ) is called a **topological vector space**, if the mappings

$$a\colon E\times E\to E,\ (x,y)\mapsto x+y,$$
$$m\colon \mathbb{K}\times E\to E,\ (\lambda,x)\mapsto \lambda x$$

are continuous.

In a topological vector space (E,τ) we will denote the neighbourhood filter of zero by $\mathcal{U}_0$ (or $\mathcal{U}_0(E)$, or $\mathcal{U}_0(\tau)$).

Examples 1.3
(a) A vector space E with the trivial topology $\tau=\{\varnothing,E\}$ is a topological vector space.

(b) A vector space $E\neq\{0\}$ with the discrete topology is not a topological vector space. Indeed, it is easy to see that the scalar multiplication m is not continuous.

(c) The scalars $\mathbb{R}$ and $\mathbb{C}$ are topological vector spaces.

(d) Normed and semi-normed spaces are topological vector spaces. △

For more explanation on Example 1.3(d) we recall that a **semi-norm** p on a vector space E is a mapping $p\colon E \to [0,\infty)$ satisfying

$p(x+y) \leqslant p(x) + p(y)$ $(x, y \in E)$, the **triangle inequality**,
$p(\lambda x) = |\lambda| p(x)$ $(x \in E,\ \lambda \in \mathbb{K})$, i.e., p is **absolutely homogeneous**.

The semi-norm p gives rise to a semi-metric d on E, defined by $d(x, y) := p(x-y)$ $(x, y \in E)$. Then the inequalities $p((x+y) - (x_0 - y_0)) \leqslant p(x - x_0) + p(y - y_0)$ and $p(\lambda x - \lambda_0 x_0) \leqslant |\lambda| p(x - x_0) + |\lambda - \lambda_0| p(x_0)$ $(x, x_0, y, y_0 \in E,\ \lambda, \lambda_0 \in \mathbb{K})$ show the continuity of addition and scalar multiplication. The semi-metric d is a metric if and only if p is a **norm**, i.e., if additionally $p(x) = 0$ implies $x = 0$, for $x \in E$.

In the following theorem we collect some basic properties of topological vector spaces.

Theorem 1.4
Let (E, τ) be a topological vector space. Then:
(a) *For all $x \in E$ the mapping $a_x\colon E \to E$, $y \mapsto x + y$ is a homeomorphism. The topology τ is determined by a neighbourhood base of zero.*
(b) *For all $\lambda \in \mathbb{K} \setminus \{0\}$ the mapping $m_\lambda\colon E \to E$, $x \mapsto \lambda x$ is a homeomorphism.*
(c) *Each $U \in \mathcal{U}_0(E)$ is **absorbing**, i.e., for all $x \in E$ there exists $\alpha > 0$ such that $x \in \lambda U$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \geqslant \alpha$.*
(d) *For all $U \in \mathcal{U}_0(E)$ there exists $V \in \mathcal{U}_0(E)$ such that $V + V \subseteq U$.*

Proof
(a) It is sufficient to show that the mapping a_x is continuous. It is a consequence of Theorem 1.2 (and the definition of the product topology on $E \times E$) that the mapping

$$j_x\colon E \to E \times E,\ \ y \mapsto (x, y)$$

is continuous. Then $a_x = a \circ j_x$ is continuous, because the addition a is continuous. The last statement is then obvious. (Note that the topology is determined if for each point in the space one knows a neighbourhood base.)

(b) Similarly to (a), we note that the mapping

$$j_\lambda\colon E \to \mathbb{K} \times E,\ \ x \mapsto (\lambda, x)$$

is continuous. Then the continuity of $m_\lambda = m \circ j_\lambda$ follows from the continuity of the scalar multiplication m.

(c) Similarly to part (a) one shows that the mapping $\mathbb{K} \ni \lambda \mapsto \lambda x \in E$ is continuous. Therefore there exists $\alpha > 0$ such that $\lambda x \in U$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \leqslant \alpha$.

(d) Let $U \in \mathcal{U}_0(E)$. Then, by the continuity of the addition at the point $(0, 0)$, there exist $V_1, V_2 \in \mathcal{U}_0(E)$ such that $V_1 + V_2 \subseteq U$. Then $V := V_1 \cap V_2$ is as asserted. □

Next we introduce the concept of dual pairs of vector spaces, a central notion in our treatment.

A **dual pair** $\langle E, F\rangle$ consists of two vector spaces E, F over the same field $\mathbb{K}$ and a bilinear mapping $b = \langle\cdot\,,\,\cdot\rangle\colon E \times F \to \mathbb{K}$. The mapping b gives rise to mappings

$$b_1\colon E \to F^*, \text{ defined by } b_1(x) := \langle x, \cdot\rangle \quad (x \in E),$$
$$b_2\colon F \to E^*, \text{ defined by } b_2(y) := \langle\cdot\,, y\rangle \quad (y \in F),$$

where E^*, F^* denote the algebraic duals of E, F, respectively. The dual pair is **separating in** E if

$$x \in E,\ \langle x, y\rangle = 0\ (y \in F) \text{ implies that } x = 0, \text{ i.e., } b_1 \text{ is injective},$$

separating in F if

$$y \in F,\ \langle x, y\rangle = 0\ (x \in E) \text{ implies that } y = 0, \text{ i.e., } b_2 \text{ is injective},$$

and **separating**, if it is separating in E and F.

The **weak topology** $\sigma(E, F)$ on E with respect to the dual pair $\langle E, F\rangle$ is defined as the initial topology with respect to the family $(\langle\cdot, y\rangle;\ y \in F)$; the weak topology $\sigma(F, E)$ on F is defined analogously.

If $B \subseteq F$ is finite, then

$$U_B := \big\{x \in E;\ |\langle x, y\rangle| < 1\ (y \in B)\big\}$$

is a $\sigma(E, F)$-neighbourhood of zero. A $\sigma(E, F)$-neighbourhood base of zero is given by

$$\big\{U_B;\ B \subseteq F \text{ finite}\big\};$$

see Remark 1.6.

The following theorem is basic for the theory and important for the construction of topological vector spaces; it shows (amongst other facts) that $\sigma(E, F)$ is a linear topology.

Theorem 1.5

Let E be a vector space, $\big((E_\iota, \tau_\iota);\ \iota \in I\big)$ a family of topological vector spaces, $f_\iota\colon E \to E_\iota$ linear maps ($\iota \in I$), τ the initial topology on E with respect to $(f_\iota;\ \iota \in I)$. Then (E, τ) is a topological vector space.

Proof

First we show the continuity of the scalar multiplication $m\colon \mathbb{K} \times E \to E$. By Theorem 1.2 it is sufficient to show that $f_\iota \circ m\colon \mathbb{K} \times E \to E_\iota$ is continuous for all $\iota \in I$. For $\lambda \in \mathbb{K}$, $x \in E$, one has

$$f_\iota \circ m(\lambda, x) = f_\iota(\lambda x) = \lambda f_\iota(x) = m_\iota(\lambda, f_\iota(x)),$$

with m_ι denoting the scalar multiplication in E_ι; therefore $f_\iota \circ m = m_\iota \circ (\mathrm{id}_{\mathbb{K}} \times f_\iota)$. Noting that Theorem 1.2 implies that $\mathrm{id}_{\mathbb{K}} \times f_\iota \colon \mathbb{K} \times E \to \mathbb{K} \times E_\iota$ is continuous we obtain the assertion.

The continuity of the addition a in E is proved analogously: For $\iota \in I$, the continuity of $f_\iota \circ a = a_\iota \circ (f_\iota \times f_\iota)$ follows from the continuity of $f_\iota \times f_\iota \colon E \times E \to E_\iota \times E_\iota$ and the addition a_ι in E_ι. □

Remark 1.6 If, in the situation of Theorem 1.5, $\mathcal{U}_\iota$ is a neighbourhood base of zero, for all $\iota \in I$, then a neighbourhood base of zero for the initial topology on E is given by

$$\Big\{ \bigcap_{\iota \in F} f_\iota^{-1}(U_\iota);\ F \subseteq I \text{ finite},\ U_\iota \in \mathcal{U}_\iota\ (\iota \in F) \Big\}.$$

This follows from (1.3) △

Examples 1.7
(a) The weak topologies $\sigma(E, F)$ and $\sigma(F, E)$, for a dual pair $\langle E, F\rangle$, are linear topologies.

(b) Let E be a vector space, P a set of semi-norms on E. Then the initial topology τ_P on E with respect to the mappings $\mathrm{id} \colon E \to (E, p)$ $(p \in P)$ is called the **topology generated** by P. Theorem 1.5 implies that τ_P is a linear topology.

(c) Let I be an index set. Then $\mathbb{K}^I$, with the product topology τ, the initial topology with respect to the projections $\mathrm{pr}_\kappa \colon \mathbb{K}^I \to \mathbb{K}$, $(x_\iota)_{\iota\in I} \mapsto x_\kappa$, is a topological vector space, by Theorem 1.5. With

$$c_c(I) := \big\{ (y_\iota)_{\iota \in I} \in \mathbb{K}^I;\ \{\iota \in I;\ y_\iota \neq 0\} \text{ finite} \big\}$$

we form the dual pair $\langle \mathbb{K}^I, c_c(I)\rangle$ by defining the duality bracket

$$\langle x, y\rangle := \sum_{\iota \in I} x_\iota y_\iota \qquad (x = (x_\iota)_{\iota\in I} \in \mathbb{K}^I,\ y = (y_\iota)_{\iota \in I} \in c_c(I)).$$

Then $\tau = \sigma(\mathbb{K}^I, c_c(I))$. Indeed, it is evident that $\tau \subseteq \sigma(\mathbb{K}^I, c_c(I))$, because $\mathrm{pr}_\kappa x = \langle x, \delta_\kappa\rangle$, where $\delta_\kappa \in c_c(I)$ is defined by $\delta_{\kappa\kappa} := 1$, $\delta_{\kappa\iota} := 0$ if $\iota \neq \kappa$. On the other hand, for each $y \in c_c(I)$, the mapping $x \mapsto \langle x, y\rangle$ is a finite linear combination of canonical projections, hence continuous with respect to τ.

The product topology is also generated by the family of semi-norms $(p_\kappa)_{\kappa \in I}$, where $p_\kappa(x) := |x_\kappa|$ $(x = (x_\iota)_{\iota \in I} \in \mathbb{K}^I)$.

(d) Let X be a topological space, $E := C(X)$ the space of continuous functions $f \colon X \to \mathbb{K}$. For compact $K \subseteq X$ we define the semi-norm p_K, by

$$p_K(f) := \sup_{x \in K} |f(x)| \qquad (f \in C(X)),$$

and set

$$P := \{p_K\,;\ K \subseteq X \text{ compact}\}.$$

Then τ_P is the topology of **compact convergence**; it is a linear topology.

(e) Let $((E_\iota, \tau_\iota)\,;\ \iota \in I)$ be a family of topological vector spaces, and let $E := \prod_{\iota\in I} E_\iota$. Then E, with the product topology, is a topological vector space. △

For a topological vector space (E, τ), the **dual**, or **dual space**, $(E, \tau)'$ is defined as the vector space of all continuous linear functionals on E. We will not always explicitly specify the topology of a topological vector space E, and accordingly, we will denote the dual of E by E' if it is clear from the context to which topology on E we refer.

By definition, every linear functional $\langle\cdot, y\rangle$, for $y \in F$, is continuous for $\sigma(E, F)$; the following result shows that the converse is also true.

Theorem 1.8

Let $\langle E, F\rangle$ be a dual pair. Let $\eta \in (E, \sigma(E, F))'$. Then there exists $y \in F$ such that $\eta(x) = \langle x, y\rangle$ $(x \in E)$. Expressed differently, one has $(E, \sigma(E, F))' = b_2(F)$.

For the proof we need a preparatory lemma from linear algebra.

Lemma 1.9 *Let E be a vector space, $\eta, \eta_1, \ldots, \eta_n \in E^*$,*

$$\bigcap_{j=1}^{n} \ker \eta_j \subseteq \ker \eta.$$

Then there exist $c_1, \ldots, c_n \in \mathbb{K}$ such that $\eta = \sum_{j=1}^n c_j\eta_j$.

Proof

(i) We start with a preliminary tool. Let F, G be vector spaces, $f\colon E \to F$ and $g\colon E \to G$ linear, g surjective, and $\ker g \subseteq \ker f$. Then there exists $\hat{f}\colon G \to F$ linear, such that $f = \hat{f} \circ g$.

In fact, $\hat{f}(g(x)) := f(x)$ $(x \in E)$ is well-defined: If $g(x) = g(x_1)$, then $x - x_1 \in \ker g \subseteq \ker f$, and therefore $f(x) = f(x_1)$. The linearity of $\hat{f}$ is then easy.

(ii) Apply (i) with $f = \eta$, $g = (\eta_1, \ldots, \eta_n)\colon E \to g(E) \subseteq \mathbb{K}^n$, to obtain $\hat{f}\colon g(E) \to \mathbb{K}$. There exists a linear extension $\hat{f}\colon \mathbb{K}^n \to \mathbb{K}$, and this extension is of the form

$$\hat{f}(y) = \sum_{j=1}^{n} c_j y_j \quad (y \in \mathbb{K}^n),$$

with suitable $(c_1, \ldots, c_n) \in \mathbb{K}^n$. Then $\eta = \hat{f} \circ (\eta_1, \ldots, \eta_n) = \sum_{j=1}^n c_j\eta_j$. □

Proof of Theorem 1.8
As η is continuous with respect to $\sigma(E, F)$, there exists a finite set $B \subseteq F$ such that

$$\eta(U_B) = \eta\big(\{x \in E;\ |\langle x, y\rangle| < 1\ (y \in B)\}\big) \subseteq B_{\mathbb{K}}(0, 1)$$

(the open unit ball in $\mathbb{K}$), or expressed differently,

$$|\eta(x)| \leqslant \max_{y \in B} |\langle x, y\rangle| \quad (x \in E).$$

For $x \in E$ with $\langle x, y\rangle = 0$ $(y \in B)$ one concludes that $\eta(x) = 0$. From Lemma 1.9 we conclude that there exist $c_y \in \mathbb{K}$ $(y \in B)$ such that

$$\eta = \sum_{y \in B} c_y \langle \cdot\,, y\rangle = \langle \cdot\,, \sum_{y \in B} c_y y\rangle.$$ □

Example 1.10
Coming back to $E = \mathbb{K}^I$ – see Example 1.7(c) – we note that Theorem 1.8 implies that $E' = \big(\mathbb{K}^I, \sigma(\mathbb{K}^I, c_c(I))\big)' = c_c(I)$. △

From the definition it is clear that $\sigma(E, E')$ is the coarsest linear topology on E such that $E' \supseteq b_2(F)$, and Theorem 1.8 expresses that for this topology one even has $E' = b_2(F)$. Later we will also obtain a finest locally convex topology with this property; see Chapter 5.

Notes The material of the present chapter is standard, and it is rather impossible to give precise information where the contents originated. For the fundamental notions of topology we refer to [Bou07c]; in particular, our Theorem 1.2 is as in [Bou07c, Chap. 1, § 3, Proposition 4].

Concerning topological vector spaces and in particular locally convex spaces we include at this place a list of treatises on the subject, in principle in historical order: [Ban32], [Edw65], [Köt66], [Hor66], [Sch71] (first edition 1966), [Trè67], [Gro73], [RoRo73], [Rud91], [Wil78], [Bou07a] (new edition from 1981 of [Bou64a], [Bou64b]), [Jar81] [MeVo97], [Osb14], [BoSm17]. The beginning is marked by Banach's pioneering book. As mentioned in the preface, it was in the 1960s that the topic became "fashionable" also for teaching, and the treatises are of varying character, volume and focus. Wilansky's contribution is notable for its richness of exercises and examples, and we add Khaleelulla's Lecture Notes [Kha82] to the list as an abundant and well structured source of counterexamples.

The list indicated above contains only texts in which the main emphasis is on locally convex topological vector spaces. Many books on Banach space theory, functional analysis or operator theory contain also substantial parts on topological vector spaces. As examples, we mention the encyclopedic volume [DuSc58] and the treatises [Yos80], [Con90] and [Wer18].