Chapter 7
Locating Dimensional Facilities in a Continuous Space

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Abstract Many applications in data analysis such as regression, projective clustering, or support vector machines can be modeled as location problems in which the facilities to be located are not represented by points but as dimensional structures. Examples for one-dimensional facilities are straight lines, line segments, or circles while boxes, strips, or balls are two-dimensional facilities. In this chapter we discuss the location of lines and circles in the plane, the location of hyperplanes and hyperspheres in higher dimensional spaces and the location of some other dimensional facilities. We formulate the resulting location problems and point out applications in statistics, operations research and data analysis. We identify important properties and review the basic solution techniques and algorithmic approaches. Our focus lies on presenting a unified understanding of the common characteristics these problems have, and on reviewing the new findings obtained in this field within the last years.

7.1 Introduction

Within the locational context, the problem of locating a dimensional facility was first posed in Wesolowsky (1972, 1975) where the location of a line minimizing the sum of rectangular or Euclidean distances to a set of data points was introduced. Since this time, the subject of locating lines and hyperplanes, circles, spheres, and other dimensional facilities has been intensively studied. Surveys are given in Martini and Schöbel (1998), Díaz-Bánez et al. (2004), an extensive list of papers dealing with the location of dimensional structures is also given in Blanquero et al. (2009).

Within the last 10 years, the topic has received new focus in the field of data science leading to new results and approaches. In this chapter, we review the new findings and present a unified understanding of the subject which is now possible

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since the field has become more mature. We hence not only present a list of problems treated in the literature, but point out common characteristics and common solution techniques which are used for many different types of such location problems.

Applications in the location of dimensional facilities are various: These range from real-world applications in location theory and operations research to applications in robust statistics, computational geometry, and data science. Particular applications are mentioned at the beginning of the respective sections.

The chapter is organized as follows. We start with a general introduction into the topic in Sect. 7.2 where we introduce the basic notation, define the problems to be considered and mention the properties on which we will focus later on. We then discuss the two most extensively researched structures in dimensional facility location: The location of lines and hyperplanes in Sect. 7.3 and the location of circles and hyperspheres in Sect. 7.4. We finally review other interesting extensions and problem variations in Sect. 7.5. The chapter is ended by some conclusion in Sect. 7.6 summarizing the findings and pointing out lines for further research.

7.2 Location of Dimensional Facilities

In classical facility location one looks for a point-shaped new facility. In our case we look for a dimensional facility X such as a line, a hyperplane, a circle or a square. The location of a dimensional facility is a natural generalization of locating a point. As in classical location problems we have given

- a finite set $V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^D$ of *data points* (also called *existing facilities*) with positive weights $w_j > 0$, $j = 1, \ldots, n$, and
- a distance measure $d : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{R}$ evaluating the distance for each pair of points in $\mathbb{R}^D$. The distance measure is used for determining the *residuals*, i.e., the distances from the data points to the new facility X. Finally, we need
- a *globalizing function* $g : \mathbb{R}^n \to \mathbb{R}$ combining the weighted residuals to one global number.

We look for a new facility X which minimizes the globalizing function g of the weighted distances to the data points.

$$\text{minimize } f(X) = g \begin{pmatrix} w_1 d(X, v_1) \\ w_2 d(X, v_2) \\ \vdots \\ w_n d(X, v_n) \end{pmatrix}, \tag{7.1}$$

where the most common globalizing functions g are the sum, i.e., $g_1(y_1, \ldots, y_n) = \sum_{j=1}^{n} y_j$ or the maximum $g_{\max}(y_1, \ldots, y_n) = \max_{j=1,\ldots,n} y_j$. The resulting problems are called *minsum* (or *median*) location problem and *minmax* (or *cen-*

ter) location problem, respectively. Also, other globalizing functions such as the centdian, or more general, ordered median functions g_λ (see Chap. 10) are possible.

If the new facility X is required to be a point, or a set of points, we are in the situation of classical continuous facility location, see Drezner et al. (2001). In this chapter, however, we assume that X is not a point but a dimensional structure such as a line, a circle, a hyperplane, a hypersphere, a polygonal line, etc. This, in turn, means that the distance $d(X, v)$ in (7.1) is the distance between a set X (which represents the dimensional facility) and a (data) point v. As common in the literature the distance between a point v and a set X is determined by projecting the point v on the set X and then taking the distance from v to the projected point, i.e.,

$$d(X, v) = \min_{x \in X} d(x, v). \tag{7.2}$$

Note that in some applications $d(X, v)$ is defined as $\max_{x \in X} d(x, v)$, and that the average distance to all points in the set also is a reasonable definition; however, (7.2) is the most common model in this context.

We now specify the distances d which have mostly been studied in the literature. The most common distances in location theory are *norm distances*. A norm distance is derived from a norm, i.e., $d : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{R}$ is given as $d(x, y) := \|x - y\|$ for some norm $\| \cdot \|$. Moreover, *gauge distances* which are derived from a gauge $\gamma : \mathbb{R}^D \to \mathbb{R}$ given through $d(x, y) = \gamma(y - x)$ have also been used in the location of dimensional facilities. Note that gauge distances are no metrics since they are in general not symmetric, and that norms are special gauges. In particular in statistics, the *vertical distance* is used which is neither a norm nor a gauge. We will see that it gives nevertheless insight into the problem, in particular for the location of lines and hyperplanes. For two points $x = (x^1, \ldots, x^D)$, $y = (y^1, \ldots, y^D) \in \mathbb{R}^D$ the vertical distance is given as

$$d_{ver}(x, y) = \begin{cases} |x^D - y^D| & \text{if } x^i = y^i,\ i = 1, \ldots, D-1 \\ \infty & \text{otherwise.} \end{cases} \tag{7.3}$$

This distance leads to trivial location problems if X is required to be a point but constitutes the most common definition of residuals in regression.

Figure 7.1 presents two examples on how distances are computed, and optimal dimensional structures may look like. In both examples we have given six data points, all of them with unit weights. The left part of Fig. 7.1 shows a line minimizing the maximum vertical distance to the set of data points. In the right part a circle minimizing the sum of Euclidean distances to the data points is depicted. The lengths of the thin lines in both examples correspond to the residuals, i.e., to the distances from the data points to the line (or to the circle, respectively). Note that the distance between $v \in X$ and X is zero—this happens twice in the right part of the figure where the minsum circle passes through two of the data points.

In the following sections we discuss different types of dimensional facilities to be located. Most of the resulting optimization problems are multi-modal and

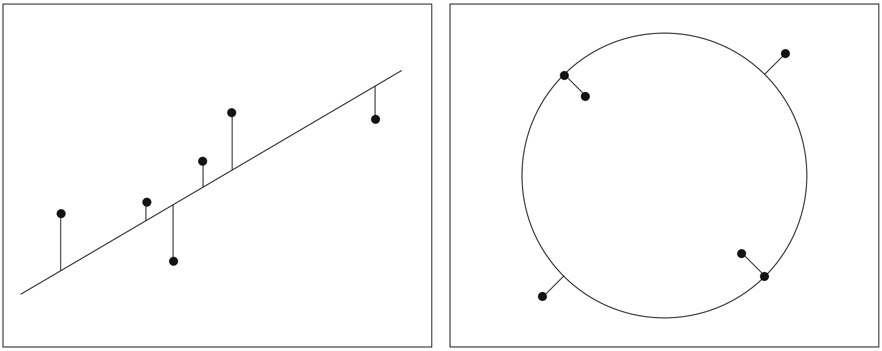

Fig. 7.1 Two illustrations for locating dimensional facilities, both with six demand points. Left: A line minimizing the maximum vertical distance. Right: A circle minimizing the sum of Euclidean distances

neither convex nor concave. Hence, methods of global optimization are required. However, in many of these location problems it is possible to exploit one or more of the following properties showing that they have much more structure than just an arbitrary global optimization problem.

LP properties:	Some of the problems become piecewise linear, sometimes even resulting in linear programming (LP) approaches which can be solved highly efficiently.
FDS properties:	A finite dominating set (FDS) is a finite set of possible solutions from which it is known that it contains an optimal solution to the problem. This allows an enumeration approach by evaluating all possible elements of the FDS.
Halving properties:	In many cases, any optimal facility to be located splits the data points into two sets of nearly equal weights. This allows to enhance enumeration approaches.

In our conclusion we provide a summary on these properties and give some general hints when they hold and why they are algorithmically useful.

7.3 Locating Lines and Hyperplanes

Given a set of data points $V \subseteq \mathbb{R}^D$ the hyperplane location problem is to find a hyperplane H minimizing the distances to the data points in V. In this section we consider such hyperplane location problems for different types of distances and different globalizing functions.

Note that line location deals with finding a line in $\mathbb{R}^2$ minimizing the distances to a set of two-dimensional data points and is included in our discussion as the special case $D = 2$.

7.3.1 Applications

The location of lines and hyperplanes has many applications in different fields: Operations research, computational geometry, and in statistics and data science. Applications in *operations research* are various. The new facility to be located may be, e.g., a highway (see Díaz-Bánez et al. 2013), a train line (see Espejo and Rodríguez-Chía 2011), a conveyor belt, or a mining shaft (e.g., Brimberg et al. 2002). Line location has also been mentioned in connection with the planning of pipelines, drainage or irrigation ditches, or in the field of plant layout (see Morris and Norback 1980).

In *computational geometry*, the width of a set is defined as the smallest possible distance between two parallel hyperplanes enclosing the set (Houle and Toussaint 1985). If the set is a polyhedron with extreme points $V = \{v_1, \ldots, v_n\}$ determining the width of this set is equivalent to finding a hyperplane minimizing the maximum distance to V. The relation between hyperplane location and transversal theory is mentioned in Sect. 7.3.4.1. In machine learning, a *support vector machine* is a hyperplane (if it exists) separating red from blue data points and maximizing the minimal distance to these points (see Bennet and Mangasarian 1992; Mangasarian 1999; Baldomero-Naranjo et al. 2018). If the set of red and blue data points are not linearly separable, one may look for a hyperplane which minimizes the maximum distance to the data points on the wrong side. This problem can be solved as a restricted hyperplane location problem (see Carrizosa and Plastria 2008; Plastria and Carrizosa 2012).

In *statistics*, classical linear regression asks for a hyperplane which minimizes the residuals of a set of data points, usually the sum of squared vertical distances between the data points and the hyperplane. Orthogonal regression (also called total least squares, see Golub and van Loan 1980) calls for a hyperplane minimizing the sum of squared *Euclidean* distances as residuals.

However, these estimators are usually not considered as robust. This gives a reason for computing L_1-estimators minimizing the sum of absolute vertical (or orthogonal) differences, since the median of a set is considered more robust than its mean. We refer to Narula and Wellington (1982) for a survey on absolute errors regression. More general, many *robust estimators* can be found as optimal solutions to ordered hyperplane location problems, i.e., hyperplane location problems minimizing an ordered median function (see Chap. 10 for the definition of ordered median functions). Such problems are treated in Sect. 7.3.6. An example are *trimmed* estimators which neglect the k largest distances assuming that these belong to outliers, or the least quantiles of squares, introduced in Bertsimas and Shioda (2007). We list some of the most popular estimators and their corresponding hyperplane location problems in Table 7.1. For each of them we specify the distance function d which is used to define the residuals, i.e., which is used to measure the distance from the data points to the hyperplane. The vector $\lambda \in \mathbb{R}^n$ specifies the ordered median function g_λ used for modeling the respective estimator. The meaning of the λ notation is extensively discussed in Nickel and Puerto (2005) or in

Table 7.1 Correspondence between line and hyperplane location problems and robust estimators

Estimator	Distance	Weights of ordered median function
Least squares	$d = d_{ver}^2$	$\lambda = (1, \ldots, 1)$
Total least squares	$d = \ell_2^2$	$\lambda = (1, \ldots, 1)$
Least trimmed squares	$d = d_{ver}^2$	$\lambda = (1, \ldots, 1, 0, \ldots, 0)$
Least absolute deviation	$d = d_{ver}$	$\lambda = (1, \ldots, 1)$
Least trimmed absolute deviation	$d = d_{ver}$	$\lambda = (1, \ldots, 1, 0, \ldots, 0)$
Least median of squares	$d = d_{ver}^2$	$\lambda = (0, \ldots, 0, 1, 0, \ldots, 0)$ (n odd)
		$\lambda = (0, \ldots, 0, 1, 1, 0, \ldots, 0)$ (n even)
Least r-quantile of squares	$d = d_{ver}^2$	$\lambda = (\underbrace{0, \ldots, 0}_{r-1}, 1, \underbrace{0, \ldots, 0}_{n-1})$

Chap. 10 of this book. More applications to classification and regression are pointed out in Bertsimas and Shioda (2007), Blanco et al. (2018).

7.3.2 Ingredients for Analyzing Hyperplane Location Problems

7.3.2.1 Distances Between Points and Hyperplanes

A hyperplane is given by its normal vector $a = (a^1, \ldots, a^D) \in \mathbb{R}^D$ and a real number $b \in \mathbb{R}$:

$$H_{a,b} = \{x \in \mathbb{R}^D : a^t x + b = 0\}.$$

Given a distance $d : \mathbb{R}^D \times \mathbb{R}^D \to \mathbb{R}$, the distance between a point $v \in \mathbb{R}^D$ and a hyperplane $H_{a,b}$ is given as $d(H_{a,b}, v) = \min\{d(x, v) : a^t x + b = 0, x \in \mathbb{R}^D\}$. For the vertical distance (see again the left part of Fig. 7.1) the following formula can easily be computed:

Lemma 7.1 (Schöbel 1999a)

$$d_{ver}(H_{a,b}, v) = \begin{cases} \frac{|a^t v + b|}{a^D} & \text{if } a^D \neq 0 \\ 0 & \text{if } a^D = 0 \text{ and } a^t v + b = 0 \\ \infty & \text{if } a^D = 0 \text{ and } a^t v + b \neq 0 \end{cases}$$

The second and the third case comprise the case of a hyperplane which is vertical itself. Its distance to a point v is infinity unless the hyperplane passes through v. If not all data points lie in one common vertical hyperplane, this means that a vertical hyperplane can never be an optimal solution to the hyperplane location problem. Hence, without loss of generality we can assume the hyperplane $H_{a,b}$ to be non-vertical if the vertical distance is used. We remark that the vertical distance is the most commonly used measure for determining the size of the residuals in regression theory and in statistics.

If d is derived from a norm or a gauge $\gamma : \mathbb{R}^D \to \mathbb{R}$, the following formula for computing $d(H_{a,b}, v)$ has been presented in Plastria and Carrizosa (2001).

Lemma 7.2 (Plastria and Carrizosa 2001)

$$d(H_{a,b}, v) = \begin{cases} \frac{a^t v + b}{\gamma^\circ(a)} & \text{if } a^t v + b \geq 0 \\ \frac{-a^t v - b}{\gamma^\circ(-a)} & \text{if } a^t v + b < 0, \end{cases}$$

where $\gamma^\circ : \mathbb{R}^D \to \mathbb{R}$ *is the dual (polar) norm common in convex analysis (e.g., Rockafellar 1970), i.e.,*

$$\gamma^\circ(v) = \sup\{v^t x : \gamma(x) \leq 1, x \in \mathbb{R}^D\}.$$

Note that $d(H_{a,b}, v) = \frac{|a^t v + b|}{\gamma^\circ(a)}$ if γ is a norm.

7.3.2.2 Dual Interpretation

The following geometric interpretation is helpful when dealing with hyperplane location problems: A non-vertical hyperplane $H_{a,b}$ (with $a^D = 1$) may be interpreted as point $(a^1, \ldots, a^{D-1}, b)$ in $\mathbb{R}^D$. Vice versa, any point $v = (v^1, \ldots, v^D)$ may be interpreted as a hyperplane. Formally, we use the following transformation.

Definition 7.1

Transforming a point to a hyperplane: $T_H(v^1, \ldots, v^D) := H_{v^1,\ldots,v^{D-1},1,v^D}$

Transforming a hyperplane to a point: $T_P(H_{a^1,\ldots,a^{D-1},1,b}) := (a^1, \ldots, a^{D-1}, b)$

It can easily be verified that

$$d_{ver}(H_{a,b}, v) = d_{ver}(T_H(v), T_P(H_{a,b}))$$

for non-vertical hyperplanes with $a^D = 1$. In particular, we obtain the following result.

Lemma 7.3 *Let* H *be a non-vertical hyperplane and* $v \in \mathbb{R}^D$ *be a point. Then*

$$v \in H \iff T_P(H) \in T_H(v).$$

This means that $H_{a,b}$ passes through a point v if and only if $T_H(v)$ passes through $(a^1, \ldots, a^{D-1}, b)$.

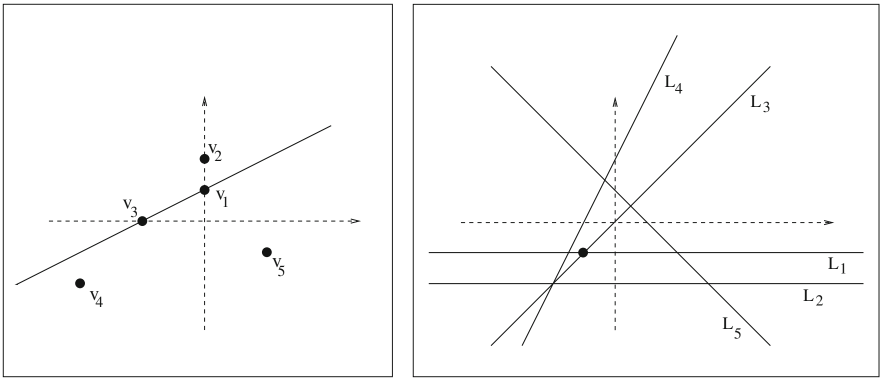

Fig. 7.2 Left: Five data points and a line in primal space. Right: The same situation in dual space corresponds to five lines and one point

In the resulting *dual space* the goal is to locate a point which minimizes the sum of distances to a set of given hyperplanes $\{T_H(v) : v \in V\}$. In the results of the next sections it will become clear that this is a helpful interpretation.

Figure 7.2 shows an example of the dual interpretation in $\mathbb{R}^2$. We consider five data points (depicted in the left part of the figure), namely $v_1 = (0, \frac{1}{2})$, $v_2 = (0, 1)$, $v_3 = (-1, 0)$, $v_4 = (-2, -1)$ and $v_5 = (1, -\frac{1}{2})$. In the dual interpretation the data points are transferred to the five lines in the right part of the figure.

$$L_1 = H_{0,1,\frac{1}{2}} = \{(x^1, x^2) : x^2 = -\frac{1}{2}\}$$
$$L_2 = H_{0,1,1} = \{(x^1, x^2) : x^2 = -1\}$$
$$L_3 = H_{-1,1,0} = \{(x^1, x^2) : x^2 = x^1\}$$
$$L_4 = H_{-2,1,-1} = \{(x^1, x^2) : x^2 = 2x^1 + 1\}$$
$$L_5 = H_{1,1,-\frac{1}{2}} = \{(x^1, x^2) : x^2 = -x^1 + \frac{1}{2}\}$$

It can also be seen that the line $H_{-\frac{1}{2},1,-\frac{1}{2}}$ through the two data points v_1 and v_3 is transformed to the point $v = (-\frac{1}{2}, -\frac{1}{2})$ in dual space which lies on the intersection of L_1 and L_3. Furthermore, note that in the point $(-1, -1)$ in dual space three of the lines meet, namely, L_2, L_3, and L_4. Hence, this point corresponds to the line $H_{-1,1,-1} = \{(x^1, x^2) : x^2 = x^1 + 1, x \in \mathbb{R}^D\}$ which passes through the three data points v_2, v_3, and v_4.

7.3.3 *The Minsum Hyperplane Location Problem*

Let us now start with the *minsum hyperplane location problem* in which we use the sum of all residuals as globalizing function. It is defined as follows: Given a set of data points $V = \{v_1, \dots, v_n\} \subseteq \mathbb{R}^D$ with positive weights $w_j > 0$, $j = 1, \dots, n$, find a hyperplane $H_{a,b}$ which minimizes

$$f_1(H_{a,b}) = \sum_{j=1}^{n} w_j d(H_{a,b}, v_j).$$

A hyperplane H minimizing $f_1(H)$ is called *minsum hyperplane (or median hyperplane) w.r.t. the distance* d. Let us assume throughout this section that there are $n > D$ affinely independent data points, otherwise an optimal solution is the hyperplane containing all of them.

7.3.3.1 Minsum Hyperplane Location with Vertical Distance

We first look at the problem with vertical distance d_{ver}. As explained after Lemma 7.1 we may without loss of generality assume that $a^D = 1$. This simplifies the problem formulation to the question of finding $a^1, \dots, a^{D-1}, b \in \mathbb{R}$ such that

$$f_1(a, b) = \sum_{j=1}^{n} w_j |v_j^t a + b| \tag{7.4}$$

is minimal (with $a^D = 1$). In order to get rid of the absolute values, we define the following index sets

$$\begin{aligned} J_{a,b}^{>} &:= \{j \in \{1, \dots, n\} : v_j^t a + b > 0\} \\ J_{a,b}^{<} &:= \{j \in \{1, \dots, n\} : v_j^t a + b < 0\} \\ J_{a,b}^{=} &:= \{j \in \{1, \dots, n\} : v_j^t a + b = 0\}. \end{aligned} \tag{7.5}$$

We furthermore set

$$W_{a,b}^{>} := \sum_{j \in J_{a,b}^{>}} w_j, \quad W_{a,b}^{=} := \sum_{j \in J_{a,b}^{=}} w_j, \quad W_{a,b}^{<} := \sum_{j \in J_{a,b}^{<}} w_j$$

and let $W := \sum_{j=1}^{n} w_j$ be the sum of all weights. Since $f_1(a, b)$ is piecewise linear in b we receive the following property which says that every minsum hyperplane splits the data points into two sets of almost equal weights.

Theorem 7.1 (Halving Property for Minsum Hyperplanes) *(Schöbel 1999a; Martini and Schöbel 1998) Let $H_{a,b}$ be a minsum hyperplane w.r.t. the vertical distance d_{ver}. Then*

$$W^{>}_{a,b} \leq \frac{W}{2} \text{ and } W^{<}_{a,b} \leq \frac{W}{2} \tag{7.6}$$

Note that the halving property (7.6) is equivalent to

$$W^{>}_{a,b} \leq W^{<}_{a,b} + W^{=}_{a,b} \text{ and } W^{<}_{a,b} \leq W^{>}_{a,b} + W^{=}_{a,b}. \tag{7.7}$$

Looking again at (7.4), note that f_1 is not only piecewise linear in b but is also convex and piecewise linear in the D variables $a^1, \ldots, a^{D-1}, b$. The latter yields the following *incidence property*: There exists an optimal minsum hyperplane which passes through at least D of the data points and these points are affinely independent. Since D affinely independent points uniquely determine a hyperplane, the set of all $\binom{n}{D}$ such hyperplanes contains at least one optimal hyperplane and hence is a finite dominating set.

Theorem 7.2 (FDS for Minsum Hyperplanes with Vertical Distance) *Let d_{ver} be the vertical distance and let $n \geq D$. Then there exists a minsum hyperplane w.r.t. d_{ver} that passes through D affinely independent data points.*

Proof (Sketch of Proof) We can rewrite the objective function $f_1(H_{a,b})$ to

$$f_1(H_{a,b}) = \sum_{j \in J^{>}_{a,b}} w_j(v_j^t a + b) + \sum_{j : \in J^{<}_{a,b}} w_j(-v_j^t a - b) \tag{7.8}$$

which is easily seen to be linear as long as the signs of $v_j^t a + b$ do not change, i.e., on any polyhedral *cell* given by disjoint sets $J^{\geq}, J^{\leq} \subseteq \{1, \ldots, n\}$ specifying which data points should be below (or on) and above (or on) the hyperplane:

$$R(J^{\geq}, J^{\leq}) := \Big\{(a^1, \ldots, a^{D-1}, b) \,:\, v_j^t a + b \geq 0 \text{ for all } j \in J^{\geq}$$
$$v_j^t a + b \leq 0 \text{ for all } j \in J^{\leq}\Big\}.$$

Note that these polyhedra can be constructed in dual space by using the arrangement of hyperplanes $T_H(v_j)$, $j = 1, \ldots, n$, i.e., the right hand side of Fig. 7.2 shows exactly the polyhedra in dual space on which the objective function is linear. The fundamental theorem of linear programming then yields an optimal solution at a vertex of some of the cells $R(J^{\geq}, J^{\leq})$, i.e., a hyperplane satisfying $v_j^t a + b = 0$ for at least D indices from $\{1, \ldots, n\}$.

Note that many papers mention this result. For $D = 2$, it was shown in Wesolowsky (1972), Morris and Norback (1983), Megiddo and Tamir (1983) and generalized to higher dimensions, e.g., in Schöbel (1999a).

In our example of Fig. 7.2 the depicted line is an optimal solution.

7.3.3.2 Minsum Hyperplane Location with Norm Distance

We now turn our attention to the location of hyperplanes with respect to a norm $\|\cdot\|$, i.e., the residuals are given as $d(v, H) = \min\{\|v - x\| : x \in H\}$. We can use Lemma 7.2 for computing the residuals and obtain the following objective function

$$f_1(H_{a,b}) = \sum_{j=1}^{n} w_j \frac{|v^t a + b|}{\|a\|^\circ} \tag{7.9}$$

where $\|\cdot\|^\circ$ denotes the dual norm of $\|\cdot\|$. Still, the objective function is piecewise linear in b, hence the halving property holds again:

Theorem 7.3 (Halving Property for Minsum Hyperplanes) *(Schöbel 1999a; Martini and Schöbel 1998) Let d be a norm distance and $H_{a,b}$ be a minsum hyperplane w.r.t. the distance d. Then*

$$W^+_{a,b} \leq \frac{W}{2} \text{ and } W^-_{a,b} \leq \frac{W}{2}$$

Also the incidence property of Theorem 7.2 still holds.

Theorem 7.4 (FDS for Minsum Hyperplanes) *(Schöbel 1999a; Martini and Schöbel 1998, 1999) Let d be a norm distance derived from norm $\|\cdot\|$ and let $n \geq D$. Then there exists a minsum hyperplane w.r.t. the distance d that passes through D affinely independent data points. If and only if $\|\cdot\|$ is a smooth norm, we have that* all *minsum hyperplanes pass through D affinely independent data points.*

Proof (Sketch of Proof) Different proofs for this property exist. Here, we use the cell structure of the proof of Theorem 7.2 for the vertical distance. The idea is to use piecewise quasiconcavity instead of piecewise linearity on these cells. Neglecting vertical hyperplanes, we again look at the regions $R(J^{\leq}, J^{\geq})$ in dual space. On any such region we obtain that the objective function (7.9) can be rewritten as

$$\begin{aligned} f_1(H_{a,b}) &= \sum_{j \in J^{>}_{a,b}} w_j \frac{v_j^t a + b}{\|a\|^\circ} + \sum_{j : \in J^{<}_{a,b}} w_j \frac{-v_j^t a - b}{\|a\|^\circ} \\ &= \frac{1}{\|a\|^\circ} \left(\sum_{j \in J^{>}_{a,b}} w_j (v_j^t a + b) + \sum_{j : \in J^{<}_{a,b}} w_j (-v_j^t a - b) \right), \end{aligned}$$

i.e., it is a positive linear function divided by a positive convex function and hence is quasiconcave. Consequently, it takes its minimum at a vertex of a region $R(J^{\leq}, J^{\geq})$, i.e., again at a hyperplane passing through D affinely independent data points.

Note that this theorem has been known for a long time for line location problems ($D = 2$) in the case of rectangular or Euclidean distances (Wesolowsky 1972, 1975; Morris and Norback 1980, 1983; Megiddo and Tamir 1983), and has been generalized to line location problems with arbitrary norms in Schöbel (1998, 1999a) and to D-dimensional hyperplane location problems with Euclidean distance in Korneenko and Martini (1990, 1993). The extension to hyperplanes with arbitrary norms is due to Schöbel (1999a) and Martini and Schöbel (1998).

7.3.3.3 Minsum Hyperplane Location with Gauge Distance

In general, the results of Theorems 7.4 and 7.3 do not hold for gauges. There exist counterexamples showing that optimal hyperplanes need not be halving, see, e.g. Schöbel (1999a). However, redefining the halving property by taking into account the non-symmetry on both sides of a hyperplane, the following similar result (based on formulation (7.7)) may be transferred to gauge distances.

Theorem 7.5 (Halving Property for Minsum Hyperplanes with Gauges) *(Plastria and Carrizosa 2001) Let d be a gauge distance and $H(a, b)$ be a minsum hyperplane w.r.t. the distance d. Then we have*

$$\sum_{j \in H^{<}_{a,b}} \frac{w_j}{\gamma^\circ(a)} \leq \sum_{j \in H^{>}_{a,b} \cup H^{=}_{a,b}} \frac{w_j}{\gamma^\circ(a)}$$

$$\sum_{j \in H^{>}_{a,b}} \frac{w_j}{\gamma^\circ(-a)} \leq \sum_{j \in H^{<}_{a,b} \cup H^{=}_{a,b}} \frac{w_j}{\gamma^\circ(-a)}.$$

For gauge distances it does also not hold that there always exists an optimal minsum hyperplane passing through D of the data points, for a counterexample see again (Schöbel 1999a). However, the following weaker result holds.

Theorem 7.6 (Incidence Property for Minsum Hyperplanes) *(Plastria and Carrizosa 2001) Let d be a gauge distance and let $n \geq D$. Then there exists a minsum hyperplane w.r.t. the distance d that passes through $D-1$ affinely independent data points.*

Note that this incidence property does not define an FDS.

7.3.4 The Minmax Hyperplane Location Problem

We now turn our attention to the *minmax hyperplane location problem* in which use the maximum of the residuals as globalizing function. That is, we look for a hyperplane $H_{a,b}$ which minimizes

$$f_{\max}(H_{a,b}) = \max_{j=1,\ldots,n} w_j d(H_{a,b}, v_j).$$

A hyperplane H minimizing $f_{\max}(H)$ is called *minmax hyperplane (or center hyperplane)* w.r.t. the distance d. Again, let us assume $n > D$. Since the main results for the location of minmax hyperplanes are similar for different types of distance functions, we do not distinguish between vertical, norm- and gauge distances here.

Minmax point location problems often rely on Helly's theorem (Helly 1923). For the location of hyperplanes, this result can only be applied for the vertical distance, since the sets $\{(a, b) : d(H_{a,b}, v) \le \alpha\}$ are non-convex in general if $d \neq d_{ver}$. Instead, relations to transversal theory may be exploited. We hence start with a link to computational geometry.

7.3.4.1 Relation to Transversal Theory

Definition 7.2 Given a family of sets $\mathscr{M}$ in $\mathbb{R}^D$, a hyperplane H is called a *hyperplane transversal with respect to* $\mathscr{M}$ if $M \cap H \neq \emptyset$ for all $M \in \mathscr{M}$.

Using this definition it is directly clear that $f_{\max}(H) \le r$ if and only if H is a hyperplane transversal for the set $\mathscr{M} = \{M_j(r), j = 1, \ldots, n\}$ with

$$M_j(r) = \{x \in \mathbb{R}^D : w_j d(x, v_j) \le r\}.$$

Instead of looking for a hyperplane minimizing the maximum distance to a set of data points, we can hence equivalently look for the smallest possible $r \ge 0$ such that a hyperplane transversal for the sets $M_j(r)$, $j = 1, \ldots, n$ exists. As an example, in Fig. 7.3 we search a line minimizing the maximum rectangular distance to the five given data points, each of them with unit weight. Since it is a line transversal for the five sets $M_j(r)$, the depicted line l satisfies $f_{\max}(l) \le r$.

7.3.4.2 The Finite Dominating Set Property

The main result for minmax hyperplane location is the following *blockedness property*.

Theorem 7.7 (FDS for Minmax Hyperplanes) *(Schöbel 1999a; Martini and Schöbel 1998, 1999; Plastria and Carrizosa 2012) Let d be derived from a norm or a gauge and let $n \ge D + 1$. Then there exists a minmax hyperplane w.r.t. the*

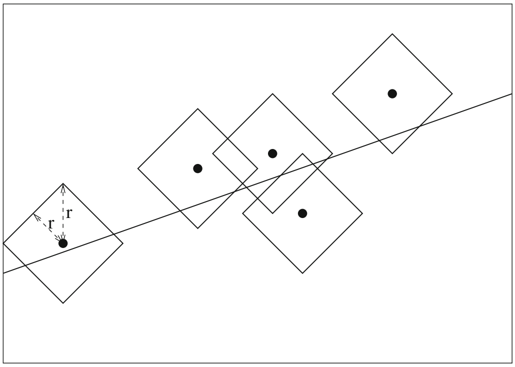

Fig. 7.3 A line transversal l to the five sets (each of them with radius r) exists, hence the objective function value of this line satisfies $f_{\max}(l) \leq r$

distance d that is at the same (maximum) distance from $D+1$ affinely independent data points. If and only if the norm or the gauge is smooth, we have that all *minmax hyperplanes are at maximum distance from $D+1$ affinely independent data points.*

Proof (Sketch of Proof for Norms) Similar to the proof for median hyperplanes we look at the case for vertical distances first. Here, the objective function is linear as long as the maximum distance does not change (if $n > 1$). We hence may use a type of farthest Voronoi diagram in the dual space, i.e., a partition of the dual space into (not necessarily connected) polyhedral cells

$$\begin{aligned} C(v_j) := & \quad \{(a,b) : d(H_{a,b}, v_j) \geq d(H_{a,b}, v) \text{ for all } v \in V\} \\ & = \{(a^1, \ldots, a^{D-1}, b) : |v_j^t a + b| \geq |v_i^t a + b| \text{ for all } i = 1, \ldots, n\} \end{aligned}$$

and it can be shown that an extreme point of such a cell is an optimal solution for the case of the vertical distance. Note that the cell structure does not change when we replace the vertical distance by a distance d derived from a norm, since we have

$$\begin{aligned} C'(v_j) := & \quad \{(a,b) : d(H_{a,b}, v_j) \geq d(H_{a,b}, v) \text{ for all } v \in V\} \\ & = \{(a^1, \ldots, a^{D-1}, b) : \frac{|v_j^t a + b|}{\gamma^\circ(a)} \geq \frac{|v_i^t a + b|}{\gamma^\circ(a)} \text{ for all } i = 1, \ldots, n\} \\ & = C(v_j), \end{aligned}$$

and using again that the objective function on these cells is quasiconcave, the result follows.

Note that in contrast to minsum hyperplane location problems, this result also holds for gauges. This was shown for $D = 2$ in Schöbel (1999a) and for arbitrary finite dimensions D in Plastria and Carrizosa (2012). Using transversal theory, it can furthermore be extended to metrics (under some mild conditions of monotonicity), see Schöbel (1999a) for the case of $D = 2$.

A geometric point of view is taken in Nievergelt (2002) for the Euclidean case. He interprets the minmax hyperplane location problem as follows: locate two parallel hyperplanes such that the set of data points lies completely between these two hyperplanes and minimize the distance between these parallel hyperplanes. He shows that in an optimal solution the two hyperplanes are *rigidly supported* by the data points in V, i.e., there does not exist any other pair of parallel hyperplanes enclosing all data points and passing through the same data points of V. This property coincides with the blockedness property of Theorem 7.7. The algorithm proposed in Nievergelt (2002) uses projective shifts to improve a solution in a finite number of steps.

7.3.5 *Algorithms for Minsum and Minmax Hyperplane Location*

We describe the main approaches used for computing minsum hyperplanes.

7.3.5.1 Enumeration

Theorems 7.2, 7.4, and 7.7 specify a finite dominating set for both the minsum and the minmax hyperplane location problem. The trivial approach is to enumerate all candidates in the FDS. For the minsum case these are just the hyperplanes passing through D of the data points. More effort is necessary to determine the hyperplanes being at maximum distance from $D+1$ of the data points for the minmax case. For $D = 2$ and norm distances these are parallel to one edge of the convex hull of the data points (Schöbel 1999a).

7.3.5.2 Linear Programming for Hyperplane Location with Vertical and Block Norm Distance

For the vertical distance d_{ver} the hyperplane location problem can be formulated as a linear program. To this end, we define additional variables $d_j \geq 0$ which contain the distances $d(H, v_j)$, $j = 1, \ldots, n$. For the minsum problem we then obtain

$$\text{minimize} \quad \sum_{j=1}^{n} w_j d_j \tag{7.10}$$

$$\text{subject to} \quad d_j \geq v_j^T a + b \text{ for } j = 1, \ldots, n \tag{7.11}$$

$$d_j \geq -v_j^T a - b \text{ for } j = 1, \ldots, n \tag{7.12}$$

$$d_j \geq 0 \text{ for } j = 1, \ldots, n \tag{7.13}$$

$$a^D = 1 \tag{7.14}$$

$$b, a^i \in \mathbb{R} \text{ for } i = 1, \ldots, D-1. \tag{7.15}$$

For the minmax problem, the objective (7.10) has to be replaced by the minmax function $f_{\max}$, i.e., by

$$\text{minimize} \max_{j=1,\ldots,n} w_j d_j,$$

which can be rewritten as linear program by using a bottleneck variable z and then replacing the objective by minimize z and adding $w_j d_j \leq z$ for $j = 1, \ldots, n$ as constraints. It is also possible to use other types of globalizing functions. For the minsum problem (see Zemel 1984) and for the minmax problem (see Megiddo 1984), the above LP formulation can be solved in O(n) time.

Now consider a block norm γ_B with unit ball $B = conv\{e_1, \ldots, e_G\}$, i.e., e_g, $g = 1, \ldots, G$ are the *fundamental directions* of the block norm. The idea is to solve the problem for each of the fundamental directions separately. To this end, we extend the vertical distance d_{ver} to a distance d_t, $t \in \mathbb{R}^D$ as follows.

$$d_t(u, v) := \begin{cases} |\alpha| & \text{if } u - v = \alpha t \text{ for some } \alpha \in \mathbb{R} \\ \infty & \text{otherwise.} \end{cases}$$

We then know the following result.

Lemma 7.4 (Schöbel 1999a) *Let H be a hyperplane and let d be derived from a block norm γ_B with fundamental directions $e_1, \ldots, e_G$. Then for any point $v \in \mathbb{R}^D$ there exists $\bar{g} \in \{1, \ldots, G\}$ such that*

$$d(H, v) = d_{e_{\bar{g}}}(H, v) = \min_{g=1,\ldots,G} d_{e_g}(H, v),$$

i.e., the fundamental direction $e_{\bar{g}}$ is independent of the point v.

This result allows to solve the problem with block norm distance in O(Gn) time in the planar case by iteratively solving the minmax hyperplane location problem with respect to distance d_{e_g}, $g = 1, \ldots, G$, and taking the best solution. Note that the G problems may be solved by transformation to the vertical distance as follows: Choose a linear (invertible) transformation T with $T(e_g) = (0, 0, \ldots, 0, 1)$. Transform all data points $v'_j = T(v_j)$, $j = 1, \ldots, n$. We obtain that

$$d_{ver}(T(H), T(v)) = d_{e_g}(H, v)$$

for any hyperplane H and any point $v \in \mathbb{R}^D$, i.e., we have transformed the problem with distance d_{e_g} to a problem with vertical distance which can be solved by linear programming (in linear time) as above. Transforming an optimal hyperplane H' for the resulting problem back to $T^{-1}(H')$ gives an optimal solution to the problem with distance d_{e_g}. Details can be found in Schöbel (1999a, 1996).

The problem of locating a hyperplane with respect to a block norm distance can also be formulated as one large integer linear program (instead of the mentioned G linear programs) as done in Blanco et al. (2018).

7.3.5.3 Enhancing the Enumeration for Line Location with Euclidean Distance

For the Euclidean distance, the minsum straight line problem has received a lot of attention. Many of the ideas to be described here could be used for other distance functions (see Schieweck and Schöbel 2012); nevertheless they have been investigated mainly for the Euclidean case. Algorithms rely on Theorems 7.3 and 7.4 and use the representation of the problem in the dual space.

The Euclidean minsum straight line problem with unit weights can be solved by sweeping along the so called *median trajectory* in the dual space (see Yamamoto et al. 1988). The median trajectory is the point-wise median of the lines $T_H(v_j)$, $j = 1, \ldots, n$, see Fig. 7.4 for the median trajectory in our example. The breakpoints on the median trajectory coincide with lines passing through two of the data points and satisfying the halving property. Hence, the complexity of the approach depends on the number $h(n)$ of halving lines. In Yamamoto et al. (1988) the complexity of the approach is given as $O(\log^2(n)h(n))$ which can be improved to $O(\log(n)h(n))$ (see Schieweck and Schöbel 2012) by substituting the algorithm for dynamic convex hulls of Overmars and van Leeuwen (1981) by the newer $O(\log(n))$ algorithm of Brodal and Jacob (2002).

Note that the order of $h(n)$ is not known yet. It has been shown that the number of halving lines is in $O(n^{4/3})$ (see Dey 1998) yielding an $O(n^{4/3}\log(n))$ approach

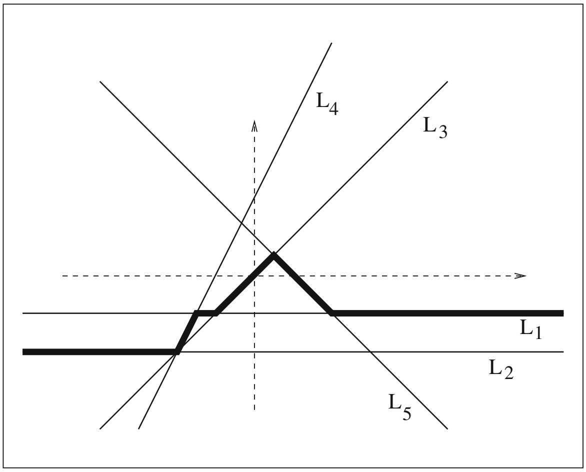

Fig. 7.4 The median trajectory for the example of Fig. 7.2

for the line location problem with Euclidean distance. The best known lower bound for the Euclidean minsum line location problem is $\Omega(n \log n)$ using reduction from the uniform-gap on a circle problem (Yamamoto et al. 1988). That is, the order of $h(n)$ is at least $O(n \log n)$. The question for an optimal algorithm for this problem is still open.

The Euclidean line location problem with arbitrary weights can be solved in $O(n^2)$, see Lee and Ching (1985).

For the Euclidean minmax line location problem the relation to transversal theory is exploited leading to an optimal $O(n \log n)$ algorithm for the case with arbitrary weights (Edelsbrunner 1985).

7.3.6 Ordered Median Line and Hyperplane Location Problem

A rather general globalizing function in location theory is the ordered median function (see Nickel and Puerto 2005, or Chap. 10). For tackling ordered median line location problems, one can combine the ideas of the preceding results on minsum and minmax location.

Theorem 7.8 (FDS for Ordered Line Location) *(see Lozano and Plastria 2009 for the planar Euclidean case) Let d be a norm distance and let $n \geq 2$. Then there exists a solution l^* to the ordered line location problem w.r.t. the distance d that satisfies at least one of the following conditions:*

- *l^* passes through two of the data points.*
- *l^* passes through one of the data points and is at same weighted distance from two of the data points.*
- *l^* is at the same weighted distance from three of the data points.*
- *There exist two pairs of data points $v_j, v_{j'} \in V$ and $v_k, v_{k'} \in V$ such that*

$$w_j d(l^*, v_j) = w_{j'} d(l^*, v_{j'}) \text{ and } w_k d(l^*, v_k) = w_{k'} d(l^*, v_{k'}),$$

  i.e., l^* is at the same weighted distance from both data points of each of the two pairs.

Proof (Sketch of Proof) The theorem has been shown in Lozano and Plastria (2009) for the ordered Euclidean line location problem, but also holds for all norm distances: Again, we look at the regions in dual space in which the order of the distances from the line to the data points does not change, i.e., in which

$$d(H_{a,b}, v_j) = d(H_{a,b}, v_i)$$

does not hold for any $j \neq i$. These regions are hence bounded by the affine linear sets

$$\left\{(a,b) : \frac{w_j|a^t v_j + b|}{\gamma^\circ(a)} = \frac{w_i|a^t v_i + b|}{\gamma^\circ(a)}\right\} = \{(a,b) : w_j|a^t v_j + b| = w_i|a^t v_i + b|\}$$

in dual space and may be interpreted as the weighted bisectors of the lines $T_H(v_j)$ and $T_H(v_i)$. Taking the intersection of these regions with the regions $R(J^{\geq}, J^{\leq})$ of the proof of Theorem 7.4, we obtain quasiconcavity on the resulting (smaller) cells. This yields that the data points of these new cells are a finite dominating set.

This FDS allows an algorithm to solve the ordered line location problem in $O(n^4)$, see Lozano and Plastria (2009) for the Euclidean case. The problem of locating a hyperplane minimizing the Euclidean ordered median function has been investigated in Kapelushnik (2008) where its equivalence to searching within the levels of an arrangement is shown. The resulting algorithm runs in $O(n^{2D})$ where its complexity is reduced to $O(n^{D+\min\{D-1,K+1\}})$ if $K = |\{j = 1, \ldots, n : \lambda_j \neq 0\}|$.

Recently, a formulation with second-order cone constraints for ordered hyperplane location problems with arbitrary norm distances has been developed in Blanco et al. (2018). In the same paper, the authors also propose a formulation as mixed-integer linear program for the special case of ordered median hyperplane location problems with block norm distances.

A special case concerns the k-centrum line location problem, in which the sum of distances from the line to the k most distant data points is minimized. It is also an ordered median problem and has been treated in Lozano et al. (2010). The methodology is similar to the approach of the general ordered median problem and exploits quasiconcavity of the objective function in the cells mentioned above. For smooth norms, it is shown that the resulting finite dominating set consists of lines either passing through two data points or being at equal weighted distance from three of them. Based on this, an $O((k + \log n)n^3)$ algorithm is proposed for computing all t-centrum lines for $1 \leq t \leq k$. For unweighted data points (Kapelushnik 2008), suggests an algorithm that finds a k-centrum line in the plane in time $O(n \log n + nk)$.

7.3.7 Some Extensions of Line and Hyperplane Location Problems

7.3.7.1 Obnoxious Line and Hyperplane Location

Instead of *minimizing* the distances to the data points, one may also consider an obnoxious problem in which the new facility should be as far away from the data points as possible. A rather general approach for obnoxious line location is presented in Lozano et al. (2015) in which a weighted ordered median function is maximized. More precisely, the problem treated is the following: Given a connected polygonal set S in the plane, the goal is to find a line which intersects S and maximizes the sum of ordered weighted Euclidean distances to the data points. For such problems, the authors are again able to derive a finite dominating set which yields an $O(n^4)$ algorithm for the general Euclidean anti-ordered median case, and an $O(n^2)$ algorithm for the case of of the Euclidean anti-median line. The case of locating an obnoxious plane (i.e., finding the widest empty slab through a set of data points V) has been considered in Díaz-Bánez et al. (2006a). Also here, a finite dominating set could be identified leading to an algorithm in time $O(n^3)$.

7.3.7.2 Locating *p* Lines or Hyperplanes

As in point facility location it is also possible to study the problem of locating p lines or hyperplanes $H_1, \ldots, H_P$. In this setting, every data point is served by its closest line. We may minimize the sum of distances

$$f_1(H_1, \ldots, H_p) = \sum_{j=1}^{n} w_j \min_{q=1,\ldots,p} d(H_q, v_j) \tag{7.16}$$

or the maximum distance

$$f_{\max}(H_1, \ldots, H_p) = \max_{j=1,\ldots,n} w_j \min_{q=1,\ldots,p} d(H_q, v_j) \tag{7.17}$$

from the data points to their closest hyperplanes, or we may use any other globalizing function. Minimizing the sum of distances is called *p-minsum-hyperplane location problem* and minimizing the maximum distance to a set of p hyperplanes is called *p-minmax-hyperplane location problem*. Locating p hyperplanes has important applications in statistics with latent classes, and also provides an alternative approach for clustering, called *projective clustering* (see, e.g., Har-Peled and Varadarajan 2002; Deshpande et al. 2006).

Both problems are known to be NP-hard for most reasonable distance measures (see Megiddo and Tamir 1982). However, since each of the p hyperplanes $H_1, \ldots, H_p$ to be located is a minsum (or minmax) hyperplane for the set of data points

$$V_q = \{v \in \{v_1, \ldots, v_n\} : d(H_q, v) \leq d(H_{q'}, v) \text{ for all } q' = 1, \ldots, p\}$$

the results on the finite dominating sets of Theorems 7.4 and 7.7 still hold:

Theorem 7.9 *Given $p \in \mathbb{N}$ and a set of data points V. Let d be the vertical distance or a norm distance.*

- *If $n \geq D$ then there exists an optimal solution to the p-minsum-hyperplane location problem in which each hyperplane passes through D data points.*

- *If $n \geq D + 1$ then there exists an optimal solution to the p-minmax-hyperplane location problem in which each of hyperplane is at maximum distance from $D+1$ data points.*

Hence, enumeration approaches based on such an FDS are possible, however, the number of candidates to be enumerated is of order $O(n^D)$.

Based on the FDS, another approach is possible: The problem may be transformed to a p-median or p-center problem on a bipartite graph with $O(|FDS|)$ nodes. The two node sets of the graph are given by the data points V and by the potential hyperplanes in the FDS. Every node v from V is connected to every node H from the FDS where the edge (v, H) is weighted by the distance, the node v has from the hyperplane H. The goal is to serve all customers in V by installing p new locations in the FDS.

Another possible approach is to use blockwise coordinate descent similar to the idea of Cooper's algorithm (Cooper 1964) and proceed iteratively: Start with a random set of p hyperplanes, determine the sets V_q for all $q = 1, \ldots, p$, re-optimize within these sets and repeat. The procedure converges to a local optimum. For a more detailed analysis of the convergence properties we refer to Jäger and Schöbel (2018).

Finally, the problem of finding p lines in the plane is studied as classification problem in Bertsimas and Shioda (2007) where it is formulated as an integer program. Binary variables $x_{j,q}$ determine to which of the $q = 1, \ldots, p$ lines the data point v_j is assigned. Applying their basic formulation to the linear program (7.10)–(7.15) of Sect. 7.3.5 gives

$$\begin{aligned}
\text{minimize} \quad & \sum_{j=1}^{n} w_j d_j \\
\text{subject to} \quad & d_j \geq v_j^T a_q + b_q - M(1 - x_{j,q}) \ \text{ for } j = 1, \ldots, n, \ q = 1, \ldots, p \\
& d_j \geq -v_j^T a_q - b_q - M(1 - x_{j,q}) \ \text{ for } j = 1, \ldots, n, \ q = 1, \ldots, p \\
& \sum_{q=1}^{p} x_{j,q} = 1 \ \text{ for } j = 1, \ldots, n \\
& x_{j,q} \in \{0, 1\} \ \text{ for } j = 1, \ldots, n, \ q = 1, \ldots, p \\
& d_j \geq 0 \ \text{ for } j = 1, \ldots, n \\
& a_q^D = 1 \ \text{ for } q = 1, \ldots, p \\
& b_q, a_q^i \in \mathbb{R} \text{ for } i = 1, \ldots, D-1, \ q = 1, \ldots, p.
\end{aligned}$$

Solving the integer program in its basic form is not possible in reasonable time; in Bertsimas and Shioda (2007) clustering algorithms are performed in a preprocessing step. The above integer program can also be used for solving the minmax version of the problem, if $\sum$ is replaced by max as globalizing function in its objective.

7.3.7.3 Restricted Line Location

Line location problems in which the line is not allowed to pass through a specified set $R \subseteq \mathbb{R}^2$ can be tackled by looking at the dual space and transforming the restriction to a forbidden set there. Since the problem is convex for vertical distances, techniques from location theory can be used, e.g., the boundary theorem saying that there exists a solution on the boundary of the restricted set whenever the restriction is not redundant (see Hamacher and Nickel 1995). Results of this type have been generalized to block norms and to arbitrary norms, see Schöbel (1999b).

In some statistical applications it is preferable to restrict the slope of the line (or the norm of a) as done in types of RLAD approaches (Wang et al. 2006). Such restrictions on the parameters of the hyperplane can again be treated and solved in dual space, see Krempasky (2012).

Another type of restriction is to force a subset of data points of V to lie on, above or below the hyperplane. Also for such problems, finite dominating sets have been derived, see Schöbel (2003) for hyperplane location problems in which the hyperplane is forced to pass through a subset of data points. Plastria and Carrizosa (2012) consider the more general case of requiring a specified subset of data points below or above the hyperplane with applications in support vector machines.

7.3.7.4 Line Location in $\mathbb{R}^D$

Locating a line in $\mathbb{R}^D$ turns out to be a difficult problem since all of the structure of line and hyperplane location problems gets lost. In Brimberg et al. (2002, 2003) some special cases are investigated for the case $D = 3$, such as locating a vertical line, or locating a line where the distance measure is given as the lengths of horizontal paths. If these lengths are measured with the rectangular distance, the problem can be reduced to two planar line location problems with vertical distance. For the general case of locating a minsum line in $\mathbb{R}^3$, global optimization methods such as Big-Cube-Small-Cube (Schöbel and Scholz 2010) have been successfully used, see Blanquero et al. (2011). The case of locating a minmax line in $\mathbb{R}^D$ is known in computational geometry as smallest enclosing cylinder problem. It has been mainly researched in $\mathbb{R}^3$ (Schömer et al. 2000; Chan 2000).

7.4 Locating Circles and Spheres

We now turn our attention to the location of hyperspheres. Again, we have given a set of data points $V \subseteq \mathbb{R}^D$ with positive weights $w_j > 0, j = 1, \ldots, n$. The *hypersphere location problem* is to find the center point and the radius of a hypersphere S which minimizes the distances to the data points in V. The most common hypersphere is the surface of the Euclidean unit ball (i.e., a classical circle in two dimensions), but the problem is also interesting for more general hyperspheres derived from unit balls of other norms. In this section we consider such hypersphere location problems for different types of norms and different globalizing functions.

Note that circle location deals with finding a circle in $\mathbb{R}^2$ minimizing the distances from its circumference to a set of data points in the plane. For circle location, more and stronger results are known than for general hypersphere location; it will hence be treated separately where appropriate.

7.4.1 Applications

Hyperspheres and circles are mathematical objects which are well-known for hundreds of years. The Rhind Mathematical Papyrus, written around 1650 BC by Egyptian mathematicians, already contains a method for approximating a circle, see Robins and Shute (1987). The problem of fitting a circle or a sphere to a set of data points has also been mentioned in the fourth century BC by notes of Aristotle on the earth's sphericity, see Dicks (1985).

Also nowadays, the location of circles and spheres has applications in different fields. The Euclidean version of the problem is of major interest in measurement science, where it is used as a model for the out-of-roundness problem which occurs in quality control and consists of deciding whether or not the roundness of a manufactured part is in the normal range (see, e.g., Farago and Curtis 1994; Ventura and Yeralan 1989; Yeralan and Ventura 1988). To this end, measurements are taken along the boundary of the manufactured part. In order to evaluate the roundness of the part, a circle is searched which fits the measurements. Mathematical models for different variants of the out-of-roundness problem are studied for instance in Le and Lee (1991), Swanson et al. (1995), Sun (2009).

Circle and hypersphere location problems have also applications in other disciplines, e.g., in particle physics (Moura and Kitney 1992; Crawford 1983) when fitting a circular trajectory to a large number of electrically charged particles within uniform magnetic fields, or in archaeology where minmax circles are used to estimate the diameter of an ancient shard (Chernov and Sapirstein 2008). In Suzuki (2005), the construction of ring roads is mentioned as an application. Many further applications are collected in Nievergelt (2010). They include

- the analysis of the design and layout of structures in archaeology,
- the analysis of megalithic monuments in history,
- the identification of the shape of planetary surfaces in astronomy,
- computer graphics and vision,
- calibration of microwave devices in electrical engineering,
- measurement of the efficiency of turbines in mechanical engineering,
- monitoring of deformations in structural engineering, or
- the identification of particles in accelerators in particle physics.

There is also a relation to equity problems (see Gluchshenko 2008; Drezner and Drezner 2007) of point facility location and to a problem in computational geometry which is to find an annulus of smallest width. These relations are specified in Sect. 7.4.4.1.

In statistics, the problem is also of interest. As Nievergelt (2002) points out, many attempts have been made of transferring total least squares algorithms from hyperplane location problems to hypersphere location problems (e.g., Kasa 1976; Moura and Kitney 1992; Crawford 1983; Rorres and Romano 1997; Späth 1997, 1998; Coope 1993; Gander et al. 1994; Nievergelt 2004).

7.4.2 Distances Between Points and Hyperspheres

Let d be a distance derived from some norm $\|\cdot\|$, i.e., $d(x, y) = \|y - x\|$. A *hypersphere of the norm* $\|\cdot\|$ is given by its center point $x = (x^1, \ldots, x^D) \in \mathbb{R}^D$ and its radius $r > 0$:

$$S_{x,r} = \{y \in \mathbb{R}^D : d(x, y) = r\}.$$

The distance between a sphere $S = S_{x,r}$ and a point $v \in \mathbb{R}^D$ is defined as the distance from v to its closest point on S, i.e.,

$$d(S, v) = \min_{y \in S} d(y, v)$$

and can be computed as

$$d(S_{x,r}, v) = |d(x, v) - r|.$$

The following properties of the distance can easily be shown.

Lemma 7.5 (Körner et al. 2012; Körner 2011) *Given a distance d derived from a norm, and a point $v \in \mathbb{R}^D$, the following hold:*

- *$d(S_{x,r}, v)$ is convex and piecewise linear in r,*
- *$d(S_{x,r}, v)$ is locally convex in (x, r) if v is a point outside the sphere, and*
- *$d(S_{x,r}, v)$ is concave in (x, r) if v is inside the sphere.*

Before analyzing minsum or minmax circles or hyperspheres, let us remark that even the special case with only $n = 3$ data points in the plane ($D = 2$) is a surprisingly interesting problem. Within a wider context it has been studied in Alonso et al. (2012a,b). Here, the circumcircle of a set of three data points is investigated (which is the optimal minmax or minsum circle for the three data points). Dependent on the norm considered, such a circumcircle need not exist, and need not be unique. Among other results on covering problems, the work focuses on a complete description of possible locations of the center points of such circumcircles.

7.4.3 *The Minsum Hypersphere Location Problem*

We start with the minsum hypersphere location problem, i.e., we use the sum of all residuals between the data points and the hypersphere as globalizing function. Given a distance d derived from norm $\|\cdot\|$, the goal hence is to find a hypersphere $S = S_{x,r}$ of norm $\|\cdot\|$ which minimizes

$$f_1(S_{x,r}) = \sum_{j=1}^{n} w_j d(S_{x,r}, v_j) = \sum_{j=1}^{n} w_j |d(x, v_j) - r|. \tag{7.18}$$

For the Euclidean case in the plane, (7.18) reduces to the location of a circle in the plane. It has been defined and treated in Drezner et al. (2002). This has then been generalized to the location of a (norm-)circle in the plane in Brimberg et al. (2009b), and later to the location of a hypersphere of some norm in $\mathbb{R}^D$ (Körner et al. 2012). The Euclidean case in dimension d has been also extensively analyzed in Nievergelt (2010).

We start by presenting some general properties of minsum hypersphere location problems. In contrast to hyperplanes, it is not obvious in which cases a minsum hypersphere exists, since a hypersphere can degenerate to a point (for $r = 0$) and to a hyperplane (for $r \to \infty$). The following results are known.

Lemma 7.6 (Brimberg et al. 2011a; Körner et al. 2012) *Consider the hyperplane location problem* (7.18) *with respect to a norm. Then the following hold.*

- *No hypersphere with $r = 0$ can be a minsum hypersphere.*
- *For any smooth norm there exist instances for which no minsum hypersphere exists.*
- *For any elliptic norm and any block norm a minsum hypersphere exists for all instances with $n \geq D + 1$.*

Since no optimal solution degenerates to a point, we need not bother with existence results if we restrict r to an upper bound and solve the problem then.

Let us now discuss the halving property. Similar to the index sets (7.5) used for hyperplane location, we define index sets to distinguish data points outside, on, and inside the hypersphere

$$J^{>}_{x,r} := \{j \in \{1, \ldots, n\} : d(x, v_j) > r\}$$
$$J^{<}_{x,r} := \{j \in \{1, \ldots, n\} : d(x, v_j) < r\}$$
$$J^{=}_{x,r} := \{j \in \{1, \ldots, n\} : d(x, v_j) = r\}$$

and let

$$W^{>}_{x,r} := \sum_{j \in J^{>}_{x,r}} w_j, \quad W^{=}_{x,r} := \sum_{j \in J^{=}_{x,r}} w_j, \quad W^{<}_{x,r} := \sum_{j \in J^{<}_{x,r}} w_j.$$

As before, let $W = \sum_{j=1}^{n} w_j$ be the sum of all weights.

Theorem 7.10 (Halving Property for Minsum Hyperspheres) *(Brimberg et al. 2011a; Körner et al. 2012) Let $S_{x,r}$ be a minsum hypersphere w.r.t. a norm distance. Then*

$$W^{>}_{x,r} \leq \frac{W}{2} \text{ and } W^{<}_{x,r} \leq \frac{W}{2} \tag{7.19}$$

Proof (Sketch of Proof) If we increase the radius from r to $r + \epsilon$ the distance to data points with indices in $J^{>}_{x,r}$ decreases by ϵ, and the distance to data points with indices in $J^{<}_{x,r}$ increases by ϵ. This means, if $W^{>}_{x,r} > \frac{W}{2}$ we can improve the objective function by increasing the radius. (Analogously, if $W^{<}_{x,r} > \frac{W}{2}$ we can improve the objective function by reducing the radius.)

While the halving property can be nicely generalized from hyperplane location problems to hypersphere location problems, this is unfortunately not true for the determination of a finite dominating set. This can already bee seen in the Euclidean case for $D = 2$, i.e., for locating a circle in the plane: Here, the generalization of Theorem 7.4 would be that there always exists an optimal Euclidean circle passing through three of the data points. However, this turns out to be wrong, even in the unweighted case (see Fig. 7.1 for a counter-example). For most distances it is not even guaranteed that there exists an optimal circle passing through two of the data points. The only incidence property that can be shown for arbitrary norms is the following.

Lemma 7.7 *Let d be a norm distance. Then there exists a minsum hypersphere w.r.t. the distance d which passes through at least one point $v \in V$.*

Proof (Sketch of Proof) Let $S_{x,r}$ be a hypersphere. Fix its center point x and assume without loss of generality that the data points are ordered such that $d(x, v_1) \leq d(x, v_2) \leq \ldots \leq d(x, v_n)$. Then the objective function $f'(r) := f_1(S_{x,r})$ in (7.18) is piecewise linear in r on the intervals $I_j := \{r : d(x, v_j) \leq r \leq d(x, v_{j+1}\}$, $j = 1, \ldots, n-1$, and hence takes a minimum at a boundary point, i.e., there exists an optimal radius $r = d(x, v_j)$ for some $v_j \in V$.

The proof uses that the radius of an optimal circle is the median of the distances $d(x, v_1), \ldots, d(x, v_n)$ which was already recognized in Drezner et al. (2002).

Not much more can be said in the general case. The only (again, weak) property into this direction we are aware of is the following:

Lemma 7.8 (Körner et al. 2012) *Let $S = S_{x,r}$ be a minsum hypersphere with radius $r < \infty$. Then S intersects the convex hull of the data points in at least two data points, i.e., $|S \cap conv(V)| \geq 2$.*

Furthermore, if $|S \cap conv(V)| < \infty$, then $S \cap conv(V) \subseteq V$.

7.4.3.1 Location of a Euclidean Minsum Circle

For the Euclidean distance and the planar case $D = 2$ it is possible to strengthen the incidence property of Lemma 7.7.

Theorem 7.11 (Brimberg et al. 2009b) *Let d be the Euclidean distance, and consider the planar case, i.e., let $D = 2$. Then there exists a minsum circle which passes through two data points of V.*

The result can be shown by looking at the second derivatives of the objective function (in an appropriately defined neighborhood) which reveal that a circle passing through exactly one or none of the data points cannot be a local minimum.

An algorithmic consequence of the Theorem 7.11 is that there exists an optimal circle with center point x being on a bisector of two of the data points, hence a line search along the bisectors is possible. Using Theorem 7.10 a large amount of bisectors may be excluded beforehand. Figure 7.5 shows the Euclidean bisectors for five data points where the relevant parts (which contain center points of circles having the halving property) are marked in bold.

Another approach was followed in Drezner and Brimberg (2014): Here the unweighted case is shown to be an ordered median *point* location problem with weights $\lambda = (-1, \ldots, -1, 1, \ldots, 1)$ with equal number of -1's and 1's if n is even, and with weights $\lambda = (-1, \ldots, -1, 0, 1, \ldots, 1)$ with equal number of -1's and 1's if n is odd. The resulting ordered median point location problem was then solved using the Big-Triangle-Small-Triangle method (Drezner and Suzuki 2004) with the d.c. bounding technique proposed in Brimberg and Nickel (2009).

7.4.3.2 Location of Minsum Circles and Hyperspheres with Block Norm Distance

If d is derived from a block norm, a finite dominating set can be constructed for the center point of the minsum circle. To this end, graph all fundamental directions $\{e_1, \ldots, e_G\} \subseteq \mathbb{R}^2$ of the block norm through any of the data points $v \in V$ and add the bisectors for all pairs of data points in V. The intersection points of these lines form a finite dominating set which can be tested within $O(n^3)$ time, see Körner (2011), Brimberg et al. (2011a).

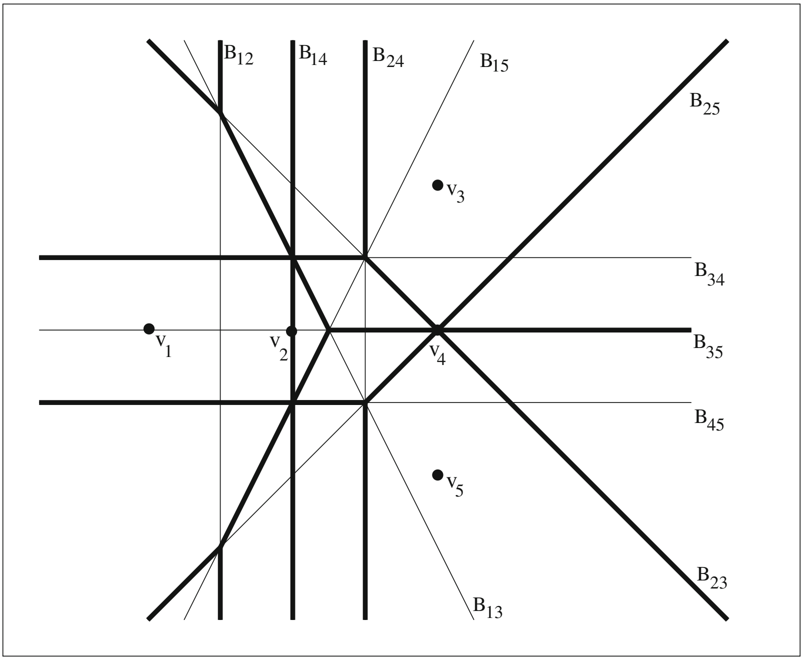

Fig. 7.5 The Euclidean bisectors for five data points. The notation B_{ij} indicates that the corresponding line is the bisector for data points v_i and v_j. The parts of the bisectors which may contain a center point of a minsum circle are marked in bold

Using that the block norm of a point y is given as

$$\|y\| = \min\{\sum_{g=1}^{G} \alpha_g : y = \sum_{g=1}^{G} \alpha_g e_g, \alpha_g \geq 0 \text{ for } g = 1, \ldots, G\}$$

the problem can in the case of block norm distances alternatively be formulated as the following linear program with $nG + 2n + D + 1$ variables, see Brimberg et al. (2011a) for the planar case and (Körner et al. 2012) for the case of hyperspheres.

$$\text{minimize} \quad \sum_{j=1}^{n} w_j \left(z_j^+ + z_j^-\right)$$

$$\text{subject to} \quad \sum_{g=1}^{G} \alpha_{g,j} = r + z_j^+ - z_j^- \text{ for } j = 1, \ldots, n$$

$$\sum_{g=1}^{G} \alpha_{g,j} e_g = x - v_j \text{ for } j = 1, \ldots, n$$

$$z_j^+, z_j^- \geq 0 \text{ for } j = 1, \ldots, n$$
$$\alpha_{g,j} \geq 0 \text{ for } g = 1, \ldots, G,\ j = 1, \ldots, n$$
$$r \geq 0$$
$$x \in \mathbb{R}^D.$$

7.4.4 The Minmax Hypersphere Location Problem

We now turn our attention to the location of a minmax hypersphere using the maximum of the residuals as globalizing function. That is, we look for a hypersphere which minimizes the maximum weighted distance to the set V of data points. Given a norm distance d, the goal hence is to find a hypersphere $S = S_{x,r}$ which minimizes

$$f_{\max}(S_{x,r}) = max_{j=1}^{n} w_j d(S_{x,r}, v_j) = \sum_{j=1}^{n} w_j |d(x, v_j) - r|. \tag{7.20}$$

Note that the problem of locating a Euclidean minmax circle in the plane is older than the corresponding Euclidean minsum circle problem; a finite dominating set has already been identified in Rivlin (1979). Its rectangular version is due to Gluchshenko et al. (2009). In $\mathbb{R}^D$ the Euclidean minmax hypersphere location problem has been analyzed mainly in the Euclidean case, see Nievergelt (2002).

7.4.4.1 Relation to Minimal Covering Annulus Problem and Equity Problem

The problem of locating a minmax circle has a nice geometric interpretation. For equally weighted data points it may be interpreted as finding an annulus of minimal width covering all data points. This problem has been studied in computational geometry, hence results on minmax circle location have been obtained independently in location theory and in computational geometry.

In location science the minmax hypersphere location problem has an interesting application as a point location problem. Namely, the (unweighted) center point x of an optimal hypersphere $S_{x,r}$ minimizes the difference

$$\max_{j=1,\ldots,n} d(x, v_j) - \min_{j=1,\ldots,n} d(x, v_j),$$

i.e., it minimizes the *range* to the set V. We conclude that minmax hypersphere location problems can be interpreted as ordered median point location problems. The point x may be interpreted as a fair location for a service facility as used in equity problems, see Gluchshenko (2008) for further results.

7.4.4.2 Location of a Euclidean Minmax Circle

Let us start with the Euclidean case in dimension $D = 2$: In this case, the problem has been discussed extensively in the literature, mainly in computational geometry under the name of finding an annulus of smallest width. In contrast to the Euclidean minsum circle problem, where an FDS could not be found, the following result shows that an FDS for the (Euclidean) minmax hypersphere exists.

Theorem 7.12 (FDS for the Euclidean Minmax Circle) *(e.g., Rivlin 1979; Brimberg et al. 2009a) Let $D = 2$ and let C be a minmax circle with finite radius. Let $h := \max_{j=1,\ldots,n} w_j d(C, v_j)$. Then there exist four data points having distance h to the circle C, two of them inside the circle and two of them outside the circle.*

The theorem was shown for the unweighted case independently in many papers, among others in Rivlin (1979), Ebara et al. (1989), García-López et al. (1998) and it was generalized to the weighted case in Brimberg et al. (2009a). The result can be interpreted in different ways:

- In the geometric interpretation, the result means that the annulus of minimal width covering all data points has two data points on its inner circumference and two data points on its outer circumference (Rivlin 1979).
- It also shows that the center point of a minimax circle is either a vertex of the (nearest neighbor) Voronoi diagram or of the farthest neighbor Voronoi diagram or lies at an intersection point of both diagrams (Le and Lee 1991; García-López et al. 1998).

For the unweighted problem (Ebara et al. 1989), use this result and present an enumeration algorithm with runtime in $O(n^2)$. If the data points in V are given in an angular order (García-López et al. 1998), present an algorithm which runs in $O(n \log n)$ and which can even be improved to $O(n)$ if the data points in V are the vertices of a convex polygon. This is in particular helpful for solving the out-of-roundness problem (see Sect. 7.4.1), since the measurements are taken along the manufactured part in angular order in this case. A gradient search heuristic is provided in Drezner et al. (2002) and global optimization methods were used in Drezner and Drezner (2007) who use the Big-Triangle-Small-Triangle method (based on Drezner and Suzuki 2004) for its solution. Randomized and approximation algorithms are also possible, see Agarwal et al. (2004, 1999).

More references on the computation of Euclidean minmax circles can be found in García-López et al. (1998) and Brimberg et al. (2009a).

7.4.4.3 Location of a Minmax Circle with Rectangular Distance

Gluchshenko (2008) and Gluchshenko et al. (2009) consider the minimal annulus problem for the rectangular distance. This means, the circle to be located is a diamond, and the distances from the given data points to the circle are measured in the rectangular norm. The following is an important result.

Theorem 7.13 (FDS for the Rectangular Minmax Circle) *(Gluchshenko et al. 2009) Let d be the rectangular distance. Then there exists a minmax circle whose center point is a center point of a smallest enclosing square of the data points.*

This means the set of all center points of smallest enclosing squares (which can be determined easily) is an FDS. Based on this (Gluchshenko et al. 2009), develop an optimal $O(n \log n)$ algorithm for finding a minmax circle with respect to the rectangular norm.

More recently, the problem in which the annulus may also be rotated has been considered in Mukherjee et al. (2013) where an $O(n^2 \log n)$ algorithm has been proposed.

7.4.4.4 Location of a Euclidean Minmax Hypersphere

The problem of finding a minmax hypersphere in dimension $D \geq 3$ was considered in García-López et al. (1998). The authors give necessary and sufficient conditions for a point to be the center point of a *locally* minimal hypersphere with respect to $f_{\max}$. Independently, also Nievergelt (2002) considers the problem of locating a hypersphere in $\mathbb{R}^D$ with Euclidean distance. Analogously to his approach for minmax hyperplanes, he interprets the problem as the location of two concentric hyperspheres with minimal distance which enclose the set V of data points. This results in a generalization of Theorem 7.12 to higher dimensions.

Theorem 7.14 (FDS for the Euclidean Minmax Hypersphere) *(Nievergelt 2002) There exists a Euclidean minmax hypersphere S which is rigidly supported by the point set V, i.e., there does not exist any other pair of concentric hyperspheres enclosing all data points of V and passing through the same data points of V as S.*

Based on this property (Nievergelt 2002), derives a finite algorithm finding a minmax hypersphere with respect to the Euclidean distance. A linear time $(1 + \epsilon)$ factor approximation algorithm for finding a Euclidean minmax hypersphere is given in Chan (2000).

7.4.5 Some Extensions of Circle Location Problems

7.4.5.1 Minimizing the Sum of Squared Distances

An earlier variant of the hypersphere location problem minimizes the sum of *squared residuals as globalizing function*, i.e., it considers

$$f_2^2(S_{x,r}) = \sum_{j=1}^{n} w_j \left(d(S_{x,r}, v_j)\right)^2$$

as objective function. In Drezner et al. (2002) it is shown that the least squares objective is equivalent to minimizing the variance of the distances. The problem is (like the minsum and minmax problem) non-convex; heuristic solution approaches are suggested. In Drezner and Drezner (2007) the Big-Triangle-Small-Triangle global optimization algorithm is successfully applied.

Minimizing the sum of squared distances from the data points in V to a circle has been also considered within statistics in Kasa (1976), Crawford (1983), Moura and Kitney (1992), Coope (1993), Gander et al. (1994), Rorres and Romano (1997), Späth (1997, 1998), Nievergelt (2004).

7.4.5.2 Locating Euclidean Concentric Circles

In a recent paper (Drezner and Brimberg 2014), introduce the following extension of the circle location problem: They look for p concentric circles with different radii $r_1, \ldots, r_p$ which minimize the distances to a given set of data points. In their paper they assume a partition of V into sets $V_1, \ldots, V_p$ and require that each point in V_i is served by the circle with radius r_i. This means the variables to be determined are the center point $x \in \mathbb{R}^2$ and the radii $r_1, \ldots, r_p$ of the p circles. The model is considered for the least squares globalizing function, as well as for using minsum and minmax. Using that

$$d(S_{x,r_j}, v_j) = |d(x, v_j) - r|$$

the objective functions which are considered are given as

$$f_2^2(x, r_1, \ldots, r_p) = \sum_{q=1}^{p} \sum_{v_j \in V_q} w_j \left(d(x, v_j) - r\right)^2$$

$$f_1(x, r_1, \ldots, r_p) = \sum_{q=1}^{p} \sum_{v_j \in V_q} w_j |d(x, v_j) - r|$$

$$f_{\max}(x, r_1, \ldots, r_p) = \max_{q=1,\ldots,p} \max_{v_j \in V_q} w_j |d(x, v_j) - r|.$$

Drezner and Brimberg (2014) solve the problem by global optimization methods, using a reformulation of the circle location problem as an ordered median point location problem (see the location of a Euclidean minsum circle in Sect. 7.4.3) and applying the Big-Triangle-Small-Triangle method (Drezner and Suzuki 2004).

7.4.5.3 Location of a Circle with Fixed Radius

The location of a circle with fixed radius is considered in Brimberg et al. (2009a). In this case, it can be shown that considering every triple of data points separately

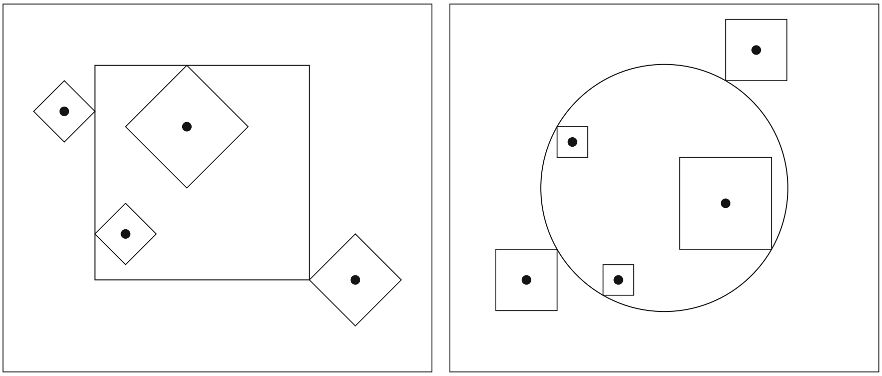

Fig. 7.6 Locating a circle of norm k_1 with respect to another norm k_2. Left: The unit circle of the maximum norm is to be located, distances are measured w.r.t. the rectangular norm. Right: The Euclidean circle is to be located, distances are measured w.r.t. the maximum norm

yields an optimal solution, i.e., a finite dominating set can be derived by solving $\binom{n}{3}$ smaller optimization problems.

7.4.5.4 Locating a Hypersphere of One Norm Measuring Distances with Respect to Another Norm

In two dimensions, the circle location problem is to translate and scale a circle $S = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ (derived from norm $\|\cdot\|$) in such a way that the distances to the data points in a set V are minimized, where the residuals are measured with respect to the same norm $\|\cdot\|$. In Körner et al. (2009, 2011) this problem is studied for two *different* norms under the name *generalized circle location*.

More precisely, given two norms k_1 and k_2 and a set of data points V in the plane with positive weights $w_j > 0$, the goal of *generalized hypersphere location* is to locate and scale a hypersphere of norm k_1 such that the sum of weighted distances to the data points is minimized, where the distances are measured by the other norm k_2. Figure 7.6 shows two possible situations. In the left part of the figure, the new facility is the scaled and translated unit circle of the $k_1 := \|\cdot\|_{\max}$ norm and the distances to the four given data points are measured by the $k_2 := \|\cdot\|_1$ norm. In the right part, $k_1 := \|\cdot\|_2$ and $k_2 := \|\cdot\|_{\max}$.

In Körner et al. (2011), properties of minsum generalized circle location in $D = 2$ dimensions are investigated, and it is shown that not much of the properties for minsum circle location still hold. There is neither an easy formula for computing the distance between a point and such a generalized circle, nor does any of the incidence criteria hold. In fact, there are examples in which no optimal circle passes through any of the data points. However, if both norms k_1 and k_2 are block norms, a finite dominating set can be identified (see Körner et al. 2009). The problem of locating a general circle is interesting for many special cases, e.g., if a box should be located. Such cases have been studied in Brimberg et al. (2011b).

7.5 Locating Other Types of Dimensional Facilities

7.5.1 Locating Line Segments

The *line segment location problem* looks for a line segment with specified length which minimizes the distances to the set V of data points.

Location of line segments has been considered in Imai et al. (1992), Agarwal et al. (1993), Efrat and Sharir (1996) for the Euclidean minmax problem, and in Schöbel (1997) for the minsum problem with vertical distances. In both the cases it is possible to determine a finite dominating set; the latter case can be transformed to a restricted line location problem.

Locating line segments received new interest within the following problem: A line segment and a point facility are to be located simultaneously. In this setting, the line segment can be used to speed up traveling in the plane in which a new point facility should be built. The problem has been treated in the plane, using rectangular distances in Espejo and Rodríguez-Chía (2011, 2012) where a characterization of optimal solutions was used to derive an algorithm. This could be improved in Díaz-Bánez et al. (2013) to an $O(n^3)$ approach. These approaches are based on a finite dominating set which can be obtained by reduction of the location problem to a finite number of simpler optimization problems.

7.5.2 The Widest Empty 1-Corner Corridor in the Plane

An *empty corridor* in the plane is an open region bounded by two parallel polygonal chains that does not contain any of the data points $V = \{v_1, \ldots, v_n\}$, and that partitions the data points into two non-empty parts. This can be interpreted as an obnoxious dimensional location problem: locate a polygonal chain maximizing the minimum distance to the data points. Empty corridors have been of interest in computational geometry (see e.g., Janardan and Preparata 1996). An empty corridor is called a *1-corner empty corridor* if each of the two bounding polygonal chains has exactly one corner point. The problem in which the angle at the corner point is given and fixed has been studied in Cheng (1996). Díaz-Bánez et al. (2006b) considered the problem of locating a widest 1-corner corridor using techniques of facility location: they were able to derive a finite dominating set consisting of locally widest 1-corner corridors among which a solution may be chosen. Their approach needs $O(n^4 \log n)$ time. It was further improved to $O(n^3 \log^2 n)$ time in Das et al. (2009).

7.5.3 Two-Dimensional Facilities

Covering problems are the most common problems in which the location of full-dimensional facilities is considered. There exist many papers about covering data points by a circle (i.e., locating one point x such that all data points are in a given threshold distance from x), by a set of circles, or even by a set of aligned circles (occurring when the center points of the circles to be located are forced to lie on a common straight line), or circles satisfying other restrictions. Covering problems are not reviewed here, we refer to Plastria (2001) or to Chap. 5.

However, also the location of a two-dimensional facility X such that the minsum or minmax globalizing function is minimized, has been considered in the literature. If there exists a location for X such that all data points are covered, this location is clearly an optimal solution with objective value zero both for the minsum and for the minmax problem. If it is not possible to cover all data points, the minsum and the minmax problem usually have different solutions.

A paper dealing with the location of a two-dimensional facility is Brimberg and Wesolowsky (2000) where the rectangular distance is considered and special cases could be transformed to classical point location problems. In the context of facility layout the location of a rectangular office with minsum and minmax globalizing function has been studied in Savas et al. (2002), Kelachankuttu et al. (2007) and Sarkar et al. (2007). In these papers, existing offices are treated as barriers. Various problem variations for the location of an axis-parallel rectangle (with fixed circumference, with fixed area, with fixed aspect ratio, or with fixed shape and size) have been considered in Brimberg et al. (2011b). For most cases, a finite dominating set could be derived.

The location of a two-dimensional ball

$$B_x = \{y \in \mathbb{R}^2 : d(x, y) \leq r\}$$

with given and fixed radius r has been considered in Brimberg et al. (2015a) both for the minsum and the minmax globalizing function. Note that the distance between B_x and v

$$d(B_x, v) = \min_{y \in B_x} d(y, v)$$

is measured as the closest distance to any point in B, and not only to data points on its circumference $S_{x,r}$. This means that

$$d(B_x, v) = \begin{cases} 0 & \text{if } v \in B_x \\ d(S_{x,r}, v) & \text{otherwise.} \end{cases}$$

Hence, Lemma 7.5 yields that $d(B_x, v)$ is a convex function and consequently, the resulting optimization problems are much easier to solve than the circle location problems of Sects. 7.4.3 and 7.4.4. We remark that the location of a full-dimensional ball has the following interesting interpretation as a point location problem with *partial coverage*:

Assume that we are looking for a new facility $x \in \mathbb{R}^2$ for which we know that little or no service cost (or inconvenience) is associated with data points that are within an acceptable travel distance r from x. Thus, costs will be associated only to those data points that are further away from the facility than this threshold distance r. If we assume that these costs are proportional to the distance in excess of r, the resulting problem is equivalent to the location of a ball with radius r, and its center point is the optimal location x we are looking for. This has been pointed out in Brimberg et al. (2015a) where the behavior of the optimal solution with respect to the threshold distance r is studied.

Line location with the partial coverage globalizing function is equivalent to locating a strip of given width and has recently been considered in Brimberg et al. (2015b).

7.5.4 General Approaches for Locating Dimensional Facilities

Blanquero et al. (2009) and Mallozzi et al. (2019) both deal with the location of a variety of dimensional facilities such as segments, arcs of circumferences, arbitrary convex and non-convex sets, their complements, or their boundaries. The idea is to fix the shape of the dimensional facility and to look for a shift vector and/or an angle of rotation. The objective they follow is very general, including most globalizing functions used in location theory.

Blanquero et al. (2009) also allow to model obnoxious or semi-obnoxious location problems as follows: The set of data points is split into a subset V^+ for which the new facility is attractive and a subset V^- for which the new facility has negative effects. The distance from the new facility to a data point should be small when the point is in V^+ and large when it is in V^-. In order to combine the distances within the same set V^+ and V^- Blanquero et al. (2009) propose to evaluate the norm (or the gauge) of the resulting single distances. Using that the Euclidean distance $d(S, v)$ between a point and a set can be written as difference of convex functions (Blanquero et al. 2009), solve the model by d.c.-programming methods, outer approximation and branch and bound.

Mallozzi et al. (2019) deal with the location of p dimensional facilities of very general shapes and the allocation of them to some given demand. Instead of a distance measure, utility functions are used. The resulting location-allocation problem is discretized and tools from optimal mass transport are used for its solution.

7.6 Conclusions

For the location of dimensional facilities we can draw the following conclusions.

- The location of a zero-dimensional facility (i.e., a point) and of a full-dimensional facility of convex shape with respect to a norm is a convex problem.
- In contrast, the location of a one-dimensional facility with respect to a norm is a non-convex problem which usually has many locally optimal solutions. Only the vertical distance leads to convex hyperplane location problems (if also the globalizing function g is convex).
- However, many of the investigated problems of locating a one-dimensional facility are piecewise quasiconcave on a cell structure in dual space. This leads to a finite dominating set. Another possibility for deriving an FDS is via Helly-type theorems.
- When distances are measured w.r.t. a block norm, hyperplane and hypersphere location problems with ordered median globalizing function are piecewise linear and can hence be solved by linear programming methods.
- The halving property holds when the problem is linear with respect to one of its variables.

The main properties pointed out in this chapter are summarized in Table 7.2. They have the following algorithmic consequences.

The FDS property gives the straightforward possibility of enumerating the candidate set. Also for the location of p facilities the FDS property is still helpful, although the number of candidates increases to $O(|FDS|^p)$. As demonstrated for the p-minsum line location problem in Sect. 7.3.7, an FDS also allows to transfer the problem of locating p facilities to a p-location problem on a bipartite graph with $O(|FDS|)$ nodes. It is ongoing work to test such approaches numerically.

Table 7.2 Summary of properties for some of the considered location problems

Problem	FDS	Halving	LP
Minsum hyperplane with $d = d_{ver}$	Yes	Yes	Yes
Minsum hyperplane with norm	Yes	Yes	No
Minsum hyperplane with block norm	Yes	Yes	Yes
Minsum hyperplane with gauges	No	(Yes)	No
Minmax hyperplane with norm	Yes	No	No
Minmax hyperplane with block norm	Yes	No	Yes
Minmax hyperplane with gauges	Yes	No	No
Ordered minsum hyperplane with norm	Yes	Yes	No
Minsum line in $\mathbb{R}^3$	No	No	No
Line may not pass through a polyhedral set	Yes	No	No
Minsum/minmax p-line with norm	Yes	No	No
Minsum hypersphere with norm	No	Yes	No
Minsum hypersphere with block norm	Yes	Yes	Yes
Minmax hypersphere with Euclidean norm	Yes	No	No
Minmax circle with rectangular norm	Yes	No	Yes

Enumeration may be enhanced by the halving property which can be used to directly discard candidates of an FDS. Such discarding tests are also useful in other approaches, even if no FDS is known, since the halving property allows to discard whole regions when searching for an optimal solution. An example is the search along bisectors which can be reduced to the relevant parts in the Euclidean minsum circle location problem. Also in geometric branch & bound approaches such as Big-Square-Small-Square (Plastria 1992), Big-Triangle-Small-Triangle (Drezner and Suzuki 2004), Big-Cube-Small-Cube (Schöbel and Scholz 2010) or Big-Arc-Small-Arc (Drezner et al 2018) discarding tests motivated by the halving property may be interesting.

Using linear programming methods is an efficient way of solving facility location problems, in particular if the number of variables is not too large. This is the case for block norms with not too many fundamental directions.

While many questions in the location of lines and hyperplanes seem to be solved, there are still questions remaining in the location of hyperspheres. These concern, on one hand, general properties about the location of hyperspheres with other than the minsum globalizing function and with arbitrary norms or gauges. On the other hand, there are also many special cases waiting to be investigated, in particular if the sphere is defined with respect to another norm as the distance function.

Concerning the location of new types of dimensional structures, researchers should look for shapes which are of interest for other disciplines or for applications. Similarly, identifying additional restrictions and particularities arising in applications in operations research, statistics, and computational geometry and including them in the models is a future challenge.

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