Chapter 14
Competitive Location Models

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Abstract This chapter first provides a review of the foundations of competitive location models. It then traces subsequent developments through time under special consideration of customer behavior. After developing a general framework for customers' decision making, the main results are cast within this framework. The conclusion outlines a number of areas, in which existing models can be refined and made more realistic.

14.1 The Basic Model: The First 50 Years

Competitive location models were first discussed by Hotelling (1929) in his seminal paper. It has spawned hundreds of contributions (for a summary until the early 1990s, see Eiselt et al. 1993) that investigate many different aspects of the basic model. A recent summary of Hotelling-style models was provided by Eiselt (2011), for details we refer to that work. This chapter will first introduce the basic model, followed by an outline of some of the main components of competitive location models. We then discuss the main aspects and types of consumer behavior, and then review the work on competitive location models under special consideration of customer behavior.

The basic model is easy to describe: consider a line segment, a so-called "linear market," which Hotelling referred to as "main street," along which customers are uniformly distributed. (The often-mentioned "ice cream vendors on the beach" were

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actually introduced by Lösch 1954). Each customer has a fixed and inelastic demand for a given homogeneous good. Duopolists are now attempting to independently enter the market, offering identical products. The competitors are profit maximizers, and they attempt to achieve their objective by determining their respective locations and prices; first both competitors choose their respective locations, followed by the simultaneous choice of prices. It is assumed that both competitors employ mill (or f.o.b.) pricing (a pricing policy in which customers pay a price set by the facility and take care of the transportation themselves) and that transportation costs between customers and facilities are linear. Customers will patronize the facility that offers the good for the lowest full price, i.e., the smallest sum of mill price and transportation costs. For simplicity, it is commonly assumed that the costs of the firms have been normalized to zero.

Already in his original paper, Hotelling did not restrict himself to the aforementioned "main street" with customers in search for inexpensive physical goods from brick-and-mortar retailers. One of the nonphysical applications he mentioned was what we today refer to as brand positioning, *viz.*, the location of a brand in some feature space. More specifically, Hotelling used the example of ciders offered by two firms, whose single distinguishing characteristic is their respective sweetness. Given that a brand is sweeter (more sour) if it is located more to the right (left) side of the market segment, the two firms will determine optimal locations and prices so as to maximize their respective profits.

Similar, albeit with a marked difference, is the political positioning model that was also mentioned in Hotelling's original paper. The idea was very simply for each of two political parties to each locate their own candidate, so as to maximize the number of votes (i.e., the number of customers, or the market share) that the candidate would obtain. The line segment was used to mimic the traditional left-right scale in politics, voters (i.e., their "ideal points," which symbolize their most favored position on the line) were again assumed to be uniformly distributed on the line segment, and the candidates would not have any inherent stand on the issues, they would simply position themselves at a point, where it would win them the largest number of votes. However, in contrast to all other previously mentioned applications, there are no prices in this model.

The main focus of Hotelling's original paper is the existence (or the lack) of a stable solution, i.e., an equilibrium. Hotelling asserts that an equilibrium would exist with both firms locating next to each other at the center of the market. This result is often dubbed the "principle of minimum differentiation," in reference to products or political candidates being very similar to each other. Even though in a footnote, Hotelling cautions that his result would not hold in highly competitive situation (which is precisely what occurs when the two firms locate very close to each other), he presented his agglomeration result as his major finding. Other authors, such as Lerner and Singer (1937) and Eaton and Lipsey (1975) obtained different results, but their contributions were based on Hotelling-style models albeit with fixed and equal prices. Hotelling's original result was not disputed until d'Aspremont et al. (1979) demonstrated 50 years later that no equilibrium exists in Hotelling's model. In order to follow the argument, first consider a graphical representation of Hotelling's

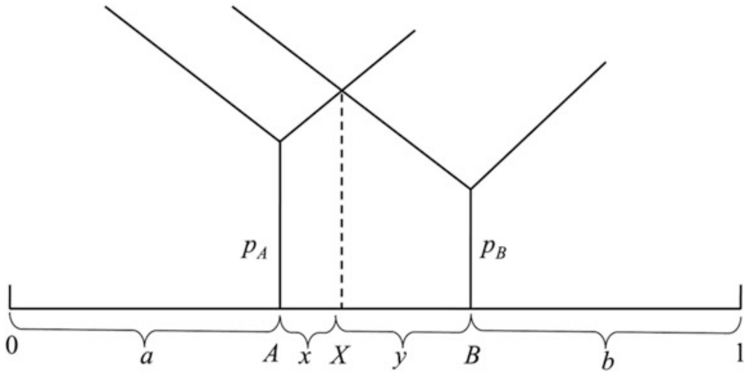

Fig. 14.1 Hotelling's duopoly on a linear market

scenario as shown in Fig. 14.1. Here, the linear market extends from 0 to 1, and the locations of the two competitors are shown as A and B, respectively. They charge mill prices p_A and p_B, respectively, and transportation costs are linear, resulting in full prices to the customers shown in the two "V" shaped functions. The two functions intersect at some point X, which is usually referred to as the *marginal customer*, i.e., the customer who pays the same *full price* (i.e., the mill price plus transportation costs) purchasing from firm A as he does purchasing from firm B. As a matter of fact, the function that describes the full price for all customers on the line segment is the lower envelope of the two "V"-shaped functions. Furthermore, the market can now be subdivided into the following parts: The first piece of length a is firm A's *hinterland*, which A captures in its entirety. Similarly, the stretch b on the right is firm B's hinterland, which is captured by B. The remaining area is the *competitive region* between firms A and B. (The terms "hinterland" and "competitive region" appear to have been introduced by Smithies 1941). This is subdivided into parts x and y, such that x is the part in which customers can purchase more cheaply from firm A, while in y, customers can purchase the good more cheaply from firm B.

This allows us to determine the market shares of the two firms simply as $M(A) = a + x$ for firm A and $M(B) = b + y$ for firm B. This depiction of the scenario also permits us to examine the two forces that govern the process. The *market share force* pushes the two facilities towards each other. The reason is that—given that his opponent does not react, at least temporarily—a facility can move towards its competitor and, in doing so, not lose market in its own hinterland, while gaining in the competitive region. This force applies, as long as customers do not have finite (and reasonably low) *reservation prices*, i.e., an upper bound on the full price they are able or willing to pay for the good. On the other hand, there is the competitive *pricing force* that pushes the two facilities apart. The reason is that if the two firms locate very close to each other, whatever price one of them sets, his competitor can undercut him slightly and thus capture the entire market. This results in facilities moving apart so as to position themselves in a region with less competitive pressure.

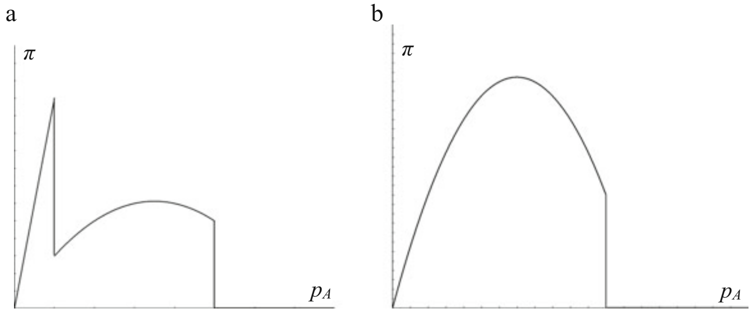

Fig. 14.2 Competitor *A*'s profit functions with linear transportation costs. (a) *A* and *B* are close to each other and (b) *A* and *B* are far apart

The obvious question is whether or not there exists a locational arrangement and a price structure, which represents a stable solution, i.e., an equilibrium. Temporarily holding the location of both and the price of one of the competitors, say, *B*, constant, Fig. 14.2a, b show competitor *A*'s profit function π in the case of firms *A* and *B* locating close to each other (Fig. 14.2a) or a significant distance apart (Fig. 14.2b).

First consider Fig. 14.2a. From left to right, *A*'s profit function is linearly increasing for low prices p_A (as firm *B* is cut out and *A*'s profit increases proportional to the price); then, as p_A increases, at some point, *B* is no longer cut out, there is a marginal customer in the competitive region, and *A*'s profit function is an inverted ellipse. As p_A increases further, there exists a point, at which it is sufficiently high so that firm *B* cuts out firm *A*, and thus *A*'s profit drops to zero. Note that there are two local maxima, one at the first breakpoint from the left, and the second in the domain of the quadratic piece of the function. In Fig. 14.2b, the linearly increasing part is valid only for negative prices, which are nonsensical in this application. Other than that, the function is similar to that in Fig. 14.2a, but with a single maximum.

d'Aspremont et al. (1979) first demonstrated that Hotelling's model does not possess an equilibrium in pure strategies, i.e., as long as each player chooses exactly one strategy, rather than randomize. They then demonstrated that an equilibrium was restored in the model if we were to use a quadratic, rather than a linear, transportation cost function. Later, Gabszewicz et al. (1986) pointed out that the lack of the existence of equilibria in Hotelling's model is due to the lack of quasiconcavity of the profit functions of the duopolists (see again Fig. 14.2a). Fig. 14.3a, b show again competitor *A*'s profit π, given a quadratic, rather than linear transportation cost function: Fig. 14.3a for competitors' locations that are close to each other, and Fig. 14.3b for locations far apart. Note that the functions are both quasiconcave.

In general, many competitive location models have shown major signs of instability: Hotelling's original model with variable prices and linear cost functions has no equilibrium, the same model with quadratic transportation costs has one—with firms located at opposite ends of the market. Hotelling's model with a

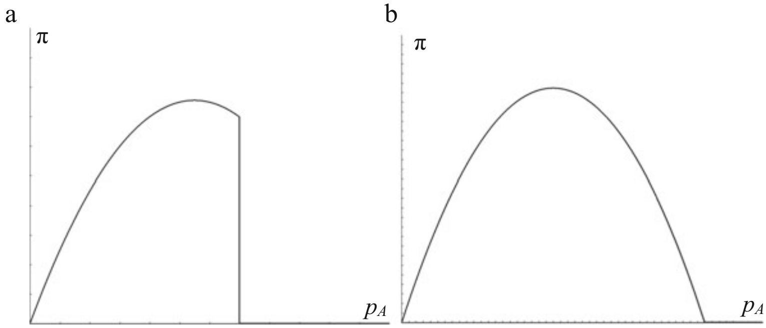

Fig. 14.3 Competitor *A*'s profit functions with quadratic transportation costs. (a) *A* and *B* are close to each other and (b) *A* and *B* are far apart

linear-quadratic cost function (see, e.g., Gabszewicz and Thisse 1986, or Anderson 1988) does not have equilibria, as long as the linear part, no matter how small, exists. Hotelling's model with fixed and equal prices (see, e.g., Lerner and Singer 1937 or Eaton and Lipsey 1975) has an equilibrium with minimal differentiation, while the same model with three firms has no equilibrium; the duopoly with fixed and unequal prices, regardless how small the difference between the prices, has no equilibrium.

Consider now the locational arrangement that minimizes the total transportation costs to the customers. Using the notational convention in Fig. 14.1 and unit transportation costs t, the total transportation costs to all customers can be written as

$$TC = t\left[\int_{\Phi=0}^{A} (A-\Phi)\,d\Phi + \int_{\Phi=A}^{X} (\Phi-A)\,d\Phi + \int_{\Phi=X}^{B} (B-\Phi)\,d\Phi + \int_{\Phi=B}^{1} (\Phi-B)\,d\Phi\right]$$
$$= t\left[3A^2/4 + 3B^2/4 - AB/2 - B + \tfrac{1}{2}\right]$$

Partial differentiation $\frac{\partial TC}{\partial A} = 0$ and $\frac{\partial TC}{\partial B} = 0$ results in the optimal points $A = ¼$ and $B = ¾$, a configuration at which the total transportation costs are $t/8$. In contrast, central agglomeration results in transportation costs of $t/4$, i.e., costs that are twice as high. As the point $(A, B) = (¼, ¾)$ minimizes the total transportation costs (which are, given mill pricing, borne by the customers), this point is often referred to as *social optimum*.

Before investigating the key elements of competitive location models, we would like to draw attention to some surveys of the subject. Brown (1989) provides a critique of Hotelling's work and points out various directions, which would make the original model more realistic. Eiselt et al. (1993) provide a taxonomy and a short evaluation of the literature up to that point. Plastria (2001) looks at the optimization aspect of the subject, while Drezner and Eiselt (2002) focus on customer behavior and its consequences on the solution. Kress and Pesch (2012) surveyed the subject, but concentrate on problems on networks, while Drezner (2014) surveys problems in the plane. Similar to the aforementioned contribution by Eiselt et al. (1993), Ashtiani (2016) first outlines some of the main characteristics of competitive location problems and then reviews individual papers published in 2000–2014. While Karakitsiou and Migdalas (2017) survey competitive location problems with respect to Nash equilibria, Aras and Küçükaydın (2017) review contributions that focus on von Stackelberg solutions. Finally, Marianov and Eiselt (2016) investigate existing competitive location models with respect to the tendencies of facilities to agglomerate or disperse.

14.2 Elements of Competitive Location Models

The subject of competitive location models, as pioneered by Hotelling, has become a rich research area. Since research has moved into many different directions, it is useful to classify models, e.g., by using the taxonomy proposed by Eiselt et al. (1993). Rather than describe it in detail, we will outline its major components here.

One aspect of all location models, competitive or not, is the choice of *space*. In contrast to regular, noncompetitive, location models, many authors have used much simplified spaces in their models: starting with Hotelling's original linear market, they have also investigated circular markets, which may appear rather contrived at first glance, but are designed to avoid the "end-of-line effects" of bounded linear markets.

Measures of distances are no issue when devising models in a single dimension, but they are, as soon as models in two or more dimensions are investigated. While some authors favor gauges in noncompetitive location models (see, e.g., Durier and Michelot 1985, or Plastria 1992) most contributions that look at continuous location models in the plane have used Minkowski distances, most prominently Manhattan, Euclidean, and Chebyshev distances.

A similar situation prevails in networks. Measures of distances in trees are not an issue, as, by definition, there is only one path between each pair of points. However, in general networks one could, at least theoretically, use any distance that best models reality. Assuming not only rational, but also cost-minimizing behavior, virtually all authors in the field have chosen shortest path distances. Assuming complete information, one could choose traffic choice models and assume that customers take not the shortest route with respect to distances but the shortest route with respect to time; or that not all customers use the same route selection strategy all the time. This would suggest itself particularly in highly congested (urban) areas. One concept that is used extensively by authors who deal with network models is known as *node property* or *Hakimi property*. It is based on Hakimi's work Hakimi (1964) on network location properties, in which he proved that in some classes of models, at least one optimal solution locates all facilities at the nodes of a network.

The second component concerns the *number of players* and facilities that are to be located. Traditionally, papers included duopolists who locate a single facility each, so that the terms "firm" and "facility" (the entity to be located) were synonymous. This is, of course, no longer the case once we include multiple firms or multiple facilities to be located by each of the planners. Here, we will use the game-theoretic term players for the (independently operating) firms, and "facilities" for what they are locating. The number of facilities that one or more of the players wish to locate may be preselected or unspecified. In the latter case, the cost or profit function of a player includes fixed costs for opening a facility at a site.

The third component of competitive location models concerns the *pricing policy*. One important feature of Hotelling's original model was that he investigated competition in location *and* prices. A more general model would let players also choose their pricing policy. In particular, we typically distinguish between a variety of different pricing policies. Among the most prominent such policies is *mill pricing*, where players set prices at the source, which are not necessarily the same at all of their facilities. Customers will then purchase the product at the facility they have chosen to patronize and pay for the transport costs. Almost all retail facilities use this principle. A special case of mill pricing is *uniform pricing*, a policy, in which the facility planner sets the same price at all of his facilities. This policy was used by the "Motel 6" chain in the 1980s, until they chose to charge different prices at different locales to better reflect their own cost structure.

Another principle is uniform *delivered pricing*. In this pricing policy, facility planners will deliver the goods to their customers for a fixed "full price" regardless of customers' locations. Domestic mail is a typical example of this type of pricing policy. Clearly, in such a policy, customers that are located close to the facility from which they receive the goods, will subsidize those who are located farther away. A special case of this policy is *zone pricing*, a policy, in which the firm has subdivided their market area into zones, such that a uniform delivered price is charged in each zone. Typical examples are the outdoor store L.L. Bean that sells canoes for one delivered price east of the Mississippi, and another price west of the river, or postal services that typically charge one rate for domestic mail and (at least) one for international mail. *Spatial price discrimination* is a policy that charges customers a full price according to the customer's location. Its applications have been severely limited by the Robinson-Patman Act of 1936, even though it does provide some benefits to the customers; see, e.g., Anderson et al. (1992). Note that uniform delivered prices and spatial price discrimination are boundary cases of zone pricing; the former in case there is only one zone, and the latter in case each point in space represents its own zone. Many contributions, especially those from the operations research community, assume that prices are universal and fixed, which is the case in legislated pricing or producer-administered mandatory prices.

The fourth component concerns the *rules of the game* the players adhere to. In essence, this feature describes how individual players act or react. Consider the simple case of pure location competition. In such a case, players could simultaneously choose their strategies, i.e., decide on the locations of their facilities. If at this point, none of the players has an incentive to unilaterally change his position, we say that a Nash (or Cournot-Nash) equilibrium has been obtained. Such a situation indicates some stability. Note that all players have, at least potentially, the same information available to them, even though perceptions may differ, indicating some asymmetry among players.

Things are getting somewhat more involved, if players have not only locations, but also prices as variables. In such a case, we can employ a refinement of Nash equilibria, *viz*., Selten's (1975) *subgame perfection*. Loosely speaking, a subgame perfect equilibrium exists, if every subgame of a given game has a Nash equilibrium. Applied to our type of problem, players may choose a "first location, then price" strategy (see, e.g., Anderson and Palma 1992), i.e., all payers simultaneously choose their locations, and in a second phase, they simultaneously choose their prices. Many authors have chosen this route. At this point, we need to define the concepts of *pure and mixed strategies*. A pure strategy prescribes a certain course of action (i.e., a decision) for a decision maker, while a mixed strategy will provide a schedule of decision, associated with probabilities that indicate with what likelihood a decision maker should use this strategy. The work by Caplin and Nalebuff (1991) outlines conditions under which a pure-strategy price equilibrium exists in a locational game, while Dasgupta and Maskin (1986), who deal with discontinuous payoff functions, describe conditions for the existence of mixed strategies.

A full sequential strategy has one player, the so-called *leader*, locate first, followed by all other players, the *followers*, which locate later. This asymmetric situation has originally been described by the economist von Stackelberg (1943). The leader, when choosing his locations, will have to guard against the followers. If all players have the same objective and the same perception of the demand structure, this means that the leader will use a strategy to maximize the minimal market share or profit he will obtain. On the other hand, the followers will have a chance to observe the action of the leader and then react accordingly, meaning that they solve a conditional optimization problem, in which they maximize their own market share or profit, given that the leader has already located. Note that the problem of the follower is much easier to solve mathematically, as it is a simple optimization problem. The problem of the leader, however, is a bilevel optimization problem, as it requires the solution of the follower's problem as an input parameter.

The last major descriptor of competitive location models concerns *customer behavior*. As a matter of fact, this aspect is the main leitmotif of this paper. The first major distinction between different classes of models is between demand allocation models and customer choice models. As the name suggests, in allocation models the firm decides which facility is allocated to a customer. A typical example would be the delivery of furniture to customers, who will receive the goods from whatever warehouse the firm decides to deliver from. (Note that, strictly speaking, the purchase of, say, a sofa, typically involves a mix of allocation and choice models: when customers drive to a store to purchase the sofa is a choice model, while the actual delivery of the sofa is an allocation model). In scenarios of customer choice, on the other hand, customers choose which facility or firm they want to deal with. Often, the two models are referred to shipping and shopping. We would like to point out, though, that there are a number of instances, in which allocation and customer choice models are quite similar. If a firm delivers goods to customers, it may ship from the facility closest to the customer. Similarly, the same customer, in case he purchases the good from a facility and transports it home, may also choose the closest facility. The main difference between the two cases is that in the former, transportation costs appear explicitly in the firm's objective function, whereas they do not in the latter, where proximity enters in the form of which facility is chosen by a customer, but not in the form of transportation costs. This paper deals exclusively with customer choice models.

The manner in which customers choose which facility they patronize, is the main subject of this contribution. The next section will provide a framework for this decision. At this point, suffice it to say that while many, or even most, papers use the "patronize the closest facility" (or cheapest, in case prices are different and mill pricing is assumed), other models have been suggested. For instance, some models include a (single-dimensional) parameter that measures the attractiveness of a facility in contrast to other, competing facilities. Furthermore, an important and fairly recent strand of research uses probabilistic choice rules, according to which customers at the same location do not all behave in the same way. Similarly, it is able to capture the fact that a customer, even if he and all of the competing facilities remain in the same positions, will not always patronize the same facility.

14.3 Consumer Behavior in Competitive Location Models

Consumer behavior is one of the most important aspects in any user-focused models, yet it is crucial to many such models. Some references are Raiport and Sviokla (1994), who identified content, context, and infrastructure as major determinants of customer behavior, Song et al. (2001) and Giudici and Passerone (2002), who use data mining in their analyses of identifying changes in consumer behavior, and Liou (2009), who presents decision rules that foster customer retention in the airline industry.

The three-stage process below presents a decision-making framework that customers use when making their choices. We will discuss the individual stages and demonstrate how they encompass the rules and assumptions made in the literature.

Stage 1 is the *evaluation stage*. In it, customers determine utilities to each of the stores. For the purpose of this paper, we assume that customers actually have complete and correct information, an assumption that may be justified by Internet searches or similar fact-finding processes, together with past experience with the facilities. The utilities created in this stage will be based on all components that typical customers deem important. In the retail context, this may include, but not be restricted to, the price charged at the facility, the distance to the facility, the parking at the facility, the friendliness of the staff, and others. Formally, we can define u_{ij} as the utility a customer located at site i (for simplicity, we will refer to "customer i") associates with goods or services at a facility at site j (called "facility j" for short). Furthermore, we define d_{ij} as the distance between customer i and facility j, while t denotes the unit transportation cost, i.e., the conversion from distance to money. We also need to define p_j as the price charged by facility j, and the basic attractiveness A_j of facility j. The basic attractiveness is a composite parameter that includes different measures, such as floor space of a retail establishment (as a proxy expression for variety), the quality of service, and other features. It is not important to find an exact aggregate measure, it is only important to find an expression that captures the differences between facilities. For simplicity, we will restrict ourselves to a single homogeneous product, such as a brand that can easily be compared between facilities. As an aside, some firms make such comparisons difficult by assigning different model numbers to the same product, one for department stores, and a different one when it is sold through specialty retail outlets.

The simplest (deterministic) utility function is

$$\text{UD1a} : u_{ij} = -td_{ij},$$

i.e., the utility of customer i regarding facility j equals the negative distance between them. Hence, maximizing the utility, such a customer will patronize the facility closest to him. Such a utility function has been used by early contributors, such as Lerner and Singer (1937), Eaton and Lipsey (1975), and later by operations researchers such as Hakimi (1983), ReVelle (1986), Serra et al. (1999a, b).

An extension is the utility function

$$\text{UD1b} : u_{ij} = -p_j - td_{ij}.$$

Maximizing such a utility is equivalent to minimizing the full price of the good, i.e., the mill price plus the transportation costs. Hotelling's own contribution falls into this category, and so do the papers by Serra and ReVelle (1999) and Pelegrín et al. (2006). Note that the utility UD1a is a special case of the utility UD1b with zero prices (or prices that are equal at all existing facilities).

Consider now the utility function

$$\text{UD1} : u_{ij} = R_i - p_j - td_{ij},$$

where R_i denotes the *reservation price* customer i assigns to one unit of the good in question, an upper bound customers are prepared to pay for one unit of the good. Given that, the utility is an expression of the amount of money that the customer "saved," i.e., the amount that he was prepared to, but did not have to, spend on a unit of the product. Some authors refer to R_i as the valuation of the product, other refer to it as income, while still others think of it as the budget. In all cases, $R_i - p_j - td_{ij}$ is an expression of the money that was available for the purpose, but did not have to be paid for the product. It is apparent that the utility functions UD1a and UD1b are special cases of the function UD1: Given equal reservation prices $R_i = R_k$, $i \neq k$, maximizing the utility UD1 reduces to UD1b, which, in turn, reduces to UD1a for fixed and equal prices p_j. One important feature of the utility function UD1 is that when the utility u_{ij} is nonpositive, it allows customer i to refrain from making any purchases.

Finally, there exists a variety of other deterministic utility functions used by some authors. Among them is Lane (1980), who uses a Cobb-Douglas-style function that expresses the utility as the product of three components: a measure of a characteristic raised to a power, another measure of the facility raised to some power, and the available income of the individual. Neven (1987) frames his discussion in the context of brand positioning, and his utility function is the difference between a (very high) reservation price, and the price plus the square of the customer-facility distance (which, in this context, is actually the difference between the customer's ideal point and the actual feature of the product). Finally, Kohlberg (1983) uses a utility function that includes the sum of travel time and waiting time, a utility that is important in the context of facilities that feature congestion, such as health-care facilities. Such a utility function can be written as

$$\text{UD1c} : u_{ij} = R_i - p_j - td_{ij} - W_i,$$

where W_i denotes the waiting time. One pertinent example in the context of health services is found in Marianov et al. (2008).

Another utility function incorporates not only distances, which are present in all spatial models—after all, they are what makes a model "spatial"—but also the "attractiveness" of the facilities. As already briefly alluded to above, this one-dimensional measure attempts to capture differences between facilities the way they are perceived by customers: floor space as a proxy for selection (even though the models under consideration just deal with a single homogeneous good), friendliness of staff, parking, lighting, temperature, cleanliness of the facility, and many others. A simple utility function that incorporates the basic attractiveness of facility j as the parameter A_j is

$$\text{UD2a} : u_{ij} = \frac{A_j}{d_{ij}^{\lambda}}$$

with some decay parameter λ. For $\lambda = 2$, the relation reverts to the well-known gravity model, first proposed by Reilly (1931) for the determination of trading areas. This function has been used by authors, such as Aboolian et al. (2007), Drezner and Drezner (1997), Eiselt and Laporte (1991), and Suárez-Vega et al. (2014), the last using the slightly more general function "basic attractiveness divided by some increasing continuous function of distance." Clearly, given the absence of prices, these models assume that prices are fixed and equal among facilities.

An alternative treatment that involves an attractiveness parameter is

$$\text{UD2b} : u_{ij} = A_j e^{-\beta d_{ij}}$$

with some parameter $\beta > 0$ that indicates the customers' sensitivity to differences in distances. Aboolian et al. (2008) use a function of this type, but go one step beyond: their base attraction A_j is a negative exponential function of the price charged at the facility.

Consider now utility functions that include probabilistic components. There are considerably fewer probabilistic location models than there are deterministic models. The probabilistic counterpart of the above deterministic function UD1 is

$$\text{UP1}: u_{ij} = R_i - p_j - td_{ij} + \varepsilon_i \mu,$$

where ε_i is, usually, a Weibull-distributed random variable, while μ is typically interpreted as a coefficient of heterogeneity of customer tastes.

On the other hand, a probabilistic version of the utility function UD2a is

$$\text{UP2}: u_{ijk},$$

defined as the utility a customer at site i has for feature k of facility j. This multidimensional version of the attraction function leads to the probabilistic allocation rule AP1 defined below.

Stage 2 in the decision-making process involves the *allocation* of a customer's demand. The most natural thing to use would be the deterministic allocation rule

$$\text{AD1}: \text{winner} - \text{take} - \text{all},$$

which allocates all of customer's demand to the facility he is most attracted to. Most of the contributions in the literature follow this rule. Actually, if the utility function is assumed to include all of a customer's wishes, this rule would be the only logical choice. However, even when considering a single customer, he may opt logically for a facility that is second-best or has an even lower ranking based on its utility. The reason could be that the customer, having patronized on facility, wants some variety, even though it is probably not as good. Alternatively, if a customer point represents actually a group of customers (meaning that customer i is actually an aggregate, typically of a census tract or some other group of customers), some members among the group may have different rankings and prefer what, on average, is a higher-ranking facility.

This heterogeneity of customer tastes can be dealt with in different ways. One such possibility is to use a

$$\text{AD2}: \text{proportional allocation.}$$

This allocation rule will allocate a customer's demand according to the relative utility a customer has for a facility. For instance, the proportion of customer i's demand to facility j according to Hakimi's (1990) "proportional" rule equals $u_{ij} / \sum_k u_{ik}$. As an example, if a customer faces a duopoly, for whose facilities he has computed utilities of 3 and 7, respectively, he will satisfy 30% and 70% of his total demand at the two respective facilities. Hakimi (1990) also designed a hybrid rule based on AD1 and AD2. He refers to it as a "partially binary" allocation. According to this rule, customers consider only the closest facility or branch of each of the competing firms, and they then distribute their demand proportionally among those branches. Suárez-Vega et al. (2004) investigated AD1, AD2, and the aforementioned hybrid in detail.

Consider now probabilistic allocation functions. A natural extension of Reilly's (1931) argument of attraction functions was Huff's (1964) allocation function, which allocates a proportion of a customer's demand to a firm based on the firm's attractiveness and its distance to the customer,

$$\text{AP1a} : p_{ij} = \frac{A_j/d_{ij}^{\lambda}}{\sum\limits_k A_k/d_{ik}^{\lambda}}$$

Huff suggested the selection of a location from a pre-specified set of locations, whereas Drezner (1994a, 1995) proposed a model for finding the best location anywhere in the plane. A multidimensional generalization of this idea was proposed by Nakashani and Cooper (1974), the so-called multiplicative competitive interaction model, or *MCI* for short. Assuming that u_{ijk} denotes the utility customer i has for feature k of store j, let p_{ij} denote the probability that a customer at site i makes a purchase at store j. The parameter α reflects how sensitive is p_{ij} to feature k. The *MCI* model then asserts that

$$\text{AP1} : p_{ij} = \frac{\prod\limits_k u_{ijk}^{\alpha_k}}{\sum\limits_j \prod\limits_\ell u_{ij\ell}^{\alpha_\ell}}.$$

Following the arguments of McFadden (1974), the use of the probabilistic utility function UP1 leads to the demand allocation rule

$$\text{AP2} : \quad p_{ij} = \frac{e^{(R_i - p_j - td_{ij})/\mu}}{\sum\limits_k e^{(R_i - p_k - td_{ik})/\mu}}.$$

Note that whereas any of the deterministic utility function could be followed by any of the allocation functions, the allocation function AP2 is a direct consequence of the utility function UP1.

Finally, in the third stage in the decision-making process, customers determine the quantity that they are going to purchase from the chosen facility/facilities. Most authors opt for the quantity choice rule

$$\text{Q1} : \text{fixed},$$

in which the quantity customers purchase is fixed. This is typically justified by asserting that the good in question is essential. While such an assumption is convenient, there are actually relatively few essential goods in real life: butter can be replaced by margarine, private transportation can—at least within reason—be replaced by public transportation; potatoes could be replaced by pasta, and so forth. Yet, true essential goods exist, such as electric power (which cannot be replaced in the short run), or medical care. Typical examples for the use of this rule include almost all contributions in the literature, starting with Hotelling (1929), Eaton and Lipsey (1975), and d'Aspremont et al. (1979) to Drezner and Drezner (1997), Fernández et al. (2007), Braid (2013), and others.

A very general alternative rule is

$$\mathrm{Q2}: q_{ij} = f\left(p_j + td_{ij}, u_{ij}\right),$$

where q_{ij} denotes the quantity customer i purchases at facility j. This rule states that the quantity that customer i purchases from facility j is a function of the full price to be paid for purchases at that facility and of the utility customer i achieves from purchases at facility j. While a customer's utility is likely to include the full price as one of its components, the quantity purchased by a customer is often assumed to depend on the (full) price of the product, rather than on a customer's utility. The early contribution by Rothschild (1979) uses a negative exponential distribution to relate a customer's demand and the customer-facility distance, while Aboolian et al.'s (2008) work includes not only distance, but also price, in their negative exponential relation. The contributions by Penn and Kariv (1989) and Matsumura and Shimizu (2006) assume respectively that the demand at a point is the difference between a constant and the travel distance, and the difference between a constant and the price paid for the product. Both cases are designed so as to express the amount of money a customer has left over after his purchase.

Once customers have gone through the three stages of their decision-making process, they have decided how much to purchase and whom to purchase it from. This can then be used as input by the competing planners of the facilities. Drezner et al. (1996) analyzed an anomaly in the decision making process that occurs if customers reevaluate their purchasing decision along the way to the chosen facility. The authors also delineated areas in which this phenomenon occurs.

14.4 Results for Different Behavioral Assumptions

This section is organized along the lines of customer choice rules outlined in the previous section. Each subsection will examine one customer choice rule, given a specific space in which customers and facilities are (going to be) located in, and the type of solutions that are investigated, *viz.*, followed by results in the literature regarding Nash equilibria, and von Stackelberg solutions. To avoid too much fragmentation, we will list those contributions that deal with some discrete space under the header "plane."

14.4.1 UD1a, Linear Market, Nash Equilibria

Stevens (1961) appears to have been the first to use game theory to reestablish Hotelling result of minimal differentiation for fixed and equal prices. Recognizing the complexity of the problem described in Hotelling's (1929) paper, some contributors decided to simplify matters. Eaton and Lipsey (1975) used fixed and equal prices. While this assumption appears somewhat contrived, it is usually justified by legislated pricing for essential goods. With this assumption, customer choice rule UD1a (the "closest" rule) is applied. Given this assumption, Hotelling's result of minimal differentiation is reestablished, as by moving towards its opponent, a firm gains customers in the competitive region and does not lose customers in its hinterland. The authors also extend the analysis to more than two firms. In particular, they determine that for more than five firms, multiple equilibria exist, and the only case without equilibria is the instance with three facilities. In particular, the two outside facilities will push inwards so as to gain additional market shares, thus squeezing the market of the inside firm to zero. This firm will counteract by "leapfrogging" to the outside, become an outside facility itself, and start moving inwards. Teitz (1968) referred to this behavior as "dancing equilibria." Shaked (1975) investigates the usual Hotelling model with fixed and equal prices, but three facilities that employ mixed strategies. It turns out that an equilibrium exists, in which all facilities randomize their strategies in the central half of the market.

In a follow-up paper, Shaked (1982) investigates the Hotelling model with three firms locating one facility each, with fixed and equal prices, allowing mixed strategies. It turns out that all firms will chose locations in the central half of the market with equal probability. Cancian et al. (1995) consider a Hotelling model with directional constraints, i.e., customers can only walk in one direction towards the firm they want to patronize. The authors determine that with random arrival times of the customers and two or more facilities, no equilibrium exists.

14.4.2 UD1a, Linear Market, von Stackelberg Solution

The first author to introduce sequential (and final) location decisions into the discussion appears to have been Hay (1976). However, it was the contribution of Prescott and Visscher (1977) that popularized the methodology and the results. In one of their examples, the authors look at a duopoly on a linear market—the simplest possible case—and determine that the leader will locate at the center of the market, while the follower will locate next to the leader, thus resulting in central agglomeration. The authors then extend their analysis to the case of three firms. After considering many cases and subcases (see, e.g., Younies and Eiselt 2011), it is determined that one of the outcomes (arguable the most likely one) is that the three facilities locate at $^1/_4$, $^3/_4$ and $^1/_2$ of the market, capturing $^3/_8$, $^3/_8$, and $^1/_4$ of the market, respectively. The fact that the first two facilities to locate earn 50% more than the last entrant into the market is, however, troublesome: having established that it takes capability and incentive to be a leader (see, e.g., Younies and Eiselt 2011), we can consider the second and third firms to enter the market as followers. However, why would any follower accept being the third rather than the second entrant, if the latter course of action is much more profitable? A similar result had already been obtained by Teitz (1968), who considered duopolists, so that the location leader would locate two facilities, while the location follower would locate a single facility. He suggested “conservative optimization,” i.e., a minimax strategy. While the leader locates his two facilities at ¼ and ¾ of the market, the follower will locate his single facility anywhere between the leader’s facilities.

An interesting extension is provided by Thisse and Wildasin (1995), who locate private facilities alongside a centrally located public facility. Households have incomes, which they spend on trips to the facilities and paying land rent. In the first stage of the game, all firms locate, followed by stage two, in which customers locate. The result is that high travel costs yield maximal differentiation, while low travel costs result in minimal differentiation. Bhadury (1996) considers a Hotelling model on the line with fixed and equal mill prices, in which the leader does not have perfect information regarding the follower’s variable costs. For a general demand distribution, the author shows that market failure is possible (i.e., the leader may not wish to locate any facilities) and that a greedy strategy is not bad (optimal for an atomistic leader, i.e., one who wishes to locate only a small number of facilities). Osborne and Pitchik (1986) allow the demand distribution to be not necessarily uniform. Allowing mixed strategies, the result for a three-firm problem has all three firms randomize over the central half of the market. Dasci and Laporte (2005) allow facilities to have different cost functions. The paper is novel in that it does not deal with exact facility locations, but with the density of retails branches that are located.

14.4.3 UD1a, Plane, Nash Equilibrium

In two-dimensional space, Okabe and Aoyagi (1991) attempt to prove a conjecture by Eaton and Lipsey (1975) in the two-dimensional plane. With fixed demand and equal mill prices, customers patronize the closest facility. In the infinite two-dimensional plane with Euclidean distances and an infinite number of independent firms, the market area of each of the firms is a cell in a Voronoi diagram. Each firm attempts to maximize the area of its Voronoi cell. The global equilibrium is reached when Voronoi cells form a regular hexagonal pattern. It is noted that results in one-and two-dimensional spaces are markedly different: the pairing in one dimension does not carry over to the two-dimensional plane. Another attempt in the two dimensional plane was reported by Okabe and Suzuki (1987). The authors use the same concept as in the previous paper, but locate finite numbers of facilities (32–256) in a bounded market the shape of a square. Global optimization techniques are sequentially and repeatedly applied. The result is a honeycomb-type pattern that, however, self-destructs again and rebuilds. The instability is likely to be the result of "boundary effects" that distort the results.

Aoyagi and Okabe (1993) consider a Hotelling model in the plane with totally inelastic demand, identical facilities, and customers who purchase the good from the closest facility. Customers are assumed to be located in a compact and convex subset Z of the two-dimensional Euclidean plane. The authors demonstrate that for $n = 2$, an equilibrium exists if and only if the market is point-wise symmetric with respect to some point in Z. The firms will then locate at that point. For three facilities, no global equilibrium exists, except maybe in the case of a equilateral triangle.

14.4.4 UD1a, Plane, von Stackelberg Solution

The first author to discuss competitive location problems in the plane given location leaders and followers appears to have been Drezner (1981, 1982). His contribution first considers the simple case, in which each firm locates a single facility in the presence of n demand points. The follower's best location is arbitrarily close to that of the leader. The sorting of angles from the leader's point to the demand points yields an $O(n \log n)$ algorithm for the follower's problem. The leader's problem (given he locates one facility and expects the follower to do the same) is shown to be solvable in $O(n^4 \log n)$ time. In case a minimum separation of some prespecified distance R is required between leader and follower, the complexity of the two problems is still $O(n \log n)$ and $O(n^5 \log n)$, respectively. Other cases include the problem in which the leader locates one facility, and the follower locates $r > 1$ facilities. This problem is easy: the leader is wedged in and his optimal strategy is to locate right on the point with the largest demand, as that is all he will get. If the leader locates $p > 1$ facilities and the follower locates one facility, then the follower's problem can be solved in $O(n^2 \log n)$ time.

Shigehiro et al. (1995) consider a duopoly with firms A and B in a bounded subset of the two-dimensional plane. Given fixed and equal prices, both firms are market share maximizers. Given demand at grid points and the one of A's two facilities being already located, firm B locates a single facility, followed by firm A locating its second facility. It turns out that firm A will locate its second facility next to it competitor's facility, thus re-establishing the pairing of facilities known from one-dimensional markets. An algorithm for the centroid problem is also described. Infante-Macias and Muñoz-Perez (1995) discuss medianoid locations in the plane with customer demand occurring at discrete points, and Manhattan distances are used. A given parameter specifies how much closer a new facility must be to a customer to be considered comparable, i.e., equally desirable. For the location of a single new facility, the paper describes an $O(n^3)$ algorithm, for a given number p of new facilities, an $O(n^5)$ algorithm is suggested. Following the asymmetry of objectives already mentioned by Eiselt and Marianov (2017), Gentile et al. (2018) consider three scenarios, each with a specific combination of objectives by leader and followers, in a discrete space. Pelegrín et al. (2015) explore the effects of tie-breaking rules in customer choice on the solutions, while Santos-Peñate et al. (2017) suggest a heuristic for the solution of the centroid problem. Seyhan et al. (2018) consider the leader-follower problem in a discrete space and, in order to make the reaction function of the follower more tractable, suggest a greedy heuristic for that purpose. Xue et al. (2017) have the follower maximize his revenue, but allow the total demand to increase in case additional facilities locate. Zhang et al. (2016) study the usual leader-follower model, but include the possibility of disruptions of service. Finally in this category, a number of authors study competitive hub location problems, as there are Sasaki et al. (2014), Mahmutogullari and Kara (2015), Niknamfar et al. (2017), and Ghaffarinasab et al. (2018).

14.4.5 UD1a, Networks, Nash Equilibria

Bhadury and Eiselt (1995) investigate duopoly models with fixed and equal prices on tree networks. They describe locational Nash equilibria for the cases where co-location (i.e., the location of both facilities at the same node) is permitted or not, and they describe a measure of stability of the equilibrium, rather than applying the usual equilibrium-no equilibrium dichotomy. In another paper, the same authors (Eiselt and Bhadury 1998) discuss the reachability of Nash equilibria (assuming that at least one such equilibrium exists) on trees. Starting with arbitrary locations of the duopolists, they apply sequential and repeated short-term optimization to investigate whether or not an equilibrium will be reached. The answer is that it will, provided an appropriate tie-breaking rule is employed. Eiselt and Laporte (1993) describe conditions, under which a three-facility problem on a tree has agglomerated, dispersed, and no equilibria.

14.4.6 UD1a, Networks, von Stackelberg Solution

Among the early contributions, Slater's (1975) work stands out. The author introduces leader and follower, respectively, but does not make the connection to von Stackelberg's work. The paper proves that on a tree network, the leader will locate at the median. In his contribution, Hakimi (1983) first introduces von Stackelberg games by referring to the locations of the leader(s) of the sequential game as *centroids* (based on their maximin objective), while the locations of the follower(s) are termed *medianoids* (as their objective is of the "minisum" type). In particular, if the leader has already located p facilities in a pattern denoted by X_p, and if the follower is poised to locate r facilities, the follower's problems is an $(r|X_p)$ medianoid. On the other hand, if a leader wants to locate p facilities, assuming that the follower will locate r facilities, we talk about an $(r|p)$ centroid. Hakimi discusses a number of results of special cases regarding the node property, i.e., the question whether or not at least one optimal location pattern naturally has locations at the nodes of the given network. In addition, he proves the *NP*-hardness of $(r|X_1)$ medianoid of general networks as well as the *NP*-hardness of the $(1|p)$ centroid. In the same year, Megiddo et al. (1983) show a polynomial $O(n^2 r)$ algorithm for the $(r|X_p)$ medianoid problem on trees. Benati and Laporte (1994) devise a tabu search algorithm for the solution of these difficult problems. Penn and Kariv (1989) require facilities to be located at the nodes of the tree, but allow a customer's demand to be linearly decreasing in the distance to the closest facility. Both firms are assumed to locate a single facility. Characterizations of the solutions, especially with respect to the median(s) of the tree are described. Hansen and Labbé (1988) present a polynomial algorithm for the $(1|1)$ centroid problem on tree networks. García Pérez and Pelegrín (2003) follow the analysis of Eiselt (1992) and determine all von Stackelberg solutions on a tree with parametric, but possibly different, prices. They also discuss the "first entry paradox" (see Ghosh and Buchanan 1988), according to which the leader in a von Stackelberg game would typically have the advantage.

ReVelle (1986) was the first to formulate the highly influential MAXCAP problem on networks, i.e., the problem, in which the follower locates facilities. By modifying the objective, he reduced the formulation to a *p*-median problem. In follow-up papers, Serra and ReVelle (1994, 1995) present the PRECAP problem that solves the leader's $(r|p)$ centroid problems. The authors design heuristic algorithms for the (bilevel) problem of the leader, and report computational experience. The main contribution in the Hakimi (1990) book chapter is the introduction of three allocation rules: binary (i.e., winner-take-all), partially binary (a customer distributes his demand proportional to the inverse distances to the closest facilities of the two firms), and the (fully) proportional rules, in which customers allocate their demand inversely proportional to the distances to the facilities. The author also presents results with these allocation rules with respect to the node property. Suárez-Vega et al. (2004) expand on Hakimi's discussion of the three allocation rules for essential and unessential demand at the nodes of the network. The authors also derive finite dominating sets, including those for concave capture functions. The work by Serra et al. (1999a, b) discusses the MAXCAP problem with different rules for the location of the entering firm. The rules, both of which belong to the class of proportion models, are based on different assumption concerning customer behavior.

Serra et al. (1999a, b) discuss the usual MAXCAP problem, but with an additional constraint that ensures that each facility has at least a market share of a certain size. This is done so as to guarantee the viability of the firm. Some computational testing is provided; the rule checks viability first and then locates and reallocates demand; if any store is not viable at this point, the one with least demand is deleted. This process is repeated until it converges. To solve the problem, heuristic concentration is the method of choice.

Spoerhase and Wirth (2008) tackle the notoriously difficult problem of $(r|p)$ centroids. In order to obtain any results (as Beckmann 1972 stated: "As everyone knows, in location theory one is forced to work with simple assumptions in order to get any results at all"), they restrict themselves to paths and trees. Along similar lines, Eiselt (1998) investigates a von Stackelberg problem on a tree, given that the perceptions of leader and follower regarding the demands at the nodes are different. Solutions to the bimatrix game (in which each player has full knowledge about the perception of his opponent) and the hypergame (in which neither competitor knows about the perception of his competitor) are characterized. In general, if a firm can assume that its competitor has researched the demand diligently, it can gain little by finding out about the exact perception of its competitor. Marianov et al. (1999) extend the MAXCAP to the location of hubs by a follower firm, assuming that passengers choose the airline which offers the shortest route (distance) between their origin and destination. Marianov and Taborga (2001) address the problem of locating public health centers competing with private ones for affluent customers, assuming that the closest center captures the demand. Marianov et al. (2004) extend these results to facilities with waiting lines. Ruiz-Hernández et al. (2017) discuss the case of delocation, "i.e., the possibility of optimally closing facilities. The usual customer choice is applied, except with some degree of loyalty. The authors investigate whether or not the first mover advantage occurs, and also study Nash equilibria.

14.4.7 UD1b, Linear Market, Nash Equilibria

Consider now models that employ the customer choice rule UD1b, i.e., models in which customers patronize the least expensive facility. Hotelling's original model belongs to this group, which, with its linear transportation costs, does not exhibit an equilibrium. This was pointed out by d'Aspremont et al. (1979) who also demonstrated that as soon as quadratic transportation costs are used, an equilibrium does exist with maximum differentiation, i.e., the two facilities locate at opposite ends of the market. Anderson (1988) provided further insight into the case: he demonstrated that for linear-quadratic transportation cost functions, i.e., cost functions that have a quadratic and a linear component, equilibria only exist if there is no linear component and the cost function is purely quadratic. Hamoudi and Moral (2005) extend the analysis and investigate linear-quadratic transportation cost functions with different parameters, which result in convex and concave transportation cost functions, respectively. The authors then define profit functions for the two cases. Because a price equilibrium does not exist for all pairs of locations, the authors delineate pairs of locations for which such an equilibrium does exist. It turns out that the region in which price equilibria exist in the concave case is complete enclosed in the region, in which equilibria exist in the convex case.

Tabuchi and Thisse (1995) analyze Hotelling's model with a quadratic transport cost function and triangular customer density. Again, a subgame-perfect equilibrium is sought. It turns out that no symmetric location equilibrium exists. Instead, asymmetric equilibria exist at $\left(0, \frac{\sqrt{33}-3}{2\sqrt{2\sqrt{33}+2}}\right)$ and $\left(1-\frac{\sqrt{33}-3}{\sqrt{2\sqrt{33}+2}}, 1\right)$, i.e., (0, 0.3736) and (0.2527, 1), given that we restrict facility locations to the inside of the market. Cremer et al. (1991) locate n facilities on a linear market. Given quadratic transportation costs and the usual Hotelling assumptions (including the "first simultaneous choice of location, then simultaneous choice of mill prices"), the model includes m public and $n—m$ private firms. While private firms maximize their individual profits, public firms maximize the social surplus that, with the assumption of inelastic demand, reduces to the minimization of transportation costs. For $n = 2$, one public and one private firm perform best. The two facilities will locate at the social optimum of ¼ and ¾, respectively. For $n = 3$ and one public facility, profits of the private firms are higher and general welfare is lower than in the all-private case. With two public facilities, the social optimum is reached. Some additional combinations of public and private facilities are also investigated.

An important strand of research considers the original Hotelling model, but allows mixed strategies on prices and pure strategies for the location subgame. Among the earlier attempts is the contribution by Osborne and Pitchik (1987), who determine that facilities will locate at about 0.27 away from the ends of the market of unit length. Matsumura and Matsushima (2009) use heterogeneity in the form of different production costs, and if those result in pure strategy equilibria not to exist, then mixed strategy equilibria are used. Location equilibria with minimal and maximal differentiation appear each with probability of ½.

Anderson (1987) showed that in the "first location, then price" two-stage game if facility A were to lead in the first-stage location game, then it would be best for its opponent B to be a leader in the second-stage pricing game. As a result, firm A would locate at the center at the market, while firm B will locate at 0.131 (or, symmetrically, at 0.869). Anderson and Neven (1989) use the usual Hotelling assumptions, including duopolists on a linear market, mill pricing and "first location, then price" competition, but allow customers to purchase goods from both firms according to some loss function and the use of a quadratic transportation cost function. The result is maximal differentiation with the duopolists locating at the two ends of the market. In another contribution, the same authors (Anderson and Neven 1991) employ spatial price discrimination in a two stage "first location, than quantity" procedure. The result is an equilibrium with minimum differentiation. The authors also demonstrate that for more than two firms, given linear transportation costs and a regularity condition, all firms will locate at the center of the market. Such agglomeration is often observed in practice, see, e.g., Marianov and Eiselt (2016). Hamilton et al. (1989) describe a Hotelling model with spatial price discrimination and a linear price-quantity relation. The authors compare the results of Cournot (i.e., quantity) and Bertrand (i.e., price) competition. Throughout, Cournot prices are higher than those in Bertrand competition, and aggregate welfare (i.e., total surplus—total transport costs) is higher under Bertrand than under Cournot.

Anderson et al. (1997) drop the assumption of uniform demand and consider logconcave demand functions, coupled with quadratic transportation costs. It turns out that if customers are more spread out, prices are higher, and that symmetric demand densities lead to symmetric locations of firms. Bester et al. (1996) reexamine d'Aspremont et al.'s (1979) Hotelling game without coordination (firm A is assumed to locate to the left of firm B) and allow mixed strategies. An infinite number of mixed-strategy Nash equilibria exist, and without coordination, the result of maximum differentiation is invalidated. Eaton (1972) follows Smithies (1941) by considering a model that includes a linearly sloping price-demand function. The author also uses a modified zero conjectural variation assumption, according to which a firm will react unless undercut. In case of a short market, the result will be agglomeration of the firms, as the length of the market grows, duopoly locations approach the social optimum. Behavior in case of a triopoly is similar: as the length of the market grows, agglomeration forces get weaker. The paper by Kohlberg and Novshek (1982) examines a similar model. Eiselt and Marianov (2017) determine the line between existence and nonexistence of locational Nash equilibria for location problems with asymmetries, such as those with different transportation costs, different production costs, and those that have different objective functions. The reference also investigates von Stackelberg solutions for these problems. While the work by Eiselt and Marianov (2017) focuses on asymmetric competitive location models, Colombo (2016) investigates equilibria on a linear market in the presence of three cities given Cournot (i.e., quantity) and Bertrand (i.e., price) competition.

There are a few contributions that examine spaces similar to a line: Eaton's (1976) model allows free entry on a circle, Kats's (1995) model locates duopolists on a circular market, whereas Tsai and Lai (2005) investigate the case of a market, in which customers are distributed along the sides of a triangle, and Braid (1989, 2013) looks at the case of intersecting roadways, i.e., intersecting lines.

14.4.8 UD1b, Plane, Nash Equilibria

Hurter Jr. and Lederer (1985) appear to have been among the few investigators to look at the subgame-perfect Nash equilibrium on the plane. Their contribution includes different cost functions for the firms and transportation costs that are proportional to Euclidean distances. Firms are supposed to locate in a given convex set. The authors show that there are no peripheral equilibrium locations. They also demonstrate that the locations that minimize the social costs for serving the entire market are a proper subset of equilibrium locations. Similarly, Tabuchi (1994) locates two firms in the two-dimensional space and uses quadratic transportation costs. The paper determines that for any convex set, there are no interior locational Nash equilibria. The author then shows that in a rectangle, Nash equilibrium has the facilities locate on opposite sides of the rectangle at their respective midpoints. If the rectangle is very long, the Nash equilibrium is unique.

This is not the same as d'Aspremont et al. (1979) result. While this result shows maximum differentiation in one direction, it has minimum differentiation in the other. Lederer and Hurter Jr. (1986) consider customers located in a subset of the two-dimensional plane with some typically nonuniform demand distribution and firms facing different production and transportation costs. Firms use spatial price discrimination and customer purchase goods from the cheapest source (a number of tie-breaking rules are specified). The resulting "location, then price" game has an equilibrium, and it is shown that identical firms (i.e., those with different production and transportation costs) do not co-locate. The analysis is then extended to nonidentical forms that locate on a disk, and again, there is no co-location. The model by Fernández et al. (2014) is the usual two-phase "first location, then price" game with delivered pricing, for which the authors demonstrate that a price equilibrium exists, which reduces the game to a pure location game. The paper then describes a branch-and-bound approach for small to medium problems and a heuristic for larger problems. Rohaninejad et al. (2017) present two models, in which firms maximize profits, and minimize the maximum deviation from the highest possible profit, respectively. Computational evidence is provided.

14.4.9 UD1b, Networks, Nash Equilibria

Lederer and Thisse (1990) examine a competitive network location model, in which firms determine their respective locations and chosen technologies in stage 1, and the prices in stage 2. The authors use spatial price discrimination. In the usual backward recursion, the paper proves that for all first stage location and technology choices, the second stage pricing game has an equilibrium. The socially optimal location and technology choices of the first stage are also a Nash equilibrium. However, locational Nash equilibria may exist that are not socially optimal. An important feature is that if the transport cost function is concave, then the equilibrium locations will satisfy the node property. Labbé and Hakimi (1991) also use delivered pricing and, in addition, a linear price-quantity relation. The two-stage game locates facilities in stage 1, and determined quantities in stage 2. It turns out that for any fixed pair of locations, the quantity game has an equilibrium. If it is required that it is always profitable to supply any market of the graph with a positive quantity of goods, then a location equilibrium exists at the nodes of the graph. If this condition is not satisfied, then either a locational Nash equilibrium does not exist, or it exists on the edges of the graph. The paper by Berglund and Kwon (2014) has a von Stackelberg firm competing with Cournot-Nash firms given capacities at the facilities. Equilibrium results are presented and the computational method of choice is a simulated annealing heuristic.

14.4.10 UD1, Linear Market, Nash Equilibria

Among the earliest papers to follow Hotelling's lead is the work by Lerner and Singer (1937). The authors keep Hotelling's linear market and the assumption on linear transportation costs, but introduced a finite reservation price, and assert that each firm assumes that its competitor's location and price is fixed, and a firm only reacts if undercut. In such a case, equilibria do exist. The authors also extend their analysis to spatial price discrimination, which results in social optima. The contribution by Economides (1986) is most interesting, as it includes Hotelling's

(1929) and d'Aspremont et al.'s (1979) results as special cases. The utility function includes a budget and the utility inherent in the product. The transportation costs are the facility—customer distance raised to some power α. The main result is that for α less than about 1.26 (which includes Hotelling's original case with $\alpha = 1$), no subgame-perfect Nash equilibrium exists, whereas for α greater than about 1.26, it does exist (which includes d'Aspremont et al.'s case of $\alpha = 2$). More specifically, for $\alpha \in [1.26, 1.6667]$, the equilibrium locations are strictly interior, while for $\alpha \geq 1.6667$, they are at the endpoints of the market.

Zhang (1995) discusses the case of a duopoly with quadratic transportation costs and reservation prices, in which decision makers make their decisions in three phases: locate first, then decide whether or not to adopt a price-matching policy, and then determine the price. The paper shows that if both players use price matching, high reservation prices lead to a unique Nash equilibrium "with tacit collusion on prices." Equilibrium locations for high reservation prices lie at the center of the market (minimum differentiation). Not surprisingly, they find that price matching reduces price competition. The paper of Smithies (1941), which has spawned many followers, discusses a Hotelling model with elastic demand and reservation prices. The author appears to have been the first to use "push" and "pull" forces (see also Eiselt and Laporte 1995). He also found that higher transportation costs lead to less competition, and as unit transportation costs increase, firm will move farther apart. Finally, the interesting contribution by Guo and Lai (2014) adds an online dealer to the brick-and-mortar duopolists. While customers purchasing from the latter, face the usual transportation costs, consumers who deal with the online firm have a waiting inconvenience cost. The authors demonstrate that an equilibrium does indeed exist given a relation between the unit transportation costs and the unit inconvenience cost. In Guo and Lai's (2017) simultaneous location-and-price game on a linear market, firms face a non-uniform distribution of demand. Another feature is the inclusion of an online e-tailer. The long run will see the brick-and-mortar retailers more densely agglomerated than without the online competition, and they will serve with urban population, while the e-tailer will specialize in the rural population.

14.4.11 UD1, Linear Market, von Stackelberg Solution

Bonanno's (1987) model examines location, which an incumbent can use to deter future entry of competitors. His model uses quadratic transportation costs, fixed setup costs for new stores and finite reservation prices. The proposed three-stage procedure has the incumbent decide how many stores to open, followed by the potential entrant who must decide whether or not to enter and, if so, where to locate his store (the choices of the follower are limited to zero or one store as to ensure tractability), followed by price competition. Given high setup costs, the leader is a monopolist and further entry is blocked. For moderate setup costs, the incumbent locates two stores at the social optimum, and entry is deterred. For even lower setup costs, entry can no longer be deterred by the incumbent.

Meza and Tombak's (2009) model uses uniform distribution, "sufficiently high" reservation prices, quadratic transportation costs, and potentially different production costs. The paper suggests a three-stage model, in which timing (of entry), location, and price are determined. The low-cost firm is the leader. It is possible for a higher-priced firm that is driven from the market, to re-enter at a later stage. With a small difference in costs, firms enter the market immediately with maximal differentiation. For a somewhat larger cost difference, the low-cost leader enters immediately, soon followed by the higher-cost firm, still maintaining maximal differentiation. For an even larger cost difference, the low-cost leader locates at an interior point, followed by its competitor that locates as far away as possible from the leader. With a very high cost difference, the low-cost leader locates at the center of the market and effective blocks all further entry.

14.4.12 UD1, Plane, Nash Equilibria

The paper by Irmen and Thisse (1998) considers a duopoly in d-dimensional real space with weighted squared Euclidean distances. Customers have a utility function that includes a reservation price, the product's price, and the sum of weighted distances between customer and the firm (the customer's ideal point and the product features, as this model is discussed in feature space). The key result is that if there is a main characteristic of the product, then there is a unique equilibrium in the location game, in which the two products exhibit maximum differentiation in that feature, while otherwise being identical. The authors cite an interesting application of their result in the news magazines *Time* and *Newsweek*, whose main difference is in the cover story. The similarity of this result and that by Tabuchi (1994) should also be noted.

14.4.13 UD1, Plane, von Stackelberg Solution

Panin et al. (2014) uses price discrimination in a sequential "first location, then price" game. Customers in their model have reservation prices and firms are assumed to have budget constraints. While the Phase 1 location competition uses the standard leader-follower concept, the Phase 2 pricing game searches for Nash equilibria. The work formulates the problem as a bilevel optimization problem and devises heuristic algorithms of the "alternating" type to solve the problem. Customers in Kononov et al. (2018) have a budget and finite demand. The study concentrates on complexity results and solvable cases.

14.4.14 UD2a, Linear Market, Nash Equilibria

The contribution by Eiselt (1991) appears to have been the first to use attraction function of the type “facility attractiveness divided by an increasing function of distance” for the purpose of locating competitive facilities. It is shown that as long as the weights are unequal, no equilibrium exists. The author then allows repeated sequential relocation. It turn out that facilities shuttle but converge towards fixed points whose location depends exclusively on the weights: if weights are similar, the fixed points are close to center, otherwise they are close to the boundaries of the market. The paper then introduces fixed and variable relocation costs, which are subsequently used to force an equilibrium.

14.4.15 UD2a, Plane, von Stackelberg Solution

This special field has been very active in the last few years. Earlier work by Drezner (1994b) locates a single new facility in the Euclidean plane with a winner-take-all allocation rule. For each customer, the paper determines a circle around the customer location, so that any facility located inside that circle will capture the customer. Such circles are then constructed for all customer points. This is then used to optimally locate a new facility with given attraction. The contributions by Fernández et al. (2017a, b) both investigate the effects of different choice rules have on locational patterns. In particular, they “rediscover” Hakimi’s (1983) binary and partially binary choice rules and solve the resulting problems with branch-and-bound methods and heuristics, respectively. Hendrix (2016) includes different costs for leader and follower as he determines optimal locations and qualities. It turns out that there is no equilibrium in qualities, so that a von Stackelberg solution for qualities is determined. Qi et al. (2017) apply the usual leader-follower concept, but will serve only customers, if they are within a prespecified distance from the facility. The work by Bagherinejad and Niknam (2018) follows similar lines in that the competitors do not just locate their facilities, but choose qualities of the facilities as well. The model under consideration allows the closing of facilities. Similarly, the contribution by Arrondo et al. (2014) choose locations and qualities and investigates exact and heuristic solution techniques. The papers by Rahmani (2016) and Sadjadi et al. (2016) both allow adjustments of a facility’s attractiveness in addition to its location. The former contribution relaxes part of the problem and uses an exact algorithm to solve the problem, while the latter work uses a methheuristic, which is then applied to some real data.

14.4.16 UD2a, Network, Nash Equilibria

Eiselt and Laporte (1991) investigate the existence of locational Nash equilibria on a tree, given an attraction function of the type facility attraction divided by distance to some power greater than or equal to one. When the base attractions of the facilities are equal, equilibria always exist with either both facilities at the median of the tree (in case co-location is permitted) or with one facility at the median and the other adjacent to it in the largest subtree spanned by the median. For unequal base attractions, if co-location is permitted and the winner-take-all allocation rule applies, then an equilibrium never exists; otherwise (i.e., with co-location permitted and an allocation proportional to the attractions and in case location at the same vertex is prohibited), equilibria may or may not exist.

14.4.17 UD2a, Network, von Stackelberg Solution

von Stackelberg problems in networks enjoy quite some popularity among operations researchers. The main reasons are their relative tractability (the problems can, at least in their basic form, be formulated as integer linear programming problems). This is very much in contrast to the leader's problem, which is a bilevel integer programming problem. Suárez-Vega et al. (2007) employ an attraction function, defined as facility weight divided by an increasing concave function of the distance. Customers purchase proportionally from the facilities they are most attracted to, *provided* they are attracted to them by a measure that exceeds a minimally acceptable threshold. The authors describe a finite dominating set. They deal with the case of a single new facility, but the results generalize to multiple facilities (even though the computations will be more complex). Benati (2003) does not fix the number of facilities the follower is going to locate. Customer behavior is modeled by a function that relates a customer's attraction to a facility to the sum of this customer's attractions to all facilities. This leads to a concave fractional problem, which is solved by a branch-and-bound method and heuristic concentration techniques.

14.4.18 UD2b, Plane, von Stackelberg Solution

Drezner et al. (2015) discuss a model, in which facilities attract customers that are located within a "sphere of influence." Given that the follower will react by maximizing its market share, the leader's objective is to maximize his own market share after the follower has reacted. A summary of leader-follower models in the plane is provided by Drezner and Drezner (2017). Levanova and Gnusarev (2018) consider the follower's problem, in which the follower has a limited budget, which can be used to locate a facility with a given attractiveness. The authors develop an ant colony algorithm (the main piece of this paper), which they use to solve randomly generated instances of the problem.

14.4.19 UD2b, Network, von Stackelberg Solution

Aboolian et al. (2008) investigate a follower problem on a network with an exponential attraction function. In order to capture a customer's demand, the follower must be more attractive than the incumbent by a positive constant. The variable production costs are the same everywhere, and the fixed location costs are location-dependent. Co-location is not permitted. The model is loosely based on work by Serra and ReVelle (1999). The node property does not hold. The authors conjecture that there is a finite dominating set, but are unable to determine it in this nonlinear integer program. Marianov et al. (2008) replace the distance with travel time, and add waiting time as a competitive factor. Shan et al. (2017) consider the follower's location-pricing game with mill pricing and a budget that limits the construction of stores. The lower-level pricing game represents a Nash equilibrium. The proposed algorithm for the follower problem is tabu search, and a numerical example concludes the paper.

Consider now results relating to the probabilistic choice rules introduced in the previous section. Most papers are written by economists, who are mainly interested in the existence of Nash equilibria on a linear market.

14.4.20 UP1, Linear Market, Nash Equilibria

In all of these contributions, the parameter μ can be interpreted as the heterogeneity of the customer tastes with respect to the product under consideration. de Palma et al. (1987a) use fixed and equal prices and unit transportation costs t (in a linear cost function) in their triopoly model. Their main result is that for small values of μ/t, there are no symmetric equilibria. As the value of μ/t increases, there are symmetric dispersed equilibria, a further increase results in dispersed and agglomerated equilibria, while for large values of μ/t, only agglomerated equilibria exist. de Palma et al. (1985) consider the usual "first location, then price" game with a linear transport cost function, and n facilities located on a linear market of length L. The key result is that for large values of μ/tL, there is clustering of the facilities at equilibrium, while small values of μ/tL lead to dispersion. Braid (1988) locates n firms on a line segment, on which the demand occurs at five even spaced the facilities. de Palma et al. (1987b) discuss a duopoly under delivered pricing in their model with linear transportation costs with parameter t. Under sufficient heterogeneity (i.e., $\mu > t/8$), a centrally agglomerated location-price equilibrium exists. The result generalizes to n firms.

Finally in this category, we find the contribution by Anderson et al. (1992), which compares the three main pricing strategies in a duopoly setting. Transportation costs are assumed to be linear, and social surplus is defined as the sum of customer surplus and the profits of both firms. Starting with small values of the heterogeneity factor μ, there is no equilibrium for mill pricing, and as μ increases, there are first symmetric dispersed equilibria, and finally, for large values of μ, there is a unique centrally agglomerated equilibrium. The case of uniform delivered demand just has no equilibrium for small μ, and centrally agglomerated equilibria for larger values of μ, and spatial discriminatory pricing has equilibria everywhere: outside the quartiles for very small values of μ that move towards a central agglomeration for sufficiently large values of μ.

14.4.21 UP1, Plane, Nash Equilibria and von Stackelberg Solutions

Choi et al. (1990) frame their discussion in the context of product positioning. Customers have a stochastic utility function that results in a logit model, and firms maximize their profit. It is known that as long as the profit functions are pseudoconcave, the game possesses a Nash equilibrium. The paper uses variational inequalities to analyze computational aspects. The key contribution is a von Stackelberg game with one leader and multiple followers. The solution of a von Stackelberg game in continuous space cannot be a Nash equilibrium, as is often the case in discrete spaces. The thesis by Tuan (2017) considers the follower problem in a discrete setting and evaluates different probabilistic choice rules.

14.4.22 UP1, Network, Nash Equilibria

de Palma et al. (1989) investigate a very general model, in which n firms compete with each other, and each locates n_i facilities. Customers first choose a firm they want to patronize, and then they patronize the closest facility of that firm. (Note the similarity of this rule and Hakimi's "partially binary" choice rule). The main result is that if consumer tastes are "sufficiently heterogeneous," then firm i will locate its n_i facilities at the n_i-median. If a stronger condition on taste heterogeneity is satisfied, then the resulting pattern—all firms locate their facilities at the n_i-medians—is the unique noncooperative Nash equilibrium. A special case is when all firms have the same number of facilities to locate, in which case all firms will locate their facilities at the same nodes, a case of minimum differentiation.

14.4.23 UP1, Network, von Stackelberg Solution

Benati (1999) discusses a maximum capture problem in the presence of heterogeneous customers. Given fixed demand, fixed and equal prices, as well as p leaders on the market whose locations are known, The paper demonstrates that the follower's objective function is submodular, and that, given appropriate redefining of the problem's parameters, the problem can be formulated as an r-median model. Čvokić et al. (2016) considers leader and follower, who locate their respective hubs. Both firms are profit maximizers. The problem is formulated, and the follower part of the formulation is solved by way of an "alternate" heuristic. Kress and Pesch (2016) also consider the follower's problem. Their formulation of the problem includes conditions for a price equilibrium. The authors then state conditions for the existence of a price equilibrium, followed by *NP*-hardness results, and a method to compute equilibrium prices, and some computational experiments.

14.4.24 UP2, Plane, von Stackelberg Solution

Drezner et al. (2002) discuss a medianoid problem in the plane, in which customers' choices are modeled in probabilistic fashion and are based on attraction functions. The follower's objective is to minimize the probability that the new facility's revenue falls short of a given threshold. The optimal locations tend to markedly differ from those that are the result of the maximization of the expected market share, especially in those cases, in which the probability of failure is relatively small.

14.4.25 UP2, Network, von Stackelberg Solution

The main contribution of the work by Serra and Colomé (2001) is the comparison of various customer choice models. The basic setting includes fixed demand at the nodes of a network, one homogeneous good, and two profit-maximizing firms with identical cost structures. There are presently q facilities on the market. One new firm enters the market and attempts to locate p new facilities. Customer behavior is modeled as follows. Model 1 is the usual all-or-nothing assumption based on the closest facility, while Model 2 is a multiplicative competitive interaction Model Nakashani and Cooper 1974, which assumes that the proportion of demand of customer i captured by facility j equals 1/(customer-facility distance) raised to the power of a parameter that indicates a customer's sensitivity with respect to distance, divided by the sum of such expressions, taken over all facilities. Model 3 is the standard proportional model, and Model 4 assumes partially binary preferences. It turns out that the simple Model 1 appears to be most robust, meaning that it has never more than an 8% deviation from the solution that is based on the correct customer behavior.

14.5 Summary, Extensions, and Outlook

This chapter has described the basic Hotelling model, outlined its major components, described a three-stage procedure that models customer behavior, and has surveyed the literature regarding results of different models. While many different features have been included, most models, which have some explanatory power, lack many facets of customer decision-making.

The most prominent difference between actual and assumed customer behavior involves the customers' trips to the chosen facility. In particular, all competitive models assume that customers make their individual purchases on a *special-single-purpose trip*, while this type of trip appears fairly rare in practice (with the exception of those trips related to work or emergency). However, a significant proportion of trips are multistop or multipurpose, since for some types of products consumers perform comparison shopping, visiting more than one facility selling the same item; or use the same trip to purchase more than one type or good. This is particularly true in a situation with high costs of fuel or long commuting distances.

One alternative is a *planned multipurpose trip with full information*. In such a case, a customer has set out with a plan, full knowledge about what to purchase at the individual stores (based, e.g., on advertisements or on-line information) and the distances between home base and individual stores (based on past experience). Typically, such a trip resembles a traveling salesman tour, see, e.g., Applegate et al. (2007), or a traveling purchaser problem, as described in Laporte et al. (2003). Planning multi-purpose shopping trips has been shown to foster the agglomeration of facilities; see, e.g., Marianov et al. (2018).

A much more difficult extension concerns *trips without full information*. The main aspect of this single- or multi-purpose trip involves *feature search*. On such a trip, a customer will first patronize a store, obtain information about the features of the desired product (often, but not exclusively, its price), and will then decide, whether to purchase the product, or continue to some other store in order to potentially obtain a better deal. Such a search will incur certain costs (in terms of transportation costs and time), while expecting potential advantages in terms of better features, such as a lower price, better quality, or additional features. How long such searches will be will certainly depend, at least in part, on the amount of money involved and on the expected utility of a continued search, as compared to that of an immediate purchase on the basis of the information gathered up to this point. Houses, vehicles, furniture and similar high-priced items are typically purchased in this manner. Narula et al. (1983) present a model that includes price search, while Braid's (1996) noncompetitive location model that locates a main facility that has the desired product, and branch facilities, which have the product with a given probability. Customers can obtain information by means of phone search, Internet search, and visit search, respectively.

An interesting strand of research involves *flow capturing*, or *flow interception models* has been developed by Hodgson (1990), Berman et al. (1995), and Berman and Krass (1998). These models replace the assumption of customers making single trips to the chosen facility by assuming that they make purchases on their way to work. Considering work as one part of shopping, this model is a multipurpose shopping model with one fixed stop (work). Competing facilities will attempt to maximize their capture of the flow of customers to work. One of the main issues in these models involves the avoidance of double counting, i.e., customers who have made a purchase at one facility, have their demand drop to zero and they will not make another purchase on their trip. Typical applications for this type of behavior include child care facilities and gas stations.

Additional behavioral patterns involve window shopping and showrooming (the practice of getting advice and information about a product at local stores and the subsequent purchase at a presumably cheaper no-frills Internet dealer). The latter behavior has already caused some problems among local stores, even though the aforementioned detrimental effects may be, at least partially, offset by the fact that customers typically obtain detailed technical information online, alleviating the local store from having (expensive) specialized sales staff. This webrooming effect, i.e., the practice of obtaining information online and then shopping locally, counteracts the effects of showrooming, at least to some extent.

A different aspect that appears to be very promising deals not with the development of more realistic models, but with their visualization, which may provide insight and increase acceptability by decision makers. A good survey of the use of geographical information systems in location analysis is provided in Chap. 19 of this volume.

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