Chapter 12
Hub Location Problems

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Abstract *Hub Location Problems* (HLPs) lie at the heart of network design planning in transportation and telecommunication systems. They constitute a challenging class of optimization problems that focus on the location of hub facilities and on the design of hub networks. This chapter overviews the key distinguishing features, assumptions and properties commonly considered in HLPs. We highlight the role location and network design decisions play in the formulation and solution of HLPs. We also provide a concise overview of the main developments and most recent trends in hub location research. We cover various topics such as hub network topologies, flow dependent discounted costs, capacitated models, uncertainty, dynamic and multi-modal models, and competition and collaboration. We also include a summary of the most successful integer programming formulations and efficient algorithms that have been recently developed for the solution of HLPs.

12.1 Introduction

Transportation, telecommunications and computer networks frequently employ hub-and-spoke architectures efficiently to route flows between many origins and destinations. Their key feature lies in the use of transshipment, consolidation, or sorting points, called *hub facilities*, to connect a large number of origin/destination (O/D) pairs by using a small number of links. Flows having the same origin but different destinations are consolidated when routed to the hubs and are then combined with other flows having different origins but the same destination. This helps reduce setup costs, centralize commodity handling and sorting operations,

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G. Laporte et al. (eds.), *Location Science*,
https://doi.org/10.1007/978-3-030-32177-2_12

and achieve economies of scale on routing costs through the consolidation of flows. Broadly speaking, *Hub Location Problems* (HLPs) consist of locating hub facilities and of designing hub networks so as to optimize a cost-based (or service-based) objective.

HLPs constitute a challenging class of NP-hard problems involving joint location and network design decisions. Their main difficulty stems from the inherent interrelation between two levels of the decision process. The first level considers the selection of a set of nodes to locate hub facilities, whereas the second level deals with the design of the hub network, by selecting the links to connect origins, destinations and hubs, as well as the routing of flows through the network.

HLPs lie at the heart of network design planning in transportation and telecommunication systems. Application areas of HLPs in transportation are abundant. These include express package delivery, air freight and passenger travel, postal delivery, trucking, and rapid transit systems. Demand corresponds to commodities (i.e. express packages, passengers, mail, goods) carried by vehicles (i.e. trucks, trains, airplanes, vessels) moved on physical networks such as roads and railways or through the air or water. Hub facilities correspond to sorting centers or transportation terminals in which one or more transportation modes interact. Consolidation of flows at hubs enables economies of scale on the transportation costs, not only on the routing of flows between hubs, but also between O/D nodes and hubs.

Applications of HLPS in telecommunications arise in the design of various distributed data networks, where demand corresponds to electronic data that are routed over a variety of physical links (i.e. fiber optic links and co-axial cables) or through the air (i.e. satellite channels and microwave links). Hub facilities are hardware such as switches, concentrators, multiplexors, and routers. Economies of scale in data transmission and network utilization, in combination with large setup costs for hub facilities and communication links, motivate the use of hub-and-spoke architectures.

The study of HLPs began with the work of O'Kelly (1986a), for continuous models, and O'Kelly (1986b, 1987), for discrete models, and has since evolved into a rich research area. Over the last three decades hub location has been studied by researchers around the globe from different disciplines such as location science, geography, regional science, network optimization, transportation, telecommunications, and computer science. There exist several reviews and surveys on HLPs, each one focusing on different aspects of these problems. The early reviews dealing with HLPs, by O'Kelly and Miller (1994) and Campbell (1994a), contain classification schemes for fundamental models and for the topological structures applicable to hub networks. Klincewicz (1998) concentrates on the design of hub networks in the context of telecommunication networks, and Bryan and O'Kelly (1999) present a survey focused on air transportation networks. Campbell et al. (2001) wrote a comprehensive survey of HLPs in which the location of hubs is the key decision. Alumur and Kara (2008) provide a classification scheme and review of the growing literature on network hub location models before 2008. Campbell and O'Kelly (2012) provide an insight into early motivations for analyzing HLPs and highlight recent research directions. Zanjirani Farahani et al. (2013) review solution methods and applications for several classes of HLPs.

This chapter focuses on the role location and network design decisions play in the formulation and solution of HLPs. It overviews features and assumptions commonly considered in discrete HLPs, and provides insights on their modeling implications. We point out how these assumptions simplify network design decisions by creating a first generation of HLPs that focuses mostly on the location and allocation decisions. We also show how network decisions become more involved when relaxing some of these assumptions.

We start with an introduction to the fundamentals of HLPs, including their distinguishing features, assumptions, properties, as well as commonly used objectives. A review of the most interesting and useful mixed integer programming (MIP) formulations for fundamental HLPs considering cost-based objectives is then presented. We also highlight some of the main developments and most recent trends in hub location. We would like to clarify that, due to space limitations, this is not intended to be a comprehensive survey of all diverse topics associated with hub location research (Campbell and O'Kelly 2012), but rather our personal treatment on some of the most interesting research on this field. In particular, we include hub network topologies, flow dependent discounted cost models, capacitated models, models dealing with uncertainty, dynamic and multi-modal models, and competition and collaboration. A summary of successful integer programming methods that have given rise to efficient approximate and exact solution algorithms for solving HLPs is also presented.

This chapter does not cover continuous HLPs or models in which locational decisions are not present. The reader is referred to O'Kelly (1986a), O'Kelly and Miller (1991), Aykin (1988), Campbell (1990, 2013), Saberi and Mahmassani (2013), and references therein for continuous variants of HLPs, and to Klincewicz (1998), Gendron et al. (1999), Wieberneit (2008), and Saito et al. (2009) for hub-and-spoke network design models in which the set of hub facilities is given a priori. The reader is also referred to Contreras and Fernández (2012) for a survey of other general network design problems that also combine location and network design decisions.

12.2 Fundamentals

HLPs are closely related to classical *Facility Location Problems* (FLPs). As a result, for several classical facility location problems such as p-median, uncapacitated facility location, p-center, and covering problems, analogous HLPs have been studied: p-hub median, uncapacitated hub location, p-hub center, and hub covering problems. Due to their multiple applications, inside these classes of HLPs there exist several variants that differ with respect to a number of assumptions like their topological structure, the allocation pattern of O/D nodes to hubs, and capacity constraints on the hub network, among others.

The key difference between FLPs and HLPs lies in the type of service demand required by the users and on the function the facilities provide. In the case of FLPs, service is given at the facilities and flows thus originate at demand nodes and their destination are the facilities. Network design and routing decisions are usually determined by the assignment pattern of demand nodes to their allocated facilities. In HLPs, service demand is between O/D nodes and hub facilities are intermediate nodes in the O/D paths which act as transshipment and consolidation points. When a hub serves as transshipment (switching or sorting) point, it allows flows to be processed and redirected to other hubs or O/D nodes with many fewer links than would be needed with direct connections. As a consolidation (concentration or breakbulk) point, a hub allows flows to be aggregated and disaggregated, creating economies of scale in the transportation or communication cost between hubs and between O/D nodes and hubs. The interaction of hub facilities and O/D nodes increases the complexity of network design and routing decisions since these are not necessarily determined by the assignment pattern of O/D nodes to hubs.

Another difference between FLPs and HLPs is that when dealing with uncapacitated hub location models, a single assignment pattern of non-hub nodes to hubs is not necessarily an optimal allocation strategy. In most uncapacitated FLPs, once the facility locations are known the flow cost is minimized by assigning each demand node to its nearest (or least costly) open facility. In the case of HLPs, once the hub locations are known, the flow cost is minimized by finding the shortest path on the network induced by the selected hubs for each O/D pair, resulting in a multiple allocation pattern of O/D nodes to hubs. For this reason, both single and multiple assignments versions of HLPs exist. In a hub location problem with single assignments, O/D nodes must be assigned to exactly one hub facility which is more difficult. All demand flows from the same origin or to the same destination, are thus routed via the same hub. In a hub location problem with multiple assignments, each O/D node can be allocated to more than one hub facility. Multiple assignment patterns simplify the routing decisions and provide greater flexibility on hub networks, allowing lower flow cost solutions. However, they can considerably increase the network design cost as a larger number of links must be activated on the hub network.

12.2.1 Features, Assumptions and Properties

The key distinguishing features of HLPs can be summarized as follows: (1) service demand is associated with flows between O/D pairs, (2) hub facilities are intermediate nodes in the O/D paths which act as transshipment or consolidation points, (3) there is a benefit (or requirement) of routing flows via hubs, (4) there is a cost-based (or service-based) objective that depends on the design of the hub network (location of hubs and selection of links) and the routing of flows.

We can provide a description of a generic hub location problem as follows. Consider a complete graph $G = (N, E)$, where N is the set of nodes representing the origins and destinations of flows, and E is the set of edges. Let N be the set of potential hub locations as well. For each node pair (i, j), let $W_{ij} \geq 0$ and $d_{ij} \geq 0$ denote the amount of flow to be routed and the distance, respectively, from the origin $i \in N$ to the destination $j \in N$. For each node $i \in N$, f_i is the fixed setup cost for locating a hub, whereas for each $e \in E$, g_e denotes the fixed setup cost for locating a hub arc. A hub arc $e = (i, j) \in E$ connects two different hub nodes i and j and has a unit flow cost of αd_{ij}. The parameter α $(0 \leq \alpha \leq 1)$ is used as a discount factor to provide reduced unit flow costs on hub arcs to reflect economies of scale resulting from consolidation of flows between hubs. The unit flow cost between O/D pairs is given by the length of the path between the origin and destination nodes in the solution network. Each O/D path has a *collection* leg from the origin node to the first hub, possibly a *transfer* leg between the first and the last hubs, and a *distribution* leg from the last hub to the destination node. A generic hub location problem consists of locating a set of hub facilities and a set of hub arcs, and of determining the routing of flows through the hub network, with the objective of minimizing the total setup and flow cost.

Most of the hub location literature has focused on *Hub Node Location Problems* (HNLPs), which consider the location of a set of hub facilities and the assignment of O/D nodes to these facilities. Arc selection and routing decisions are usually determined by the assumptions made on the cost structure and the assignment pattern. The network induced by the solution of a HNLP consists of three types of arcs: (1) *hub arcs* connecting two hubs, (2) *access arcs* connecting non-hub nodes and hubs, and (3) *direct arcs* connecting two non-hub nodes. A more general class of hub location models, known as *Hub Arc Location Problems* (HALPs), have received less attention in the literature. HALPs consider the location of a set of hub arcs, that induce a set of hub nodes, and the assignment of O/D nodes to these hub arcs. In HALPs, the possibility of connecting two hub nodes with a fourth type of arc arises. A *bridge arc* is an arc that connects two different hub nodes, without benefiting from the reduced unit flow cost of a hub arc. HNLPs can be seen as particular cases of HALPs in which additional conditions are imposed.

Four common assumptions underlie most HLPs:

1. Flows have to be routed via a set of hubs.
2. Access arcs and bridge arcs have no setup cost.
3. The discount factor α is the same for all hub arcs and does not depend on the amount of flow that is actually routed on each hub arc.
4. Distances d_{ij} satisfy the triangle inequality.

A consequence of Assumption 1 is that direct connections between O/D nodes which are not hubs are not allowed and thus, O/D paths must include at least one hub node. In most HNLPs an additional fifth assumption stating that the setup cost of hub arcs is equal to zero (i.e., $g_e = 0$ for each $e \in E$) is also considered. This allows hubs to be interconnected at no extra cost and, together with Assumptions 3 and 4, an important resulting property in solution networks of HNLPs is that the set of hub arcs define a complete subgraph on the set of hub nodes (i.e. hubs are fully interconnected). As a consequence, hub arc selection decisions become trivial once the location of hub nodes is known. Another important property, obtained when combining all assumptions, is that paths between O/D pairs will contain at least one and at most two hubs. However, it is important to note that whenever Assumption 4 is not satisfied, paths may contain more than two hubs and more than one hub arc.

The above properties simplify the network design decisions and characterize the structure of O/D paths. In HNLPs, all O/D paths include either a single hub node and no hub arc, or two hub nodes and a single hub arc. Moreover, because of Assumptions 2 and 4, each collection and distribution leg, if present, contains only one access arc. O/D paths are thus of the form (i, k, m, j), where $(k, m) \in N \times N$ is the ordered pair of hubs to which i and j are allocated, respectively. Note that these paths contain one, two or at most three arcs, depending on the number of visited hubs and on the function of origins and destinations (i.e. hub or non-hub nodes). For each O/D pair, the flow cost of routing W_{ij} along the path (i, k, m, j) is then given by $F_{ijkm} = W_{ij}\left(\chi d_{ik} + \alpha d_{km} + \delta d_{mj}\right)$, where χ, α, and δ represent the collection, transfer and distribution costs along the path. To reflect economies of scale between hubs, we assume that $\tau < \chi$ and $\tau < \delta$. We note that in the literature, the path (i, k, m, j) is sometimes written with the alternative order (i, j, k, m).

Figure 12.1a shows an example of a solution network of a HNLP in which different structures on O/D paths arise (squares represent hub nodes and circles represent non-hub nodes). The path (1, 2, 9, 10) is a two-hub path formed by the access arcs (1, 2), (9, 10) and the hub arc (2, 9). The path (2, 2, 9, 6) is also a two-hub path but containing only the access arc (9, 6) and the hub arc (2, 9). The path (3, 3, 9, 9) is yet another two-hub path formed only by the hub arc (3, 9). The path (1, 2, 2, 8) is a one-hub path containing only the access arcs (1, 2) and (2, 8). The path (7, 8, 8, 8) is also a one-hub path containing the single access arc (7, 8).

In HALPs, hubs are not necessarily fully interconnected due to the set up cost on the hub arcs or because additional conditions on the network topology are imposed. This causes O/D paths to become more involved, since they may use more than three arcs and visit more than two hub nodes. Similar to HNLPs, because of Assumptions 2 and 4, each collection and distribution leg, if present, employs either one access arc or one bridge arc. However, the transfer leg can now use several bridge and hub

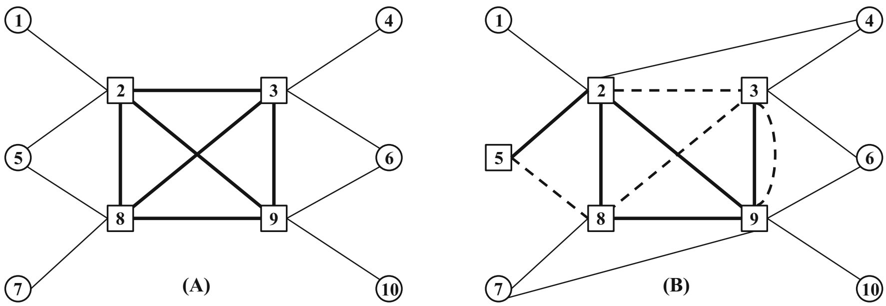

Fig. 12.1 Solution network of a hub node location problem (**a**) and a hub arc location problem (**b**)

arcs, depending on the particular assumptions considered on the structure of O/D paths.

To simplify the routing decisions in HALPs, an additional assumption stating that O/D paths contain at most one hub arc can be imposed. This limits paths to have at most three arcs, being the first and last ones either access or bridge arcs and the intermediate arc, if it exists, a hub arc. As mentioned in Campbell et al. (2005a), this assumption is used to increase service level in classical HLPs and is also consistent with practice. In air transportation, for example, it ensures that a passenger will never have to change flights more than twice. In ground transportation, it is convenient to restrict the number of hub facilities that each route has to pass through so as to reduce handling and congestion at hubs and to provide a form of performance guarantee. O/D paths are once more of the form (i, k, m, j), and thus, defining their flow cost as F_{ijkm}.

Figure 12.1b shows an example of a solution network of a HALP in which different structures on O/D paths arise (dashed lines represent bridge arcs). The path (5, 8, 2, 3) is a four-hub path formed by the bridge arcs (5, 8), (2, 3) and the hub arc (8, 2). The path (5, 8, 9, 10) is a three-hub path containing the bridge arc (5, 8), the hub arc (8, 9) and the access arc (9, 10).

12.2.2 Supermodular Properties

We next show how a general class of HLPs can be stated as the minimization of a real-valued supermodular set function. This fundamental property, which is also known for other types of classical facility location problems (p-median, uncapacitated and capacitated facility location), can be exploited to develop mathematical formulations and solution algorithms with worst case bounds.

This class of HLPs, referred to as *Supermodular Hub Location Problems* (SHLPs), considers Assumptions 1–4 and the additional assumption that limits O/D paths to contain at most one hub arc. SHLPs consist of locating a set of at most q hub arcs ($q \geq 1$), that induce a set of at most p hub nodes ($p \geq 2$), and of determining the routing of commodity flows through the hub network, with the objective of minimizing the total setup and flow cost. We can state SHLPs as the following combinatorial problem. Let $U = N \cup E$ be a finite set containing both the set of nodes N and the set of edges E of G. For each non-empty subset $(S, R) \subseteq U$, where $S \subseteq E$ and $R \subseteq N$, define

$$c(S, R) = \sum_{i \in R} c_i; \qquad g(S, R) = \sum_{e \in S} g_e; \qquad h(S, R) = \sum_{i,j \in N} h^{ij}(S) = \sum_{i,j \in N} \min_{(k,m) \in S} F_{ijkm},$$

and

$$f(S, R) = c(S, R) + g(S, R) + h(S, R) = \sum_{i \in R} c_i + \sum_{e \in S} g_e + \sum_{i,j \in N} \min_{(k,m) \in S} F_{ijkm}, \tag{12.1}$$

and $f(\emptyset) = 0$. For nonempty sets of hub nodes $R \subseteq N$ and hub arcs $S \subseteq E$, $c(S, R)$ is the total setup costs for setting hub nodes, $g(S, R)$ is the total setup cost of the hub arcs, and $h(S, R)$ is the total cost for routing the flows when the set of hub arcs S is chosen. Thus, $f(S, R)$ is the objective function value associated with the set of hub nodes R and the set of hub arcs S. Therefore, SHLPs can be stated as the problem of finding a set of arcs $S \subseteq E$ of cardinality at most q ($q \leq |E|$) and R of cardinality at most p ($p \leq |N|$) such that $f(S, R)$ is minimum, i.e.,

$$\min_{(S,R)\subseteq U} \{f(S, R) : |S| \leq q,\ |R| \leq p,\ N(S) = R\}, \tag{12.2}$$

where $N(S) = \{i \in N : (i, j) \in S \text{or} (j, i) \in S\}$ is the set of nodes incident with some edge in S. In order to deal only with feasible problems, we assume that $p \geq \lceil \frac{q}{2} \rceil$. When $p \geq \min\{|N|, 2q\}$ the maximum cardinality constraint on the number of hub nodes becomes redundant. Similarly, if $q \geq \min\{|E|, \binom{p}{2}\}$ the maximum cardinality constraint on the number of hub arcs becomes redundant. A fundamental property of f is that, for $(S, R) \subset (T, Q)$ and $e \in E \setminus T$, adding e to T will decrease f by no more than by adding e to S. A real-valued set function with such property is called *supermodular set function*.

Proposition 12.1

(a) $h(S, R) = \sum_{i,j \in N} h^{ij}(S, R)$ *is supermodular and nonincreasing.*

(b) $f(S, R) = c(S, R) + g(S, R) + h(S, R)$ *is supermodular.*

Problem (12.2) can thus be stated as the minimization of a supermodular set function, which is known to be in the class of NP-hard problems. We use SHLP to describe any problem that can be formulated as (12.2). SHLPs are a quite general class of HLPs and include several special cases which are of particular interest such as p-hub median, uncapacitated hub location, and q-hub arc location. Other classical facility location problems, such as the p-median or the uncapacitated facility location problem, are also relevant special cases of SHLPs. However, we note that not every HLP can be stated as problem (12.2). For instance, when a single assignment pattern is imposed the flow cost associated with a given set of hub arcs S is no longer $h(S, R)$, since all flow with the same origin (destination) must be routed through the same collection (transfer) leg. That is, HLPs with single assignments cannot be formulated as SHLPs. Moreover, even if multiple allocation is allowed, the addition of capacity constraints also preclude the supermodularity property when commodities cannot be split.

12.2.3 Objectives

Most of the hub location research has focused on HLPs that consider either a cost-based or a service-based objective. Transportation applications tend to focus on the flow transportation costs and travel times, whereas telecommunication applications focus more on the setup costs of the hub network. Analogously to facility location, HLPs can be classified based on the type of objective they use.

- *p-Hub Median Problems* O'Kelly (1987), Campbell (1996) assume that the number of hubs to locate is given as an input of the problem. They consist of locating a set of p hub facilities with the objective of minimizing the total flow cost for routing the flows through the hub network.
- *Hub Location Problems* O'Kelly (1992), Campbell (1994b) consider that the number of hubs to locate is not known a priori, but a fixed setup cost for each hub is considered. The objective is to minimize the sum of hub fixed costs and of demand flow costs over the hub network.
- *p-Hub Center Problems* Campbell (1994b), Kara and Tansel (2000) are minimax problems that focus on the minimization of a maximum service or cost measure between O/D pairs. Some of these measures are: (1) the maximum flow cost (or travel time) of all O/D pairs, (2) the maximum flow cost (or travel time) of all arcs of the hub network, and (3) the maximum flow cost (or travel time) associated with an access arc.
- *Hub Covering Problems* Campbell (1994b), Kara and Tansel (2003) impose a maximum threshold value on the service level (travel time) and focus on the minimization of the setup cost of the hub network. They assume demand is covered if both origin and destination nodes are within a specified distance of a hub node. They differ on their considered coverage criteria. An O/D pair (i, j) is covered by hubs k and m if: (1) the length of the path (i, k, m, j) is within a specified value, (2) the length of each arc in the path (i, k, m, j) does not exceed a specified value, or (3) each of the access arcs meet different specified values.

Both single and multiple assignment models, as well as uncapacitated and capacitated models have been considered in the literature for most of these classical objectives. We refer to Campbell (1994a), Campbell et al. (2001), and Alumur and Kara (2008) for a detailed overview of these models.

HLPs with more complex classes of objective functions have also been studied. Costa et al. (2008) and Köksalan and Soylu (2010) consider HLPS with multiple objectives. Puerto et al. (2011, 2016) study a general class of HLPs that consider an ordered median function (see Chap. 10) for which the above mentioned objectives (and others) are particular cases. O'Kelly (2012) considers objectives related to the fuel burn and environmental impact in airline hub networks. Campbell and O'Kelly (2012) review some HLPs that integrate both cost and service objectives. Alibeyg et al. (2016, 2018) introduce hub location problems with profit-oriented objectives that measure the tradeoff between the revenue derived from served commodities and the overall network design and flow costs.

12.3 Formulating Hub Location Problems

One of the major modeling challenges in HLPs is due to the fact that knowing the hub network structure is not necessarily sufficient to evaluate the objective function. Formulations must be able to model the path used for routing each flow to determine the flow cost. Significant progress has been made toward the development of MIP formulations for fundamental HLPs. These exploit the structure of the solution network obtained when considering the modeling assumptions presented in Sect. 12.2.1. We next introduce the most important families of MIP formulations for both single and multiple assignment variants of p-hub median and hub location problems. These have been successfully used in combination with sophisticated solution algorithms to obtain optimal solutions for large-scale instances. They have also been extended to model more complex variants of HLPs including additional features of real applications. We refer to Campbell et al. (2007), Alumur and Kara (2008), Wagner (2008a), Ernst et al. (2009), Hwang and Lee (2013), and Lowe and Sim (2013) for formulations of p-hub center and hub covering problems.

12.3.1 Single Assignments

A *natural* way of formulating HLPs with single assignments is to consider them as facility location problems with additional quadratic costs associated with the interaction between hub facilities. For each pair $i, k \in N$, we define location-allocation variables z_{ik}, equal to one if and only if node i is assigned to hub k. When $i = k$, variable z_{kk} represents the establishment or not of a hub at node k. The *Uncapacitated Hub Location Problem with Single Assignments* (UHLPSA) can be stated as the following quadratic mixed integer program (O'Kelly 1987):

$$\text{minimize} \sum_{k \in N} f_k z_{kk} + \sum_{i,k \in N} (\chi O_i + \delta D_i)\, d_{ik} z_{ik} + \sum_{i,j,k,m \in N} \alpha W_{ij} d_{km} z_{ik} z_{jm} \tag{12.3}$$

$$\text{subject to} \sum_{k \in N} z_{ik} = 1 \qquad i \in N \tag{12.4}$$

$$z_{ik} \le z_{kk} \qquad i, k \in N \tag{12.5}$$

$$z_{ik} \in \{0, 1\} \qquad i, k \in N, \tag{12.6}$$

where $O_i = \sum_{j \in N} W_{ij}$ and $D_i = \sum_{j \in N} W_{ji}$. The first term of the objective function represents the total setup cost of the hub facilities, whereas the second and third terms are the flow cost on the access and hub arcs, respectively. Constraints (12.4) guarantee that every O/D node is assigned to exactly one hub, whereas constraints (12.5) impose that they can only be assigned to open hubs. Note that constraints (12.4)–(12.6) define the set of feasible solutions of the *Uncapacitated Facility Location Problem* (see Chap. 2). However, objective (12.3) contains an additional quadratic term associated with the inter-hub flow cost. Several linearized formulations have been proposed to overcome this added difficulty of UHLPSAs.

An important family of formulations, referred to as *path-based formulations*, use decision variables to characterize O/D paths visiting either one or two hub nodes. We introduce binary routing variables x_{ijkm}, $i, j, k, m \in N$, equal to 1 if and only if the flow originated at i and destination j transits via a first hub node k and a second hub node m. The UHLPSA can be stated as follows (Skorin-Kapov et al. 1997):

$$\begin{aligned}
\text{minimize} \quad & \sum_{k \in N} f_k z_{kk} + \sum_{i,j,k,m \in N} F_{ijkm} x_{ijkm} \\
\text{subject to} \quad & (12.4)\text{–}(12.6) \\
& \sum_{m \in N} x_{ijkm} = z_{ik} \qquad i, j, k \in N && (12.7) \\
& \sum_{k \in N} x_{ijkm} = z_{jm} \qquad i, j, m \in N && (12.8) \\
& x_{ijkm} \geq 0 \qquad i, j, k, m \in N. && (12.9)
\end{aligned}$$

Constraints (12.7) state that if node i is assigned to hub k then all the flow from node i to any other node j must go through some other hub m. Constraints (12.8) have a similar interpretation relative to the flow arriving at a node j assigned to hub m from some node i. There is no need to state explicitly the integrality on the x_{ijkm} variables given that constraints (12.7)–(12.8), in combination with (12.6), ensure that for each node pair $i, j \in N$ exactly one variable x_{ijkm} equals to one and the rest of them to zero. One of the attractive features of this formulation is that it usually provides tight linear programming (LP) relaxation bounds, at the expense of requiring $O(n^4)$ variables and $O(n^3)$ constraints. Saito et al. (2009) study the polyhedral structure of the quadratic semi-assignment polytope, a relaxation of this formulation, and provides strong valid inequalities to further improve its LP bound.

It is possible to project out the path-based variables x_{ijkm} to obtain a formulation with fewer variables (see Labbé and Yaman 2004; Labbé et al. 2005). We define continuous variables y_{km}, $k, m \in N$, equal to the amount of flow routed on hub arc (k, m). The UHLPSA can be formulated as

$$\begin{aligned}
\text{minimize} \quad & \sum_{k \in N} f_k Z_k + \sum_{i,k \in N} (\chi O_i + \delta D_i) d_{ik} z_{ik} + \sum_{k,m \in N} \alpha d_{km} y_{km} \\
\text{subject to} \quad & (12.4)\text{–}(12.6) \\
& y_{km} \geq \sum_{(i,j) \in K} W_{ij} \left(z_{ik} + z_{jm} - 1 \right) \qquad k, m \in N, K \subseteq N \times N && (12.10) \\
& y_{km} \geq 0 \qquad k, m \in N. && (12.11)
\end{aligned}$$

For each arc (k, m), constraints (12.10) and (12.11) imply

$$y_{km} = \max_{K \subseteq N \times N} \sum_{(i,j) \in K} W_{ij} \left(z_{ik} + z_{jm} - 1 \right) = \sum_{(i,j) \in K_{km}} W_{ij} \left(z_{ik} + z_{jm} - 1 \right),$$

where K_{km} is the set of all demands which are routed on hub arc (k, m). This formulation contains only $O(n^2)$ variables but an exponential number of constraints. Labbé and Yaman (2004) show that constraints (12.10) are a particular case of a more general class of facet defining inequalities which can be separated in polynomial time.

Another important family of formulations, referred to as *flow-based formulations*, use continuous variables to compute the amount of flow routed on a particular arc originated at a given node. In the case of single assignments, we only need to use one set of flow variables associated with the hub arcs. We thus define continuous variables Y_{ikm}, $i, j, k \in N$, equal to the amount of flow originated at node i and passing through hub arc (k, m). The UHLPSA can be formulated as follows (Ernst and Krishnamoorthy 1996):

$$\text{minimize} \quad \sum_{k \in N} f_k z_{kk} + \sum_{i,k \in N} (\chi O_i + \delta D_i) d_{ik} z_{ik} + \sum_{i,k,m \in N} \alpha d_{km} Y_{ikm}$$

subject to (12.4)–(12.6)

$$\sum_{j \in N} W_{ij} z_{jk} + \sum_{m \in N} Y_{ikm} = \sum_{m \in N} Y_{imk} + O_i z_{ik} \qquad i, k \in N \tag{12.12}$$

$$Y_{ikm} \geq 0 \qquad i, k, m \in N. \tag{12.13}$$

Constraints (12.12) are the well-known flow conservation constraints for each O/D node i at each (potential) hub node k, where the supply and demand at each node is determined by the allocation pattern. The above formulation contains $O(n^3)$ variables and $O(n^2)$ constraints and thus, fewer variables and constraints as compared with the path-base formulation. However, it usually produces weaker LP bounds. Contreras et al. (2010, 2017) present some families of extended cut-set inequalities that can help improve the LP bounds.

12.3.2 Multiple Assignments

Given that in HLPs with multiple assignments O/D nodes can be connected to more than one hub facility, we can exploit the properties on the structure of O/D paths to obtain path-based formulations with less variables than the ones required for single assignment models. In particular, it is known that every flow uses at most one direction of a hub arc, the one with lower flow cost (Hamacher et al. 2004).

Hence we define an *undirected* flow cost F_{ije} for each $e = (k, m) \in E$ and $i, j \in N$ as $F_{ije} = \min\{F_{ijkm}, F_{ijmk}\}$. We also define binary location variables Z_i, $i \in N$, equal to 1 if and only if a hub is located at node i. The *Uncapacitated Hub Location Problem with Multiple Assignments* (UHLPMA) can be stated as follows (Hamacher et al. 2004; Marín 2005a):

$$\text{minimize} \quad \sum_{k \in N} f_k Z_k + \sum_{i,j \in N} \sum_{e \in E} F_{ije} x_{ije}$$

$$\text{subject to} \quad \sum_{e \in E} x_{ije} = 1 \qquad i, j \in N \tag{12.14}$$

$$\sum_{e \in E : k \in e} x_{ije} \le z_k \qquad i, j, k \in N \tag{12.15}$$

$$x_{ije} \ge 0 \qquad i, j, k \in N \tag{12.16}$$

$$Z_i \in \{0, 1\} \qquad i \in N. \tag{12.17}$$

Constraints (12.14) guarantee that there exists a single path connecting the origin and destination nodes of every commodity. Constraints (12.15) prohibit commodities from being routed via a non-hub node. Similar to UHLPSA, there is no need to explicitly state the integrality on the x_{ije} variables because there always exists an optimal solution of (12.14)–(12.17) in which all x_{ije} variables are integer. When solving this formulation, it may be possible to find an optimal solution in which, for a subset of node pairs $i, j \in N$, more than one x_{ije} variable is strictly positive (i.e., two or more paths are used to route flow between i and j). This would imply that there is more than one OD path with the same route cost. In that case, one can recover an integer solution by arbitrarily selecting one of such paths for each node pair $i, j \in N$ with multiple paths and making the associated x_{ije} variable equal to one and the others equal to zero. This path-based formulation has $O(n^4)$ variables and $O(n^3)$ constraints and usually provides tight LP bounds. Hamacher et al. (2004) and Marín (2005a) independently prove that constraints (12.15) are indeed facet-defining inequalities. Marín (2005a) provide other classes of inequalities associated with the set-packing polytope which also define facets.

The number of routing variables x_{ije} can be further reduced by defining a set of candidate hub arcs for each O/D pair (see Contreras et al. 2011b). This is done by using the property that no flow will be routed through a hub arc containing two different hubs whenever it is cheaper to route it through only one of them (Boland et al. 2004; Marín 2005a).

In HLPs with multiple assignments it is also possible to completely eliminate the undirected routing variables x_{ije} by exploiting the supermodular properties presented in Sect. 12.2.2. We define binary hub arc location variables y_e, $e \in E$, equal to 1 if and only if a hub arc is located at e. For each $i, j \in N$, we order the elements of E by non-decreasing values of their coefficients F_{ije}, and we denote e_{ijr} to the r-th element according to that ordering. That is, $F_{ije_1} \le F_{ije_2} \le \cdots \le F_{ije_{|E|}} \le F_{e_{ij|E|+1}}$, where $F_{e_{ij|E|+1}} = F_{ije^*}$ is the cost for the fictitious edge e^* such that (1) $F_{ije^*} > \max_{e \in E} F_{ije}$, for all $i, j \in N$; and (2) $\sum_{i,j \in N} F_{ije*} > \max_{e \in E}(f_e + \sum_{i,j \in N} F_{ije})$. This assumption guarantees that at least one hub variable y_e is at value one in any optimal solution. The UHLPMA can be stated as the following MIP (see Contreras and Fernández 2014):

$$\text{minimize} \quad \sum_{k \in N} f_k Z_k + \sum_{i,j \in N} \eta_{ij}$$

$$\text{subject to} \quad \eta_{ij} \geq F_{ije_r} + \sum_{e \in E} (F_{ije} - F_{ije_r})^{-} y_e \quad r = 1, \ldots, |E| + 1, \; i, j \in N \tag{12.18}$$

$$y_e \leq z_k \qquad e = (k, m) \in E \tag{12.19}$$

$$y_e \leq z_m \qquad e = (k, m) \in E \tag{12.20}$$

$$y_e, z_i \in \{0, 1\} \qquad e \in E, i \in N, \tag{12.21}$$

where η_{ij} are continuous decision variables used to evaluate the flow cost of O/D pair (i, j) and $(x)^{-} = \min\{0, x\}$. This new formulation has $O(n^2)$ variables and $O(n^4)$ constraints. It is interesting to note that, for the particular case of the p-hub median problem, the above supermodular formulation coincides with the *radius-based formulation* of García et al. (2012).

As in the case of single assignments, we can also use flow-based formulations to model the UHLPMA. However, we now need additional flow variables for the collection and distribution legs. We define continuous variables X_{ijm}, $i, j, m \in N$, equal to the amount of flow from hub m to destination j that originates at node i. We also define continuous variables Z_{ik}, $i, k \in N$ equal to the amount of flow from origin node i to hub k. Using these sets of decision variables, we can formulate the UHLPMA as follows (Ernst and Krishnamoorthy 1998b):

$$\text{minimize} \quad \sum_{k \in N} f_k Z_k + \sum_{i,k \in N} \chi d_{ik} Z_{ik} + \sum_{i,k,m \in N} \alpha d_{km} Y_{ikm} + \sum_{ijm} \delta d_{jm} X_{ijm}$$

subject to (12.17)–(12.13)

$$\sum_{k \in N} Z_{ik} = O_i \qquad i \in N \tag{12.22}$$

$$\sum_{m} X_{ijm} = W_{ij} \qquad i, j \in N \tag{12.23}$$

$$Z_{ik} + \sum_{m \in N} Y_{ikm} = \sum_{m \in N} Y_{imk} + \sum_{j} X_{ijm} \qquad i, k \in N \tag{12.24}$$

$$Z_{ik}, X_{ijm} \geq 0 \qquad i, j, m \in N. \tag{12.25}$$

Constraints (12.22) ensure that all flow from each origin is sent to a subset of hubs. Constraints (12.23) forces the flow of each O/D pair to arrive at its destination.

Constraints (12.24) are the flow conservation constraints at hub facilities. The above formulation contains $O(n^3)$ variables and $O(n^2)$ constraints. Boland et al. (2004) presents some preprocessing procedures that can be used to reduce the number of variables and constraints, and some valid inequalities to improve the LP bounds of capacitated variants.

12.4 Main Developments and Recent Trends

Early hub location research focused mostly on a first generation of HLPs which consider the assumptions introduced in Sect. 12.2.1. In this section we present some research areas that have attracted most attention in the literature over the last decade, leading to more realistic models that relax some of these assumptions and incorporate additional features of real applications. We focus on six particular areas: hub network topologies, flow dependent discounted costs, capacitated models, models dealing with uncertainty, dynamic and multi-modal models, and competition and collaboration.

12.4.1 Hub Network Topologies

Full interconnection between hub nodes may be prohibitive in applications where there is a considerable setup cost associated with the hub arcs (see O'Kelly and Miller 1994; Klincewicz 1998; O'Kelly et al. 2015a). To overcome this difficulty, several models considering incomplete hub networks have been studied. HALPs, originally introduced in Campbell et al. (2005a,b), relax the assumption of full interconnection between hubs and consider the location of a set of hub arcs that may (or may not) require a particular topological structure of their induced network. Some of these models do not even require the hub arcs to define a single connected component. Alumur et al. (2009), Tanash et al. (2017), O'Kelly et al. (2015a), Miranda et al. (2017), and Martins de Sá et al. (2018a,b), among others, study the design of incomplete hub networks in which no network structure other than connectivity is imposed on the backbone network. Miranda et al. (2017) also consider a variant in which hop constraints are used to limit the number of arcs in OD paths. Other works study models that do not consider a complete backbone network but rather, a particular topological structure. Figure 12.2 shows some examples of different hub network structures.

Kim and Tcha (1992), Contreras et al. (2009b, 2010) and Martins de Sá et al. (2013), study the design of tree-star hub networks in which the hubs are connected by means of a tree and the O/D nodes are assigned to exactly one hub. Labbé and Yaman (2008) and Yaman (2008) consider the design of star-star networks in which hub nodes are directly connected to a central node (i.e. star backbone network) and the O/D nodes are assigned to exactly one hub node. Martins de Sá et al. (2015)

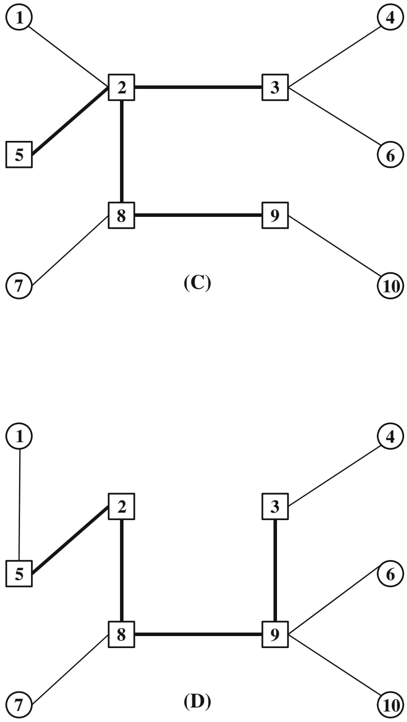

Fig. 12.2 Structure of a cycle-star (**a**), star-star (**b**), tree-star (**c**), and line-star (**d**) hub network

study the problem of designing a hub-line network in which hubs are connected by means of a line and the aim is to minimize the total service time between pairs of nodes. Martins de Sá et al. (2014b) present and extension of this problem to the case in which multiple hub-lines are to be located. Lee et al. (1993) and Contreras et al. (2017) focus on the design of cycle-star networks in which the hubs are connected by means of a cycle. O'Kelly et al. (2015a) analyze the role of setup costs for link activation decisions in the design of hub networks. The proposed model allows particular versions of hub networks to emerge from the cost structure, rather than assuming a predefined network structure.

Some papers focus on the design of more complex access networks that are no longer determined by a single or multiple assignment pattern of O/D nodes to hubs. Figure 12.3 depicts some examples of various access network structures. Aykin (1994, 1995) and Sung and Jin (2001) present models that explicitly consider direct connections between non-hub nodes (i.e. they relax Assumption 1). Klincewicz (1998) and Yaman et al. (2007) consider multi-stop access paths that may visit more than one O/D nodes on the way to a hub node. Nagi and Salhi (1998), Camargo et al. (2013), Rodríguez-Martín et al. (2014), and Rieck et al. (2014) study problems in which collection and distribution tours have to be designed. Thomadsen and Larsen (2007) and Saboury et al. (2013) describe HLPs in which both the backbone and

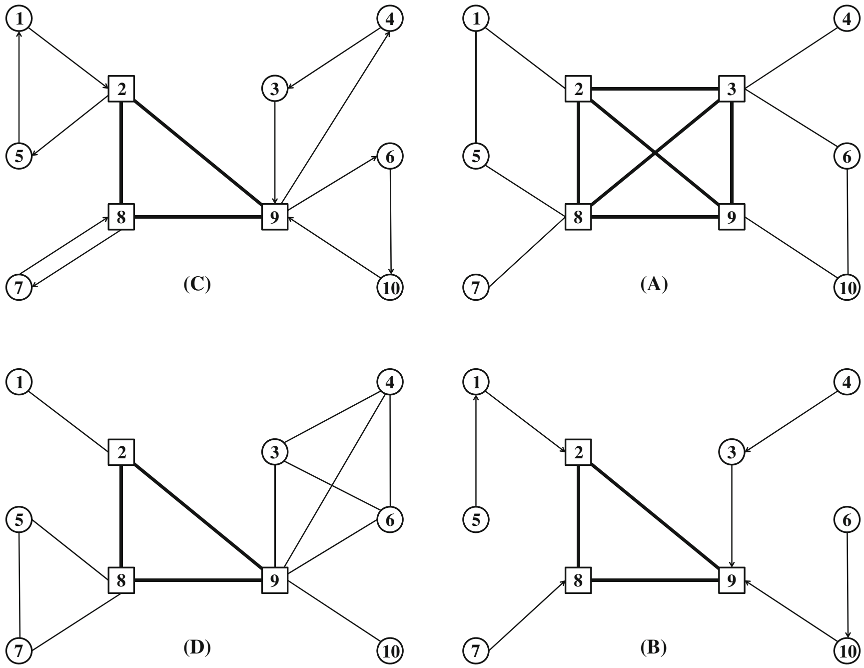

Fig. 12.3 Access network with direct connections (**a**), multi-stops (**b**), tours (**c**), and complete subgraphs (**d**)

access networks are fully interconnected. Finally, we refer to Chap. 14 for references considering hub network topologies arranged in a hierarchical structure.

12.4.2 Modeling Flow Costs

The assumption of flow-independent discounted costs (Assumption 3) is most appropriate in applications where hub arcs are associated with faster transportation modes. However, this can be an oversimplification in applications where the costs represent the economies of scale due to the bundling of flows on the hub arcs. For instance, this assumption could lead to solution networks where hub arcs send considerable less flow than access arcs, yet the flow cost is only discounted on the hub arcs. It may also happen that the amount of flow that is actually routed on each hub arc is quite variable, yet the same discount factor is always applied. For these reasons, the use of flow-independent costs may not only miscalculate the overall flow cost of the hub network, but could also erroneously select the optimal set of hub nodes and the assignment pattern of O/D nodes to hubs.

Several authors have pointed out these anomalies and different hub location models able to capture the flow-dependency of discounted costs have been proposed. The first hub location model that explicitly accounts for economies of scale by allowing discount factors on hub arcs to be a function of flows was introduced in O'Kelly and Bryan (1998). This model, referred to as FLOWLOC, uses a non-linear cost function, in which costs increase at a decreasing rate as flows increase, to compute the flow cost in each hub arc. For any amount of flow, the cost is assumed to be always less than the linear cost associated with a constant discount factor. This function is approximated with a piecewise linear function to obtain a linear integer programming formulation for the problem. Bryan (1998) provides some extensions of the FLOWLOC model that relax the assumption of full interconnection between hubs, by using a minimum threshold value to activate a hub arc, and that incorporate a flow-dependent cost function for both the hub and access arcs. Klincewicz (2002) shows that, once the location of the hubs is known, the FLOWLOC model can be reduced to a classical UFLP. Horner and O'Kelly (2001) present a different non-linear flow cost function based on link performance functions commonly used in urban transportation planning. This function is used to model flow-dependent costs in both hub and access arcs.

Racunica and Wynter (2005) study an extension of HLPs arising in the design of intermodal transportation networks for freight rail. Their model uses another type of non-linear concave function to model flow-dependent discounted costs only on the transfer and distribution legs. In contrast to the FLOWLOC model, this function is based on an efficiency threshold that considers that discounted flow costs should be higher than the linear cost up to a threshold, and less costly thereafter. Cunha and Silva (2007) and Lüer-Villagra and Marianov (2019) consider an alternative linear flow cost function in which a threshold for switching cost lines is used. In this case, the flow-independent discount rate applies only when the amount of flow on a link exceeds the threshold.

Kimms (2006) introduces a different approach for modeling flow-dependent discounted costs in all the arcs of the network, which is based on fixed-charge cost functions commonly used in other network design problems. This function consists of a fixed flow-independent setup cost and of a variable flow-dependent (or marginal) cost. This paper presents three different models: an uncapacitated model, a capacitated model, and a multimodal model with different capacities for each mode of transportation. O'Kelly et al. (2015a) study a similar uncapacitated problem in which fixed and variables flow costs for arcs are incorporated to provide a flow-dependent cost rate. Yaman and Carello (2005), Tanash et al. (2017), and Hoff et al. (2017) study modular hub location problems with single assignments in which a stepwise function is used to model flow-dependent costs on hub arcs. Contrary to fixed-charge cost functions, stepwise functions do not consider a variable cost component and are frequently used to model transportation costs in vehicle routing and pick-up and delivery problems (see Laporte 2009).

12.4.3 Capacitated Models

Similar to FLPs, an important extension to HLPs is the incorporation of capacity considerations when designing hub networks. However, in the case of HLPs the capacity constraints may arise not only at the hub facilities but also on the arcs of the network. Moreover, when considering capacitated models with multiple assignment patterns, commodities may be split over several paths and thus, splittable and non-splittable commodity variants arise. In the former case, commodities are allowed to be split over several paths between their origins and destinations. However, in the latter case the commodities cannot be split, meaning that each commodity will be routed through the network from its origin to its destination through a unique path. Note that a multiple assignment pattern that allows splitting is highly desirable when minimizing the total flow cost. However, splitting commodities may not be feasible in some applications.

Capacitated versions of HLPs with multiple assignments are studied by Campbell (1994b), Ebery et al. (2000), Boland et al. (2004), and Puerto et al. (2016) with capacity constraints on the incoming or outgoing flow at the hubs. Bryan (1998) introduces a model in which capacities are associated with the hub arcs rather than with the hub nodes. Marín (2005b) studies a capacitated model in which commodities are splittable. Rodríguez-Martín and Salazar-González (2008) study another model where commodities can be split into several routes. Capacity constraints are imposed on the incoming flow of each hub, whether it originated from non-hub nodes or from hub nodes. In addition, an upper limit is imposed on the flow traversing any link of the network.

Capacitated versions of HLPs with single assignment have also been studied by Campbell (1994b), Ernst and Krishnamoorthy (1999), Labbé et al. (2005), Correia et al. (2010), Contreras et al. (2009a, 2011d). All these models only consider capacity constraints on the incoming or outgoing flow at the hub nodes. Aykin (1994, 1995) have considered HLPs with capacity constraints on the incoming flow at the hubs as well as on direct O/D links. Carello et al. (2004), Yaman and Carello (2005) and Yaman (2008) have studied capacitated HLPs with modular link capacities. They considered capacity constraints on the incoming and outgoing flow at hubs.

All of the above mentioned capacitated models consider that both hub and arc capacities are exogenous, i.e. capacity levels for potential hub nodes and hub arcs are determined a priori. Given that capacities can have a determining impact on locational and routing decisions, some researchers have started studying more realistic capacitated models in which the amount of installed capacity is part of the decision process. Correia et al. (2010) studied an extension of capacitated HLPs with single assignment in which the hub capacity is a decision variable. Elhedhli and Wu (2010) introduced a capacitated model in which hub capacity is also a decision variable. Contreras et al. (2012) presented models with multiple assignments in which the amount of capacity installed at the hubs is part of the decision process, for both splittable and non-splittable commodity cases.

Alumur et al. (2016) introduced models with single and multiple assignments in which capacities at hub nodes can be gradually expanded over a finite planning horizon. Serper and Alumur (2016) presented a more comprehensive capacitated model in which capacities have to be determined in both hubs and arcs of the network. Given that alternative transportation modes and different types of vehicles are considered, the design of the hub network is done by explicitly determining the number of vehicles of each type to operate on each link of the network.

12.4.4 Uncertainty in Hub Location

The design of hub networks corresponds to long-term strategic decisions which are typically made within an uncertain environment. That is, costs, demands, distances, and other parameters may change after location and network design decisions have been made. Nevertheless, most HLPs treat data as known and deterministic. This can result in highly sub-optimal solutions given the inherent uncertainty surrounding future conditions. Some researchers have studied how different uncertainty aspects can be taken into account when designing hub networks.

Marianov and Serra (2003) is probably the first paper dealing with uncertainty, focusing on stochasticity at the hub nodes by representing hub airports as $M/D/c$ queues and limiting through chance constraints the number of airplanes that can queue at an airport. Sim et al. (2009) introduce the stochastic p-hub center problem and employ a chance-constrained formulation to model the minimum service-level requirement. This model takes into account the variability in travel times when designing the hub network so that the maximum travel time through the network is minimized.

Contreras et al. (2011a) study how the classical UHLPMA can be modeled as a two-stage integer stochastic program with recourse in the presence of uncertainty on demands and flow costs. In particular, three different stochastic versions are introduced. The first considers the flow between O/D nodes to be stochastic. The second assumes that uncertainty is given by a single parameter equally influencing the flow cost for all links of the network. The third considers the more general case in which the uncertainty of transportation costs is independent for each link of the network. The authors show that the first to variants are equivalent to their associated expected value problem in which uncertain amount of flows and flow costs are replaced with their expected value. However, this equivalence does not hold for the third case. Alumur et al. (2012b) consider HLPs under uncertainty in the setup cost for the location of hubs and in the demand flows for both single and multiple assignments models. The first class of models deals with uncertainty on the setup costs in the absence of a known probability distribution for these random parameters. The authors propose the use of a minimax regret model in which the objective is to minimize the worst-case regret over a finite set of scenarios. The second class considers uncertainty on the demand flows and uses a two-stage stochastic program with recourse. However, as shown in Contreras et al. (2011a) these problems are equivalent to their associated expected value problem. The third class considers uncertainty in both setup costs and demand flows and are modeled as two-stage minimax regret programs with recourse. Correia et al. (2018) introduce a more general two-stage stochastic multi-period capacitated hub location problem in which uncertainty is assumed for the demands. The first-stage decisions deal with the location of the hubs over the planning horizon and their initial installed capacity. The second-stage decisions concern the assignment of non-hubs to hubs, the routing of flows, and the capacity expansion for existing hubs.

Merakli and Yaman (2016) introduce robust uncapacitated p-hub median problems with multiple assignments under polyhedral demand uncertainty. They employ a hose model and a hybrid model to characterize demand uncertainty. The former assumes that the only available information is an upper bound on the total flow adjacent to each node, while the latter incorporates in addition lower and upper bounds on each OD flow. Merakli and Yaman (2017) extends the hose uncertainty model to a more challenging scenario in which capacities at hubs are considered, impacting the feasibility of solutions. Zetina et al. (2017) present robust counterparts for UHLPMAs in which the level of conservatism is controlled with a budget of uncertainty. The proposed models incorporate both independently and jointly demand and flow costs uncertainties when the only available information is an interval of uncertainty. The considered objectives aim at a minimizing the sum of the hub setup costs and of demand flow costs in the worst-case scenario.

Martins de Sá et al. (2018a) study a robust counterpart of an incomplete hub location problem with multiple assignments in which link activation decisions are taken into account. The model considers uncertainty in setup costs for hub nodes and hub arcs as well as demand and uses a budget of uncertainty to control the level of conservatism. Martins de Sá et al. (2018b) address another robust incomplete hub location problem in which service time constraints for each demand flow are incorporated. In this case, the uncertainty is related to travel times between nodes and the goal is to obtain cost-effective solutions with a high probability of being feasible with respect to the service time constraints.

Demand uncertainty has also been studied in hub location from a congestion perspective. When demand flows increase unexpectedly within a short time, they are likely to congest the hub network. This causes an increase in the operational cost of the network due to delays at hub facilities. Elhedhli and Hu (2005) present a single allocation hub location model that considers hub congestion-related costs as an exponential function of the hub flow. Camargo et al. (2009) propose the multiple allocation analogue of the previous model. Elhedhli and Wu (2010) study a different approach in which the hub network is modeled as a network of $M/M/1$ queues where each hub behaves as a single server with a given exponential service rate determined by its capacity. The congestion cost is modeled using a Kleinrock average delay function. Camargo and Miranda (2012) provide extensions to the previous single allocation models by considering two different perspectives: a network owner perspective in which the goal is to design a hub network with the least congestion cost, and a user perspective in which the goal is to minimize the maximum congestion effect. Aziz et al. (2018) consider the design of hub networks under stochastic demand and congestion. Hubs are modeled as spatially distributed $M/G/1$ queues and congestion is captured using the expected queue lengths at hubs.

An important uncertainty aspect neglected until recently is the reliability of hub networks. Kim and O'Kelly (2009) present a reliable p-hub location problem arising in the design of telecommunication networks. This problem considers the reliability of O/D paths by taking into account the probability of successful communication to deliver traffic without congestion or loss between O/D pairs. It focuses on maximizing the total network flow that can be routed when incorporating the reliability of O/D paths. An et al. (2015), Aziz et al. (2016), and Rostami et al. (2018) study models in which disruptions at hub nodes are taken into account when designing the hub network. The proposed models mitigate the resulting hub unavailability (one at a time) by using backup hubs and alternative routes for demand flows. The objective of these models is to minimize the total expected flow cost considering both the regular and the disruptive situation. Tran et al. (2016) assume that more than one hub can simultaneously fail, each of which can fail with a site-specific probability. Ramamoorthy et al. (2018) present multiple allocation hub interdiction and hub protection problems. In the hub interdiction problem, the goal is to determine a set of r critical hubs from an existing set of p hubs such that when interdicted results in the maximum post-interdiction flow cost. In the hub protection problem, the decision maker seeks to fortify a set of u hubs from an existing set of p hubs against interdiction. These models lead to complex bi-level and tri-level optimization problems which are known to be extremely difficult to solve.

12.4.5 Dynamic and Multi-Modal Models

One common feature of real applications is the dynamic nature of the problem. Parameters such as costs, demand, and resources often vary over the planning horizon. From the location point of view this gives rise to different types of multi-period, or dynamic problems. In this type of problems, not only a routing plan has to be made, but the times at which facilities are opened or closed must be determined.

Campbell (1990) develops a continuous approximation model to locate transportation terminals (hubs) for a general freight carrier serving an increasing demand in a fixed region. It can be seen as a continuous dynamic hub location model in which it is assumed that the O/D points are scattered randomly over the service region. Contreras et al. (2011c) study a dynamic model with multiple assignments which includes strategic decisions related to the location, operation and closing of hub facilities over time. It is assumed that the forecast demand between O/D pairs is known with certainty but varies over the time horizon. Moreover, the proposed model allows hubs to be opened and closed at different time periods to provide a flexible hub network. Gelareh et al. (2015) presents another multi-period hub location model arising in the design of public transportation networks in which the full interconnection assumption is relaxed and thus, additional hub arc selection are considered. Alumur et al. (2016) study multi-period models with single and multiple assignments in which capacities at hub facilities can be gradually expanded over a planning horizon.

Another important feature in some applications is the presence of strategic decisions related to the choice for mode of transportation. Most HLPs consider that only one mode of transportation is available and hence there is only one type of hub facility. However, global hub networks usually employ a mixture of air, ground and water transportation modes. In a multi-modal hub network, each mode can be characterized by its flow cost structure, modal connectivity, availability of transfer points, and service time performance.

Racunica and Wynter (2005) address the design of hub networks for inter-modal freight transport on dedicated or semi-dedicated freight rail lines which could make use of shuttle trains on the hub arcs. Groothedde et al. (2005) develop a multi-modal hub location model that focus on the design of a collaborative hub network for the distribution of fast moving consumer goods using a combination of trucking and inland barges. Ishfaq and Sox (2011) present a multiple allocation model to design a rail-road inter-modal network. It considers the location of two different types of hubs with different modal connectivity costs and the incorporation of service time requirements. Meng and Wang (2011) study the design of an inter-modal hub network for multi-type container transportation with multiple stakeholders: the network planner, carriers, hub operations and inter-modal operators. The proposed model incorporates the user equilibrium behavior of inter-modal operators in route choice. Alumur et al. (2012a) introduce a more general hub network design problem in which the full interconnection of hubs assumption is relaxed and hub arc location decisions, that include the selection of the type of transportation mode, are considered. This model incorporates setup costs, transportation costs and service levels when designing the multi-modal hub network. Alumur et al. (2012c) study a related hub covering problem to locate two types of hub nodes and hub arcs associated with ground and air transportation. The model uses a cost-oriented objective while ensuring time-definite deliveries. Serper and Alumur (2016) present capacitated models considering alternative transportation modes and different types of vehicles. The models select an optimal number of vehicles of each type to operate on each link of the network. Dukkanci and Kara (2017) study a hub covering problem with service time constraints. They propose a hierarchical multimodal hub network structure in which different types of vehicles can be used in each layer.

12.4.6 Competition and Collaboration

Most HLPs studies assume that the decision maker is a monopolist firm in a market and thus can capture all demand flow in the market, regardless of the design of the hub network. As a result, location and network design decisions are usually determined by the firm's cost-based objective without taking into account customer preferences. However, in practice many telecommunication and transportation networks operate in a competitive environment where several firms exist in a market and compete to provide service to customers. Customers must determine which competing firm to use based on several criteria such as the travel time and the costs charged. Competitive hub location models focus on the design of hub networks so as to maximize the market share of competing firms. In these models, customers (or demand flows) are captured from competitor's hub networks whenever the new hub network offers a reduction of the travel time or distance needed by the customers to go from their origins to their destinations.

Most competitive hub location models use a sequential location approach, in which an existing company (the leader) serves the demand flow in a region, and a new company (the follower) wants to enter the market and will attempt to capture the maximum possible demand and thus, maximize its market share. Marianov et al. (1999) introduce competitive hub location models in which the follower wants to locate a set of hub nodes so as to maximize the captured demand flow. In the first proposed model it is assumed that demand is fully captured when the flow cost does not exceed the current competitor's cost. The second model considers a more realistic version in which a stepwise linear function is used to model the proportion of demand captured depending on the new flow cost as compared to the competitor's cost. In both models, at most one path is used to route flow between each O/D pair. Wagner (2008b) points out that if the new company is assumed to capture demand flow when its flow cost is equal to the current competitor's cost, then the optimal solution is always to locate a hub node in each location where the leader has one, making the new company capture all demand. Therefore, the author suggests modifying the definition of the problem so that demand is captured by the follower if and only if the new cost is strictly smaller than the competitor's cost. Eiselt and Marianov (2009) provide an extension to the models presented in Marianov et al. (1999), in which each competitor can have more than one path between O/D pairs. The proportion of flow captured on a particular path is modeled through a gravity-like attraction function that does not only depend on the flow cost but also on the travel time. Gelareh et al. (2010) present a competitive model arising in liner shipping networks, where a new liner service provider wants to design a hub network to maximize its market share, using an stepwise attraction function which depends on the service time and flow cost. This model allows O/D paths to contain more than one hub arc or to have direct connections between origins and destinations. Lüer-Villagra and Marianov (2013) study a competitive model in which an existing firm uses a hub network and charges its flow costs plus a fixed additional percentage to their customers. A new company wants to enter into the same market using an incomplete hub network and to determine prices so as to maximize its profit, rather than its market share. The profit comes from the revenues derived from captured flows, minus the a fixed and variable costs. Customer preferences on selected firm and route are modeled using a logit model. Mahmutogullari and Kara (2016) propose other competitive models in which two decision-makers sequentially determine the location of their hubs and then customers choose one firm with respect to provided service levels. The goal of each firm is to maximize its own market share. O'Kelly et al. (2015b) introduce a model with price-sensitive demands. It considers three different service levels for routing flow between OD pairs that use either two-hub OD paths, on-hub OD paths or direct connections. The authors model the problem as an economic equilibrium problem that maximizes a nonlinear concave utility function, minus the flow cost and setup cost for the location of the hubs.

Using a game theoretic framework, Sasaki and Fukushima (2001) introduce a continuous Stackelberg hub location model where a large company competes with several medium-size companies to maximize its profit. The large company first locates a new hub on a plane as a leader, and the other companies then locate their new hubs. The authors use a nonlinear logit function to model the level of captured customers and formulate the leader's problem as a bilevel program and the follower's problems as lower level programs. Sasaki (2005) provides an extension to the discrete case assuming there is a leader and only one follower. The proposed model considers that companies cannot provide any service whose captured market share does not reach to a threshold lower limit value. Sasaki et al. (2009) study a more general model in which the full interconnection assumption is relaxed and a set of hub arcs must be located. As in Sasaki (2005), two firms compete for customers in a Stackelberg framework, where the leader firm locates hub arcs to maximize its market share, knowing that the follower will later locate its own hub arcs to maximize its market share.

Instead of considering a pure competitive environment, some studies have looked at hub network alliances and mergers, as well as user cooperation employing a game theoretic approach. In Skorin-Kapov (1998) a cooperative game theory is used to analyze several cost allocation problems referred to as hub network games. In particular, the flow routing cost is distributed among the hub network users with possibly conflicting interests, but their cooperation is essential for the exploitation of economies of scale on the routing of flows. Lin and Lee (2010) propose a non-cooperative game theoretic model to study the competition hub network design in an oligopolistic market with few dominant firms. In this model, each firm will first observe the hub network and demand flows of other firms and will then simultaneously determine its hub network, demand, and routing plan in order to maximize its profits. The firms' decisions jointly determine the market prices, which include the reassessment and redesign of hub networks of all other firms. The process of observation, design and reassessment will continue until a long-term Cournot-Nash equilibrium is established.

Adler and Smilowitz (2007) present hub location models to analyze global alliances and mergers in the airline industry under competition. In particular, the authors develop a game theoretic approach in which merger and hub location decisions are considered to evaluate hub networks under competition. The proposed problems are modeled as games played among multiple airlines, consisting of selecting the optimal hubs to develop, expand or remove in the newly merged hub network. Asgari et al. (2013) study a game theoretic hub network design model that investigates the competition and cooperation amongst two major hub ports and the shipping companies, with the objective of minimizing the shipping companies' cost and maximizing the hub ports' revenue.

12.5 Solving Hub Location Problems

The interrelation of location and network design decisions make HLPs particularly difficult to solve. A considerable effort has thus been made over the past two decades to develop algorithms capable of obtaining high quality solutions of various classes of HLPs, particularly when considering more realistic, large-scale instances. Some of these algorithms are able to provide an estimation of the quality of the obtained solutions and some them are able to prove that the obtained solution is optimal. In this section, we point out recent papers describing the most effective solution algorithms for various classes of HLPs. The interested reader is referred to Alumur and Kara (2008) and Zanjirani Farahani et al. (2013) for a detailed survey of approximate and exact algorithms for HLPs.

12.5.1 Complexity Results

Most HLPs are known to be NP-hard. However, very little research has been done to analyze the complexity and polynomial-time approximability of particular classes of HLPs. In the case of fundamental HLPs with single assignments, in which the full interconnection assumption is used, even if the location of the hub nodes is given the remaining subproblem is still NP-hard. This problem is known as the *quadratic semi-assignment problem* or the *single allocation problem* (see Saito et al. (2009), Sohn and Park (2000), and references therein). Sohn and Park (1997) show that for the particular case of the *uncapacitated p-hub median problem with single assignments* (UpHLPSA), when $p = 2$ the problem can be polynomially solved by reducing it to $n(n-1)/2$ independent minimum cut problems. Sohn and Park (2000) prove that the single allocation problem becomes NP-hard as soon as the number of hubs is three and hence, the UpHLPSA is NP-hard for $p \geq 3$. Iwasa et al. (2009) describe a deterministic 3-approximation algorithm and a randomized 2-approximation algorithm for the single allocation problem. Moreover, they provide a $(5/4)$-approximation algorithm for the particular case in which the number of hubs is three.

When considering HLPs with incomplete hub networks, even if the location of hubs and the assignment of O/D nodes to hubs is given, the subproblem associated with the location of hub arcs remains challenging. For instance, when considering tree-star topologies the design of a tree spanning the set of hub nodes is equivalent to the so-called *optimum communication spanning tree problem*, known to be NP-hard (Contreras et al. 2010). In the case of cycle-star topologies, connecting the hub nodes by means of a cycle is equivalent to the *minimum flow cost Hamiltonian cycle problem*, known to be NP-hard (Contreras et al. 2017).

In the case of uncapacitated HLPs with multiple assignments, in which the full interconnection assumption is used, once the location of the hubs is known the allocation subproblem is equivalent to an *all pairs shortest path problem* and thus, can be solved in polynomial time (Ernst and Krishnamoorthy 1998a). When considering capacities on the hub nodes and commodities can be split, Contreras et al. (2012) show that the allocation subproblem remains polynomially solvable as it is equivalent to a classical *transportation problem*. However, when commodities cannot be split the subproblem is equivalent to a *generalized assignment problem* and thus becomes NP-hard.

Contreras and Fernández (2014) show that a general class of HLPs with multiple assignments, known as *supermodular hub location problems* (Sect. 12.2.2), is NP-hard. We recall that SHLPs include several special cases such as p-hub median, uncapacitated hub location, and q-hub arc location. The authors also present worst-case performance results for simple greedy and local improvement heuristics for particular classes of SHLPs in which the objective functions are also non-increasing, as in p-hub median and q-hub arc location problems.

Kara and Tansel (2003) show that *hub set-covering problems with single assignments* are NP-hard. Kara and Tansel (2000) prove that the *uncapacitated p-hub center problem with single assignments* is also NP-hard for $p < n - 1$. Ernst et al. (2009) show that the multiple assignments version of this problem is also NP-hard. They also prove that the single allocation subproblem with respect to a given set of hubs is already NP-hard, whereas for the multiple assignment case is not. Liang (2013) considers the *star p-hub center problem* and shows that is strongly NP-hard and that there is no $(5/4-\epsilon)$-approximation algorithm for it for any $\epsilon > 0$, unless P $=$ NP. This paper also provides a $7/2$-approximation algorithm for this problem.

12.5.2 Heuristic Algorithms

A considerable amount of hub location research on heuristic algorithms has focused on fundamental HLPs. To the best of our knowledge, the best heuristic for the *uncapacitated p-hub location problem with single assignments* is the variable neighborhood search algorithm of Ilić et al. (2010). It outperforms all previous heuristics and yields solutions for very large-scale instances with up to 1000 nodes and $p = 20$ within reasonable CPU times. The best results for the UHLPSA seem to be obtained using the learning-based probabilistic tabu search recently designed by Guan et al. (2018). This heuristic has the best performance when compared with other heuristics, especially on large instances with up to 900 nodes. Contreras et al. (2011d) provide GRASP heuristics for capacitated versions of this problem. Contreras et al. (2011b) design a GRASP heuristic for the UHLPMA capable of obtaining high quality solutions for instances with up to 500 nodes within reasonable CPU times. Meyer et al. (2009) present an ant colony optimization algorithm for the *p-hub center problem with single assignments* which is able to obtain high quality solutions for large-scale instances with up to 400 nodes.

Some researchers have recently focused on the development of efficient heuristic algorithms for more realistic extensions of HLPs. Calık et al. (2009) describe a tabu search to solve hub covering problems over incomplete hub networks. Köksalan and Soylu (2010) study evolutionary algorithms for two bicriteria uncapacitated p-hub location problems considering congestion-related costs. Contreras et al. (2017) describe a GRASP algorithm for the design of incomplete hub networks with a cycle-star topology. Saboury et al. (2013) present two hybrid heuristics to design of hub networks with fully interconnected backbone and access networks. Martins de Sá et al. (2014b) propose an adaptive large neighborhood search and GRASP algorithms to design hub networks with multiple hub lines. Tran et al. (2016) develop a parallel tabu search to solve reliable hub location problems. Hoff et al. (2017) present a metaheuristic based on adaptive memory programming and path-relinking to solve a capacitated modular hub location problem.

12.5.3 Lower Bounding Procedures and Exact Algorithms

Dual ascent and dual adjustments techniques have been used to efficiently obtain the LP bound of MIP formulations for various HLPs. Yoon and Current (2008) use dual based heuristics to solve HLPs with additional arc selection decisions. Cánovas et al. (2007) present a branch-and-bound (BB) algorithm based on dual techniques to obtain optimal solutions to uncapacitated HLPs with multiple assignments. Meyer et al. (2009) develop a two-phase exact algorithm for the p-hub center problem with single assignments. In this algorithm the BB method presented in Ernst and Krishnamoorthy (1998a) is used during the first phase to obtain a set of potential optimal hub locations. This algorithm seems to be the best exact algorithm for hub center problems, being able to solve to optimality large-scale instances with up to 400 nodes.

Lagrangian relaxation (LR) has been successfully used to obtain tight lower and upper bounds on the value of the optimal solution of several classes of HLPs. Pirkul and Schilling (1998) present efficient LR heuristics to approximately solve uncapacitated HLPs with single assignments, whereas Yaman (2008), Contreras et al. (2009a,b), and Elhedhli and Wu (2010) propose LR heuristics to solve various capacitated HLPs. Exact BB methods based on LR have also been developed to optimally solve HLPs. Marín (2005a) propose a relax-and-cut algorithm for the UHLPMA, which adds violated facet-defining inequalities to a LR of the path-based formulation presented in Sect. 12.3.2, to optimally solve instances with up to 50 nodes. Contreras et al. (2011c) present an exact BB method, that uses a LR of an extension of the path-based formulation presented in Sect. 12.3.2, to obtain optimal solutions for uncapacitated dynamic hub location problems with up to 100 nodes and 10 time periods. Alibeyg et al. (2018) develop an exact BB algorithm that uses a LR to solve hub location problems with profits involving up to 100 nodes.

Benders decomposition (BD) is another successful method used to optimally solve several classes of HLPs. Camargo et al. (2009) use a BD algorithm to solve large-scale instances of the challenging flow-dependent cost (FLOWLOC) model. Contreras et al. (2011b) describe an exact algorithm for the UHLPMA which applies an enhanced BD to the path-based formulation presented in Sect. 12.3.2, to obtain optimal solutions for large-scale instances with up to 500 nodes. Contreras et al. (2012) provide an extension of the previous BD to solve multi-capacity HLPs with multiple assignments, with splittable and non-splittable commodities, for instances with up to 300 nodes. Contreras et al. (2011a) develops a Monte Carlo simulation-based algorithm that integrates a BD to solve uncapacitated HLPs having stochastic flow costs. Camargo et al. (2013) describe a BD algorithm to solve hub location-routing problems, in which additional routing decisions to serve O/D nodes are considered. This algorithm can solve instances with up to 100 nodes. Several BD algorithms have also been implemented for HLPs with congestion costs for both multiple (Camargo et al. 2009) and single (Camargo et al. 2011; Camargo and Miranda 2012) assignments versions, HALPs with particular topological structures such as tree-start networks (Martins de Sá et al. 2013) and hub-line networks (Martins de Sá et al. 2015, 2014b), HLPs arising in public transportation networks (Gelareh and Nickel 2011), liner shipping applications (Gelareh and Nickel 2011; Gelareh and Pisinger 2011), and incomplete hub networks (Miranda et al. 2017; Martins de Sá et al. 2018a,b).

Branch-and-cut (BC) methods have also been developed to optimally solve various HLPs. Labbé et al. (2005) develop a BC algorithm based on the two-index formulation presented in Sect. 12.3.1 for various classes of capacitated HLPs with single assignments. This method is able to solve to optimality instances with up to 50 nodes. García et al. (2012) presents a BC algorithm for the uncapacitated p-hub median problem with multiple assignments. This algorithm uses an extension of the two-index formulation presented in Sect. 12.3.2 and is able to optimally solve large-scale instances with up to 200 nodes with very large values of p. Contreras and Fernández (2014) also introduce a BC algorithm based on the two-index formulation for the general class of *supermodular hub location problems* presented in Sect. 12.2.2. This method is able to solve q-hub arc location problems with up to 125 nodes. Contreras et al. (2010, 2017) use an adaptation of the flow-based formulation introduced in Sect. 12.3.1 to develop BC algorithms to solve HLPs with tree-star and cycle-star topologies, respectively. Contreras et al. (2017) is able to solve to optimality instances with up to 100 nodes. Catanzaro et al. (2011) study a incomplete hub network design problem with additional graph partitioning and routing decisions. Rodríguez-Martín et al. (2014) introduce a BC algorithm for a hub location-routing problem, which is able to solve instances with up to 50 nodes. Meier and Clausen (2018) present a novel linearization technique together with a cutting plane algorithm to solve uncapacitated and capacitated hub location problems with single assignments. This linearization, requiring only two-index variables, is applicable in the case of Euclidean data and can be used to solve instances with up to 200 nodes.

Column generation (CG) is the method that has received the least attention in the hub location literature. Thomadsen and Larsen (2007) present a branch-and-price method for solving a HLP with fully interconnected access networks. Contreras et al. (2011d) develop an exact algorithm, that combines LR and CG methods as a bounding procedure, to obtain optimal solutions of large-scale capacitated HLPs with single assignments with up to 200 nodes. Rothenbcher et al. (2016) propose an exact branch-and-price-and-cut algorithm for the service network design and hub location problem. It uses a path-based formulation solved via column generation as a bounding procedure at the nodes of the tree. It also uses several families of valid inequalities to strengthen the LP bounds.

12.6 Conclusions

We have provided an overview of hub location problems in which both the location of hubs and the design of the hub network are key decisions. We have highlighted how the commonly used assumptions presented in Sect. 12.2.1 simplify network design decisions, which have created a first generation of *idealized* hub location models focusing mostly on location and allocation decisions. Several researchers have exploited the rich structure of these models and as a consequence, significant progress has been made on the development of strong MIP formulations and efficient algorithms for their solution.

Strong path-based formulations, used in combination with sophisticated decomposition methods, have proven to be among the most effective formulations to solve to optimality large-scale instances (with hundreds of nodes) for several classes of hub location problems. Flow-based formulations, having fewer variables and constraints, have been particularly useful when used with general purpose MIP solvers to solve small to medium-size instances (containing usually no more than 50 nodes) for a wide range of problems without having to develop ad hoc solution algorithms. These formulations have also been strengthened with the addition of valid inequalities and have been used within a cutting plane framework to solve challenging hub location variants. Over the past few years, promising two-index (integer linear) formulations have started to arise. However, a substantial amount of work still needs to be done to analyze how these can be used as a basis for sophisticated algorithms.

We have also pointed out how location and network design decisions become more involved when relaxing some of the *simplifying* assumptions presented in Sect. 12.2.1. In particular, Sect. 12.4.1 described several classes of hub network topologies, arising from different areas of application, which have started to be studied. The resulting hub location problems contain additional hub arc and access arc selection decisions, making them substantially more difficult to model and solve than first generation problems considering full interconnection between hubs and access networks characterized by single or multiple assignment patterns. Section 12.4.2 focused on more realistic models with discounting levels that depend on the amount of flow passing through each arc to better model the flow cost. Although some flow-dependent models have already been presented in the literature, alternative modeling approaches need to be studied to represent more accurately flow costs, specially on transportation applications. Section 12.4.3 reviewed several capacitated hub location models, most of which focus on capacity restrictions on the hub nodes and only a few of them on the links. More complex problems combining both types of capacities need to be studied. Section 12.4.4 described some models in which specific sources of uncertainty were considered, mostly from a stochastic programming perspective. However, additional aspects such as congestion on hubs and arcs, reliability, and disruptions, among other things, need to be further studied. Very few models considering dynamic and multi-modal features have been proposed (Sect. 12.4.5). Additional models need to be developed to better represent the optimal evolution of hub networks and the choice for mode of transportation. Given that most companies using hub networks are not monopolists in a market and are also not redesigning their network from scratch, competition and collaboration are very important aspects in most hub location applications (Sect. 12.4.6). For this reason, additional models that consider a competitive environment, collaborations, mergers, acquisitions, and divestments of companies, need to be further studied.

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