Chapter 10
Ordered Median Location Problems

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Abstract This chapter analyzes the ordered median location problem in three different frameworks: continuous, discrete and networks; where some classical but also new results have been collected. For each solution space we study general properties that lead to solution algorithms. In the continuous case, we present two solution approaches for the planar case with polyhedral norms (the most intuitive case) and a novel approach applicable for the general case based on a hierarchy of semidefinite programs that can approximate up to any degree of accuracy the solution of any ordered median problem in finite dimension spaces with polyhedral or ℓ_p-norms. We also cover the problem on networks deriving finite dominating sets for some particular classes of λ parameters and showing the impossibility of finding a FDS with polynomial cardinality for general lambdas in the multifacility case. Finally, we present a covering based formulation for the capacitated discrete ordered median problem with binary assignment which is rather promising in terms of gap and CPU time for solving this family of problems.

10.1 Introduction

The Ordered Median location problem, see Nickel and Puerto (2005), has been recognized as a powerful tool from a modeling point of view within the field of Location Analysis. Actually, this problem provides a common framework for most of the classical location problems (median, center, k-centrum, centdian, trimmed-mean, among others) as well as for others which have not been studied before. As an illustrative example, in the well-known case of logistics supply chain networks, this modeling tool allows to distinguish the roles played by the different parties

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in the network inducing new type of distribution patterns, see Kalcsics et al. (2010a,b). This type of formulation incorporates flexibility through rank dependent compensation factors, and it allows one to model situations where the driving force in a distribution problem can fall in any of its different parties.

The goal of the ordered median location problem is to minimize the ordered weighted average of the distances or transportation costs, between the clients/demand points and the server, once we have applied rank dependent compensation factors on them. These rank dependent weights allow us, for instance, to compensate unfair situations. Indeed, if a solution places a set of facilities so that the accessibility cost of a demand point at j is in the s-th position in the ordered sequence of cost between each client and its corresponding server and the cost of a demand point at j' is in the t-th position with $s < t$, the model tries to favor j' with respect to j by assigning to the demand point in the s-th position a smaller weight than the one assigned to demand point in the t-th position. (Note that these weights do not penalize site j but instead they compensate site j' because these weights reduce the dispersion of the costs.) In order to incorporate this ordinal information in the overall transportation cost, the objective function applies a correction factor to the transportation cost for each demand point (to reach the facility) which is dependent on the position of that cost relative to similar costs from other demand points. For example, a different penalty might be applied if the transportation cost of a demand point at j was the 5th-most expensive cost rather than the 2nd-most expensive, see Boland et al. (2006), Marín et al. (2009), Nickel and Puerto (2005), Puerto and Fernández (2000), Rodríguez-Chía et al. (2000). It is even possible to neglect some costs by assigning a zero penalty. This adds a "sorting"-problem to the underlying location problem, making its formulation and solution more challenging.

This type of objective function has been extensively studied and successfully applied in a variety of problems within the literature of Location Analysis. Puerto and Fernández (2000) and Papini and Puerto (2004) characterize the structure of optimal solutions sets. Rodríguez-Chía et al. (2000), Blanco et al. (2013, 2014), Espejo et al. (2009), Nickel et al. (2005), Drezner (2007), Drezner and Nickel (2009a,b) and Rodríguez-Chía et al. (2010), among others, develop algorithms for different continuous ordered median location problems. In addition, there are nowadays some successful approaches available when the framework space is either discrete (see Boland et al. 2006; Domínguez-Marín et al. 2005; Espejo et al. 2009; Labbé et al. 2017; Martínez-Merino et al. 2017; Deleplanque et al. 2018; Marín et al. 2009, 2010; Puerto et al. 2011, 2014, 2013; Redondo et al. 2016; Turner et al. 2015) or a network (see Berman et al. 2009; Kalcsics et al. 2003, 2002; Nickel and Puerto 1999; Puerto and Tamir 2005; Puerto and Rodríguez-Chía 2005; Rozanov and Tamir 2018; Turner and Hamacher 2011). The interested reader is also referred to Chap. 7 in this book and Blanco et al. (2018) for some applications to the location of extensive facilities.

The aim of this chapter is to introduce the reader into the field of ordered median location providing some modeling tools and properties. These elements will allow one to formulate and solve location problems in different solution spaces (continuous, networks and discrete) using this unifying tool. To achieve this goal, in the next section we formally introduce the family of ordered median functions (OMf). Sections 10.3.2, 10.4 and 10.5 are devoted to analyze the ordered median location problem in three different solution spaces: continuous, networks and discrete, respectively. The chapter ends with some concluding remarks.

10.2 The Ordered Median Function

As mentioned above, the structure of Ordered Median Functions involves a nonlinearity in the form of an ordering operation that introduces a degree of complication but at the same time gives an extra freedom which allows one a lot of flexibility in modeling. In this section, we will review interesting properties of these functions in a first step to understand their behavior and then, we shall give a characterization of this objective function.

We start defining the ordered median function. This function is a weighted average of ordered elements. For any $x \in \mathbb{R}^n$ denote $x_{ord} = (x_{(1)}, \ldots, x_{(n)})$ where $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$. We consider the function:

$$\begin{aligned} sort_n : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x &\longrightarrow x_{ord}. \end{aligned} \tag{10.1}$$

Definition 10.1 The function $f_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}$ is an ordered median function, for short $f_\lambda \in \mathsf{OMf}(n)$, if $f_\lambda(x) = \langle \lambda, sort_n(x) \rangle$ for some $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, where $\langle , \rangle$ denotes the usual scalar product in $\mathbb{R}^n$.

It is clear that ordered median functions are nonlinear. Whereas the nonlinearity is induced by the sorting. One of the consequences of this sorting is that the pseudo-linear representation given in Definition 10.1 is pointwise defined. Nevertheless, one can identify its linearity domains. (See Puerto and Fernández 2000; Nickel and Puerto 2005; Rodríguez-Chía et al. 2000.) The identification of these regions provides us with a subdivision of the framework space where in each of its cells the function is linear. Obviously, the topology of these regions depends on the space and on the lambda vector. A detailed discussion can be found in Puerto and Fernández (2000). As mentioned in the introduction, different choices of lambda lead also to different functions within the same family: $\lambda = (1/n, \ldots, 1/n)$ is the mean average, $\lambda = (0, \ldots, 0, 1)$ is the center, $\lambda = (\alpha, \ldots, \alpha, \alpha, 1)$ is the α-centdian, $\alpha \in [0, 1]$, $\lambda = (0, \ldots, 0, 1, \stackrel{k}{\ldots}, 1)$ is the k-centrum or $\lambda = (\alpha, 0, \ldots, 0, 1 - \alpha)$ is Hurwicz's criterion, see Chaps. 1, 2 and 4 for further details.

These functions are not new and some operators related to them have been developed by other authors independently. This is the case of the ordered weighted operators (OWA) studied by Yager (1988) to aggregate semantic preferences in the context of artificial intelligence; as well as SAND functions (isotone and sublinear functions) introduced by Francis et al. (2000) to study aggregation errors in multifacility location models.

First, we recall some simple properties and remarks concerning ordered median functions. Most of them are natural questions that appear when a family of functions is considered. Partial answers are summarized in the following proposition.

Proposition 10.1 *Let $f_\lambda(x), f_\mu(x) \in \mathit{OMf}(n)$.*

(1) $f_\lambda(x)$ is a continuous function.
(2) $f_\lambda(x)$ is a symmetric function , i.e., for any $x \in \mathbb{R}^n$ $f_\lambda(x) = f_\lambda(sort_n(x))$.
(3) $f_\lambda(x)$ is a convex function iff $\lambda_1 \le \ldots \le \lambda_n$.
(4) If c_1 and c_2 are constants, then the function $c_1 f_\lambda(x) + c_2 f_\mu(x) \in \mathit{OMf}(n)$.
(5) If $\{f_{\lambda^r}(x)\}$ is a sequence of ordered median functions that pointwise converges to a function f, then $f \in \mathit{OMf}(n)$.
(6) If $\{f_{\lambda^r}(x)\}$ is a set of ordered median functions, all bounded above in each point x of $\mathbb{R}^n$, then the pointwise maximum (or sup) function defined at each point x is in general not an OMf.
(7) Let $p < n - 1$ and $x^p = (x_1, \ldots, x_p)$, $x^{\setminus p} = (x_{p+1}, \ldots, x_r)$. If $f_\lambda(x) \in \mathit{OMf}(n)$ then $f_{\lambda^p}(x^p) + f_{\lambda^{\setminus p}}(x^{\setminus p}) \lesseqqgtr f_\lambda(x)$.
(8) Every ordered median function $\mathit{OMf}(n)$ is a difference of two positively homogeneous convex functions and has a representation

$$f_\lambda(x) = \sum_{i=1}^{n} \lambda_1 \varphi_i(x),$$

where
$\varphi_r(x) = \min\left\{\max\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\} | i_1 < i_2 < \ldots < i_r \text{ and } i_1, i_2, \ldots, i_r \in \{1, \ldots, n\}\right\}$.

Proof The proof of (1) can be found in Rosenbaum (1950). The proof of (3) and (8) are in Grzybowski et al. (2011). The proofs of items (2) and (4) are straightforward and therefore are omitted. A proof of (5) and counterexamples for (6) and (7) are given in Nickel and Puerto (2005, Examples 1.1 and 1.2). □

In order to continue the analysis of the ordered median function we need to introduce some notation that will be used in the following. Let $\mathscr{P}(1 \ldots n)$ be the set of all the permutations of the first n natural numbers,

$$\mathscr{P}(1 \ldots n) = \{\pi : \pi \text{ is a permutation of } 1, \ldots, n\}. \tag{10.2}$$

We write $\pi = (\pi(1), \ldots, \pi(n))$.

The next result, that we include for the sake of completeness, is well-known and its proof can be found in the book by Hardy et al. (1952).

Lemma 10.1 *Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two vectors in $\mathbb{R}^n$. Suppose that $x \le y$, then $x_{ord} = (x_{(1)}, \ldots, x_{(n)}) \le y_{ord} = (y_{(1)}, \ldots, y_{(n)})$*

To understand the nature of the OMf we need a precise characterization. This will be done in the following two results using the concepts of symmetry and sublinearity.

Theorem 10.1 *A function f defined over $\mathbb{R}^n_+$ is continuous, symmetric and linear over $\{x : 0 \leq x_1 \leq \ldots \leq x_n\}$ if and only if $f \in$ OMf(n).*

Proof Since f is linear over $X^{\leq} := \{x \geq 0 : 0 \leq x_1 \leq \ldots \leq x_n\}$, there exists $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that for any $x \in X^{\leq}$ $f(x) = \langle \lambda, x \rangle$. Now, let us consider any $y \notin X^{\leq}$. There exists a permutation $\pi \in \mathscr{P}(1 \ldots n)$ such that $y_\pi \in X^{\leq}$. By the symmetry property it holds $f(y) = f(y_\pi)$. Moreover, for y_π we have $f(y_\pi) = \langle \lambda, y_\pi \rangle$. Hence, we get that for any $x \in \mathbb{R}^n$

$$f(x) = \langle \lambda, x_{ord} \rangle.$$

Finally, the converse is trivially true. □

There are particular instances of the λ vector that make their analysis interesting. One of them is the convex case, i.e., $\lambda_1 \leq \ldots \leq \lambda_n$, where we can obtain a characterization without the explicit knowledge of a linearity region.

Theorem 10.2 *Given $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$; and $\lambda_\pi = (\lambda_{\pi(1)}, \ldots, \lambda_{\pi(n)})$ with $\pi \in \mathscr{P}(1 \ldots n)$, a symmetric function f defined over $\mathbb{R}^n$ is the support function of the set $S_\lambda = conv\{\lambda_\pi : \pi \in \mathscr{P}(1 \ldots n)\}$ if and only if f is the convex ordered median function*

$$f_\lambda(x) = \sum_{i=1}^{n} \lambda_i x_{(i)}. \tag{10.3}$$

Proof Let us assume that f is symmetric and the support function of S_λ. Then,

$$f(x) = \sup_{s \in S_\lambda} \langle s, x \rangle = \sup_{\pi \in \mathscr{P}(1 \ldots n)} \langle \lambda_\pi, x \rangle = \sup_{\pi \in \mathscr{P}(1 \ldots n)} \langle \lambda, x_\pi \rangle = \sum_{i=1}^{n} \lambda_i x_{(i)}.$$

Conversely, it suffices to apply Theorem 368 in Hardy et al. (1952) to (10.3). □

Convexity is an important property within the scope of continuous optimization. Thus, it is crucial to know the conditions that ensure this property. Nevertheless, in the context of discrete optimization convexity cannot even be defined. Nevertheless, in this case submodularity plays a similar role. (The interested reader is referred to the chapter of the Handbook Discrete Optimization by McCormick 2005.) In the following, we also recall a submodularity property of the convex ordered median function, Puerto and Tamir (2005).

Let $x = (x_i)$, $y = (y_i)$, be vectors in $\mathbb{R}^n$. Define the *meet* of x, y to be the vector $x \bigwedge y = (\min\{x_i, y_i\})$, and the *join* of x, y by $x \bigvee y = (\max\{x_i, y_i\})$. The meet and join operations define a lattice on $\mathbb{R}^n$.

Theorem 10.3 (Submodularity Theorem) *Given $\lambda = (\lambda_1, \ldots, \lambda_n)$, satisfying $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$, $f_\lambda(x)$ is submodular over the lattice defined by the above meet and join operations, i.e.,*

$$f_\lambda(x \bigvee y) + f_\lambda(x \bigwedge y) \leq f_\lambda(x) + f_\lambda(y), \quad \forall x, y \in \mathbb{R}^n.$$

10.3 The Continuous Ordered Median Problem

This section is devoted to the analysis of the Ordered Median Location Problem in a continuous framework. For the ease of understanding, we have divided this section in two main parts. In the first one, we restrict ourselves to the polyhedral gauges emphasizing the planar case. In this setting, one can derive nice geometrical properties that help to capture the main elements of the problem, namely its linearity domains, ordered regions and intuitive algorithms for obtaining the optimal solutions. Second, we address a general case where we shall apply a new global optimization technique that allows us to handle and solve a wide range of ordered median location problems.

10.3.1 The Single Facility Polyhedral Ordered Median Location Problem

Consider a set of demand points $A = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{R}^n$ (representing existing facilities or clients) and two sets of non negative scalars $w = (w_1, \ldots, w_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$. The element w_i is the weight assigned to the existing facility a_i and it represents the importance of this demand point. The elements of λ allow us to choose between different kinds of objective functions. We also consider a gauge $\gamma(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}$ to measure distances. Recall that any gauge is defined by the Minkowski functional of a compact, convex set with the zero in its interior (see Nickel and Puerto 2005).

The ordered median problem is given by:

$$\min_{x \in \mathbb{R}^n} F(x) = \langle \lambda, sort_n((w_1\gamma(x - a_1), \ldots, w_n\gamma(x - a_n))) \rangle. \tag{10.4}$$

Note that the problem is well-defined even if ties occur. In that case any order of the tied positions gives the same value.

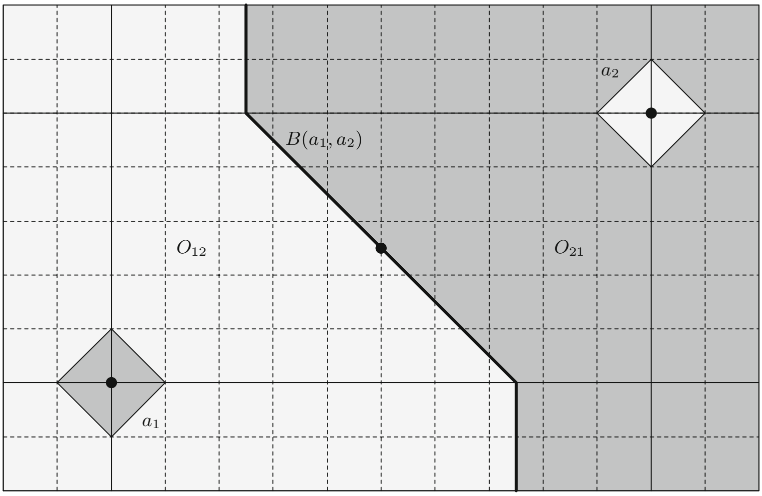

Fig. 10.1 Two regions where the function of Example 10.1 has different linear representation

Example 10.1 Consider two demand points $a_1 = (0, 0)$ and $a_2 = (10, 5)$, $\lambda_1 = 100$ and $\lambda_2 = 1$ with ℓ_1-norm as gauge and $w_1 = w_2 = 1$. We obtain only two optimal solutions to Problem (10.4), lying in each demand point. Observe that a linear representation of the objective function is regionwise defined and that the objective function is not convex since we have a nonconvex optimal solution set, see Fig. 10.1,

$$F(a_1) = 100 \times 0 + 1 \times 15 = 15$$
$$F(a_2) = 100 \times 0 + 1 \times 15 = 15$$
$$F(\frac{1}{2}(a_1 + a_2)) = 100 \times 7.5 + 1 \times 7.5 = 757.5.$$

In this section, for the sake of presentation, we restrict ourselves to study the particular case where the distances are measured with polyhedral gauges, i.e., the unit balls associated with these gauges are convex polytopes. For this reason we will assume in this subsection that $B \subseteq \mathbb{R}^n$ is a bounded polytope whose interior contains the zero and we denote the set of extreme points of B by $Ext(B) = \{e_g : g = 1, \ldots, \mathcal{G}\}$. The polar set B^0 of B is given by $B^0 = \{x \in \mathbb{R}^n : \langle x, p \rangle \leq 1 \quad \forall p \in B\}$. In the polyhedral case, B^0 is also a polytope, see Ward and Wendell (1985) and Durier and Michelot (1985). The normal cone to B at x is given by $N(B, x) := \{p \in \mathbb{R}^n : \langle p, y - x \rangle \leq 0 \quad \forall y \in B\}$ and the boundary of B is denoted by $bd(B)$.

In what follows, we recall some geometrical properties of the planar formulation of Problem (10.4) which give us specific insights into the considered model. In this case we define fundamental directions as the halflines defined by 0 and the extreme points of B. Let $\pi = (p_i)_{i=1,\ldots,n}$ be a family of elements of $\mathbb{R}^n$ such that $p_i \in B^0$ for each $i \in \{1,\ldots,n\}$ and let $C_\pi = \bigcap_{i=1}^n (a_i + N(B^o, p_i))$. A nonempty convex set C is called an elementary convex set (e.c.s.) if there exists a family π such that $C_\pi = C$.

It should be noted that if the unit balls are polytopes we can obtain the elementary convex sets as intersections of cones generated by fundamental directions of these balls pointed at each demand point. Therefore each elementary convex set is a polyhedron whose vertices are called intersection points (see Fig. 10.1). Finally, we recall that in the planar case an upper bound of the number of elementary convex sets is $O(n^2\mathscr{G}^2)$ where $\mathscr{G}$ is the number of extreme points of B (see Durier and Michelot (1985) for further details).

Although the objective function of Problem (10.4) may look like the one of the Weber problem we do not have a unified linear representation of such a function in the whole space. From the definition of the objective function, it is easy to see, that the representation may change every time $\gamma(x-a_i)-\gamma(x-a_j)$ becomes 0 for some $i, j \in \{1,\ldots,n\}$ with $i \neq j$. Next, we analyze the sets where the representation of the objective function as a weighted sum stays unchanged.

Definition 10.2 The set $B_\gamma(a_i, a_j)$ consisting of points $\{x \ : w_i\gamma(x-a_i) = w_j\gamma(x-a_j), i \neq j\}$ is called bisector of a_i and a_j with respect to γ.

As an illustration of Definition 10.2 one can see in Fig. 10.1 the bisector line for the points a_1 and a_2 with the ℓ_1-norm. The set of bisectors builds a subdivision of the plane (very similar to the well-known order$-k$ Voronoi diagrams, see the book Okabe et al. 1992). The cells of this subdivision will be called from now on ordered regions. We formally introduce this concept.

Definition 10.3 Given a permutation $\sigma \in \mathscr{P}(1,\ldots,n)$, the ordered region O_σ is the following set

$$O_\sigma = \{x \in \mathbb{R}^n : w_{\sigma_1}\gamma(x-a_{\sigma_1}) \leq \ldots \leq w_{\sigma_n}\gamma(x-a_{\sigma_n})\}.$$

Observe that these regions need not be convex sets, see Fig. 10.1. The ordered regions play a very important role in the algorithmic approach developed for solving the problem. Moreover, under the above hypothesis the overall number of ordered regions in the planar case is $O(n^4\mathscr{G}^2)$, see Rodríguez-Chía et al. (2000) for further details. The importance of these regions is that the ordered median function has a unique linear representation in the interior of the intersection of any ordered region with any elementary convex set. The sets resulting of these intersections are called generalized elementary convex sets and it is known that the entire set of optimal solutions of Problem (10.4) always coincides with some generalized elementary convex sets, see Puerto and Fernández (2000) for further details.

Although the set of optimal solutions of Problem (10.4) always coincides with a generalized elementary convex set, the large number of these regions and their intricate geometry requires some kind of good generation and enumeration schemes to derive an algorithm. This approach is doable in the plane for polyhedral gauges, where one can easily derive an appealing geometrical algorithm to solve these problems. Compute the subdivision of the plane induced by the lines defining the fundamental directions of the gauges and the bisectors. Observe that this construction can be efficiently performed using any algorithm to generate subdivisions induced by arrangements of hyperplanes, see Edelsbrunner (1987). The complexity of computing the ordered regions and its number is $O(n^4\mathcal{G}^2)$. Next, one needs to evaluate the objective function in each vertex of the subdivision. Each evaluation can be done in $O(n\mathcal{G}\log n\mathcal{G})$. This results in an algorithm that solves the problem in the plane with a complexity of $O(n^5\mathcal{G}^3\log n\mathcal{G})$.

In what follows we present an alternative, intuitive solution approach for the polyhedral version of the ordered median problem that consists in a enumerative algorithm that solves a linear program per visited ordered region. In order to do that, we first obtain some interesting properties of the following linear program where O_σ is an ordered region defined by the permutation σ:

$$\begin{array}{lll} \text{minimize} & \sum_{i=1}^{n}\lambda_i z_{\sigma_i} & \\ \text{subject to} & w_i\langle e_g^0, x-a_i\rangle \le z_i,\ e_g^o\in B^o, i=1,2,\ldots,n & \\ & z_{\sigma_i}\le z_{\sigma_{i+1}} \qquad\qquad i=1,2,\ldots,n-1 & \end{array} \qquad (P_\sigma)$$

where e_g^0 are the extreme points of B^0.

Lemma 10.2 *Let X^* be an optimal solution of P_σ.*

(i) *If $X^*\in O_\sigma$ then X^* is also an optimal solution to the ordered median problem constrained to O_σ.*

(ii) *If $X^*\in O_{\sigma'}\neq O_\sigma$ then the optimal solution of the ordered median problem constrained to $O_{\sigma'}$ is better than the optimal solution of the ordered median problem constrained to O_σ.*

Proof

(i) At an optimal point X^* in O_σ we have

$$w_i\langle e_{g_i}^o, X^*-a_i\rangle = z_i, i=1,2,\ldots,n\ ,\ \text{for some } g_i,$$

which means that $z_i = w_i\gamma(X^*-a_i)$ and the result follows.

(ii) At an optimal point X^* of P_σ in $O_{\sigma'}$ we have

$$\langle e_g^o, X^*-a_i\rangle < z_i \qquad \text{for all } g$$

for at least one i. This means that we can decrease the objective function by moving from O_σ to $O_{\sigma'}$ and the result follows. □

Based on Lemma 10.2 we develop another algorithm for this problem. For each ordered region we solve the problem as a linear program which geometrically means either finding the locally best solution in this ordered region or finding out that this region does not contain the global optimum by Lemma 10.2. In the former case two situations may occur. First, if the solution lies in the interior of the considered region (in $\mathbb{R}^n$) then we move to a different one not yet processed and secondly, if the solution is on the boundary we do a local search in the neighborhood regions where this point belongs to. It is worth noting that to accomplish this search a list $\mathscr{L}$ containing the already visited neighborhood regions is used in the algorithm. Besides, it is also important to realize that neither Step 2 nor Step 5 of the next algorithm need to explicitly construct the corresponding ordered region. It suffices to evaluate and to sort the distances to the demand points. In addition, this algorithm can be improved in the interesting, important case where $\lambda_1 \leq \ldots \leq \lambda_n$. In this situation the objective function is globally convex and this fact can be exploited to reduce the enumeration of the entire list of ordered regions. Indeed, if one optimal solution of any Problem P_σ is interior to the ordered region O_σ or this solution cannot be improved in adjacent regions then by the global convexity property of the objective function, it is the global minimum. Otherwise, one can follow a descent iterative scheme moving from one region to another one not previously visited. The above arguments justify the validity of the following algorithm for the convex case. Alternatively, one could simply resort to general randomized subgradient descent algorithms which, under mild conditions (see Ruszczynski and Syski 1986) will converge to the global optimal solution due to the finiteness of the linearity regions of these problems.

Algorithm 10.1

Step 1. Choose x^o as an appropriate starting point. Initialize $\mathscr{L} := \emptyset$, $y^* = x^o$.
Step 2. Consider O_{σ^o} which y^* belong to, where σ^o determines the order.
Step 3. Solve the linear program P_{σ^0}. Let $u^0 = (x_1^0, x_2^0, z_\sigma^0)$ be an optimal solution. If $x^0 = (x_1^0, x_2^0) \notin O_{\sigma^o}$ then let O_{σ^o} be such that $x^0 \in O_{\sigma^o}$ and go to Step 3.
Step 4. Let $y^o = (x_1^0, x_2^0)$.
Step 5. If y^o belongs to the interior of O_{σ^o} then set $y^* = y^0$ and go to Step 8.
Step 6. If $F(y^o) \neq F(y^*)$ then $\mathscr{L} := \{\sigma^0\}$
Step 7. If there exist i and j verifying $\gamma(y^o - a_{\sigma_i^o}) = \gamma(y^o - a_{\sigma_j^o})$ with $i < j$ such that $(\sigma_1^o, \ldots, \sigma_j^o, \ldots, \sigma_i^o, \ldots, \sigma_n^o) \notin \mathscr{L}$ then do

(a) $y^* := y^o, \sigma^o := (\sigma_1^o, \sigma_2^o, \ldots, \sigma_j^o, \ldots, \sigma_i^o, \ldots, \sigma_n^o)$
(b) $\mathscr{L} := \mathscr{L} \cup \{\sigma^o\}$
(c) go to Step 3

else go to Step 8 (Optimum found)
Step 8. Output y^*

The above algorithm is efficient in the sense that it is polynomially bounded in fixed dimension. Once the dimension of the problem is fixed, its complexity is dominated by the complexity of solving a linear program for each ordered region. Since the number of ordered regions is polynomially bounded, Algorithm 10.1 is polynomial.

The nice geometry of the problem in the plane allows us to derive the two above algorithms. Nevertheless, this geometry in higher dimension is rather intricate and the above approach, based on building ordered regions, is very difficult since no efficient algorithm for computing bisectors is available in dimension greater than 2.

In spite of that, we will present an alternative algorithm for solving the single facility ordered median problem in any dimension d. For this, we shall introduce a valid MILP model that provides the optimal solution of the problem. Indeed, consider the following set of binary variables

$$z_{ij} := \begin{cases} 1 & \text{if the distance induced by facility } i \\ & \text{goes in sorted position } j \\ 0 & \text{otherwise.} \end{cases}$$

and the continuous variable

$$\theta_j = \text{distance between a facility and its server in the } j\text{-th position in the ordered sequence of distances between each facility and its corresponding server.}$$

In order to minimize the ordered median function for a given set of nonnegative lambda parameters $\lambda_1, \ldots, \lambda_n$, we define the following problem.

$$\text{minimize} \sum_{j=1}^{n} \lambda_j \theta_j \tag{10.5}$$

$$\text{subject to } (1 - z_{ij})M + \theta_j \geq w_i \langle e_g^0, x - a_i \rangle, e_g^o \in B^o, \; i, j = 1, 2, \ldots, n \tag{10.6}$$

$$\sum_{i=1}^{n} z_{ij} = 1, \quad j = 1, \ldots, n \tag{10.7}$$

$$\sum_{j=1}^{n} z_{ij} = 1, \quad i = 1, \ldots, n \tag{10.8}$$

$$\theta_j \leq \theta_{j+1}, \quad j = 1, \ldots, n-1 \tag{10.9}$$

$$\theta_j \geq 0, \quad j = 1, \ldots, n \tag{10.10}$$

$$z_{ij} \in \{0, 1\}, \quad i, j = 1, \ldots, n \tag{10.11}$$

$$x \in \mathbb{R}^d. \tag{10.12}$$

Constraints (10.7) and (10.8) define a permutation by placing a single distance to a facility at each position and each distance to a facility at a single sorted position. Constraints (10.6) relate distance values with the values placed in a sorted sequence. Constraint (10.9) imposes that the sorted values are ordered non-increasingly. Finally, (10.10)–(10.12) define the range of variables of the model.

The above approach solves efficiently the problem in any dimension provided that the gauges used to measure distances are polyhedral since Problem (10.5)–(10.12) is a MILP that can be handled with any of the nowadays available MIP solvers.

We would like to conclude this section with some comments on several extensions of the considered problem. On the one hand, the multicriteria planar version of the above problem was analyzed in Nickel et al. (2005). On the other hand, the planar case of the ordered median problem using an ℓ_p-norm was also studied by Drezner and Nickel (2009a,b) where techniques of global optimization were used for solving it. In addition, Espejo et al. (2009), Rodríguez-Chía et al. (2010) proposed an adaptation of the Weiszfeld algorithm for the convex version of this problem, i.e., $0 \leq \lambda_1 \leq \ldots \leq \lambda_n$. Finally, we would like to mention some references that consider the multifacility version of particular classes of ordered median problems. These references can be seen as a starting point to dig into this challenging topic. The interested reader is referred to Blanco et al. (2016), Ben-Israel and Iyigun (2010), Brimberg et al. (2000), Schöbel and Scholz (2010) for different approaches to the continuous multifacility location problem.

10.3.2 Generalized Continuous Ordered Median Location Problems

This section extends the analysis presented above, in Sect. 10.3.1, to the case of non-polyhedral norms and any dimension d. In doing that we shall cast that problem within the more general paradigm of polynomial programming. This approach allows us to apply powerful tools borrowed from the theory of global optimization to solve our original problem, see Blanco et al. (2013). This section contains advanced material which is self-contained. For this reason those nonspecialized readers not interested in global optimization techniques may decide to skip it without losing continuity with the remaining sections of this chapter.

We are given a set $A = \{a_1, \ldots, a_n\} \subset \mathbb{R}^d$ endowed with a ℓ_τ-norm (here ℓ_τ stands for the norm $\|x\|_\tau = \left(\sum_{i=1}^d |x_i|^\tau\right)^{1/\tau}$, for all $x \in \mathbb{R}^d$); and a feasible domain $\mathbf{K} := \{x \in \mathbb{R}^d : g_j(x) \geq 0, \quad j = 1, \ldots, \ell\} \subset \mathbb{R}^d$, assumed to be a closed semi-algebraic set, i.e., a set defined by a finite number of polynomial inequalities, where each $g_j(x) \in \mathbb{R}[x]$ is a polynomial, being $\mathbb{R}[x]$ the ring of real polynomials in $(x_1, \ldots, x_d)$. Since we are interested in solving location problems we shall assume without loss of generality that we wish to solve the problem in a bounded domain so that $\mathbf{K}$ is compact. The goal is to find a point $x^* \in \mathbf{K}$ minimizing some globalizing function of the distances to the set A. Here, we consider that the globalizing function is rather general and that it is given as an ordered weighted average of polynomials (the reader may observe that the same approach also extends to rational functions, Blanco et al. 2013).

Some well-known examples, that are formulated in the above terms, are the following (see, e.g., Blanquero and Carrizosa 2009, Drezner 2007, Espejo et al. 2009, Kalcsics et al. 2015, López-de-los-Mozos et al. 2008 or Nickel and Puerto 2005): $f(u_1,\ldots,u_n) = \sum_{i<j}^{n} |u_i - u_j|$, is the absolute deviation or envy criterion, $f(u_1,\ldots,u_n) = \sum_{i=1}^{n}(u_i - 1/n\sum_{j=1}^{n} u_j)^2$, is the variance function, $f(u_1,\ldots,u_n) = \sum_{j=1}^{n} w_j/u_j^2$, where w_j are scalar weights, is the obnoxious facility criterion and $f(u_1,\ldots,u_n) = \sum_{j=1}^{n} b_j/(1 + h_j|u_j|^{\lambda})$, with b_j and h_j appropriate weights, is the Huff competitive location objective function.

The main feature and what distinguishes location problems from other general purpose optimization problems, is that the dependence of the decision variables is given through the norms to the demand points in A, i.e., $\|x - a_i\|_\tau$. In this section, we consider a generalized version of the ordered continuous single facility location problem over closed semi-algebraic feasible sets, i.e., the Ordered Median of Polynomial Functions problem:

$$\rho_\lambda := \text{minimize}\,\{\sum_{j=1}^{m} \lambda_j \tilde{f}_{(j)}(x) : x \in \mathbf{K}\}, \qquad \textbf{(OMPF)}$$

where:

- $\lambda_j \in \mathbb{R}$ $j = 1,\ldots,m$ are modeling weights.
- $f_j(u) : \mathbb{R}^n \mapsto \mathbb{R}$, with $f_j(u) \in \mathbb{R}[u_1,\ldots,u_n]$ (the ring of real polynomials in $(u_1,\ldots,u_n)$), $x \in \mathbf{K}$ for all $j = 1,\ldots,m$. We shall define the dependence of f_j to the decision variable $x \in \mathbb{R}^d$ via $u = (u_1,\ldots,u_n)$, where $u_i : \mathbb{R}^d \mapsto \mathbb{R}$, $u_i(x) := \|x - a_i\|_\tau$, $i = 1,\ldots,n$. Therefore, the j-th component of the ordered median objective function of our problems reads as:

$$\begin{aligned} \tilde{f}_j(x) : \mathbb{R}^d &\mapsto \mathbb{R} \\ x &\mapsto \tilde{f}_j(x) := f_j(\|x - a_1\|_\tau,\ldots,\|x - a_n\|_\tau). \end{aligned}$$

In the classical ordered median problem these functions correspond with the distances from the demand points to the service facility, i.e. $f_j(\|x - a_1\|_\tau,\ldots,\|x - a_n\|_\tau) = \|x - a_j\|_\tau$; thus, in our application to the ordered median problem we will always assume to have $m = n$ and functions $\tilde{f}_j(x) := \|x - a_j\|_\tau$.

- $\mathbf{K} := \{x \in \mathbb{R}^d : g_j(x) \geq 0,\ j = 1,\ldots,\ell\} \subset \mathbb{R}^d$ satisfies Archimedean property. (See Lasserre (2009) for a detail discussion on the Archimedean property and its implications in real algebraic geometry and global optimization. In our setting this property is essentially equivalent to assume compact feasible regions.)
- $\tau := r/s$, $r, s \in \mathbb{N}$, $r \geq s$ and $gcd(r,s) = 1$.

First of all, since **K** is compact there exist $M' > 0$ such that $\|x\|_2 \leq M'$ for all $x \in \mathbf{K}$. Then, we observe that any feasible solution of (**OMPF**) satisfies $\|x - a_i\|_2 \leq M' + \|a_i\|_2 \leq M' + \max_{1\leq i\leq n} \|a_i\|_2 := M$. Then, since all norms are equivalent in $\mathbb{R}^d$, there exists $\gamma > 0$ such that $\|x\|_{2\tau}/\|x\|_2 \leq \gamma$, for all $x \in \mathbb{R}^d$. Hence, $\|x - a_i\|_{2\tau} \leq \gamma M =: \bar{M}$. This bound will allow us to derive the constraints (10.21) of our reformulation of Problem (**OMPF**). These constraints ensure that the feasible region is bounded which in our framework is sufficient to imply compactness. For this reason, we will call them from now on *compactness* constraints.

Next, our goal is to cast the above problem within the framework of polynomial optimization. Associated with the above minimization problem we introduce an equivalent formulation that will be useful to apply the moment tools to solve the ordered median problem. For each $i = 1, \ldots, m$, $j = 1, \ldots, m$ consider the following family of decision variables for each $x \in \mathbf{K}$

$$w_{ij} = \begin{cases} 1 \text{ if } \tilde{f}_i(x) = \tilde{f}_{(j)}(x), \\ 0 \ \text{ otherwise.} \end{cases} .$$

However, we observe that ℓ_τ-norms are not, in general, polynomials. To avoid this inconvenience, we introduce the following auxiliary problem. Observe that this formulation lifts the original problem in a higher dimensional space to represent the piecewise polynomials that appear in (**OMPF**) as polynomials in the new set of variables.

$$\overline{\rho}_\lambda = \text{minimize} \sum_{j=1}^{m} \lambda_j \sum_{i=1}^{m} f_i(u) w_{ij} := p_\lambda(x, u, v, w) \tag{10.13}$$

$$\text{subject to} \sum_{j=1}^{m} w_{ij} = 1, i = 1, \ldots, m, \tag{10.14}$$

$$\sum_{i=1}^{m} w_{ij} = 1, j = 1, \ldots, m, \tag{10.15}$$

$$\sum_{i=1}^{m} w_{ij} f_i(u) \leq \sum_{i=1}^{m} w_{ij+1} f_i(u), j = 1, \ldots, m-1, \tag{10.16}$$

$$w_{ij}^2 - w_{ij} = 0, i, j = 1, \ldots, m, \tag{10.17}$$

$$v_{k\ell}^{2s} = (x_\ell - a_{k\ell})^{2r}, \ k = 1, \ldots, n, \ \ell = 1, \ldots, d, \tag{10.18}$$

$$u_k^r = (\sum_{\ell=1}^{d} v_{k\ell})^s, \ k = 1, \ldots, n, \tag{10.19}$$

$$\sum_{j=1}^{m} w_{ij}^2 \leq 1, \; i = 1, \dots, m, \tag{10.20}$$

$$\sum_{j=1}^{d} v_{ij}^2 \leq \bar{M}^{2\tau}, \; i = 1, \dots, n, \tag{10.21}$$

$$w_{ij} \in \mathbb{R}, \; i, j = 1, \dots, m, \tag{10.22}$$

$$v_{k\ell} \geq 0, u_k \geq 0, k = 1, \dots, n, \ell = 1, \dots, d, \tag{10.23}$$

$$x \in \mathbf{K}. \tag{10.24}$$

By means of the w variables, the objective function (10.13) is the ordered weighted sum of the f_i polynomials which can be written as the polynomial p_λ. The first set of constraints (10.14) ensures that for each x, $\tilde{f}_i(x)$ is sorted in a unique position. The second set (10.15) ensures that the jth position is only assigned to one polynomial function. The next constraints (10.16) state that $f_{(1)}(u) \leq \cdots \leq f_{(m)}(u)$. Constraints (10.17) are added to assure that $w_{ij} \in \{0, 1\}$. Next, the two families of constraints (10.18) and (10.19) set u_k^r as the correct value of $\|a_k - x\|_\tau$ (recall that $\tau = r/s$). The last set of constraints (10.20) and (10.21) ensure that Archimedean property holds for the new feasible region $\overline{\mathbf{K}}$ of the above auxiliary problem. (Note that this last set of constraints are redundant but it is convenient to add them for a better description of the feasible set.)

We also observe that the above problem simplifies for those cases where r is even. In these cases, we can replace the constraints (10.18) by the simplest constraints

$$v_{k\ell}^s = (x_k - a_{k\ell})^r, \quad \forall\, k, \ell.$$

This reformulation reduces the degree of the polynomials defining the feasible set.

We illustrate the above formulation with a standard model in location analysis: the k-centrum problem in the plane.

Example 10.2 Let us assume that we are given a set of demand points $A = \{a_1, \dots, a_n\} \subset \mathbb{R}^2$, where $a_i = (a_{i1}, a_{i2})$, for $i = 1, \dots, n$. We wish to model the k-centrum ($k < n$) with ℓ_3-distance, i.e., $r = 3$ and $s = 1$, with respect to the demand points in A and a feasible region defined by a set $\mathbf{K}$. It is clear that in this case $d = 2$, $m = n$ and each function $\tilde{f}_i(x) := \|x - a_i\|_3, \; i = 1, \dots, n$.

According to the model above this problem can be formulated as follows:

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{j=n-k+1}^{n}\sum_{i=1}^{n} u_i w_{ij} & \\
\text{subject to} & \displaystyle\sum_{i=1}^{n} w_{ij} = 1, & j = 1, \ldots, n, \\
& \displaystyle\sum_{i=1}^{n} w_{ij} = 1, & j = 1, \ldots, n, \\
& \displaystyle\sum_{i=1}^{n} w_{ij} u_i \leq \sum_{i=1}^{n} w_{ij+1} u_i, & j = 1, \ldots, n-1 \\
& w_{ij}^2 - w_{ij} = 0, & i, j = 1, \ldots, n, \\
& v_{k\ell}^2 = (x_\ell - a_{k\ell})^6, & k = 1, \ldots, n, \ \ell = 1, \ldots, 2, \\
& u_k^3 = (\displaystyle\sum_{\ell=1}^{d} v_{k\ell}), & k = 1, \ldots, n, \\
& \displaystyle\sum_{j=1}^{n} w_{ij}^2 \leq 1, & i = 1, \ldots, n, \\
& \displaystyle\sum_{j=1}^{2} v_{ij}^2 \leq \bar{M}^6, & i = 1, \ldots, n, \\
& w_{ij} \in \mathbb{R}, & i, j = 1, \ldots, m, \\
& v_{k\ell} \geq 0, u_k \geq 0, & k = 1, \ldots, n, \ell = 1, \ldots, d, \\
& x \in \mathbf{K} &
\end{array}
$$

Next, we get a result that shows the equivalence between the above polynomial optimization formulation and our location problem (**OMPF**).

Theorem 10.4 *Let x be a feasible solution of (**OMPF**) then there exists a solution (x, u, v, w) for (10.13)–(10.24) such that their objective values are equal. Conversely, if (x, u, v, w) is a feasible solution for (10.13)–(10.24) then there exists a solution (x) for (**OMPF**) having the same objective value. In conclusion, $\rho_\lambda = \overline{\rho}_\lambda$. Moreover, if $\mathbf{K} \subset \mathbb{R}^d$ satisfies the Archimedean property then $\overline{\mathbf{K}} \subset \mathbb{R}^{d+m^2+n(d+1)}$ also satisfies the Archimedean property.*

The interested reader is referred to Blanco et al. (2013, Theorem 4) for a detailed proof.

Now, we can prove a convergence result that allows us to solve, up to any degree of accuracy, the above class of problems. In order to proceed further we need to introduce some additional material related to the Theory of Moments, Lasserre (2009).

Recall that by $\mathbb{R}[x]$ we denote the ring of real polynomials in the variables $x = (x_1, \ldots, x_d)$, for $d \in \mathbb{N}$ ($d \geq 1$), and by $\mathbb{R}[x]_r \subset \mathbb{R}[x]$ the space of polynomials of degree at most $r \in \mathbb{N}$ (here $\mathbb{N}$ denotes the set of non-negative integers). We also denote by $\mathscr{B} = \{x^\alpha : \alpha \in \mathbb{N}^d\}$ a canonical basis of monomials for $\mathbb{R}[x]$, where $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, for any $\alpha \in \mathbb{N}^d$. Note that $\mathscr{B}_r = \{x^\alpha \in \mathscr{B} : \sum_{i=1}^d \alpha_i \leq r\}$ is a basis for $\mathbb{R}[x]_r$. For any sequence indexed in the canonical monomial basis $\mathscr{B}$, $\mathbf{y} = (y_\alpha)_{\alpha \in \mathbb{N}^d} \subset \mathbb{R}$, let $\mathrm{L}_\mathbf{y} : \mathbb{R}[x] \to \mathbb{R}$ be the linear functional defined, for any $f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha \, x^\alpha \in \mathbb{R}[x]$, as $\mathrm{L}_\mathbf{y}(f) := \sum_{\alpha \in \mathbb{N}^d} f_\alpha \, y_\alpha$.

The *moment* matrix $\mathrm{M}_r(\mathbf{y})$ of order r associated with $\mathbf{y}$, has its rows and columns indexed by (x^α) and $\mathrm{M}_r(\mathbf{y})(\alpha, \beta) := \mathrm{L}_\mathbf{y}(x^{\alpha+\beta}) = y_{\alpha+\beta}$, for $|\alpha|, |\beta| \leq r$ (here $|a|$ stands for the sum of the coordinates of $a \in \mathbb{N}^d$). For $g = \sum_{\gamma \in \mathbb{N}^d} g_\gamma x^\gamma \in \mathbb{R}[x]$, the *localizing* matrix $\mathrm{M}_r(g\mathbf{y})$ of order r associated with $\mathbf{y}$ and g, has its rows and columns indexed by (x^α) and $\mathrm{M}_r(g\mathbf{y})(\alpha, \beta) := \mathrm{L}_\mathbf{y}(x^{\alpha+\beta} g(x)) = \sum_\gamma g_\gamma y_{\gamma+\alpha+\beta}$, for $|\alpha|, |\beta| \leq r$. Let $\mathbf{y} = (y_\alpha)$ be a real sequence indexed in the monomial basis $(x^\beta u^\gamma v^\delta w^\zeta)$ of $\mathbb{R}[x, u, v, w]$ (with $\alpha = (\beta, \gamma, \delta, \zeta) \in \mathbb{N}^d \times \mathbb{N}^n \times \mathbb{N}^{nd} \times \mathbb{N}^{m^2}$). Let $h_0(x, u, v, w) := p_\lambda(x, u, v, w)$, and denote $\xi_j := \lceil (\deg g_j)/2 \rceil$ and $\nu_j := \lceil (\deg h_j)/2 \rceil$, where $\{g_1, \ldots, g_\ell\}$, and $\{h_1, \ldots, h_{3m+m^2+n(d+3)}\}$ are the polynomial constraints that define $\mathbf{K}$ and $\overline{\mathbf{K}} \setminus \mathbf{K}$ in (10.13)–(10.24), respectively. For

$$r \geq r_0 := \max\{\max_{k=1,\ldots,\ell} \xi_k, \max_{j=0,\ldots,3m+m^2+n(d+3)} \nu_j\},$$

we introduce the hierarchy of semidefinite programs:

$$\begin{array}{lll} \text{minimize}_\mathbf{y} & \mathrm{L}_\mathbf{y}(p_\lambda) & \\ \text{subject to} & \mathrm{M}_r(\mathbf{y}) \succeq 0, & \\ & \mathrm{M}_{r-\xi_k}(g_k, \mathbf{y}) \succeq 0, & k = 1, \ldots, \ell, \\ & \mathrm{M}_{r-\nu_j}(h_j, \mathbf{y}) \succeq 0, & j = 1, \ldots, 3m + m^2 + n(d+3), \end{array} \qquad (\overline{\mathbf{Q}}_r)$$

with optimal value denoted $\min \overline{\mathbf{Q}}_r$.

Theorem 10.5 *Let $\overline{\mathbf{K}} \subset \mathbb{R}^{d+m^2+n(d+1)}$ be the feasible domain of Problem* (10.13)–(10.24)*. Then, with the notation above:*

(a) $\min \overline{\mathbf{Q}}_r \uparrow \rho_\lambda$ *as* $r \to \infty$.
(b) Let $\mathbf{y}^r$ *be an optimal solution of the SDP relaxation* ($\overline{\mathbf{Q}}_r$)*. If*

$$\operatorname{rank} \mathrm{M}_r(\mathbf{y}^r) = \operatorname{rank} \mathrm{M}_{r-r_0}(\mathbf{y}^r) = t$$

then $\min \overline{\mathbf{Q}}_r = \rho_\lambda$ *and one may extract t points* $(x^*(k), u^*(k), v^*(k), w^*(k))_{k=1}^t \subset \overline{\mathbf{K}}$*, all global minimizers of Problem (***OMPF***).*

Proof The convergence of the semidefinite relaxation ($\overline{\mathbf{Q}}_r$) follows from a result by Jibetean and de Klerk (2006, Theorem 9) that is applied here to the polynomial function in (10.13) and the closed semi-algebraic set $\overline{\mathbf{K}}$. The second assertion on the rank condition, for extracting optimal solutions, follows from applying (Lasserre 2009, Theorem 5.7) to the SDP relaxation ($\overline{\mathbf{Q}}_r$). □

We also observe that one can exploit the block diagonal structure of the problem (10.13)–(10.21) since the only monomials that appear in that formulation are of the form $x^\alpha u_i^\beta \prod_{j=1}^m v_{ij}^{\gamma_j}$ for all $i = 1, \ldots, m$. Hence, a result similar to Theorem 12 in Blanco et al. (2013) about a sparse reformulation also holds for this problem.

Tables 10.1 and 10.2 present some computational results obtained applying the above technique for different planar ordered median problems. Programs were coded in MATLAB R2010b and executed in a PC with an Intel Core i7 processor at 2×2.93 GHz and 8 GB of RAM. The semidefinite programs were solved by calling SDPT3 4.0, Kim-Chuan et al. (2006). We report the CPU times for computing solutions as well as the gap, ϵ_{obj}, with respect to upper bounds obtained with the battery of functions in `optimset` of MATLAB, which only provide approximations on the exact solutions (optimality cannot be certified). In order to compute the accuracy of an obtained solution, we use the following measure for the error (see Blanco et al. 2013):

$$\epsilon_{\text{obj}} = \frac{|\text{the optimal value of the SDP } - \text{fopt}|}{\max\{1, \text{fopt}\}}, \tag{10.25}$$

where fopt is the approximated optimal value obtained with the functions in `optimset`. The interested reader is referred to Blanco et al. (2013, Section 5) for further details and computational results using the tools in this section applied to location problems.

10.4 The Ordered Median Problem on Networks

Let $N = (G, \ell)$ denote a network with underlying graph $G = (V, E)$, with node set $V = \{v_1, \ldots, v_n\}$ and edge set $E = \{e_1, \ldots, e_m\}$. We restrict ourselves to undirected graphs. Therefore, we write every edge $e \in E$ as $\{i, j\}$, v_i, $v_j \in V$.

Each edge $e \in E$ is associated with a positive length by means of the function $\ell : E \to \mathbb{R}_+$. By $d(v_i, v_j)$, we denote the length of the shortest path between v_i and v_j measured by ℓ. Through $w : V \to \mathbb{R}_+ \cup \{0\}$, every vertex is assigned to a nonnegative weight. A point x on an edge $e = \{i, j\}$ is defined as a pair $x = (e, t)$, $t \in [0, 1]$, with

$$d(v_k, x) := d(x, v_k) := \min\{d(v_k, v_i) + t\ell(e), d(v_k, v_j) + (1 - t)\ell(e)\}. \tag{10.26}$$

The set of all the points of a network (G, ℓ) is denoted by $P(G)$. It should be noted that this set also contains the nodes V.

Table 10.1 Computational results for different location problems in $\mathbb{R}^2$ with ℓ_2-norm

	Weber		Center		k-Centrum k = 0.1*n		k-Centrum k = 0.5*n		Range		Trimmed-mean	
	ℓ_2		ℓ_2		ℓ_2		ℓ_2		ℓ_2		ℓ_2	
n	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}
10	0.31	0.00000127	1.33	0.00000978	1.34	0.00001760	1.34	0.00000455	1.26	−0.11849865	2.98	0.00018438
20	0.68	0.00000005	3.08	0.00001456	3.31	0.00000598	3.18	0.00000111	2.21	−0.06784203	6.34	0.00018729
30	1.00	0.00000003	5.35	0.00046734	6.34	0.00000465	5.50	0.00000123	3.10	−0.02626473	9.96	0.00013896
50	1.70	0.00000005	10.61	0.00001725	11.97	0.00000425	13.22	0.00000048	6.57	−0.07291619	20.89	0.00015183
100	3.55	0.00000004	30.83	0.00000542	38.59	0.00000292	37.58	0.00000020	14.58	−0.02572793	46.62	0.00015415
200	7.05	0.00000004	84.16	0.00001519	99.55	0.00000093	100.39	0.00000044	31.34	−0.03714671	118.09	0.00014847
300	10.66	0.00000003	139.36	0.00000386	164.28	0.00000055	159.49	0.00000005	74.49	−0.03314587	188.91	0.00014136
400	14.27	0.00000003	216.28	0.00000337	240.42	0.00000057	211.09	0.00000010	94.59	−0.04756016	304.58	0.00014574
500	17.74	0.00000003	305.36	0.00000336	328.64	0.00000028	285.02	0.00000012	172.06	−0.05599743	391.78	0.00014832

Table 10.2 Computational results for different location problems in $\mathbb{R}^2$ with ℓ_3-norm

	Weber		Center		k-Centrum k = 0.1*n		k-Centrum k = 0.5*n		Range		Trimmed-mean	
	ℓ_3		ℓ_3		ℓ_3		ℓ_3		ℓ_3		ℓ_3	
n	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}	CPU time	ϵ_{obj}
10	0.44	0.00000029	1.70	0.00000441	1.46	0.00000998	1.45	0.00000512	1.38	−0.10196862	2.87	0.00026887
20	1.01	0.00000007	3.59	0.00001389	3.71	0.00001100	4.15	0.00000065	2.70	−0.02628318	6.75	0.00017690
30	1.50	0.00000044	6.33	0.00001259	6.46	0.00000321	6.93	0.00000056	5.35	−0.09088091	11.19	0.00019343
50	2.50	0.00000018	12.91	0.00000947	13.92	0.00000554	16.20	0.00000048	10.51	−0.07220939	20.62	0.00021732
100	5.21	0.00000012	34.07	0.00000690	42.11	0.00000256	34.41	0.00000040	24.30	−0.03754705	52.83	0.00017720
200	10.73	0.00000010	87.18	0.00000663	111.38	0.00000043	98.39	0.00000028	55.67	−0.04069077	128.14	0.00018684
300	16.07	0.00000008	173.36	0.00001240	180.18	0.00000067	157.35	0.00000017	92.37	−0.07366743	191.46	0.00016696
400	21.30	0.00000015	240.12	0.00001163	262.77	0.00000053	233.61	0.00000010	154.74	−0.02080770	312.34	0.00020440
500	27.46	0.00000010	299.41	0.00000498	341.34	0.00000035	291.80	0.00000006	168.54	−0.01652014	391.24	0.00019197

10.4.1 The Single Facility Ordered Median Problem

In this section we deal with the simplest version of the ordered median problem on networks where just a single location is to be placed. In order to do that, we consider the following notation. Let

$$d(x) := (w_1 d(v_1, x), \dots, w_n d(v_n, x))$$

and

$$d_{\leq}(x) := (w_{(1)} d(v_{(1)}, x), \dots, w_{(n)} d(v_{(n)}, x))$$

a permutation of the elements of $d(x)$, verifying

$$w_{(1)} d(v_{(1)}, x) \leq w_{(2)} d(v_{(2)}, x) \leq \dots \leq w_{(n)} d(v_{(n)}, x).$$

For the sake of simplicity, let $d_{(i)}(x) := w_{(i)} d(v_{(i)}, x)$. The ordered median problem on N is defined as

$$f_\lambda(d(x)) := \sum_{i=1}^{n} \lambda_i d_{(i)}(x) \quad \text{with} \quad \lambda = (\lambda_1, \dots, \lambda_n) \geq 0\ , \tag{10.27}$$

and

$$M(\lambda) := \min_{x \in P(G)} f_\lambda(d(x)). \tag{10.28}$$

In this section we state the fundamental properties of Problem (10.28). We will present a localization result which generalizes the well-known results by Hakimi on finite dominating sets for the center and median problems on networks (Hakimi 1964) and gives some insight in the connection between median and center problems.

For all $v_i,\ v_j \in V,\ i \neq j$ define

$$EQ_{ij} := \{x \in P(G) : w_i d(v_i, x) = w_j d(v_j, x)\} \tag{10.29}$$

and let $EQ := \bigcup\{EQ_{ij} : i, j \text{ with } i \neq j\}$.

The points in EQ are called equilibria points of N. Two points $a, b \in EQ$ are called consecutive, if there is no other $c \in EQ$ on the shortest path between a and b. The points in EQ establish a partition on N with the property that for two consecutive elements $a, b \in EQ$ the permutation which gives the order of the vector $d_{\leq}(x)$ is the same for all $x \in [a, b]$.

Now we will give a finite dominating set (FDS) for the optimal locations of Problem (10.28), see Nickel and Puerto (1999) for further details.

Theorem 10.6 *An optimal solution for Problem* (10.28) *can always be found in the set* $Cand := EQ \cup V$.

Proof Starting from the original graph G, build a set of new graphs $G_1, \ldots, G_K$ by inserting all points of EQ as new nodes. Now every subgraph G_i is defined by either

I. Two consecutive elements of EQ on an edge or
II. An element $v_i \in V \backslash EQ$ and the adjacent elements of EQ

and the corresponding edges. In this situation for every subgraph G_i the permutation of $d_{\leq}(x)$ is constant (by definition of EQ). Therefore for all $x \in P(G_i)$ we have

$$\sum_{i=1}^{n} \lambda_i d_{(i)}(x) = \sum_{i=1}^{n} \lambda_i w_{\pi(i)} d(v_{\pi(i)}, x) \ ,$$

where $\pi \in P(1, \ldots, n)$, and $P(1, \ldots, n)$ is defined as the set of all permutations of $\{1, \ldots, n\}$. Therefore we can replace the objective by a classical median-objective. Now we can apply Hakimi's node dominance result in every G_i and the result follows. □

Theorem 10.6 also gives rise to some geometrical subdivision of the network N. Like indicated in the proof of Theorem 10.6 we can assign to every subgraph $G_i, i = 1, \ldots, k$ a n-tuple giving in the i-th position the i-th nearest vertex to all points in G_i. As an example we have in Fig. 10.2 a graph with 3 nodes and all weights w_i and all lengths are 1.

This partition can be seen as a kind of higher order Voronoi diagram of N quite related to the Voronoi partition of networks introduced in Hakimi et al. (1992).

For algorithmic purposes one should note that the set EQ can be computed by intersection of all distance functions, see (10.26), on all edges. Since a distance function has maximally one breakpoint on every edge we can use a line sweep

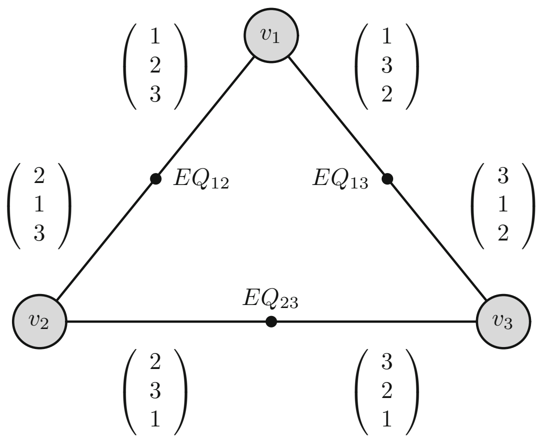

Fig. 10.2 A 3-node network with $EQ = \{EQ_{12}, EQ_{13}, EQ_{23}, v_1, v_2, v_3\}$ and the geometrical subdivision

technique to determine EQ on one edge in $O((n+k)\log n)$, where $k \le n^2$ is the number of intersection points. Therefore we can compute EQ for the whole network in $O(m(n+k)\log n)$ time. Of course, this is a worst-case bound and the set of candidates can be further reduced by some domination arguments: Take for two candidates x, y the corresponding weighted (and sorted) distance vectors $d_{\le}(x)$, $d_{\le}(y)$. If $d_{\le}(x)$ is in every component strictly smaller than $d_{\le}(y)$ then there is no positive λ with which $f_\lambda(d(y)) \le f_\lambda(d(x))$. This domination argument can be integrated in any line sweep technique reducing, in most cases, the number of candidates.

Example 10.3 Consider the network given in Fig. 10.3 with $w_1 = w_2 = w_5 = 1$ and $w_3 = w_4 = w_6 = 2$. Table 10.3 lists the set EQ, where the labels of the rows EQ_{ij} indicate that i, j are the vertices under consideration and the columns indicate

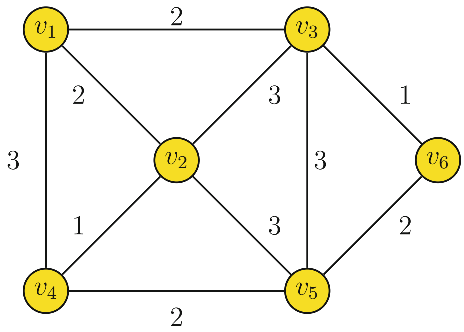

Fig. 10.3 A 6-node network used in Example 10.3 where the numbers on the edges represent their length

Table 10.3 List of the set EQ for Example 10.3

	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{2,5\}$	$\{3,5\}$	$\{3,6\}$	$\{4,5\}$	$\{5,6\}$
EQ_{12}	$\frac{1}{2}$		$\frac{2}{3}$	$\frac{5}{6}$			$\frac{2}{3}$			$\frac{1}{2}$
EQ_{13}		$\frac{2}{3}$		$\frac{4}{9}$			$\frac{2}{3}$			$\frac{1}{2}$
EQ_{14}	1		$\frac{2}{3}$	0	0	$\frac{8}{9}$	$\frac{8}{9}$			$\frac{1}{6}$
EQ_{15}			$\frac{5}{6}$		$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$		
EQ_{16}		1		1		$\frac{8}{9}$	$\frac{8}{9}$	0	$\frac{5}{6}$	
EQ_{23}		$\frac{1}{3}$		$\frac{2}{3}$			$\frac{2}{3}$			$\frac{1}{2}$
EQ_{24}			$\frac{2}{3}$		$\frac{2}{3}$				$\frac{1}{2}$	
EQ_{25}		$[\frac{3}{4}, 1]$		1		$\frac{1}{2}$	0	0	$\frac{1}{4}$	
EQ_{26}		$\frac{2}{3}$		$\frac{8}{9}$			$\frac{1}{3}$			$\frac{1}{6}$
EQ_{34}	$\frac{1}{4}$		$\frac{1}{6}$	$\frac{1}{3}$			$\frac{5}{6}$			$\frac{1}{4}$
EQ_{35}	$\frac{1}{6}$		$\frac{1}{9}$	$\frac{1}{3}$			$\frac{1}{3}$	1		1
EQ_{36}			$[\frac{5}{6}, 1]$		1	$\frac{1}{3}$	$\frac{5}{6}$	$\frac{1}{2}$	0	
EQ_{45}	$\frac{1}{2}$		$\frac{1}{3}$	$\frac{1}{3}$		$\frac{1}{9}$			$\frac{1}{3}$	
EQ_{46}	0	0	0	$\frac{1}{2}$		$[\frac{2}{3}, 1]$	$[\frac{2}{3}, 1]$		1	0
EQ_{56}		$\frac{1}{2}$		$\frac{2}{3}$			$\frac{1}{9}$			$\frac{2}{3}$

the edge $e = \{r, s\}$. The entry in the table gives for a point $x = (e, t)$ the value of t (if t is not unique an interval of values is shown).

Now we only have to evaluate the objective function with a given set of λ-values for EQ and determine the optima. Table 10.4 gives the solutions for some specific choices for λ. To describe the solution set we use the notation EQ_{kl}^{ij} to denote the part of EQ_{kl} which lies on the edge $\{i, j\}$.

Kalcsics et al. (2002) gives an FDS for the single facility ordered median problem with general node weights, i.e., the w-weights can be negative. Moreover, for the case of a directed network with non-negative w-weights, they prove that there is always an optimal solution in V.

10.4.2 The p-Facility Ordered Median Problem

In this section we deal with the multi-facility extension of the ordered median problem. The p-facility ordered median problem consists of finding a set $X_p = \{x_1, \ldots, x_p\}$ that minimizes the following objective function

$$\text{minimize}_{X_p} \sum_{i=1}^{n} \lambda_i d_{(i)}(X_p) \tag{10.30}$$

where $d(v, X_p) := \min_{i=1,\ldots,p} d(v, x_i)$ for all $v \in V$; $d(X_p) := (w_1 d(v_1, X_p), \ldots, w_n d(v_n, X_p))$ and $d_{\le}(X_p) := (w_{(1)} d(v_{(1)}, X_p), \ldots, w_{(n)} d(v_{(n)}, X_p))$ a permutation of the elements of $d(X_p)$, verifying:

$$w_{(1)} d(v_{(1)}, X_p) \le \ldots \le w_{(n)} d(v_{(n)}, X_p).$$

The main result of this section establishes a generalization of the well-known theorem of Hakimi which states that always exists an optimal solution in V.

Theorem 10.7 *If* $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$ *then Problem* (10.30) *has always an optimal solution* X_p^* *contained in* V.

Proof Since by hypothesis $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$ we have that

$$d_\lambda(d(X_p)) = \sum_{i=1}^{n} \lambda_i d_{(i)}(X_p) = \text{minimize}\{\sum_{i=1}^{n} \lambda_i d_{\pi(i)}(X_p) : \pi \in \Pi(\{1, \ldots, n\})\}.$$

Assume that $X_p \not\subset V$. Then there must exist $x_i \in X_p$ with $x_i \notin V$. Let $e = \{v, w\}$ be the edge containing x_i and $\ell(e)$ its length. Denote by $X_p(s) = X_p \setminus \{x_i\} \cup \{x(s)\}$ where $x(s)$ is the point on e with $d(v, x(s)) = s$, $s \in [0, l(e)]$.

Table 10.4 Solutions for some specific choices for λ in Example 10.3

Obj. function	Corresponding λ	Set of optimal solutions	Obj. value
Center	$\lambda = (0, 0, 0, 0, 0, 1)$	$EQ_{46}^{23}, EQ_{46}^{35}, EQ_{34}^{56}$	5
2-Centra	$\lambda = (0, 0, 0, 0, \frac{1}{2}, \frac{1}{2})$	$[EQ_{35}^{23}, EQ_{56}^{23}]$, $[EQ_{36}^{35}, EQ_{14}^{35}]$, $[EQ_{14}^{56}, EQ_{13}^{56}]$	5
3-Centra	$\lambda = (0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	EQ_{26}^{23}	$\frac{40}{9}$
Median	$\lambda = (1, 1, 1, 1, 1, 1)$	$EQ_{16}^{23} = v_3$	18
Cent-dian	$\lambda = (\frac{\hat{\lambda}}{6}, \frac{\hat{\lambda}}{6}, \frac{\hat{\lambda}}{6}, \frac{\hat{\lambda}}{6}, \frac{\hat{\lambda}}{6}, \frac{6-5\hat{\lambda}}{6})$	EQ_{34}^{56}, $0 \le \hat{\lambda} \le \frac{36}{43}$, v_3 otherwise	$-\frac{17}{12}\hat{\lambda} + 5$, $-5\hat{\lambda} + 8$
Noname	$\lambda = (1, 1, 0, 0, 1, 1)$	$EQ_{14}^{23}, EQ_{12}^{56}$	13

The function g defined as $g(s) = \sum_{i=1}^{n} \lambda_i d_{(i)}(X_p(s))$ is concave for all $s \in [0, \ell(e)]$ because it is the composition of a concave and a linear function, i.e.,

$$g(s) = \min_{\pi \in \Pi(\{1,\dots,n\})} \left\{ \sum_{i=1}^{n} \lambda_i d_{\pi(i)}(X_p(s)) \right\}$$

and each

$$d_{\pi(j)}(X_p(s)) = \min\{d(v_{\pi(j)}, x_1), \dots, \min\{d(v_{\pi(j)}, a) + s, d(v_{\pi(j)}, b) + \ell(e) - s\}, \dots, \\ d(v_{\pi(j)}, x_n)\}$$

is concave. Hence, $g(s) = F(X_p(s)) \geq \min\{F(X_p(0)), F(X_p(\ell(e))\}$ and the new solution set $X_p(s)$ contains one vertex of V instead of x_i. Repeating this scheme a finite number of times the result follows. □

In the previous section we proved that the set $V \cup EQ$ always contains the set of optimal solutions of the single facility problem (independent of the structure of λ). It may seem natural to expect that the same result holds for the p-facility case as it happens for the p-center problem. However, Example 10.4 shows that this property fails to be true.

This easy example shows the limit for the set $Cand = V \cup EQ$ to be a FDS (finite dominating set) for the multifacility extension of our model. In the literature we can find some characterizations of FDS for particular cases of the p-facility ordered median problem. For instance, Kalcsics et al. (2003) studies the multifacility ordered median problem where the λ-weights are defined as:

$$a = \lambda_1 = \dots = \lambda_k \neq \lambda_{k+1} = \dots = \lambda_n = b,$$

for a fixed k, such that, $1 \leq k < n$. They prove that the set Y, defined by (10.31), is a FDS for this problem.

However, none of these papers deals with the general case of the multifacility ordered median problem. In fact, these papers impose very restrictive hypotheses such that their respective results can not be extended further, see Puerto et al. (2018) for an updated review. In the following section we characterize a FDS for the general 2-facility ordered median problem.

10.4.2.1 A Finite Set of Candidates for the Two Facility Case

In this section we identify a finite set of candidates to be optimal solutions of the 2-facility ordered median problem. In order to consider the set of equilibrium points as a finite set we will assume that EQ only contains the equilibrium points that are isolated and the extreme points of the subedges in equilibrium, see Rodríguez-Chía et al. (2005) for further details.

Theorem 10.8 *Consider the following sets:*

$$R = \{r \,:\, r = w_i d(v_i, y),\ v_i \in V,\ y \in V \cup EQ\},$$
$$Y(r) = \{y \in P(G) \,:\, w_i d(v_i, y) = r,\ v_i \in V\} \quad \textit{with } r \in R,$$
$$Y = \bigcup_{r \in R} Y(r), \tag{10.31}$$

$T = \{X_2 = (x_1, x_2) \in P(G) \times P(G) \,:\, \exists v_r, v_s$ served by x_1 and $v_{r'}, v_{s'}$ served by x_2, such that $w_r d(v_r, x_1) = w_{r'} d(v_{r'}, x_2)$ and $w_s d(v_s, x_1) = w_{s'} d(v_{s'}, x_2)$. Moreover, if $w_r = w_{r'}$ and $w_s = w_{s'}$, then the slopes of the functions $d(v_r, \cdot)$ and $d(v_s, \cdot)$, in the edge that x_1 belongs to, must have the same (different) signs at x_1 and the slopes of the functions $d(v_{r'}, \cdot)$ and $d(v_{s'}, \cdot)$, in the edge that x_2 belongs to, must have different (the same) signs at x_2 \}.

$$F = ((EQ \cup V) \times Y) \cup T \subset P(G) \times P(G). \tag{10.32}$$

The set F is a finite set of candidates to be optimal solutions of the 2-facility ordered median problem in the network N.

Remark 10.1 The structure of the set F is different from previous FDS which appeared in the literature. Indeed, the set F is itself a set of candidates for optimal solutions because it is a set of pairs of points. That means that we do not have to choose the elements of this set by pairs to enumerate the whole set of candidates. The candidate solutions may be either a pair of points belonging to $(EQ \cup V) \times Y$ or a pair belonging to T, but they never can be one point of Y and another point of any pair in T.

The following examples show that the set F can not be shrunk because even in easy cases on the real line all the points are needed. The first example shows a graph where the optimal solution $X_2 = (x_1, x_2)$ verifies that x_1 is an equilibrium point and x_2 is not an equilibrium point which belongs to $Y(r) \setminus (EQ \cup V)$ for a given r. In the second example the optimal solution $X_2 = (x_1, x_2)$ belongs to the set T.

Example 10.4 Let $N = (G, \ell)$ be a network with underlying graph $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$. The length function is given by $\ell(\{1, 2\}) = 3$, $\ell(\{2, 3\}) = 20$, $\ell(\{3, 4\}) = 6$. The w-weights are all equal to one and the λ-weights are $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, $\lambda_3 = 0.4$, $\lambda_4 = 0.3$, see Fig. 10.4.

It should be noted that this example can not have optimal solutions on the edge $\{2, 3\}$ because any point of this edge is dominated by v_2 or v_3. In addition, using the symmetry of the problem we have omitted the evaluation of some of the elements of Y.

In this example the optimal solution is given by $x_1 = p(\{1, 2\}, 1.5)$ and $x_2 = p(\{3, 4\}, 1.5)$ (see Table 10.5). It is easy to check that x_1 is an equilibrium point between v_1 and v_2, and $x_2 \in Y(1.5)$. It is worth noting that the radius 1.5 is given by the distance from the equilibrium point, $p(\{1, 2\}, 1.5)$, generated by v_1 and v_2 to any of these nodes.

Fig. 10.4 Network of Example 10.4 where the dots, the ticks and the small ticks are the nodes, the equilibrium points and the elements of Y, respectively. Observe that in this case there are no pairs in T

Table 10.5 Evaluation of the candidate pairs of Example 10.4

Candidate pair X_2	Value	Candidate pair X_2	Value
$p(\{1,2\},0), p(\{3,4\},0)$	3	$p(\{1,2\},1.5), p(\{3,4\},0)$	2.7
$p(\{1,2\},0), p(\{3,4\},1.5)$	2.85	$p(\{1,2\},1.5), p(\{3,4\},1.5)$	2.4
$p(\{1,2\},0), p(\{3,4\},3)$	2.7	$p(\{1,2\},1.5), p(\{3,4\},3)$	2.55

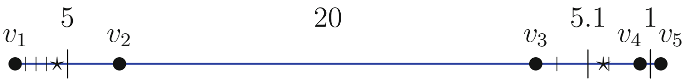

Fig. 10.5 Network of Example 10.5 where the dots, the ticks, the small ticks and the stars are the nodes, the equilibrium points, the elements of Y and T, respectively. By domination and symmetry arguments not all the candidates are necessary and therefore, they are not depicted

Table 10.6 Evaluation of the candidate pairs of Example 10.5

Candidate pair X_2	Value	Candidate pair X_2	Value
$p(\{1,2\},0), p(\{3,4\},0)$	11.81	$p(\{1,2\},2.05), p(\{3,4\},3.05)$	8.455
$p(\{1,2\},0), p(\{3,4\},2.55)$	11.6	$p(\{1,2\},2.45), p(\{3,4\},2.55)$	9.005
$p(\{1,2\},0), p(\{3,4\},3.05)$	10.6	$p(\{1,2\},2.5), p(\{3,4\},0)$	14.31
$p(\{1,2\},0), p(\{4,5\},0)$	10.61	$p(\{1,2\},2.5), p(\{3,4\},2.5)$	9.06
$p(\{1,2\},0), p(\{4,5\},0.5)$	11.66	$p(\{1,2\},2.5), p(\{3,4\},2.55)$	8.955
$p(\{1,2\},0), p(\{4,5\},1)$	11.71	$p(\{1,2\},2.5), p(\{3,4\},2.6)$	8.95
$p(\{1,2\},0.5), p(\{4,5\},0.5)$	11.16	$p(\{1,2\},2.5), p(\{3,4\},3.05)$	8.905
$p(\{1,2\},1), p(\{4,5\},0)$	10.61	$p(\{1,2\},2.5), p(\{3,4\},3.6)$	8.96
$p(\{1,2\},1), p(\{4,5\},1)$	11.71	$p(\{1,2\},2.5), p(\{4,5\},0)$	9.11
$p(\{1,2\},1.45), p(\{3,4\},2.55)$	10.005	$p(\{1,2\},2.5), p(\{4,5\},0.5)$	9.16
$p(\{1,2\},1.95), p(\{3,4\},3.05)$	8.455	$p(\{1,2\},2.5), p(\{4,5\},1)$	10.21
$p(\{1,2\},2), p(\{3,4\},3.1)$	8.41		

Example 10.5 Let $N = (G, \ell)$ be a network with underlying graph $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\}\}$. The length function is given by $\ell(\{1,2\}) = 5$, $\ell(\{2,3\}) = 20$, $\ell(\{3,4\}) = 5.1$, $\ell(\{4,5\}) = 1$. The w-weights are all equal to one and the λ-weights are $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 0, \lambda_4 = 1, \lambda_5 = 1.1$, see Fig. 10.5.

In this example the optimal solution is given by $x_1 = p(\{1,2\},2)$ and $x_2 = p(\{3,4\},3.1)$ (see Table 10.6). Therefore the optimal pair (x_1, x_2) belongs to the set T. Indeed, $d(v_1, x_1) = d(v_4, x_2)$ and $d(v_2, x_1) = d(v_5, x_2)$ and the slopes of

$d(v_1, \cdot), d(v_2, \cdot)$ in the edge $\{1, 2\}$ at x_1 are $1, -1$ respectively; and the slopes of $d(v_4, \cdot), d(v_5, \cdot)$ in the edge $\{3, 4\}$ at x_2 are $-1, -1$ respectively.

Once we have proved that F is an essential set to describe the set of optimal solutions of the 2-facility ordered median problem we want to know its cardinality.

Proposition 10.2 *The cardinality of F is $O(m^3n^6)$.*

Proof In each edge there are at most two equilibrium points associated with each pair of nodes. Thus $|EQ| = O(mn^2)$ and $|R| = O(mn^3)$. The maximum degree of a node $v_i \in V$ is m (the star network) so $|Y(r)| = O(mn)$ with $r \in R$. Thus, $|Y| = O(m^2n^4)$. On the second hand, on each edge, each pair of nodes may determine an element of a pair in T. Therefore, the set T has a cardinality $O((n^2m)^2)$. In conclusion $|F| = O(m^3n^6 + m^2n^4) = O(m^3n^6)$. □

It is worth noting that F is an actual set of finite elements to be optimal solutions of Problem (10.30). The difference with previous approaches is that this set is not a set of candidates for each individual facility but it is the set of candidate pairs to be optimal solutions.

10.4.2.2 A Discouraging Result for the *p*-Facility Case

It is well-known that FDS of polynomial size exist for the classical p-median, p-center, p-centdian and p-k-centrum problems (see Hooker et al. 1991; Kalcsics et al. 2003). In addition, our previous section has shown a finite set of candidates to be optimal solutions of the 2-facility ordered median problem in a network. However, despite the similarity existing between those problems and the general p-facility ordered median problem, these results can not be extended to our model.

The reason for this is the following. For the 1-facility ordered median problem we have that the set of candidates to be optimal solutions is EQ, that means, the equilibrium points (see Nickel and Puerto 1999). For the 2-facility ordered median problem we have obtained that the set of candidates to be optimal solutions is $EQ \times Y \cup T$, that means, the points generated by the distances between each node and each equilibrium point and the set T. It should be noted that in this case we have added these points because there may exist ties which do not allow to move the service facility improving the objective function. In the 3-facility ordered median problem, the previous candidate set is not enough because if $x_1 \in EQ$ and $x_2 \in Y \setminus EQ$, the distances between each node and x_2 do not need to be included in the set of radius, R. Therefore, it may occur that there exists a tie between two nodes and the service facilities x_2 and x_3 respectively, so that there is no movement of the facilities at x_2 and x_3 which improves the objective function (see Example 10.6).

Example 10.6 Let $N = (G, \ell)$ be a network with underlying graph $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}\}$. The length function is given by $\ell(\{1, 2\}) = 3, \ell(\{2, 3\}) = 50, \ell(\{3, 4\}) = 6, \ell(\{4, 5\}) = 50, \ell(\{5, 6\}) = 10$. The w-weights are all equal to one and the λ-

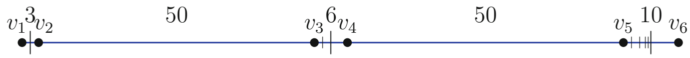

Fig. 10.6 Network of Example 10.6, using the same notation as in Fig. 10.4

modeling weights are $\lambda_1 = 0.1, \lambda_2 = 0.2, \lambda_3 = 0.4, \lambda_4 = 0.3, \lambda_5 = 0.6, \lambda_6 = 0.55$, see Fig. 10.6.

In this example the optimal solution is given by $x_1 = p(\{1,2\}, 1.5)$, $x_2 = p(\{3,4\}, 1.5)$ and $x_3 = p(\{4,5\}, 4.5)$ (see Table 10.7). It can be seen that x_1 is an equilibrium point, $x_2 \in Y(1.5)$ and x_3 neither belongs to Y nor is a component of a pair of T.

This example illustrates that in order to obtain the optimal solution for the 3-facility problem new points have to be added. Our conjecture is that these points can be generated using recursively the construction of the set of radii but now regarding the distances from the points in $\pi_2(F) := \{x_2 : (x_1, x_2) \in F\}$, that is, the points in $P(G)$ which correspond to the second candidate of any pair in F, and the node set:

$$R_1 = \{r : r = w_i d(v_i, y), v_i \in V, y \in \pi_2(F)\},$$

$$Y_1(r) = \{y : y \in P(G), w_i d(v_i, y) = r, v_i \in V\},$$

$$Y_1 = \bigcup_{r \in R_1} Y_1(r).$$

The same situation occurs in the p-facility case, so that in general this construction must be repeated p-times in order to obtain a finite candidate set to be optimal solutions for that problem. Therefore the structure of the candidate set defined in the previous section depends on the number of facilities to be located. Actually, Puerto and Rodríguez-Chía (2005) prove that there is no polynomial size FDS for the general ordered p-median problem even on path networks. The proof consists of building a family of $O(n^n)$ problems on the same graph with different solutions (each solution contains at least one point not included in the remaining), n being the number of nodes.

For the case of locating extensive facilities on the line, in Rozanov and Tamir (2018), it is proved a nestedness property (given any two facility lengths $t_1, t_2, 0 \leq t_1 < t_2$, there is an optimal solution with length t_1 which lies within some optimal solution with length t_2). In addition, in Schnepper (2017), Schnepper et al. (2019), it is analyzed the p-k-max problem on networks, a particular case of the ordered median problem. The reader is referred to Puerto et al. (2018) for an updated review of results on location of extensive facilities on networks.

Table 10.7 Evaluation of the candidate solutions of Example 10.6

Candidate pair X_3	Val.	Candidate pair X_3	Val.
$p(\{1,2\},0), p(\{3,4\},0), p(\{4,5\},0)$	10	$p(\{1,2\},1.5), p(\{3,4\},0), p(\{4,5\},0)$	10.1
$p(\{1,2\},0), p(\{3,4\},0), p(\{4,5\},1.5)$	9.77	$p(\{1,2\},1.5), p(\{3,4\},0), p(\{4,5\},1.5)$	9.62
$p(\{1,2\},0), p(\{3,4\},0), p(\{4,5\},3)$	9.55	$p(\{1,2\},1.5), p(\{3,4\},0), p(\{4,5\},3)$	9.25
$p(\{1,2\},0), p(\{3,4\},0), p(\{4,5\},4)$	9.3	$p(\{1,2\},1.5), p(\{3,4\},0), p(\{4,5\},4)$	9
$p(\{1,2\},0), p(\{3,4\},0), p(\{4,5\},4.5)$	9.15	$p(\{1,2\},1.5), p(\{3,4\},0), p(\{4,5\},4.5)$	8.85
$p(\{1,2\},0), p(\{3,4\},0), p(\{4,5\},5)$	9	$p(\{1,2\},1.5), p(\{3,4\},0), p(\{4,5\},5)$	8.75
$p(\{1,2\},0), p(\{3,4\},1.5), p(\{4,5\},0)$	9.7	$p(\{1,2\},1.5), p(\{3,4\},1.5), p(\{4,5\},0)$	9.55
$p(\{1,2\},0), p(\{3,4\},1.5), p(\{4,5\},1.5)$	9.17	$p(\{1,2\},1.5), p(\{3,4\},1.5), p(\{4,5\},1.5)$	8.87
$p(\{1,2\},0), p(\{3,4\},1.5), p(\{4,5\},3)$	8.95	$p(\{1,2\},1.5), p(\{3,4\},1.5), p(\{4,5\},3)$	8.5
$p(\{1,2\},0), p(\{3,4\},1.5), p(\{4,5\},4)$	8.7	$p(\{1,2\},1.5), p(\{3,4\},1.5), p(\{4,5\},4)$	8.25
$p(\{1,2\},0), p(\{3,4\},1.5), p(\{4,5\},4.5)$	8.57	$p(\{1,2\},1.5), p(\{3,4\},1.5), p(\{4,5\},4.5)$	8.12
$p(\{1,2\},0), p(\{3,4\},1.5), p(\{4,5\},5)$	8.6	$p(\{1,2\},1.5), p(\{3,4\},1.5), p(\{4,5\},5)$	8.15
$p(\{1,2\},0), p(\{3,4\},3), p(\{4,5\},0)$	11.2	$p(\{1,2\},1.5), p(\{3,4\},3), p(\{4,5\},0)$	9.1
$p(\{1,2\},0), p(\{3,4\},3), p(\{4,5\},1.5)$	8.87	$p(\{1,2\},1.5), p(\{3,4\},3), p(\{4,5\},1.5)$	8.42
$p(\{1,2\},0), p(\{3,4\},3), p(\{4,5\},3)$	8.35	$p(\{1,2\},1.5), p(\{3,4\},3), p(\{4,5\},3)$	8.2
$p(\{1,2\},0), p(\{3,4\},3), p(\{4,5\},4)$	8.4	$p(\{1,2\},1.5), p(\{3,4\},3), p(\{4,5\},4)$	8.25
$p(\{1,2\},0), p(\{3,4\},3), p(\{4,5\},4.5)$	8.42	$p(\{1,2\},1.5), p(\{3,4\},3), p(\{4,5\},4.5)$	8.27
$p(\{1,2\},0), p(\{3,4\},3), p(\{4,5\},5)$	8.45	$p(\{1,2\},1.5), p(\{3,4\},3), p(\{4,5\},5)$	8.3

10.5 The Capacitated Discrete Ordered Median Problem

In this section our goal is to introduce the family of discrete ordered median location problems. As we have seen in previous sections, the main feature of these models is their flexibility to generalize the most popular objective functions studied in the location analysis literature and to allow modeling a wide variety of new problems appearing in logistics and manufacturing.

The uncapacitated version of the discrete ordered median location problem has been analyzed in several papers, Boland et al. (2006), Nickel (2001), Nickel and Puerto (2005), Marín et al. (2009, 2010), Puerto et al. (2011, 2013), Labbé et al. (2017), Deleplanque et al. (2018), and different formulations and algorithms to solve medium sized problems have been developed. Recently, these models were extended to deal with capacities in Kalcsics et al. (2010a,b). However, although the approach in the initial papers leads to satisfactory results concerning motivations, applications and interpretations the solution times of larger problem instances need further improvements.

The goal of this section is to present, first, an intuitive formulation of the problem based on three-indexed variables, see Boland et al. (2006); and second, a formulation which makes use of the coverage ideas in Marín et al. (2009, 2010), applied to the capacitated version of the Discrete Ordered Median Problem, CDOMP, with binary assignment, see Puerto (2008), Puerto et al. (2011, 2013). To perform this task, first we introduce the Capacitated Discrete Ordered Median Problem formally and give these two mathematical programming formulations. Then, the last part of this section is devoted to test the efficiency of the last approach by providing some preliminary numerical experiments.

10.5.1 A Three-Index Formulation

In order to introduce this formulation let A denote the given set of n sites and identify these with the integers $1, \ldots, n$, i.e., $A = \{1, \ldots, n\}$. We assume without loss of generality that the set of candidate sites for new facilities is identical to the set of clients. Let $C = (c_{ij})_{i,j=1,\ldots,n}$ be the given non-negative $n \times n$ cost matrix, where c_{ij} denotes the cost of satisfying the demand of client i from a facility located at site j. Let $p \leq n$ be the number of facilities to be located. Each client i has a demand a_i that must be served and each server j has an upper bound b_j on the capacity that it can fulfill. We assume further that assignment is binary, that is, the demand of each client must be served by a unique server.

A solution to the location problem is given by a set of p sites; we use $X \subseteq A$, with $|X| = p$, to denote a solution. Then, the problem consists of finding the set of sites X with $|X| = p$, which can supply the overall demand at a minimum cost with respect to the ordered median objective function.

A natural way to attack the formulation of the discrete ordered median problem is to use variables that keep track of the order of the transportation costs from each client and its server. This approach gives rise to a formulation with three-index variables, one for the order and the remaining two indices, for the client-server allocation. In order to formulate this model we consider a set of λ-weights, where λ_i can be seen as a correction factor to the ith-position with $i = 1, \ldots, n$. In addition, we define the following set of variables:

$$x_{ij}^k = \begin{cases} 1, & \text{if client } i \text{ is supplied by server } j \text{ and is the } k\text{-th} \\ & \text{cheapest cost allocation} \\ 0, & \text{otherwise,} \end{cases} \qquad i, j, k = 1, \ldots, n,$$

$$y_j = \begin{cases} 1, & \text{if the server at } j \text{ is open} \\ 0, & \text{otherwise,} \end{cases} \qquad j = 1, \ldots, n.$$

Hence, the formulation of the model is:

$$\text{minimize} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_k c_{ij} x_{ij}^k \tag{10.33}$$

$$\text{subject to} \sum_{j=1}^{n}\sum_{k=1}^{n} x_{ij}^k = 1, \quad i = 1, \ldots, n \tag{10.34}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}^k = 1, \quad k = 1, \ldots, n \tag{10.35}$$

$$\sum_{i=1}^{n}\sum_{k=1}^{n} a_i x_{ij}^k \le b_j y_j, \quad j = 1, \ldots, n, \tag{10.36}$$

$$\sum_{j=1}^{n} y_j = p, \tag{10.37}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij}^k \le \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} x_{ij}^{k+1}, \quad k = 1, \ldots, n-1. \tag{10.38}$$

$$x_{ij}^k \in \{0, 1\}, \quad i, j, k = 1, \ldots, n; \tag{10.39}$$

$$y_j \in \{0, 1\}, \quad j = 1, \ldots, n\,. \tag{10.40}$$

The objective function accounts for the weighted sum of the transportation cost using the lambda parameters. Constraints (10.34) ensure that each origin site i is allocated exactly to one server j. Constraints (10.35) guarantee that any position in the sorted vector of *client-server* costs is allocated to just one pair. Constraints (10.36) are the capacity constraints and also ensure that one origin may be allocated to a specific server only if it is open. Constraint (10.37) fixes the number of facilities to be located. Finally, constraints (10.38) ensure that the transportation cost assigned to the k-position is smaller than the one assigned to the $(k+1)$-position.

10.5.2 A Covering Formulation and Some Properties

In this subsection, we introduce a formulation for the binary assignment capacitated discrete ordered median problem based on covering variables. This formulation was first presented in Puerto (2008).

We first define H as the number of different non–zero elements of the cost matrix C. Hence, we can order the different values of C in non–decreasing sequence: $c_{(0)} := 0 < c_{(1)} < c_{(2)} < \cdots < c_{(H)} := \max_{1\le i,j\le n}\{c_{ij}\}$.

Given a feasible solution, we can use this ordering to perform the sorting process of the allocation costs. This can be done by the following variables ($j = 1, \ldots, n$ and $k = 1, \ldots, H$):

$$u_{jk} := \begin{cases} 1, & \text{if the } j\text{–th smallest allocation cost is at least } c_{(k)}, \\ 0, & \text{otherwise.} \end{cases} \tag{10.41}$$

With respect to this definition the j–th smallest cost element is equal to $c_{(k)}$ if and only if $u_{jk} = 1$ and $u_{j,k+1} = 0$. Therefore, we can reformulate the objective function of the CDOMP (i.e., the capacitated ordered median problem), using the variables u_{jk}, as $\sum_{j=1}^{n}\sum_{k=1}^{H}\lambda_j \cdot (c_{(k)} - c_{(k-1)}) \cdot u_{jk}$.

First of all, we need to impose the following group of sorting constraints on the u_{jk}–variables: $u_{j+1,k} \ge u_{jk},\ \ j = 1, \ldots, n-1;\ \ k = 1, \ldots, H$. To guarantee that exactly p servers will be opened among the n possibilities, we consider constraint (10.37) defined in the previous formulation.

Then, we need to ensure that demand and capacities are satisfied. For these reasons we introduce: (1) the variables x_{ij} (binary allocation) :

$$x_{ij} = \begin{cases} 1, & \text{if the client } i \text{ is allocated to server } j \\ 0, & \text{otherwise} \end{cases} \tag{10.42}$$

and (2) the constraints $\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \ldots, n$ (each client is just assigned to one server) and $\sum_{i=1}^{n} a_i x_{ij} \le b_j y_j, \quad j = 1, \ldots, n$ (all the demand and capacity requirements must be satisfied and clients can only be assigned to servers which are open).

In addition, the relationship that links the variables u and x is: $\sum_{j=1}^{n} u_{jk} = \sum_{i=1}^{n}\sum_{j:c_{ij}\ge c_{(k)}} x_{ij}$. The meaning being clear. The number of allocations with a cost at least $c_{(k)}$ must be equal to the number of servers that support demand from facilities at a cost greater than or equal to $c_{(k)}$.

Summing up all these constraints and the objective function, the CDOMP can be formulated as

$$\text{minimize} \sum_{j=1}^{n}\sum_{k=1}^{H} \lambda_j (c_{(k)} - c_{(k-1)}) u_{jk} \tag{10.43}$$

$$\text{subject to} \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \ldots, n \tag{10.44}$$

$$\sum_{i=1}^{n} a_i x_{ij} \leq b_j y_j, \quad j = 1, \ldots, n, \tag{10.45}$$

$$x_{ij} \leq y_j \quad i, j = 1, \ldots, n \tag{10.46}$$

$$\sum_{j=1}^{n} y_j = p \tag{10.47}$$

$$\sum_{j=1}^{n} u_{jk} = \sum_{i=1}^{n} \sum_{\substack{j=1\ldots,n \\ c_{ij} \geq c_{(k)}}} x_{ij}, \quad k = 1, \ldots, H \tag{10.48}$$

$$u_{j+1k} \geq u_{jk}, \quad j = 1, \ldots, n-1; \; k = 1, \ldots, H \tag{10.49}$$

$$u_{jk} \in \{0, 1\}, \quad j = 1, \ldots, n; \; k = 1, \ldots, H \tag{10.50}$$

$$x_{ij}, y_j \in \{0, 1\}, \quad i, j = 1, \ldots, n; \tag{10.51}$$

Since the proposed formulation contains $O(nH)$ binary variables and $O(nH)$ constraints, fast solution times for larger problem instances, using standard software tools, are very unlikely. In this sense, the following proposition states that we can relax the y_j variables to be continuous and the solution will not change.

Proposition 10.3 (CDOMP) *admits a formulation with* $y_j \in [0, 1]$ *and for each optimal solution of the relaxed problem one can obtain an optimal solution of the original problem.*

Proof Use (10.46) and (10.47) to ensure that any fractional y solution can be modified to be binary and feasible without increasing the objective value.

□

The above formulation admits some valid inequalities that, at times, reinforce the linear relaxation improving the lower bound and reducing the computation time to solve the problem. In the following, we list three families of them.

The first one are the natural inequalities $u_{jk} \geq u_{jk+1}, \quad j = 1, \ldots, n, \quad k = 1, \ldots, H-1$. They come from the fact that the rows of the u-matrix are sorted. We have observed in our experiments that these constraints are not always satisfied by the optimal solution of the linear relaxation and thus they are useful in improving the formulation. This family of inequalities were introduced in Marín et al. (2009) for tightening the formulation of the Uncapacitated Discrete Ordered Median Problem.

Our next set of inequalities state that the number of assignments done by the x-variables at a cost at least $c_{(j)}$ for clients in S cannot exceed the number of ones in the last $|S| = r$ rows of the j-th column of the u-matrix. Then, if there are r allocations of demand points in S at a costs at least $c_{(j)}$, since the columns in the u-matrix are ordered in non-decreasing sequence, we get the following: $\sum_{i\in S}\sum_{k:c_{ik}\geq c_{(j)}} x_{ik} \leq \sum_{i=n-r+1}^{n} u_{ij}, \quad \forall\ S \subseteq \{1,\ldots,n\}, \quad |S| = r, \quad r = 1,\ldots,n,\ j = 1,\ldots,H$. Note that there is an exponential number of inequalities in this family.

Another set of valid inequalities are those stating that either client i is allocated at a cost at least $c_{(k)}$ or there must exist an open server j such that the allocation cost of client i is smaller than $c_{(k)}$. This results in: $\sum_{j:c_{ij}\geq c_{(k)}} x_{ij} + \sum_{j:c_{ij}<c_{(k)}} y_j \geq 1,\ i = 1,\ldots,n$.

In addition, we mention the staircase inequalities introduced by Labbé et al. (2017), where several new formulations for the Uncapacitated Discrete Ordered Median Problem (DOMP) based on its similarity with some scheduling problems are presented (some of them with a considerably smaller number of constraints).

The rest of this section presents some computational results for this formulation of the capacitated discrete ordered problem. We restrict ourselves to consider just the second formulation, because although the first one is very intuitive and good to have a better understanding of the problem, its running times are much bigger than those obtained by the second one, see e.g., Puerto (2008). In order to test the performance of the considered formulation, we report on an experimental design that consists of the following factors: (1) *Size of the problem:* The number of sites, n, determines the dimensions of the cost matrix and the λ vectors. Moreover, it is an upper bound of the number of suppliers (p) to be located. We consider five different levels of $n = 10,\ 20,\ 30,\ 40,\ 60$. (2) *Number of suppliers:* p is the second factor with three levels for each choice of n: $p = \lfloor n/5\rfloor + 1,\ \lfloor n/2\rfloor,\ 4\times\lfloor n/5\rfloor$. (3) *Type of problem:* Each λ-vector is associated with a different objective function. Its levels are designed depending on the value of n as follows: (a) λ-vector corresponding to the p-median problem, i.e., $\lambda = (1,\ldots,1) \in \mathbb{R}^n$; (b) λ-vector corresponding to the p-center problem, i.e., $\lambda = (0,\ldots,0,1) \in \mathbb{R}^n$; (c) λ-vector corresponding with the $\lfloor n/4\rfloor$-centrum problems; and (d) λ-vector corresponding to the (k_1,k_2)-trimmed mean problem, i.e., $\lambda = (0,\ldots,0,1,\ldots,1,0,\ldots,0) \in \mathbb{R}^n$ where $k_1 = \lfloor 0.2n\rfloor$, $k_2 = \lfloor 0.2n\rfloor$. (4) *Demand of facilities:* Each demand is considered integer and uniformly drawn from [10, 20]. (5) *Capacity of suppliers:* We consider that the capacities are uniformly discrete random variables in the interval $[1.1\sum_{i=1}^{n} a_i/p,\ 1.4\sum_{i=1}^{n} a_i/p]$. This choice ensures feasibility of the considered problems. (6) *Transportation cost:* We assume free self service and integer costs. The values c_{ij}, $i \neq j$, are drawn uniformly in [0, 200].

We solve five instances for each possible combination of levels and we report the average and maximum: running time, gap at the root node and number of nodes in the branch-and-bound tree for this formulation. All computational studies were performed on a PC with a *Genuine Intel(R) CPU U4100* with two processors at 1.30 GHz and 4 GB of RAM. To solve the different instances of the problems we used XPRESS-IVE solver version 7.5, with a code implemented in XPRESS-MOSEL version 3.4.2.

The information of our computational test is reported in Table 10.8 that summarizes the results for the four considered problems types. The organization of the table is the following: columns show the results for the different sizes of n and p. A superindex in some values of p states the number of instances for the corresponding combination of n and p exceeding the CPU time limit (1 h). Each block of rows reports the results of the instances based on the formulation (10.43)–(10.51). Within each block of rows we report on the *gap* at the root node [average (Ag) and maximum (Mg)], CPU time to solve the integer problems [average (At) and maximum (Mt)] and number of *nodes* in the branch-and-bound tree [average (An) and maximum (Mn)].

We observe, from the results in Table 10.8 that we could solve most of the instances, even medium sized $n = 60$, within 1 h of CPU time. This fact shows a good performance of the formulation. In addition, it is worth noting that the quality of the lower bounds provided by this formulation depends on the type of problem. In general, the lower bounds are rather poor for larger values of p relative to n. On the other hand, for small to medium values of p relative to n the performance of the lower bounds are good for median and trimmed mean problems, reasonable for k-centrum (less than 50%) and poor for the center problem. These results show that there is room for further investigation on the polyhedral structure of this formulation in order to develop valid inequalities that could be integrated in a Branch and Cut algorithm to solve faster and hence larger problem sizes.

In conclusion, the formulation of the CDOMP based on covering, (10.43)–(10.51), is a promising approach. Moreover, it can be also strengthen with known valid inequalities, as for instance in Puerto et al. (2011), leading to solve larger problem sizes of capacitated discrete ordered median problems.

Finally, we would like to mention that two ad-hoc solution procedures have been developed for the uncapacitated DOMP, the first one based on a parallelized Lagrangian relaxation approach, see Redondo et al. (2016) and the second one is a Branch-Price-and-Cut procedure, see Deleplanque et al. (2018). These two approaches could also be adapted to tackle the capacitated version of this problem.

10.6 Conclusions

This chapter provides an overview of the ordered median function and its corresponding Ordered Median Location Problem as a powerful tool from a modeling point of view within the area of Location Analysis. We have included some of their most important insights considering three different solution spaces: continuous, networks and discrete. Our goal has been to structure this chapter as an useful tool for those readers that wish to start the study of the ordered functions and their related

Table 10.8 Numerical results obtained with the covering formulation for the median, center, *k*-centrum and Trimmed-mean problems

Median															
n	10			20			30			40			60		
p	3	5	8	5	10	16	7	15	24	9	20	32	13	30[2]	48[2]
At	0.6	4.3	2.5	5.6	11.1	31.4	41.4	44.8	23	38.7	116.2	718.4	213.9	1939.3	2092.6
Mt	1.7	9.9	4.5	12.3	20.2	66.6	102	135.9	59.6	63.5	198.3	2644.6	427.6	3600.8	3600.4
An	47	10.2	51.6	12.2	191	1557.8	344	1055.8	607.4	124.4	2357	31,512.6	716.4	23,586.4	42,081.6
Mn	231	31	127	39	451	4077	767	4865	1647	485	4221	129,917	1848	46,523	91,324
Ag	2.3	21.2	77.3	6	15.3	83.6	6.7	23.3	71.8	5.2	25.3	76.4	7.7	27	76.9
Mg	4.1	41.2	89.8	8.7	25.3	92.1	10.1	43.3	77.7	7.6	35.8	91.9	10.8	35.3	88
Center															
n	10			20			30			40			60		
p	3	5	8	5	10	16	7	15[1]	24[2]	9	20	32[2]	13[1]	30[1]	48[3]
At	13.7	9.4	2.4	47.3	19.8	236.3	90.1	874.6	1786.9	338.9	260.2	2162.7	1977.3	1416.2	2578.6
Mt	16.7	13.9	4.3	81.4	34.4	629.9	130.6	3599.7	3599.3	568	713	3600	3599.9	3600	3600.4
An	17.4	391	65	605.6	1558	21,804	685.8	51,162.4	82,292.4	3052.4	7549	68,845.4	16,741.2	22,036.8	41,283.6
Mn	37	925	123	1189	3467	47,542	1391	211,405	167,386	4465	25,227	105,734	43,411	40,891	64,023
Ag	74.2	78.8	94.7	69.2	80.9	96.9	70.1	80.9	97.5	70.6	81.6	97.6	71.5	85.4	97.6
Mg	77.6	83.2	97.6	74.5	83.3	99	76.3	85.9	99.2	72	82.9	98.3	72.9	99.4	98.6

k-Centrum															
n	10			20			30			40			60		
p	3	5	8	5	10	16	7	15	24	9	20	32[3]	13	30	48[1]
At	10.8	6.1	5.3	14.1	33.2	57.7	74.6	84.6	921.2	173	442.2	2336.9	676.1	1155.4	2222
Mt	18.5	10.6	9.2	22.7	115.3	191.7	162	209.2	3012.1	394.9	1051.8	3600	814.2	2265	3599.7
An	20.6	101.4	32.2	140.6	1933.8	3527.2	1662.2	3552.8	44,945	2563.4	14,941.6	71,892.6	5663.4	24,109.4	29,005.6
Mn	81	487	127	493	8047	12,769	5623	9626	171,469	7745	40,269	125,596	11,227	52,339	46,295
Ag	28	37	60.2	26.6	46.2	74.5	29.3	42.1	86.2	30.3	42.6	87.8	30.3	39.7	79.2
Mg	37.9	70.3	84.2	28.9	58.8	92.3	33.3	48.3	91.9	37.8	52.1	92	32.2	48.2	88.2

Trimmed mean															
n	10			20			30			40			60		
p	3	5	8	5	10	16	7	15	24	9	20	32	13	30	48
At	6.8	0.9	1	5.1	7	17.6	22.9	31.5	4.8	24.5	84.3	36.8	79	179.9	339
Mt	12.6	2.4	2.9	11.9	11.9	34.2	51.6	46	7.5	48.3	296.9	97.3	195.4	252.8	520.5
An	2.6	1	3.4	38.2	147.6	890.6	108	561	6.2	9	4487.6	998.2	225	2540.6	6100.2
Mn	8	1	13	187	335	2235	406	1191	27	29	19,803	3223	753	3806	11,259
Ag	26.8	25	0	25.2	29.3	0	26.6	36.1	0	26.1	31.7	0	25.5	37	0
Mg	33.2	25	0	26.2	42.3	0	28.2	48.3	0	29.7	39.1	0	26.5	43.5	0

ordered median location problems. Moreover, the extensive list of references that have been included may result in an interesting source, for expert readers, to carry out a deeper study of this topic.

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