On Selections of Some Generalized Set-Valued Inclusions

Bahman Hayati, Hamid Khodaei, and Themistocles M. Rassias

Abstract We present some results on the existence of a unique selection of a set-valued function satisfying some generalized set-valued inclusions.

1 Introduction

For a nonempty set Y we denote by $\mathfrak{F}_0(Y)$ the family of all nonempty subsets of Y. In a linear normed space Y we define the following families of sets:

$$ccl(Y) := \{A \in \mathfrak{F}_0(Y) : \ A \text{ is closed and convex set}\},$$

$$cclz(Y) := \{A \in \mathfrak{F}_0(Y) : \ A \text{ is closed and convex set containing } 0\},$$

$$ccz(Y) := \{A \in \mathfrak{F}_0(Y) : \ A \text{ is compact and convex set containing } 0\}.$$

The diameter of a set $A \in \mathfrak{F}_0(Y)$ is defined by

$$\delta(A) := \sup\{\| a - b \|: \ a, b \in A\}.$$

Let K be a nonempty set. We say that a set-valued function $F : K \to \mathfrak{F}_0(Y)$ is with bounded diameter if the function $K \ni x \mapsto \delta(F(x)) \in \mathbb{R}$ is bounded. Finally recall that a selection of a set-valued map $F : K \to \mathfrak{F}_0(Y)$ is a single-valued map $f : K \to Y$ with the property $f(x) \in F(x)$ for all $x \in K$.

B. Hayati · H. Khodaei (✉)
Department of Mathematics, Malayer University, Malayer, Iran
e-mail: hayati@malayeru.ac.ir; hkhodaei@malayeru.ac.ir

T. M. Rassias
Department of Mathematics, Zografou Campus, National Technical University of Athens, Athens, Greece
e-mail: trassias@math.ntua.gr

T. M. Rassias, P. M. Pardalos (eds.), *Mathematical Analysis and Applications*, Springer Optimization and Its Applications 154,
https://doi.org/10.1007/978-3-030-31339-5_7

Smajdor [1] and Gajda and Ger [2] proved that if $(S, +)$ is a commutative semigroup with zero and Y is a real Banach space, then $F : S \to ccl(Y)$ is a subadditive set-valued function; i.e.,

$$F(x + y) \subset F(x) + F(y), \quad x, y \in S,$$

with bounded diameter admits a unique additive selection (i.e., a unique mapping $f : S \to Y$ such that $f(x + y) = f(x) + f(y)$ and $f(x) \in F(x)$ for all $x, y \in S$). In 2001, Popa [3] proved that if $K \neq \emptyset$ is a convex cone in a real vector space X (i.e., $sK + tK \subseteq K$ for all $s, t \geq 0$) and $F : K \to ccl(Y)$ (where Y is a real Banach space) is a set-valued function with bounded diameter fulfilling the inclusion

$$F(\alpha x + \beta y) \subset \alpha F(x) + \beta F(y), \quad x, y \in K,$$

for $\alpha, \beta > 0$, $\alpha + \beta \neq 1$, then there exists exactly one additive selection of F.

Set-valued functional equations have been investigated by a number of authors and there are many interesting results concerning this problem (see [4–14]).

We determine the conditions for which a set-valued function $F : K \to \mathfrak{F}_0(Y)$ satisfying one of the following inclusions

$$\sigma_{y,z} F(\alpha x) + 8\alpha^{-1} F(x) \subseteq 2\alpha^{-1} \left(\sigma_y F(x) + \sigma_z F(x)\right) + 4\alpha F(x),$$

$$\sigma_{y,z} F(\alpha x) + 8F(x) \subseteq 2 \left(\sigma_y F(x) + \sigma_z F(x)\right) + 4\alpha^2 F(x),$$

$$\sigma_{y,z} F(\alpha x) + 8\alpha F(x) \subseteq 2\alpha \left(\sigma_y F(x) + \sigma_z F(x)\right) + 4\alpha^3 F(x),$$

$$\begin{aligned} \sigma_{y,z} F(\alpha x) + 4\alpha^2 (2F(x) + F(y) + F(z)) \subseteq 2\alpha^2 \left(\sigma_y F(x) + \sigma_z F(x)\right) \\ +2\sigma_z F(y) + 4\alpha^4 F(x) \end{aligned} \tag{1}$$

for all $x, y, z \in K$ and any fixed positive integers $\alpha > 1$ admits a unique selection satisfying the corresponding functional equation. Here $\sigma_y F(x)$ denotes $\sigma_y F(x) = F(x + y) + F(x - y)$, and $\sigma_{y,z} F(x)$ denotes $\sigma_{y,z} F(x) = \sigma_z \left(\sigma_y F(x)\right) = \sigma_z F(x + y) + \sigma_z F(x - y)$.

2 Selections of Set-Valued Mappings

In what follows we give some notations and present results which will be used in the sequel.

Definition 1 Let X be a real vector space. For $A, B \in \mathfrak{F}_0(X)$, the (Minkowski) addition is defined as

$$A + B = \{a + b : a \in A, b \in B\}$$

and the scalar multiplication as

$$\lambda A = \{\lambda a : a \in A\}$$

for $\lambda \in \mathbb{R}$.

Lemma 1 (*Nikodem* [15]) *Let X be a real vector space and let λ, μ be real numbers. If $A, B \in \mathfrak{F}_0(X)$, then*

$$\lambda(A + B) = \lambda A + \lambda B,$$

$$(\lambda + \mu)A \subseteq \lambda A + \mu A.$$

In particular, if A is convex and $\lambda\mu \geq 0$, then

$$(\lambda + \mu)A = \lambda A + \mu A.$$

Lemma 2 (Rådström's Cancelation Law) *Let Y be a real normed space and $A, B, C \in \mathfrak{F}_0(Y)$. Suppose that $B \in ccl(Y)$ and C is bounded. If $A + C \subseteq B + C$, then $A \subseteq B$.*

The above law has been formulated by Rådström [16], but the proof given there is valid in topological vector spaces (see [17, 18]).

Corollary 1 *Let Y be a real normed space and $A, B, C \in \mathfrak{F}_0(Y)$. Assume that $A, B \in ccl(Y)$, C is bounded, and $A + C = B + C$. Then $A = B$.*

Nikodem and Popa in [9] and Piszczek in [12] proved the following theorem.

Theorem 1 *Let K be a convex cone in a real vector space X, Y a real Banach space and $\alpha, \beta, p, q > 0$. Consider a set-valued function $F : K \to ccl(Y)$ with bounded diameter fulfilling the inclusion*

$$F(\alpha x + \beta y) \subset pF(x) + qF(y), \quad x, y \in K.$$

If $\alpha + \beta < 1$, then there exists a unique selection $f : K \to Y$ of F satisfying the equation

$$f(\alpha x + \beta y) = pf(x) + qf(y), \quad x, y \in K.$$

If $\alpha + \beta > 1$, then F is single valued.

The case of $p + q = 1$ was investigated by Popa in [14], Inoan and Popa in [5]. By means of the inclusion relation, Park et al. [7, 11] investigated the approximation of some set-valued functional equations.

We now present some examples. A constant function $F : K \to ccl(Y)$, $F(x) = M$ for $x \in K$, where $K \subseteq X$ is a cone and $M \in ccl(Y)$ is fixed, satisfies the equation

$$F(\alpha x + \beta y) = pF(x) + qF(y), \quad x, y \in K,$$

and each constant function $f : K \to Y$, $f(x) = m$ for $x \in K$, where $m \in M$ is fixed, satisfies

$$f(\alpha x + \beta y) = pf(x) + qf(y), \quad x, y \in K.$$

The set-valued function $F : \mathbb{R} \to ccl(\mathbb{R})$ given by

$$F(x) = [x - 1, x + 1], \quad x \in \mathbb{R},$$

satisfies the equation

$$F\left(\frac{x + y}{2}\right) = \frac{F(x) + F(y)}{2}, \quad x, y \in \mathbb{R},$$

and each function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = x + c, \quad x \in \mathbb{R},$$

where $c \in [-1, 1]$ is fixed, is a selection of F and satisfies

$$f\left(\frac{x + y}{2}\right) = \frac{f(x) + f(y)}{2}, \quad x, y \in \mathbb{R}.$$

In the rest of this paper, unless otherwise explicitly stated, we will assume that $(K, +)$ is a commutative group, Y is a real Banach space, and k is a positive integer less than or equal to 3.

Theorem 2 *Let $F : K \to cclz(Y)$ be a set-valued function with bounded diameter.*

(1) *If*

$$\alpha^{2-k}\sigma_{y,z}F(\alpha x) + 8F(x) \subseteq 2\left(\sigma_y F(x) + \sigma_z F(x)\right) + 4\alpha^2 F(x), \qquad (2)$$

for all $x, y, z \in K$, then there exists a unique selection $f : K \to Y$ of F such that, for all $x, y \in K$, (i) $f(x + y) = f(x) + f(y)$ if $k = 1$; (ii) $\sigma_y f(x) = 2f(x) + 2f(y)$ if $k = 2$; (iii) $\sigma_y f(2x) = 2\sigma_y f(x) + 12f(x)$ if $k = 3$.

(2) *If*

$$2\left(\sigma_y F(x)+\sigma_z F(x)\right)+4\alpha^2 F(x)\subseteq \alpha^{2-k}\sigma_{y,z}F(\alpha x)+8F(x) \tag{3}$$

for all $x, y, z \in K$, then F is single-valued.

Proof

(1) Letting $y = z = 0$ in (2), we have

$$\begin{aligned}\alpha^{2-k}\left(F(\alpha x)+F(\alpha x)+F(\alpha x)+F(\alpha x)\right)+8F(x)\\ \subseteq 2\left(F(x)+F(x)+F(x)+F(x)\right)+4\alpha^2 F(x)\end{aligned}$$

for all $x \in K$. Since the set $F(x)$ is convex, we can conclude from Lemma 1 that

$$4\alpha^{2-k}F(\alpha x)+8F(x)\subseteq 8F(x)+4\alpha^2 F(x)$$

for all $x \in K$. Using the Rådström's cancelation law, one obtains

$$F(\alpha x)\subseteq \alpha^k F(x)$$

for all $x \in K$. Replacing x by $\alpha^n x$, $n \in \mathbb{N}$, in the last inclusion, we see that

$$\alpha^{-k(n+1)}F(\alpha^{n+1}x)\subseteq \alpha^{-kn}F(\alpha^n x)$$

for all $x \in K$. Thus $\left(\alpha^{-kn}F(\alpha^n x)\right)_{n\in\mathbb{N}_0}$ is a decreasing sequence of closed subsets of the Banach space Y. We also get

$$\delta\left(\alpha^{-kn}F(\alpha^n x)\right)=\alpha^{-kn}\delta\left(F(\alpha^n x)\right)$$

for all $x \in K$. Now since $\sup_{x\in K}\delta\left(F(x)\right)<+\infty$, we get that

$$\lim_{n\to+\infty}\delta\left(\alpha^{-kn}F(\alpha^n x)\right)=0$$

for all $x \in K$. Hence

$$\lim_{n\to+\infty}\alpha^{-kn}F(\alpha^n x)=\bigcap_{n\in\mathbb{N}_0}\alpha^{-kn}F(\alpha^n x)=: f(x)$$

is a singleton. Thus we obtain a function $f : K \to Y$ which is a selection of F.

We will now prove that f for $m = 1$, 2, and 3 is additive, quadratic, and cubic, respectively. We have

$$\alpha^{2-k(n+1)}\sigma_{\alpha^n y,\alpha^n z}F(\alpha^{n+1}x) + 8\alpha^{-kn}F(\alpha^n x) \\ \subseteq 2\alpha^{-kn}\left(\sigma_{\alpha^n y}F(\alpha^n x) + \sigma_{\alpha^n z}F(\alpha^n x)\right) + 4\alpha^{-kn+2}F(\alpha^n x)$$

for all $x, y, z \in K$ and $n \in \mathbb{N}$. By the definition of f, we get

$$\begin{aligned} &\alpha^{2-k}\sigma_{y,z}f(\alpha x) + 8f(x) \\ &\quad = \alpha^{2-k}\sigma_{\alpha^n y,\alpha^n z}\bigcap\nolimits_{n\in\mathbb{N}_0}\alpha^{-kn}F(\alpha^{n+1}x) + 8\bigcap\nolimits_{n\in\mathbb{N}_0}\alpha^{-kn}F(\alpha^n x) \\ &\quad = \bigcap\nolimits_{n\in\mathbb{N}_0}\left(\alpha^{2-k(n+1)}\sigma_{\alpha^n y,\alpha^n z}F(\alpha^{n+1}x) + 8\alpha^{-kn}F(\alpha^n x)\right) \\ &\quad \subseteq \bigcap\nolimits_{n\in\mathbb{N}_0}\left(2\alpha^{-kn}\left(\sigma_{\alpha^n y}F(\alpha^n x) + \sigma_{\alpha^n z}F(\alpha^n x)\right) + 4\alpha^{-kn+2}F(\alpha^n x)\right) \end{aligned}$$

for all $x, y, z \in K$. Thus we obtain

$$\begin{aligned} &\left\|\alpha^{2-k}\sigma_{y,z}f(\alpha x) + 8f(x) - 2\sigma_y f(x) - 2\sigma_z f(x) - 4\alpha^2 f(x)\right\| \\ &\quad \le \delta\left(2\alpha^{-kn}\left(\sigma_{\alpha^n y}F(\alpha^n x) + \sigma_{\alpha^n z}F(\alpha^n x)\right) + 4\alpha^{-kn+2}F(\alpha^n x)\right) \\ &\quad = 2\delta\left(\alpha^{-kn}\sigma_{\alpha^n y}F(\alpha^n x)\right) + 2\delta\left(\alpha^{-kn}\sigma_{\alpha^n z}F(\alpha^n x)\right) + 4\alpha^2\delta\left(\alpha^{-kn}F(\alpha^n x)\right) \end{aligned}$$

which tends to zero as n tends to ∞. Thus

$$\alpha^{2-k}\sigma_{y,z}f(\alpha x) = 2\left(\sigma_y f(x) + \sigma_z f(x)\right) + 4\left(\alpha^2 - 2\right)f(x) \tag{4}$$

for all $x, y, z \in K$. Setting $x = y = z = 0$ in (4), we have $f(0) = 0$. Putting $y = 0$ in (4) and using $f(0) = 0$, one gets

$$\alpha^{2-k}\sigma_z f(\alpha x) = \sigma_z f(x) + 2\left(\alpha^2 - 1\right)f(x)$$

for all $x, z \in K$. Based on Theorem 2.1 of [19] (also see [20, 21]), we conclude that, for all $x, y \in K$, if $k = 1$, then $f(x + y) = f(x) + f(y)$, if $k = 2$, then $\sigma_y f(x) = 2f(x) + 2f(y)$ and if $k = 3$, then $\sigma_y f(2x) = 2\sigma_y f(x) + 12f(x)$.

Next, let us prove the uniqueness of f. Suppose that f and g are selections of F. We have $(kn)^k f(x) = f(knx) \in F(knx)$ and $(kn)^k g(x) = g(knx) \in F(knx)$ for all $x \in K$ and $n \in \mathbb{N}$. Then we get

$$\begin{aligned} (kn)^k\|f(x) - g(x)\| &= \left\|(kn)^k f(x) - (kn)^k g(x)\right\| \\ &= \|f(knx) - g(knx)\| \\ &\le 2\delta\left(F(knx)\right) \end{aligned}$$

for all $x \in K$ and $n \in \mathbb{N}$. It follows from $\sup_{x\in K} \delta(F(x)) < +\infty$ that $f(x) = g(x)$ for all $x \in K$.

(2) Letting $y = z = 0$ in (3) and using the Rådström's cancelation law, one gets $F(x) \subseteq \alpha^{-k} F(\alpha x)$ for all $x \in K$. Hence,

$$F(x) \subseteq \alpha^{-kn} F(\alpha^n x) \subseteq \alpha^{-k(n+1)} F(\alpha^{n+1} x)$$

for all $x \in K$. It follows that $\left(\alpha^{-kn} F(\alpha^n x)\right)_{n\in\mathbb{N}_0}$ is an increasing sequence of sets in the Banach space Y. It follows from $\sup_{x\in K} \delta(F(x)) < +\infty$ that

$$\lim_{n\to+\infty} \delta\left(\alpha^{-kn} F(\alpha^n x)\right) = \lim_{n\to+\infty} \alpha^{-kn} \delta\left(F(\alpha^n x)\right) = 0$$

for all $x \in K$. Then, for all $n \in \mathbb{N}_0$ and $x \in K$, $\alpha^{-kn} F(\alpha^n x)$ is single-valued and

$$\alpha^{2-k} \sigma_{y,z} F(\alpha x) = 2\left(\sigma_y F(x) + \sigma_z F(x)\right) + 4\left(\alpha^2 - 2\right) F(x)$$

for all $x, y, z \in K$. By adopting the method used in case (1), we see that, for all $x, y \in K$, if $k = 1$, then $F(x + y) = F(x) + F(y)$, if $k = 2$, then $\sigma_y F(x) = 2F(x) + 2F(y)$ and if $k = 3$, then $\sigma_y F(2x) = 2\sigma_y F(x) + 12F(x)$.

Theorem 3 *Let $F : K \to cclz(Y)$ be a set-valued function with bounded diameter.*

(1) *If F satisfies the inclusion (1), then there exists a unique selection $f : K \to Y$ of F such that $\sigma_y f(2x) = 4\sigma_y f(x) + 24 f(x) - 6 f(y)$ for all $x, y \in K$.*

(2) *If*

$$\begin{aligned} &2\alpha^2 \left(\sigma_y F(x) + \sigma_z F(x)\right) + 2\sigma_z F(y) + 4\alpha^4 F(x) \\ &\subseteq \sigma_{y,z} F(\alpha x) + 4\alpha^2 \left(2F(x) + F(y) + F(z)\right) \end{aligned} \tag{5}$$

for all $x, y, z \in K$, then F is single-valued.

Proof

(1) Letting $y = z = 0$ in (1), we have

$$\begin{aligned} &F(\alpha x) + F(\alpha x) + F(\alpha x) + F(\alpha x) + 4\alpha^2 \left(2F(x) + F(0) + F(0)\right) \\ &\subseteq 2\alpha^2 \left(F(x) + F(x) + F(x) + F(x)\right) + 2\left(F(0) + F(0)\right) + 4\alpha^4 F(x) \end{aligned}$$

for all $x \in K$. Hence, from the convexity of $F(x)$ and Lemma 1, we see from that

$$F(\alpha x) + 2\alpha^2 F(x) + 2\alpha^2 F(0) \subseteq 2\alpha^2 F(x) + F(0) + \alpha^4 F(x) \tag{6}$$

for all $x \in K$. Setting $x = 0$ in (6), we have

$$\left(4\alpha^2 + 1\right) F(0) \subseteq \left(\alpha^4 + 2\alpha^2 + 1\right) F(0),$$

and using the Rådström's cancelation law, one obtains

$$\{0\} \subseteq F(0). \tag{7}$$

Again applying (6) and the Rådström's cancelation law, one gets

$$F(\alpha x) + (2\alpha^2 - 1)F(0) \subseteq \alpha^4 F(x) \tag{8}$$

for all $x \in K$. It follows from (7) and (8) that

$$F(\alpha x) \subseteq F(\alpha x) + (2\alpha^2 - 1)F(0) \subseteq \alpha^4 F(x)$$

for all $x \in K$. Hence

$$\alpha^{-4(n+1)} F(\alpha^{n+1} x) \subseteq \alpha^{-4n} F(\alpha^n x)$$

for all $x \in K$. In the same way as in Theorem 2, we obtain a function $f : K \to Y$ which is a selection of F and

$$\sigma_{y,z} f(\alpha x) = 2\alpha^2 \left(\sigma_y f(x) + \sigma_z f(x) + 2\left(\alpha^2 - 2\right) f(x)\right) + 2\left(\sigma_z f(y) - 2\left(\alpha^2\right) f(y)\right) - 4\alpha^2 f(z) \tag{9}$$

for all $x, y, z \in K$. Setting $x = y = z = 0$ in (9), we have $f(0) = 0$. Putting $y = 0$ in (9) and using $f(0) = 0$, one gets

$$\sigma_z f(\alpha x) = \alpha^2 \sigma_z f(x) + 2\alpha^2(\alpha^2 - 1) f(x) + 2(1 - \alpha^2) f(z)$$

for all $x, z \in K$. Based on Theorem 2.1 of [22], we conclude that f is quartic; i.e., $\sigma_y f(2x) = 4\sigma_y f(x) + 24 f(x) - 6 f(y)$ for all $x, y \in K$.

(2) Letting $y = z = 0$ in (5) and using the convexity of $F(x)$ and the Rådström's cancelation law, we obtain

$$\alpha^4 F(x) + F(0) \subseteq F(\alpha x) + 2\alpha^2 F(0)$$

for all $x \in K$. Substituting x, y, and z by zero in (5) yields

$$F(0) \subseteq \{0\}.$$

From the last two inclusions, it follows that

$$\alpha^4 F(x) \subseteq F(\alpha x) + (2\alpha^2 - 1)F(0) \subseteq F(\alpha x)$$

for all $x \in K$. Hence,

$$F(x) \subseteq \alpha^{-4n} F(\alpha^n x) \subseteq \alpha^{-4(n+1)} F(\alpha^{n+1} x)$$

for all $x \in K$. In the same way, as in Theorem 2, we deduce that F is single-valued and $\sigma_y F(2x) = 4\sigma_y F(x) + 24F(x) - 6F(y)$ for all $x, y \in K$.

3 Set-Valued Dynamics and Applications

In this section we present a few applications of the results presented in the previous sections.

Theorem 4 *If $W \in ccz(Y)$ and $f : K \to Y$ satisfies*

$$\alpha\sigma_{y,z} f(\alpha x) - 2\sigma_y f(x) - 2\sigma_z f(x) + 4\left(2 - \alpha^2\right) f(x) \in W \tag{10}$$

for all $x, y, z \in K$, then there exists a unique function $T : K \to Y$ such that

$$\begin{cases} \alpha\sigma_{y,z} T(\alpha x) = 2\left(\sigma_y T(x) + \sigma_z T(x)\right) + 4\left(\alpha^2 - 2\right) T(x), \\ T(x) - f(x) \in \frac{1}{4(\alpha^2 - \alpha)} W \end{cases}$$

for all $x, y, z \in K$.

Proof Let $F(x) := f(x) + \frac{1}{4\alpha(\alpha-1)} W$ for $x \in K$. Then

$$\begin{aligned} \alpha\sigma_{y,z} F(\alpha x) + 8F(x) &= \alpha\sigma_{y,z} f(\alpha x) + 8f(x) + \tfrac{\alpha+2}{\alpha(\alpha-1)} W \\ &\subseteq 2\sigma_y f(x) + 2\sigma_z f(x) + 4\alpha^2 f(x) + \tfrac{\alpha+2}{\alpha(\alpha-1)} W + W \\ &= 2\left(\sigma_y f(x) + \tfrac{1}{2\alpha(\alpha-1)} W\right) + 2\left(\sigma_z f(x) + \tfrac{1}{2\alpha(\alpha-1)} W\right) \\ &\quad +4\alpha^2\left(f(x) + \tfrac{1}{4\alpha(\alpha-1)} W\right) \\ &= 2\left(\sigma_y F(x) + \sigma_z F(x)\right) + 4\alpha^2 F(x) \end{aligned}$$

for all $x, y, z \in K$. Now, according to Theorem 2 with $k = 1$, there exists a unique function $T : K \to Y$ such that

$$\alpha\sigma_{y,z}T(\alpha x) = 2\left(\sigma_y T(x) + \sigma_z T(x)\right) + 4\left(\alpha^2 - 2\right)T(x)$$

for all $x, y, z \in K$ and $T(x) \in F(x)$ for all $x \in K$.

Corollary 2 *Suppose $W \in ccz(Y)$ and $f : K \to Y$ satisfies (10) for all $x, y, z \in K$. Then there exists a unique additive function $T : K \to Y$ such that, for all $x \in K$,*

$$T(x) - f(x) \in \frac{1}{4(\alpha^2 - \alpha)}W.$$

We recall that a semigroup $(S, +)$ is called left (right) amenable if there exists a left (right) invariant mean on the space $B(S, \mathbb{R})$ of all real bounded functions defined on S. By a left (right) invariant mean we understand a linear functional M satisfying

$$\inf_{x\in S} f(x) \le M(f) \le \sup_{x\in S} f(x),$$

and

$$M({}_a f) = M(f) \qquad (M(f_a) = M(f))$$

for all $f \in B(S, \mathbb{R})$ and $a \in S$, where ${}_a f$ (f_a) is the left (right) translate of f defined by ${}_a f(x) = f(a + x)$, $(f_a(x) = f(x + a))$, $x \in S$. If, on the space $B(S, \mathbb{R})$, there exists a real linear functional which is simultaneously a left and right invariant mean, then we say that S is two-sided amenable or just amenable.

One can prove that every Abelian semigroup is amenable. For the theory of amenability see, for example, Greenleaf [23]. Finally, let us see a result in [24].

Theorem 5 *Let $(S, +)$ be a left amenable semigroup and let X be a Hausdorff locally convex linear space. Let $F : S \to \mathfrak{F}_0(X)$ be set-valued function such that $F(s)$ is convex and weakly compact for all $s \in S$. Then F admits an additive selection if, and only if, there exists a function $f : S \to X$ such that*

$$f(s + t) - f(t) \in F(s) \tag{11}$$

for all $s, t \in S$.

As a consequence of the above theorem, we have the following corollaries.

Corollary 3 *Let $(S, +)$ be a left amenable semigroup and let X be a reflexive Banach space. In addition, let $\rho : S \to [0, \infty)$ and $g : S \to X$ be arbitrary functions. Then there exists an additive function $a : S \to X$ such that*

$$\| a(s) - g(s) \| \le \rho(s) \tag{12}$$

for all $s \in S$, if, and only if, there exists a function $f : S \to X$ such that

$$\| f(s+t) - f(t) - g(s) \| \leq \rho(s) \tag{13}$$

for all $s, t \in S$.

Proof Define a set valued map $F : S \to \mathfrak{F}_0(X)$ by

$$F(s) = \{x \in X : \ \| x - g(s) \| \leq \rho(s)\}$$

for all $s \in S$. Then, due to the reflexivity of X, F has weakly compact nonempty convex values. It follows from (12) that a is a selection of F, and (13) is equivalent to (11). Now, the result follows from Theorem 5.

Corollary 4 (*Ger* [25]) *Let $(S, +)$ be a left amenable semigroup, let X be a reflexive Banach space, and let $\rho : S \to [0, \infty)$ be an arbitrary function. If the function $f : S \to X$ satisfies $\| f(s+t) - f(s) - f(t) \| \leq \rho(s)$ for all s, t in S, then there exists an additive function $a : S \to X$ such that $\| f(s) - a(s) \| \leq \rho(s)$ holds for all s in S.*

References

1. W. Smajdor, Superadditive set-valued functions. Glas. Mat. **21** (1986), 343–348
2. Z. Gajda, R. Ger, Subadditive multifunctions and Hyers-Ulam stability. Numer. Math. **80**, 281–291 (1987)
3. D. Popa, Additive selections of (α, β)-subadditive set valued maps. Glas. Mat. Ser. III **36**, 11–16 (2001)
4. J. Brzdęk, D. Popa, B. Xu, Selections of set-valued maps satisfying a linear inclusion in a single variable. Nonlin. Anal. **74**, 324–330 (2011)
5. D. Inoan, D. Popa, On selections of generalized convex set-valued maps. Aequat. Math. **88**, 267–276 (2014)
6. H. Khodaei, On the stability of additive, quadratic, cubic and quartic set-valued functional equations. Results Math. **68**, 1–10 (2015)
7. G. Lu, C. Park, Hyers-Ulam stability of additive set-valued functional equations. Appl. Math. Lett. **24**, 1312–1316 (2011)
8. K. Nikodem, On quadratic set-valued functions. Publ. Math. Debrecen **30**, 297–301 (1984)
9. K. Nikodem, D. Popa, On selections of general linear inclusions. Publ. Math. Debrecen **75**, 239–249 (2009)
10. K. Nikodem, D. Popa, On single-valuedness of set-valued maps satisfying linear inclusions. Banach J. Math. Anal. **3**, 44–51 (2009)
11. C. Park, D. O'Regan, R. Saadati, Stability of some set-valued functional equations. Appl. Math. Lett. **24**, 1910–1914 (2011)
12. M. Piszczek, On selections of set-valued inclusions in a single variable with applications to several variables. Results Math. **64**, 1–12 (2013)
13. M. Piszczek, The properties of functional inclusions and Hyers-Ulam stability. Aequat. Math. **85**, 111–118 (2013)
14. D. Popa, A property of a functional inclusion connected with Hyers-Ulam stability. J. Math. Inequal.**4**, 591–598 (2009)

15. K. Nikodem, K-Convex and K-Concave Set-Valued Functions, Zeszyty Naukowe, Politech, Krakow, 1989
16. H. Rådström, An embedding theorem for space of convex sets. Proc. Am. Math. Soc. **3**, 165–169 (1952)
17. W. Smajdor, Subadditive and subquadratic set-valued functions, Prace Nauk. Uniw. Śląsk, 889, Katowice (1987)
18. R. Urbański, A generalization of the Minkowski-Rådström-Hörmander theorem. Bull. Polish Acad. Sci. **24**, 709–715 (1976)
19. M.E. Gordji, Z. Alizadeh, H. Khodaei, C. Park, On approximate homomorphisms: a fixed point approach. Math. Sci. **6**, 59 (2012)
20. J.H. Bae, W.G. Park, A functional equation having monomials as solutions. Appl. Math. Comput. **216**, 87–94 (2010)
21. M.E. Gordji, Z. Alizadeh, Y.J. Cho, H. Khodaei, On approximate C^*-ternary m-homomorphisms: a fixed point approach. Fixed Point Theory Appl. (2011), Art. ID 454093
22. Y.S. Lee, S.Y. Chung, Stability of quartic functional equation in the spaces of generalized functions. Adv. Differ. Equ. (2009), Art. ID 838347
23. F.P. Greenleaf, *Invariant Mean on Topological Groups and their Applications*, Van Nostrand Mathematical Studies, vol. 16, New York/Toronto/London/Melbourne, 1969
24. R. Badora, R. Ger, Z. Páles, Additive selections and the stability of the Cauchy functional equation. ANZIAM J. **44**, 323–337 (2003)
25. R. Ger, The singular case in the stability behaviour of linear mappings. Grazer Math. Ber. **316**, 59–70 (1992)