On the Study of Circuit Chains Associated with a Random Walk with Jumps in Fixed, Random Environments: Criteria of Recurrence and Transience

Chrysoula Ganatsiou

Abstract By considering a nonhomogeneous random walk with jumps (with steps -1 or $+1$ or in the same position having a right-elastic barrier at 0) we investigate the unique representations by directed circuits and weights of the corresponding Markov chains (circuit chains) in fixed, random environments. This will give us the possibility to find suitable criteria regarding the properties of recurrence and transience of the above-mentioned circuit chains in fixed, random environments.

2010 AMS Mathematics Subject Classification 60J10, 60G50, 60K37

1 Introduction

In recent years a systematic research has been developed (Kalpazidou [10], MacQueen [12], Qian Minping and Qian Min [13], Zemanian [16] and others) in order to investigate representations of the finite-dimensional distributions of Markov processes (with discrete or continuous parameter) having an invariant measure, as decompositions in terms of the *circuit* (or cycle) *passage functions*

$$J_c(i, j) = \begin{cases} 1, & \text{if } i, j \text{ are consecutive states of c,} \\ 0, & \text{otherwise,} \end{cases}$$

for any directed sequence $c = (i_1, i_2, \ldots, i_v, i_1)$ (or $\hat{c} = (i_1, i_2, \ldots, i_v)$) of states, called a *circuit* (or a *cycle*), $v > 1$ of the corresponding Markov process. This research has stimulated a motivation towards the representation of Markov processes through directed circuits (or cycles) and weights in terms of circuit (or cycle) passage functions in fixed or random environments as well as the study of specific problems associated with Markov processes in a different way. The

C. Ganatsiou (✉)
Department of Digital Systems, University of Thessaly, Larissa, Greece
e-mail: chganats@uth.gr; ganatsio@otenet.gr

T. M. Rassias, P. M. Pardalos (eds.), *Mathematical Analysis and Applications*, Springer Optimization and Its Applications 154,
https://doi.org/10.1007/978-3-030-31339-5_6

representations are called *circuit (or cycle) representations* while the corresponding discrete parameter Markov chains generated by directed weighted circuits are called *circuit chains* [1, 10].

In parallel, *random walks* are one of the most basic and well-studied topics in probability theory and one of the most fundamental types of stochastic processes formed by successive summation of independent, identically distributed random variables. For random walks on the integer lattice Z^d the main reference is the classic book by F. Spitzer [15]. They have a long rich history [2, 3, 8] which has been advanced according to many directions of investigation. The term "random walk" was coined by Karl Pearson [14], and the study of random walks dates back to the "Gamblers Ruin" problem analyzed by Pascal, Fermat, Huygens, Bernoulli, and others. Theoretical developments of random walks have involved mathematics (especially probability theory), computer science, statistical physics, operations research, and more. Random walk models have also been applied in various domains, ranging from locomotion and foraging of animals, the dynamics of neuronal firing and decision-making in the brain to population genetics, polymer chains, descriptions of financial markets, rankings systems, dimension reduction, and feature extraction from high-dimensional data (e.g., in the form of "diffusion maps"), sports statistics, prediction of the arrival times of diseases spreading on networks, etc.

Usually they are studied from the Markov chain point of view, where the random mechanism of spatial motion is determined by the given transition probabilities (probabilities of jumps) at each state in a *non-random (fixed) environment*. Although random walks provide a simple conventional model to describe various transport processes in many cases, the medium where the system evolves is highly irregular due to many irregularities (defects, fluctuations, etc.) known as random environments which lead to the choice of the local characteristics of the motion at random according to certain probability distribution. Such models are referred to as *random walks in random environments*. The definition of these random walks involves two special ingredients: the *environment* (randomly chosen but still fixed throughout the time evolution) and the *random walk* (whose transition probabilities are determined by the environment) [8].

It is known also that in various applications (physics, chemistry, genetics, etc.) we are led to study Markov chains obtained by restricting the motion of a "particle" which performs a random walk. This is done by introducing barriers. In this case the Markov chain defined in this way having no longer independent increments is called a *random walk with barriers* while its state space is a proper subset of Z. Furthermore except for the *homogeneous random walks* with independent and identically distributed increments there is a class of random walks formed by successive summation of independent random variables which are no longer identically distributed. This means that they still have independent increments which are no longer identically distributed. These random walks are called *nonhomogeneous* and they can be investigated from the Markov chain point of view which in general coincides with that for chains with independent increments.

Let us consider the nonhomogeneous random walk with state space S=N, right-elastic barrier at 0 [7] and transition probabilities given by $p_{ij} = 0$, if $|i - j| > 1$, $p_{i,i-1} = q_i$, $p_{i,i} = r_i$, $p_{i,i+1} = p_i$, $p_i + q_i + r_i = 1$, $i \geq 1$, $p_{00} = r_0$, $p_{01} = p_0 = 1 - r_0$, $p_i > 0$, $q_{i+1} > 0$, $r_i \geq 0$, $i \geq 0$, which expresses the movement of a particle depending on the time that the particle begins to move. It is obvious that all states form an essential class. It is known that regarding the classification of the states through the use of proper theorems ensuring a bounded solution of the system of equations

$$z_i = \sum_{j=1}^{\infty} p_{ij} z_j, \quad i \geq 1$$

we have that: the states are *positive recurrent* if and only if

$$\sum_{i=1}^{\infty} r_i = +\infty \qquad \text{and} \qquad \sum_{i=1}^{\infty} \frac{1}{r_i p_i} < +\infty$$

and *null recurrent* if and only if

$$\sum_{i=1}^{\infty} r_i = \sum_{i=1}^{\infty} \frac{1}{r_i \cdot p_i} = +\infty \qquad \text{where} \qquad r_i = \frac{q_1 \cdots q_i}{p_1 \cdots p_i}, \; i \geq 1 \; [9]$$

The main purpose of this work is to bring together the two subjects—random walks and circuit chains—by discussing their interconnection. In particular following the context of the theory of Markov processes'cycle-circuit representation, the present work arises as an attempt to study the circuit and weight representation of the above-mentioned nonhomogeneous random walk with jumps in fixed, random environments as well as to investigate proper criteria regarding recurrence and transience of the corresponding "adjoint" Markov chains (circuit chains) describing uniquely the above-mentioned random walk by directed circuits and weights in fixed, random environments giving a new perspective in the whole study and especially in the classification of states.

The work is organized as follows. In Section 2, we give a brief account of certain concepts of circuit-cycle representation theory of Markov processes that we shall need throughout the paper. In Section 3, the above-mentioned nonhomogeneous random walk with jumps (having one right-elastic barrier at 0) is considered and the unique representations by directed circuits and weights of the corresponding Markov chains (circuit chains) are investigated in fixed, random environments. These representations will give us the possibility to find proper criteria regarding positive/null recurrence and transience of the above-mentioned circuit chains in fixed, random environments [4–6], as it is described in Section 4.

Throughout the paper, we shall need the following notations:

$$\mathbb{N} = \{0, 1, 2, \dots\}, \qquad \mathbb{N}^* = \{1, 2, \dots\}, \qquad \mathbb{Z} = \{\dots, -1, 0, 1, \dots\},$$
$$\mathbb{Z}_+^* = \{1, 2, 3, \dots\}, \qquad \mathbb{Z}_-^* = \{\dots, -2, -1\}$$

2 Preliminaries

Let S be a denumerable set. The directed sequence $c = (i_1, i_2, \dots, i_v, i_1)$ modulo the cyclic permutations, where $i_1, i_2, \dots, i_v \in S, v > 1$, completely defines a directed circuit in S. The ordered sequence $\hat{c} = (i_1, i_2, \dots, i_v)$ associated with the given directed circuit c is called a *directed cycle* in S. A directed circuit may be considered as $c = (c(m), c(m+1), \dots, c(m+v-1), c(m+v))$, if there exists an $m \in \mathbb{Z}$, such that $i_1 = c(m+0), i_2 = c(m+1), \dots, i_v = c(m+v-1), i_1 = c(m+v)$, that is a periodic function from $\mathbb{Z}$ to S. The corresponding directed cycle is defined by the ordered sequence $\hat{c} = (c(m), c(m+1), \dots, c(m+v-1))$. The values $c(k)$ are the *points* of c, while the directed pairs , $(c(k), c(k+1)), k \in Z$, are the *directed edges* of c. The smallest integer $p \equiv p(c) \geq 1$ satisfying the equation $c(m+p) = c(m)$, for all $m \in \mathbb{Z}$, is the *period* of c. A directed circuit c such that $p(c) = 1$ is called a *loop*. (In the present work, we shall use directed circuits with distinct point elements.)

Let a directed circuit c (or a directed cycle $\hat{c}$) with period $p(c) > 1$. Then we may define by

$$J_c^{(n)}(i, j) = \begin{cases} 1, & \text{if there exists an } m \in \mathbb{Z} \text{ such that } i = c(m),\ j = c(m+n), m \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

the *n-step passage function* associated with the directed circuit c, for any $i, j \in S, n \geq 1$.

We may also define by

$$J_c(i) = \begin{cases} 1, & \text{if there exists an m } \in \mathbb{Z} \text{such that } i = c(m), \\ 0, & \text{otherwise} \end{cases}$$

the *passage function* associated with the directed circuit c, for any $i \in S$. The above definitions are due to MacQueen [12] and Kalpazidou [10].

Given a denumerable set S and an infinite denumerable class C of overlapping directed circuits (or directed cycles) with distinct points (except for the terminals) in S such that all the points of S can be reached from one another following paths of circuit-edges, that is, for each two distinct points i and j of S there exists a finite sequence $c_1, c_2 \dots, c_k, k \geq 1$, of circuits (or cycles) of C such that i lies on c_1 and j lies on c_k and any pair of consecutive circuits (c_n, c_{n+1}) have at least one point in common. We may assume also that the class C contains, among its elements, circuits (or cycles) with period greater than or equal to 2. With each directed circuit (or directed cycle) let us associate a *strictly positive weight* w_c which must be independent of the choice of the representative of c, that is, it must satisfy the consistency condition, $w_{c\circ t_k} = w_c$, $k \in \mathbb{Z}$, where t_k is the translation of length k.

For a given class C of overlapping directed circuits (or cycles) and for a given sequence $(w_c)_{c\in C}$ of weights we may define by

$$p_{ij} = \frac{\sum_{c\in C} w_c \cdot J_c^{(1)}(i, j)}{\sum_{c\in C} w_c \cdot J_c(i)} \tag{2.1}$$

the elements of a Markov transition matrix on S, if and only if $\sum_{c\in C} w_c \cdot J_c(i) < \infty$, for any $i \in S$. This means that a given Markov transition matrix $P = (p_{ij})$, $i, j \in S$ can be represented by directed circuits (or cycles) and weights if and only if there exists a class of overlapping directed circuits (or cycles) C and a sequence of positive weights $(w_c)_{c\in C}$ such that the formula (2.1) holds. In this case, the representation of the distribution of Markov process (with discrete or continuous parameter) having an invariant measure as decomposition in terms of the circuit (or cycle) passage functions is called *circuit (or cycle) representation* while the corresponding discrete parameter Markov chain generated by directed circuits (or cycles) is called *circuit (or cycle) chain* with Markov transition matrix P given by (2.1) and unique stationary distribution p (a solution of $p.P = p$) defined by

$$p(i) = \sum_{c\in C} w_c \cdot J_c(i), i \in S.$$

It is known that the following classes of Markov chains may be represented uniquely by directed circuits (or cycles) and weights:

(i) *the recurrent Markov chains* [13],
(ii) *the reversible Markov chains*.

3 Circuit and Weight Representations

3.1 Fixed Environments

Let us consider the Markov chain $(X_n)_{n\in N}$ on $\mathbb{N}$ (X_n expresses the location of a particle at time n, $n \in N$) which describes the nonhomogeneous random walk with jumps having a right-elastic barrier at 0, with transitions $k \to (k+1)$, $k \to (k-1)$ and $k \to k$, in a fixed environment, whose elements of the corresponding Markov transition matrix (transition probabilities) are defined by

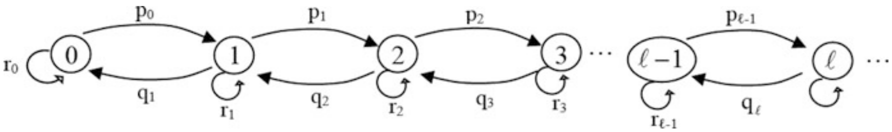

Fig. 1 The Markov chain $(X_n)_{n\in\mathbb{N}}$ (fixed environments)

$$P(X_{n+1} = 0/X_n = 0) = r_0,$$
$$P(X_{n+1} = 1/X_n = 0) = p_0,\ p_0 = 1 - r_0$$
$$P(X_{n+1} = k+1/X_n = k) = p_k,\ k \geq 1$$
$$P(X_{n+1} = k/X_n = k) = r_k,\ k \geq 1$$
$$P(X_{n+1} = k-1/X_n = k) = q_k,\ k \geq 1$$

such that $p_k + q_k + r_k = 1$, $p_k > 0, q_{k+1} > 0, r_k \geq 0$, for every $k \in \mathbb{N}$, as it is shown in Figure 1.

Assume that $(p_k)_{k\in N}$ and $(r_k)_{k\in N}$ are arbitrary fixed sequences with $0 < p_0 = 1 - r_0 \leq 1, p_k > 0, q_{k+1} > 0, r_k \geq 0$, for every $k \in \mathbb{N}$. If we consider the directed circuits $c_k = (k, k+1, k), c'_k = (k, k), k \in \mathbb{N}$ and the collections of weights $(w_{c_k})_{k\in N}$ and $(w_{c'_k})_{k\in N}$ respectively, then we may obtain the corresponding transition probabilities

$$p_k = \frac{w_{c_k}}{w_{c_{k-1}} + w_{c_k} + w_{c'_k}},$$

with

$$p_0 = \frac{w_{c_0}}{w_{c_0} + w_{c'_0}}$$

and

$$q_k = \frac{w_{c_{k-1}}}{w_{c_{k-1}} + w_{c_k} + w_{c'_k}}, \qquad r_k = \frac{w_{c'_k}}{w_{c_{k-1}} + w_{c_k} + w_{c'_k}}$$

such that $p_k + q_k + r_k = 1$, for every $k \geq 1$, with $r_0 = 1 - p_0 = \dfrac{w_{c'_0}}{w_{c_0} + w_{c'_0}}$.

Here the class $C(k)$ contains the directed circuits $c_k = (k, k+1, k)$, $c'_k = (k, k)$ and $c_{k-1} = (k-1, k, k-1)$.

Equivalently the transition matrix $P = (p_{ij})$ with

$$p_{ij} = \frac{\sum_{k\in N} w_{c_k} \cdot J^{(1)}_{c_k}(i,j)}{\sum_{k\in N}\left[w_{c_k}\cdot J_{c_k}(i) + w_{c'_k}\cdot J_{c'_k}(i)\right]}, \text{ for } i \neq j, \tag{3.1}$$

$$p_{ii} = \frac{\sum_{k\in N} w_{c'_k} \cdot J^{(1)}_{c'_k}(i,i)}{\sum_{k\in N}\left[w_{c_k}\cdot J_{c_k}(i) + w_{c'_k}\cdot J_{c'_k}(i)\right]}, \tag{3.2}$$

where $J^{(1)}_{c_k}(i, j) = 1$, if i, j are consecutive points of the circuit c_k, $J_{c_k}(i) = 1$, if i is a point of the circuit c_k, and $J_{c'_k}(i) = 1$, if i is a point of the circuit c'_k, expresses the representation of the Markov chain $(X_n)_{n\in N}$ by directed circuits and weights.

Furthermore let us consider also the "*adjoint*" Markov chain $(X'_n)_{n\in N}$ on N whose elements of the corresponding Markov transition matrix are defined by

$$\begin{aligned}
&P(X'_{n+1} = 0/X'_n = 0) = r'_0,\\
&P(X'_{n+1} = 1/X'_n = 0) = q'_0, q'_0 = 1 - r'_0,\\
&P(X'_{n+1} = k-1/X'_n = k) = p'_k, k \geq 1,\\
&P(X'_{n+1} = k/X'_n = k) = r'_k, k \geq 1,\\
&P(X'_{n+1} = k+1/X'_n = k) = q'_k, k \geq 1
\end{aligned}$$

such that $p'_k + q'_k + r'_k = 1$,$p'_{k+1} > 0$,$q'_k > 0$, $r'_k \geq 0$ for every $k \in \mathbb{N}$, as it is shown in Figure 2.

Assume that $(q'_k)_{k\in N}$, $(r'_k)_{k\in N}$ are arbitrary fixed sequences with $0 < q'_0 = 1 - r'_0 \leq 1$, $p'_{k+1} > 0$, $q'_k > 0$, $r'_k \geq 0$, for every $k \in \mathbb{N}$. If we consider the directed circuits $c''_k = (k + 1, k, k + 1)$, $c'''_k = (k, k)$ $k \in \mathbb{N}$, and the collections of weights $(w_{c''_k})_{k\in N}$, $(w_{c'''_k})_{k\in N}$, respectively, then we may have that

$$q'_k = \frac{w_{c''_k}}{w_{c''_{k-1}} + w_{c''_k} + w_{c'''_k}},$$

with

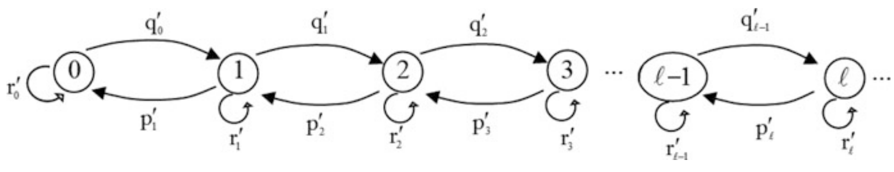

Fig. 2 The "adjoiont" Markov chain $(X'_n)_{n\in\mathbb{N}}$ (fixed environments)

$$q_0' = \frac{w_{c_0''}}{w_{c_0''} + w_{c_0'''}}$$

and

$$p_k' = \frac{w_{c_{k-1}''}}{w_{c_{k-1}''} + w_{c_k''} + w_{c_k'''}}, \qquad r_k' = \frac{w_{c_k'''}}{w_{c_{k-1}''} + w_{c_k''} + w_{c_k'''}}$$

such that $p_k' + q_k' + r_k' = 1$, for every $k \geq 1$, with $r_0' = 1 - q_0' = \dfrac{w_{c_0'''}}{w_{c_0''} + w_{c_0'''}}$.

Here the class $C'(k)$ contains the directed circuits $c_k'' = (k+1, k, k+1)$,$c_{k-1}'' = (k, k-1, k)$ and $c_k''' = (k, k)$. As a consequence, the transition matrix $P' = (p_{ij}')$ with elements equivalent to that given by the above-mentioned formulas (3.1), (3.2) expresses also the representation of the "adjoint" Markov chain $(X_n')_{n \in N}$ by directed circuits and weights.

Consequently we have the following:

Proposition 1 *The Markov chain* $(X_n)_{n \in N}$ *defined as above has a unique representation by directed circuits and weights.*

Proof Let us consider the set of directed circuits $c_k = (k, k+1, k)$ and $c_k' = (k, k)$, for every $k \in \mathbb{N}$, since only the transitions from k to $k+1$, k to $k-1$ and k to k are possible. There are three circuits through each point $k \geq 1 c_{k-1}, c_k, c_k'$, and two circuits through $0 : c_0, c_0'$.

The problem we have to manage is the definition of the weights. We may symbolize by w_k the weight w_{c_k} of the circuit c_k and by w_k' the weight $w_{c_k'}$ of the circuit c_k', for any $k \in \mathbb{N}$. The sequences $(w_k)_{k \in N}$,$(w_k')_{k \in N}$ must be a solution of

$$p_k = \frac{w_k}{w_{k-1} + w_k + w_k'}, k \geq 1 \qquad \text{with} \qquad p_0 = \frac{w_0}{w_0 + w_0'},$$

$$r_k = \frac{w_k'}{w_{k-1} + w_k + w_k'}, k \geq 1 \qquad \text{with} \qquad r_0 = \frac{w_0'}{w_0 + w_0'},$$

$$q_k = 1 - p_k - r_k, k \geq 1.$$

Let us take by $b_k = \dfrac{w_k}{w_{k-1}}$, $\gamma_k = \dfrac{w_k'}{w_{k-1}'}$, $k \geq 1$. As a consequence we may have

$$b_k = \frac{p_k}{q_k} = \frac{p_k}{1 - p_k - r_k}, \gamma_k = \frac{r_k}{r_{k-1}} \frac{p_{k-1}}{p_k} b_k, \text{ for every } k \geq 1.$$

Given the sequences $(p_k)_{k \in N}$ and $(r_k)_{k \in N}$ it is clear that the above sequences $(b_k)_{k \geq 1}$,$(\gamma_k)_{k \geq 1}$ exist and are unique. This means that the sequences $(w_k)_{k \in N}$,$(w'_k)_{k \in N}$ are defined uniquely, up to multiplicative constant factors, by

$$w_k = w_0 \cdot b_1 \dots b_k,$$
$$w'_k = w'_0 \cdot \gamma_1 \dots \gamma_k$$

(the unicity is understood up to the constant factors w_0, w'_0).

Proposition 2 *The "adjoint" Markov chain $(X'_n)_{n \in N}$ defined as above has a unique representation by directed circuits and weights.*

Proof Following an analogous way of that given in the proof of Proposition 1 the problem we have also to manage here is the definition of the weights. To this direction we may symbolize by w''_k the weight $w_{c''_k}$ of the circuit c''_k and by w'''_k the weight $w_{c'''_k}$ of the circuit c'''_k , for every $k \in \mathbb{N}$. The sequences $(w''_k)_{k \in N}$,$(w'''_k)_{k \in N}$ must be solutions of

$$q'_k = \frac{w''_k}{w''_{k-1} + w''_k + w'''_k}, k \geq 1 \qquad \text{with} \qquad q'_0 = \frac{w''_0}{w''_0 + w'''_0},$$
$$r'_k = \frac{w'''_k}{w''_{k-1} + w''_k + w'''_k}, k \geq 1 \qquad \text{with} \qquad r'_0 = \frac{w'''_0}{w''_0 + w'''_0},$$
$$p'_k = 1 - q'_k - r'_k, \qquad k \geq 1$$

By considering the sequences $(s_k)_k$,$(t_k)_k$ where $s_k = \dfrac{w''_{k-1}}{w''_k}$, $t_k = \dfrac{w'''_{k-1}}{w'''_k}$, $k \geq 1$ we may obtain that

$$s_k = \frac{1 - q'_k - r'_k}{q'_k}, t_k = \frac{r'_{k-1}}{r'_k} \cdot \frac{q'_k}{q'_{k-1}} \cdot s_k, \qquad \text{for every} \qquad k \geq 1.$$

For given sequences $(q'_k)_{k \in N}$, $(r'_k)_{k \in N}$ it is obvious that $(s_k)_{k \geq 1}$,$(t_k)_{k \geq 1}$ exist and are unique for those sequences, that is, the sequences $(w''_k)_{k \in N}$,$(w'''_k)_{k \in N}$ are defined uniquely, up to multiplicative constant factors, by

$$w''_k = \frac{w''_0}{s_1 \cdot s_2 \dots s_k}$$
$$w'''_k = \frac{w'''_0}{t_1 \cdot t_2 \dots t_k}$$

(the unicity is based on the constant factors w''_0, w'''_0).

3.2 Random Environments

Let us consider the random walk on $\mathbb{Z}$, with transitions $k \to (k+1)$, $k \to (k-1)$ and $k \to k$ whose transition probabilities $(p_k)_{k\in\mathbb{Z}}$, $(r_k)_{k\in\mathbb{Z}}$ constitute stationary ergodic sequences. A realization of these stationary ergodic sequences is called a *random environment* for this random walk. In order to investigate the unique circuit and weight representation of this random walk in random environments, for almost every environment, let us consider a probability space $(\Omega, \mathscr{F}, \mu)$, a measure preserving ergodic automorphism of this space $m : \Omega \mapsto \Omega$ and the measurable functions $p : \Omega \mapsto (0, 1)$, $r : \Omega \mapsto (0, 1)$ such that every $\omega \in \Omega$ generates the random environment $p_k \equiv p(m^k\omega)$, $r_k \equiv r(m^k\omega)$, $k \in \mathbb{Z}$. Since m is measure preserving and ergodic, the sequences $(p_k)_{k\in\mathbb{Z}}$, $(r_k)_{k\in\mathbb{Z}}$ are stationary ergodic sequences of random variables.

Let also $S = \mathbb{Z}^N$ be the infinite product space with coordinates $(X_n)_{n\in N}$. Then we may define a family $(\mathbb{P}^\omega)_{\omega\in\Omega}$ of probability measures such that, for every $\omega \in \Omega$, the sequence $(X_n)_{n\in N}$ forms a Markov chain on $\mathbb{Z}$ whose elements of the corresponding Markov transition matrix are defined by

$$\mathbb{P}^\omega(X_0 = 0) = 1,$$

$$\mathbb{P}^\omega(X_{n+1} = k + 1/X_n = k) = p(m^k\omega),$$

$$\mathbb{P}^\omega(X_{n+1} = k/X_n = k) = r(m^k\omega),$$

$$\mathbb{P}^\omega(X_{n+1} = k - 1/X_n = k) = 1 - p(m^k\omega) - r(m^k\omega) \equiv q(m^k\omega), k \in \mathbb{Z},$$

as it is shown in Figure 3.

We have the following:

Proposition 3 *For μ almost every environment $\omega \in \Omega$ the chain $(X_n)_{n\in N}$ has a unique circuit and weight representation.*

Proof Following an analogous way of that given in Section 3.1, let us consider the set of directed circuits $c_k = (k, k + 1, k)$ and $c'_k = (k, k)$, for every $k \in \mathbb{Z}$, since only the transitions from k to $k + 1$, k to $k - 1$ and k to k are possible. There are three circuits through each point $k \in \mathbb{Z} : c_{k-1}, c_k$ and c'_k.

The problem we have to manage is the definition of the weights of the circuits. We may symbolize by $w_k(\omega)$ the weight of the circuit c_k and by $w'_k(\omega)$ the weight

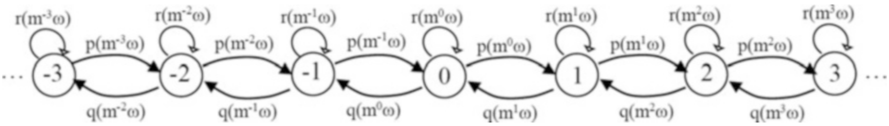

Fig. 3 The Markov chain $(X_n)_{n\in\mathbb{N}}$ (random environments)

of the circuit c'_k , for every $k \in \mathbb{Z}$. For the definition of weights let us consider the sequences $(b_k(\omega))_{k\in\mathbb{Z}}$, $(\gamma_k(\omega))_{k\in\mathbb{Z}}$ defined by

$$b_k(\omega) = \frac{w_k(\omega)}{w_{k-1}(\omega)}, \gamma_k(\omega) = \frac{w'_k(\omega)}{w'_{k-1}(\omega)}, k \in \mathbb{Z}.$$

As a consequence, we may have

$$b_k(\omega) = \frac{p(m^k\omega)}{1 - p(m^k\omega) - r(m^k\omega)} = \frac{p(m^k\omega)}{q(m^k\omega)} \equiv \frac{p}{q}(m^k\omega), \tag{3.3}$$

$$\gamma_k(\omega) = \frac{r(m^k\omega)}{r(m^{k-1}\omega)} \cdot \frac{p(m^{k-1}\omega)}{p(m^k\omega)} \cdot b_k(\omega), \text{ for every } k \in \mathbb{Z}. \tag{3.4}$$

Given the stationary ergodic sequences $(p_k)_{k\in\mathbb{Z}}$,$(r_k)_{k\in\mathbb{Z}}$, for which every $\omega \in \Omega$ generates the random environment $p_k \equiv p(m^k\omega), r_k \equiv r(m^k\omega), k \in \mathbb{Z}$, we have that the preceding equations (3.3), (3.4) give a unique definition of the sequences $(b_k(\omega))_{k\in\mathbb{Z}}$, $(\gamma_k(\omega))_{k\in\mathbb{Z}}$ for μ-almost every ω, by the ergodicity of m. Then the sequences of weights $(w_k(\omega))_{k\in\mathbb{Z}}$ and $(w'_k(\omega))_{k\in\mathbb{Z}}$ are defined uniquely by

$$w_k(\omega) = w_0(\omega)b_1(\omega) \cdot b_2(\omega) \ldots b_k(\omega), k \in \mathbb{Z}^*_+,$$

$$w_k(\omega) = \frac{w_0(\omega)}{b_0(\omega) \cdot b_{-1}(\omega) \cdot b_{-2}(\omega) \cdots b_{k+1}(\omega)}, k \in \mathbb{Z}^*_-,$$

and

$$w'_k(\omega) = w'_0(\omega)\gamma_1(\omega) \cdot \gamma_2(\omega) \cdots \gamma_k(\omega), k \in \mathbb{Z}^*_+,$$

$$w'_k(\omega) = \frac{w'_0(\omega)}{\gamma_0(\omega) \cdot \gamma_{-1}(\omega) \cdot \gamma_{-2}(\omega) \cdots \gamma_{k+1}(\omega)}, k \in \mathbb{Z}^*_-.$$

(the unicity of the weight sequences $(w_k(\omega))_{k\in\mathbb{Z}}$, $(w'_k(\omega))_{k\in\mathbb{Z}}$ is understood up to the constant factors $w_0(\omega)$ and $w'_0(\omega)$).

Let us now introduce the "adjoint" random walk in random environment $(X'_n)_{n\in N}$. For every $\omega \in \Omega$ and for the family $(\mathbb{P}^\omega)_{\omega\in\Omega}$ of probability measures, the sequence $(X'_n)_{n\in N}$ is a Markov chain on $\mathbb{Z}$ whose elements of the corresponding Markov transition matrix are defined by

$$\mathbb{P}^\omega(X'_0 = 0) = 1,$$

$$\mathbb{P}^\omega(X'_{n+1} = k - 1/X'_n = k) = p(m^k\omega),$$

$$\mathbb{P}^\omega(X'_{n+1} = k/X'_n = k) = r(m^k\omega),$$

$$\mathbb{P}^\omega(X'_{n+1} = k + 1/X'_n = k) = 1 - p(m^k\omega) - r(m^k\omega) \equiv q(m^k\omega), k \in \mathbb{Z},$$

as it is shown in Figure 4.

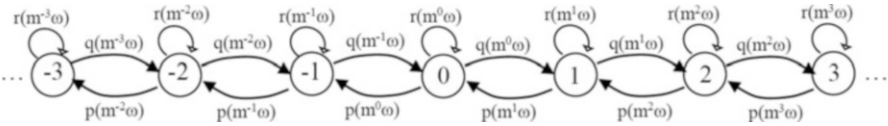

Fig. 4 The "adjoint" Markov chain $(X'_n)_{n\in\mathbb{N}}$ (random environments)

So we have the following:

Proposition 4 *For μ almost every environment $\omega \in \Omega$, the chain $(X'_n)_{n\in N}$ has a unique circuit and weight representation.*

Proof As in Proposition 3, the problem we have also to manage here is the definition of the weights of the circuits. To this direction we may denote by $w''_k(\omega)$ the weight of the circuit $c''_k = (k+1, k, k+1)$ and by $w'''_k(\omega)$ the weight of the circuit $c'''_k = (k, k)$, for every $k \in \mathbb{Z}$. By using an analogous way of that given before for the chain $(X_n)_{n\in N}$, let us consider the sequences $(\ell_k(\omega))_{k\in\mathbb{Z}}$, $(t_k(\omega))_{k\in\mathbb{Z}}$, defined by

$$\ell_k(\omega) = \frac{w''_{k-1}(\omega)}{w''_k(\omega)}, \qquad t_k(\omega) = \frac{w'''_{k-1}(\omega)}{w'''_k(\omega)},$$

such that

$$\ell_k(\omega) = \frac{p(m^k\omega)}{1 - p(m^k\omega) - r(m^k\omega)} = \frac{p(m^k\omega)}{q(m^k\omega)} \equiv \frac{p}{q}(m^k\omega), \tag{3.5}$$

$$t_k(\omega) = \frac{r(m^{k-1}\omega)}{r(m^k\omega)} \cdot \frac{1 - p(m^k\omega) - r(m^k\omega)}{1 - p(m^{k-1}\omega) - r(m^{k-1}\omega)} \cdot \ell_k(\omega), \tag{3.6}$$

for every $k \in \mathbb{Z}$.

Then the sequences of weights $(w''_k(\omega))_{k\in\mathbb{Z}}$, $(w'''_k(\omega))_{k\in\mathbb{Z}}$ are defined uniquely by

$$w''_k(\omega) = \frac{w''_0(\omega)}{\ell_1(\omega)\cdot\ell_2(\omega)\cdot\ell_3(\omega)\cdots\ell_k(\omega)}, k \in \mathbb{Z}^*_+,$$

$$w''_k(\omega) = w''_0(\omega)\cdot\ell_0(\omega)\cdot\ell_{-1}(\omega)\cdot\ell_{-2}(\omega)\cdots\ell_{k+3}(\omega)\cdot\ell_{k+2}(\omega)\cdot\ell_{k+1}(\omega), \quad k \in \mathbb{Z}^*_-$$

and

$$w'''_k(\omega) = \frac{w'''_0(\omega)}{t_1(\omega)\cdot t_2(\omega)\cdots t_k(\omega)}, \qquad k \in \mathbb{Z}^*_+$$

$$w'''_k(\omega) = w'''_0(\omega) t_0(\omega)\cdot t_{-1}(\omega)\cdot t_{-2}(\omega)\cdots t_{k+3}(\omega)\cdot t_{k+2}(\omega)\cdot t_{k+1}(\omega), \qquad k \in \mathbb{Z}^*_-$$

(the unicity of the weight sequences $(w''_k(\omega))_{k\in\mathbb{Z}}$, $(w'''_k(\omega))_{k\in\mathbb{Z}}$ is understood up to the constant factors $w''_0(\omega)$, $w'''_0(\omega)$).

4 Recurrence and Transience

4.1 Fixed Environments

We have that for the chain $(X_n)_{n\in N}$ there is a unique invariant measure up to a multiplicative constant factor $\mu_k = w_{k-1} + w_k + w'_k$, $k \geq 1$, $\mu_0 = w_0 + w'_0$, while for the chain $(X'_n)_{n\in N}$, $\mu'_k = w''_{k-1} + w''_k + w'''_k$, $k \geq 1$ with $\mu'_0 = w''_0 + w'''_0$. In the case that an irreducible chain is recurrent there is only and only one invariant measure (finite or not), so we may obtain the following:

Proposition 5

(i) The chain $(X_n)_{n\in N}$ defined as above is positive recurrent if and only if

$$\sum_{k=1}^{\infty}(b_1 b_2 \ldots b_k) < +\infty \left(or\ \frac{1}{w_0} \cdot \sum_{k=1}^{\infty} w_k < +\infty \right),$$

$$\sum_{k=1}^{\infty}(\gamma_1 \cdot \gamma_2 \cdots \gamma_k) < +\infty \left(or\ \frac{1}{w'_0} \cdot \sum_{k=1}^{\infty} w'_k < +\infty \right).$$

(ii) The chain $(X'_n)_{n\in N}$ defined as above is positive recurrent if and only if

$$\sum_{k=1}^{\infty} \frac{1}{s_1 \cdot s_2 \cdots s_k} < +\infty \left(or\ \frac{1}{w''_0} \cdot \sum_{k=1}^{\infty} w''_k < +\infty \right),$$

$$\sum_{k=1}^{\infty} \frac{1}{t_1 \cdot t_2 \cdots t_k} < +\infty \left(or\ \frac{1}{w'''_0} \cdot \sum_{k=1}^{\infty} w'''_k < +\infty \right).$$

In order to obtain recurrence and transience criteria for the chains $(X_n)_{n\in N}$, $(X'_n)_{n\in N}$ we shall need the following proposition [11]:

Proposition 6 *Let us consider a Markov chain on $\mathcal{N}$ which is irreducible. Then if there exists a strictly increasing function that is harmonic on the complement of a finite interval and that is bounded, then the chain is transient. In the case that there exists such a function which is unbounded the chain is recurrent.*

Following this direction we shall use a well-known method-theorem based on the Foster-Kendall theorem ([11]) by considering the harmonic function $g = (g_k, k \geq 1)$. For the chain $(X_n)_{n\in N}$ this is a solution of

$$p_0 \cdot g_1 + r_0 \cdot g_0 = g_0,$$

$$p_k \cdot g_{k+1} + q_k \cdot g_{k-1} + r_k \cdot g_k = g_k, k \geq 1.$$

Since $\Delta g_k = g_k - g_{k-1}$, for every $k \geq 1$, we obtain that

$$p_k \cdot g_{k+1} + q_k \cdot g_k - q_k \cdot g_k + q_k \cdot g_{k-1} + r_k \cdot g_k = g_k$$

or

$$p_k \cdot (\Delta g_{k+1} + g_k) + q_k \cdot g_k - q_k \cdot g_k + q_k \cdot g_{k-1} + r_k \cdot g_k = g_k$$

or

$$p_k \cdot \Delta g_{k+1} + (p_k + q_k + r_k) \cdot g_k - q_k \cdot g_k + q_k \cdot g_{k-1} = g_k$$

or

$$p_k \cdot \Delta g_{k+1} - q_k \cdot (g_k - g_{k-1}) = 0$$

or

$$p_k \cdot \Delta g_{k+1} = q_k \cdot \Delta g_k.$$

If we put $\alpha_k = \dfrac{\Delta g_k}{\Delta g_{k+1}}$ we get $\alpha_k = \dfrac{p_k}{q_k}$ (with $p_k = 1 - q_k - r_k$), $k \geq 1$, which is the equation of the definition of the sequences $(s_k)_{k\geq 1}$ and $(t_k)_{k\geq 1}$ (as a multiplicative factor of the $(s_k)_{k\geq 1}$) for the chain $(X'_n)_{n\in N}$ such that $q'_k = q_k$, $r'_k = r_k$, for every $k \geq 1$. This means that the strictly increasing harmonic functions of the chain $(X_n)_{n\in N}$ are in correspondence with the weight representations of the chain $(X'_n)_{n\in N}$ such that

$$\begin{aligned} q'_k &= P(X'_{n+1} = k+1/X'_n = k) = P(X_{n+1} = k-1/X_n = k) = q_k, \\ r'_k &= P(X'_{n+1} = k/X'_n = k) = P(X_{n+1} = k/X_n = k) = r_k, \\ p'_k &= 1 - q'_k - r'_k = 1 - q_k - r_k = p_k, \qquad \text{for every } k \geq 1. \end{aligned} \tag{4.1}$$

To express this kind of duality we shall call the chain $(X'_n)_{n\in N}$, the *adjoint* of the chain $(X_n)_{n\in N}$ and reciprocally in the case that the relation (4.1) is satisfied.

Equivalently for the chain $(X'_n)_{n\in N}$ the harmonic function $g' = (g'_k, k \geq 1)$ satisfies the equation

$$\begin{aligned} &r'_0 \cdot g'_0 + q'_0 \cdot g'_1 = g'_0, \\ &q'_k \cdot g'_{k+1} + p'_k \cdot g'_{k-1} + r'_k \cdot g'_k = g'_k, k \geq 1. \end{aligned}$$

Since $\Delta g'_k = g'_k - g'_{k-1}$, for every $k \geq 1$, we have that

$$q'_k \cdot (\Delta g'_{k+1} + g'_k) + p'_k \cdot g'_k - p'_k \cdot g'_k + p'_k \cdot g'_{k-1} + r'_k \cdot g'_k = g'_k$$

or

$$(p'_k + q'_k + r'_k) \cdot g'_k + q'_k \cdot \Delta g'_{k+1} - p'_k \cdot g'_k + p'_k \cdot g'_{k-1} = g'_k$$

or

$$q'_k \cdot \Delta g'_{k+1} = p'_k \cdot (g'_k - g'_{k-1}) = p'_k \cdot \Delta g'_k.$$

If we put $\beta_k = \dfrac{\Delta g'_{k+1}}{\Delta g'_k}$ we get $\beta_k = \dfrac{p'_k}{q'_k}$ (with $q'_k = 1 - p'_k - r'_k$),$k \geq 1$, which is the equation of the definition of the sequences $(b_k)_{k \geq 1}$ and $(\gamma_k)_{k \geq 1}$ (as a multiplicative factor of the $(b_k)_{k \geq 1}$) for the chain $(X_n)_{n \in N}$ such that $p'_k = p_k, r'_k = r_k$ for every $k \geq 1$. By considering a similar approximation of that given before for the chain $(X_n)_{n \in N}$ we may say that the strictly increasing harmonic functions of the chain $(X'_n)_{n \in N}$ are in correspondence with the weight representations of the chain $(X_n)_{n \in N}$ such that equivalent equations of (4.1) are satisfied.

So we may have the following:

Proposition 7 *The chain $(X_n)_{n \in N}$ defined as above is transient if and only if the adjoint chain $(X'_n)_{n \in N}$ is positive recurrent and reciprocal. Moreover the adjoint chains $(X_n)_{n \in N}, (X'_n)_{n \in N}$ are null recurrent simultaneously.*

In particular

(i) The chain $(X_n)_{n \in N}$ defined as above is transient if and only if

$$\frac{1}{w''_0} \cdot \sum_{k=1}^{\infty} w''_k < +\infty \textit{ and } \frac{1}{w'''_0} \cdot \sum_{k=1}^{\infty} w'''_k < +\infty.$$

(ii) The chain $(X'_n)_{n \in N}$ defined as above is transient if and only if

$$\frac{1}{w_0} \cdot \sum_{k=1}^{\infty} w_k < +\infty \textit{ and } \frac{1}{w'_0} \cdot \sum_{k=1}^{\infty} w'_k < +\infty.$$

(iii) The adjoint chains $(X_n)_{n \in N}, (X'_n)_{n \in N}$ are null recurrent if

$$\frac{1}{w_0} \cdot \sum_{k=1}^{\infty} w_k = \frac{1}{w'_0} \cdot \sum_{k=1}^{\infty} w'_k = +\infty \textit{ and } \frac{1}{w''_0} \cdot \sum_{k=1}^{\infty} w''_k = \frac{1}{w'''_0} \cdot \sum_{k=1}^{\infty} w'''_k = +\infty.$$

Proof The proof of Proposition 7 is an application mainly of Proposition 6 as well as of Proposition 5.

4.2 Random Environments

Regarding the criteria of recurrence and transience in the case of fixed environments, we have already proved that the behaviors of recurrence and transience for the "adjoint" chains $(X_n)_{n\in N}$,$(X'_n)_{n\in N}$ are tied together and depend on the convergence or not of the series

$$\sum_{k=1}^{+\infty} w_k, \sum_{k=1}^{+\infty} w'_k, \sum_{k=1}^{+\infty} w''_k \text{ and } \sum_{k=1}^{+\infty} w'''_k.$$

In the case of random environments the recurrence and transience are properties which are true for μ almost every environment $\omega \in \Omega$ or for μ almost no environment, because the system $(\Omega, \mathscr{F}, \mu, m)$ is supposed to be ergodic. This is true in general for a random walk in a random environment which is irreducible.

In order to investigate suitable criteria for the transience and recurrence of the corresponding uniquely defined circuit chains describing the above-mentioned random walk with jumps in a random environment, we may use the criteria given in the study for fixed environments for the chains $(X_n)_{n\in N}$, $(X'_n)_{n\in N}$ restricted to the half-lines $[i, +\infty)$ with reflection in i. According to the criterion in the case that

$$\sum_{k=1}^{+\infty} w_k(\omega) < +\infty \qquad \text{and} \qquad \sum_{k=1}^{+\infty} w'_k(\omega) < +\infty, \quad \mu - a.e.$$

we have that the restricted chain $(X_n)_{n\in N}$ is positive recurrent on $[i, +\infty)$, while the restricted "adjoint" chain $(X'_n)_{n\in N}$ is transient on $[i, +\infty)$, since it is known that the chain $(X_n)_{n\in N}$ defined as above is positive recurrent if and only if its "adjoint" chain $(X'_n)_{n\in N}$ is transient and reciprocal. An analogous result is obtained in the case of the half-lines $(-\propto, j]$ with reflection in j.

Therefore we have the following:

Proposition 8 *The random walk $(X_n)_{n\in N}$ in random environments defined as above is transient, for $\mu - a.e.$ environment $\omega \in \Omega$, if and only if its "adjoint" random walk $(X'_n)_{n\in N}$ is positive recurrent and reciprocal. Moreover the adjoint random walks $(X_n)_{n\in N}$ and $(X'_n)_{n\in N}$ are null recurrent simultaneously.*

Proof Taking into account the Birkoff's ergodic theorem for the sequences $(b_k(\omega))_{k\in\mathbb{Z}}$, $(\gamma_k(\omega))_{k\in\mathbb{Z}}$ for μ-almost every ω (see relations (3.3), (3.4)), we may write

$$w_k(\omega) = w_0(\omega) \prod_{d=1}^{k} b_d(\omega) \sim e^{kc}, \ k \in \mathbb{Z}_+^*,$$

$$w_k(\omega) = w_0(\omega) \left[\prod_{d=0}^{-(k+1)} b_{-d}(\omega) \right]^{-1} \sim e^{-kc}, \ k \in \mathbb{Z}_-^*,$$

$$w'_k(\omega) = w'_0(\omega) \prod_{d=1}^{k} \gamma_d(\omega) \sim e^{kc},\ k \in \mathbb{Z}_+^*,$$

$$w'_k(\omega) = w'_0(\omega) \left[\prod_{d=0}^{-(k+1)} \gamma_{-d}(\omega) \right]^{-1} \sim e^{-kc},\ k \in \mathbb{Z}_-^*$$

for the sequences of weights $(w_k(\omega))_{k\in\mathbb{Z}}$, $(w'_k(\omega))_{k\in\mathbb{Z}}$ of the chain $(X_n)_{n\in N}$. Following an analogous way for the "adjoint" chain $(X'_n)_{n\in N}$ we have

$$w''_k(\omega) = w''_0(\omega) \left[\prod_{d=1}^{k} \ell_d(\omega) \right]^{-1} \sim e^{-kc},\ k \in \mathbb{Z}_+^*,$$

$$w''_k(\omega) = w''_0(\omega) \prod_{d=0}^{-(k+1)} \ell_{-d}(\omega) \sim e^{kc},\ k \in \mathbb{Z}_-^*,$$

$$w'''_k(\omega) = w'''_0(\omega) \left[\prod_{d=1}^{k} t_d(\omega) \right]^{-1} \sim e^{-kc},\ k \in \mathbb{Z}_+^*,$$

$$w'''_k(\omega) = w'''_0(\omega) \prod_{d=0}^{-(k+1)} t_{-d}(\omega) \sim e^{kc},\ k \in \mathbb{Z}_-^*,$$

for the sequences of weights $(w''_k(\omega))_{k\in\mathbb{Z}}$, $(w'''_k(\omega))_{k\in\mathbb{Z}}$, of the chain $(X'_n)_{n\in N}$. We take into account the following cases:

(i) $c < 0$. We get

$$\sum_{k=1}^{+\infty} w_k(\omega) < +\infty,\ \sum_{k=-\infty}^{0} w_k(\omega) < +\infty, \sum_{k=1}^{+\infty} w'_k(\omega) < +\infty,\ \sum_{k=-\infty}^{0} w'_k(\omega) < +\infty,$$

$$\sum_{k=1}^{+\infty} w''_k(\omega) = +\infty,\ \sum_{k=-\infty}^{0} w''_k(\omega) = +\infty, \sum_{k=1}^{+\infty} w'''_k(\omega) = +\infty,\ \sum_{k=-\infty}^{0} w'''_k(\omega) = +\infty.$$

By using the criterion given in subsection 4.1 for the chains $(X_n)_{n\in N}$ and $(X'_n)_{n\in N}$ restricted

(a) to the half-lines $[i, +\propto)$ with reflection in i, we have that the restricted chain $(X_n)_{n\in N}$ is positive recurrent on $[i, +\propto)$, while the restricted chain $(X'_n)_{n\in N}$ is transient,
(b) to the half-lines $(-\propto, j]$ with reflection in j, we have also that the restricted chain $(X_n)_{n\in N}$ is positive recurrent on $(-\propto, j]$, while its adjoint chain $(X'_n)_{n\in N}$ is transient.

(ii) $c > 0$. We get

$$\sum_{k=1}^{+\infty} w_k(\omega) = +\infty, \sum_{k=-\infty}^{0} w_k(\omega) = +\infty, \sum_{k=1}^{+\infty} w'_k(\omega) = +\infty, \sum_{k=-\infty}^{0} w'_k(\omega) = +\infty,$$

$$\sum_{k=1}^{+\infty} w''_k(\omega) < +\infty, \sum_{k=-\infty}^{0} w''_k(\omega) < +\infty, \sum_{k=1}^{+\infty} w'''_k(\omega) < +\infty, \sum_{k=-\infty}^{0} w'''_k(\omega) < +\infty.$$

Regarding the criterion given in subsection 4.1 for the chains $(X_n)_{n\in N}$ and $(X'_n)_{n\in N}$ restricted

(a) to the half-lines $[i, +\propto)$ with reflection in i, we have that the restricted chain $(X_n)_{n\in N}$ is transient on $[i, +\propto)$, while the restricted chain $(X'_n)_{n\in N}$ is positive recurrent,
(b) to the half-lines $(-\propto, j]$ with reflection in j, we have also that the restricted chain $(X_n)_{n\in N}$ is transient on $(-\propto, j]$, while its adjoint chain $(X'_n)_{n\in N}$ is positive recurrent.

(iii) $c = 0$. Regarding the ergodic theorem, it is well-known that the averages $\frac{1}{k}\sum_{n=0}^{k-1}(\text{fom}^n)$ take infinitely many values greater than the limit and infinitely many values smaller than the limit. This means that in the sequences of weights

$$(w_k(\omega))_{k\in\mathbb{Z}}, (w'_k(\omega))_{k\in\mathbb{Z}}, (w''_k(\omega))_{k\in\mathbb{Z}}, (w'''_k(\omega))_{k\in\mathbb{Z}},$$

for a.e. $\omega \in \Omega$, infinitely many values in both directions are greater than 1. As a consequence, we may have that

$$\sum_{k=1}^{+\infty} w_k(\omega) = +\infty, \sum_{k=-\infty}^{0} w_k(\omega) = +\infty, \sum_{k=1}^{+\infty} w'_k(\omega) = +\infty, \sum_{k=-\infty}^{0} w'_k(\omega) = +\infty,$$

$$\sum_{k=1}^{+\infty} w''_k(\omega) = +\infty, \sum_{k=-\infty}^{0} w''_k(\omega) = +\infty, \sum_{k=1}^{+\infty} w'''_k(\omega) = +\infty, \sum_{k=-\infty}^{0} w'''_k(\omega) = +\infty.$$

By using the criterion of null recurrence for the chains $(X_n)_{n\in N}$, $(X'_n)_{n\in N}$ restricted to the half-lines $[i, +\infty)$ and $(-\propto, j]$ with reflection in i, j respectively, in the case of fixed environments, we may have also that both chains are null recurrent on $\mathbb{Z}$, for $\mu - a.e.\omega \in \Omega$.

References

1. Yv. Derriennic, Random walks with jumps in random environments (examples of cycle and weight representations), in *Probability Theory and Mathematical Statistics: Proceedings of the 7th Vilnius Conference 1998*, ed. by B. Grigelionis, J. Kubilius, V. Paulauskas, V.A. Statulevicius, and H. Pragarauskas (VSP, Vilnius, 1999), pp. 199–212
2. W. Feller, An Introduction to Probability Theory and Its Applications, vol. I, 3rd edn. (Wiley, Hoboken, NJ, 1968)
3. W. Feller, An Introduction to Probability Theory and Its Applications, vol. II, 2nd edn. (Wiley, Hoboken, NJ, 1971)
4. Ch. Ganatsiou, On cycle representations of random walks in fixed, random environments, in *Proceedings of the 58th World Congress of the International Statistical Institute*, Dublin, 21–26 August 2011
5. Ch. Ganatsiou, On the study of transience and recurrence of the Markov chain defined by directed weighted circuits associated with a random walk in fixed environment. J. Prob. Stat. **2013** (2013), Article ID 424601, 5 pp.
6. Ch. Ganatsiou, On the study of transience and recurrence of circuit chains associated with a random walk with jumps in random environments, in *Proceedings of the First Congress of Greek Mathematicians (FCGM-2018), Hellenic Mathematical Society (HMS)*, Special Session in Probability and Statistics, Athens, 25–30 June 2018
7. T.E. Harris, First passage and recurrence distributions. Trans. Am. Math. Soc. **73**, 471–486 (1952)
8. B.D. Hughes, *Random Walks and Random Environments, Volume 1: Random Walks* (Oxford University Press, Oxford, 1995)
9. M. Iosifescu, P. Tăutu, *Stochastic Processes and Applications in Biology and Medicine I*, 1st edn. (Springer, Berlin/Heidelberg/New York, 1973)
10. S. Kalpazidou, *Cycle Representations of Markov Processes* (Springer, New York, NY, 1995)
11. S. Karlin, H. Taylor, *A First Course in Stochastic Processes* (Academic, New York, NY, 1975)
12. J. MacQueen, Circuit processes. Ann. Probab. **9**, 604–610 (1981)
13. Q. Minping, Q. Min, Circulation for recurrent Markov chains. Z.Wahrsch. Verw. Gebiete **59**(2), 203–210 (1982)
14. K. Pearson, The problem of the random walk, Nature **72**, 294 (1905)
15. F. Spitzer, *Principles of Random Walks*, 2nd edn. (Spinger, New York, NY, 1976)
16. A.H. Zemanian, *Infinite Electrical Networks* (Cambridge University Press, Cambridge, 1991)