On Hyperstability of the Two-Variable Jensen Functional Equation on Restricted Domain

Iz-iddine EL-Fassi

Abstract We present a method that allows to study approximate solutions to the two-variable Jensen functional equation

$$2f\left(\frac{x+z}{2}, \frac{y+w}{2}\right) = f(x, y) + f(z, w)$$

on a restricted domain. Namely, we show that (under some weak natural assumptions) functions that satisfy the equation approximately (in some sense) must be actually solutions to it. The method is based on a quite recent fixed point theorem in some functions spaces and can be applied to various similar equations in many variables. Our outcomes are connected with the well-known issues of Ulam stability and hyperstability.

2010 Mathematics Subject Classifications Primary 39B82, 39B62; Secondary 47H14, 47H10

1 Introduction

In this paper, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$ denote the sets of all positive integers, real numbers, and non-negative real numbers, respectively; $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Moreover, X and Y always stand for normed spaces. The next definition describes the notion of hyperstability that we apply here (A^B denotes the family of all functions mapping a set $B \neq \emptyset$ into a set $A \neq \emptyset$).

Definition 1 Let A be a nonempty set, (Z, d) be a metric space, $\chi : A^n \to \mathbb{R}_+$, $B \subset A^n$ be nonempty, and $\mathscr{F}_1, \mathscr{F}_2$ map a nonempty $\mathscr{D} \subset Z^A$ into Z^{A^n}. We say that the conditional equation

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T. M. Rassias, P. M. Pardalos (eds.), *Mathematical Analysis and Applications*, Springer Optimization and Its Applications 154,
https://doi.org/10.1007/978-3-030-31339-5_5

$$\mathscr{F}_1\varphi(x_1,\ldots,x_n)=\mathscr{F}_2\varphi(x_1,\ldots,x_n),\qquad (x_1,\ldots,x_n)\in B, \tag{1}$$

is χ-hyperstable provided every $\varphi_0\in\mathscr{D}$, satisfying

$$d\big(\mathscr{F}_1\varphi_0(x_1,\ldots,x_n),\mathscr{F}_2\varphi_0(x_1,\ldots,x_n)\big)\le\chi(x_1,\ldots,x_n),\qquad (x_1,\ldots,x_n)\in B, \tag{2}$$

is a solution to (1).

That notion is strictly connected with the well-known issue of Ulam's stability for various (e.g., difference, differential, functional, integral, operator) equations. Let us recall that the study of such problems was motivated by the following question of Ulam (cf. [24, 39]) asked in 1940.

Ulam's question *Let $(G_1,\cdot)$ and $(G_2,\cdot)$ be two groups and $d:G_2\times G_2\to[0,\infty)$ be a metric. Given $\epsilon>0$, does there exist $\delta>0$ such that if a mapping $g:G_1\to G_2$ satisfies the inequality*

$$d(g(xy),g(x)g(y))\le\delta$$

for all $x,y\in G_1$, then there is a homomorphism $h:G_1\to G_2$ with

$$d(g(x),h(x))\le\epsilon$$

for all $x\in G$?

In 1941, Hyers [24] solved the well-known Ulam stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. The following theorem is the most classical result concerning the Hyers-Ulam stability of the Cauchy equation

$$f(x+y)=f(x)+f(y),\qquad x,y\in X. \tag{3}$$

Theorem 1 *Let $f:X\to Y$ satisfy the inequality*

$$\|f(x+y)-f(x)-f(y)\|\le\theta(\|x\|^p+\|y\|^p) \tag{4}$$

for all $x,y\in X\backslash\{0\}$, where θ and p are real constants with $\theta>0$ and $p\neq1$. Then the following two statements are valid.

(a) *If $p\ge0$ and Y is complete, then there exists a unique solution $T:X\to Y$ of (3) such that*

$$\|f(x)-T(x)\|\le\frac{\theta}{|1-2^{p-1}|}\|x\|^p,\qquad x\in X\backslash\{0\}. \tag{5}$$

(b) *If $p<0$, then f is additive, i.e., (3) holds.*

Note that Theorem 1 reduces to the first result of stability due to Hyers [24] if $p = 0$, Aoki [3] for $0 < p < 1$ (see also Th.M. Rassias' paper [35] in which it is proved for the first time the stability of the linear mapping). Afterward, Gajda [22] obtained this result for $p > 1$ and gave an example to show that Theorem 1 fails whenever $p = 1$. Also, Rassias [36] proved Theorem 1 for $p < 0$ (see [38, page 326] and [7]). Now, it is well known that the statement (b) is valid, i.e., f must be additive in that case, which has been proved for the first time in [32] and next in [8] on the restricted domain. For related results, concerning stability of the homomorphism equation on restricted domains, we refer to [1, 13–16, 25, 26, 29, 30, 34, 37, 38].

We say that a function $f : X \to Y$ satisfies the Jensen equation if

$$2f\Big(\frac{x+y}{2}\Big) = f(x) + f(y), \qquad x, y \in X. \tag{6}$$

The stability of the Jensen equation has been investigated at first by Kominek [31]. In 2006, Bae and Park [4] obtained the generalized Hyers-Ulam stability of a bi-Jensen function. Moreover, the stability problem for the bi-Jensen functional equation was discussed by a number of authors (see [27, 28]).

Recently Aghajani and Zahedi [2] investigated stability of the two-variable Jensen functional equation of the following form:

$$2f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) = f(x, y) + f(z, w), \qquad x, y, z, w \in X. \tag{7}$$

The term hyperstability was used for the first time probably in [33]; however, it seems that the first hyperstability result was published in [6] and concerned the ring homomorphisms. For further information concerning the notion of hyperstability we refer to the survey paper [11] (for recent related results see, e.g., [5, 8–10, 17–21, 23]).

The purpose of this work is to prove hyperstability results for the equation of the form (7) on restricted domains, that is some conditional versions of that equation. The method is based on a quite recent fixed point theorem in some functions spaces from [12]. In the same way, we can study approximate solutions on restricted domains to various functional equations (in many variables) that are sufficiently similar to (7).

Let U be a nonempty subset of X. We say that a function $f : U^2 \to Y$ fulfills equation (7) on U (or is a solution to (7) on U) provided

$$2f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) = f(x, y) + f(z, w), \tag{8}$$

$$x, y, z, w \in U, \quad \frac{x+z}{2}, \frac{y+w}{2} \in U;$$

if $U = X$, then we simply say that f fulfills (or is a solution to) Equation (7).

We consider functions $f : U^2 \to Y$ fulfilling (8) approximately, i.e., satisfying the inequality

$$\Big\| 2f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) - f(x,y) - f(z,w) \Big\| \le \gamma(x,y,z,w), \tag{9}$$
$$x, y, z, w \in U, \quad \frac{x+z}{2}, \frac{y+w}{2} \in U,$$

with a given $\gamma : U^4 \to \mathbb{R}_+$. We prove that, for some natural particular forms of γ (and under some additional assumptions on U), the conditional functional equation (8) is γ-hyperstable in the class of functions $f : U^2 \to Y$, i.e., each $f : U^2 \to Y$ satisfying inequality (9) with such γ must fulfill Equation (8).

2 Auxiliary Results

One of the methods of proof is based on a fixed point result that can be derived from [12]. To present it we need the following three hypothesis:

(H1) W is a nonempty set, Y is a Banach space, $f_1, \ldots . f_k : W \to W$ and $L_1, \ldots . L_k : W \to \mathbb{R}_+$ are given.

(H2) $\mathscr{T} : Y^W \to Y^W$ is an operator satisfying the inequality

$$\|\mathscr{T}\xi(x) - \mathscr{T}\mu(x)\| \le \sum_{i=1}^{k} L_i(x) \|\xi(f_i(x)) - \mu(f_i(x))\|, \quad \xi, \mu \in Y^W, \ x \in W.$$

(H3) $\Lambda : {\mathbb{R}_+}^W \to {\mathbb{R}_+}^W$ is a linear operator defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)), \qquad \delta \in {\mathbb{R}_+}^W, \ x \in W.$$

The mentioned fixed point theorem is stated in [12] as follows.

Theorem 2 *Let hypotheses* **(H1)–(H3)** *be valid and functions* $\varepsilon : W \to \mathbb{R}_+$ *and* $\varphi : W \to Y$ *fulfill the following two conditions:*

$$\|\mathscr{T}\varphi(x) - \varphi(x)\| \le \varepsilon(x), \qquad x \in W,$$

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \qquad x \in W.$$

Then, there exists a unique fixed point ψ of $\mathscr{T}$ with

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x), \qquad x \in W.$$

Moreover

$$\psi(x) = \lim_{n\to\infty} \mathscr{T}^n \varphi(x)$$

for all $x \in W$.

3 Hyperstability Results for Equation (8)

The following theorems are the main results in this paper and concern the γ-hyperstability of (8). Namely, for

$$\gamma(x, y, z, w) = c\|x\|^p\|y\|^q\|z\|^r\|w\|^s,$$

with suitable $c, p, q, r, s \in \mathbb{R}$, and

$$\gamma(x, y, z, w) = c(\|x\|^{p_1} + \|y\|^{p_2} + \|z\|^{p_3} + \|w\|^{p_4})^t$$

with suitable $c, p_1, p_2, p_3, p_4, t \in \mathbb{R}$, under some additional assumptions on nonempty $U \subset X$, we show that the conditional functional equation (8) is γ-hyperstable in the class of functions f mapping U^2 to a normed space.

In the remaining part of the paper, X and Y are normed spaces, $X_0 := X\setminus\{0\}$, and $\mathbb{N}_{m_0}$ denotes the set of all positive integers greater than or equal to a given $m_0 \in \mathbb{N}$.

Theorem 3 *Assume that $U \subset X_0$ is nonempty and there is $m_0 \in \mathbb{N}$, $m_0 > 3$, with*

$$-x, nx \in U, \qquad x \in U,\ n \in \mathbb{N},\ n \geq m_0 - 1. \tag{10}$$

If $f : U \times U \to Y$ satisfies

$$\left\|2f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) - f(x, y) - f(z, w)\right\| \leq c\|x\|^p\|y\|^q\|z\|^r\|w\|^s, \tag{11}$$
$$x, y, z, w \in U, \quad \frac{x+z}{2}, \frac{y+w}{2} \in U,$$

with some $c \geq 0$ and $p, q, r, s \in \mathbb{R}$ such that $p+r < 0$ or $q+s < 0$, then (8) holds.

Proof Without loss of generality we can assume that Y is complete, because if this is not the case, then we can simply replace Y by its completion. Assume that $p + r < 0$ (the case $q + s < 0$ is analogous) and fix $l \in \mathbb{N}_{m_0}$.

Replacing (x, z, y, w) by $(mx, (2-m)x, ly, (2-l)y)$ in (11), we get

$$\left\| \frac{1}{2} f(mx, ly) + \frac{1}{2} f((2-m)x, (2-l)y) - f(x, y) \right\| \\ \leq \frac{cm^p (m-2)^r l^q (l-2)^s}{2} \|x\|^{p+r} \|y\|^{q+s} \tag{12}$$

for all $m \in \mathbb{N}_{m_0}$ and $x, y \in U$. Fix $m \in \mathbb{N}_{m_0}$ and write

$$\mathscr{T}_m \xi(x, y) := \frac{1}{2} \xi(mx, ly) + \frac{1}{2} \xi((2-m)x, (2-l)y),$$

$$\varepsilon_m(x, y) := \frac{cm^p (m-2)^r l^q (l-2)^s}{2} \|x\|^{p+r} \|y\|^{q+s}$$

for every $\xi \in Y^{U \times U}$ and $x, y \in U$. Then inequality (12) takes the form

$$\left\| \mathscr{T}_m f(x, y) - f(x, y) \right\| \leq \varepsilon_m(x, y), \qquad x, y \in U.$$

Let

$$\Lambda_m \delta(x, y) := \frac{1}{2} \delta(mx, ly) + \frac{1}{2} \delta((2-m)x, (2-l)y)$$

for $x, y \in U$ and $\delta \in \mathbb{R}_+{}^{U \times U}$. Then the operator Λ_m has the form described in (**H3**) with $k = 2$,

$$f_1(x, y) \equiv (mx, ly), \qquad f_2(x, y) \equiv ((2-m)x, (2-l)y),$$

$$L_1(x, y) \equiv L_2(x, y) \equiv 1/2$$

for all $x, y \in U$. Moreover, for every $\xi, \mu \in Y^{U \times U}$ and $x, y \in U$, we obtain

$$\begin{aligned} \left\| \mathscr{T}_m \xi(x, y) - \mathscr{T}_m \mu(x, y) \right\| & \\ = & \left\| \frac{1}{2} \xi(mx, ly) + \frac{1}{2} \xi((2-m)x, (2-l)y) \right. \\ & \left. - \frac{1}{2} \mu(mx, ly) - \frac{1}{2} \mu((2-m)x, (2-l)y) \right\| \\ \leq & \frac{1}{2} \|(\xi - \mu)(mx, ly)\| + \frac{1}{2} \|(\xi - \mu)((2-m)x, (2-l)y)\| \\ = & \sum_{i=1}^{2} L_i(x, y) \left\| (\xi - \mu)(f_i(x, y)) \right\|. \end{aligned}$$

with $(\xi - \mu)(x, y) \equiv \xi(x, y) + \mu(x, y)$. So, **(H2)** is valid for $\mathscr{T}_m$. Note yet that

$$\Lambda_m \varepsilon_m(x, y) \le a_m \varepsilon_m(x, y), \qquad m \in \mathbb{N}_{m_0},\ x, y \in U, \tag{13}$$

with

$$a_m := \frac{1}{2} m^{p+r} l^{q+s} + \frac{1}{2}(m-2)^{p+r}(l-2)^{q+s}.$$

Clearly, there is $m_1 \in \mathbb{N}_{m_0}$, such that

$$a_m < 1, \qquad m \in \mathbb{N}_{m_1}.$$

Therefore, by (13), we obtain that

$$\begin{aligned} \varepsilon_m^*(x, y) := \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x, y) &\le \varepsilon_m(x, y) \sum_{n=0}^{\infty} (a_m)^n \\ &= \frac{\varepsilon_m(x, y)}{1 - a_m}, \qquad x, y \in U,\ \ m \in \mathbb{N}_{m_1}. \end{aligned}$$

Thus, according to Theorem 2, for each $m \in \mathbb{N}_{m_1}$ the function $J_m : U \times U \to Y$, given by $J_m(x, y) = \lim_{n\to\infty} \mathscr{T}_m^n f(x, y)$ for $x, y \in U$, is a unique fixed point of $\mathscr{T}_m$, i.e.,

$$J_m(x, y) = \frac{1}{2} J_m(mx, ly) + \frac{1}{2} J_m((2-m)x, (2-l)y)$$

for all $x, y \in U$; moreover

$$\| J_m(x, y) - f(x, y) \| \le \frac{\varepsilon_m(x, y)}{1 - a_m}, \qquad x, y \in U.$$

We show that

$$\left\| 2\mathscr{T}_m^n f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) - \mathscr{T}_m^n f(x, y) - \mathscr{T}_m^n f(z, w) \right\| \le c a_m^n \|x\|^p \|y\|^q \|z\|^r \|w\|^s \tag{14}$$

for every $n \in \mathbb{N}_0$ and $x, y, z, w \in U$ with $\frac{x+z}{2}, \frac{y+w}{2} \in U$.

Clearly, if $n = 0$, then (14) is simply (11). So, fix $n \in \mathbb{N}_0$ and suppose that (14) holds for n and every $x, y, z, w \in U$ with $\frac{x+z}{2}, \frac{y+w}{2} \in U$. Then, for every $x, y, z, w \in U$ with $\frac{x+z}{2}, \frac{y+w}{2} \in U$,

$$\begin{aligned}
&\left\|2\mathscr{T}_m^{n+1} f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) - \mathscr{T}_m^{n+1} f(x,y) - \mathscr{T}_m^{n+1} f(z,w)\right\| \\
&\quad = \Big\| 2\Big(\frac{1}{2}\mathscr{T}_m^n f\Big(m\frac{x+z}{2}, l\frac{y+w}{2}\Big) + \frac{1}{2}\mathscr{T}_m^n f\Big((2-m)\frac{x+z}{2}, (2-l)\frac{y+w}{2}\Big)\Big) \\
&\qquad - \frac{1}{2}\mathscr{T}_m^n f(mx, ly) - \frac{1}{2}\mathscr{T}_m^n f((2-m)x, (2-l)y) \\
&\qquad - \frac{1}{2}\mathscr{T}_m^n f(mz, lw) - \frac{1}{2}\mathscr{T}_m^n f((2-m)z, (2-l)w)\Big\| \\
&\quad \leq \frac{1}{2}\Big\| 2\mathscr{T}_m^n f\Big(m\frac{x+z}{2}, l\frac{y+w}{2}\Big) - \mathscr{T}_m^n f(mx, ly) - \mathscr{T}_m^n f(mz, lw)\Big\| \\
&\qquad + \frac{1}{2}\Big\| 2\mathscr{T}_m^n f\Big((2-m)\frac{x+z}{2}, (2-l)\frac{y+w}{2}\Big) - \mathscr{T}_m^n f((2-m)x, (2-l)y) \\
&\qquad - \mathscr{T}_m^n f((2-m)z, (2-l)w)\Big\| \\
&\quad \leq \frac{1}{2} c a_m^n \,\|mx\|^p \,\|ly\|^q \,\|mz\|^r \,\|lw\|^s \\
&\qquad + \frac{1}{2} c a_m^n \|(2-m)x\|^p \|(2-l)y\|^q \|(2-m)z\|^r \|(2-l)w\|^s \\
&\quad = c a_m^n \Big[\frac{1}{2} m^{p+r} l^{q+s} + \frac{1}{2}(m-2)^{p+r}(l-2)^{q+s}\Big] \|x\|^p \|y\|^q \|z\|^r \|w\|^s \\
&\quad = c(a_m)^{n+1} \|x\|^p \|y\|^q \|z\|^r \|w\|^s.
\end{aligned}$$

Thus, by induction, we have shown that (14) holds for all $x, y, z, w \in U$ such that $\frac{x+z}{2}, \frac{y+w}{2} \in U$ and for all $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (14), we obtain that

$$2J_m\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) = J_m(x,y) + J_m f(z,w) \tag{15}$$

for every $x, y, z, w \in U$ with $\frac{x+z}{2}, \frac{y+w}{2} \in U$.

In this way, for each $m \in \mathbb{N}_{m_0}$, we obtain a function J_m such that (15) holds for $x, y, z, w \in U$ with $\frac{x+z}{2}, \frac{y+w}{2} \in U$ and

$$\big\| f(x,y) - J_m(x,y)\big\| \leq \frac{\varepsilon_m(x,y)}{1-a_m}, \qquad x, y \in U, \; m \in \mathbb{N}_{m_1}.$$

Since

$$\lim_{m\to\infty} a_m = 0, \qquad \lim_{m\to\infty} \varepsilon_m(x,y) = 0, \qquad x, y \in U,$$

it follows, with $m \to \infty$, that f fulfills (8). □

In a similar way we can prove the following theorems.

Theorem 4 *Assume that $U \subset X_0$ is nonempty and there is $m_0 \in \mathbb{N}$, with*

$$\frac{1}{n}x,\ \frac{1}{2}\Big(1+\frac{1}{n}\Big)x \in U, \qquad x \in U,\ n \in \mathbb{N},\ n \geq m_0. \tag{16}$$

If $f : U \times U \to Y$ satisfies (11) with some $c \geq 0$ and $p, q, r, s \in \mathbb{R}$ such that $''p + r > 1$ and $q + s \geq 0''$ or $''q + s > 1$ and $p + r \geq 0''$, then (8) holds.

Proof Without loss of generality we can assume that Y is complete, because if this is not the case, then we can simply replace Y by its completion. Assume that $p + r > 1$ with $q + s \geq 0$ (the case $q + s > 1$ with $p + r \geq 0$ is analogous) and fix $l \in \mathbb{N}_{m_0}$.

Replacing (z, w) by $\left(\frac{1}{m}x, \frac{1}{l}y\right)$ in (11), we get

$$\Big\|2f\Big(\frac{m+1}{2m}x, \frac{l+1}{2l}y\Big) - f(x, y) - f\Big(\frac{x}{m}, \frac{y}{l}\Big)\Big\| \leq \frac{c}{m^r l^s}\,\|x\|^{p+r}\,\|y\|^{q+s} \tag{17}$$

for all $m \in \mathbb{N}_{m_0}$ and $x, y \in U$. Fix $m \in \mathbb{N}_{m_0}$ and we define

$$\mathscr{T}_m\xi(x, y) := 2\xi\Big(\frac{m+1}{2m}x, \frac{l+1}{2l}y\Big) - \xi\Big(\frac{x}{m}, \frac{y}{l}\Big), \quad \xi \in Y^{U\times U}$$

$$\varepsilon_m(x, y) := \frac{c}{m^r l^s}\,\|x\|^{p+r}\,\|y\|^{q+s}$$

$$\Lambda_m\delta(x, y) := 2\delta\Big(\frac{m+1}{2m}x, \frac{l+1}{2l}y\Big) + \delta\Big(\frac{x}{m}, \frac{y}{l}\Big), \quad \delta \in \mathbb{R}_+^{U\times U}$$

for every $x, y \in U$. Then inequality (17) takes the form

$$\big\|\mathscr{T}_m f(x, y) - f(x, y)\big\| \leq \varepsilon_m(x, y), \qquad x, y \in U,$$

and the operator Λ_m has the form described in (**H3**) with $k = 2$,

$$f_1(x, y) \equiv \Big(\frac{m+1}{2m}x, \frac{l+1}{2l}y\Big), \qquad f_2(x, y) \equiv \Big(\frac{x}{m}, \frac{y}{l}\Big),$$

$$L_1(x, y) \equiv 2, \qquad L_2(x, y) \equiv 1$$

for all $x, y \in U$. Moreover, for every $\xi, \mu \in Y^{U\times U}$ and $x, y \in U$, we obtain

$$\begin{aligned}\big\|\mathscr{T}_m\xi(x, y) - \mathscr{T}_m\mu(x, y)\big\| &\leq 2\big\|(\xi-\mu)(f_1(x, y))\big\| + \big\|(\xi-\mu)(f_2(x, y))\big\| \\ &= \sum_{i=1}^{2} L_i(x, y)\big\|(\xi-\mu)(f_i(x, y))\big\|.\end{aligned}$$

So, **(H2)** is valid for $\mathcal{T}_m$. Note yet that

$$\Lambda_m \varepsilon_m(x, y) \le b_m \varepsilon_m(x, y), \qquad m \in \mathbb{N}_{m_0}, \quad x, y \in U, \tag{18}$$

with

$$b_m := 2\Big(\frac{1+m}{2m}\Big)^{p+r}\Big(\frac{1+l}{2l}\Big)^{q+s} + \frac{1}{m^{p+r}l^{q+s}}.$$

Clearly, there is $m_1 \in \mathbb{N}_{m_0}$, such that

$$b_m < 1, \qquad m \in \mathbb{N}_{m_1}.$$

Therefore, by (18), we obtain that

$$\begin{aligned}\varepsilon_m^*(x, y) :&= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x, y) \le \varepsilon_m(x, y) \sum_{n=0}^{\infty} (a_m)^n \\ &= \frac{\varepsilon_m(x, y)}{1 - b_m}, \qquad x, y \in U, \quad m \in \mathbb{N}_{m_1}.\end{aligned}$$

Hence, according to Theorem 2, for each $m \in \mathbb{N}_{m_1}$ the function $J_m : U \times U \to Y$, given by $J_m(x, y) = \lim_{n\to\infty} \mathcal{T}_m^n f(x, y)$ for $x, y \in U$, is a unique fixed point of $\mathcal{T}_m$, i.e.,

$$J_m(x, y) = 2J_m\Big(\frac{1+m}{2m}x, \frac{1+l}{2l}y\Big) - J_m\Big(\frac{x}{m}, \frac{y}{l}\Big)$$

for all $x, y \in U$; moreover

$$\big\| J_m(x, y) - f(x, y)\big\| \le \frac{\varepsilon_m(x, y)}{1 - b_m}, \qquad x, y \in U.$$

Similarly as in the proof of Theorem 3, we show that

$$\Big\|2\mathcal{T}_m^n f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) - \mathcal{T}_m^n f(x, y) - \mathcal{T}_m^n f(z, w)\Big\| \le c b_m^n \|x\|^p \|y\|^q \|z\|^r \|w\|^s \tag{19}$$

for every $n \in \mathbb{N}$ and $x, y, z, w \in U$ with $\frac{x+z}{2}, \frac{y+w}{2} \in U$.

Moreover, we obtain a function J_m satisfies (8) and

$$\big\| f(x, y) - J_m(x, y)\big\| \le \frac{\varepsilon_m(x, y)}{1 - b_m}, \qquad x, y \in U, \ m \in \mathbb{N}_{m_1}.$$

Since $p+r>1$, one of p, r must be positive, let $r>0$, then we obtain

$$\lim_{m\to\infty} b_m < 1, \qquad \lim_{m\to\infty} \varepsilon_m(x,y) = 0, \qquad x, y \in U,$$

it follows, with $m \to \infty$, that f fulfills (8). □

Theorem 5 *Assume that $U \subset X_0$ is nonempty and there is $m_0 \in \mathbb{N}$, with*

$$\Big(2+\frac{1}{n}\Big)x, -\frac{1}{n}x, \in U, \qquad x \in U,\ n \in \mathbb{N},\ n \geq m_0. \tag{20}$$

If $f : U \times U \to Y$ satisfies (11), with some $c \geq 0$ and $p, q, r, s \in \mathbb{R}$ such that $''0 < p+r < 1$ and $q+s \leq 0''$ or $''0 < q+s < 1$ and $p+r \leq 0''$, then (8) holds.

Proof Assume that Y is complete, $0 < p+r < 1$ and $q+s \leq 0$ (the case $0 < q+s < 1$ and $p+r \leq 0$ is analogous) and fix $l \in \mathbb{N}_{m_0}$. Then, one of p, r must be positive, let $p > 0$.

Replacing (x, z, y, w) by $\Big(-\frac{1}{m}x, (2+\frac{1}{m})x, -\frac{1}{l}y, (2+\frac{1}{l})y\Big)$ in (11), we get

$$\Big\|\frac{1}{2}f\Big(-\frac{x}{m}, -\frac{y}{l}\Big) + \frac{1}{2}f\Big(\Big(2+\frac{1}{m}\Big)x, \Big(2+\frac{1}{l}\Big)y\Big) - f(x,y)\Big\| \\ \leq \frac{c}{2m^p l^q}\Big(2+\frac{1}{m}\Big)^r\Big(2+\frac{1}{l}\Big)^s \|x\|^{p+r}\,\|y\|^{q+s} \tag{21}$$

for all $m \in \mathbb{N}_{m_0}$ and $x, y \in U$. Fix $m \in \mathbb{N}_{m_0}$ and similarly as previously we define

$$\mathscr{T}_m\xi(x,y) := \frac{1}{2}\xi\Big(-\frac{x}{m}, -\frac{y}{l}\Big) + \frac{1}{2}\xi\Big(\Big(2+\frac{1}{m}\Big)x, \Big(2+\frac{1}{l}\Big)y\Big), \quad \xi \in Y^{U\times U}$$

$$\varepsilon_m(x,y) := \frac{c}{2m^p l^q}\Big(2+\frac{1}{m}\Big)^r\Big(2+\frac{1}{l}\Big)^s \|x\|^{p+r}\,\|y\|^{q+s}$$

$$\Lambda_m\delta(x,y) := \frac{1}{2}\delta\Big(-\frac{x}{m}, -\frac{y}{l}\Big) + \frac{1}{2}\delta\Big(\Big(2+\frac{1}{m}\Big)x, \Big(2+\frac{1}{l}\Big)y\Big), \quad \delta \in \mathbb{R}_+^{U\times U}$$

for every $x, y \in U$. Then inequality (21) takes the form

$$\big\|\mathscr{T}_m f(x,y) - f(x,y)\big\| \leq \varepsilon_m(x,y), \qquad x, y \in U.$$

Obviously Λ_m has the form described in (**H3**) with $k=2$,

$$f_1(x,y) \equiv \Big(-\frac{x}{m}, -\frac{y}{l}\Big), \qquad f_2(x,y) \equiv \Big(\Big(2+\frac{1}{m}\Big)x, \Big(2+\frac{1}{l}\Big)y\Big),$$

$$L_1(x,y) \equiv L_2(x,y) \equiv 1/2$$

for all $x, y \in U$. It is clear that, for every $\xi, \mu \in Y^{U\times U}$ and $x, y \in U$, we obtain

$$\left\|\mathcal{T}_m\xi(x,y) - \mathcal{T}_m\mu(x,y)\right\| \leq \sum_{i=1}^{2} L_i(x,y)d\left\|(\xi-\mu)(f_i(x,y))\right\|.$$

So, **(H2)** is valid for $\mathcal{T}_m$. Note yet that

$$\Lambda_m\varepsilon_m(x,y) \leq d_m\varepsilon_m(x,y), \qquad m \in \mathbb{N}_{m_0}, \quad x, y \in U, \tag{22}$$

with

$$d_m := \frac{1}{2}\Big(2+\frac{1}{m}\Big)^{p+r}\Big(2+\frac{1}{l}\Big)^{q+s} + \frac{1}{2m^{p+r}l^{q+s}}.$$

Clearly, there is $m_1 \in \mathbb{N}_{m_0}$, such that

$$d_m < 1, \qquad m \in \mathbb{N}_{m_1}.$$

Therefore, by (22), we obtain that

$$\varepsilon_m^*(x,y) \leq \varepsilon_m(x,y)\sum_{n=0}^{\infty}(a_m)^n = \frac{\varepsilon_m(x,y)}{1-d_m}, \qquad x, y \in U, \quad m \in \mathbb{N}_{m_1}.$$

The remaining reasonings are analogous as in the proof of that Theorem 3. □

Remark 1 Let $c \geq 0$ and $p, q, r, s \in \mathbb{R}$ such that $p+q+r+s \in \mathbb{R}\backslash\{0,1\}$. If $U = X_0$ and $f : X \to Y$ satisfies (11) on X_0, then f satisfies (8) on X_0.

Theorem 6 *Let U be a nonempty subset of $X\backslash\{0\}$ fulfilling condition (10) with some $m_0 \in \mathbb{N}$. Let $c \geq 0$ and $p_1, p_2, p_3, p_4, t \in \mathbb{R}$ be such that $tp_i < 0$ for $i = 1, 2, 3, 4$. If $f : U^2 \to Y$ satisfies the functional inequality*

$$\left\|2f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) - f(x,y) - f(z,w)\right\| \leq c(\|x\|^{p_1} + \|y\|^{p_2} + \|z\|^{p_3} + \|w\|^{p_4})^t, \tag{23}$$

$$x, y, z, w \in U, \quad \frac{x+z}{2}, \frac{y+w}{2} \in U,$$

then (8) holds.

Proof As in the proof of Theorem 3, without loss of generality we can assume that Y is complete. Write $p(m) = m^{tp_0}$ for $m \in \mathbb{N}_3$, where

$$p_0 := \begin{cases} \max\{p_1, p_2, p_3, p_4\} & \text{if } t > 0; \\ \min\{p_1, p_2, p_3, p_4\} & \text{if } t < 0. \end{cases}$$

Clearly, if $t > 0$, then $p_i < 0$ for $i = 0, \dots, 4$ and consequently

$$\max\{m^{p_1}, m^{p_2}, m^{p_3}, m^{p_4}\} = m^{p_0}, \qquad m \in \mathbb{N}_3. \tag{24}$$

Analogously, if $t < 0$, then $p_i > 0$ for $i = 0, \dots, 4$ and

$$\min\{m^{p_1}, m^{p_2}, m^{p_3}, m^{p_4}\} = m^{p_0}, \qquad m \in \mathbb{N}_3. \tag{25}$$

Replacing (x, z, y, w) by $(mx, (2-m)x, my, (2-m)y)$ in (23), we get

$$\begin{aligned}\Big\|\frac{1}{2}f(mx, my) + \frac{1}{2}f((2-m)x, (2-m)y) - f(x, y)\Big\| \\ \leq \frac{c}{2}\big(\|mx\|^{p_1} + \|my\|^{p_2} + \|(2-m)x\|^{p_3} + \|(2-m)y\|^{p_4}\big)^t\end{aligned} \tag{26}$$

for all $x, y \in U$ and $m \in \mathbb{N}_{m_0}$. Let

$$\varepsilon_m(x, y) := \frac{c}{2}\big(\|mx\|^{p_1} + \|my\|^{p_2} + \|(2-m)x\|^{p_3} + \|(2-m)y\|^{p_4}\big)^t,$$

$$\mathscr{T}_m\xi(x) := \frac{1}{2}\xi(mx, my) + \frac{1}{2}\xi((2-m)x, (2-m)y)$$

for $x, y \in U$, $m \in \mathbb{N}_{m_0}$ and $\xi \in Y^{U\times U}$. Then, by (24) (if $t > 0$) and (25) (if $t < 0$), we get

$$\varepsilon_m(\pm mx, \pm my) \leq p(m)\varepsilon(x, y), \qquad x, y \in U, m \in \mathbb{N}_{m_0}, \tag{27}$$

and inequality (26) takes the form

$$\|\mathscr{T}_m f(x, y) - f(x, y)\| \leq \varepsilon_m(x, y), \qquad x, y \in U, m \in \mathbb{N}_{m_0}.$$

Write

$$\Lambda_m\delta(x, y) = \frac{1}{2}\delta(mx, my) + \frac{1}{2}\delta((2-m)x, (2-m)y)$$

for $x, y \in U$, $m \in \mathbb{N}_{m_0}$ and $\delta \in {\mathbb{R}_+}^{U\times U}$. Then, for each $m \in \mathbb{N}_{m_0}$, operator Λ_m has the form described in **(H3)** with $k = 3$ and

$$f_1(x, y) \equiv (mx, my), \qquad f_2(x, y) \equiv ((2-m)x, (2-m)y), \qquad L_1(x, y) \equiv L_2(x, y) \equiv 1/2.$$

Moreover, for every $\xi, \mu \in Y^{U\times U}$, $m \in \mathbb{N}_{m_0}$ and $x, y \in U$, we have

$$\|\mathscr{T}_m\xi(x,y)-\mathscr{T}_m\mu(x,y)\| \leq \sum_{i=1}^{3} L_i(x,y)\,\|(\xi-\mu)(f_i(x,y))\|\,.$$

So, **(H2)** is valid. Next, it is easily seen that, by induction on n, from (27) we obtain

$$\Lambda_m^n\varepsilon_m(x,y) \leq \alpha_m^n\varepsilon(x,y), \qquad n,m\in\mathbb{N}_{m_0}, x,y\in U, \tag{28}$$

where $\alpha_m := \frac{1}{2}p(m)+\frac{1}{2}p(m-2)$. Note that we can find $m_1\in\mathbb{N}_{m_0}$ with

$$\alpha_m < 1, \qquad m\in\mathbb{N}_{m_1},$$

which means that

$$\varepsilon_m^*(x,y) := \sum_{n=0}^{\infty}\Lambda_m^n\varepsilon(x,y) \leq \varepsilon_m(x,y)\sum_{n=0}^{\infty}(\alpha_m)^n = \frac{\varepsilon_m(x,y)}{1-\alpha_m}$$

for all $x,y\in U$ and $m\in\mathbb{N}_{m_1}$.

Similarly as in the proof of Theorem 3, we show that

$$\left\|2\mathscr{T}_m^n f\Big(\frac{x+z}{2},\frac{y+w}{2}\Big)-\mathscr{T}_m^n f(x,y)-\mathscr{T}_m^n f(z,w)\right\| \leq c\alpha_m^n(\|x\|^{p_1}+\|y\|^{p_2}+\|z\|^{p_3}+\|w\|^{p_4})^t \tag{29}$$

for every $n\in\mathbb{N}$ and $x,y,z,w\in U$ with $\frac{x+z}{2},\frac{y+w}{2}\in U$. Also the remaining reasonings are analogous as in the proof of that theorem. □

The next theorem shows the hyperstability of the two-variable Jensen functional equation on the set containing 0.

Theorem 7 *Assume that Y is complete and $U\subset X$ is nonempty with 0, such that $2U\subset U$ and $\frac{1}{2}U\subset U$. If $f: U\times U\to Y$ satisfies*

$$\left\|2f\Big(\frac{x+z}{2},\frac{y+w}{2}\Big)-f(x,y)-f(z,w)\right\| \leq c\|x\|^p\|y\|^q\|z\|^r\|w\|^s, \tag{30}$$
$$x,y,z,w\in U,\ \frac{x+z}{2},\frac{y+w}{2}\in U,$$

with some $c\geq 0$ and $p,q,r,s>0$ such that $p+q+r+s\neq 1$, then (8) holds.

Proof Putting $z=w=0$ in (30), we obtain

$$2f\Big(\frac{x}{2},\frac{y}{2}\Big) = f(x,y)+f(0,0), \qquad x,y\in U,$$

i.e.,

$$2\Big(f\Big(\frac{x}{2},\frac{y}{2}\Big)-f(0,0\Big)=f(x,y)-f(0,0),\qquad x,y\in U.$$

Thus g defined as $g(x,y)\equiv f(x,y)-f(0,0)$ satisfies (30) and

$$2g\Big(\frac{x}{2},\frac{y}{2}\Big)=g(x,y),\qquad x,y\in U. \tag{31}$$

Next we divide the proof into two cases.

Case 1: $p+q+r+s<1$. Using (31) to (30) we can prove by induction that for every $n\in\mathbb{N}_0$

$$\Big\|2g\Big(\frac{x+z}{2},\frac{y+w}{2}\Big)-g(x,y)-g(z,w)\Big\|\leq c\Big(\frac{2^{p+q+r+s}}{2}\Big)^n\|x\|^p\|y\|^q\|z\|^r\|w\|^s, \tag{32}$$

$$x,y,z,w\in U,\quad \frac{x+z}{2},\frac{y+w}{2}\in U,$$

Indeed, if $n=0$, then (32) is simply (30). So, fix $n\in\mathbb{N}_0$ and assume that (32) holds for n. Then using (31) to (32) we have

$$\Big\|4g\Big(\frac{x+z}{4},\frac{y+w}{4}\Big)-2g\Big(\frac{x}{2},\frac{y}{2}\Big)-2g\Big(\frac{z}{2},\frac{w}{2}\Big)\Big\|\leq c\Big(\frac{2^{p+q+r+s}}{2}\Big)^n\|x\|^p\|y\|^q\|z\|^r\|w\|^s, \tag{33}$$

$$x,y,z,w\in U,\quad \frac{x+z}{2},\frac{y+w}{2}\in U,$$

dividing by 2 and replacing (x,y,z,w) by $(2x,2y,2z,2w)$ in the last inequality we obtain

$$\Big\|2g\Big(\frac{x+z}{2},\frac{y+w}{2}\Big)-g(x,y)-g(z,w)\Big\|\leq c\Big(\frac{2^{p+q+r+s}}{2}\Big)^{n+1}\|x\|^p\|y\|^q\|z\|^r\|w\|^s, \tag{34}$$

$$x,y,z,w\in U,\quad \frac{x+z}{2},\frac{y+w}{2}\in U,$$

so (32) holds for every $n\in\mathbb{N}_0$. As $p+q+r+s<1$, letting $n\to\infty$ in (32), we obtain that g satisfies (8) on U. Obviously f satisfies (8) on U, too.

Case 2: $p+q+r+s>1$. Replacing (x,y) by $(2x,2y)$ in (31) we get

$$2g(x,y)=g(2x,2y),\qquad x,y\in U. \tag{35}$$

Similarly as in 1) using (30), (35) and induction we obtain

$$\Big\|2g\Big(\frac{x+z}{2},\frac{y+w}{2}\Big)-g(x,y)-g(z,w)\Big\|\le c\Big(\frac{2}{2^{p+q+r+s}}\Big)^n\|x\|^p\|y\|^q\|z\|^r\|w\|^s, \tag{36}$$

$$x,y,z,w\in U,\quad \frac{x+z}{2},\frac{y+w}{2}\in U,$$

for every $n\in\mathbb{N}_0$. With $n\to\infty$ in the last inequality we have

$$2g\Big(\frac{x+z}{2},\frac{y+w}{2}\Big)=g(x,y)+g(z,w),\quad x,y,z,w\in U,\quad \frac{x+z}{2},\frac{y+w}{2}\in U$$

Thus f also satisfies (8) on U. □

4 Some Applications and Examples

The above theorems imply in particular the following corollary, which shows their simple application.

Corollary 1 *Let $U\subset X$ be nonempty and $F:U^4\to Y$ be a function such that $F(x_0,y_0,z_0,w_0)\neq 0$ for some $x_0,y_0,z_0,w_0\in U$ with $\frac{x_0+z_0}{2},\frac{y_0+w_0}{2}\in U$ and*

$$\|F(x,y,z,w)\|\le c\|x\|^p\|y\|^q\|z\|^r\|z\|^s,\quad x,y,z,w\in U,\ \frac{x+z}{2},\frac{y+w}{2}\in U, \tag{37}$$

or

$$\|F(x,y,z,w)\|\le c\big(\|x\|^{p_1}+\|y\|^{p_2}+\|z\|^{p_3}+\|w\|^{p_4}\big)^t,$$
$$x,y,z,w\in U,\ \frac{x+z}{2},\frac{y+w}{2}\in U, \tag{38}$$

where $c\ge 0$ and $p,q,r,s,p_1,p_2,p_3,p_4,t\in\mathbb{R}$. Assume that one of the conditions (i)–(iv) *is valid in a case F satisfies* (37) *and* (v) *is valid in a case F satisfies* (38)*, where*

(i) *$p+r<0$ or $q+s<0$, $0\notin U$, and* (10) *holds with some $m_0\in\mathbb{N}_4$,*
(ii) *$p+r>1$ and $q+s\ge 0$ (or $q+s>1$ and $p+r\ge 0$), $0\notin U$ and* (16) *holds with some $m_0\in\mathbb{N}$,*
(iii) *$0<p+r<1$ and $q+s\le 0$ (or $0<q+s<1$ and $p+r\le 0$), $0\notin U$ and* (20) *holds with some $m_0\in\mathbb{N}$,*
(iv) *$p,q,r,s>0$ such that $p+q+r+s\neq 1$, $0\in U$, $2U\subset U$ and $\frac{1}{2}U\subset U$,*
(v) *$tp_i<0$ for $i=1,2,3,4$, $0\notin U$ and* (10) *holds with some $m_0\in\mathbb{N}\backslash\{1,2\}$.*

Then the functional equation

$$2f_0\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) = f_0(x, y) + f_0(z, w) + F(x, y, z, w), \tag{39}$$

$$x, y, z, w \in U \quad \frac{x+z}{2}, \frac{y+w}{2} \in U,$$

has no solution in the class of functions $f_0 : U \to Y$.

Proof Suppose that there exists a solution $f_0 : U \to Y$ to (39). Then (11) or (23) holds, and consequently, according to the above theorems, f_0 is a solution to (8), which means that $F(x_0, y_0, z_0, w_0) = 0$ for some $x_0, y_0, z_0, w_0 \in U$ with $\frac{x_0+z_0}{2}, \frac{y_0+w_0}{2} \in U$. This is a contradiction. □

Now, we give some examples which show that in the above theorems the additional assumption on U are necessary.

Example 1 Let $X = Y = \mathbb{R}$, $U = [-1, 1]\backslash\{0\}$, $p, q, r, s < 0$, $c = 4$ and $f : U^2 \to \mathbb{R}$ be defined by $f(x, y) = |x + y|$. Then f satisfies

$$\Big|2f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) - f(x, y) - f(z, w)\Big| \le 4|x|^p|y|^q|z|^r|w|^s, \quad x, y, z, w \in U$$

but f is not a solution of equation (8) on U. We see that $0 \notin U$ and U does not satisfy the assumption of Theorem 3 .

Example 2 Let $X = Y = \mathbb{R}$, $U = [1, \infty)$, $p, q, r, s > 0$, $c = 4$ and $f : U^2 \to \mathbb{R}$ be defined by $f(x, y) = \frac{1}{x} + \frac{1}{y}$. Then f satisfies

$$\Big|2f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) - f(x, y) - f(z, w)\Big| \le 4|x|^p|y|^q|z|^r|w|^s, \quad x, y, z, w \in U$$

but f is not a solution of equation (8) on U. It is easy to check that the assumptions of Theorems 4, 5, and 7 are not satisfied.

In this example, we show that the condition $-x \in U$ for every $x \in U$ in Theorem 6 is necessary.

Example 3 Let $X = Y = \mathbb{R}$, $U = (0, \infty)$, $t = 1$, $p_i = p < 0$ for $i = 1, \dots, 4$, and $f : U^2 \to \mathbb{R}$ be defined by $f(x, y) = x^p + y^p$. Then f satisfies

$$\Big|2f\Big(\frac{x+z}{2}, \frac{y+w}{2}\Big) - f(x, y) - f(z, w)\Big| \le 2^{1-p}(|x|^p + |y|^p + |z|^p + |w|^p),$$

$$x, y, z, w \in U$$

but f is not a solution of equation (8) on U, which shows that in Theorem 6 the assumption that $-x \in U$ for every $x \in U$ is necessary.

We end the paper with an open problem.

Remark 2 For the cases $p + r = q + s = 0$ and $tp_i = 0$ for $i = 1, \dots, 4$, the method used in the proofs of the above theorems cannot be applied, thus this is still an open problem.

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