A Variational Approach to the Financial Problem with Insolvencies and Analysis of the Contagion

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Abstract In this chapter we improve some results in literature on the general financial equilibrium problem related to individual entities, called sectors, which invest in financial instruments as assets and as liabilities. Indeed the model, studied in the chapter, takes into account the insolvencies and we analyze how these insolvencies affect the financial problem. For this improved model we describe a variational inequality for which we provide an existence result. Moreover, we study the dual Lagrange problem, in which the Lagrange variables, which represent the deficit and the surplus per unit, appear and an economical indicator is provided. Finally, we perform the contagion by means of the deficit and surplus variables. As expected, the presence of the insolvencies makes it more difficult to reach the financial equilibrium and increases the risk of a negative contagion for all the systems.

1 Introduction

The term "insolvency" is often used to denote that an individual or an organization can no longer meet its financial obligations with its lender. Usually, before getting involved in insolvency proceedings, some informal arrangements with creditors are attempted. Insolvency can be caused by poor cash management, a reduction in cash inflow forecasts or by an increase in expenses.

When insolvent, the credit loans are revoked both at the credit institution concerned and at all the institutions and banks to which the customer has had debts;

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further, it becomes impossible for the client/company to obtain liquidity from other institutions.

In the USA the number of bankruptcies decreased to 23,106 companies in the first quarter of 2018 from 23,157 companies in the fourth quarter of 2017. According to The Guardian, the number of people who went bankrupt in 2017 in the United Kingdom rose to the highest level after the financial crisis, revealing the devastating toll of rising debts for the families. According to the Insolvency Service 99,196 people were declared insolvent in 2017, with an increase of 9.4% with respect to the year before and very close to the peak recorded during the recession. Lots of households (about 59,220 in 2017) are turning to "bankruptcy-lite" debt deals, where individuals reschedule their debts and agree to much lower payments. Italy confirms the unenviable leadership in the ranking of companies in difficulty among the main Western European countries. According to the surveys of Coface, a group at the top in credit insurance, in Italy there are 7.2% of companies in difficulty, in Spain 6.3%, in France 5.7%, and in Germany 4.9%. The percentage takes into account the insolvent companies and those indebted, unprofitable, who struggle to honor the payments at maturity. In Italy, the current levels of insolvency are more than double that of 2007, with one of the worst performances recorded at the European level. In general, the trend of insolvencies at global level is almost stable in 2017. The modest decline that was expected last year, equal to about a -1%, is in fact the weakest result since 2009.

Some financial network models have already been studied in the literature. The first authors to develop a multi-sector, multi-instrument financial equilibrium model using the variational inequality theory were Nagurney et al. [35]. Recently, in [1, 7, 8, 11] more general models have been studied allowing that the data are evolving over time.

In this chapter we improve the previous results, including the insolvencies of the financial institutions.

We obtain such a result, considering in the utility function the presence of the term $\sum_{j=1}^{n} r_j(t)(1-\tau_{ij}(t))c_j(t)(1+h_j(t))y_{ij}(t)$, which represents, by means of the insolvency coefficients $c_j(t)$, the portion of liabilities that are not reimbursed. Since a big number of critic situations have been caused by the fact that the banks or the financial institutions were not able to recover a part of their debts, we focus our attention on this more complete model, deriving the variational formulation, applying the infinite-dimensional duality theory and examining the contagion effect on the economy. In this context, a particular attention is devoted to the problem of the contagion, in order to know when it happens and also to establish how the insolvencies contribute to the occurrence of the contagion. We are able to control the contagion, using the dual Lagrange problem and the dual Lagrange variables, which represent the deficit and the surplus per unit, arising from instrument j. Considering the dual problem, we can examine the financial model both from the *Point of View of the Sectors* and from the *System Point of View* (see Section 3.3) and we can clearly see that liabilities from the point of view of the sectors are investments for the economic system, namely a positive factor, upon which to base the development of the economy. As expected, the presence of the insolvencies, that we are able to quantify, makes it more difficult to reach the financial equilibrium, since reduced income has to balance all the expenditure of the system.

The chapter is organized as follows: in Section 2 we present the detailed financial model, together with the evolutionary variational inequality formulation of the equilibrium conditions, and an existence result is provided; in Section 3 we apply the duality to the general financial equilibrium problem, deriving the Deficit Formula, the Balance Law, and the Liability Formula, we give the dual formulation of the financial problem, we study the regularity of the Lagrange variables, deficit and surplus, and we analyze, by means of these variables, the financial contagion; in Section 4 we provide a numerical financial example and, finally, in Section 5 we summarize our results and conclusions.

It is worth mentioning that the methods applied in this chapter may be used in the study of many other equilibrium problems [4, 5, 10, 19–26, 34].

2 The Financial Model and the Equilibrium Conditions

2.1 Presentation of the Model

For the reader's convenience, we present the detailed financial model (see also [1]). We consider a financial economy consisting of m sectors, for example households, domestic business, banks and other financial institutions, as well as state and local governments, with a typical sector denoted by i, and of n instruments, for example mortgages, mutual funds, saving deposits, money market funds, with a typical financial instrument denoted by j, in the time interval $[0, T]$. Let $s_i(t)$ denote the total financial volume held by sector i at time t as assets, and let $l_i(t)$ be the total financial volume held by sector i at time t as liabilities. Further, we allow markets of assets and liabilities to have different investments $s_i(t)$ and $l_i(t)$, respectively. Since we are working in the presence of uncertainty and of risk perspectives, the volumes $s_i(t)$ and $l_i(t)$ held by each sector cannot be considered stable with respect to time and may decrease or increase. For instance, depending on the crisis periods, a sector may decide not to invest on instruments and to buy goods as gold and silver. At time t, we denote the amount of instrument j held as an asset in sector i's portfolio by $x_{ij}(t)$ and the amount of instrument j held as a liability in sector i's portfolio by $y_{ij}(t)$. The assets and liabilities in all the sectors are grouped into the matrices $x(t),\ y(t) \in \mathbb{R}^{m\times n}$, respectively. At time t we denote the price of instrument j held as an asset and as a liability by $r_j(t)$ and by $(1 + h_j(t))r_j(t)$, respectively, where h_j is a nonnegative function defined into $[0, T]$ and belonging to $L^\infty([0, T], \mathbb{R})$. We introduce the term $h_j(t)$ because the prices of liabilities are generally greater than or equal to the prices of assets. In this manner we describe, in a more realistic way, the behavior of the markets for which the liabilities are more expensive than the assets. We group the instrument prices held as an asset and as a liability into the vectors $r(t) = [r_1(t), r_2(t), \dots, r_i(t), \dots, r_n(t)]^T$ and $(1 + h(t))r(t) = [(1 + h_1(t))r_1(t), (1 + h_2(t))r_2(t), \dots, (1 + h_i(t))r_i(t), \dots, (1 + h_n(t))r_n(t)]^T$, respectively. In our problem the prices of each instrument appear as unknown variables. Under the assumption of perfect competition, each sector will behave as if it has no influence on the instrument prices or on the behavior of the other sectors, but on the total amount of the investments and the liabilities of each sector.

We choose as a functional setting the very general Lebesgue space

$$L^2([0,T],\mathbb{R}^p) = \left\{ f : [0,T] \to \mathbb{R}^p \text{ measurable} : \int_0^T \|f(t)\|_p^2 dt < +\infty \right\},$$

with the norm

$$\|f\|_{L^2([0,T],\mathbb{R}^p)} = \left(\int_0^T \|f(t)\|_p^2 dt \right)^{\frac{1}{2}}.$$

Then, the set of feasible assets and liabilities for each sector $i = 1, \dots, m$ becomes

$$P_i = \left\{ (x_i(t), y_i(t)) \in L^2([0,T], \mathbb{R}_+^{2n}) : \\ \sum_{j=1}^n x_{ij}(t) = s_i(t), \quad \sum_{j=1}^n y_{ij}(t) = l_i(t) \text{ a.e. in } [0,T] \right\}$$

and the set of all feasible assets and liabilities becomes

$$P = \left\{ (x(t), y(t)) \in L^2([0,T], \mathbb{R}^{2mn}) : (x_i(t), y_i(t)) \in P_i,\ i = 1, \dots, m \right\}.$$

Now, we introduce the ceiling and the floor price associated with instrument j, denoted by $\overline{r}_j$ and by $\underline{r}_j$, respectively, with $\overline{r}_j(t) > \underline{r}_j(t) \geq 0$, a.e. in $[0, T]$. The floor price $\underline{r}_j(t)$ is determined on the basis of the official interest rate fixed by the central banks, which, in turn, take into account the consumer price inflation. Then the equilibrium prices $r_j^*(t)$ cannot be less than these floor prices. The ceiling price $\overline{r}_j(t)$ derives from the financial need to control the national debt arising from the amount of public bonds and of the rise in inflation. It is a sign of the difficulty on the recovery of the economy. However it should be not overestimated because it produced an availability of money.

In detail, the meaning of the lower and upper bounds is that to each investor a minimal price $\underline{r}_j$ for the assets held in the instrument j is guaranteed, whereas each investor is requested to pay for the liabilities in any case a minimal price $(1+h_j)\underline{r}_j$. Analogously each investor cannot obtain for an asset a price greater than $\overline{r}_j$ and as a liability the price cannot exceed the maximum price $(1 + h_j)\overline{r}_j$.

We denote the given tax rate levied on sector i's net yield on financial instrument j, as τ_{ij}. Assume that the tax rates lie in the interval $[0, 1)$ and belong to

$L^\infty([0,T],\mathbb{R})$. Therefore, the government in this model has the flexibility of levying a distinct tax rate across both sectors and instruments.

We group the instrument ceiling and floor prices into the column vectors $\overline{r}(t) = (\overline{r}_j(t))_{j=1,\dots,n}$ and $\underline{r}(t) = (\underline{r}_j(t))_{j=1,\dots,n}$, respectively, and the tax rates τ_{ij} into the matrix $\tau(t) \in L^2([0,T],\mathbb{R}^{m\times n})$.

The set of feasible instrument prices is:

$$\mathscr{R} = \{r \in L^2([0,T],\mathbb{R}^n) : \underline{r}_j(t) \leq r_j(t) \leq \overline{r}_j(t), \quad j = 1,\dots,n, \text{ a.e. in } [0,T]\},$$

where $\underline{r}$ and $\overline{r}$ are assumed to belong to $L^2([0,T],\mathbb{R}^n)$.

In order to determine for each sector i the optimal distribution of instruments held as assets and as liabilities, we consider, as usual, the influence due to risk-aversion and the optimality conditions of each sector in the financial economy, namely the desire to maximize the value of the asset holdings while minimizing the value of liabilities. In the current economic situation there is a serious problem caused by the suffering that undermines the whole system. For this reason we intend to address the study of the financial problem in the presence of insolvencies.

Hence, in order to meet this need, we take into account the non-performing loans, introducing the insolvency coefficients $c_j(t)$, $j = 1,\dots,n$. We assume that the insolvency coefficients $c_j(t)$ lie in the interval $[0,1)$ and belong to $L^\infty([0,T],\mathbb{R})$.

Then, we introduce the utility function $U_i(t, x_i(t), y_i(t), r(t))$, for each sector i, defined as follows:

$$U_i(t, x_i(t), y_i(t), r(t)) = u_i(t, x_i(t), y_i(t))$$

$$+\sum_{j=1}^{n} r_j(t)(1-\tau_{ij}(t))[x_{ij}(t) - (1-c_j(t))(1+h_j(t))y_{ij}(t)],$$

where the term $-u_i(t, x_i(t), y_i(t))$ represents a measure of the risk of the financial agent, the term $\sum_{j=1}^{n} r_j(t)(1-\tau_{ij}(t))[x_{ij}(t) - (1+h_j(t))y_{ij}(t)]$ represents the value of the difference between the asset holdings and the value of liabilities, and the term $\sum_{j=1}^{n} r_j(t)(1-\tau_{ij}(t))c_j(t)(1+h_j(t))y_{ij}(t)$ represents, by means of the insolvency coefficients $c_j(t)$, the portion of liabilities that are not reimbursed. Such a term appears as a positive contribute for sector i and a loss for the system.

We suppose that the sector's utility function $U_i(t, x_i(t), y_i(t))$ is defined on $[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$, is measurable in t, and is continuous with respect to x_i and y_i. Moreover we assume that $\frac{\partial u_i}{\partial x_{ij}}$ and $\frac{\partial u_i}{\partial y_{ij}}$ exist and that they are measurable in t and continuous with respect to x_i and y_i. Further, we require that $\forall i = 1,\dots,m$, $\forall j = 1,\dots,n$, and a.e. in $[0,T]$ the following growth conditions hold true:

$$|u_i(t,x,y)| \leq \alpha_i(t)\|x\|\|y\|, \quad \forall x, y \in \mathbb{R}^n, \tag{1}$$

and

$$\left|\frac{\partial u_i(t,x,y)}{\partial x_{ij}}\right| \le \beta_{ij}(t)\|y\|, \quad \left|\frac{\partial u_i(t,x,y)}{\partial y_{ij}}\right| \le \gamma_{ij}(t)\|x\|, \tag{2}$$

where $\alpha_i, \beta_{ij}, \gamma_{ij}$ are nonnegative functions of $L^\infty([0,T],\mathbb{R})$. Finally, we suppose that the function $u_i(t,x,y)$ is concave.

An example of measure of the risk aversion is given by a generalization to the evolutionary case of the well-known Markowitz quadratic function based on the variance-covariance matrix denoting the sector's assessment of the standard deviation of prices for each instrument (see [31, 32]). This evolutionary measure of Markowitz type can be refined in such a way that it can incorporate the adjustment in time which depends on the previous equilibrium states.

In Section 2.4 we define a utility function of Markowitz type.

2.2 The Equilibrium Flows and Prices

Now, we establish the equilibrium conditions for the prices, which express the equilibration of the total assets, the total liabilities, and the portion of financial transactions per unit F_j employed to cover the expenses of the financial institutions, including possible dividends and manager bonus. Indeed, the equilibrium condition for the price r_j of instrument j is the following:

$$\sum_{i=1}^{m}(1-\tau_{ij}(t))\left[x^*_{ij}(t)-(1-c_j(t))(1+h_j(t))y^*_{ij}(t)\right]+F_j(t)$$
$$\begin{cases} \ge 0 \text{ if } r^*_j(t)=\underline{r}_j(t) \\ = 0 \text{ if } \underline{r}_j(t)<r^*_j(t)<\overline{r}_j(t) \\ \le 0 \text{ if } r^*_j(t)=\overline{r}_j(t) \end{cases} \tag{3}$$

where (x^*, y^*, r^*) is the equilibrium solution for the investments as assets and as liabilities and for the prices. In other words, the prices are determined taking into account the amount of the supply, the demand of an instrument, and the charges F_j, namely if there is an actual supply excess of an instrument as assets and of the charges F_j in the economy, then its price must be the floor price. If the price of an instrument is greater than the floor price, but not at the ceiling, then the market of that instrument must clear. Finally, if there is an actual demand excess of an instrument as liabilities in the economy, then the price must be at the ceiling.

Now, we can give different but equivalent equilibrium conditions, each of which is useful to illustrate particular features of the equilibrium.

Definition 1 A vector of sector assets, liabilities, and instrument prices $(x^*(t), y^*(t), r^*(t)) \in P \times \mathscr{R}$ is an equilibrium of the dynamic financial model if and only if $\forall i = 1, \ldots, m$, $\forall j = 1, \ldots, n$, and a.e. in $[0, T]$, it satisfies the system of inequalities

$$-\frac{\partial u_i(t, x^*, y^*)}{\partial x_{ij}} - (1 - \tau_{ij}(t))r_j^*(t) - \mu_i^{(1)*}(t) \geq 0, \tag{4}$$

$$-\frac{\partial u_i(t, x^*, y^*)}{\partial y_{ij}} + (1 - \tau_{ij}(t))(1 - c_j(t))(1 + h_j(t))r_j^*(t) - \mu_i^{(2)*}(t) \geq 0, \tag{5}$$

and equalities

$$x_{ij}^*(t)\Big[- \frac{\partial u_i(t, x^*, y^*)}{\partial x_{ij}} - (1 - \tau_{ij}(t))r_j^*(t) - \mu_i^{(1)*}(t)\Big] = 0, \tag{6}$$

$$y_{ij}^*(t)\Big[-\frac{\partial u_i(t, x^*, y^*)}{\partial y_{ij}} + (1 - \tau_{ij}(t))(1 - c_j(t))(1 + h_j(t))r_j^*(t) - \mu_i^{(2)*}(t)\Big] = 0, \tag{7}$$

where $\mu_i^{(1)*}(t)$, $\mu_i^{(2)*}(t) \in L^2([0, T], \mathbb{R})$ are Lagrange multipliers, and verifies conditions (3) a.e. in $[0, T]$.

We associate with each financial volumes s_i and l_i held by sector i the functions $\mu_i^{(1)*}(t)$ and $\mu_i^{(2)*}(t)$, related, respectively, to the assets and to the liabilities and which represent the "equilibrium disutilities" per unit of sector i. Then, (4) and (6) mean that the financial volume invested in instrument j as assets x_{ij}^* is greater than or equal to zero if the j-th component $-\frac{\partial u_i(t, x^*, y^*)}{\partial x_{ij}} - (1 - \tau_{ij}(t))r_j^*(t)$ of the disutility is equal to $\mu_i^{(1)*}(t)$, whereas if $-\frac{\partial u_i(t, x^*, y^*)}{\partial x_{ij}} - (1 - \tau_{ij}(t))r_j^*(t) > \mu_i^{(1)*}(t)$, then $x_{ij}^*(t) = 0$. The same occurs for the liabilities.

The functions $\mu_i^{(1)*}(t)$ and $\mu_i^{(2)*}(t)$ are the Lagrange multipliers associated a.e. in $[0, T]$ with the constraints $\sum_{j=1}^{n} x_{ij}(t) - s_i(t) = 0$ and $\sum_{j=1}^{n} y_{ij}(t) - l_i(t) = 0$, respectively. They are unknown a priori, but this fact has no influence because we will prove in the following theorem that Definition 1 is equivalent to a variational inequality in which $\mu_i^{(1)*}(t)$ and $\mu_i^{(2)*}(t)$ do not appear (see [1, Theorem 2.1]).

Theorem 1 *A vector $(x^*, y^*, r^*) \in P \times \mathscr{R}$ is a dynamic financial equilibrium if and only if it satisfies the following variational inequality:*

Find $(x^*, y^*, r^*) \in P \times \mathscr{R}$:

$$\sum_{i=1}^{m} \int_0^T \Bigg\{ \sum_{j=1}^{n} \Big[- \frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial x_{ij}} - (1 - \tau_{ij}(t))r_j^*(t)\Big] \times [x_{ij}(t) - x_{ij}^*(t)]$$

$$+\sum_{j=1}^{n}\Big[-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial y_{ij}}+(1-\tau_{ij}(t))(1-c_j(t))r_j^*(t)(1+h_j(t))\Big]$$

$$\times[y_{ij}(t)-y_{ij}^*(t)]\Big\}dt$$

$$+\sum_{j=1}^{n}\int_0^T\sum_{i=1}^{m}\Big\{(1-\tau_{ij}(t))\Big[x_{ij}^*(t)-(1-c_j(t))(1+h_j(t))y_{ij}^*(t)\Big]+F_j(t)\Big\}$$

$$\times\Big[r_j(t)-r_j^*(t)\Big]dt\geq 0,\qquad \forall(x,y,r)\in P\times\mathscr{R}. \tag{8}$$

Remark 1 We would like to explicitly remark that our definition of equilibrium conditions (Definition 1) is equivalent to the equilibrium definition given by a vector $(x^*, y^*, r^*)\in P\times\mathscr{R}$ satisfying (3) and, $\forall i=1,\ldots,m$:

$$\max_{P_i}\int_0^T\Big\{u_i(t,x_i(t),y_i(t))+\sum_{j=1}^{n}(1-\tau_{ij}(t))r_j^*(t)[x_{ij}(t)-(1-c_j(t))(1+h_j(t))y_{ij}(t)]\Big\}dt=$$
$$\int_0^T\Big\{u_i(t,x_i^*(t),y_i^*(t))+\sum_{j=1}^{n}(1-\tau_{ij}(t))r_j^*(t)[x_{ij}^*(t)-(1-c_j(t))(1+h_j(t))y_{ij}^*(t)]\Big\}dt.$$

We prefer to use Definition 1, since it is expressed in terms of equilibrium disutilities.

2.3 Existence Theorem

Now, we would like to give an existence result. First of all, we remind some definitions. Let X be a reflexive Banach space and let $\mathbb{K}$ be a subset of X and X^* be the dual space of X.

Definition 2 A mapping $A:\mathbb{K}\to X^*$ is pseudomonotone in the sense of Brezis (B-pseudomonotone) iff

1. For each sequence u_n weakly converging to u (in short $u_n\rightharpoonup u$) in $\mathbb{K}$ and such that $\limsup_n\langle Au_n, u_n-v\rangle\leq 0$ it results that:

$$\liminf_n\langle Au_n, u_n-v\rangle\geq\langle Au, u-v\rangle,\quad \forall v\in\mathbb{K}.$$

2. For each $v\in\mathbb{K}$ the function $u\mapsto\langle Au, u-v\rangle$ is lower bounded on the bounded subset of $\mathbb{K}$.

Definition 3 A mapping $A : \mathbb{K} \to X^*$ is hemicontinuous in the sense of Fan (F-hemicontinuous) iff for all $v \in \mathbb{K}$ the function $u \mapsto \langle Au, u - v\rangle$ is weakly lower semicontinuous on $\mathbb{K}$.

The following existence result does not require any kind of monotonicity assumptions.

Theorem 2 *Let $\mathbb{K} \subset X$ be a nonempty closed convex bounded set and let $A : \mathbb{K} \subset E \to X^*$ be B-pseudomonotone or F-hemicontinuous. Then, variational inequality*

$$\langle Au, v - u\rangle \geq 0 \quad \forall v \in \mathbb{K} \tag{9}$$

admits a solution.

In the following subsection we shall present an example of a function, which satisfies the above assumptions.

2.4 An Example of a Markowitz-Type Risk Measure

We generalize and provide an evolutionary Markowitz-type measure of the risk proposed with a memory term. This function is effective, namely an existence theorem for the general financial problem holds (see [17]). In this way we cover a lack, providing the existence of a significant evolutionary measure of the risk. The particular, but significant, example of utility function is:

$$u_i(x_i(t), y_i(t))$$

$$= \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix}^T Q^i \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} + \int_0^t \begin{bmatrix} x_i(t-z) \\ y_i(t-z) \end{bmatrix}^T Q^i \begin{bmatrix} x_i(t-z) \\ y_i(t-z) \end{bmatrix} dz, \tag{10}$$

where Q^i denotes the sector i's assessment of the standard deviation of prices for each instrument j.

In [17] it has been proven that Markowitz function verifies all the assumptions of the existence theorem, hence a problem with a function like this admits solutions.

3 The Duality for the Financial Equilibrium Problem

In this section we study the duality for the financial equilibrium problem (see also [6]).

To this end, for reader's convenience, we recall here some definitions and results of the infinite dimensional duality theory.

3.1 The New Infinite-Dimensional Duality Theory

In order to obtain the strong duality, we need that some delicate conditions, called "constraints qualification conditions," hold. In the infinite dimensional settings the next assumption, the so-called *Assumption S*, results to be a necessary and sufficient condition for the strong duality (see [3, 9, 12, 13, 33]).

Let $f : S \to \mathbb{R}$, $g : S \to Y$, $h : S \to Z$ be three mappings, where S is a convex subset of a real normed space X, Y is a real normed space ordered by a convex cone C, Z is a real normed space and consider the optimization problem:

$$\begin{cases} f(x_0) = \min_{x \in \mathbb{K}} f(x) \\ x_0 \in \mathbb{K} = \{x \in S : g(x) \in -C,\ \ h(x) = \theta_Z\}, \end{cases} \tag{11}$$

where θ_Z is the zero element in the space Z.

Its Lagrange dual problem is:

$$\max_{\lambda \in C^*,\, \mu \in Z^*} \inf_{x \in S} \left[f(x) + \langle \lambda, g(x) \rangle + \langle \mu, h(x) \rangle \right], \tag{12}$$

where

$$C^* := \left\{ u \in Y^* : \langle u, y \rangle \geq 0,\ \ \forall y \in C \right\}$$

is the dual cone of C and Z^* is the dual space of Z. Then, we say that the strong duality holds for problems (11) and (12) if and only if problems (11) and (12) admit a solution and the optimal values coincide.

Some classical results due to Rockafellar [36], Holmes [27], Borwein and Lewis [2] give sufficient conditions in order that the strong duality between problems (11) and (12) holds, which use concepts such as the core, the intrinsic core, the strong quasi-relative interior of C. Such concepts (see [2, 27, 29, 36]) require the nonemptiness of the ordering cone, which defines the cone constraints in convex optimization and variational inequalities. However, the ordering cone of almost all the known problems, stated in infinite dimensional spaces, has the interior (and all the above generalized interior concepts) empty. Hence, the above interior conditions cannot be used to guarantee the strong duality.

Only recently, in [12] the authors introduced a new condition called S, which turns out to be a necessary and sufficient condition for the strong duality and really useful in the applications. This condition does not require the nonemptiness of the interior of the ordering cone. This new strong duality theory was then refined in [9, 13, 15, 28, 33].

Now we present in detail these new conditions.

Let us first recall that for a subset $C \subseteq X$ and $x \in X$ the tangent cone to C at x is defined as

$$T_C(x) = \{y \in X :\ y = \lim_{n \to \infty} \lambda_n (x_n - x),\ \lambda_n > 0,\ x_n \in C,\ \ \lim_{n \to \infty} x_n = x\}.$$

If $x \in clC$ (the closure of C) and C is convex, we have

$$T_C(x) = clcone(C - \{x\}),$$

where the $coneA = \{\lambda x : x \in A, \lambda \in \mathbb{R}^+\}$ denotes the cone hull of a general subset A of the space.

Definition 4 (Assumption S) Given the mappings f, g, h and the set $\mathbb{K}$ as above, we shall say that *Assumption S* is fulfilled at a point $x_0 \in \mathbb{K}$ if it results to be

$$T_{\widetilde{M}}(0, \theta_Y, \theta_Z) \cap \Big( ]-\infty, 0[\times \{\theta_Y\} \times \{\theta_Z\} \Big) = \emptyset, \tag{13}$$

where

$$\widetilde{M} = \{(f(x) - f(x_0) + \alpha, g(x) + y, h(x)) : x \in S \setminus \mathbb{K}, \alpha \geq 0, y \in C\}.$$

The following theorem holds (see Theorem 1.1 in [13] for the proof):

Theorem 3 *Under the above assumptions on f, g, h, and C, if problem (11) is solvable and Assumption S is fulfilled at the extremal solution $x_0 \in \mathbb{K}$, then also problem (12) is solvable, the extreme values of both problems are equal, namely, if $(x_0, \lambda^*, \mu^*) \in \mathbb{K} \times C^* \times Z^*$ is the optimal point of problem (12),*

$$\begin{aligned} f(x_0) &= \min_{x \in \mathbb{K}} f(x) = f(x_0) + \langle \lambda^*, g(x_0) \rangle + \langle \mu^*, h(x_0) \rangle \\ &= \max_{\substack{\lambda \in C^* \\ \mu \in Z^*}} \inf_{x \in S} \{ f(x) + \langle \lambda, g(x) \rangle + \langle \mu, h(x) \rangle \} \end{aligned} \tag{14}$$

and, it results to be:

$$\langle \lambda^*, g(x_0) \rangle = 0.$$

3.2 Existence of Lagrange Multipliers

Now, we can apply the infinite-dimensional duality for the financial equilibrium problem expressed by variational inequality (8), which ensures the existence of the Lagrange multipliers. To this end, let us set:

$$\begin{aligned} f(x, y, r) = \int_0^T \Bigg\{ \sum_{i=1}^m \sum_{j=1}^n \Bigg[-\frac{\partial u_i(t, x^*(t), y^*(t))}{\partial x_{ij}} - (1 - \tau_{ij}(t)) r_j^*(t) \Bigg] \\ \times [x_{ij}(t) - x_{ij}^*(t)] \end{aligned}$$

$$+ \sum_{i=1}^{m}\sum_{j=1}^{n}\left[-\frac{\partial u_i(t, x^*(t), y^*(t))}{\partial y_{ij}} + (1-\tau_{ij}(t))(1-c_j(t))(1+h_j(t))r_j^*(t)\right]$$
$$\times\ [y_{ij}(t) - y_{ij}^*(t)]$$
$$+ \sum_{j=1}^{n}\left[\sum_{i=1}^{m}(1-\tau_{ij}(t))\left[x_{ij}^*(t) - (1-c_j(t))(1+h_j(t))y_{ij}^*(t)\right] + F_j(t)\right]$$
$$\times \left[r_j(t) - r_j^*(t)\right]\Bigg\}\, dt.$$

Then, the Lagrange functional is

$$\mathscr{L}(x, y, r, \lambda^{(1)}, \lambda^{(2)}, \mu^{(1)}, \mu^{(2)}, \rho^{(1)}, \rho^{(2)}) = f(x, y, r)$$
$$-\sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^T \lambda_{ij}^{(1)}(t)x_{ij}(t)\,dt - \sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^T \lambda_{ij}^{(2)} y_{ij}(t)\,dt$$
$$-\sum_{i=1}^{m}\int_0^T \mu_i^{(1)}(t)\left(\sum_{j=1}^{n} x_{ij}(t) - s_i(t)\right) dt \tag{15}$$
$$-\sum_{i=1}^{m}\int_0^T \mu_i^{(2)}(t)\left(\sum_{j=1}^{n} y_{ij}(t) - l_i(t)\right) dt$$
$$+\sum_{j=1}^{n}\int_0^T \rho_j^{(1)}(t)(\underline{r}_j(t) - r_j(t))\,dt + \sum_{j=1}^{n}\int_0^T \rho_j^{(2)}(t)(r_j(t) - \overline{r}_j(t))\,dt,$$

where $(x, y, r) \in L^2([0, T], \mathbb{R}^{2mn+n})$, $\lambda^{(1)}, \lambda^{(2)} \in L^2([0, T], \mathbb{R}_+^{mn})$, $\mu^{(1)}, \mu^{(2)} \in L^2([0, T], \mathbb{R}^m)$, $\rho^{(1)}, \rho^{(2)} \in L^2([0, T], \mathbb{R}_+^n)$ and $\lambda^{(1)}, \lambda^{(2)}, \rho^{(1)}, \rho^{(2)}$ are the Lagrange multipliers associated, a.e. in $[0, T]$, with the sign constraints $x_i(t) \geq 0$, $y_i(t) \geq 0$, $r_j(t) - \underline{r}_j(t) \geq 0$, $\overline{r}_j(t) - r_j(t) \geq 0$, respectively, whereas the functions $\mu^{(1)}(t)$ and $\mu^{(2)}(t)$ are the Lagrange multipliers associated, a.e. in $[0, T]$, with the equality constraints $\sum_{j=1}^{n} x_{ij}(t) - s_i(t) = 0$ and $\sum_{j=1}^{n} y_{ij}(t) - l_i(t) = 0$, respectively.

Applying the new strong duality theory, the following theorem holds.

Theorem 4 *Let $(x^*, y^*, r^*) \in P \times \mathscr{R}$ be a solution to variational inequality (8) and let us consider the associated Lagrange functional (15). Then, the strong duality holds and there exist $\lambda^{(1)*}, \lambda^{(2)*} \in L^2([0, T], \mathbb{R}_+^{mn})$, $\mu^{(1)*}, \mu^{(2)*} \in L^2([0, T], \mathbb{R}^m)$, $\rho^{(1)*}, \rho^{(2)*} \in L^2([0, T], \mathbb{R}_+^n)$ such that $(x^*, y^*, r^*, \lambda^{(1)*}, \lambda^{(2)*}, \mu^{(1)*}, \mu^{(2)*}, \rho^{(1)*}, \rho^{(2)*})$ is a saddle point of the Lagrange functional, namely*

$$\begin{aligned}&\mathscr{L}(x^*, y^*, r^*, \lambda^{(1)}, \lambda^{(2)}, \mu^{(1)}, \mu^{(2)}, \rho^{(1)}, \rho^{(2)})\\ \le\,&\mathscr{L}(x^*, y^*, r^*, \lambda^{(1)*}, \lambda^{(2)*}, \mu^{(1)*}, \mu^{(2)*}, \rho^{(1)*}, \rho^{(2)*}) = 0\\ \le\,&\mathscr{L}(x, y, r, \lambda^{(1)*}, \lambda^{(2)*}, \mu^{(1)*}, \mu^{(2)*}, \rho^{(1)*}, \rho^{(2)*})\end{aligned} \tag{16}$$

$\forall (x, y, r) \in L^2([0, T], \mathbb{R}^{2mn+n})$, $\forall \lambda^{(1)}, \lambda^{(2)} \in L^2([0, T], \mathbb{R}^{mn}_+)$, $\forall \mu^{(1)}, \mu^{(2)} \in L^2([0, T], \mathbb{R}^m)$, $\forall \rho^{(1)}, \rho^{(2)} \in L^2([0, T], \mathbb{R}^n_+)$ *and, a.e. in* $[0, T]$,

$$-\frac{\partial u_i(t, x^*(t), y^*(t))}{\partial x_{ij}} - (1-\tau_{ij}(t))r_j^*(t) - \lambda_{ij}^{(1)*}(t) - \mu_i^{(1)*}(t) = 0,$$

$$\forall i = 1, \dots, m, \ \forall j = 1 \dots, n;$$

$$-\frac{\partial u_i(t, x^*(t), y^*(t))}{\partial y_{ij}} + (1-c_j(t))(1-\tau_{ij}(t))(1+h_j(t))r_j^*(t) - \lambda_{ij}^{(2)*}(t) - \mu_i^{(2)*}(t) = 0,$$

$$\forall i = 1, \dots, m, \ \forall j = 1 \dots, n;$$

$$\sum_{i=1}^m (1-\tau_{ij}(t)) \left[x_{ij}^*(t) - (1-c_j(t))(1+h_j(t))y_{ij}^*(t)\right] + F_j(t) + \rho_j^{(2)*}(t) = \rho_j^{(1)*}(t), \tag{17}$$

$$\forall j = 1, \dots, n;$$

$$\lambda_{ij}^{(1)*}(t)x_{ij}^*(t) = 0,\ \lambda_{ij}^{(2)*}(t)y_{ij}^*(t) = 0, \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n \tag{18}$$

$$\mu_i^{(1)*}(t)\left(\sum_{j=1}^n x_{ij}^*(t) - s_i(t)\right) = 0, \quad \mu_i^{(2)*}(t)\left(\sum_{j=1}^n y_{ij}^*(t) - l_i(t)\right) = 0, \tag{19}$$

$$\forall i = 1, \dots, m$$

$$\rho_j^{(1)*}(t)(\underline{r}_j(t) - r_j^*(t)) = 0,\ \rho_j^{(2)*}(t)(r_j^*(t) - \overline{r}_j(t)) = 0, \quad \forall j = 1, \dots, n. \tag{20}$$

Formula (17) represents the Deficit Formula. Indeed, if $\rho_j^{(1)*}(t)$ is positive, then the prices are minimal and there is a supply excess of instrument j as an asset and of the charge $F_j(t)$, namely the economy is in deficit and, for this reason, $\rho_j^{(1)*}(t)$ is called *the deficit variable* and represents the deficit per unit.

Analogously, if $\rho_j^{(2)*}(t)$ is positive, then the prices are maximal and there is a demand excess of instrument j as a liability, namely there is a surplus in the economy. For this reason $\rho_j^{(2)*}(t)$ is called *the surplus variable* and represents the surplus per unit.

From (17) it is possible to obtain the Balance Law

$$\sum_{i=1}^{m} l_i(t) = \sum_{i=1}^{m} s_i(t) - \sum_{i=1}^{m}\sum_{j=1}^{n} \tau_{ij}(t)\left[x_{ij}^*(t) - y_{ij}^*(t)\right]$$
$$-\sum_{i=1}^{m}\sum_{j=1}^{n}(1-\tau_{ij}(t))h_j(t)y_{ij}^*(t) + \sum_{i=1}^{m}\sum_{j=1}^{n}(1-\tau_{ij}(t))c_j(t)(1+h_j(t))y_{ij}^*(t)$$
$$+\sum_{j=1}^{n} F_j(t) - \sum_{j=1}^{n} \rho_j^{(1)*}(t) + \sum_{j=1}^{n} \rho_j^{(2)*}(t). \tag{21}$$

Finally, assuming that the taxes $\tau_{ij}(t)$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, have a common value $\theta(t)$, the increments $h_j(t)$, $j = 1, \ldots, n$, have a common value $i(t)$, and the insolvency coefficients $c_j(t)$, $j = 1, \ldots, n$, have a common value $c(t)$, otherwise we can consider the average values (see Remark 7.1 in [1]), the significant Liability Formula follows

$$(1-c(t))\sum_{i=1}^{m} l_i(t) = \frac{(1-\theta(t))\sum_{i=1}^{m} s_i(t) + \sum_{j=1}^{n} F_j(t) - \sum_{j=1}^{n} \rho_j^{(1)*}(t) + \sum_{j=1}^{n} \rho_j^{(2)*}(t)}{(1-\theta(t))(1+i(t))}. \tag{22}$$

From (22) we can deduce that in this situation to reach the equilibrium is even more difficult than in the case of absence of insolvencies, because only a portion of liabilities must balance all the expenses.

3.3 The Viewpoints of the Sector and of the System

The financial problem can be considered from two different perspectives: one from the *Point of View of the Sectors*, which try to maximize the utility and a second point of view, that we can call *System Point of View*, which regards the whole equilibrium, namely in respect of the previous laws. For example, from the point of view of the sectors, $l_i(t)$, for $i = 1, \ldots, m$, are liabilities, whereas for the economic system they are investments and, hence, the Liability Formula, from the system point of view, can be called "*Investments Formula*." The system point of view coincides with the dual Lagrange problem (the so-called shadow market) in which $\rho_j^{(1)}(t)$ and $\rho_j^{(2)}(t)$ are the dual multipliers, representing the deficit and the surplus per unit arising from instrument j. Formally, the dual problem is given by

Find $(\rho^{(1)*}, \rho^{(2)*}) \in L^2([0, T], \mathbb{R}_+^{2n})$ such that

$$\sum_{j=1}^{n} \int_0^T (\rho_j^{(1)}(t) - \rho_j^{(1)*}(t))(\underline{r}_j(t) - r_j^*(t))dt \tag{23}$$

$$+\sum_{j=1}^{n}\int_0^T (\rho_j^{(2)}(t)-\rho_j^{(2)*}(t))(r_j^*(t)-\overline{r}_j(t))dt \le 0,$$

$$\forall(\rho^{(1)},\rho^{(2)})\in L^2([0,T],\mathbb{R}_+^{2n}).$$

Indeed, taking into account inequality (16), we get

$$\begin{aligned}
-\sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^T (\lambda_{ij}^{(1)}(t)-\lambda_{ij}^{(1)*}(t))x_{ij}^*(t)\,dt-\sum_{i=1}^{m}\sum_{j=1}^{n}\int_0^T (\lambda_{ij}^{(2)}-\lambda_{ij}^{(2)*})y_{ij}^*(t)\,dt \\
-\sum_{i=1}^{m}\int_0^T (\mu_i^{(1)}(t)-\mu_i^{(1)*}(t))\left(\sum_{j=1}^{n}x_{ij}^*(t)-s_i(t)\right)dt \\
-\sum_{i=1}^{m}\int_0^T (\mu_i^{(2)}(t)-\mu_i^{(2)*}(t))\left(\sum_{j=1}^{n}y_{ij}^*(t)-l_i(t)\right)dt \\
+\sum_{j=1}^{n}\int_0^T (\rho_j^{(1)}(t)-\rho_j^{(1)*}(t))(\underline{r}_j(t)-r_j^*(t))\,dt \\
+\sum_{j=1}^{n}\int_0^T (\rho_j^{(2)}(t)-\rho_j^{(2)*}(t))(r_j^*(t)-\overline{r}_j(t))\,dt \le 0
\end{aligned}$$

$\forall\lambda^{(1)},\lambda^{(2)} \in L^2([0,T],\mathbb{R}_+^{mn})$, $\mu^{(1)},\mu^{(2)} \in L^2([0,T],\mathbb{R}^m)$, $\rho^{(1)},\rho^{(2)} \in L^2([0,T],\mathbb{R}_+^n)$.

Choosing $\lambda^{(1)}=\lambda^{(1)*}$, $\lambda^{(2)}=\lambda^{(2)*}$, $\mu^{(1)}=\mu^{(1)*}$, $\mu^{(2)}=\mu^{(2)*}$, we obtain the dual problem (23)

Note that, from the *System Point of View*, also the expenses of the institutions $F_j(t)$ are supported from the liabilities of the sectors.

Remark 2 Let us recall that from the Liability Formula we get the following index $E(t)$, called "Evaluation Index," that is very useful for the rating procedure:

$$E(t)=\frac{(1-c(t))\sum_{i=1}^{m}l_i(t)}{\sum_{i=1}^{m}\tilde{s}_i(t)+\sum_{j=1}^{n}\tilde{F}_j(t)}, \tag{24}$$

where we set

$$\tilde{s}_i(t)=\frac{s_i(t)}{1+i(t)},\qquad \tilde{F}_j(t)=\frac{F_j(t)}{(1+i(t))(1-\theta(t))}.$$

From the Liability Formula we obtain

$$E(t)=1-\frac{\sum_{j=1}^{n}\rho_j^{(1)*}(t)}{(1-\theta(t))(1+i(t))\left(\sum_{i=1}^{m}\tilde{s}_i(t)+\sum_{j=1}^{n}\tilde{F}_j(t)\right)}$$

$$+\frac{\sum_{j=1}^{n}\rho_j^{(2)*}(t)}{(1-\theta(t))(1+i(t))\left(\sum_{i=1}^{m}\tilde{s}_i(t)+\sum_{j=1}^{n}\tilde{F}_j(t)\right)} \qquad (25)$$

If $E(t)$ is greater than or equal to 1, the evaluation of the financial equilibrium is positive (better if $E(t)$ is proximal to 1), whereas if $E(t)$ is less than 1, the evaluation of the financial equilibrium is negative.

The term $(1-c(t))\sum_{i=1}^{m}l_i(t)$ in (24) represents the effective liabilities (or the effective investments from the system point of view). The evaluation index (25) is less than the one in the model in [1], where the insolvency coefficients are not considered, and this means that, in presence of insolvency, it is more difficult to reach the financial equilibrium.

3.4 Regularity Results

In [16] a regularity result of $\rho_j^{(1)*}(t)$, $\rho_j^{(2)*}(t)$, has been proved. Let us set

$$F(t)=[F_1(t),F_2(t),\ldots,F_n(t)]^T;$$

$$\nu=(x,y,r)=\left(\left(x_{ij}\right)_{\substack{i=1,\ldots,m\\ j=1,\ldots,n}},\left(y_{ij}\right)_{\substack{i=1,\ldots,m\\ j=1,\ldots,n}},\left(r_j\right)_{j=1,\ldots,n}\right);$$

$$A(t,\nu)=\left(\left[-\frac{\partial u_i(t,x,y)}{\partial x_{ij}}-(1-\tau_{ij}(t))r_j(t)\right]_{\substack{i=1,\ldots,m\\ j=1,\ldots,n}},\right.$$

$$\left[-\frac{\partial u_i(t,x,y)}{\partial y_{ij}}+(1-\tau_{ij}(t))(1-c_j(t))(1+h_j(t))r_j(t)\right]_{\substack{i=1,\ldots,m\\ j=1,\ldots,n}}, \qquad (26)$$

$$\left[\sum_{i=1}^{m}(1-\tau_{ij}(t))\left(x_{ij}(t)-(1-c_j(t))(1+h_j(t))y_{ij}(t)\right)+F_j(t)\right]_{j=1,\dots,n}\Bigg);$$

$$A:\mathscr{K}\to L^2([0,T],\mathbb{R}^{2mn+n}),$$

with

$$\mathscr{K}=P\times\mathscr{R}.$$

Let us note that $\mathscr{K}$ is a convex, bounded, and closed subset of $L^2([0,T],\mathbb{R}^{2mn+n})$. Moreover assumption (2) implies that A is lower semicontinuous along line segments.

The following result holds true (see [16, Theorem 2.4]):

Theorem 5 *Let $A \in C^0([0,T],\mathbb{R}^{2mn+n})$ be strongly monotone in x and y, monotone in r, namely, there exists α such that, for $t\in[0,T]$,*

$$\langle\langle A(t,\nu_1)-A(t,\nu_2),\nu_1-\nu_2\rangle\rangle\geq\alpha(\|x_1-x_2\|^2+\|y_1-y_2\|^2), \tag{27}$$

$\forall\nu_1=(x_1,y_1,r_1)$, $\nu_2=(x_2,y_2,r_2)\in\mathbb{R}^{2mn+n}$.

Let $\underline{r}(t)$, $\overline{r}(t)$, $h(t)$, $F(t)=[F_1(t),F_2(t),\dots,F_n(t)]^T$, $C(t)=[c_1(t),c_2(t),\dots,c_n(t)]^T\in C^0([0,T],\mathbb{R}^n_+)$, let $\tau(t)\in C^0([0,T],\mathbb{R}^{mn})$ and let $s,l\in C^0([0,T],\mathbb{R}^m)$, satisfying the following assumption (β):

- *there exists $\delta_1(t)\in L^2([0,T])$ and $c_1\in\mathbb{R}$ such that, for a.a. $t\in[0,T]$:*

$$\|s(t)\|\leq\delta_1(t)+c_1;$$

- *there exists $\delta_2(t)\in L^2([0,T])$ and $c_2\in\mathbb{R}$ such that, for a.a. $t\in[0,T]$:*

$$\|l(t)\|\leq\delta_2(t)+c_2.$$

Then the Lagrange variables, $\rho^{(1)*}(t)$, $\rho^{(2)*}(t)$, which represent the deficit and the surplus per unit, respectively, are continuous too.

3.5 The Contagion Problem

In this section we want to show that it is possible to establish when the economy becomes negative by means of the dual variables $\rho^{(1)*}(t)$, $\rho^{(2)*}(t)$ (see also [14]).

Contagion can be explained as a situation when a crisis in a particular economy or region spreads out and affects others (see [18] for a complete survey on the financial contagion). The Lehman Brothers' failure in the USA is an example of contagion. Fundamental problems in the contagion are to try to know when it can happen, to give a measure of it, and to understand why it occurs. In the particular financial problem we are dealing with, which is based on portfolio flows and investment positions, namely on assets and liabilities of different sectors, we perform the contagion by using the deficit and the surplus variables as well as the balance law. Specifically, we recall that $\rho^{(1)*}(t)$ represents the deficit variable and $\rho^{(2)*}(t)$ represents the surplus variable. For our purpose it is useful to recall also the balance law:

$$\sum_{i=1}^{m} l_i(t) - \sum_{i=1}^{m} s_i(t) + \sum_{i=1}^{m}\sum_{j=1}^{n} \tau_{ij}(t)\left[x^*_{ij}(t) - y^*_{ij}(t)\right]$$
$$+ \sum_{i=1}^{m}\sum_{j=1}^{n}(1-\tau_{ij}(t))h_j(t)y^*_{ij}(t) - \sum_{i=1}^{m}\sum_{j=1}^{n}(1-\tau_{ij}(t))c_j(t)(1+h_j(t))y^*_{ij}(t) - \sum_{j=1}^{n} F_j(t)$$
$$= -\sum_{j=1}^{n}\rho_j^{(1)*}(t) + \sum_{j=1}^{n}\rho_j^{(2)*}(t). \tag{28}$$

We realize that when the left-hand side is negative, it means that the sum of the liabilities, namely the investments of the system, cannot cover the expenses incurred. The sign of the left-hand side depends on the difference

$$-\sum_{j=1}^{n}\rho_j^{(1)*}(t) + \sum_{j=1}^{n}\rho_j^{(2)*}(t).$$

When such a difference is negative, from (28) it follows that the whole system is at a loss. In this case we say that a negative contagion is determined and we can assume that the insolvencies of individual entities propagate through the entire system. It is sufficient that only one deficit component $\rho_j^{(1)*}(t)$ is very large to obtain, even if the other $\rho_j^{(2)*}(t)$ are lightly positive, a negative balance for the whole system. In addition, if even only one $\rho_j^{(1)*}(t)$ is positive, then for that instrument j all the sectors are already in crisis.

When

$$\sum_{j=1}^{n}\rho_j^{(1)*}(t) > \sum_{j=1}^{n}\rho_j^{(2)*}(t),$$

namely the sum of all the deficit exceeds the sum of all the surplus, we get $E(t) \leq 1$ and, hence, also $E(t)$ is a significant indicator that the financial contagion happens. Causes of contagion are the lack of investments, the financial insolvency, or the excess in the expenses.

4 A Numerical Example

Let us analyze a numerical financial example in which we consider as the risk aversion function an evolutionary measure of Markowitz type, which expresses at each instant $t \in [0, T]$ the risk aversion by means of variance-covariance matrices denoting the sector's assessment of the standard deviation of prices for each instrument.

Let us consider an economy with two sectors and two financial instruments, as shown in Figure 1, and choose as variance-covariance matrices of the two sectors the following ones:

$$Q^1(t) = \begin{bmatrix} 1 & 0 & -0.5t & 0 \\ 0 & 1 & 0 & 0 \\ -0.5t & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Q^2(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -0.5t & 0 \\ 0 & -0.5t & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We define the feasible set as follows:

$$\mathbb{K} = \Big\{(x_{11}(t), x_{12}(t), x_{21}(t), x_{22}(t), y_{11}(t), y_{12}(t), y_{21}(t), y_{22}(t), r_1(t), r_2(t)) \in L^2([0,1], \mathbb{R}^{10}_+) :$$
$$x_{11}(t) + x_{12}(t) = t + 2, \quad x_{21}(t) + x_{22}(t) = 2t + 3, \quad \text{a.e. in } [0,1]$$
$$y_{11}(t) + y_{12}(t) = 2t, \quad y_{21}(t) + y_{22}(t) = 3t, \quad \text{a.e. in } [0,1]$$

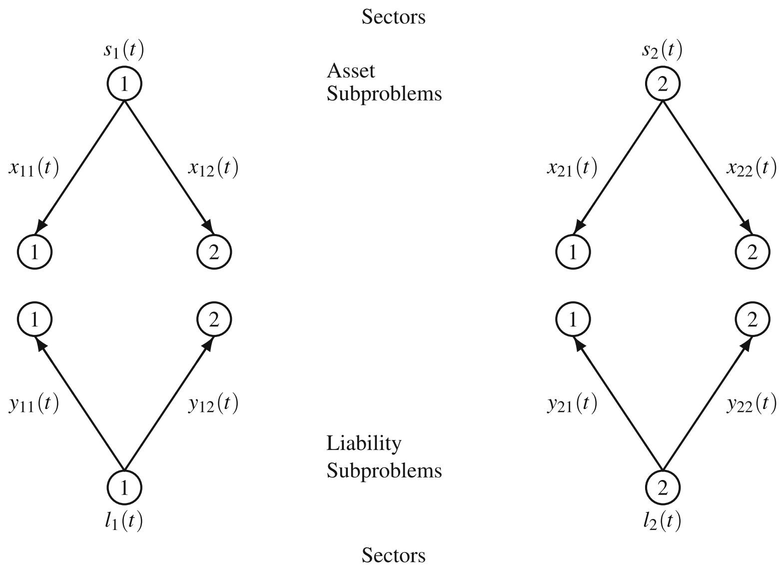

Fig. 1 Two sectors and two financial instruments network a.e. in [0, 1]

$$4t \le r_1(t) \le 5t+12, \quad t \le r_2(t) \le 6t+5, \quad \text{a.e. in } [0,1]\Big\}.$$

Let us assume that

$$h_1(t) = \frac{3}{2}t, \text{ and } h_2(t) = \frac{t}{2}.$$

Finally, let us consider

$$\tau_{11}(t) = \frac{t}{2}, \quad \tau_{12}(t) = \frac{3}{4}t, \quad \tau_{21}(t) = \frac{t}{2}, \quad \tau_{22}(t) = \frac{t}{4}$$

and

$$c_1(t) = 0.1 \quad c_2(t) = 0.15.$$

Then, variational inequality (8) becomes:

$$\int_0^1 \Big\{\big[2x^*_{11}(t) - ty^*_{11}(t) - \left(1-\frac{t}{2}\right) r^*_1(t)\big]\big(x_{11}(t) - x^*_{11}(t)\big)$$

$$+\big[2x^*_{12}(t) - \left(1-\frac{3}{4}t\right) r^*_2(t)\big]\big(x_{12}(t) - x^*_{12}(t)\big)$$

$$+\big[2x^*_{21}(t) - \left(1-\frac{t}{2}\right) r^*_1(t)\big]\big(x_{21}(t) - x^*_{21}(t)\big)$$

$$+\big[2x^*_{22}(t) - ty^*_{21}(t) - \left(1-\frac{t}{4}\right) r^*_2(t)\big]\big(x_{22}(t) - x^*_{22}(t)\big)$$

$$+\big[2y^*_{11}(t) - tx^*_{11}(t) + 0.9\left(1+\frac{3}{2}t\right)\left(1-\frac{t}{2}\right) r^*_1(t)\big](y_{11}(t) - y^*_{11}(t))$$

$$+\big[2y^*_{12}(t) + 0.85\left(1+\frac{t}{2}\right)\left(1-\frac{3}{4}t\right) r^*_2(t)\big](y_{12}(t) - y^*_{12}(t))$$

$$+\big[2y^*_{21}(t) - tx^*_{22}(t) + 0.9\left(1+\frac{3}{2}t\right)\left(1-\frac{t}{2}\right) r^*_1(t)\big](y_{21}(t) - y^*_{21}(t))$$

$$+\big[2y^*_{22}(t) + 0.85\left(1+\frac{t}{2}\right)\left(1-\frac{t}{4}\right) r^*_2(t)\big](y_{22}(t) - y^*_{22}(t))$$

$$+\big[\left(1-\frac{t}{2}\right) x^*_{11}(t) + \left(1-\frac{t}{2}\right) x^*_{21}(t) - 0.9\left(1+\frac{3}{2}t\right)$$

$$\left[\left(1-\frac{t}{2}\right) y^*_{11}(t) + \left(1-\frac{t}{2}\right) y^*_{21}(t)\right] + F_1(t)\big]\big(r_1(t) - r^*_1(t)\big)$$

$$+\Big[\left(1-\frac{3}{4}t\right)x^*_{12}(t)+\left(1-\frac{t}{4}\right)x^*_{22}(t)-0.85\left(1+\frac{t}{2}\right)$$
$$\left[\left(1-\frac{3}{4}t\right)y^*_{12}(t)+\left(1-\frac{t}{4}\right)y^*_{22}(t)\right]+F_2(t)\Big]\big(r_2(t)-r^*_2(t)\big)\Big\}dt\geq 0,$$
$$\forall(x,y,r)\in\mathbb{K}. \tag{29}$$

Using the direct method we get the following solution:

$$\begin{aligned}
x^\star_{11}(t)&=-\frac{381t^4-610t^3-600t^2+3200t+2560}{160(t^2-16)}; & x^\star_{12}(t)&=\frac{381t^4-450t^3-280t^2+640t-2560}{160(t^2-16)}\\
x^\star_{21}(t)&=-\frac{-415t^4+222t^3-2120t^2+4480t+3840}{160(t^2-16)}; & x^\star_{22}(t)&=\frac{-415t^4+542t^3-1640t^2-640t-3840}{160(t^2-16)}\\
y^\star_{11}(t)&=-\frac{331t^3-410t^2+360t}{40(t^2-16)}; & y^\star_{12}(t)&=\frac{411t^3-410t^2-920t}{40(t^2-16)}\\
y^\star_{21}(t)&=-\frac{485t^3-502t^2+760t}{40(t^2-16)}; & y^\star_{22}(t)&=\frac{605t^3-502t^2-1160t}{40(t^2-16)}\\
r^\star_1(t)&=4t; & r^\star_2(t)&=t
\end{aligned} \tag{30}$$

Since $r^\star_1(t)$ e $r^\star_2(t)$ are the floor prices then $\rho^{(2)}_1(t)=\rho^{(2)}_2(t)=0$. From the Deficit Formula (17) we obtain that:

$$\rho^{(1)}_1(t)=\frac{(2-t)}{160(t^2-16)}[2220.2t^4-799.6t^3+2742.4t^2-1824t-3200]+F_1(t),$$

$$\rho^{(1)}_2(t)=\frac{1}{160(t^2-16)}[(2396.6t^5-2932t^4-17320.8t^3+4259.2t^2+39808t-25600)]+F_2(t).$$

$\rho^{(1)*}_1(t)$ is strictly positive for each $F_1(t)\geq 0$, whereas, for each $F_2(t)$ nonnegative, $\rho^{(1)*}_2(t)$ is positive in the interval $\left[0,\bar{t}\right]$ $\bar{t}=0.827636$. In such an interval the solution of the problem is given by (30).

The deficits can be reduced only if $F_1(t)$ and $F_2(t)$ decrease, even if we cannot obtain the financial equilibrium.

In the interval $\left[\bar{t},1\right]$ it is possible that the financial equilibrium can be reached obtaining also a surplus. A suggestion in this sense is given by the Evaluation Index, which gives complete information on the behavior of the economy and of the contagion.

Actually we have

$$\theta(t)=\frac{t}{2};\quad i(t)=t;\quad c(t)=0.125;$$

$$\sum_{i=1}^{2}l_i(t)=5t;\quad \sum_{i=1}^{2}s_i(t)=3t+5;$$

$$\sum_{i=1}^{2} \tilde{s_i}(t) = \frac{3t+5}{1+t}; \quad \sum_{j=1}^{2} \tilde{F}_j(t) = \frac{F_1(t)+F_2(t)}{(1+t)(1-\frac{t}{2})}.$$

Thus, the Evaluation Index is:

$$E(t) = \frac{(1-c(t)) \sum_{i=1}^{2} l_i(t)}{\sum_{i=1}^{2} \tilde{s_i}(t) + \sum_{j=1}^{2} \tilde{F}_j(t)} = \frac{(4.375t)(1+t)(2-t)}{(2-t)(5+3t)+2(F_1(t)+F_2(t))}. \tag{31}$$

In the interval $\left[\frac{\sqrt{5721}-11}{70}, 1\right]$ (where $3t^2+11t-40>0$), the economy has a positive average evaluation, if the condition

$$F_1(t)+F_2(t) \le \frac{2-t}{16}\,(35t^2+11t-40)$$

is verified.

This result has been obtained considering the average $\theta(t)$ and $i(t)$, however it seems convenient and desirable that the data $\tau_{ij}(t)$ and $h_j(t)$ are not too different.

In our model, which takes into account the insolvencies, the Evaluation index (31) is less than the one obtained in [1], in which the insolvencies are not considered. Then, as expected, in the presence of insolvencies the economy gets worse. If we do not take into account the insolvencies, the Evaluation index (31) coincides with the one in [1].

5 Conclusions

In the chapter, we assessed the influence of the insolvencies on the financial model and on the financial contagion. Our results show that the risk of contagion increases with the presence of insolvencies, with decreasing investments and increasing expenditure. Then, our conclusion is that it is necessary to focus on these three factors, in order to improve the financial equilibrium. The suggestion to the governments, that follows from our analysis, is to reduce the insolvencies, deferring in time the payment of the liabilities, and supporting the sectors.

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