Finite Element Analysis in Fluid Mechanics

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Abstract In the last decades, the finite element method (FEM) in fluid mechanics applications has gained substantial momentum. FE analysis was initially introduced to solid mechanics. However, the progress in fluid mechanics problems was slower due to the non-linearities of the equations and inherent difficulties of the classical FEM to deal with instabilities in the solution of these problems. The main goal of this review is to analyze FEM and provide the theoretical basis of the approach mainly focusing on parabolic type of problems applied in fluid mechanics. Initially, we analyze the basics of FEM for the Stokes problem and we provide theorems for uniqueness and error estimates of the solution. We further discuss FE approaches for the solution of the advection–diffusion equation such as the stabilized FEM, the variational multiscale method, and the discontinuous Galerkin method. Finally, we extend the analysis on the non-linear Navier–Stokes equations and introduce recent FEM advancements.

1 Introduction

Finite element method (FEM) has gained substantial momentum in the last decades. FEM was initially introduced as an answer to solid mechanics problems that were difficult to solve until then. Most of them would be encountered in aeronautics or civil engineering due to the need of solving problems related to the construction of complicated structures. The method was extended to fluid mechanics applications where the convective terms play important role leading to a non-linear formulation of the problem. The progress in fluid mechanics was slower due to the non-linearities and instabilities of the solution of these problems.

The basic principles of the FEM were developed by the German mathematician Ritz in 1909. In 1915 Galerkin worked on the theoretical aspects of the method.

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The absence of computers delayed further advancement of the method. Later on, when computers were introduced, the method was further developed. Hrenikoff, 1941, introduced the framework method, in which a plain elastic medium could be replaced by an equivalent system of sticks and rods. In 1943 Courant solved the torsion problem by using triangular elements based on the principle of minimum potential energy introducing the Rayleigh–Ritz method. Courant's theory could not be implemented due to the unavailability of computers at the time [82].

Argyris, 1955, in the book "Energy Theorems and Structural Analysis" introduced the principles of the finite element method [3, 85]. In 1956 Clough, Turner, Martin, and Top calculated the stiffness matrix of rod and other elements. Argyris and Kelsey, 1960, published their work which was based on the finite element principles. In the same year, the term finite element method was introduced by Clough in his paper and the term has been used extensively in the literature until today. Zienkiewicz and Chung wrote the first book on finite elements method, in 1967. Other notable researchers in the FEM field are Samuel Levy, Borje Langefors, Paul Denke, Baudoin Fraejis De Veubeke, L. Brandeis Wehle Jr., Theodore Pian, Warner Lansing, Bertran Klein, John Archer, Robert Melosh, John Przemieniecki, Ian Taig, Richard Gallagher, Bruce Irons, and others.

As mentioned before the progress of FEM in fluid mechanics applications had several drawbacks due to the non-linear convective terms and instabilities of the solution based on the element selection. For these reasons many researchers studied the advection–diffusion equation. The Galerkin method was introduced as a natural extension of the weak formulation of the PDEs under consideration. One of the reasons why finite elements have been less popular in the past than other numerical techniques such as finite differences is the lack of upwind techniques. However, accurate upwind methods have been constructed. The most popular of these upwind approaches is the Streamline Upwind Petrov–Galerkin method (SUPG) [89]. It can be shown that upwinding may increase the quality of the solution considerably. Another important aspect of upwinding is that it makes the systems of equations appropriate for the utilization of iterative methods. As a consequence both the number of iterations and the computation time substantially decrease.

The advection–diffusion equation represents diffusion of a scalar variable while convected by a velocity field. In this respect, the equation by itself applies in several physical phenomena and is a precursor to studying the non-linear Navier–Stokes equations that represents in a simplifying manner the transport of velocity itself. In any case, the development of accurate and stable numerical formulations for the advection–diffusion equation is quite challenging. For example, the classical Galerkin method is known to perform poorly for advection-dominated transport problems. Spurious oscillations emerge in the solution due to the truncation error inherently introduced in the discretized Galerkin approximation. The literature suggests numerous strategies to overcome this problem. The addition of artificial diffusion is a standard strategy, another is the employment of a non-centered discretization of the advection operator, the so-called upwind schemes [45]. Other strategies involve multiscale models using bubble functions or wavelets [72], while in many cases, these methods are equivalent [17]. In the relevant section of this chapter, more information is provided regarding some of the strategies in the context of finite element methods that have been developed to address the problems that standard discretizations face.

Studying the advection–diffusion equation helps in understanding more complicated problems such as the Navier–Stokes equations. For the discretization of the incompressible Navier–Stokes equations, since the pressure is an unknown in the momentum but not in the continuity equation, the discretization must satisfy some special requirements. In fact one is no longer free to choose any combination of pressure and velocity approximation but the finite elements must be constructed such that the Ladyzhenskaya–Brezzi–Babuska (LBB) condition is satisfied. This condition provides a relation between pressure and velocity approximation. In finite differences and finite volumes the equivalent of the LBB condition is satisfied if staggered grids are applied.

The solenoidal (divergence free) approach has been introduced where in this method, the elements are constructed in such a way that the approximate divergence freedom is satisfied explicitly. This method seems very attractive, however, the extension to three-dimensional problems is difficult. Stabilized and multiscale formulations are among the most fundamental method for fluid mechanics problems. The SUPG is one of the first finite element approaches for studying fluid mechanics applications. However, due to the advancement in research nowadays, new finite element approaches have emerged such as the variational multiscale method (VMS), the characteristic base split (CBS) method, the gradient smoothed method (GSM), discontinuous Galerkin (DG) and adaptive FEM.

In this review we initially present the basic analysis focused on the Stokes problem providing error estimates. We further analyze the advection–diffusion equation introducing several FEM advancements. We conclude this chapter with a brief analysis on FEM for the non-linear Navier–Stokes equations.

2 Preliminaries and Basic Theorems

We begin this chapter with the main steps of the finite element method. In advance, we formulate basic definitions and theorems about the existence and uniqueness of the solution in these problems. More details can be found in the textbooks by Brenner and Scott and Brezzi [15, 16].

Definition 1 Let $a(\cdot,\cdot)$ be a bilinear form on a normed linear space, H. The bilinear form is said to be *bounded* (or *continuous*) if exists $C < \infty$ such that

$$| a(u, v) | \leq C \|u\|_H \|v\|_H \quad \forall\, u, v \in H,$$

and *coercive* on subspace $V = \left\{v \in H^1(0, 1) : v(0) = 0\right\}$, $V \subset H$ if exists $\delta > 0$ such that

$$a(v, v) \geq \delta \|v\|_H^2, \quad \forall\, v \in V,$$

where $a(u, v) = \int_0^1 u'\, v'\, dx$, $\|\cdot\|_H$ is the norm in space H.

Focusing our attention on the non-symmetric variational problem, that is more general, the following conditions are valid:

$$\begin{cases} (H, (\cdot,\cdot)) \text{ is a Hilbert space.} \\ V \text{ is a (closed) subspace of } H. \\ a(\cdot,\cdot) \text{ is a bilinear form on } V. \\ a(\cdot,\cdot) \text{ is continuous (bounded) on } V. \\ a(\cdot,\cdot) \text{ is coercive on } V. \end{cases}$$

Then the non-symmetric variational problem is the following, given $F \in V'$, find $u \in V$, such that $a(u, v) = F(v)$, $\forall\, v \in V$, where V' is the dual space of V.

The discrete form or the Galerkin approximation of this problem is the following, given a finite dimensional subspace $V_h \subset V$ and $F \in V'$, find $u_h \in V_h$ such that

$$a(u_h, v) = F(v), \quad \forall\, v \in V_h. \tag{1}$$

Theorem 1 (Lax-Milgram) *Given a Hilbert space* $(V, (\cdot,\cdot))$*, a continuous, coercive bilinear form* $a(\cdot,\cdot)$ *and a continuous linear functional* $F \in V'$*, there exists a unique solution* $u \in V$*, such that*

$$a(u, v) = F(v), \quad \forall\, v \in V. \tag{2}$$

This theorem guarantees existence and uniqueness of the solution for both the variational and the approximation problems under the conditions mentioned previously and its proof can be found in [15, 16]. We define the *energy norm*, $\|\cdot\|_E$ as

$$\|v\|_E = \sqrt{a(v.v)}, \quad \forall\, v \in V. \tag{3}$$

Based on the above definition for the energy norm and with the use of the Schwartz' inequality the error estimate for the previous problem (2) is proven to be

$$\|u - u_h\|_E = \inf\{\|u - v\|_E : v \in S\}, \tag{4}$$

where u is the solution and u_h the approximate one and $v \in S$, S a finite dimensional subspace of V. This is the basic error estimate and is optimal in the energy norm. Moreover, in some cases it can be proved that we can replace "infimum" with "minimum," more details can be found elsewhere [15],

$$\|u - u_h\|_E = \min\{\|u - v\|_E : v \in S\}. \tag{5}$$

We further focus our attention on a specific linear parabolic problem the Stokes problem.

3 The Stokes Problem

Initially, we consider the stationary Stokes problem for incompressible flow. Ω is a bounded open set of $\mathbb{R}^n$ (where $n = 2, 3$) with regular boundary and $\mathbf{f}$ is a square integrable function on Ω. We seek a solution $(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times (L^2(\Omega)/\mathbb{R})$ of the problem,

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f} \;\; in \;\; \Omega, \\ div\, \mathbf{u} = 0 \;\; in \;\; \Omega, \\ \mathbf{u} = 0 \;\; on \;\; \partial\Omega. \end{cases} \tag{6}$$

Based on this problem, we will introduce the error estimates (*a priori* and *a posteriori*) and we briefly discuss about the uniqueness of the solution for this problem [10]. Our goal is to extend these arguments for the non-stationary case.

According to the finite element analysis we end up with the following weak form:

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), & \forall\, \mathbf{v} \in H_0^1(\Omega)^n, \;\; \mathbf{u} \in H_0^1(\Omega)^n, \\ b(\mathbf{u}, q) = 0 & \forall\, q \in H^1(\Omega), \;\; p \in H^1(\Omega), \end{cases} \tag{7}$$

where $a(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla\mathbf{u}\; \nabla\mathbf{v}\; d\Omega$ and $b(p, \mathbf{v}) = \int_\Omega p\; \nabla\mathbf{v}\; d\Omega$.

Given two finite dimensional subspaces $V_h \subset H^1(\Omega)^n$ and $Q_h \subset H^1(\Omega)$ the corresponding discrete form is

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + b(p_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), & \forall\, \mathbf{v}_h \in V_{0h}, \;\; \mathbf{u}_h \in V_{0h}, \\ b(\mathbf{u}_h, q_h) = 0, & \forall\, q_h \in Q_h, \;\; p_h \in Q_h, \end{cases} \tag{8}$$

where $V_{0h} = \{\mathbf{v}_h \in V_h : \mathbf{v}_h\, |_{\partial\Omega} = 0\}$.

Two cases are analyzed for both triangular and quadrilateral elements depending on the number of nodes on each element [10]. We focus only on the Taylor–Hood method (six node triangular elements), second order polynomials for the velocity and first order polynomials for the pressure at each element ($P_2 - P_1$).

After finding a solution, for the problem under consideration, it is important to study whether the stability of the problem is affected by the input data. This can be done using the inf–sup condition, the Ladyzhenskaya–Babuska–Brezzi (LBB) condition. This is a condition for saddle point problems, i.e. problems arise in different types of discretization of equations. Convergence is ensured for most discretization schemes for positive definite problems but for saddle point problems there are still discretizations that are unstable, due to spurious oscillations [89]. In these cases a better approach is the adaptation of the computational grid [78]. We further discuss for the BB condition, introducing the following theorem.

Theorem 2 *If Ω is polygonal and $\Omega_h = \Omega$, $\Omega_h = \bigcup_i T_i$, where T_i are the triangles and h denotes the length of greatest triangle side, if all triangles have at least one vertex which is not on $\partial\Omega$, if V_h, Q_h are chosen as in the Taylor–Hood method, then there exists a constant C, independent of h, such that*

$$sup_{\mathbf{v}_h \in V_{0h}} \frac{(\mathbf{v}_h, \nabla q_h)}{(\mathbf{v}_h, \mathbf{v}_h)^{\frac{1}{2}}} \geq C\,(\nabla q_h, \nabla q_h)^{\frac{1}{2}}\,, \quad \forall q_h \in Q_h. \tag{9}$$

This theorem follows the idea of the BB condition and the proof depends on the choice of the elements and can be found in [10]. One of the most important questions in solving such a problem is that of existence and uniqueness of the solution. In this case we focus on the discrete form of the problem under consideration, (8) where we can ensure the previous with the following theorem.

Theorem 3 *Under the conditions of Theorem 2 the discrete form, Equation (8), has a unique solution $(\mathbf{u}_h, p_h)$ in $V_{0h} \times (Q_h/\mathbb{R})$.*

Additionally, we are interested in error estimates of the Stokes problem as discussed in the following sections.

3.1 A Priori Error Estimates

The *a priori* error estimates depend only on the exact solution, but not on the approximated one. On the other hand, the *a posteriori* error estimates require computation of the solution. *A posteriori* error estimates can also provide results on which element size gives a larger error contribution leading to conclusions about grid adaptation [78]. A theorem that provides *a priori* error estimates for the discrete form of the stationary Stokes problem using Taylor–Hood elements ($P_2 - P_1$) is as follows.

Theorem 4 *Let Ω be a polygon and $\Omega_h = \Omega$ for all h. We assume that each element of $\mathscr{T}_h$ (set of triangles) has at least one vertex not on the boundary. Then the following inequalities are valid:*

$$\begin{aligned} \|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0 &\leqslant h^2 K\left(\|\mathbf{u}\|_{H^3(\Omega)^N}+\|p\|_{H^2(\Omega)/R}\right), \\ \|\nabla(p-p_h)\|_0 &\leqslant hK\left(\|\mathbf{u}\|_{H^3(\Omega)^N}+\|p\|_{H^2(\Omega)/R}\right). \end{aligned} \tag{10}$$

Similar inequalities can be found in the case where we have quadrilaterals [10].

Expanding previous arguments for the non-stationary problem we find that there are not as many studies as in the previous case. According to Kemmochi [59] for the non-stationary Stokes problem,

$$\begin{cases} \mathbf{u}_t - \Delta\mathbf{u} + \nabla p = \mathbf{f} \;\; in \;\; \Omega \times [0,T], \\ div\,\mathbf{u} = 0 \;\; in \;\; \Omega \times [0,T], \\ \mathbf{u} = 0 \;\; on \;\; \partial\Omega \times [0,T], \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0(\cdot) \;\; in \;\; \Omega, \end{cases} \tag{11}$$

the error estimates for the velocity $\mathbf{u}$ and pressure p are

$$\begin{aligned} \|\mathbf{u}-\mathbf{u}_h\|_H &\leqslant Ch^2t^{-1}\,\|\mathbf{u}_0\|_H, \\ \|p-p_h\|_Q &\leqslant Cht^{-1}\,\|\mathbf{u}_0\|_H. \end{aligned} \tag{12}$$

Remark 1 The difference between the *a priori* error estimates for the stationary and the non-stationary Stokes problem is the introduction of the time variable in the results. Additional results can be obtained for the time derivative for the non-stationary Stokes problem.

In many cases, of the classic finite element approach, the LBB condition is not satisfied, thus it is necessary to find a way to solve the problems and also satisfy this condition. An effective way to overcome this problem is to utilize the adaptive FEM. In the following, we analyze the method suggested by Arnold, Brezzi, and Fortin for the Stokes problem [4]. The discrete form is

$$\begin{cases} \displaystyle\sum_{i,j=1}^{2}\int_\Omega \epsilon_{ij}(u)\epsilon_{ij}(v)\,dx - \int_\Omega p\,\nabla\cdot u\,dx = \int_\Omega fv\,dx \;\; \forall v \in (H_0^1(\Omega))^2, \\ \displaystyle\int_\Omega q\,\nabla\cdot u\,dx = 0 \;\; \forall q \in L^2(\Omega)/R, \end{cases} \tag{13}$$

where $\epsilon_{ij}(u) = \dfrac{(\partial_i u_j + \partial_j u_i)}{2}$. This method is based on using the MINI element as a way to satisfy the inf–sup condition introducing an operator $\Pi_h : (H_0^1(\Omega))^2 \to V_h$. Thus the second equation can be written as

$$\int_\Omega q_h div\,(\Pi_h v - v)\;dx = 0, \;\; \forall q_h \in Q_h, \forall v \in \left(H_0^1\right)^2 \tag{14}$$

and

$$\|\Pi_h v\|_1 \le c\,\|v\|_1 \quad \forall v \in \left(H_0^1\right)^2 . \tag{15}$$

For the MINI element the space is

$$V_h = \left(\mathring{M}_0^1\right)^2 \oplus \left(B^3\right)^2, \quad Q_h = M_0^1, \tag{16}$$

where

$$M_0^k(\mathscr{T}_h) = \left\{v \mid v \in C^0(\Omega), v_{|T} \in P_k(T),\ \forall\, T \in T_h\right\},\ \mathring{M}_0^k(\mathscr{T}_h) = M_0^k(\mathscr{T}_h) \cap H_0^1(\Omega) \tag{17}$$

for $k \ge 1$ and

$$B^k(T_h) = \left\{v \mid v_{|T} \in P_k(T) \cap H_0^1(T),\ \ \forall\, T \in T_h\right\}, \tag{18}$$

for $k \ge 3$ and T the triangular elements of $\mathscr{T}_h$. For the problem based on the MINI elements, the following argument is valid:

$$\|u - u_h\|_1 + \|p + p_h\|_{0/R} \le C \inf\left\{\|u - v\|_1 + \|p + q\|_{0/R}\right\} \le Ch\,\|f\|_0 , \tag{19}$$

where C is independent of h. These spaces can be further extended leading to other methods [4]. For example, there is a case where it can be seen as an enriched version of Taylor–Hood method where convergence is simpler than the classical Taylor–Hood method. In other methods discontinuous approximation of the pressure is used as mentioned in Crouzeix–Raviart [4, 30].

3.2 A Posteriori Error Estimates

In this section we focus our attention on *a posteriori* estimates for the approximation of time dependent Stokes equations. We introduce the notion of the Stokes reconstruction operator and present the error equation that satisfies the exact divergence-free condition described in detail in [57].

The energy technique for *a posteriori* error analysis of finite element discretizations of parabolic problems provides suboptimal rates in the $L^\infty(0, T; L^2(\Omega))$ norm. Makridakis and Nochetto in their study combine energy techniques with appropriate pointwise representation of the error based on an elliptic reconstruction operator which restores optimal order and regularity for piecewise polynomials of degree higher than one [68]. Additionally, Lakkis and Makridakis based on the previous work derive *a posteriori* error estimates for fully discrete approximations of the solutions of linear parabolic equations. The discretization uses finite element spaces that change in time [62]. Akrivis and collaborators presented a refined analysis for quasilinear parabolic problems applying implicit-explicit multistep finite element schemes [1]. Let us consider the non-stationary Stokes problem for incompressible flow. These equations are discretized in space by the finite elements or the finite volumes method. This problem is still open and directly related to Navier–Stokes equations. This is due to the fact that the *a posteriori* error theory is still in progress as reported by several researchers [11, 34, 57, 62, 68]. We assume the availability of *a posteriori* estimator for the Stokes problem, expressed by the following assumption.

Assumption Let $(\mathbf{w}, q) \in \mathbf{Z} \times \Pi$ and $(\mathbf{w}_h, q_h) \in \mathbf{Z}_h \times \Pi_h$ be the exact solution and its finite element approximation. For the space X (equal to $\mathbf{H} = (L^2(\Omega))^d$, $\mathbf{V} = (H_0^1(\Omega))^d$, $d = 2, 3$ or $\mathbf{V}'$ the dual space of $\mathbf{V}$), we assume that there exists *a posteriori* estimator function, $\mathscr{E}((\mathbf{w}_h, q_h)$ and $\mathscr{E}_{pres}((\mathbf{w}_h, q_h)$, which depend on $(\mathbf{w}_h, q_h)$, $\mathbf{g}$ and the corresponding norm, such that

$$\|\mathbf{w} - \mathbf{w}_h\|_X \leq \mathscr{E}((\mathbf{w}_h, q_h), \mathbf{g};\ X) \quad \textit{and} \quad \|q - q_h\|_\Pi \leq \mathscr{E}_{pres}((\mathbf{w}_h, q_h), \mathbf{g};\ \Pi). \tag{20}$$

It can be shown that the discrete solution coincides with the continuous solution [57]. In order to define the Stokes reconstruction as introduced by Karakatsani and Makridakis, 2007, we provide the following definitions [46, 57],

Definition 2 (Stokes Operator) Let $\bar{\Delta} :\ \mathbf{H}^2 \cap \mathbf{Z} \subset \mathbf{J} \to \mathbf{J}$ be the Stokes operator, meaning, the L^2-projection of the Laplace operator onto $\mathbf{J}$. Then introducing the discrete version of the Stokes operator $\bar{\Delta}_h :\ \mathbf{Z}_h \to \mathbf{Z}_h$ by,

$$\left\langle \bar{\Delta}_h \mathbf{v}, \boldsymbol{\chi} \right\rangle = -a(\mathbf{v}, \boldsymbol{\chi}), \quad \forall \boldsymbol{\chi} \in \mathbf{Z}_h. \tag{21}$$

Definition 3 (Stokes Reconstruction) For fixed $t \in [0, T]$, let $(\mathbf{U}, P) \in \mathbf{V} \times \Pi$ be the solution of the stationary Stokes problem,

$$\begin{cases} a(\mathbf{U}, \mathbf{v}) + b(\mathbf{v}, P) = \langle \mathbf{g}_h(t), \mathbf{v} \rangle, & \forall\ \mathbf{v} \in \mathbf{V}, \\ \qquad\qquad b(\mathbf{v}, P) = 0, & \forall\ q \in \Pi, \end{cases} \tag{22}$$

where

$$\mathbf{g}_h = -\Delta_h \mathbf{u}_h - \mathbf{f}_h + \mathbf{f}. \tag{23}$$

We call $(\mathbf{U}, P) = (\mathbf{U}(t), P(t))$ the Stokes reconstruction of the discrete velocity and pressure fields, $(\mathbf{u}_h(t), p_h(t))$.

Based on the above definitions Karakatsani and Makridakis, 2007, introduce the following theorem, where it provides the error equations based on the *a posteriori* estimator function introduced before [57].

Theorem 5 (Error Equation) *Let* $(\mathbf{U}, P)$ *be the Stokes reconstruction and* $(\mathbf{u}, p)$ *the solution of the Stokes problem which is assumed to be sufficiently regular. If* $\mathbf{e} = \mathbf{U} - \mathbf{u}$ *and* $\varepsilon = P - p$*, then* $(\mathbf{e}, \varepsilon)$ *is the weak solution of the problem,*

$$\begin{cases} \mathbf{e}_t - \Delta \mathbf{e} + \nabla \varepsilon = (\mathbf{U} - \mathbf{u}_h)_t, \\ \quad div\ \mathbf{e} = 0. \end{cases} \tag{24}$$

Additionally, $\mathbf{U} - \mathbf{u}_h$ *and* $(\mathbf{U} - \mathbf{u}_h)_t$ *satisfy the following estimates:*

$$\left\| \partial_t^{(j)} (\mathbf{U} - \mathbf{u}_h) \right\|_X \leq \mathscr{E}((\partial_t^{(j)} \mathbf{u}_h, \partial_t^{(j)} p_h), \ \partial_t^{(j)} \mathbf{g}_h; \ X), \quad j = 0, \ 1, \tag{25}$$

where X *is one of the spaces,* $\mathbf{H}$*,* $\mathbf{V}$ *or* $\mathbf{V}'$*, discussed before and* $\mathscr{E}$ *is the a posteriori estimator function defined in previous assumption. The proof of this theorem can be found in [57].*

Theorem 6 **(**$L^\infty(\mathbf{H})$ **and** $L^2(\mathbf{V})$ **Norm Error Estimates)** *Let us assume that* $(\mathbf{u}, p)$ *is the solution of the time dependent Stokes problem, Equation* (11)*, and* $(\mathbf{u}_h, p_h)$ *is the finite element approximation. Let* $(\mathbf{U}, P)$ *be the solution of the stationary Stokes problem and* $\mathscr{E}$ *is the a posteriori estimator function defined previously. Then the following a posteriori error bounds hold for,* $0 < t \leq T$*,*

$$\begin{aligned} &\|\mathbf{u}(t) - \mathbf{U}(t)\|_{\mathbf{H}}^2 + \int_0^t \|(\mathbf{u} - \mathbf{U})(s)\|_{\mathbf{V}}^2\, ds \\ &\qquad \leqslant \|\mathbf{u}(0) - \mathbf{U}(0)\|_{\mathbf{H}}^2 + \int_0^t \mathscr{E}((\mathbf{u}_{h,t}, p_{h,t}), \mathbf{g}_{h,t}; \mathbf{V}')^2 ds. \end{aligned} \tag{26}$$

Additional inequalities and the proof of this theorem can be found in [57].

They additionally provide a theorem for $L^\infty(\mathbf{V})$ norm error estimates and at the same study they discuss about estimates using the parabolic duality argument [34, 90]. In this study two related applications of the reconstruction of the Stokes problem are discussed [57].

3.3 Crouzeix–Raviart Finite Element Discretization and Finite Volume Scheme

An *a posteriori* bound for the time dependent Stokes problem under the Crouzeix–Raviart finite element approximation is derived. However, further detailed work is required related to the specific form of possible singularities of the exact solution for this problem [57]. The finite volume (FV) scheme approximations is the Crouzeix–Raviart couple $\mathbf{V}_h \times \Pi_h$. The FV methods rely on local conservation properties of the differential equations under consideration over the "control volume." Integrating over a region $b \subset \Omega$ and utilizing the Green's formula, we obtain the following system for the Stokes problem in the discrete form,

$$\begin{cases} \int_{b_e} \mathbf{u}_{h,t} - \int_{\partial b_e} \nabla \mathbf{u}_h \mathbf{n} + \int_{\partial b_e} p_h \mathbf{n} = \int_{b_e} \mathbf{f}, \quad \forall\ e \in E_h, \\ \int_K div\ \mathbf{u}_h = 0, \quad \forall\ K \in \mathscr{T}_h. \end{cases} \tag{27}$$

where z_K is an inner point of $K \in \mathscr{T}_h$, connecting the point with line segments to the vertices of the triangle K, we partition it into three segments K_e, where $e \in E_h(K)$, then each side e is associated with a quadrilateral, b_e, which is the union of the subregions K_e. Chatzipandelidis et al. have introduced *a priori* and *a posteriori* error estimates for the FV methods and for the stationary Stokes problem with the admission that FV scheme provides a variational formulation similar to the FE scheme [21]. These studies highlight the importance of *a posteriori* error estimates on a theoretical basis especially for parabolic problems such as the Stokes problem [12, 21].

We highlight the main finding from Karakatsani and Makridakis study for the FV scheme that is the following theorem,

Theorem 7 (Residual Based $L^2(H^1)$ and $L^\infty(H^1)$ Norm Error Estimates) *Let us assume that* $(\mathbf{u}, p)$ *is the solution of the time dependent Stokes problem and* $(\mathbf{u}_h, p_h)$ *is the finite volume approximation. The following a posteriori error bounds hold for,* $0 < t \leq T$,

$$\begin{aligned} \|\nabla(\mathbf{u}-\mathbf{u}_h)(t)\|_H \ \leq\ & \left\|\mathbf{u}_0 - \mathbf{u}_h^0\right\|_V + C\left(\int_0^t \eta_1(\mathbf{u}_{h,t}(s))^2 ds\right)^{1/2} \\ & + C\,\eta_1\,(\mathbf{u}_h(0)) + C\,\eta_1(\mathbf{u}_h(t)), \end{aligned} \tag{28}$$

Additional inequalities and the proof of this theorem can be found in [57].

Further, Larson and Malqvist derived a residual based *a posteriori* error estimates for parabolic problems on mixed form using Raviart-Thomas-Nedelec (RTN) finite elements in space and backward Euler in time [63]. In their study an *a posteriori* error estimate for the divergence of the flux in a weak norm is derived. The concept of elliptic reconstruction has been used to derive *a posteriori* error estimates for parabolic problems as briefly described before [57, 68]. In this framework, Larson and Malqvist use known *a posteriori* error estimates for the corresponding elliptic problem to derive error bounds for the parabolic problem [63]. However, the literature on FEM for parabolic problems on mixed form is less extensive and the development of the theory is still in progress [31, 90].

4 Advection–Diffusion Equation

The steady-state advection and diffusion of a scalar field is described by the partial differential equation (assuming homogeneous Dirichlet boundary condition),

$$\alpha \cdot \nabla u - \nabla \cdot (D \nabla u) = f \quad \text{in } \Omega, \tag{29}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{30}$$

where α is the velocity that the quantity, u, is moving with, which is considered to be divergent-free, $\nabla \cdot \alpha = 0$ [17]. For example, take as quantity the concentration of a chemical species that diffuses in a river while moving with its velocity α. The diffusion coefficient of the quantity is denoted with D and f represents sources or sinks.

The advection–diffusion problems are frequently treated as the point of departure for the study of the non-linear Navier–Stokes equations, at the level of developing discretization methods. The Peclet number, defined as the ratio of the advection and diffusion rates, $Pe = |a|h/D$, is a characteristic dimensionless number for such problems. A small Peclet number ($Pe \ll 1$) indicates diffusion-dominated flows while a large one ($Pe \gg 1$) indicates advection-dominated flows. In the diffusion-dominated regime, the standard Galerkin finite element method provides a good approximation of the solution [14].

The standard variational formulation arises by requesting the residual of Equation (29) to be orthogonal to a basis of the function space, H_0^1. The task is to find $u \in H_0^1(\Omega)$ such that

$$(\alpha \cdot \nabla u, v) - (\nabla \cdot (D \nabla u), v) = (f, v), \tag{31}$$

is satisfied for any test function $v \in H_0^1(\Omega)$. The Sobolev space, H_0^1, consists of functions that are one time weakly differentiable and also satisfy the zero Dirichlet boundary condition. In this respect, the second order term of the weak formulation can be integrated by parts, leading to,

$$(\alpha \cdot \nabla u, v) + (D \nabla u, \nabla v) = (f, v). \tag{32}$$

4.1 The Galerkin Formulation

To approximately solve Equation (32) using the Finite Element method, Ω is discretized in non-overlapping triangle element domains Ω_e with boundaries $\Gamma_e,\ e = 1, 2 \ldots K$, such that,

$$\Omega = \bigcup_{k=0}^{K} \overline{\Omega_k}.$$

The standard Galerkin formulation is retrieved by searching a solution in a finite-dimensional linear polynomial function space, $V_h \subset H_0^1(\Omega)$,

$$V_h = \{v_h \in H_0^1(\Omega) \mid v_h(\Omega_k) \in P_1(\Omega_k),\ \Omega_k \in \Omega\}$$

The problem now states, find $u_h \in V_h(\Omega)$ such that,

$$(\alpha \cdot \nabla u_h, v_h) + (D\nabla u_h, \nabla v_h) = (f, v_h),\ \ \forall\, v_h \in V_h(\Omega). \tag{33}$$

4.2 The Stabilized Finite Element Methods

It is well known that for advection-dominated flows, where the Peclet number is large, the solution involves non-physical oscillations [17]. To address the deficiency of the standard polynomial finite element method for advection-dominated flow problems, various approaches have been proposed, such as the streamline upwind Petrov–Galerkin (SUPG) method [18], Galerkin least squares (GLS) method [54], and the unusual stabilized FEM (USFEM) [38]. The common characteristic of the aforementioned methods is the introduction of artificial diffusion in the solution process while preserving the consistency of the discretization. Such methods are commonly referred to as stabilized finite element methods (SFEM).

The SFEM for the stationary advection–diffusion problem can be grouped as follows: find $u_h \in V_h(\Omega)$ such that,

$$B(u_h, v_h) = F(v_h),\ \ \forall\, v_h \in V_h(\Omega), \tag{34}$$

where,

$$B(u_h, v_h) = (\alpha \cdot \nabla u_h, v_h) + (D\nabla u_h, \nabla v_h) + Q(u_h, v_h), \tag{35}$$

$$F(v_h) = (f, v_h), \tag{36}$$

where $Q(u_h, v_h)$ indicates the additional terms added to the standard variational formulation. These are added to preserve consistency and enhance numerical stability. For instance, the stability term corresponding to the SUPG method is,

$$Q_{SUPG}(u_h, v_h) = \sum_K \tau_k(\alpha \cdot \nabla u_h - \nabla \cdot (k\nabla u_h) - f, \alpha \cdot \nabla u_h)_k, \tag{37}$$

where $(\cdot, \cdot)_k$ denotes element wise integration and τ_k is the stability coefficient for the SUPG method, as defined in [39],

$$\begin{cases} \tau_k = \dfrac{h_k}{2|\alpha|_p}\xi(Pe_k), \\ Pe_k = \dfrac{m_k|a|_p h_k}{2k}, \\ \xi(Pe_k) = \begin{cases} Pe_k, & 0 \leq Pe_k < 1 \\ 1, & Pe_k \geq 1 \end{cases} \\ |\alpha|_p = \left(\displaystyle\sum_{i=1}^{N}|\alpha_i|_p\right)^{\frac{1}{p}}, \ 1 \leq p < \infty, \\ m_k = \min\left\{\dfrac{1}{3},\ 2C_k\right\}, \\ C_k \displaystyle\sum_K h_k^2 \, \|\Delta v_h\|_{0,K}^2 \leqslant \|\nabla v_h\|_0^2, \ v_h \in V_h. \end{cases} \tag{38}$$

Accordingly, the stability terms added to the standard variational formulation for the GLS and the USFEM methods are,

$$Q_{GLS}(u_h, v_h) = \sum_K \tau_k(\alpha \cdot \nabla u_h - \nabla \cdot (D\nabla u_h) - f, \alpha \cdot \nabla u_h - \nabla \cdot (D\nabla u_h))_k, \tag{39}$$

$$Q_{USFEM}(u_h, v_h) = \sum_K \tau_k(\alpha \cdot \nabla u_h - \nabla \cdot (D\nabla u_h) - f, \alpha \cdot \nabla u_h + \nabla \cdot (D\nabla u_h))_k. \tag{40}$$

The stability of the SUPG method for transient convection–diffusion equations is studied in [13]. In the work by Onate [79], it was proven that the stabilization terms can be interpreted as a natural contribution to the governing differential equations of advection–diffusion problems. By considering the concept of flow equilibrium, the stabilization terms emerging in methods such as SUPG, subgrid scale (SS), GLS, Lax–Wendroff, characteristic Galerkin, Laplacian pressure operator, are not introduced as correction terms at the discretization level but rather derive naturally. For a comprehensive analysis of SFEM for the stationary or non-stationary advection-diffusion–reaction equation, the review by Codina [28] is recommended.

Writing the advection–diffusion equation in its first-order form via introduction of the flux of the scalar field as an additional unknown, is suited for many problems where higher accuracy of the flux is important, such as flow in porous media. Masud et al. studied the first-order form of the advection–diffusion equation in the framework of SFEM [74] .

Based on the partition of unity framework that is an instance of the generalized finite element method (GFEM), Turner et al. improved the performance of the Galerkin formulation designing enrichment functions using *a priori* knowledge about the qualitative behavior of solution to make better choices for the local approximation space [91]. The proposed method differs from the standard stabilization strategies as stability is not achieved by adding terms but by multiplying the polynomial with the enrichment functions.

4.3 The Variational Multiscale Method

Stabilized SUPG-type methods have several advantages, such as applicability to a wide range of problems and simplicity in computer implementations. However, spurious oscillations are often observed in regions around sharp layers even with the enhanced stability provided by the SUPG method [84]. To overcome the sharp features of the solution of advection-dominated problems, a higher resolution of the grid is usually employed which is, however, impractical in many cases. The multiscale approach comes at hand when fine scales cannot be captured by a given discretization in space [72].

The variational multiscale method was introduced in [49, 51] as a procedure for deriving numerical methods capable of dealing with multiscale phenomena that the straightforward application of the Galerkin's method with standard bases cannot address. It can be considered as a procedure to rebuild the error term in the weak form of the problem, yielding a stabilized form of the problem with higher accuracy on coarse grids.

The union of element interiors and element boundaries is denoted by Ω' and Γ', respectively,

$$\Omega' = \bigcup_{k=0}^{K} \Omega_k, \qquad \Gamma' = \bigcup_{k=0}^{K} \Gamma_k. \tag{41}$$

The appropriate function spaces for the coarse and the fine scale fields are introduced via a direct sum decomposition,

$$V = \overline{V} \oplus V', \tag{42}$$

where $\overline{V}$ is the space of trial and test functions for the coarse scale field,

$$\overline{V} = \{\overline{v} \in H_0^1(\Omega) \mid \overline{v}(\Omega_k) \in P_n(\Omega_k),\ \Omega_k \in \Omega\}, \tag{43}$$

where $P_n(\Omega_e)$ denotes polynomials of order n over the element interior.

In the discrete case, V' can contain various finite dimensional approximations such as bubble functions or p-refinements that further satisfy the assumption that the fine scales vanish identically over the element boundaries. Consequently,

$$V' = \left\{v' \mid v' = 0 \quad \text{on} \quad \Gamma'\right\}. \tag{44}$$

In this respect, the scalar field is decomposed in the coarse and fine scales denoted by $\overline{u}_h$ and u'_h, respectively,

$$u_h(x) = \overline{u_h}(x) + u'_h(x). \tag{45}$$

Likewise, the trial function is decomposed in its coarse and the fine scale components indicated as $\overline{v_h}$ and v'_h, respectively,

$$v_h(x) = \overline{v_h}(x) + v'_h(x). \tag{46}$$

Remark 2 Alternatively, the decomposition can be interpreted as the split of the solution in the part obtained on a given mesh and the part that is lost because its scale is smaller than the characteristic length of this mesh, representing the error in the solution.

The decomposed trial and test functions are substituted in the standard variational form (33), leading to,

$$(\alpha \cdot \nabla(\overline{u_h} + u'_h), (\overline{v_h} + v'_h)) + (k\nabla(\overline{u_h} + u'_h), \nabla(\overline{v_h} + v'_h)) = (f, (\overline{v_h} + v'_h)). \tag{47}$$

Employing the linearity of the weighting function, the problem can be split into the coarse and the fine scale parts, indicated as $\overline{v_h}$ and v'_h. The coarse scale sub-problem can be written as

$$(\alpha \cdot \nabla(\overline{u_h} + u'_h), \overline{v_h}) + (D\nabla(\overline{u}_h + u'_h), \nabla\overline{v_h}) = (f, \overline{v_h}). \tag{48}$$

The fine scale sub-problem can be written as

$$(\alpha \cdot \nabla(\overline{u_h} + u'_h), v'_h) + (D\nabla(\overline{u_h} + u'_h), \nabla v'_h) = (f, v'_h). \tag{49}$$

When compared with the standard Galerkin method, the multiscale approach involves additional integrals that are evaluated element wise. These additional terms represent the effects of the subgrid scales in terms of the residuals of the coarse scales of the problem. The architecture of the method is simple: u'_h is determined analytically and is eliminated from the $\overline{u_h}$ problem that is computed numerically. $\overline{u_h}$ and u_h may overlap or be disjoint, and u_h may be globally or locally defined, while the effect of u_h on the $\overline{u_h}$ problem is nonlocal [49].

Hughes et al. generalized the problem working in the context of an abstract Dirichlet problem involving a second-order differential operator which enables the study of equations of practical interest, such as the advection–diffusion equation [55, 56]. After introducing the variational formulation of the Dirichlet problem, the authors took advantage of the multiscale approach.

An overview of finite element approximations to deal with oscillations near layers using the variational multiscale formulation is presented in [29], where the time-discretization of the sub-grid scales is also addressed. Recently, Sendur et al.

used the pseudo residual-free bubbles (PRFB) method to achieve discretization in space and the fractional-step θ-scheme for the discretization in time [84] . The discontinuous enrichment method augments the polynomial field by free-space solutions of the homogeneous differential equation differentiating from the standard bubble methods in enforcing continuity across element boundaries by Lagrange multipliers [91].

4.4 The Discontinuous Galerkin (DG) Method

Another class of important methods is the discontinuous Galerkin (DG) methods that are popular in convection-dominated advection–diffusion problems due to their good stability and local conservation properties [27, 76]. The DG methods have several advantages such as high order accuracy, local data structure, and high parallelization capacity, attracting the interest of several groups [6, 22, 25, 40]. Moreover, the DG methods can cover meshes with hanging nodes and/or locally varying polynomial degrees rendering them ideally suited for *hp*-adaptivity. In contrast with the continuous approach, in the discontinuous context, the local elemental bases can be chosen freely due to the lack of inter-element continuity requirements, yielding sparse mass matrices [43].

For advection and (advection-dominated) advection–diffusion equations, *hp*-finite element approximations have been investigated by Houston, Schwab, and Suli for interior penalty discontinuous finite elements [47], leading to the so-called *hp*-streamline diffusion method and the *hp*-discontinuous Galerkin method [19, 20].

To capture detailed features of the solution near singular points or sharp layers, a very fine mesh is required. However, the computation of the solution is very challenging due to the amount of computer memory and time needed. For quicker convergence and to reduce the computational cost, the mesh can be refined locally at suitable locations. For stationary convection–diffusion equations, the quest for robust *a posteriori* error estimators that are independent of the Peclet number has advanced in various contexts [83, 93, 97]. For instance, *a posteriori* estimates using the reconstruction of the flux term can be found in [36]. In non-stationary convection–diffusion equations, as time progresses, the nature of the solution may vary throughout the domain rendering the use of adaptive algorithms an attractive proposition for the accurate and efficient numerical approximation of such problems. As adaptive algorithms are usually based on suitable *a posteriori* error estimators, robust estimation of the temporal and spatial error depends on their formulation. For non-stationary linear convection–diffusion equations, *a posteriori* error estimators have been developed for various discretizations [2, 35, 42, 92].

Chung and Enquist [23] in 2006 conceived the staggered DG (SDG) method that is a sub-class of the DG method. The introduction of the staggered mesh approach automatically satisfies the preservation of the physical laws arising from the corresponding partial differential equations. The SDG method can be continuous along some of the faces and discontinuous along other faces. An SDG scheme for the convection-diffusion equation was proposed by Chung and Lee [24] in 2012. Recently, an adaptive SDG method to solve the steady state convection–diffusion equation was presented by Du et al. [33]. The study by Cockburn et al. [27] is devoted to some new DG methods for convection–diffusion–reaction problems, called local discontinuous Galerkin–hybridizable (LDG–H) methods. Three novel features render these methods attractive. Namely, the first is that they are hybridizable and hence efficiently implementable, the second is that they provide approximations for the flux which are optimally convergent when both the flux and the scalar variables are approximated by polynomials of the same degree on each element. Finally, the third feature is that the approximations to the scalar variable super converge.

5 The Navier–Stokes Problem

In finite element formulation and computation of incompressible flows there are two main sources of instabilities associated with the classical Galerkin formulation of the Navier–Stokes problem. One source of instabilities is due to the presence of advection terms leading to spurious oscillations mainly in the velocity field, as discussed in the previous section. The other source of instability is due to an inappropriate combination of interpolation functions for the velocity and pressure field. These instabilities usually appear as oscillations primarily in the pressure field [89]. Below, we present the most interesting FE methodologies for solving the Navier–Stokes problem.

5.1 Streamline–Upwind/Petrov–Galerkin (SUPG)

The most popular stabilized method, the Streamline–Upwind/Petrov–Galerkin (SUPG) formulation, was introduced in 1979 for the incompressible Navier–Stokes equations [9, 50]. By augmenting the Galerkin formulation with residual-based terms, the SUPG formulation addressed the instability of the Galerkin technique for convection dominated flows, leading to a stable method with optimal convergence properties. For compressible flows, the SUPG formulation was initially introduced in 1982 [87], but a more thorough presentation of the method with additional examples published in [52]. The compressible flow SUPG formulation was initially introduced for conservation variables, and later for primitive variables. For more details on these developments, the interested reader is referred to a recent paper on stabilized methods for compressible flows [9].

The Pressure-Stabilizing/Petrov–Galerkin (PSPG) formulation for the Navier–Stokes equations of incompressible flows in the framework of residual-based methods was introduced in [86, 89]. This method allowed the use of equal-order interpolation functions for the velocity and pressure variables and assured numerical stability and optimal accuracy. An earlier version of the PSPG formulation for the Stokes problem was introduced in [53]. The SUPG and PSPG stabilizations were combined under a single name, the SUPS stabilization method [8, 9].

5.2 Variational Multiscale Method (VMS) and Stabilized FEM

Stabilized and multiscale formulations are among the most fundamental and important methodologies for finite element computations of complex fluid mechanics problems. Tezduyar et al. have proposed certain stabilized formulations with bilinear and linear equal-order-interpolation elements for the computation of dynamic and steady incompressible flows [89]. In their study, the stabilization procedure involves a modified Galerkin/least-squares formulation of the steady-state equations. The results from the considered test problems show that the $Q_1 - Q_1$ element is slightly less dissipative than the $P_1 - P_1$ element. The solutions obtained with these elements compare well with the solutions obtained from other studies [88]. The incompressible Navier–Stokes equations are written as

$$\begin{cases} \dot{\mathbf{v}} + \mathbf{v} \cdot \nabla \mathbf{v} - 2\,\nu\,\nabla \cdot \varepsilon(\mathbf{v}) + \nabla p = \mathbf{f}, \quad in\ \Omega \times [0, \mathrm{T}], \\ div\,\mathbf{v} = 0, \quad in\ \Omega \times [0, \mathrm{T}], \\ \mathbf{v} = \mathbf{g}, \quad on\ \Gamma_{\mathbf{g}} \times [0, \mathrm{T}], \\ \boldsymbol{\sigma} \cdot \mathbf{n} = (2\,\nu\,\nabla^s \mathbf{v} - p\mathbf{I}) \cdot \mathbf{n} = \mathbf{h}, \quad on\ \Gamma_{\mathbf{h}} \times [0, \mathrm{T}], \\ \mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), \quad on\ \Omega_0, \end{cases} \tag{50}$$

where $\mathbf{v}$ is the velocity vector, p is the kinematic pressure, $\mathbf{f}$ is the body force vector, ν is the kinematic viscosity, $\nabla^s \mathbf{v}$ is the symmetric part of the velocity gradient, $\mathbf{I}$ is the identity tensor, and $\varepsilon(\mathbf{v})$ is the strain rate tensor which is defined as $\varepsilon(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^{\mathrm{T}})$. Equation (50) represents the momentum and continuity equations, with the Dirichlet and Neumann boundary conditions, and the initial condition, respectively.

Discretizing the bounded domain Ω into non-overlapping regions Ω^e with boundaries Γ^e, $e = 1, 2, \ldots, n_{el}$, such that $\Omega = \cup_{e=1}^{n_{el}} \Omega^e$. The union of element interiors and element boundaries are $\Omega' = \bigcup_{e=1}^{n_{el}} (\mathrm{int})\Omega^e$ and $\Gamma' = \bigcup_{e=1}^{n_{el}} \Gamma^e$, respectively. In variational multiscale method (VMS) the velocity field is decomposed into the sum of the coarse or resolved scales and the fine or subgrid scales [70, 73],

$$\mathbf{v}(\mathbf{x}, t) = \bar{\mathbf{v}}(\mathbf{x}, t) + \mathbf{v}'(\mathbf{x}, t), \tag{51}$$

and the weighting function is decomposed in its coarse and the fine scale components indicated as $\bar{\mathbf{w}}(\mathbf{x})$ and $\mathbf{w}'(\mathbf{x})$, respectively,

$$\mathbf{w}(\mathbf{x}) = \bar{\mathbf{w}}(\mathbf{x}) + \mathbf{w}'(\mathbf{x}). \tag{52}$$

Remark 3 The main goal of the VMS method is to solve the fine-scale problem, defined over the sum of element interiors to obtain the fine scale solution. This solution is then substituted in the coarse-scale problem, eliminating the explicit appearance of the fine scales while still modeling their effects. Both coarse and fine scale equations are nonlinear equations due to the convection term, and to solve them a linearization is taking place [73].

The resulting equation is expressed in terms of the coarse scales and for the sake of simplicity the superposed bars are dropped. So, the VMS residual-based stabilized form for the incompressible Navier–Stokes equations is

$$\begin{aligned}
&(\mathbf{w}, \delta\mathbf{v}_t) + (\mathbf{w}, \delta\mathbf{v} \cdot \nabla\mathbf{v}^{(i)} + \mathbf{v}^{(i)} \cdot \nabla\delta\mathbf{v}) + \beta(\mathbf{w}, \mathbf{v}^{(i)}\nabla \cdot \delta\mathbf{v} + \delta\mathbf{v}\nabla \cdot \mathbf{v}^{(i)}) \\
&+(\nabla^S\mathbf{w}, 2\nu\,\nabla^S\delta\mathbf{v}) - (\nabla \cdot \mathbf{w}, \delta p) + (q, \nabla \cdot \delta\mathbf{v}) \\
&+(\mathbf{v}^{(i)} \cdot \nabla\mathbf{w} + 2\nu\,\Delta\mathbf{w} + \nabla q + (1-\beta)\mathbf{w}\nabla \cdot \mathbf{v}^{(i)}, \boldsymbol{\tau}\,\mathbf{r}_2) \\
&-(1-\beta)\,(\mathbf{w}, (\boldsymbol{\tau}\,\mathbf{r}_2) \cdot \nabla\mathbf{v}^{(i)}) + \beta((\boldsymbol{\tau}\,\mathbf{r}_2) \cdot \nabla\mathbf{w}, \mathbf{v}^{(i)}),
\end{aligned} \tag{53}$$

where the last two lines of the equation correspond to the stabilization terms, $\beta \in [0, 1]$, $\mathbf{r}_2$ is the residual from the linearization of the non-linear fine-scale problem, $\boldsymbol{\tau}$ is the fine-scale variational operator, and Δ is the vector Laplacian operator. A significant contribution of the VMS method is the systematic and consistent derivation of the fine-scale variational operator, $\boldsymbol{\tau}$, termed as the stabilization tensor that possesses the right order in the advective and diffusive limits, and variationally projects the fine-scale solution on the coarse-scale space [73]. The stabilization operator can be defined as [70],

$$\begin{aligned}
\boldsymbol{\tau} &= b^e \int b^e d\Omega \\
&\times \begin{bmatrix} \int (b^e)^2 \nabla^T \mathbf{v}^{(i)} d\Omega + \int b^e \mathbf{v}^{(i)} \cdot \nabla b^e d\Omega\,\mathbf{I} \\ +\beta \int b^e \mathbf{v}^{(i)} \otimes \nabla b^e d\Omega + \beta \int b^e (\nabla \cdot \mathbf{v}^{(i)})\, d\Omega\,\mathbf{I} \\ +\nu \int |\nabla b^e|^2\, d\Omega\,\mathbf{I} + \nu \int \nabla b^e \otimes \nabla b^e d\Omega \end{bmatrix}^{-1},
\end{aligned} \tag{54}$$

where $b^e(\xi)$ is a bubble function over Ω'. More details on the derivation and the obtained form of the VMS residual-based stabilized form and the fine-scale variational operator, τ, for the incompressible Navier–Stokes equations can be found in [70, 73]. Massud and collaborators have further extended the VMS methodology for shear-rate dependent non-Newtonian fluids and incompressible turbulent fluid flows [61, 71, 75].

5.3 Characteristic Based Split (CBS) Method and Two-Step Methods

The characteristic base split (CBS) method was first introduced by Zienkiewicz and Nithiarasu, 1995, in order to find a similar method to the Taylor–Galerkin, applicable in two or three dimensional problems. The algorithm is based on splitting the equations in two parts where the first would be a scalar convective-diffusion type of equations and the solution is derived from the characteristic Galerkin method [77, 98]. The second part constitutes of self-adjoined equations. There are four forms of the algorithm (fully explicit, semi-implicit, nearly implicit, fully implicit) depending on the problems we are called to solve. Here we focus only on the fully explicit and semi-implicit forms. Initially, we deal with the scalar convection–diffusion problem and the characteristic Galerkin explicit approximation. Assuming that the equation for this problem is

$$\frac{\partial V}{\partial t} = \frac{\partial F_i}{\partial x_i} + \frac{\partial G_i}{\partial x_i} + Q = 0, \tag{55}$$

where x_i is the i-th coordinate, F_i, G_i are the convected and the diffusion flux terms, respectively, and Q is the source term [98]. An alternative form of this equation is

$$\frac{\partial \phi}{\partial t} = -u_j \frac{\partial \phi}{\partial x_j} + \frac{\partial}{\partial x_i}\left(k \frac{\partial \phi}{\partial x_i}\right) - Q - \phi \frac{\partial u_j}{\partial x_j} = R(\phi). \tag{56}$$

The term, $-u_j \frac{\partial \phi}{\partial x_j}$, is not self-adjoined. Introducing a transformation we change the coordinate system, this term can be vanished and the equation will be a fully self-adjoined system. The stability condition for this problem is given as

$$\Delta t \le \Delta t_{crit} = \frac{h}{| u |}\left(\sqrt{\frac{1}{Pe^2} + \frac{1}{3}} - \frac{1}{Pe}\right), \tag{57}$$

where Pe is the Peclet number defined as $Pe = \frac{|u|h}{2k}$. For multidimensional problems such as the two-dimensional Navier-Stokes the critical time step will be

$$\Delta t_{crit} = \frac{\Delta t_\sigma \Delta t_\nu}{\Delta t_\sigma + \Delta t_\nu}, \tag{58}$$

where Δt_σ is given by Equation (57) and $\Delta t_\nu = h^2/2k$. If $\Delta t = \Delta t_{crit}$ the steady state solutions are almost identical to that from the optimal streamline upwind methods [98]. The Navier–Stokes problem can be written in a form of the convection–diffusion problem as

$$\frac{\partial \mathbf{V}}{\partial t} = \frac{\partial \mathbf{F}_i}{\partial x_i} + \frac{\partial \mathbf{G}_i}{\partial x_i} + \mathbf{Q} = 0. \tag{59}$$

Then the basic steps for this problem are highlighted below:

- Solve momentum equation without pressure terms,
- Calculate pressure from Poisson equation,
- Correct the velocities,
- Calculate additional terms as temperature, concentration, energy, etc. from the corresponding equations.

The equations after the standard Galerkin discretization are

$$\Delta\overline{\mathbf{U_i^*}} = -\mathbf{M}^{-1}\Delta t[(\mathbf{C}\overline{\mathbf{U}} + \mathbf{K}\overline{\mathbf{U}} - \mathbf{f}) - \Delta(\mathbf{K_u}\overline{\mathbf{U}} + \mathbf{f})]^n, \tag{60}$$

where $U_i = \mathbf{N}\overline{\mathbf{U_i}}$, $\Delta U_i = \mathbf{N}\Delta\overline{\mathbf{U_i}}$, $\Delta U_i^* = \mathbf{N}\Delta\overline{\mathbf{U_i}}^*$ and $p = \mathbf{N}_p\overline{\mathbf{p}}$ in the first step. This gives the solution for $\overline{U*}_i$.

We can solve the following equation to find $\Delta\overline{\mathbf{p}}$,

$$(\tilde{\mathbf{M}} + \Delta t^2\theta_1\theta_2\mathbf{H})\Delta\overline{\mathbf{p}} = \Delta t[\mathbf{Q}(\overline{\mathbf{U}}^{\mathbf{n}} + \theta_1\Delta\overline{\mathbf{U}^*}) - \Delta t\theta_1\mathbf{H}\overline{\mathbf{p}} - \mathbf{f}_p]^n, \tag{61}$$

where $\mathbf{H}$, $\tilde{\mathbf{M}}$, $\mathbf{Q}$ are matrices, and this is step two.

Further $\overline{\mathbf{U}}^{\mathbf{n+1}}$, $\overline{\mathbf{p}}^{\mathbf{n+1}}$ can be computed from

$$\Delta\overline{\mathbf{U}} = \Delta\overline{\mathbf{U}^*} - \mathbf{M}^{-1}\Delta t\left[\mathbf{Q}^T(\overline{\mathbf{p}}^{\mathbf{n}} + \theta_2\Delta\overline{\mathbf{p}}) + \frac{\Delta t}{2}\mathbf{P}\overline{\mathbf{p}}^{\mathbf{n}}\right], \tag{62}$$

and this is the third step of the process .

Finally, the last step is to solve the equation for the energy,

$$\Delta\overline{\mathbf{E}} = -\Delta t\left[\mathbf{C}\overline{\mathbf{E}}^{\mathbf{n}} + \mathbf{K}\mathbf{T}^{\mathbf{n}} + \mathbf{f}_e^n - \Delta t(\mathbf{K}_u\overline{\mathbf{E}}^{\mathbf{n}} + \mathbf{f}_e^n)\right]. \tag{63}$$

More details about the method and the terms used can be found in [77, 98]. For incompressible problems the algorithm can be used in the semi-implicit form. This form is conditionally stable if $\theta_1, \theta_2 \in \left[\frac{1}{2}, 1\right]$, where θ_1, θ_2 are variables coming from the discretization at steps two and three. For the fully explicit form we can set $\theta_2 = 0$ and θ_1 will be the same as in the semi-implicit form.

Another method based on splitting is the two step algorithm [48, 94]. The idea comes from the two level method where two different type of meshes are used, a coarse mesh for solving a nonlinear system and a fine mesh for the linear system. The two-step method is based on solving Navier–Stokes equation in two different ways but using the same computational mesh. The first step is solving a Navier–Stokes problem using a lower order element pair ($P_1 - P_1$) and the projection of the pressure onto a piecewise constant space. In step two a general Stokes problem is solved with a higher order elements ($P_2 - P_2$) using the projection of the pressure gradient onto the same space. The purpose of the first step is to find a prediction of the solution and step two is a correction for the initial approximation. The convergence for both the velocity and pressure is of order $O(h^2)$. Huang et al. compare the method with the $P_1 - P_1$ and $P_2 - P_2$ stabilized method, analyzed before. They report that the two-step method timewise is between the other two but the error is similar with the $P_2 - P_2$ stabilized method [48].

5.4 Gradient Smoothed Method (GSM)

The gradient smoothed method (GSM) was developed by combining the meshfree methods with the FEM approach [64, 65]. The main idea in the GSM is to use a finite element mesh to construct numerical models of good performance. Liu and collaborators introduced the GSM for the solution of steady-state and transient incompressible fluid flow problems [95]. The proposed method is based on irregular cells and thus can be used for problems with arbitrarily complex geometrical boundaries.

In the GSM, derivatives at various locations, such as at nodes, cell centroids, and cell–edges midpoints, are approximated using gradient smoothing operation over relevant gradient smoothing domains. For a two dimensional problem the gradients of a field variable u, at a point of interest, $\mathbf{x}_i$, in the domain Ω_i can be approximated in the form,

$$\nabla u(\mathbf{x}_i) \approx \int\limits_{\Omega_i} \nabla u(\mathbf{x})\, \bar{w}(\mathbf{x} - \mathbf{x}_i) dA, \tag{64}$$

where ∇ is the gradient operator and $\bar{w}$ is a smoothing function. For simplicity, the smoothing function can be set to be a piecewise constant over the smoothing domain. Integrating by parts or using Gauss divergence theorem and utilizing the properties of the smoothing function over the smoothing domain the following equations is obtained for the gradient,

$$\nabla u \approx \frac{1}{A_i} \oint_{\partial \Omega_i} u\, \bar{n}\, ds, \tag{65}$$

where $\bar{n}$ is the unit normal vector on $\partial \Omega_i$ and A_i is the area of the smoothing domain. Equation (65) provides a simple way to approximate gradients at a point by area-weighted integral along the boundary of a local smoothing domain, Ω_i. Similarly, by applying the gradient smoothing technique for the second-order derivatives the Laplace operator at a point of interest, $\mathbf{x}_i$, can be approximated as

$$\nabla \cdot (\Delta u_i) \approx \frac{1}{A_i} \oint_{\partial \Omega_i} \bar{n} \cdot \Delta u \, ds. \tag{66}$$

The spatial derivatives at any point of interest can be approximated over a smoothing domain that needs to be properly defined for a purpose, as presented above. The GSM can tackle the incompressible Navier–Stokes equations enhanced with artificial compressibility, in which the spatial derivatives are approximated by consistent and successive use of gradient smoothing operation over smoothing domains at various locations [66, 95, 96]. A favorable GSM scheme corresponding to a compact stencil with positive coefficients of influence has been derived in [95]. In this study, pseudo-time advancing approach is used for solving the governing equations with mixed hyperbolic–parabolic properties. The dual time stepping scheme and implicit five-stage Runge–Kutta method are implemented to enhance the efficiency and stability in the solution procedure. The obtained results show good agreement with literature [95].

5.5 Discontinuous and Adaptive Galerkin Method

In the last decades, discontinuous Galerkin (DG) methods form a class of numerical methods that combine features of the finite element and the finite volume framework, successfully applied to PDEs from a wide range of applications. An overview to DG method for elliptic problems and research directions can be found in [5, 26].

In order to use the equal order interpolation functions for velocity and pressure, the Navier–Stokes equations can be decoupled to distinct equations through the split method. The obtained equations are nonlinear hyperbolic, elliptic, and Helmholtz equations, respectively. The hybrid method combines DG and FE methods. Therefore, DG method is concerned to accomplish spatial discretization of the nonlinear hyperbolic equation to avoid using stabilization approaches in FEM. The split methods due to their decoupled schemes allows choosing equal order basis functions for velocity and pressure [32, 41, 44]. Marchandise and Remacle used an implicit pressure stabilized FEM to solve the Navier–Stokes equations, and DG method was employed to deal with the level-set equation [69]. They calculated the velocity and pressure in the coupled momentum equation together with adding stabilization terms for studying two-phase flows. Pandare and Luo proposed a coupled reconstructed discontinuous Galerkin (rDG) method and continuous Galerkin method for the solution of unsteady incompressible Navier–Stokes equations [80] .

In the paper by Gao et al., the main goal is to take full advantage of DG method and FEM on the basis of a split method [37, 58] to deal with the incompressible Navier–Stokes equations [41]. For the spatial discretization, they treat the nonlinear convection term through DG method, which can guarantee stability, accuracy and also avoid stabilization techniques used in FEM. Lomtev and Karniadakis in their study present a new DG method for simulating compressible viscous flows with shocks on standard unstructured grids [67]. This method is based on a discontinuous

Galerkin formulation both for the advective and the diffusive terms. High-order accuracy is achieved by using a recently developed hierarchical spectral basis. This basis is formed by combining Jacobi polynomials of high-order weights written in a new coordinate system. It retains a tensor-product property, and provides accurate numerical quadrature. Their formulation is conservative, and monotonicity is enforced by appropriately lowering the basis order and performing *hp*-refinement around discontinuities [67].

Bassi and Rebay introduce a high-order DG method for the numerical solution of the compressible flows [7]. The method combines two main ideas, the physics of wave propagation, accounted for by means of Riemann problems and accuracy being obtained by high-order polynomial approximations within elements. The method is suited to compute high-order accurate solution of the Navier–Stokes equations on unstructured grids. Klaij et al. in their study present a conservative arbitrary Lagrangian Eulerian (ALE) approach to deal with deforming meshes utilizing DG method for optimal flexibility on the local mesh refinement and adjustment of the polynomial order in each element (*hp*-adaptation) [60]. The numerical method allows for local grid adaptation as well as moving and deforming boundaries. Persson and colleagues introduced a method for computing time-dependent solutions to the compressible Navier–Stokes equations on variable geometries [81]. The transport equations are written as a conservation law for the independent variables in the reference configuration, the complexity introduced by variable geometry is reduced to solving a transformed conservation law in a fixed reference configuration. The spatial discretization is carried out using the DG method on unstructured meshes, while time integration is performed by a Runge–Kutta method. The problem under consideration was altered by adding an equation for the time evolution of the transformation Jacobian to the original conservation law and correcting for the accumulated metric errors. Results are discussed to present the capability of the approach to handle high-order approximations on complex geometries [81].

6 Conclusions

Finite element method (FEM) has gained substantial momentum in the last decades. FEM was initially introduced as an answer to solid mechanics problems while the progress in fluid mechanics was slower due to the non-linearities and instabilities in the solution. In this review we analyzed FEM providing the theoretical basis of the approach mainly focusing on parabolic type of problems, applied in fluid mechanics. Initially, we focused on basic FEM analysis for the Stokes problem. We further discussed FE approaches for the solution of the advection–diffusion equation such as the stabilized FEM, the variational multiscale method, and the discontinuous Galerkin method. Finally, we extended the analysis on the non-linear transport problems and discussed how FEM are utilized for the solution of the Navier–Stokes equations.

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