Graphic Contraction Principle and Applications

A. Petruşel and I. A. Rus

Abstract The purpose of this paper is to emphasize the role of the graphic contractions in metric fixed point theory. Two general results about the fixed points of graphic contractions and several related examples are given. The case of non-self graphic contractions will be also considered. Existence, uniqueness, data dependence, well-posedness, Ulam-Hyers stability, and the Ostrowski property for the fixed point equation will be discussed. Some fixed point results in metric spaces endowed with a partial ordering will be also proved. Finally, open questions and research directions are presented.

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1 Introduction

Let X be a nonempty set and $f : X \to X$ be an operator. We consider the fixed point equation

$$x = f(x), \ x \in X. \tag{1}$$

We denote by F_f the fixed point set of f, i.e., $F_f := \{x \in X \mid f(x) = x\}$.

In the same context, if $f(X_\lambda) \subset X_\lambda$ for all $\lambda \in \Lambda$ and $X = \bigcup_{\lambda \in \Lambda} X_\lambda$ is a partition of the space, then we say that $\bigcup_{\lambda \in \Lambda} X_\lambda$ is an invariant partition of X with respect to the operator f.

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If (X, d) is a metric space, then, by definition, f is a weakly Picard operator if

$$f^n(x) \to x^*(x) \in F_f \text{ as } n \to \infty, \text{ for all } x \in X.$$

Actually, the above definition generates the set retraction $f^\infty : X \to F_f$ given by $f^\infty(x) = \lim_{n\to\infty} f^n(x)$.

If we write $F_f = \{x^*\}$, then f has a unique fixed point and we denote it by x^*. A weakly Picard operator with a unique fixed point is, by definition, a Picard operator. We denote the attraction basin of a fixed point x^* of f by

$$(AB)_f(x^*) := \{x \in X |\ f^n(x) \to x^* \text{ as } n \to \infty\}.$$

We will also use the notation X_{x^*} for the above set.

A weakly Picard operator $f : X \to X$ for which there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and satisfying $\psi(0) = 0$, such that

$$d(x, f^\infty(x)) \leq \psi(d(x, f(x)), \text{ for all } x \in X,$$

is called a weakly ψ-Picard operator.

Moreover, a Picard operator for which there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and satisfying $\psi(0) = 0$, such that

$$d(x, x^*) \leq \psi(d(x, f(x)), \text{ for all } x \in X,$$

is called a ψ-Picard operator.

For more considerations on weakly Picard operator theory, see [4, 36, 44, 46–48].

Definition 1 Let (X, d) be a metric space and $f : X \to X$ be an operator. Then, f is called:

(i) a k-contraction if $k \in]0, 1[$ and

$$d(f(x), f(y)) \leq kd(x, y), \text{ for every } x, y \in X.$$

(ii) a graphic k-contraction if $k \in]0, 1[$ and

$$d(f(x), f^2(x)) \leq kd(x, f(x)), \text{ for every } x \in X.$$

To our best knowledge, the first fixed point theorem for graphic contractions on a closed subset of $\mathbb{R}^n$ was given by Rheinboldt [31]. In his paper [31], Rheinboldt named a mapping satisfying the above condition (ii) by the term of iterated contraction. See also [21] for related results and applications in $\mathbb{R}^n$.

In 1972, Rus proved in [33] (see also [48, p. 29]) that a graphic k-contraction on a complete metric space has at least one fixed point.

In 1974 a fixed point result for graphic k-contractions in the framework of a Banach space was given by Subrahmanyam in [49]. In his paper, Subrahmanyam used (see Definition 4 in [49]) the name of Banach operator of type k for the above concept. His result (see Corollary 2 in [49]) says that if $f : S \to S$ is a continuous Banach operator of type k (where S is a closed subset of a Banach space), then it has a fixed point.

Five years later, Hicks and Rhoades proved the following fixed point result in complete metric spaces.

Theorem 1 (See [11]) *Let (X, d) be a complete metric space and $k \in [0, 1[$. Suppose there exists $x \in X$ such that*

$$d(f(y), f^2(y)) \le kd(y, f(y)), \text{ for every } y \in O(x, \infty)$$
$$:= \{x, f(x), f^2(x), \cdots, f^n(x), \cdots\}.$$

Then:

(A) $\lim_{n\to\infty} f^n(x) = x^*$ *exists;*
(B) $d(f^n(x), x^*) \le \frac{k^n}{1-k} d(x, f(x))$*;*
(C) x^* *is a fixed point of* f *if and only if the functional* $G : X \to \mathbb{R}_+$ *given by* $G(x) := d(x, f(x))$ *is* f*-orbitally lower semi-continuous at* x^**, i.e., if* $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* $O(x, \infty)$ *and* $x_n \to x^*$*, then* $G(x^*) \le \liminf_{n\to\infty} G(x_n)$.

The purpose of this paper is to emphasize the role of the graphic contractions in metric fixed point theory. Two general results about the fixed points of graphic contractions and several related examples are given. The case of non-self graphic contractions will be also considered. Existence, uniqueness, data dependence, well-posedness, Ulam-Hyers stability, and the Ostrowski property for the fixed point equation will be discussed. Some fixed point results in metric spaces endowed with a partial ordering will be also proved. Finally, some open questions and research directions are presented.

More precisely, the structure of our paper is the following:

1. Introduction and preliminaries
2. Two general results on self graphic contractions
3. Relevant examples of graphic contractions
4. Some general results on non-self graphic contractions
5. Data dependence and stability results
6. Ran-Reurings type results for graphic contractions
7. Open questions and some new research directions

2 Two General Results on Graphic Contractions

We recall first a result concerning some equivalent statements in the theory of weakly Picard operators.

Theorem 2 ([34]) *Let X be a nonempty set and $f : X \to X$ be an operator. The following statements are equivalent:*

(1) $F_f = F_{f^n} \neq \emptyset$, *for all* $n \in \mathbb{N}^*$*;*
(2) There exists a metric d on X such that $f : (X, d) \to (X, d)$ is a weakly Picard operator;
(3) For each $k \in]0, 1[$ *there exists a complete metric d on X such that:*

(a) $d(f(x), f^2(x)) \leq kd(x, f(x))$, *for every* $x \in X$*.;*
(b) f is orbitally continuous on X;

(4) For each $k \in]0, 1[$ *there exist a complete metric d on X and a partition of the space* $X = \bigcup_{\lambda \in \Lambda} X_\lambda$*, such that:*

(a) $f(X_\lambda) \subset X_\lambda$*, for all* $\lambda \in \Lambda$*;*
(b) $f : X_\lambda \to X_\lambda$ *is a k-contraction, for all* $\lambda \in \Lambda$*;*
(c) $F_f \cap X_\lambda = \{x_\lambda^*\}$*, for all* $\lambda \in \Lambda$*.*

The following concepts will be important in our main results.

Definition 2 Let (X, d) be a metric space and $f : X \to X$ be an operator. Then:

(i) the fixed point equation (1) is called well-posed if $F_f = \{x^*\}$ and for any sequence $(x_n)_{n \in \mathbb{N}}$ in X for which

$$d(x_n, f(x_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$x_n \to x^* \text{ as } n \to \infty.$$

(ii) the operator f has the Ostrowski property if $F_f = \{x^*\}$ and for any sequence $(x_n)_{n \in \mathbb{N}}$ for which

$$d(x_{n+1}, f(x_n) \to 0 \text{ as } n \to \infty,$$

we have that

$$x_n \to x^* \text{ as } n \to \infty.$$

(iii) the fixed point equation (1) has the data dependence property if $F_f = \{x^*\}$ and for any operator $g : X \to X$ for which there exists $\eta > 0$ with

$$d(f(x), g(x)) \le \eta, \quad \text{for all } x \in X,$$

the following implication holds:

$$y^* \in F_g \text{ implies } d(x^*, y^*) \le \psi(\eta),$$

where the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous in zero and satisfies $\psi(0) = 0$.

In the case of $F_f \neq \emptyset$ and $r : X \to F_f$ a retraction, the above notions take the following form:

Definition 3

(iv) the fixed point equation (1) is called well-posed if for each $x^* in F_f$ and any sequence $(x_n)_{n\in\mathbb{N}}$ in $r^{-1}(x^*)$ for which

$$d(x_n, f(x_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$x_n \to x^* \text{ as } n \to \infty.$$

(v) the operator f has the Ostrowski property if for each $x^* in F_f$ and any sequence $(x_n)_{n\in\mathbb{N}}$ in $r^{-1}(x^*)$ for which

$$d(x_{n+1}, f(x_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$x_n \to x^* \text{ as } n \to \infty.$$

(vi) the fixed point equation (1) has the data dependence property if for any operator $g : X \to X$ for which there exists $\eta > 0$ with

$$d(f(x), g(x)) \le \eta, \quad \text{for all } x \in X,$$

there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in zero and satisfying $\psi(0) = 0$, such that

$$H_d(F_f, F_g) \le \psi(\eta).$$

For the above notions, see [3, 4, 41, 44, 45, 47].

The following result is well-known.

Lemma 1 (Cauchy-Toeplitz Lemma, See [21, 47, 48]) *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_+$, such that the series $\sum_{n\geq 0} a_n$ is convergent and $(b_n)_{n\in\mathbb{N}} \in \mathbb{R}_+$ be a sequence such that $\lim_{n\to\infty} b_n = 0$. Then*

$$\lim_{n\to\infty}(\sum_{k=0}^{n} a_{n-k}b_k) = 0.$$

We will present now the saturated principle of graphic contractions.

Theorem 3 ([48]) *Let (X, d) be a complete metric space and $f : X \to X$ be a graphic k-contraction. Then:*

(1) the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges in (X, d) and $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < \infty$, for each $x \in X$;

If, in addition, $\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x))$ for each $x \in X$, then we have the following conclusions:

(2) $F_f = F_{f^n} \neq \emptyset$, for all $n \in \mathbb{N}^*$;

(3) f is a weakly Picard operator and $X = \bigcup_{x^*\in F_f} X_{x^*}$ is an invariant partition of X;

(4) $d(x, f^\infty(x)) \leq \frac{1}{1-k} d(x, f(x))$, for every $x \in X$;

(5) the fixed point equation (1) is well-posed;

(6) if $k < \frac{1}{3}$, then $d(f(x), f^\infty(x)) \leq \frac{k}{1-2k} d(x, f^\infty(x))$, for every $x \in X$;

(7) if $k < \frac{1}{3}$, then f has the Ostrowski property;

3 Relevant Examples of Graphic Contractions

We will present now several examples of graphic contractions.

Example 1 Let $f \in C([a, b] \times \mathbb{R}^m, \mathbb{R}^m)$. Consider the following system of first order differential equations

$$x'(t) = f(t, x(t)). \tag{2}$$

We are looking for solutions $x \in C^1([a, b], \mathbb{R}^m)$ of the above system.

In the above conditions, it is easy to see that (2) is equivalent with the following system of functional-integral equations

$$x(t) = x(a) + \int_a^t f(s, x(s))ds, \tag{3}$$

where $x \in C([a, b], \mathbb{R}^m)$.

We denote

$$X_\lambda := \{x \in C([a,b],\mathbb{R}^m) \,|\, x(a) = \lambda\}$$

and define

$$T : C([a,b],\mathbb{R}^m) \to C([a,b],\mathbb{R}^m) \text{ by } Tx(t) := x(a) + \int_a^t f(s,x(s))ds.$$

Then the following conclusions hold:

(i) $C([a,b],\mathbb{R}^m) = \bigcup_{\lambda\in\mathbb{R}^m} X_\lambda$ is a partition;
(ii) $T(X_\lambda) \subset X_\lambda$, for every $\lambda \in \mathbb{R}^m$;
(iii) if $f(t,\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is L-Lipschitz, then $T\Big|_{X_\lambda}$ is a contraction and we have that $X_\lambda \cap F_T = \{x_\lambda^*\}$;
(iv) the operator $T : C([a,b],\mathbb{R}^m) \to C([a,b],\mathbb{R}^m)$ is a graphic contraction with respect to a suitable Bielecki type norm;
(v) the operator $T : C([a,b],\mathbb{R}^m) \to C([a,b],\mathbb{R}^m)$ has no isolated fixed points.

Example 2 Let $f \in C([a,b]\times\mathbb{R}^m\times\mathbb{R}^m,\mathbb{R}^m)$. Consider the following system of second order differential equations

$$-x''(t) = f(t,x(t),x'(t)). \tag{4}$$

We are looking for solutions $x \in C^2([a,b],\mathbb{R}^m)$ of the above system.

In the above conditions, it is easy to see that (4) is equivalent with each of the following system of functional-integral equations

$$x(t) = x(a) + \frac{x'(a)}{2}(t-a) + \int_a^t (t-s)f(s,x(s)x'(s))ds,\ t\in[a,b], \tag{5}$$

where we are looking for solutions $x \in C^1([a,b],\mathbb{R}^m)$,

and

$$x(t) = \frac{t-a}{b-a}x(b) + \frac{b-t}{b-a}x(a) + \int_a^b G(t,s)f(s,x(s)x'(s))ds,\ t\in[a,b], \tag{6}$$

where G denotes the usual Green function corresponding to (4) and we are looking for solutions $x \in C^1([a,b],\mathbb{R}^m)$.

We denote

$$X_{u,v} := \{x \in C^1([a,b],\mathbb{R}^m) \,|\, x(a) = u,\ x'(a) = v\}$$

and define

$$T : C^1([a,b],\mathbb{R}^m) \to C^1([a,b],\mathbb{R}^m),$$

by

$$Tx(t) := x(a) + \frac{x'(a)}{2}(t-a) + \int_a^t (t-s) f(s, x(s)x'(s))ds,$$

respectively

$$\tilde{X}_{u,v} := \{x \in C^1([a,b], \mathbb{R}^m) |\ x(a) = u,\ x(b) = v\}$$

and

$$S : C^1([a,b], \mathbb{R}^m) \to C^1([a,b], \mathbb{R}^m),$$

by

$$Sx(t) := \frac{t-a}{b-a}x(b) + \frac{b-t}{b-a}x(a) + \int_a^b G(t,s) f(s, x(s)x'(s))ds.$$

Then the following conclusions hold:

(i) $C^1([a,b], \mathbb{R}^m) = \bigcup_{u,v \in \mathbb{R}^m} X_{u,v} = \bigcup_{u,v \in \mathbb{R}^m} \tilde{X}_{u,v}$ are partitions;
(ii) $T(X_{u,v}) \subset X_{u,v}$ and $S(\tilde{X}_{u,v}) \subset \tilde{X}_{u,v}$, for every $u, v \in \mathbb{R}^m$;
(iii) if $f(t, \cdot, \cdot) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is L-Lipschitz, then:
    (a) T is a graphic contraction with respect to a suitable Bielecki norm;
    (b) if $(b-a)$ is sufficiently small, then S is graphic contraction with respect to the $\|\cdot\|_\infty$-norm;

Example 3 In a metric space (X, d) several generalized contractions, such as:

(a) Kannan mappings
(b) Ćirić-Reich-Rus mappings
(c) Chatterjea mappings
(d) Zamfirescu mappings
(e) Hardy-Rogers mappings
(f) Berinde mappings
(g) Suzuki mappings, etc.

are graphic contractions, see [1, 10, 12, 14, 15, 32, 35, 48].

If additionally the space is complete, then, by the theorems proved by the above authors, we obtain various conclusions about the fixed point set, such as $F_f = \{x^*\}$ or $F_f \neq \emptyset$;

Example 4 Let $(X, +, \mathbb{R}, \|\cdot\|)$ be a normed space and $\Phi : X \to \mathbb{R}$ be a nontrivial linear functional. For $\lambda \in \mathbb{R}$ we consider

$$X_\lambda := \{x \in X \mid \Phi(x) = \lambda\}.$$

Then $X = \bigcup_{\lambda \in \mathbb{R}} X_\lambda$ is a partition of the space X, which will be called the partition of X corresponding to Φ. Moreover, if Φ is continuous, then X_λ is a closed set in X.

Let X be a Banach space, $\Phi : X \to \mathbb{R}$ be a nontrivial linear continuous functional and $A : X \to X$ be a bounded linear operator. We say that Φ is an invariant functional of A if

$$\Phi(A(x)) = \Phi(x), \text{ for all } x \in X.$$

In the above conditions, if we suppose:

(i) Φ is an invariant functional of A;
(ii) $k := \|A|_{X_0}\| < 1$,

then A is a graphic k-contraction.

Example 5 Consider on the Banach space $(C[0, 1], \|\cdot\|_\infty)$ the Berstein operator

$$B_n(x)(t) := \sum_{k=0}^{n} f(\frac{k}{n}) \binom{k}{n} x^k (1-x)^{n-k}, \ n \in \mathbb{N}^*$$

Then, B_n is a graphic $(1 - \frac{1}{2^{n-1}})$-contraction.

Example 6 In a metric space (X, d) the identity $f = 1_{|X}$ is a graphic contraction. In this case $F_f = X$.

Example 7 Let $f : [-1, 1] \to [-1, 1]$ be defined by

$$f(x) := \begin{cases} \frac{x}{2}, & x \neq 0 \\ \frac{1}{2}, & x = 0, \end{cases}$$

Then f is a discontinuous graphic $\frac{1}{2}$-contraction, $f^n(x) \to 0$ as $n \to \infty$, for every $x \in [-1, 1]$ and $F_f = \emptyset$.

Example 8 Let $f : [0, 1] \to [0, 1]$ be defined by

$$f(x) := \begin{cases} 0, & x \in [0, \frac{1}{2}[\\ 1, & x \in [\frac{1}{2}, 1], \end{cases}$$

Then f is a discontinuous graphic k-contraction (with any $k \in]0, 1[$) and $F_f = \{0, 1\}$. Moreover, $f^n(x) \to 0$ as $n \to \infty$, for every $x \in [0, \frac{1}{2}[$ and $f^n(x) \to 1$ as $n \to \infty$, for every $x \in [\frac{1}{2}, 1]$.

For other examples of graphic contractions, see [1, 21, 23, 24, 38, 40, 44, 46, 48].

4 Some General Results on Non-self Graphic Contractions

We will present now a general result concerning the fixed point equation (1) for the case of a non-self graphic contraction.

Theorem 4 *Let (X, d) be a complete metric space, $k \in]0, 1[$, $x_0 \in X$, $R > 0$ and $f : \tilde{B}(x_0, R) \to X$ be an operator. We suppose that the following assumptions take place:*

(i) $d(x_0, f(x_0)) \leq (1 - k)R$;
(ii) *if* $x, f(x) \in \tilde{B}(x_0, R)$, *then* $d(f(x), f^2(x)) \leq kd(x, f(x))$;
(iii) *if* $x \in \tilde{B}(x_0, R)$ *has the property that* $f^n(x) \in \tilde{B}(x_0, R)$ *for every* $n \in \mathbb{N}^*$ *and the sequence* $(f^n(x))_{n\in\mathbb{N}}$ *is convergent, then*

$$\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x)).$$

In the above conditions, we have the following conclusions:

(a) $f^n(x_0) \in \tilde{B}(x_0, R)$ *for every* $n \in \mathbb{N}$ *and* $f^n(x_0) \to f^\infty(x_0) \in F_f$ *as* $n \to \infty$.
(b) *if for some* $y \in \tilde{B}(x_0, R)$ *we have that* $f^n(y) \in \tilde{B}(x_0, R)$ *for every* $n \in \mathbb{N}^*$, *then* $f^n(y) \to f^\infty(y) \in F_f$ *as* $n \to \infty$.
(c) *if* $y_0 \in \tilde{B}(x_0, R)$ *is such that* $d(x_0, y_0) \leq \eta_1 R$ *and* $d(x_0, f(y_0)) \leq \eta_2 R$ *with* $\eta_1 + \eta_2 \leq 1 - k$, *then* $f^n(y_0) \in \tilde{B}(x_0, R)$ *for every* $n \in \mathbb{N}^*$ *and* $f^n(y_0) \to f^\infty(y_0) \in F_f$ *as* $n \to \infty$.
(d) *if* $x^* \in F_f$ *and* $x \in (AB)_f(x^*)$, *then* $d(x, x^*) \leq \frac{1}{1-k} d(x, f(x))$.
(e) *if* $x^* \in F_f$, $y_n \in (AB)_f(x^*)$ *for every* $n \in \mathbb{N}$ *is such that* $d(y_n, f(y_n)) \to 0$ *as* $n \to \infty$, *then* $y_n \to x^*$ *as* $n \to \infty$.
(f) *if* $k < \frac{1}{3}$ *and* $x^* \in F_f$, *then* $d(f(x), x^*) \leq \frac{k}{1-2k} d(x, x^*)$, *for every* $x \in (AB)_f(x^*)$.
(g) *if* $k < \frac{1}{3}$, $x^* \in F_f$ *and* $y_n \in (AB)_f(x^*)$ *for every* $n \in \mathbb{N}$ *is such that* $d(y_{n+1}, f(y_n)) \to 0$ *as* $n \to \infty$, *then* $y_n \to x^*$ *as* $n \to \infty$.

Proof

(a) Let $x_0 \in X$ such that $d(x_0, f(x_0)) \leq (1 - k)R$. Then

$$d(x_0, f^2(x_0)) \leq d(x_0, f(x_0)) + d(f(x_0), f^2(x_0)) \leq$$

$$d(x_0, f(x_0)) + kd(x_0, f(x_0)) = (1 + k)d(x_0, f(x_0)) \leq (1 - k^2)R.$$

Hence $f^2(x_0) \in \tilde{B}(x_0, R)$. By mathematical induction, we obtain that

$$d(x_0, f^n(x_0)) \leq (1 - k^n)R, \text{ for every } n \in \mathbb{N}, n \geq 2.$$

Thus, $f^n(x_0) \in \tilde{B}(x_0, R)$, for every $n \in \mathbb{N}^*$. Denote $x_n := f^n(x_0)$ and observe that, for every $n \in \mathbb{N}$, we have

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n) \leq \cdots \leq k^n d(x_0, f(x_0)).$$

By a standard argument, we get that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy and converges, in (X, d), to an element of $\tilde{B}(x_0, R)$. By (iii) this element is a fixed point of f and we denote it by $f^\infty(x_0)$.

(b) Let $y \in \tilde{B}(x_0, R)$ such that $f^n(y) \in \tilde{B}(x_0, R)$, for every $n \in \mathbb{N}^*$. Then, in a similar way to the above proof, by (ii) and (iii), we get that the sequence $(f^n(x))_{n\in\mathbb{N}}$ is Cauchy and it converges to a fixed point $f^\infty(y) \in \tilde{B}(x_0, R)$ of f.

(c) Let $y_0 \in \tilde{B}(x_0, R)$ be such that $d(x_0, y_0) \leq \eta_1 R$ and $d(x_0.f(y_0)) \leq \eta_2 R$, where $\eta_1 + \eta_2 \leq 1 - k$. Then, since $f(y_0) \in \tilde{B}(x_0, R)$, we have

$$d(x_0, f^2(y_0)) \leq d(x_0, f(y_0)) + d(f(y_0), f^2(y_0)) \leq \eta_2 R + kd(y_0, f(y_0)) \leq$$

$$\eta_2 R + k\left(d(y_0, x_0) + d(x_0, f(y_0))\right) \leq \big(k(\eta_1 + \eta_2) + \eta_2\big) R \leq (1 - k^2)R.$$

Moreover,

$$d(x_0, f^3(y_0)) \leq d(x_0, f^2(y_0)) + d(f^2(y_0), f^3(y_0)) \leq$$

$$\big(k(\eta_1 + \eta_2) + \eta_2\big) R + k^2 d(y_0, f(y_0)) \leq \Big[(k + k^2)(\eta_1 + \eta_2) + \eta_2\Big] R \leq (1 - k^3)R.$$

By mathematical induction, we obtain

$$d(x_0, f^n(y_0)) \leq (1 - k^n)R, \text{ for every } n \in \mathbb{N}^*,$$

showing that $f^n(y_0) \in \tilde{B}(x_0, R)$ for every $n \in \mathbb{N}^*$. Then, in a similar way to the proof of (a) and (b), using (ii) and (iii), we get that the sequence $(f^n(y_0))_{n\in\mathbb{N}}$ is Cauchy and it converges to a fixed point $f^\infty(y_0) \in \tilde{B}(x_0, R)$ of f.

(d) Let $x^* \in F_f$ and $x \in (AB)_f(x^*)$. Then

$$d(x, x^*) \leq d(x, f(x)) + d(f(x), f^2(x)) + \cdots + d(f^n(x), x^*) \leq$$

$$(1 + k + \ldots k^{n-1}) d(x, f(x)) + d(f^n(x), x^*) \leq \frac{1}{1-k} d(x, f(x)) + d(f^n(x), x^*).$$

Letting $n \to \infty$ we obtain the conclusion.

(e) Let $x^* \in F_f$ and $y_n \in (AB)_f(x^*)$ for every $n \in \mathbb{N}$ be such that $d(y_n, f(y_n)) \to 0$ as $n \to \infty$. Then

$$d(y_n, x^*) \leq d(y_n, f(y_n)) + \cdots + d(f^{n-1}(y_n), f^n(y_n)) + d(f^n(y_n), x^*) \leq$$

$$(1 + k + \cdots + k^{n-1}) d(y_n, f(y_n)) + d(f^n(y_n), x^*) \leq$$

$$\frac{1}{1-k} d(y_n, f(y_n)) + d(f^n(y_n), x^*).$$

Letting $n \to \infty$ we obtain the conclusion.

(f) Suppose that $k < \frac{1}{3}$. Let $x^* \in F_f$ and $x \in (AB)_f(x^*)$. Then

$$d(f(x), x^*) \leq d(f(x), f^2(x)) + \cdots + d(f^{n-1}(x), f^n(x)) + d(f^n(x), x^*) \leq$$

$$(k+k^2+\cdots k^{n-1})d(x, f(x))+d(f^n(x), x^*) \leq \frac{k}{1-k}d(x, f(x))+d(f^n(x), x^*).$$

Using the triangle inequality, we obtain that

$$d(f(x), x^*) \leq \frac{k}{1-2k}d(x, x^*) + \frac{1-k}{1-2k}d(f^n(x), x^*).$$

The conclusion follows letting $n \to \infty$.

(g) Suppose that $k < \frac{1}{3}$. Let $x^* \in F_f$ and $y_n \in (AB)_f(x^*)$ for every $n \in \mathbb{N}$ be such that $d(y_{n+1}, f(y_n)) \to 0$ as $n \to \infty$. Then, we have

$$d(y_{n+1}, x^*) \leq d(y_{n+1}, f(y_n)) + d(f(y_n), f^2(y_n)) + \cdots + d(f^n(y_n), x^*) \leq$$

$$d(y_{n+1}, f(y_n)) + kd(y_n, f(y_n)) + \cdots + k^{n-1}d(y_n, f(y_n)) + d(f^n(y_n), x^*) \leq$$

$$d(y_{n+1}, f(y_n)) + \frac{k}{1-k}d(y_n, f(y_n)) + d(f^n(y_n), x^*) \leq$$

$$d(y_{n+1}, f(y_n)) + \frac{k}{1-k}\left[d(y_n, x^*) + d(x^*, y_{n+1}) + d(y_{n+1}, f(y_n))\right] +$$

$$d(f^n(y_n), x^*) = \frac{1}{1-k}d(y_{n+1}, f(y_n)) + \frac{k}{1-k}d(y_n, x^*) +$$

$$\frac{k}{1-k}d(x^*, y_{n+1}) + d(f^n(y_n), x^*).$$

Thus

$$d(y_{n+1}, x^*) \leq \frac{1}{1-2k}d(y_{n+1}, f(y_n)) + \frac{k}{1-2k}d(y_n, x^*) + \frac{1-k}{1-2k}d(f^n(y_n), x^*).$$

We denote $\alpha := \frac{1}{1-2k}$, $\beta := \frac{k}{1-2k}$ and $\gamma := \frac{1-k}{1-2k}$. Notice that $\alpha, \gamma > 1$ and $\beta \in]0, 1[$.

Then we get

$$d(y_{n+1}, x^*) \leq \alpha d(y_{n+1}, f(y_n)) + \beta d(y_n, x^*) + \gamma d(f^n(y_n), x^*) \leq$$

$$\alpha \left[d(y_{n+1}, f(y_n)) + \beta d(y_n, f(y_{n-1}))\right] +$$

$$\gamma\left[d(f^n(y_n),x^*)+\beta d(f^{n-1}(y_{n-1}),x^*)\right]+$$

$$\beta^2 d(y_{n-1},x^*)\leq\cdots\leq$$

$$\alpha\left[d(y_{n+1},f(y_n))+\beta d(y_n,f(y_{n-1}))+\cdots\beta^n d(y_1,f(y_0))\right]+$$

$$\gamma\left[d(f^n(y_n),x^*)+\beta d(f^{n-1}(y_{n-1}),x^*)+\cdots+\beta^n d(y_0,x^*)\right]+\beta^n d(y_0,x^*).$$

Letting $n\to\infty$ and using Cauchy-Toeplitz Lemma (see Lemma 1), we obtain the desired conclusion.

A second general result for non-self contractions is the following.

Theorem 5 *Let (X,d) be a complete metric space, $k\in]0,1[$, $x_0\in X$, $R>0$ and $f:\tilde{B}(x_0,R)\to X$ be an operator. We suppose that the following assumptions take place:*

(i) $d(x_0,f(x_0))\leq(1-k)R$;
(ii) *if* $x, f(x)\in\tilde{B}(x_0,R)$, *then* $d(f(x),f^2(x))\leq kd(x,f(x))$;
(iii) *if* $x\in\tilde{B}(x_0,R)$ *has the property that* $f^n(x)\in\tilde{B}(x_0,R)$ *for every* $n\in\mathbb{N}^*$ *and the sequence* $(f^n(x))_{n\in\mathbb{N}}$ *is convergent, then*

$$\lim_{n\to\infty} f(f^n(x))=f(\lim_{n\to\infty} f^n(x)).$$

(iv) $card(F_f)\leq 1$.

In the above conditions, we have the following conclusions:

(a) $F_f=\{x^*\}$ *and* $f^n(x_0)\to x^*$ *as* $n\to\infty$.
(b) *if for some* $y\in\tilde{B}(x_0,R)$ *we have that* $f^n(y)\in\tilde{B}(x_0,R)$ *for every* $n\in\mathbb{N}^*$, *then* $(f^n(y))\to x^*$ *as* $n\to\infty$.
(c) *if* $y_0\in\tilde{B}(x_0,R)$ *is such that* $d(x_0,y_0)\leq\eta_1 R$ *and* $d(x_0,f(y_0))\leq\eta_2 R$ *with* $\eta_1+\eta_2\leq 1-k$, *then* $f^n(y_0)\in\tilde{B}(x_0,R)$ *for every* $n\in\mathbb{N}^*$ *and* $(f^n(y_0))\to x^*$ *as* $n\to\infty$.
(d) $d(x,x^*)\leq\frac{1}{1-k}d(x,f(x))$, *for every* $x\in(AB)_f(x^*)$.
(e) *if* $y_n\in(AB)_f(x^*)$ *for every* $n\in\mathbb{N}$ *is such that* $d(y_n,f(y_n))\to 0$ *as* $n\to\infty$, *then* $y_n\to x^*$ *as* $n\to\infty$.
(f) *if* $k<\frac{1}{3}$, *then* $d(f(x),x^*)\leq\frac{k}{1-2k}d(x,x^*)$, *for every* $x\in(AB)_f(x^*)$.
(g) *if* $k<\frac{1}{3}$ *and* $y_n\in(AB)_f(x^*)$ *for every* $n\in\mathbb{N}$ *is such that* $d(y_{n+1},f(y_n))\to 0$ *as* $n\to\infty$, *then* $y_n\to x^*$ *as* $n\to\infty$.

Proof The conclusions follow by Theorem 4 and the hypothesis (iv).

For some general considerations on fixed point theory for nonself contractions, see [1–3, 6, 9, 17, 19, 21, 48].

5 Data Dependence and Stability Results for Graphic Contractions

In this section we will present some continuous data dependence and stability results for the fixed point equation (1) governed by a graphic contraction.

We will start with the data dependence problem. More precisely, let (X, d) be a metric space, $x_0 \in X$, $R > 0$ and $f, g : \tilde{B}(x_0, R) \to X$ are two operators. We suppose that the following assumptions take place:

(i) $F_f \neq \emptyset$ and, for $x^* \in F_f$ we have that $(AB)_f(x^*) \neq \{x^*\}$.
(ii) there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, for each $x \in \tilde{B}(x_0, R)$.

The problem is in which conditions there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in zero and satisfying $\psi(0) = 0$, such that

$$d(y^*, x^*) \leq \psi(\eta), \text{ for every } x^* \in F_f \text{ and every } y^* \in F_g \cap (AB)_f(x^*).$$

In this context, we have the following result.

Theorem 6 *Let (X, d) be a complete metric space, $x_0 \in X$, $R > 0$ and $f, g : \tilde{B}(x_0, R) \to X$ be two operators. We suppose that the following assumptions take place:*

(i) f satisfies all the assumptions in Theorem 4.
(ii) there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, for each $x \in \tilde{B}(x_0, R)$.

Then

$$d(x^*, y^*) \leq \frac{\eta}{1-k}, \text{ for every } x^* \in F_f \cap \tilde{B}(x_0, R) \text{ and every } y^* \in F_g \cap (AB)_f(x^*).$$

Proof By Theorem 4 there exists $x^* \in F_f \cap \tilde{B}(x_0, R)$. Suppose $F_g \cap (AB)_f(x^*)$ is nonempty (otherwise, we have nothing to prove). Then, we have

$$d(x^*, y^*) = d(f(x^*), g(y^*)) \leq d(f(x^*), f(y^*)) +$$

$$d(f(y^*), g(y^*)) \leq d(x^*, f(y^*)) + \eta \leq$$

$$d(x^*, f^n(y^*)) + d(f^n(y^*), f^{n-1}(y^*)) + \cdots + d(f^2(y^*), f(y^*)) + \eta \leq$$

$$d(x^*, f^n(y^*)) + k^{n-1} d(f(y^*), y^*) + \cdots + kd(f(y^*), y^*) + \eta \leq$$

$$d(x^*, f^n(y^*)) + (k^{n-1} + \cdots + k + 1)\eta \leq d(x^*, f^n(y^*)) + \frac{\eta}{1-k}.$$

The conclusion follows letting $n \to \infty$.

For some general considerations on data dependence problem, see [4, 6–9, 13, 18–20, 23, 26, 30, 35, 39, 43, 47, 48].

We will consider now the Ulam-Hyers stability of the fixed point equation (1).

Definition 4 Let (X, d) be a complete metric space, $x_0 \in X$, $R > 0$ and $f : \tilde{B}(x_0, R) \to X$ be an operator. Then, the fixed point equation (1) is Ulam-Hyers stable if there exists $c > 0$ such that for every $\varepsilon > 0$ and every ε-solution y^* of the fixed point equation (1), i.e.,

$$d(y^*, f(y^*)) \leq \varepsilon,$$

there exists a solution $x^* \in \tilde{B}(x_0, R)$ of the fixed point equation (1) such that

$$d(x^*, y^*) \leq c\varepsilon.$$

For the above problem, we have the following result.

Theorem 7 *Let (X, d) be a complete metric space, $x_0 \in X$, $R > 0$ and $f : \tilde{B}(x_0, R) \to X$ be an operator. We suppose that the following assumptions take place:*

(i) f satisfies all the assumptions in Theorem 4.
(ii) let $\varepsilon > 0$ and let $y^* \in \tilde{B}(x_0, R)$ be such that

$$d(y^*, f(y^*)) \leq \varepsilon \text{ and } y^* \in (AB)_f(x^*).$$

Then

$$d(x^*, y^*) \leq \frac{\varepsilon}{1-k}.$$

Proof By Theorem 4 there exists $x^* \in F_f \cap \tilde{B}(x_0, R)$. Let $\varepsilon > 0$ and let $y^* \in \tilde{B}(x_0, R)$ be such that

$$d(y^*, f(y^*)) \leq \varepsilon \text{ and } y^* \in (AB)_f(x^*).$$

Then, we have

$$d(x^*, y^*) \leq d(x^*, f^n(y^*)) + d(f^n(y^*), f^{n-1}(y^*)) + \cdots + d(f(y^*), y^*) \leq$$

$$d(x^*, f^n(y^*)) + k^{n-1}d(f(y^*), y^*) + \cdots + kd(f(y^*), y^*) + d(f(y^*), y^*) \leq$$

$$d(x^*, f^n(y^*)) + (k^{n-1} + \cdots + k + 1)d(f(y^*), y^*) \leq d(x^*, f^n(y^*)) + \frac{\varepsilon}{1-k}.$$

The conclusion follows letting $n \to \infty$.

For other considerations and results on Ulam-Hyers stability, see [5, 29, 38, 41, 42, 44].

6 Operators on a Complete Metric Space Which Are Graphic Contractions on an Invariant Subset

In this section, we will prove a fixed point theorem for an operator $f : X \to X$ satisfying the graphic contraction condition on an invariant (not necessary closed) subset of a complete metric space.

Let (X, d) be a complete metric space and $f : X \to X$ be an operator. Let us suppose that there exists a nonempty subset Y of X such that $f(Y) \subset Y$ and $f : Y \to Y$ is a graphic contraction. Then, for every $x \in Y$ the sequence of successive approximations $(f^n(x))_{n\in\mathbb{N}}$ is convergent in (X, d). Let us denote $f^\infty(x) := \lim_{n\to\infty} f^n(x)$, $x \in Y$. Notice that if $f \circ f^\infty = f^\infty \circ f$, then $f^\infty(x) \in F_f$. Moreover, the operator

$$f : Y \cup f^\infty(Y) \to Y \cup f^\infty(Y)$$

is weakly Picard and a graphic contraction too. For this operator, we are in the conditions of Theorem 3, since $Y \cup f^\infty(Y)$ is complete with respect to the sequences $(f^n(x))_{n\in\mathbb{N}}$ with $x \in Y \cup f^\infty(Y)$, see [37]. As a consequence, we have the following general result.

Theorem 8 *Let (X, d) be a complete metric space and $f : X \to X$ be an operator. We suppose that there exists a nonempty subset Y of X such that $f(Y) \subset Y$ and $f : Y \to Y$ is a graphic k-contraction. Then the following conclusions take place:*

(a) for every $x \in Y$ the sequence of successive approximations $(f^n(x))_{n\in\mathbb{N}}$ is convergent in (X, d) and

$$\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < \infty;$$

If, in addition,

$$\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x)), \text{ for each } x \in Y,$$

then we also have the following conclusions:

(b) $F_f \cap Y = F_{f^n} \cap Y \neq \emptyset$, for every $n \in \mathbb{N}^*$;

(c) $f : Y \cup f^\infty(Y) \to Y \cup f^\infty(Y)$ is a weakly Picard and

$$Z := Y \cup f^\infty(Y) = \bigcup_{x^*\in F_f\cap Z} Z_{x^*}$$

is an invariant partition of Z;

(d) $d(x, f^\infty(x)) \leq \frac{1}{1-k} d(x, f(x))$, for every $x \in Y$;

(e) *if* $y_n \in Z_{x^*}$ *for every* $n \in \mathbb{N}$ *is such that* $d(y_n, f(y_n)) \to 0$ *as* $n \to \infty$*, then* $y_n \to x^*$ *as* $n \to \infty$*, for every* $x^* \in F_f \cap Z$*;*
(f) *if* $k < \frac{1}{3}$*, then* $d(f(x), f^\infty(x)) \leq \frac{k}{1-2k} d(x, f^\infty(x))$*, for every* $x \in Y$*;*
(g) *if* $k < \frac{1}{3}$*, then* $f : Z \to Z$ *has the Ostrowski property.*

Actually, following the above ideas, the problem is in which conditions on the metric space (X, d) and on the operator f there exists such a subset Y with the above properties. For example, we can do the following construction.

Let X be a nonempty set endowed with a partial order "$\preceq$" and $d : X \times X \to \mathbb{R}_+$ be a complete metric. Let $f : X \to X$ be an operator which is increasing with respect to "$\preceq$". Then the following subsets are invariant with respect to f:

(1) $Y := \{x \in X : x \preceq f(x)\}$;
(2) $Y := \{x \in X : f(x) \preceq x\}$;
(3) $Y := \{x \in X : x \preceq f(x)$ or $f(x) \preceq x\}$.

Hence, in order to apply Theorem 8 we need to impose the following conditions:

(i) $Y \neq \emptyset$;
(ii) $f : Y \to Y$ is a graphic contraction.

For example, we have the following results.

Theorem 9 *Let* X *be a nonempty set endowed with a partial order "*$\preceq$*" and* $d : X \times X \to \mathbb{R}_+$ *be a complete metric. Let* $f : X \to X$ *be an operator which is increasing with respect to "*$\preceq$*". Suppose that there exist a constant* $k \in]0, 1[$ *and an element* $x_0 \in X$ *such that:*

(i) $x \in X$ *with* $x \preceq f(x)$ *(or reversely) implies* $d(f(x), f^2(x)) \leq kd(x, f(x))$*;*
(ii) $x_0 \preceq f(x_0)$ *(or reversely);*
(iii) $\lim_{n\to\infty} f(f^n(x_0)) = f(\lim_{n\to\infty} f^n(x_0))$.

Then $F_f \neq \emptyset$ *and the sequence of successive approximations* $(f^n(x_0))_{n\in\mathbb{N}}$ *converges to a fixed point of* f.

Proof Let $x_0 \in X$ such that $x_0 \preceq f(x_0)$. Then, the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is increasing with respect to $\preceq$. Hence, we can apply the graphic contraction condition (i) and we obtain that f is asymptotically regular at x_0, i.e.,

$$d(f^n(x_0), f^{n+1}(x_0)) \leq k^n d(x_0, f(x_0)) \to 0 \text{ as } n \to \infty.$$

Now, in order to show that $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence, we observe that, for $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$, we have

$$d(f^n(x_0), f^{n+p}(x_0)) \leq$$

$$d\left(f^n(x_0), f^{n+1}(x_0)\right) + \ldots + d\left(f^{n+p-1}(x_0), f^{n+p}(x_0)\right) \leq$$

$$k^n(1 + k + \cdots + k^{p-1})d(x_0, f(x_0))$$

Since $k < 1$, we get that

$$d(f^n(x_0), f^{n+p}(x_0)) \leq \frac{k^n}{1-k} d(x_0, f(x_0)) \to 0 \text{ as } n, p \to \infty.$$

Hence $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence and by the completeness of the metric d, there exists $x^* \in X$ such that $(f^n(x_0))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$. By (iii) it follows that $x^* \in F_f$.

A more general result is the following one.

Theorem 10 *Let X be a nonempty set endowed with a partial order "$\preceq$" and $d : X \times X \to \mathbb{R}_+$ be a complete metric. Let $f : X \to X$ be an operator such that the following assumptions are satisfied:*

(i) the set $X_\preceq := \{x \in X : x \preceq f(x)$ or $f(x) \preceq x\}$ is nonempty;
(ii) $X_\preceq$ is invariant with respect to f, i.e., $f(X_\preceq) \subseteq X_\preceq$;
(iii) there exists a constant $k \in]0, 1[$ such that

$$d(f(x), f^2(x)) \leq kd(x, f(x)), \text{ for all } x \in X_\preceq;$$

(iv) $\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x))$, for each $x \in X_\preceq$.

Then $F_f \neq \emptyset$ and, for all $x \in X_\preceq$, the sequence of successive approximations $(f^n(x))_{n\in\mathbb{N}}$ converges to a fixed point of f.

Proof For $x_0 \in X_\preceq$ we consider the sequence $(f^n(x_0))_{n\in\mathbb{N}}$. Then $f^n(x_0) \in X_\preceq$, for each $n \in \mathbb{N}$. Hence, by the graphic contraction condition (i), we obtain that

$$d(f^n(x_0), f^{n+1}(x_0)) \leq k^n d(x_0, f(x_0)) \to 0 \text{ as } n \to \infty.$$

As before, we can show that $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence. Thus, by the completeness of the metric d, there exists $x^* \in X$ such that $(f^n(x_0))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$. By (iv) it follows that $x^* \in F_f$.

Remark 1 Theorem 9 is a generalization of the well-known fixed point theorem for contraction mappings given by Ran and Reurings in [27]. Theorem 10 extends to the case of graphic contractions, one of the main result of the paper [22]. The results of this section are in connection and generalize some other known results, see, for example, [16, 28, 35] and the references therein.

7 Open Questions and New Research Directions

We will formulate now some open questions and related research directions.

7.1 *Generalized Contractions as Graphic Contractions*

Which are those generalized contractions (self or non-self) which are graphic contractions?

In the case of various applications of these results, the problem is to improve it from the saturated principle of graphic contractions point of view.

References: [1, 20, 32, 35, 48].

7.2 *Nonlinear Graphic Contractions*

The problem is to prove similar results for the case of nonlinear graphic contractions (also called graphic φ-contractions), i.e., in the case when f satisfies the following assumption

$$d(f(x), f^2(x)) \leq \varphi(d(x, f(x)), \text{ for every } x \in X(\text{ or } x \in \tilde{B}(x_0, R),$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, see [35].

References: [1, 4, 35, 48].

7.3 *Generalized Metric Spaces*

Another open question and research direction is to study the above problems in various generalized metric spaces, such as:

(1) $\mathbb{R}^m_+$-metric spaces;
(2) $s(\mathbb{R}_+)$-metric spaces;
(3) b-metric (quasi-metric) spaces;
(4) Banach spaces with f a differentiable operator.

References: [21, 35, 46, 48].

7.4 Coupled Fixed Point Problems Via Graphic Contraction Conditions

Using Theorems 9 and 10 and the approach presented in [25], the problem is to give existence results for the following coupled fixed point problem:

$$\begin{cases} x = T(x, y) \\ y = T(y, x). \end{cases}$$

7.5 Y-Contractions

The problems studied in this paper are particular cases of the following general problem.

Let (X, d) be a complete metric space and $Y \subset X \times X$ be a nonempty subset. By definition, an operator $f : X \to X$ is called a Y-contraction if there exists $k \in]0, 1[$ such that

$$d(f(x), f(y)) \leq kd(x, y), \text{ for every } (x, y) \in Y.$$

The problem is to construct a fixed point theory of Y-contractions.

For other considerations and results, see [48, pp. 282–284] and the references therein.

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