Recent Advances of Convexity Theory and Its Inequalities

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Abstract In this chapter, we introduce some new notions of generalized convex functionals in normed linear spaces. It unifies and generalizes the many known and new classes of convex functions. The corresponding Schur, Jensen, and Hermite-Hadamard type inequalities are also established.

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1 Introduction

Definition 1 A function $f : [a, b] \to \mathbb{R}$ is called convex if

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2), \tag{1}$$

$\forall x_1, x_2 \in [a, b], \forall \lambda \in [0, 1]$.

This classical inequality (1) plays an important role in analysis, optimization and in the theory of inequalities, and it has a huge literature dealing with its applications, various generalizations and refinements. Further, convexity is one of the most fundamental and important notions in mathematics. Convexity theory and its inequalities are fields of interest of numerous mathematicians and there are many paper, books, and monographs devoted to these fields and various applications (see, e.g., [1, 4, 6–14, 16, 18–22] and the references therein).

In this chapter, we introduce some new notions of generalized convex functionals in normed linear spaces in Section 2. It unifies and generalizes the many known and new classes of convex functions. Some new basic inequalities are presented in Section 3. New generalized Hermite-Hadamard type inequalities are presented in

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Section 4. In Sections 5 and 6, strongly convex functional and the corresponding inequalities in normed linear spaces are also given.

2 Generalized Convex Functionals in Normed Linear Spaces

In what follows, $(X, \|\cdot\|)$ denotes the real normed linear spaces, D be a convex subset of X, $h : (0, 1) \to (0, \infty)$ is a given function, whose h is not identical to 0.

In this section, we introduce and study a new class of generalized convex functionals, that is, $(\alpha, \beta, \lambda, \lambda_0, t, \xi, h)$ convex functionals.

Definition 2 A functional $f : D \to (0, \infty)$ is called $(\alpha, \beta, \lambda, \lambda_0, t, \xi, h)$ convex if

$$f((\lambda\|x_1\|^\alpha + \lambda_0(1-\lambda)\|x_2\|^\alpha)^{1/\alpha}) \le \{h(t^\xi) f^\beta(\|x_1\|) + \lambda_0 h(1-t^\xi) f^\beta(\|x_2\|)\}^{1/\beta}, \tag{2}$$

$\forall x_1, x_2 \in D, \forall \lambda, \lambda_0, t, \xi \in [0, 1], \alpha, \beta$ are real numbers, and $\alpha, \beta \ne 0$.

If $\lambda_0 = 1$ in (2), that is,

$$f((\lambda\|x_1\|^\alpha + (1-\lambda)\|x_2\|^\alpha)^{1/\alpha}) \le \{h(t^\xi) f^\beta(\|x_1\|) + h(1-t^\xi) f^\beta(\|x_2\|)\}^{1/\beta}, \tag{3}$$

we say that f is a $(\alpha, \beta, \lambda, t, \xi, h)$ convex functional.

If $\xi = 1$ in (3), that is,

$$f((\lambda\|x_1\|^\alpha + (1-\lambda)\|x_2\|^\alpha)^{1/\alpha}) \le \{h(t) f^\beta(\|x_1\|) + h(1-t) f^\beta(\|x_2\|)\}^{1/\beta},$$

we say that f is a $(\alpha, \beta, \lambda, t, h)$ convex functional.

For $t = \lambda$ in (2), that is,

$$f((\lambda\|x_1\|^\alpha + \lambda_0(1-\lambda)\|x_2\|^\alpha)^{1/\alpha}) \le \{h(\lambda^\xi) f^\beta(\|x_1\|) + \lambda_0 h(1-\lambda^\xi) f^\beta(\|x_2\|)\}^{1/\beta}, \tag{4}$$

we say that f is a $(\alpha, \beta, \lambda, \lambda_0, \xi, h)$ convex functional.

If $\xi = 1$ in (4), that is,

$$f((\lambda\|x_1\|^\alpha + \lambda_0(1-\lambda)\|x_2\|)^{1/\alpha}) \le \{h(\lambda) f^\beta(\|x_1\|) + \lambda_0 h(1-\lambda) f^\beta(\|x_2\|)\}^{1/\beta}, \tag{5}$$

we say that f is a $(\alpha, \beta, \lambda, \lambda_0, h)$ convex functional.

For $\lambda_0 = 1$ in (5), that is,

$$f((\lambda\|x_1\|^\alpha + (1-\lambda)\|x_2\|^\alpha)^{1/\alpha}) \le \{h(\lambda) f^\beta(\|x_1\|) + h(1-\lambda) f^\beta(\|x_2\|)\}^{1/\beta}, \tag{6}$$

we say that f is a $(\alpha, \beta, \lambda, h)$ convex functional.

In particular, if $h(\lambda) = \lambda^s$, $0 < |s| \le 1$ in (6), that is,

$$f((\lambda\|x_1\|^\alpha + (1-\lambda)\|x_2\|^\alpha)^{1/\alpha}) \le \{\lambda^s f^\beta(\|x_1\|) + (1-\lambda)^s f^\beta(\|x_2\|)\}^{1/\beta}, \tag{7}$$

we say that f is a $(\alpha, \beta, \lambda, s)$ convex functional. If $s = 1$, then (7) reduces to (α, β, λ) convex functional.

For $\alpha = \beta = 1$ in (2), that is,

$$f(\lambda\|x_1\| + \lambda_0(1-\lambda)\|x_2\|) \le h(t^{\xi})f(\|x_1\|) + \lambda_0 h(1-t^{\xi})f(\|x_2\|), \tag{8}$$

we say that f is a $(\lambda, \lambda_0, t, \xi, h)$ convex functional.

If $\lambda_0 = 1$ in (8), that is,

$$f(\lambda\|x_1\| + (1-\lambda)\|x_2\|) \le h(t^{\xi})f(\|x_1\|) + h(1-t^{\xi})f(\|x_2\|), \tag{9}$$

we say that f is a (λ, t, ξ, h) convex functional.

In particular, if $t = \lambda$ in (8), that is,

$$f(\lambda\|x_1\| + \lambda_0(1-\lambda)\|x_2\|) \le h(\lambda^{\xi})f(\|x_1\|) + \lambda_0 h(1-\lambda^{\xi})f(\|x_2\|), \tag{10}$$

we say that f is a $(\lambda, \lambda_0, \xi, h)$ convex functional.

If $\xi = 1$ in (10), that is,

$$f(\lambda\|x_1\| + \lambda_0(1-\lambda)\|x_2\|) \le h(\lambda)f(\|x_1\|) + \lambda_0 h(1-\lambda)f(\|x_2\|), \tag{11}$$

we say that f is a (λ, λ_0, h) convex functional.

If $\lambda_0 = 1$ in (11), that is,

$$f(\lambda\|x_1\| + (1-\lambda)\|x_2\|) \le h(\lambda)f(\|x_1\|) + h(1-\lambda)f(\|x_2\|), \tag{12}$$

we say that f is an h-convex functional.

In the following Examples 1–6, we make appointment that

$$X = [0, \infty),\ D \subset [0, \infty),\ f : D \to [0, \infty).$$

Then (2) reduces to

$$f((\lambda x_1^{\alpha} + \lambda_0(1-\lambda)x_2^{\alpha})^{1/\alpha}) \le \{h(t^{\xi})f^{\beta}(x_1) + \lambda_0 h(1-t^{\xi})f^{\beta}(x_2)\}^{1/\beta}, \tag{13}$$

$\forall x_1, x_2 \in D$, $\forall \lambda, \lambda_0, t, \xi \in [0, 1]$, α, β are real numbers, and $\alpha, \beta \neq 0$.

If $\xi = 1$ in (13), that is,

$$f((\lambda x_1^{\alpha} + \lambda_0(1-\lambda)x_2^{\alpha})^{1/\alpha}) \le \{h(t)f^{\beta}(x_1) + \lambda_0 h(1-t)f^{\beta}(x_2)\}^{1/\beta}, \tag{14}$$

$\forall x_1, x_2 \in D$, $\forall \lambda, t \in [0, 1]$, we say that f is a $(\alpha, \beta, \lambda, \lambda_0, t, h)$ convex function.

If $t = \lambda$ in (14), that is,

$$f((\lambda x_1^{\alpha} + \lambda_0(1-\lambda)x_2^{\alpha})^{1/\alpha}) \le \{h(\lambda)f^{\beta}(x_1) + \lambda_0 h(1-\lambda)f^{\beta}(x_2)\}^{1/\beta}, \tag{15}$$

$\forall x_1, x_2 \in D$, $\forall \lambda \in [0, 1]$, we say that f is a $(\alpha, \beta, \lambda, \lambda_0, h)$ convex function.

If $\lambda_0 = 1$ in (14), that is,

$$f((\lambda x_1^\alpha + (1-\lambda)x_2^\alpha)^{1/\alpha}) \le \{h(t)f^\beta(x_1) + h(1-t)f^\beta(x_2)\}^{1/\beta}, \tag{16}$$

$\forall x_1, x_2 \in D, \forall \lambda, t \in [0, 1]$, we say that f is a $(\alpha, \beta, \lambda, t, h)$ convex function. If $t = \lambda$ in (16), that is,

$$f((\lambda x_1^\alpha + (1-\lambda)x_2^\alpha)^{1/\alpha}) \le \{h(\lambda)f^\beta(x_1) + h(1-\lambda)f^\beta(x_2)\}^{1/\beta}, \tag{17}$$

$\forall x_1, x_2 \in D, \forall \lambda \in [0, 1]$, we say that f is a $(\alpha, \beta, \lambda, h)$ convex function.

If $h(\lambda) = \lambda^s, 0 < |s| \le 1$ in (16), (17), that is,

$$f((\lambda x_1^\alpha + (1-\lambda)x_2^\alpha)^{1/\alpha}) \le \{t^s f^\beta(x_1) + (1-t)^s f^\beta(x_2)\}^{1/\beta}, \tag{18}$$

$$f((\lambda x_1^\alpha + (1-\lambda)x_2^\alpha)^{1/\alpha}) \le \{\lambda^s f^\beta(x_1) + (1-\lambda)^s f^\beta(x_2)\}^{1/\beta}, \tag{19}$$

we say that f is a $(\alpha, \beta, \lambda, t, s)$, $(\alpha, \beta, \lambda, s)$ convex function, respectively.

In particular, if $s = 1$, then (18), (19) reduce to $(\alpha, \beta, \lambda, t)$, (α, β, λ) convex function, respectively.

Example 1 If $\alpha = \beta = 1$ in (13), then

$$f(\lambda x_1 + \lambda_0(1-\lambda)x_2) \le h(t^\xi)f(x_1) + \lambda_0 h(1-t^\xi)f(x_2), \tag{20}$$

$\forall x_1, x_2 \in D, \forall \lambda, \lambda_0, t, \xi \in [0, 1]$, we say that f is a $(\lambda, \lambda_0, t, \xi, h)$ convex function.

In particular, if $\lambda_0 = 1$ in (20), that is,

$$f(\lambda x_1 + (1-\lambda)x_2) \le h(t^\xi)f(x_1) + h(1-t^\xi)f(x_2), \tag{21}$$

$\forall x_1, x_2 \in D, \forall \lambda, t, \xi \in [0, 1]$, we say that f is a (λ, t, ξ, h) convex function.

For $h(t) = t^s, 0 < |s| \le 1, \xi = 1$ in (21), that is,

$$f(\lambda x_1 + (1-\lambda)x_2) \le t^s f(x_1) + (1-t)^s f(x_2), \tag{22}$$

$\forall x_1, x_2 \in D, \forall \lambda, t \in [0, 1]$, we say that f is a (λ, t, s) convex function.

In particular, when $s = 1$ in (22), that is,

$$f(\lambda x_1 + (1-\lambda)x_2) \le tf(x_1) + (1-t)f(x_2), \tag{23}$$

$\forall x_1, x_2 \in D, \forall \lambda, t \in [0, 1]$, we say that f is a (λ, t) convex function (see, e.g., [8]).

For $t = \lambda$ in (20), that is,

$$f(\lambda x_1 + \lambda_0(1-\lambda)x_2) \le h(\lambda^\xi)f(x_1) + \lambda_0 h(1-\lambda^\xi)f(x_2), \tag{24}$$

$\forall x_1, x_2 \in D, \forall \lambda, \lambda_0, \xi \in [0, 1]$, we say that f is a $(\lambda, \lambda_0, \xi, h)$ convex function.

If $\xi = 1$ in (24), that is,

$$f(\lambda x_1 + \lambda_0(1-\lambda)x_2) \le h(\lambda)f(x_1) + \lambda_0 h(1-\lambda)f(x_2), \tag{25}$$

$\forall x_1, x_2 \in D, \forall \lambda \in [0, 1]$, we say that f is a (λ, λ_0, h) convex function. In particular, when $\lambda_0 = 1$,(25) reduces to h-convex function (see [4, 19]), that is,

$$f(\lambda x_1 + (1-\lambda)x_2) \le h(\lambda)f(x_1) + h(1-\lambda)f(x_2). \tag{26}$$

If $h(\lambda) = \lambda$, then (26) reduces to (1).

If $h(\lambda) = \lambda^s, 0 < s \le 1$ in (26), that is,

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda^s f(x_1) + (1-\lambda)^s f(x_2), \tag{27}$$

$\forall x_1, x_2 \in D, \forall \lambda \in [0, 1]$, we say that f is a s-Breckner convex function (see, e.g., [4, 5, 8]).

If $h(\lambda) = \lambda^{-s}, 0 < s \le 1$ in (26), that is,

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda^{-s} f(x_1) + (1-\lambda)^{-s} f(x_2), \tag{28}$$

$\forall x_1, x_2 \in D, \forall \lambda \in [0, 1]$, we say that f is a s-Godunova-Levin function (see [4]). In particular, when $s = 1$, (28) reduces to Godunova-Levin function (see, e.g., [5, 8])

If $h(\lambda) = 1$ in (26), that is,

$$f(\lambda x_1 + (1-\lambda)x_2) \le f(x_1) + f(x_2), \tag{29}$$

$\forall x_1, x_2 \in D, \forall \lambda \in [0, 1]$, we say that f is a P-function (see, e.g., [5]).

If $h(\lambda) = \lambda^s, 0 < |s| \le 1$, in (24), that is,

$$f(\lambda x_1 + \lambda_0(1-\lambda)x_2) \le \lambda^{s\xi} f(x_1) + \lambda_0(1-\lambda^{\xi})^s f(x_2), \tag{30}$$

$\forall x_1.x_2 \in D, \forall \lambda, \lambda_0, \xi \in [0, 1]$, we say that f is a $(\lambda, \lambda_0, \xi, s)$ convex function. In particular, if $\xi = s = 1$ in (30), that is,

$$f(\lambda x_1 + \lambda_0(1-\lambda)x_2) \le \lambda f(x_1) + \lambda_0(1-\lambda)f(x_2), \tag{31}$$

we say that f is a λ_0-convex function (that is, m-convex function in [2]).

If $\lambda_0 = 0$ in (20), then

$$f(\lambda x) \le h(t^{\xi})f(x), \ \ x \in D. \tag{32}$$

When $t = \lambda, \xi = 1, h(t) = t$ in (32), that is,

$$f(\lambda x) \le \lambda f(x), \tag{33}$$

we say that f is a starshaped function (see [2])

Example 2 If $\beta = 1$ in (13), then

$$f((\lambda x_1^\alpha + \lambda_0(1-\lambda)x_2^\alpha)^{1/\alpha}) \le h(t^\xi)f(x_1) + \lambda_0 h(1-t^\xi)f(x_2), \tag{34}$$

$\forall x_1, x_2 \in D, \forall \lambda, \lambda_0, t, \xi \in [0,1], \alpha \ne 0$, we say that f is a $(\alpha, \lambda, \lambda_0, t, \xi, h)$ convex function.

For $t = \lambda$ in (34), that is,

$$f((\lambda x_1^\alpha + \lambda_0(1-\lambda)x_2^\alpha)^{1/\alpha}) \le h(\lambda^\xi)f(x_1) + \lambda_0 h(1-\lambda^\xi)f(x_2), \tag{35}$$

$\forall x_1, x_2 \in D, \forall \lambda, \lambda_0, t, \xi \in [0,1], \alpha \ne 0$, we say that f is a $(\alpha, \lambda, \lambda_0, \xi, h)$ convex function. When $\lambda_0 = 1, \xi = 1$, (35) reduces to (α, h) convex function (that is, (p, h) convex function in [5]). In particular, if $\lambda_0 = 1, \xi = 1, h(t) = t$, (35) reduces to α-convex function (that is, p-convex function in [5, 22])

Example 3 If $\alpha = 1, \beta = q$ in (13), then

$$f(\lambda x_1 + \lambda_0(1-\lambda)x_2) \le \{h(t^\xi)f^q(x_1) + \lambda_0 h(1-t^\xi)f^q(x_2)\}^{1/q}, \tag{36}$$

$\forall x_1, x_2 \in D, \forall \lambda, \lambda_0, t, \xi \in [0,1], q \ne 0$, we say that f is a $(q, \lambda, \lambda_0, t, \xi, h)$ convex function.

For $t = \lambda$ in (36), that is,

$$f(\lambda x_1 + \lambda_0(1-\lambda)x_2) \le \{h(\lambda^\xi)f^q(x_1) + \lambda_0 h(1-\lambda^\xi)f^q(x_2)\}^{1/q}, \tag{37}$$

$\forall x_1, x_2 \in D, \forall \lambda, \lambda_0, t, \xi \in [0,1], q \ne 0$, we say that f is a $(q, \lambda, \lambda_0, \xi, h)$ convex function. When $\lambda_0 = 1, \xi = 1, h(t) = t$, (37) reduces to q-convex function (see, e.g., [8]).

Example 4 If $\alpha = 1, \beta = -1$ in (13), then

$$f(\lambda x_1 + \lambda_0(1-\lambda)x_2) \le \{h(t^\xi)f^{-1}(x_1) + \lambda_0 h(1-t^\xi)f^{-1}(x_2)\}^{-1}, \tag{38}$$

$\forall x_1, x_2 \in D, \forall \lambda, \lambda_0, t, \xi \in [0,1]$, we say that f is a $(AH, \lambda, \lambda_0, t, \xi, h)$ convex function, where AH means the arithmetic-harmonic means.

For $t = \lambda$ in (38), that is,

$$f(\lambda x_1 + \lambda_0(1-\lambda)x_2) \le \{h(\lambda^\xi)f^{-1}(x_1) + \lambda_0 h(1-\lambda^\xi)f^{-1}(x_2)\}^{-1}, \tag{39}$$

$\forall x_1, x_2 \in D, \forall \lambda, \lambda_0, \xi \in [0,1]$, we say that f is a $(AH, \lambda, \lambda_0, \xi, h)$ convex function.

For $h(\lambda) = \lambda^s, 0 < |s| \le 1, \lambda_0 = \xi = 1$ in (39), that is,

$$f(\lambda x_1 + (1-\lambda)x_2) \le \{\lambda^s f^{-1}(x_1) + (1-\lambda)^s f^{-1}(x_2)\}^{-1}, \tag{40}$$

$\forall x_1, x_2 \in D, \forall \lambda \in [0, 1]$, we say f is a (AH, λ, s) convex function. In particular, if $s = 1$, then (40) reduces to AH convex function.

Example 5 If $\alpha = -1, \lambda_0 = 1, h(\lambda) = \lambda$ in (15), then

$$f(\frac{x_1 x_2}{\lambda x_2 + (1-\lambda)x_1}) \leq \{\lambda f^{\beta}(x_1) + (1-\lambda) f^{\beta}(x_2)\}^{1/\beta},$$

$\forall x_1, x_2 \in D, \forall \lambda \in [0, 1], \beta \neq 0$, we say that f is a harmonically β-convex functions, see [15].

Example 6 If $\alpha = -1, \beta = 1$ in (13), then

$$f((\lambda x_1^{-1} + \lambda_0(1-\lambda)x_2^{-1})^{-1}) \leq h(t^{\xi}) f(x_1) + \lambda_0 h(1-t^{\xi}) f(x_2), \tag{41}$$

$\forall x_1, x_2 \in D, \forall \lambda, \lambda_0, t, \xi \in [0, 1]$, we say that f is a $(HA, \lambda, \lambda_0, t, \xi, h)$ convex function.

For $t = \lambda$ in (41), that is,

$$f((\lambda x_1^{-1} + \lambda_0(1-\lambda)x_2^{-1})^{-1}) \leq h(\lambda^{\xi}) f(x_1) + \lambda_0 h(1-\lambda^{\xi}) f(x_2), \tag{42}$$

$\forall x_1, x_2 \in D, \forall \lambda, \lambda_0, \xi \in [0, 1]$, we say that f is a $(HA, \lambda, \lambda_0, \xi, h)$ convex function.

For $h(\lambda) = \lambda^s, 0 < |s| \leq 1, \lambda_0 = \xi = 1$ in (42), that is,

$$f((\lambda x_1^{-1} + (1-\lambda)x_2^{-1})^{-1}) \leq \lambda^s f(x_1) + (1-\lambda)^s f(x_2), \tag{43}$$

$\forall x_1, x_2 \in D, \forall \lambda \in [0, 1]$, we say f is a (HA, λ, s) convex function. In particular, if $s = 1$, then (43) reduces to HA convex function.

Example 7 If $\alpha = \beta = -2$ in (13), then

$$f((\lambda x_1^{-2} + \lambda_0(1-\lambda)x_2^{-2})^{-(1/2)}) \leq \{h(t^{\xi}) f^{-2}(x_1) + \lambda_0 h(1-t^{\xi}) f^{-2}(x_2)\}^{-(1/2)}, \tag{44}$$

$\forall x_1, x_2 \in D, \forall \lambda, \lambda_0, t, \xi \in [0, 1]$, we say that f is a $(HS, \lambda, \lambda_0, t, \xi, h)$ convex function.

For $t = \lambda$ in (44), that is,

$$f((\lambda x_1^{-2} + \lambda_0(1-\lambda)x_2^{-2})^{-(1/2)}) \leq \{h(\lambda^{\xi}) f^{-2}(x_1) + \lambda_0 h(1-\lambda^{\xi}) f^{-2}(x_2)\}^{-(1/2)}, \tag{45}$$

$\forall x_1, x_2 \in D, \forall \lambda, \lambda_0, \xi \in [0, 1]$, we say that f is a $(HS, \lambda, \lambda_0, \xi, h)$ convex function.

For $h(\lambda) = \lambda^s, 0 < |s| \leq 1, \lambda_0 = \xi = 1$ in (45), that is,

$$f((\lambda x_1^{-2} + (1-\lambda)x_2^{-2})^{-(1/2)}) \leq \{\lambda^s f^{-2}(x_1) + (1-\lambda)^s f^{-2}(x_2)\}^{-(1/2)}, \tag{46}$$

$\forall x_1, x_2 \in D, \forall\lambda \in [0, 1]$, we say that f is a (HS, λ, s) convex function, that is, f is the harmonic square s-convex function. In particular, if $s = 1$, then (46) reduces to HS convex function.

Example 8 Let X be a real normed linear space, and D be a convex subset of X, $h : (0, 1) \to (0, \infty)$ is a given function. If

$$\lambda = \frac{t_1}{t_1 + t_2}, h(\lambda) = \frac{\lambda(t_1)}{\lambda(t_1 + t_2)}, \ 0 < t_1, t_2 < \infty,$$

then

$$1 - \lambda = \frac{t_2}{t_1 + t_2}, h(1 - \lambda) = \frac{\lambda(t_2)}{\lambda(t_1 + t_2)},$$

and by (25), we get

$$f(\frac{t_1 x_1 + \lambda_0 t_2 x_2}{t_1 + t_2}) \leq \frac{\lambda(t_1) f(x_1) + \lambda_0 \lambda(t_2) f(x_2)}{\lambda(t_1 + t_2)}, \tag{47}$$

$\forall x_1, x_2 \in D$, $\forall\lambda, \lambda_0 \in [0, 1]$, we say that f is a (λ, λ_0) convex function. When $\lambda_0 = 1$,(47) reduces to λ-convex function (see, e.g., [3, 4]).

Hence, Definition 2 unifies and generalizes the many known and new classes of convex functions.

3 Some New Basic Inequalities

The classical Schur, Jensen, and Hermite-Hadamard inequalities play an important role in analysis, optimization and in the theory of inequalities, and it has a huge literature dealing with its applications, various generalizations and refinements (see, e.g., [2, 6–9, 11, 12, 18–22], and the references therein). In this and next section, we present the corresponding inequalities for $(\alpha, \beta, \lambda, \lambda_0, t, \xi, h)$ convex functionals.

Definition 3 ([19]) A function $h : (0, 1) \to (0, \infty)$ is called a super-multiplicative function if

$$h(tu) \geq h(t)h(u), \tag{48}$$

for all $t, u \in (0, 1)$.

Lemma 1 *Let $g(\|x\|) = f^{\beta}(\|x\|^{1/\alpha})$, $x \in D$, α, β are real numbers, and $\alpha, \beta \neq 0$. Then a functional $f : D \to (0, \infty)$ is $(\alpha, \beta, \lambda, \lambda_0, t, \xi, h)$ convex if and only if the functional $g : D \to (0, \infty)$ is $(\lambda, \lambda_0, t, \xi, h)$ convex. In particular, a functional $f : D \to (0, \infty)$ is $(\alpha, \beta, \lambda, \lambda_0, \xi, h)$ convex if and only if the functional $g : D \to$*

$(0,\infty)$ is $(\lambda,\lambda_0,\xi,h)$ convex, and a functional $f : D \to (0,\infty)$ is (α,β,λ,h) convex if and only if the functional $g : D \to (0,\infty)$ is h-convex.

Proof Setting $\|u\| = \|x\|^{1/\alpha}$, $x \in D$. Assume that f is $(\alpha,\beta,\lambda,\lambda_0,t,\xi,h)$ convex, then for all $x_1, x_2 \in D$, we get

$$
\begin{aligned}
&g(\lambda\|x_1\| + \lambda_0(1-\lambda)\|x_2\|) = g(\lambda\|u_1\|^\alpha + \lambda_0(1-\lambda)\|u_2\|^\alpha)\\
&= f^\beta((\lambda\|u_1\|^\alpha + \lambda_0(1-\lambda)\|u_2\|^\alpha)^{1/\alpha}) \le h(t^\xi) f^\beta(\|u_1\|) + \lambda_0 h(1-t^\xi) f^\beta(\|u_2\|)\\
&= h(t^\xi) g(\|u_1\|^\alpha) + \lambda_0 h(1-t^\xi) g(\|u_2\|^\alpha) = h(t^\xi) g(\|x_1\|) + \lambda_0 h(1-t^\xi) g(\|x_2\|),
\end{aligned}
$$

which proves that g is $(\lambda,\lambda_0,t,\xi,h)$ convex.

Conversely, if g is $(\lambda,\lambda_0,t,\xi,h)$ convex, then

$$
\begin{aligned}
f^\beta((\lambda\|u_1\|^\alpha + \lambda_0(1-\lambda)\|u_2\|^\alpha)^{1/\alpha}) &= g(\lambda\|u_1\|^\alpha + \lambda_0(1-\lambda)\|u_2\|^\alpha)\\
&= g(\lambda\|x_1\| + \lambda_0(1-\lambda)\|x_2\|) \le h(t^\xi) g(\|x_1\|) + \lambda_0 h(1-t^\xi) g(\|x_2\|)\\
&= h(t^\xi) f^\beta(\|x_1\|^{1/\alpha}) + \lambda_0 h(1-t^\xi) f^\beta(\|x_2\|^{1/\alpha})\\
&= h(t^\xi) f^\beta(\|u_1\|) + \lambda_0 h(1-t^\xi) f^\beta(\|u_2\|),
\end{aligned}
$$

which proves that f is $(\alpha,\beta,\lambda,\lambda_0,t,\xi,h)$ convex.

First of all, we establish Schur type inequalities of (α,β,λ,h) convex functionals.

Theorem 1 *Let $f : D \to (0,\infty)$ be a h-convex functional and $h : (0,1) \to (0,\infty)$ is a super-multiplicative function, then for all $x_1, x_2, x_3 \in D$, such that $\|x_1\| < \|x_2\| < \|x_3\|$, and $0 < \|x_3\| - \|x_1\| < 1$, the following generalized Schur inequality holds:*

$$
f(\|x_2\|) \le \frac{h(\|x_3\| - \|x_2\|)}{h(\|x_3\| - \|x_1\|)} f(\|x_1\|) + \frac{h(\|x_2\| - \|x_1\|)}{h(\|x_3\| - \|x_1\|)} f(\|x_3\|). \tag{49}
$$

Proof Setting

$$
\lambda = \frac{\|x_3\| - \|x_2\|}{\|x_3\| - \|x_1\|},
$$

we have $0 < \lambda < 1$,

$$
1 - \lambda = \frac{\|x_2\| - \|x_1\|}{\|x_3\| - \|x_1\|},
$$

and $\|x_2\| = \lambda\|x_1\| + (1-\lambda)\|x_3\|$. By (12), we get

$$
\begin{aligned}
f(\|x_2\|) &= f(\lambda\|x_1\| + (1-\lambda)\|x_3\|)\\
&\le h(\lambda) f(\|x_1\|) + h(1-\lambda) f(\|x_3\|).
\end{aligned} \tag{50}
$$

By (48), we get

$$h(\|x_3\| - \|x_2\|) = h(\lambda(\|x_3\| - \|x_1\|)) \geq h(\lambda)h(\|x_3\| - \|x_1\|).$$

Hence,

$$h(\lambda) \leq \frac{h(\|x_3\| - \|x_2\|)}{h(\|x_3\| - \|x_1\|)}. \tag{51}$$

Similarly, we get

$$h(1-\lambda) \leq \frac{h(\|x_2\| - \|x_1\|)}{h(\|x_3\| - \|x_1\|)}. \tag{52}$$

Therefore, (49) follows from (50), (51), and (52). The proof is complete.

Using Lemma 1, we get

Corollary 1 *Let $f : D \to (0, \infty)$ be a $(\alpha, \beta, \lambda, h)$ convex functional, and $h : (0, 1) \to (0, \infty)$ is a super-multiplicative function, then for all $x_1, x_2, x_3 \in D$, such that$\|x_1\|^\alpha < \|x_2\|^\alpha < \|x_3\|^\alpha$, and $0 < \|x_3\|^\alpha - \|x_1\|^\alpha < 1$, the following Schur-type inequalities holds:*

$$f^\beta(\|x_2\|) \leq \frac{h(\|x_3\|^\alpha - \|x_2\|^\alpha)}{h(\|x_3\|^\alpha - \|x_1\|^\alpha)} f^\beta(\|x_1\|) + \frac{h(\|x_2\|^\alpha - \|x_1\|^\alpha)}{h(\|x_3\|^\alpha - \|x_1\|^\alpha)} f^\beta(\|x_3\|). \tag{53}$$

Corollary 2 *Let $f : (0, \infty) \to (0, \infty)$ be a $(\alpha, \beta, \lambda, h)$ convex function and $h : (0, 1) \to (0, \infty)$ is a super-multiplicative function, then for all $x_1, x_2, x_3 \in (0, \infty)$, such that $x_1^\alpha < x_2^\alpha < x_3^\alpha$, and $0 < x_3^\alpha - x_1^\alpha < 1$, the following generalized Schur inequality holds:*

$$f^\beta(x_2) \leq \frac{h(x_3^\alpha - x_2^\alpha)}{h(x_3^\alpha - x_1^\alpha)} f^\beta(x_1) + \frac{h(x_2^\alpha - x_1^\alpha)}{h(x_3^\alpha - x_1^\alpha)} f^\beta(x_3). \tag{54}$$

Next by using the definition of (λ, t, ξ, h) convex functional and induction, one obtains the following new generalized Jensen inequality:

Theorem 2 *Let $f : D \to (0, \infty)$ be a (λ, t, ξ, h) convex functional and $h : (0, 1) \to (0, \infty)$ is a super-multiplicative function, then*

$$f(\sum_{k=1}^n \lambda_k \|x_k\|) \leq \sum_{k=1}^n h(t_k^\xi) f(\|x_k\|), \tag{55}$$

for any $x_k \in D$, $\lambda_k, t_k, \xi \in [0, 1]$, $1 \leq k \leq n$, with $\sum_{k=1}^n \lambda_k = 1$ and $\sum_{k=1}^n t_k^\xi = 1$.

Proof For $n = 2$, this is just the definition of (λ, t, ξ, h) convex functional, and for $n > 2$ it follows by induction. Assume that (55) is true for some positive integer $n > 2$, we shall prove that

$$f(\sum_{k=1}^{n+1}\lambda_k\|x_k\|)\le\sum_{k=1}^{n+1}h(t_k^{\xi})f(\|x_k\|), \tag{56}$$

for any $x_k \in D$, $\lambda_k, t_k, \xi \in [0,1], 1 \le k \le n+1$, with $\sum_{k=1}^{n+1}\lambda_k = 1$ and $\sum_{k=1}^{n+1}t_k^{\xi}=1$.

To show that (56) is true, we note that

$$\begin{aligned}
f(\sum_{k=1}^{n+1}\lambda_k\|x_k\|) &= f(\sum_{k=1}^{n-1}\lambda_k\|x_k\|+\lambda_n\|x_n\|+\lambda_{n+1}\|x_{n+1}\|)\\
&= f\{\sum_{k=1}^{n-1}\lambda_k\|x_k\|+(\lambda_n+\lambda_{n+1})(\frac{\lambda_n}{\lambda_n+\lambda_{n+1}}\|x_n\|+\frac{\lambda_{n+1}}{\lambda_n+\lambda_{n+1}}\|x_{n+1}\|)\}\\
&\le \sum_{k=1}^{n-1}h(t_k^{\xi})f(\|x_k\|)+h(t_n^{\xi}+t_{n+1}^{\xi})f(\frac{\lambda_n}{\lambda_n+\lambda_{n+1}}\|x_n\|+\frac{\lambda_{n+1}}{\lambda_n+\lambda_{n+1}}\|x_{n+1}\|).
\end{aligned} \tag{57}$$

By (48), we get

$$\begin{aligned}
h(t_n^{\xi}) &= h(\frac{t_n^{\xi}}{t_n^{\xi}+t_{n+1}^{\xi}}\times(t_n^{\xi}+t_{n+1}^{\xi}))\\
&\ge h(\frac{t_n^{\xi}}{t_n^{\xi}+t_{n+1}^{\xi}})h(t_n^{\xi}+t_{n+1}^{\xi}),
\end{aligned}$$

that is,

$$h(\frac{t_n^{\xi}}{t_n^{\xi}+t_{n+1}^{\xi}})\le\frac{h(t_n^{\xi})}{h(t_n^{\xi}+t_{n+1}^{\xi})}. \tag{58}$$

Similarly, we get

$$h(\frac{t_{n+1}^{\xi}}{t_n^{\xi}+t_{n+1}^{\xi}})\le\frac{h(t_{n+1}^{\xi})}{h(t_n^{\xi}+t_{n+1}^{\xi})}. \tag{59}$$

Using (9), (58), and (59), we have

$$f(\frac{\lambda_n}{\lambda_n+\lambda_{n+1}}\|x_n\|+\frac{\lambda_{n+1}}{\lambda_n+\lambda_{n+1}}\|x_{n+1}\|)$$

$$\leq h(\frac{t_n^\xi}{t_n^\xi + t_{n+1}^\xi}) f(\|x_n\|) + h(\frac{t_{n+1}^\xi}{t_n^\xi + t_{n+1}^\xi}) f(\|x_{n+1}\|)$$

$$\leq \frac{1}{h(t_n^\xi + t_{n+1}^\xi)} \{h(t_n^\xi) f(\|x_n\|) + h(t_{n+1}^\xi) f(\|x_{n+1}\|)\}. \quad (60)$$

Hence, (56) follows from (57) and (60). The proof is complete.

Corollary 3 *Let $f : D \to (0, \infty)$ be a $(\alpha, \beta, \lambda, t, \xi, h)$ convex functional and $h : (0, 1) \to (0, \infty)$ is a super-multiplicative function, then*

$$f((\sum_{k=1}^n \lambda_k \|x_k\|^\alpha)^{1/\alpha}) \leq \{\sum_{k=1}^n h(t_k^\xi) f^\beta(\|x_k\|)\}^{1/\beta},$$

for any $x_k \in D, \lambda_k, t_k, \xi \in [0, 1], 1 \leq k \leq n$, with $\sum_{k=1}^n \lambda_k = 1$ and $\sum_{k=1}^n t_k^\xi = 1$.

Corollary 4 *Let $f : (0, \infty) \to (0, \infty)$ be a $(\alpha, \beta, \lambda, t, \xi, h)$ convex function and $h : (0, 1) \to (0, \infty)$ is a super-multiplicative function, then*

$$f((\sum_{k=1}^n \lambda_k x_k^\alpha)^{1/\alpha}) \leq \{\sum_{k=1}^n h(t_k^\xi) f^\beta(x_k)\}^{1/\beta},$$

for any $x_k \in (0, \infty), \lambda_k, t_k, \xi \in [0, 1]$, $1 \leq k \leq n$, with $\sum_{k=1}^n \lambda_k = 1$ and $\sum_{k=1}^n t_k^\xi = 1$.

Corollary 5 *Let $f : (0, \infty) \to (0, \infty)$ be a $(\alpha, \beta, \lambda, t, s)$ convex function, then*

$$f((\sum_{k=1}^n \lambda_k x_k^\alpha)^{1/\alpha}) \leq \{\sum_{k=1}^n t_k^s f^\beta(x_k)\}^{1/\beta},$$

for any $x_k \in (0, \infty), \lambda_k, t_k, s \in [0, 1], 1 \leq k \leq n$, with $\sum_{k=1}^n \lambda_k = 1$, and $\sum_{k=1}^n t_k^s = 1$.

Corollary 6 *Let $f : (0, \infty) \to (0, \infty)$ be a (λ, t, s) convex function, then*

$$f(\sum_{k=1}^n \lambda_k x_k) \leq \sum_{k=1}^n t_k^s f(x_k),$$

for any $x_k \in (0, \infty), \lambda_k, t_k, s \in [0, 1], 1 \leq k \leq n$, with $\sum_{k=1}^n \lambda_k = 1$, and $\sum_{k=1}^n t_k^s = 1$.

4 New Generalized Hermite-Hadamard Type Inequalities

In this section, we present a counterpart of the Hermite-Hadamard type inequality for $(\alpha, \beta, \lambda, \lambda_0, h)$ convex functional. In what follows, we write

$$E_n(p)=\{x=(x_1, x_2, \cdots, x_n):x_k\in\mathbb{R}^1,\ 1\le k\le n,\ \|x\|_p=(\sum_{k=1}^{n}|x_k|^p)^{1/p},\ 1\le p<\infty\},$$

$$B(0,r)=\{x\in E_n(p):\|x\|_p\le r\}.$$

In particular, $E_n(2)$ is an n-dimensional Euclidean space $\mathbb{R}^n$.

Theorem 3 *Let $B(0,r_1)$ be an n-ball of radius r_1 in $E_n(p)$, $E = B(0,r_2) - B(0,r_1), 0 < r_1 < r_2 < \infty$. Let $f : E \to (0,\infty)$ be a $(\alpha, \beta, \lambda, \lambda_0, h)$ convex functional. If $\int_E \|x\|_p^{\alpha-n} f^\beta(\|x\|_p)dx < \infty$, and $h \in L(0,1)$, then*

$$\begin{aligned}
&\frac{1}{2h(1/2)} f^\beta((\frac{r_1^{\alpha/p}+\lambda_0 r_2^{\alpha/p}}{2})^{1/\alpha})\\
&\le \frac{\alpha p^{n-1}\Gamma(n/p)}{(\lambda_0 r_2^{\alpha/p}-r_1^{\alpha/p})(\Gamma(1/p))^n}\int_E \|x\|_p^{\alpha-n} f^\beta(\|x\|_p)dx\\
&\le \{f^\beta(r_1^{1/p})+\lambda_0 f^\beta(r_2^{1/p})\}\int_0^1 h(u)du,
\end{aligned}\tag{61}$$

where $\Gamma(\alpha)$ is the Gamma function:

$$\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx\ \ (\alpha>0).$$

Proof By transforming the integral to polar coordinates (see [9]), we have

$$\int_E \|x\|_p^{\alpha-n} f^\beta(\|x\|_p)dx = \frac{(\Gamma(1/p))^n}{p^n\Gamma(n/p)}\int_{r_1}^{r_2} r^{(\alpha/p)-1} f^\beta(r^{1/p})dr. \tag{62}$$

Setting $r = \left(\frac{r_2-u}{r_2-r_1} r_1^{\alpha/p} + \lambda_0 \frac{u-r_1}{r_2-r_1} r_2^{\alpha/p}\right)^{p/\alpha}$, we have

$$\begin{aligned}
\int_{r_1}^{r_2} r^{(\alpha/p)-1} f^\beta(r^{1/p})dr &= \frac{p(\lambda_0 r_2^{\alpha/p}-r_1^{\alpha/p})}{\alpha(r_2-r_1)}\\
&\times\int_{r_1}^{r_2} f^\beta((\frac{r_2-u}{r_2-r_1}r_1^{\alpha/p}+\lambda_0\frac{u-r_1}{r_2-r_1}r_2^{\alpha/p})^{1/\alpha})du.
\end{aligned}\tag{63}$$

By (5) in Definition 2, we get

$$f^{\beta}((\frac{r_2-u}{r_2-r_1}r_1^{\alpha/p}+\lambda_0\frac{u-r_1}{r_2-r_1}r_2^{\alpha/p})^{1/\alpha}) \leq h(\frac{r_2-u}{r_2-r_1})f^{\beta}(r_1^{1/p})+\lambda_0 h(\frac{u-r_1}{r_2-r_1})f^{\beta}(r_2^{1/p}). \quad (64)$$

Thus, by (62), (63), and (64), we obtain

$$\int_E \|x\|_p^{\alpha-n} f^{\beta}(\|x\|_p)dx \leq \frac{(\Gamma(1/p))^n(\lambda_0 r_2^{\alpha/p}-r_1^{\alpha/p})}{\alpha p^{n-1}(r_2-r_1)\Gamma(n/p)} \times\{f^{\beta}(r_1^{1/p})\int_{r_1}^{r_2} h(\frac{r_2-u}{r_2-r_1})du+\lambda_0 f^{\beta}(r_2^{1/p})\int_{r_1}^{r_2} h(\frac{u-r_1}{r_2-r_1})du\} = \frac{(\Gamma(1/p))^n(\lambda_0 r_2^{\alpha/p}-r_1^{\alpha/p})}{\alpha p^{n-1}\Gamma(n/p)}\{f^{\beta}(r_1^{1/p})+\lambda_0 f^{\beta}(r_2^{1/p})\}\int_0^1 h(u)du$$

which gives the right-hand inequality in (61).

To show the left-hand inequality in (61), setting $u=\frac{1}{2}(r_1+r_2)+t$, then

$$r^{\alpha/p}=\frac{r_2-u}{r_2-r_1}r_1^{\alpha/p}+\lambda_0\frac{u-r_1}{r_2-r_1}r_2^{\alpha/p} = \frac{1}{2}(r_1^{\alpha/p}+\lambda_0 r_2^{\alpha/p})+\frac{\lambda_0 r_2^{\alpha/p}-r_1^{\alpha/p}}{r_2-r_1}t. \quad (65)$$

Setting

$$\|x_1\|_p=\{\frac{1}{2}(r_1^{\alpha/p}+\lambda_0 r_2^{\alpha/p})-\frac{\lambda_0 r_2^{\alpha/p}-r_1^{\alpha/p}}{r_2-r_1}t\}^{p/\alpha},$$

$$\|x_2\|_p=\{\frac{1}{2}(r_1^{\alpha/p}+\lambda_0 r_2^{\alpha/p})+\frac{\lambda_0 r_2^{\alpha/p}-r_1^{\alpha/p}}{r_2-r_1}t\}^{p/\alpha}$$

we get

$$\|x_1\|_p^{\alpha/p}+\|x_2\|_p^{\alpha/p}=r_1^{\alpha/p}+\lambda_0 r_2^{\alpha/p}.$$

Thus, by the definition of (α,β,λ,h) convex functional, we have

$$f^{\beta}((\frac{r_1^{\alpha/p}+\lambda_0 r_2^{\alpha/p}}{2})^{1/\alpha})=f^{\beta}((\frac{1}{2}\|x_1\|_p^{\alpha/p}+\frac{1}{2}\|x_2\|_p^{\alpha/p})^{1/\alpha}) \leq h(1/2)\{f^{\beta}(\|x_1\|_p^{1/p})+f^{\beta}(\|x_2\|_p^{1/p})\}. \quad (66)$$

Hence, by (63), (65), and (66), we get

$$\int_{r_1}^{r_2} r^{\frac{\alpha}{p}-1} f^{\beta}(r^{1/p})dr = \frac{p(\lambda_0 r_2^{\alpha/p} - r_1^{\alpha/p})}{\alpha(r_2 - r_1)}$$

$$\times \int_{-(r_2-r_1)/2}^{(r_2-r_1)/2} f^{\beta}((\frac{1}{2}(r_1^{\alpha/p} + \lambda_0 r_2^{\alpha/p}) + \frac{\lambda_0 r_2^{\alpha/p} - r_1^{\alpha/p}}{r_2 - r_1} t)^{1/\alpha})dt$$

$$= \frac{p(\lambda_0 r_2^{\alpha/p} - r_1^{\alpha/p})}{\alpha(r_2 - r_1)} \int_0^{(r_2-r_1)/2} (f^{\beta}(\|x_1\|_p^{1/p}) + f^{\beta}(\|x_2\|_p^{1/p}))dt$$

$$\geq \frac{p(\lambda_0 r_2^{\alpha/p} - r_1^{\alpha/p})}{\alpha(r_2 - r_1)h(1/2)} \int_0^{(r_2-r_1)/2} f^{\beta}((\frac{r_1^{\alpha/p} + \lambda_0 r_2^{\alpha/p}}{2})^{1/\alpha})dt$$

$$= \frac{p(\lambda_0 r_2^{\alpha/p} - r_1^{\alpha/p})}{2\alpha h(1/2)} f^{\beta}((\frac{r_1^{\alpha/p} + \lambda_0 r_2^{\alpha/p}}{2})^{1/\alpha}). \tag{67}$$

By (62) and (67), we get

$$\int_E \|x\|_p^{\alpha-n} f^{\beta}(\|x\|_p)dx = \frac{(\Gamma(1/p))^n}{p^n \Gamma(n/p)} \int_{r_1}^{r_2} r^{(\alpha/p)-1} f^{\beta}(r^{1/p})dr$$

$$\geq \frac{(\Gamma(1/p))^n (\lambda_0 r_2^{\alpha/p} - r_1^{\alpha/p})}{2\alpha p^{n-1} h(1/2)\Gamma(n/p)} f^{\beta}((\frac{r_1^{\alpha/p} + \lambda_0 r_2^{\alpha/p}}{2})^{1/\alpha}),$$

which finishes the proof.

Corollary 7 *Let* $f : E_n(p) \to (0, \infty)$ *be a* $(\alpha, \beta, \lambda, \lambda_0, h)$ *convex functional. If* $h \in L(0, 1)$ *and* $\int_{B(0,r)} \|x\|_p^{\alpha-n} f^{\beta}(\|x\|_p)dx < \infty$*, then*

$$\frac{1}{2h(1/2)} f^{\beta}((\frac{\lambda_0}{2})^{1/\alpha} r^{1/p}) \leq \frac{\alpha p^{n-1}\Gamma(n/p)}{\lambda_0 r^{\alpha/p}(\Gamma(1/p))^n} \int_{B(0,r)} \|x\|_p^{\alpha-n} f^{\beta}(\|x\|_p)dx$$

$$\leq \{f^{\beta}(0) + \lambda_0 f^{\beta}(r^{1/p})\} \int_0^1 h(u)du. \tag{68}$$

Corollary 8 *Let* $f : (0, \infty) \to (0, \infty)$ *be a* $(\alpha, \beta, \lambda, \lambda_0, h)$ *convex function. If* $h \in L(0, 1)$*, and* $\int_a^b x^{\alpha-1} f^{\beta}(x)dx < \infty$*,* $0 < a < b < \infty$*, then*

$$\frac{1}{2h(1/2)} f^{\beta}((\frac{a^{\alpha} + \lambda_0 b^{\alpha}}{2})^{1/\alpha}) \leq \frac{\alpha}{\lambda_0 b^{\alpha} - a^{\alpha}} \int_a^b x^{\alpha-1} f^{\beta}(x)dx$$

$$\leq \{f^{\beta}(a) + \lambda_0 f^{\beta}(b)\} \int_0^1 h(u)du. \tag{69}$$

Corollary 9 ([4]) *Let $f : (0, \infty) \to (0, \infty)$ be an h-convex function. If $h \in L(0, 1)$, $f \in L[a, b]$, $[a, b] \subset (0, \infty)$, then*

$$\frac{1}{2h(1/2)} f(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \{f(a) + f(b)\} \int_0^1 h(u)du. \tag{70}$$

In particular, if $h(t) = t^s, 0 < s \leq 1$, then (70) reduces to (25) in [4]; if $h(t) = t^{-s}, 0 < s \leq 1$, then (70) reduces to (26) in [4].

Remark 1 If $\beta = 1$ and $\lambda_0 = 1$, then (69) reduces to Theorem 5 in [5]. For $h(t) = t, \lambda_0 = 1$ in (69), we get

$$f^\beta((\frac{a^\alpha + b^\alpha}{2})^{1/\alpha}) \leq \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b x^{\alpha-1} f^\beta(x)dx \leq \frac{1}{2}\{f^\beta(a) + f^\beta(b)\}. \tag{71}$$

For $\alpha = 1$ in (71), we get

$$f^\beta(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f^\beta(x)dx \leq \frac{1}{2}\{f^\beta(a) + f^\beta(b)\}. \tag{72}$$

If $\beta = 1$, then (72) reduces to the classical Hermite-Hadamard inequality:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{1}{2}\{f(a) + f(b)\}. \tag{73}$$

Remark 2 Inequality (72) is proved by Yang Zhen-hang, but he adds the conditions: $\beta \geq 1$ and $f(x), f''(x) > 0$. (see [11, P. 12]).

Theorem 4 *Let $f : (0, \infty) \to (0, \infty)$ be a $(\alpha, \beta, \lambda, h)$ convex function and $h : (0, 1) \to (0, \infty)$ is a super-multiplicative function, if $[a, b] \subset (0, \infty)$ and $a^\alpha = x_1^\alpha < x_2^\alpha < \cdots < x_n^\alpha = b^\alpha$ be equidistant points, then*

$$\frac{1}{h(1/n)} f^\beta((\frac{x_1^\alpha + x_n^\alpha}{2})^{1/\alpha}) \leq \sum_{k=1}^n f^\beta(x_k) \leq f^\beta(x_1) \sum_{k=1}^n h(1-\lambda_k) + f^\beta(x_n) \sum_{k=1}^n h(\lambda_k), \tag{74}$$

where $\lambda_k = \frac{k-1}{n-1}, k = 1, 2, \cdots, n, n > 1$.

Proof Since the points $x_1^\alpha, \cdots, x_n^\alpha$ are equidistant, putting $t = \frac{x_n^\alpha - x_1^\alpha}{n-1}$, we have $x_k^\alpha = x_1^\alpha + (k-1)t$, $k = 1, 2, \cdots, n$ and $\frac{1}{n}\sum_{k=1}^n x_k^\alpha = \frac{1}{2}(x_1^\alpha + x_n^\alpha)$. By Corollary 4, we get

$$\begin{aligned} f^\beta((\frac{x_1^\alpha + x_n^\alpha}{2})^{1/\alpha}) &= f^\beta((\frac{1}{n}\sum_{k=1}^n x_k^\alpha)^{1/\alpha}) \\ &\le \sum_{k=1}^n h(\frac{1}{n}) f^\beta(x_k) = h(\frac{1}{n})\sum_{k=1}^n f^\beta(x_k), \end{aligned}$$

which gives the left-hand inequality in (74).

To show the right-hand inequality in (74), we note that $x_k^\alpha = x_1^\alpha + (k-1)t$ can be written as $x_k^\alpha = (1-\lambda_k)x_1^\alpha + \lambda_k x_n^\alpha$, where $\lambda_k = \frac{k-1}{n-1}$, $k = 1, 2, \cdots, n$. By the definition of $(\alpha, \beta, \lambda, h)$ convex function, we get

$$\begin{aligned} f^\beta(x_k) &= f^\beta(((1-\lambda_k)x_1^\alpha + \lambda_k x_n^\alpha)^{1/\alpha}) \\ &\le h(1-\lambda_k) f^\beta(x_1) + h(\lambda_k) f^\beta(x_n). \end{aligned}$$

Summing up the above inequalities, we get

$$\sum_{k=1}^n f^\beta(x_k) \le f^\beta(x_1)\sum_{k=1}^n h(1-\lambda_k) + f^\beta(x_n)\sum_{k=1}^n h(\lambda_k),$$

which finishes the proof.

Corollary 10 *Let $f : (0, \infty) \to (0, \infty)$ be an h-convex function and $h : (0, 1) \to (0, \infty)$ is a super-multiplicative function. If $[a, b] \subset (0, \infty)$ and $a = x_1 < x_2 < \cdots < x_n = b$ be equidistant points, then*

$$\begin{aligned} \frac{1}{h(1/n)} f(\frac{x_1 + x_n}{2}) &\le \sum_{k=1}^n f(x_k) \\ &\le f(x_1)\sum_{k=1}^n h(1-\lambda_k) + f(x_n)\sum_{k=1}^n h(\lambda_k), \end{aligned} \tag{75}$$

where $\lambda_k = \frac{k-1}{n-1}$, $k = 1, 2, \cdots, n$, $n > 1$.

Remark 3 Using Lemma 1, we also obtain Theorem 4 from Corollary 10. For $h(t) = t^s$, $0 < |s| \le 1$ in (74), we get

$$f^\beta((\frac{x_1^\alpha + x_n^\alpha}{2})^{1/\alpha}) \le \frac{1}{n^s}\sum_{k=1}^n f^\beta(x_n) \le \frac{1}{(n(n-1))^s}(\sum_{k=1}^{n-1} k^s)\{f^\beta(x_1) + f^\beta(x_n)\}.$$

In particular, when $s = 1$, we get

$$f^{\beta}((\frac{x_1^{\alpha}+x_n^{\alpha}}{2})^{1/\alpha}) \le \frac{1}{n}\sum_{k=1}^{n} f^{\beta}(x_k) \le \frac{1}{2}\{f^{\beta}(x_1)+f^{\beta}(x_n)\}. \tag{76}$$

If $\alpha = \beta = 1$, then (76) reduces to the discrete analogous of the classical Hermite-Hadamard inequality (73) (see [13]):

$$f(\frac{x_1+x_n}{2}) \le \frac{1}{n}\sum_{k=1}^{n} f(x_k) \le \frac{1}{2}\{f(x_1)+f(x_n)\}.$$

5 Strongly Convex Functionals in Normed Linear Spaces

Strongly convex functions have been introduced by Polyak [17] and they play an important role in optimization theory, mathematical economics, and other branches of pure and applied mathematics. Many properties and applications of them can be found in the literature (see, for instance, [1, 4, 10, 13, 14, 16, 17], and the references therein).

In what follows, $(X, \|\cdot\|)$ denotes the real normed linear spaces, D be a convex subset of X, $h : (0, 1) \to (0, \infty)$ is a given function and c be a positive constant.

Definition 4 (See [13]) A function $f : D \to \mathbb{R}$ is called strongly convex with modulus c, if

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2) - c\lambda(1-\lambda)\|x_1 - x_2\|^2, \tag{77}$$

$\forall x_1, x_2 \in D, \forall \lambda \in [0, 1]$.

In this section, we introduce a new class of strongly convex functional with modulus c in real normed linear spaces, that is, $(\alpha, \beta, \lambda, t, h)$ strongly convex functional with modulus c in real normed linear spaces, and present the new Schur, Jensen, and Hermite-Hadamard type inequalities for these strongly convex functional with modulus c. They are significant generalizations of the corresponding inequalities for the classical convex functions.

Definition 5 A functional $f : D \to (0, \infty)$ is said to be a $(\alpha, \beta, \lambda, t, h)$ strongly convex with modulus c, if

$$\begin{aligned} f^{\beta}((\lambda\|x_1\|^{\alpha} + (1-\lambda)\|x_2\|^{\alpha})^{1/\alpha}) &\le h(t) f^{\beta}(\|\|x_1\|) + h(1-t) f^{\beta}(\|x_2\|) \\ &\quad - ch(t)h(1-t)|\|x_1\|^{\alpha} - \|x_2\|^{\alpha}|^2, \end{aligned} \tag{78}$$

$\forall x_1, x_2 \in D, \forall \lambda, t \in [0, 1], \alpha, \beta$ are real numbers, and $\alpha, \beta \neq 0$.

For $c = 0$ in (78), we get

$$f((\lambda\|x_1\|^\alpha + (1-\lambda)\|x_2\|^\alpha)^{1/\alpha}) \leq \{h(t)f^\beta(\|x_1\|) + h(1-t)f^\beta(\|x_2\|)\}^{1/\beta}, \tag{79}$$

that is, f reduces to $(\alpha, \beta, \lambda, t, h)$ convex functional in Section 2.

For $t = \lambda$ in (78), that is,

$$f^\beta((\lambda\|x_1\|^\alpha + (1-\lambda)\|x_2\|^\alpha)^{1/\alpha}) \leq h(\lambda)f^\beta(\|x_1\|) + h(1-\lambda)f^\beta(\|x_2\|) - ch(\lambda)h(1-\lambda)|\|x_1\|^\alpha - \|x_2\|^\alpha|^2, \tag{80}$$

then f is said to be a $(\alpha, \beta, \lambda, h)$ strongly convex functional with modulus c. If $c = 0$ in (80), then f reduces to $(\alpha, \beta, \lambda, h)$ convex functional in Section 2.

If $h(\lambda) = \lambda^s, 0 < |s| \leq 1$, then $(\alpha, \beta, \lambda, t, h)$, $(\alpha, \beta, \lambda, h)$ strongly convex functional with modulus c reduce to $(\alpha, \beta, \lambda, t, s)$, $(\alpha, \beta, \lambda, s)$ strongly convex functional with modulus c, respectively. In particular, if $s = 1$, then f is said to be a $(\alpha, \beta, \lambda, t)$, (α, β, λ) strongly convex functional with modulus c, respectively.

If $D = (0, \infty)$ in (78), that is, if a function $f : (0, \infty) \to (0, \infty)$ satisfies

$$f^\beta((\lambda x_1^\alpha + (1-\lambda)x_2^\alpha)^{1/\alpha}) \leq h(t)f^\beta(x_1) + h(1-t)f^\beta(x_2) - ch(t)h(1-t)|x_2^\alpha - x_1^\alpha|^2, \tag{81}$$

$\forall x_1, x_2 \in (0, \infty)$, $\forall \lambda, t \in [0, 1]$, α, β are real numbers, and $\alpha, \beta \neq 0$, then f is said to be a $(\alpha, \beta, \lambda, t, h)$ strongly convex function with modulus c.

For $c = 0$ in (81), we get

$$f((\lambda x_1^\alpha + (1-\lambda)x_2^\alpha)^{1/\alpha}) \leq \{h(t)f^\beta(x_1) + h(1-t)f^\beta(x_2)\}^{1/\beta}, \tag{82}$$

that is, f reduces to $(\alpha, \beta, \lambda, t, h)$ convex function in Section 2.

If $t = \lambda$ in (81), that is,

$$f^\beta((\lambda x_1^\alpha + (1-\lambda)x_2^\alpha)^{1/\alpha}) \leq h(\lambda)f^\beta(x_1) + h(1-\lambda)f^\beta(x_2) - ch(\lambda)h(1-\lambda)|x_2^\alpha - x_1^\alpha|^2, \tag{83}$$

$\forall x_1, x_2 \in (0, \infty)$, $\forall \lambda \in [0, 1]$, α, β are real numbers, and $\alpha, \beta \neq 0$, then f is said to be a $(\alpha, \beta, \lambda, h)$ strongly convex function with modulus c.

For $c = 0$ in (83), we get

$$f((\lambda x_1^\alpha + (1-\lambda)x_2^\alpha)^{1/\alpha}) \leq \{h(\lambda)f^\beta(x_1) + h(1-\lambda)f^\beta(x_2)\}^{1/\beta}, \tag{84}$$

that is, f reduces to $(\alpha, \beta, \lambda, h)$ convex function in Section 2.

If $h(\lambda) = \lambda^s$, $0 < |s| \le 1$ in (83), we get

$$f^{\beta}((\lambda x_1^{\alpha} + (1-\lambda)x_2^{\alpha})^{1/\alpha}) \le \lambda^s f^{\beta}(x_1) + (1-\lambda)^s f^{\beta}(x_2) - c\lambda^s(1-\lambda)^s|x_2^{\alpha} - x_1^{\alpha}|^2, \tag{85}$$

then f is said to be a $(\alpha, \beta, \lambda, s)$ strongly convex function with modulus c.

For $c = 0$ in (85), that is,

$$f((\lambda x_1^{\alpha} + (1-\lambda)x_2^{\alpha})^{1/\alpha}) \le \{\lambda^s f^{\beta}(x_1) + (1-\lambda)^s f^{\beta}(x_2)\}^{1/\beta}, \tag{86}$$

that is, f reduces to $(\alpha, \beta, \lambda, s)$ convex function in Section 2.

Remark 4 If $\alpha = \beta = 1$, $s = 1$, then (86) reduces to the classical convex function. In fact, the notion of $(\alpha, \beta, \lambda, t, h)$ strongly convex functional with modulus c unifies and generalizes the many known and new classes of convex functions, see, e.g., [1, 10, 13, 14, 16, 17], and the references therein.

6 New Schur, Jensen, Hermite-Hadamard Type Inequalities

In this section, we present the Schur, Jensen, and Hermite-Hadamard type inequalities for (α, β, λ) strongly convex functional with modulus c.

Lemma 2 *Let $g = \{f^{\beta} - c\|\cdot\|^{2\alpha}\}^{1/\beta}$ with $f^{\beta}(\|x\|) \ge c\|x\|^{2\alpha}$, $x \in D$, then a functional $f : D \to (0, \infty)$ is (α, β, λ) strongly convex with modulus c if and only if the functional $g : D \to (0, \infty)$ is (α, β, λ) convex.*

Proof Assume that f is (α, β, λ) strongly convex with modulus c, then

$$\begin{aligned}
&g^{\beta}((\lambda\|x_1\|^{\alpha} + (1-\lambda)\|x_2\|^{\alpha})^{1/\alpha}) \\
&\quad = f^{\beta}((\lambda\|x_1\|^{\alpha} + (1-\lambda)\|x_2\|^{\alpha})^{1/\alpha}) - c|(\lambda\|x_1\|^{\alpha} + (1-\lambda)\|x_2\|^{\alpha})^{1/\alpha}|^{2\alpha} \\
&\quad \le \lambda f^{\beta}(\|x_1\|) + (1-\lambda) f^{\beta}(\|x_2\|) - c\lambda(1-\lambda)|\|x_1\|^{\alpha} - \|x_2\|^{\alpha}|^2 \\
&\quad -c|\lambda\|x_1\|^{\alpha} + (1-\lambda)\|x_2\|^{\alpha}|^2 \\
&\quad = \lambda f^{\beta}(\|x_1\|) + (1-\lambda) f^{\beta}(\|x_2\|) - c\lambda\|x_1\|^{2\alpha} - c(1-\lambda)\|x_2\|^{2\alpha} \\
&\quad = \lambda g^{\beta}(\|x_1\|) + (1-\lambda) g^{\beta}(\|x_2\|),
\end{aligned}$$

which proves that g is (α, β, λ) convex.

Conversely, if g is (α, β, λ) convex, then

$$\begin{aligned}
&f^{\beta}((\lambda\|x_1\|^{\alpha} + (1-\lambda)\|x_2\|^{\alpha})^{1/\alpha}) \\
&= g^{\beta}((\lambda\|x_1\|^{\alpha} + (1-\lambda)\|x_2\|^{\alpha})^{1/\alpha}) + c\|(\lambda\|x_1\|^{\alpha} + (1-\lambda)\|x_2\|^{\alpha})^{1/\alpha}\|^{2\alpha} \\
&\le \lambda g^{\beta}(\|x_1\|) + (1-\lambda) g^{\beta}(\|x_2\|) + c|\lambda\|x_1\|^{\alpha} + (1-\lambda)\|x_2\|^{\alpha}|^2 \\
&= \lambda f^{\beta}(\|x_1\|) + (1-\lambda) f^{\beta}(\|x_2\|) - c\lambda(1-\lambda)|\|x_1\|^{\alpha} - \|x_2\|^{\alpha}|^2,
\end{aligned}$$

which proves that f is (α, β, λ) strongly convex with modulus c.

Using Corollary 1 (with $h(t) = t$) and Lemma 2, and the definition of (α, β, λ) strongly convex with modulus c, we get

Theorem 5 *Let a functional $f : D \to (0, \infty)$ be (α, β, λ) strongly convex with modulus c, and $h : (0, 1) \to (0, \infty)$ is a super-multiplicative function, then for all $x_1, x_2, x_3 \in D$, such that $\|x_1\|^\alpha < \|x_2\|^\alpha < \|x_3\|^\alpha$, and $0 < \|x_3\|^\alpha - \|x_1\|^\alpha < 1$, the following Schur-type inequalities holds:*

$$f^\beta(\|x_2\|) \le \frac{\|x_3\|^\alpha - \|x_2\|^\alpha}{\|x_3\|^\alpha - \|x_1\|^\alpha} f^\beta(\|x_1\|) + \frac{\|x_2\|^\alpha - \|x_1\|^\alpha}{\|x_3\|^\alpha - \|x_1\|^\alpha} f^\beta(\|x_3\|)$$
$$- c\{\|x_2\|^{2\alpha} + (\|x_1\|^\alpha - \|x_2\|^\alpha)\|x_3\|^\alpha - \|x_1\|^\alpha \|x_2\|^\alpha\}. \tag{87}$$

Using Corollary 3 (with $h(\lambda_k = \lambda_k, t_k = \lambda_k, \xi_k = 1))$ and Lemma 2, and the definition of (α, β, λ) strongly convex with modulus c, one obtains the following new Jensen-type inequality:

Theorem 6 *Let a functional $f : D \to (0, \infty)$ be (α, β, λ) strongly convex with modulus c, and $f^\beta(\|x\|) \ge c\|x\|^{2\alpha}, x \in D$, and $h : (0, 1) \to (0, \infty)$ is a super-multiplicative function, then*

$$f^\beta((\sum_{k=1}^n \lambda_k \|x_k\|^\alpha)^{1/\alpha}) \le \sum_{k=1}^n \lambda_k f^\beta(\|x_k\|)$$
$$-c\{\sum_{k=1}^n \lambda_k \|x_k\|^{2\alpha} - (\sum_{k=1}^n \lambda_k \|x_k\|^\alpha)^2\}, \tag{88}$$

for any $x_k \in D, \lambda_k \in [0, 1], 1 \le k \le n$, with $\sum_{k=1}^n \lambda_k = 1$.

We present a counterpart of the Hermite-Hadamard inequality for $(\alpha, \beta, \lambda, h)$ strongly convex functional with modulus c. In what follows, we use the notations in Section 4.

Theorem 7 *Let $B(0, r_1)$ be an n-ball of radius r_1 in $E_n(p)$, $E = B(0, r_2) - B(0, r_1), 0 < r_1 < r_2 < \infty$. Let $f : E \to (0, \infty)$ be a $(\alpha, \beta, \lambda, h)$ strongly convex functional with modulus c. If $\int_E \|x\|_p^{\alpha-n} f^\beta(\|x\|_p)dx < \infty$, and $h \in L(0, 1)$, then*

$$\frac{1}{2h(1/2)} f^\beta((\frac{r_2^{\alpha/p} + r_1^{\alpha/p}}{2})^{1/\alpha}) + \frac{c}{6}(r_2^{\alpha/p} - r_1^{\alpha/p})^2$$
$$\le \frac{\alpha p^{n-1}\Gamma(n/p)}{(r_2^{\alpha/p} - r_1^{\alpha/p})\Gamma^n(1/p)} \int_E \|x\|_p^{\alpha-n} f^\beta(\|x\|_p)dx$$
$$\le \{f^\beta(r_1^{1/p}) + f^\beta(r_2^{1/p})\} \int_0^1 h(u)du$$
$$-c|r_2^{\alpha/p} - r_1^{\alpha/p}|^2 \int_0^1 h(t)h(1-t)dt. \tag{89}$$

Proof By transforming the integral to polar coordinates (see [9]), we have

$$\int_E \|x\|_p^{\alpha-n} f^\beta(\|x\|_p)dx = \frac{(\Gamma(1/p))^n}{p^n\Gamma(n/p)}\int_{r_1}^{r_2} r^{(\alpha/p)-1} f^\beta(r^{1/p})dr. \tag{90}$$

Setting $r = \left(\frac{r_2-u}{r_2-r_1}r_1^{\alpha/p} + \frac{u-r_1}{r_2-r_1}r_2^{\alpha/p}\right)^{p/\alpha}$, we have

$$\int_{r_1}^{r_2} r^{(\alpha/p)-1} f^\beta(r^{1/p})dr = \frac{p(r_2^{\alpha/p} - r_1^{\alpha/p})}{\alpha(r_2-r_1)} \times \int_{r_1}^{r_2} f^\beta\left(\left(\frac{r_2-u}{r_2-r_1}r_1^{\alpha/p} + \frac{u-r_1}{r_2-r_1}r_2^{\alpha/p}\right)^{1/\alpha}\right)du. \tag{91}$$

By the definition of $(\alpha, \beta, \lambda, h)$ strongly convex with modulus c, we get

$$\begin{aligned} f^\beta\left(\left(\frac{r_2-u}{r_2-r_1}r_1^{\alpha/p} + \frac{u-r_1}{r_2-r_1}r_2^{\alpha/p}\right)^{1/\alpha}\right) &\le h\left(\frac{r_2-u}{r_2-r_1}\right)f^\beta(r_1^{1/p}) + h\left(\frac{u-r_1}{r_2-r_1}\right)f^\beta(r_2^{1/p}) \\ &-ch\left(\frac{r_2-u}{r_2-r_1}\right)h\left(\frac{u-r_1}{r_2-r_1}\right)|r_2^{\alpha/p} - r_1^{\alpha/p}|^2. \end{aligned} \tag{92}$$

Thus, by (90), (91) and (92), we obtain

$$\begin{aligned} \int_E \|x\|_p^{\alpha-n} f^\beta(\|x\|_p)dx &\le \frac{(\Gamma(1/p))^n(r_2^\alpha - r_1^\alpha)}{\alpha p^{n-1}(r_2-r_1)\Gamma(n/p)} \\ &\times\left\{f^\beta(r_1^{1/p})\int_{r_1}^{r_2} h\left(\frac{r_2-u}{r_2-r_1}\right)du + f^\beta(r_2^{1/p})\int_{r_1}^{r_2} h\left(\frac{u-r_1}{r_2-r_1}\right)du\right. \\ &\left.-c|r_2^{\alpha/p} - r_1^{\alpha/p}|^2\int_{r_1}^{r_2} h\left(\frac{r_2-u}{r_2-r_1}\right)h\left(\frac{u-r_1}{r_2-r_1}\right)du\right\} \\ &= \frac{(\Gamma(1/\alpha))^n(r_2^{\alpha/p} - r_1^{\alpha/p})}{\alpha p^{n-1}\Gamma(n/p)}\{(f^\beta(r_1^{1/p}) + f^\beta(r_2^{1/p})) \\ &\times\int_0^1 h(t)dt - c|r_2^{\alpha/p} - r_1^{\alpha/p}|^2\int_0^1 h(t)h(1-t)dt\}, \end{aligned}$$

which gives the right-hand inequality in (89).

To show the left-hand inequality in (89), setting $u = \frac{1}{2}(r_1+r_2) + t$, then

$$\begin{aligned} r^{\alpha/p} &= \frac{r_2-u}{r_2-r_1}r_1^{\alpha/p} + \frac{u-r_1}{r_2-r_1}r_2^{\alpha/p} \\ &= \frac{1}{2}(r_1^{\alpha/p} + r_2^{\alpha/p}) + \frac{r_2^{\alpha/p} - r_1^{\alpha/p}}{r_2-r_1}t. \end{aligned} \tag{93}$$

Setting

$$\|x_1\|_p = \{\frac{1}{2}(r_1^{\alpha/p} + r_2^{\alpha/p}) - \frac{r_2^{\alpha/p} - r_1^{\alpha/p}}{r_2 - r_1} t\}^{p/\alpha}$$
$$\|x_2\|_p = \{\frac{1}{2}(r_1^{\alpha/p} + r_2^{\alpha/p}) + \frac{r_2^{\alpha/p} - r_1^{\alpha/p}}{r_2 - r_1} t\}^{p/\alpha}$$

we get

$$\|x_1\|_p^{\alpha/p} + \|x_2\|_p^{\alpha/p} = r_1^{\alpha/p} + r_2^{\alpha/p}$$
$$\|x_2\|_p^{\alpha/p} - \|x_1\|_p^{\alpha/p} = \frac{2(r_2^{\alpha} - r_1^{\alpha})}{r_2 - r_1} t.$$

Thus, by (80), we have

$$\begin{aligned} f^{\beta}((\frac{r_1^{\alpha/p} + r_2^{\alpha/p}}{2})^{1/\alpha}) &= f^{\beta}((\frac{1}{2}\|x_1\|_p^{\alpha/p} + \frac{1}{2}\|x_2\|_p^{\alpha/p})^{1/\alpha}) \\ &\le h(\frac{1}{2}) f^{\beta}(\|x_1\|_p^{1/p}) + h(\frac{1}{2}) f^{\beta}(\|x_2\|_p^{1/p}) - c(h(\frac{1}{2}))^2 |\|x_2\|_p^{\alpha/p} - \|x_1\|_p^{\alpha/p}|^2 \\ &= h(\frac{1}{2})\{f^{\beta}(\|x_1\|_p^{1/p}) + f^{\beta}(\|x_2\|_p^{1/p})\} - 4c(h(\frac{1}{2}))^2 (\frac{r_2^{\alpha/p} - r_1^{\alpha/p}}{r_2 - r_1})^2 t^2. \end{aligned} \tag{94}$$

Hence, by (91), (93), and (94), we get

$$\begin{aligned} \int_{r_1}^{r_2} r^{(\alpha/p)-1} f^{\beta}(r^{1/p}) dr &= \frac{p(r_2^{\alpha/p} - r_1^{\alpha/p})}{\alpha(r_2 - r_1)} \\ &\times \int_{-(r_2-r_1)/2}^{(r_2-r_1)/2} f^{\beta}((\frac{1}{2}(r_1^{\alpha/p} + r_2^{\alpha/p}) + \frac{r_2^{\alpha/p} - r_1^{\alpha/p}}{r_2 - r_1} t)^{1/\alpha}) dt \\ &= \frac{p(r_2^{\alpha/p} - r_1^{\alpha/p})}{\alpha(r_2 - r_1)} \int_0^{(r_2-r_1)/2} \{f^{\beta}(\|x_1\|_p^{1/p}) + f^{\beta}(\|x_2\|_p^{1/p})\} dt \\ &\ge \frac{p(r_2^{\alpha/p} - r_1^{\alpha/p})}{\alpha(r_2 - r_1) h(1/2)} \int_0^{(r_2-r_1)/2} \{f^{\beta}((\frac{r_1^{\alpha/p} + r_2^{\alpha/p}}{2})^{1/\alpha}) + 4ch(\frac{1}{2})(\frac{r_2^{\alpha/p} - r_1^{\alpha/p}}{r_2 - r_1})^2 t^2\} dt \\ &= \frac{p(r_2^{\alpha/p} - r_1^{\alpha/p})}{2\alpha h(1/2)} f^{\beta}((\frac{r_1^{\alpha/p} + r_2^{\alpha/p}}{2})^{1/\alpha}) + c\frac{p(r_2^{\alpha/p} - r_1^{\alpha/p})^3}{6\alpha}. \end{aligned} \tag{95}$$

By (90) and (95), we get

$$\int_E \|x\|_p^{\alpha-n} f^\beta(\|x\|_p)dx = \frac{(\Gamma(1/p))^n}{p^n\Gamma(n/p)}\int_{r_1}^{r_2} r^{(\alpha/p)-1} f^\beta(r^{1/p})dr$$
$$\geq \frac{(\Gamma(1/p))^n(r_2^{\alpha/p}-r_1^{\alpha/p})}{\alpha p^{n-1}\Gamma(n/p)}$$
$$\times \{\frac{1}{2h(1/2)} f^\beta((\frac{r_1^{\alpha/p}+r_2^{\alpha/p}}{2})^{1/\alpha}) + \frac{c}{6}(r_2^{\alpha/p}-r_1^{\alpha/p})^2\},$$

which finishes the proof.

Corollary 11 *Let* $X = \mathbb{R}^n$, $B(0, r_k)$ *be an n-ball of radius* r_k *in* $\mathbb{R}^n$, $E = B(0, r_2) - B(0, r_1)$, $0 < r_1 < r_2 < \infty$. *Let a functional* $f : E \to (0, \infty)$ *be* $(\alpha, \beta, \lambda, h)$ *strongly convex with modulus c,* $\int_E \|x\|_2^{\alpha-n} f^\beta(\|x\|_2)dx < \infty$, *and* $h \in L(0, 1)$, *then*

$$\frac{1}{2h(1/2)} f^\beta((\frac{r_2^{\alpha/2}+r_1^{\alpha/2}}{2})^{1/\alpha}) + \frac{c}{6}(r_2^{\alpha/2}-r_1^{\alpha/2})^2$$
$$\leq \frac{\alpha 2^{n-1}\Gamma(n/2)}{\pi^{n/2}(r_2^{\alpha/2}-r_1^{\alpha/2})}\int_E \|x\|_2^{\alpha-n} f^\beta(\|x\|_2)dx$$
$$\leq \{f^\beta(r_1^{1/2}) + f^\beta(r_2^{1/2})\}\int_0^1 h(t)dt - c|r_2^{\alpha/2}-r_1^{\alpha/2}|^2\int_0^1 h(t)h(1-t)dt. \quad (96)$$

Corollary 12 *Let a function* $f : (0, \infty) \to (0, \infty)$ *be* $(\alpha, \beta, \lambda, h)$ *strongly convex with modulus c. If* $\int_a^b x^{\alpha-1} f^\beta(x)dx < \infty$, $0 < a < b < \infty$, *and* $h \in L(0, 1)$, *then*

$$\frac{1}{2h(1/2)} f^\beta((\frac{a^\alpha+b^\alpha}{2})^{1/\alpha}) + \frac{c}{6}(b^\alpha-a^\alpha)^2$$
$$\leq \frac{\alpha}{b^\alpha-a^\alpha}\int_a^b x^{\alpha-1} f^\beta(x)dx$$
$$\leq \{f^\beta(a) + f^\beta(b)\}\int_0^1 h(t)dt - c(b^\alpha-a^\alpha)^2\int_0^1 h(t)h(1-t)dt. \quad (97)$$

If $\alpha = -1$, then (97) reduces to

$$\frac{1}{2h(1/2)} f^\beta(\frac{2ab}{a+b}) + \frac{c}{6}(\frac{b-a}{ab})^2$$
$$\leq \frac{ab}{b-a}\int_a^b \frac{1}{x^2} f^\beta(x)dx$$
$$\leq \{f^\beta(a) + f^\beta(b)\}\int_0^1 h(t)dt - c(\frac{b-a}{ab})^2\int_0^1 h(t)h(1-t)dt.$$

In particular, if $c = 0$, $h(t) = t$, then the above inequality reduces to the mail result of [15]:

$$f^{\beta}(\frac{2ab}{a+b}) \le \frac{ab}{b-a}\int_a^b \frac{1}{x^2} f^{\beta}(x)dx \le \frac{1}{2}\{f^{\beta}(a) + f^{\beta}(b)\}.$$

If $\alpha = \beta = 1$, then (97) reduces to the Hermite-Hadamard inequality for strongly h-convex functions:

$$\frac{1}{2h(1/2)} f(\frac{a+b}{2}) + \frac{c}{6}(b-a)^2 \le \frac{1}{b-a}\int_a^b f(x)dx$$
$$\le \{f(a) + f(b)\}\int_0^1 h(t)dt - c(b-a)^2\int_0^1 h(t)h(1-t)dt. \tag{98}$$

If $h(t) = t^s$, $0 < s \le 1$, then (98) reduces to:

$$2^{s-1} f(\frac{a+b}{2}) + \frac{c}{6}(b-a)^2 \le \frac{1}{b-a}\int_a^b f(x)dx$$
$$\le \frac{f(a) + f(b)}{s+1} - c(b-a)^2 \frac{(\Gamma(1+s))^2}{\Gamma(2(1+s))}. \tag{99}$$

If $h(t) = t^{-s}$, $0 < s < 1$, then (98) reduces to:

$$2^{-(s+1)} f(\frac{a+b}{2}) + \frac{c}{6}(b-a)^2 \le \frac{1}{b-a}\int_a^b f(x)dx$$
$$\le \frac{f(a) + f(b)}{1-s} - c(b-a)^2 \frac{(\Gamma(1-s))^2}{\Gamma(2(1-s))} \tag{100}$$

If $h(t) = t$, $s = 1$, then (98) reduces to:

$$f(\frac{a+b}{2}) + \frac{c}{6}(b-a)^2 \le \frac{1}{b-a}\int_a^b f(x)dx$$
$$\le \frac{f(a) + f(b)}{2} - \frac{c}{6}(b-a)^2. \tag{101}$$

If $c = 0$, then (101) reduces to the classical Hermite-Hadamard inequality (73).

Hence, the above results are some substantial refinements and generalizations of the corresponding results obtained by Nikodem [13] and Merentes and Nikodem [10].

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