# Extensions of Kannappan's and Van Vleck's Functional Equations on Semigroups

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**Abstract** This paper treats two functional equations, the Kannappan-Van Vleck functional equation

$$\mu(y)f(x\tau(y)z_0) \pm f(xyz_0) = 2f(x)f(y),\ \ x, y \in S$$

and the following variant of it

$$\mu(y)f(\tau(y)xz_0) \pm f(xyz_0) = 2f(x)f(y),\ \ x, y \in S,$$

in the setting of semigroups $S$ that need not be abelian or unital, $\tau$ is an involutive morphism of $S$, $\mu : S \longrightarrow C$ is a multiplicative function such that $\mu(x\tau(x)) = 1$ for all $x \in S$ and $z_0$ is a fixed element in the center of $S$.

We find the complex-valued solutions of these equations in terms of multiplicative functions and solutions of d'Alembert's functional equation.

## 1 Introduction

Van Vleck [1, 2] studied the continuous solutions $f : R \longrightarrow R$, $f \neq 0$, of the following functional equation

$$f(x - y + z_0) - f(x + y + z_0) = 2f(x)f(y),\ \ x, y \in R, \tag{1}$$

where $z_0 > 0$ is fixed. He showed that any continuous solution of (1) with minimal period $4z_0$ is $f(x) = \cos(\frac{\pi}{2z_0}(x - z_0))$, $x \in R$.

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Stetkær [3, Exercise 9.18] found the complex-valued solutions of equation

$$f(xy^{-1}z_0) - f(xyz_0) = 2f(x)f(y),\ x, y \in G, \tag{2}$$

on groups that need not be abelian and $z_0$ is a fixed element in the center of $G$.

Perkins and Sahoo [4] replaced the group inversion by an involution $\tau\colon G \longrightarrow G$ and obtained the abelian, complex-valued solutions of the equation

$$f(x\tau(y)z_0) - f(xyz_0) = 2f(x)f(y),\ x, y \in G, \tag{3}$$

by means of d'Alembert's functional equation

$$g(xy) + g(x\tau(y)) = 2g(x)g(y),\ x, y \in G. \tag{4}$$

Stetkær [5] extended the results of [4] about equation (3) to semigroups and derived an explicit formula for the solutions in terms of multiplicative maps. In particular, Stetkær proved that all solutions of (3) are abelian. So, the restriction to abelian solutions in [4] is not needed.

D'Alembert's classic functional equation

$$g(x+y) + g(x-y) = 2g(x)g(y),\ x, y \in R \tag{5}$$

has solutions $g\colon R \longrightarrow C$ that are periodic, for instance $g(x) = \cos(x)$, and solutions that are not, for instance $g(x) = \cosh(x)$.

Kannappan [6] proved that any solution of the extension of (5)

$$f(x - y + z_0) + f(x + y + z_0) = 2f(x)f(y),\ x, y \in R, \tag{6}$$

where $z_0 \neq 0$ is a real constant has the form $f(x) = g(x - z_0)$, where $g\colon R \longrightarrow C$ is a periodic solution of (5) with period $2z_0$.

Perkins and Sahoo [4] considered the following version of Kannappan's functional equation

$$f(xyz_0) + f(xy^{-1}z_0) = 2f(x)f(y),\ x, y \in G \tag{7}$$

on groups and they found the form of any abelian solution $f$ of (7).

Stetkær [7] took $z_0$ in the center and expressed the complex-valued solutions of Kannappan's functional equation

$$f(xyz_0) + f(x\tau(y)z_0) = 2f(x)f(y),\ x, y \in S \tag{8}$$

on semigroups with involution $\tau$ in terms of solutions of d'Alembert's functional equation (4).

In the very special case of $z_0$ being the neutral element of a monoid $S$ equation (8) becomes (4) which has been solved by Davison [8].

Here we shall consider the following functional equations

$$f(xyz_0) + \mu(y)f(x\tau(y)z_0) = 2f(x)f(y),\ x, y \in S, \tag{9}$$

$$f(xyz_0) + \mu(y)f(\tau(y)xz_0) = 2f(x)f(y),\ x, y \in S, \tag{10}$$

$$\mu(y)f(x\tau(y)z_0) - f(xyz_0) = 2f(x)f(y),\ x, y \in S \tag{11}$$

and

$$\mu(y)f(\tau(y)xz_0) - f(xyz_0) = 2f(x)f(y),\ x, y \in S, \tag{12}$$

where $S$ is a semigroup, $\tau$ is an involutive morphism of $S$. That is, $\tau$ is an involutive automorphism: $\tau(xy) = \tau(x)\tau(y)$ and $\tau(\tau(x)) = x$ for all $x, y \in S$ or $\tau$ is an involutive anti-automorphism: $\tau(xy) = \tau(y)\tau(x)$ and $\tau(\tau(x)) = x$ for all $x, y \in S$. The map $\mu : S \longrightarrow C$ is a multiplicative function such that $\mu(x\tau(x)) = 1$ for all $x \in S$ and $z_0$ is a fixed element in the center of $S$. By algebraic methods:

(1) We find all solutions of (11) and (12). Only multiplicative functions occur in the solution formulas.
(2) We find the solutions of (10) for the particular case of $\tau$ being an involutive automorphism and
(3) We express the solutions of (9) and (10) in terms of solutions of d'Alembert's $\mu$-functional equation

$$g(xy) + \mu(y)g(x\tau(y)) = 2g(x)g(y),\ x, y \in S. \tag{13}$$

Of course we are not the first to consider trigonometric functional equations having a multiplicative function $\mu$ in front of terms like $f(x\tau(y))$ or $f(\tau(y)x)$. The $\mu$-d'Alembert's functional equation (13) which is an extension of d'Alembert's functional equation (4) has been treated systematically by Stetkær [3, 9] on groups with involution. The non-zero solutions of (13) on groups with involution are the normalized traces of certain representation of $S$ on $C^2$.

Stetkær [10] obtained the complex-valued solution of the following variant of d'ĄAlembert's functional equation

$$f(xy) + f(\tau(y)x)) = 2f(x)f(y),\ x, y \in S, \tag{14}$$

where $\tau$ is an involutive automorphism of $S$.

Elqorachi and Redouani [11] proved that the solutions of the variant of d'Alembert's functional equation

$$f(xy) + \mu(y)f(\tau(y)x)) = 2f(x)f(y),\ x, y \in S \tag{15}$$

are of the form $f(x) = \frac{\chi(x)+\mu(x)\chi(\tau(x))}{2}$, $x \in S$, where $\tau$ is an involutive automorphism of $S$ and $\chi\colon S \longrightarrow S$ is a multiplicative function.

Bouikhalene and Elqorachi [12] obtained the solutions of (11) for involutive anti-automorphism $\tau$. In the same paper they also found the solutions of (11) for involutive automorphism $\tau$, but on monoids only.

Throughout this paper $S$ denotes a semigroup with an involutive morphism $\tau$: $S \longrightarrow S$, $\mu$: $S \longrightarrow C$ denotes a multiplicative function such that $\mu(x\tau(x)) = 1$ for all $x \in S$ and $z_0$ a fixed element in the center of $S$.

In all proofs of the results of this paper we use without explicit mentioning the assumption that $z_0$ is contained in the center of $S$ and its consequence $\tau(z_0)$ is contained in the center of $S$.

## 2 Solutions of Equation (9) on Semigroups

In this section we express the solutions of (9) in terms of solutions of d'Alembert's functional equation (13). The following lemma will be used later.

**Lemma 1** *If $f$: $S \longrightarrow C$ is a solution of (9), then for all $x \in S$*

$$f(x) = \mu(x)f(\tau(x)), \tag{16}$$

$$f(x\tau(z_0)z_0) = \mu(\tau(z_0))f(z_0)f(x), \tag{17}$$

$$f(xz_0^2) = f(x)f(z_0), \tag{18}$$

$$f(z_0) \neq 0 \Longleftrightarrow f \neq 0. \tag{19}$$

***Proof*** Equation (16): By replacing $y$ by $\tau(y)$ in (9) and multiplying the result obtained by $\mu(y)$ and using $\mu(y\tau(y)) = 1$ we get by computation that

$$\mu(y)2f(x)f(\tau(y)) = \mu(y)f(x\tau(y)z_0) + \mu(y\tau(y))f(xyz_0)$$

$$= \mu(y)f(x\tau(y)z_0) + f(xyz_0) = 2f(x)f(y),$$

which implies (16).

Equation (17): Replacing $x$ by $\tau(z_0)$ in (9) and using (16) two times we get by a computation that

$$f(\tau(z_0)yz_0) + \mu(y)f(\tau(z_0)\tau(y)z_0) = 2f(\tau(z_0))f(y) = 2\mu(\tau(z_0))f(z_0)f(y)$$

and

$$f(\tau(z_0)yz_0) + \mu(y)f(\tau(z_0)\tau(y)z_0) = 2f(\tau(z_0))f(y)$$

$$= f(\tau(z_0)yz_0) + \mu(y)\mu(\tau(z_0)\tau(y)z_0)f(\tau(z_0)yz_0) = 2f(\tau(z_0)yz_0).$$

This proves (17).

Equation (18): Putting $y = z_0$ in (9) and using (17) we obtain (18).

Equation (19): Assume that $f(z_0) = 0$. By replacing $x$ by $xz_0$ and $y$ by $yz_0$ in (9) and using (17) and (18) we get by a computation that

$$2f(xz_0)f(yz_0) = f(xz_0yz_0^2) + \mu(yz_0)f(xz_0\tau(y)\tau(z_0)z_0)$$

$$= f(z_0)f(xyz_0) + \mu(y)f(z_0)f(x\tau(y)z_0) = 0 \text{ for all } x, y \in S,$$

which implies that $f(xz_0) = 0$ for all $x \in S$. So, from equation (9) we get $2f(x)f(y) = 0$ for all $x, y \in S$, and then $f(x) = 0$ for all $x \in S$. Conversely, it's clear that $f(x) = 0$ for all $x \in S$ implies that $f(z_0) = 0$.

For the rest of this section we use the following notations [7].

- $\mathscr{A}$ consists of the solutions of $g : S \longrightarrow C$ of d'Alembert's functional equation (13) with $g(z_0) \neq 0$ and satisfying the condition

$$g(xz_0) = g(z_0)g(x) \text{ for all } x \in S. \tag{20}$$

- To any $g \in \mathscr{A}$ we associate the function $Tg = g(z_0)g : S \longrightarrow C$.
- $\mathscr{K}$ consists of the non-zero solutions $f : S \longrightarrow C$ of Kannappan's functional equation (9).

In the following main result of the present section, the complex solutions of equation (9) are expressed by means of solutions of d'Alembert's functional equation (13).

**Theorem 1**

*(1) $T$ is a bijection of $\mathscr{A}$ onto $\mathscr{K}$. The inverse $T^{-1}$: $\mathscr{K} \longrightarrow \mathscr{A}$ is given by the formula*

$$(T^{-1}f)(x) = \frac{f(xz_0)}{f(z_0)}$$

*for all $f \in \mathscr{K}$ and $x \in S$.*

*(2) Any non-zero solution $f: S \longrightarrow C$ of the Kannappan's functional equation (9) is of the form $f = T(g) = g(z_0)g$, where $g \in \mathscr{A}$. Furthermore,*

$$f(x) = g(xz_0) = \mu(z_0)g(x\tau(z_0)) = g(z_0)g(x)$$

*for all $x \in S$.*

*(3) $f$ is central, i.e. $f(xy) = f(yx)$ for all $x, y \in S$ if and only if $g$ is central.*

*(4) $f$ is abelian [3, Definition B.3] if and only if $g$ is abelian.*

*(5) If $S$ is equipped with a topology, then $f$ is continuous if and only if $g$ is continuous.*

***Proof*** For any $g \in \mathscr{A}$ and for all $x, y \in S$ we have

$$T(g)(xyz_0) + \mu(y)T(g)(x\tau(y)z_0) = g(z_0)[g(xyz_0) + \mu(y)g(x\tau(y)z_0)]$$

$$= g(z_0)^2[g(xy) + \mu(y)g(x\tau(y))] = 2g(z_0)g(x)g(z_0)g(y) = 2T(g)(x)T(g)(y).$$

On the other hand, $T(g)(z_0) = g(z_0)^2 \neq 0$, so we get $T(\mathscr{A}) \subseteq \mathscr{K}$.

By adapting the proof of [7, Lemma 3] $T$ is injective. Now, we will show that $T$ is surjective. Let $f \in \mathscr{K}$. Then from (19) we have $f(z_0) \neq 0$ and we can define the function $g(x) = \frac{f(xz_0)}{f(z_0)}$. In the following we will show that $g \in \mathscr{A}$ and $T(g) = f$. By using the definition of $g$ and (17)–(18) we have

$$f(z_0)^2[g(xy) + \mu(y)g(x\tau(y))] = f(z_0)f(xyz_0) + \mu(y)f(z_0)f(x\tau(y)z_0)$$

$$= f(xyz_0^3) + \mu(y)\mu(z_0)f(x\tau(y){z_0}^2\tau(z_0))$$

$$= f(xz_0yz_0z_0) + \mu(yz_0)f(xz_0\tau(yz_0)z_0) = 2f(xz_0)f(yz_0) = 2f(z_0)^2g(x)g(y)$$

for all $x, y \in S$. This shows that $g$ is a solution of d'Alembert's functional equation (13).

By replacing $x$ by $xz_0^2$ and $y$ by $z_0$ in (9) we get

$$f(xz_0^4) + \mu(z_0)f(xz_0^3\tau(z_0)) = 2f(xz_0^2)f(z_0). \tag{21}$$

By replacing $x$ by $xz_0$ and $y$ by $z_0^2$ in (9) we have

$$f(xz_0^4) + \mu(z_0^2)f(xz_0^2\tau(z_0^2)) = 2f(z_0^2)f(xz_0). \tag{22}$$

From (17) and (18) we have

$$f(xz_0^3\tau(z_0)) = \mu(\tau(z_0))f(x)(f(z_0))^2$$

and

$$f(xz_0^2\tau(z_0^2)) = (\mu(\tau(z_0)))^2f(x)(f(z_0))^2.$$

In view of (21) and (22) we deduce that $f(z_0^2)f(xz_0) = f(xz_0^2)f(z_0)$. So, by using the definition of $g$ we obtain $g(xz_0) = g(x)g(z_0)$ for all $x \in S$. In particular, $g(z_0^2) = g(z_0)^2 = \frac{f(z_0^2z_0)}{f(z_0)} = \frac{f(z_0)f(z_0)}{f(z_0)} = f(z_0) \neq 0$. Furthermore, $T(g)(x) = g(z_0)g(x) = g(xz_0) = \frac{f(xz_0^2)}{f(z_0)} = \frac{f(x)f(z_0)}{f(z_0)} = f(x)$.

The statements (2)–(5) are obvious. This completes the proof.

Now, we extend Stetkær's result [7] from anti-automorphisms to the more general case of morphism as follows.

**Corollary 1** *Let $z_0$ be a fixed element in the center of a semigroup $S$ and let $\tau$ be an involutive morphism of $S$. Then, any non-zero solution $f\colon S \longrightarrow C$ of the functional equation (8) is of the form $f = g(z_0)g$, where $g$ is a solution of d'Alembert's functional equation (4) with $g(z_0) \neq 0$ and satisfying the condition $g(xz_0) = g(z_0)g(x)$ for all $x \in S$.*

We will in the following propositions determine all abelian (resp. central) solutions $f$ of Kannappan's functional equation (9).

**Proposition 1** *Let $z_0$ be a fixed element in the center of a semigroup $S$. Let $\tau\colon S \longrightarrow S$ be an involutive anti-automorphism of $S$ and let $\mu\colon S \longrightarrow C$ be a multiplicative function such that $\mu(x\tau(x)) = 1$ for all $x \in S$. The non-zero abelian solutions of Kannappan's functional equation (9) are the functions of the form*

$$f(x) = \frac{\chi(x) + \mu(x)\chi(\tau(x))}{2}\chi(z_0),\ x \in S,$$

*where $\chi : S \longrightarrow C$ is a multiplicative function such that $\chi(z_0) \neq 0$ and $\mu(z_0)\chi(\tau(z_0)) = \chi(z_0)$.*

***Proof*** Verifying that the function $f$ defined in Proposition 1 is an abelian solution of (9) consists of simple computations that we omit.

Let $f\colon S \longrightarrow C$ be a non-zero solution of (9). From Theorem 1(2) and (4) the function $f$ has the form $f = g(z_0)g$ where $g \in \mathscr{A}$ and $g$ is abelian. From [3, Proposition 9.31] there exists a non-zero multiplicative function $\chi\colon S \longrightarrow C$ such that $g = \frac{\chi+\mu\chi\circ\tau}{2}$. Since $g \in \mathscr{A}$, it satisfies (20). If we replace $x$ by $z_0$ in (20) we get $g(z_0^2) = g(z_0)^2$, which via computation gives that $\chi(z_0) = \mu(z_0)\chi(\tau(z_0))$. This implies that $f$ has the desired form. This completes the proof.

By using [11, Lemma 3.2] and the proof of the preceding proposition we get

**Proposition 2** *Let $z_0$ be a fixed element in the center of a semigroup $S$. Let $\tau\colon S \longrightarrow S$ be an involutive automorphism of $S$ and let $\mu\colon S \longrightarrow C$ be a multiplicative function such that $\mu(x\tau(x)) = 1$ for all $x \in S$. The non-zero central solutions of the Kannappan's functional equation (9) are the functions of the form*

$$f(x) = \frac{\chi(x) + \mu(x)\chi(\tau(x))}{2}\chi(z_0),\ x \in S,$$

*where $\chi : S \longrightarrow C$ is a multiplicative function such that $\chi(z_0) \neq 0$ and $\mu(z_0)\chi(\tau(z_0)) = \chi(z_0)$.*

## 3 Solutions of Equation (10) on Semigroups

In this section we determine the complex-valued solutions of (10) for any involutive morphism $\tau\colon S \longrightarrow S$. By help of Theorem 1 we express them in terms of solutions of d'Alembert's functional equation (13). We first prove the following two useful lemmas.

**Lemma 2** *If $f\colon S \longrightarrow C$ is a solution of (10), then for all $x \in S$*

$$f(x) = \mu(x) f(\tau(x)), \tag{23}$$

$$f(x\tau(z_0)z_0) = \mu(\tau(z_0)) f(z_0) f(x), \tag{24}$$

$$f(xz_0^2) = f(x) f(z_0), \tag{25}$$

$$f(z_0) \neq 0 \Longleftrightarrow f \neq 0. \tag{26}$$

***Proof*** Equation (23): Interchanging $x$ and $y$ in (10) and multiplying the two members of the equation by $\mu(\tau(y))$ we get

$$\mu(x)\mu(\tau(y)) f(\tau(x)yz_0) + \mu(\tau(y)) f(yxz_0) = 2f(x)\mu(\tau(y)) f(y),\ \ x, y \in S. \tag{27}$$

Replacing $y$ by $\tau(y)$ in (10) we obtain

$$\mu(\tau(y)) f(yxz_0) + f(x\tau(y)z_0) = 2f(x) f(\tau(y)),\ \ x, y \in S. \tag{28}$$

By subtracting (28) from (27) we get

$$\mu(x\tau(y)) f(\tau(x)yz_0) - f(x\tau(y)z_0) = 2f(x)[\mu(\tau(y)) f(y) - f(\tau(y)],\ \ x, y \in S. \tag{29}$$

By replacing $x$ by $\tau(x)$ in (29) we have

$$\mu(\tau(x)\tau(y)) f(xyz_0) - f(\tau(x)\tau(y)z_0) = 2f(\tau(x))[\mu(\tau(y)) f(y) - f(\tau(y)],\ \ x, y \in S. \tag{30}$$

Replacing $y$ by $\tau(y)$ in (29) and multiplying the two members of the equation by $\mu(\tau(y)\tau(x))$ we obtain

$$f(\tau(x)\tau(y)z_0) - \mu(\tau(x)\tau(y)) f(xyz_0) = 2f(x)\mu(\tau(x))[f(\tau(y)) - \mu(\tau(y)) f(y)],\ \ x, y \in S. \tag{31}$$

Now, by adding (30) and (31) we get $[f(\tau(x)) - \mu(\tau(x)) f(x)][f(\tau(y)) - \mu(\tau(y)) f(y)] = 0$ for all $x, y \in S$. This proves (23).

Equation (24): Taking $x = \tau(z_0)$ in (10) and using (23) we get

$$\mu(y) f(\tau(y)\tau(z_0)z_0) + f(\tau(z_0)yz_0) = 2\mu(\tau(z_0)) f(z_0) f(y)$$

$$= f(\tau(z_0)yz_0) + \mu(y)\mu(\tau(y)\tau(z_0)z_0) f(\tau(z_0)yz_0) = 2f(\tau(z_0)yz_0),$$

which implies (23).

Equation (25): By replacing $y$ by $z_0$ in (10) and using (24) we obtain

$$\mu(z_0)f(\tau(z_0)xz_0) + f(xz_0{}^2) = 2f(z_0)f(x)$$

$$= \mu(z_0)\mu(\tau(z_0))f(z_0)f(x) + f(xz_0{}^2).$$

So, we deduce (24).

Equation (25): The proof is similar to the proof of (19).

**Lemma 3** *Let $\mathcal{M}$ consist of the solutions $g\colon S \longrightarrow C$ of the variant d'Alembert's functional equation (15) with $g(z_0) \neq 0$ and satisfying the condition (20). Let $\mathcal{N}$ consist of the non-zero solutions $f\colon S \longrightarrow C$ of the variant Kannappan's functional equation (10); Then*

*(1) The map $J\colon \mathcal{M} \longrightarrow \mathcal{N}$ defined by $Jh := h(z_0)h\colon S \longrightarrow C$ is a bijection. The inverse $J^{-1}$; $\mathcal{N} \longrightarrow \mathcal{M}$ is given by the formula $(J^{-1}f)(x) = \frac{f(xz_0)}{f(z_0)} = g(x)$ for all $x \in S$ and for all $f \in \mathcal{N}$. Furthermore,*

*(2) If $\tau\colon S \longrightarrow S$ is an involutive automorphism, the function $g$ has the form $g = \frac{\chi+\mu\chi\circ\tau}{2}$, where $\chi : S \longrightarrow C$, $\chi \neq 0$, is a multiplicative function.*

*(3) If $\tau\colon S \longrightarrow S$ is an involutive anti-automorphism, the function $g$ satisfies the d'Alembert's functional equation (13).*

***Proof*** For all $h \in \mathcal{M}$ we have

$$Jh(xyz_0) + \mu(y)Jh(\tau(y)xz_0) = h(z_0)h(xyz_0) + \mu(y)h(z_0)h(\tau(y)xz_0)$$

$$= h(z_0)^2[h(xy) + \mu(y)h(\tau(y)x)] = 2h(z_0)h(x)h(z_0)h(y) = 2Jh(x)Jh(y).$$

Furthermore, $Jh(z_0) = h(z_0)^2 \neq 0$. So, $Jh \in \mathcal{N}$. By adapting the proof of [7, Lemma 3] $J$ is injective. Now, let $f \in \mathcal{N}$ and let $g(x) := \frac{f(xz_0)}{f(z_0)}$ for $x \in S$. By using the definition of $g$, equations (10), (24), and (25) we get

$$f(z_0)^2[g(xy) + \mu(y)g(\tau(y)x) - 2g(x)g(y)]$$

$$= f(z_0)f(xyz_0) + \mu(y)f(z_0)f(\tau(y)xz_0) - 2f(xz_0)f(yz_0)$$

$$= f(xyz_0z_0^2) + \mu(y)\mu(z_0)f(\tau(y)xz_0\tau(z_0)z_0) - 2f(xz_0)f(yz_0)$$

$$f(xz_0yz_0z_0) + \mu(yz_0)f(\tau(yz_0)xz_0z_0) - 2f(xz_0)f(yz_0) = 0.$$

Since $f(z_0) \neq 0$ then $g$ satisfies (15). By using similar computations as in the proof of Theorem 1 we get that $g(xz_0) = g(z_0)g(x)$ for all $x \in S$.

(2) If $\tau\colon S \longrightarrow S$ is an involutive automorphism then from [11, Lemma 3.2] $g$ has the form $g = \frac{\chi+\mu\chi\circ\tau}{2}$, where $\chi : S \longrightarrow C$, $\chi \neq 0$, is a multiplicative function.

(3) If $\tau: S \longrightarrow S$ is an involutive anti-automorphism then by adapting the proof of [11, Theorem 2.1(1)(i)] for $\delta = 0$ we get that $g$ satisfies the d'Alembert's functional equation (13).

**Theorem 2**

*(1) Let $\tau: S \longrightarrow S$ be an involutive automorphism. The non-zero solutions $f: S \longrightarrow C$ of the functional equation (10) are the functions of the form*

$$f = \frac{\chi + \mu\chi \circ \tau}{2}\chi(z_0), \tag{32}$$

*where $\chi : S \longrightarrow C$ is a multiplicative function such that $\chi(z_0) \neq 0$ and $\mu(z_0)\chi(\tau(z_0)) = \chi(z_0)$.*

*(2) Let $\tau: S \longrightarrow S$ be an involutive anti-automorphism. The non-zero solutions $f : S \longrightarrow C$ of the functional equation (10) are the functions of the form $f = g(z_0)g$, where $g$ is a solution of d'Alembert's functional equation (13) with $g(z_0) \neq 0$ and satisfying the condition $g(xz_0) = g(z_0)g(x)$ for all $x \in S$.*

***Proof*** Let $f: S \longrightarrow S$ be a non-zero solution of equation (10). From Theorem 1(2) $f = g(z_0)g(x) = g(xz_0)$, where $g$ is a solution of d'Alembert's functional equation (4). We will discuss two possibilities.

(1) $\tau$ is an involutive automorphism of $S$. From Lemma 3, there exists $\chi : S \longrightarrow C$ a multiplicative function such that $g = \frac{\chi+\mu\chi\circ\tau}{2}$. So,

$$f = g(z_0) = \frac{\chi + \mu\chi \circ \tau}{2}g(z_0) = \frac{\chi(z_0) + \mu(z_0)\chi \circ \tau(z_0)}{2}\frac{\chi + \mu\chi \circ \tau}{2}. \tag{33}$$

By using $g(z_0^2) = g(z_0)^2$ we get after simple computation that $\chi(z_0) = \mu(z_0)\chi(\tau(z_0))$. This proves (1).

(2) $\tau$ is an involutive anti-automorphism of $S$. Combining Theorem 1 and Lemma 3(2) we find (2). This completes the proof.

## 4 Solutions of Equation (11)

The solutions of the functional equation (11) with $\tau$ an involutive anti-automorphism are explicitly obtained by Bouikhalene and Elqorachi [12] on semigroups not necessarily abelian in terms of multiplicative functions. In this section we obtain a similar formula for the solutions of the functional equation (11) when $\tau$ was an involutive automorphism. The following lemma is obtained in [12] for the case where $\tau$ is an involutive anti-automorphism. It still holds for the case where $\tau$ is an involutive automorphism.

**Lemma 4** *Let $f \neq 0$ be a solution of (11). Then for all $x \in S$ we have*

$$f(x) = -\mu(x)f(\tau(x)), \tag{34}$$

$$f(z_0) \neq 0, \tag{35}$$

$$f(z_0^2) = 0, \tag{36}$$

$$f(x\tau(z_0)z_0) = \mu(\tau(z_0))f(x)f(z_0), \tag{37}$$

$$f(xz_0^2) = -f(z_0)f(x), \tag{38}$$

$$\mu(x)f(\tau(x)z_0) = f(xz_0). \tag{39}$$

*The function $g(x) = \frac{f(xz_0)}{f(z_0)}$ is a non-zero solution of d'Alembert's functional equation (13).*

Now, we are ready to prove the main result of this section.

In [12] we used [3, Proposition 8.14] to prove that the function $g$ defined in Lemma 4 is an abelian solution of (13), where $\tau$ is an involutive anti-automorphism of $S$. This reasoning no longer works for the present situation. We will use another approach.

**Theorem 3** *The non-zero solutions $f : S \longrightarrow C$ of the functional equation (11), where $\tau$ is an involutive morphism of $S$ are the functions of the form*

$$f = \chi(z_0)\frac{\mu\chi \circ \tau - \chi}{2}, \tag{40}$$

*where $\chi : S \longrightarrow C$ is a multiplicative function such that $\chi(z_0) \neq 0$ and $\mu(z_0)\chi(\tau(z_0)) = -\chi(z_0)$.*

*If $S$ is a topological semigroup and that $\tau : S \longrightarrow S$, $\mu : S \longrightarrow C$ are continuous, then the non-zero solution $f$ of equation (11) is continuous if and only if $\chi$ is continuous.*

***Proof*** Let $f$ be a non-zero solution of (11). Replacing $x$ by $xz_0$ in (11) and using (38) we get

$$-\mu(y)f(x\tau(y)) + f(xy) = 2f(y)g(x), \; x, y \in S, \tag{41}$$

where $g$ is the function defined in Lemma 4.

If we replace $y$ by $yz_0$ in (11) and use (37) and (38) we get

$$\mu(yz_0)\mu(\tau(z_0))f(x\tau(y)) + f(xy) = 2f(x)g(y) = \mu(y)f(x\tau(y)) + f(xy), \; x, y \in S. \tag{42}$$

By adding (41) and (42) we get that the pair $f, g$ satisfies the sine addition law

$$f(xy) = f(x)g(y) + f(y)g(x) \text{ for all } x, y \in S.$$

Now, in view of [13, Lemma 3.4], [3, Theorem 4.1] $g$ is abelian. Since $g$ is a non-zero solution of d'Alembert's functional equation (13), then from [3, Proposition 9.31] there exists a non-zero multiplicative function $\chi: S \longrightarrow C$ such that $g = \frac{\chi+\mu\chi\circ\tau}{2}$. The rest of the proof is similar to the one used in [12].

## 5 Solutions of Equation (12)

The solutions of (12) were obtained in [12] on monoids for $\tau$ an involutive automorphism. In this section we determine the solutions of (12) for the general case where $S$ is assumed to be a semigroup and $\tau$ an involutive morphism of $S$.

The following useful lemmas will be used later.

**Lemma 5** *Let $f: S \longrightarrow C$ be a solution of equation (12). Then for all $x, y \in S$ we have*

$$f(x) = -\mu(x)f(\tau(x)), \tag{43}$$

$$f \neq 0 \Longleftrightarrow f(z_0) \neq 0, \tag{44}$$

$$\mu(y)f(\tau(y)x) = -\mu(x)f(\tau(x)y), \tag{45}$$

$$f(x\tau(z_0)z_0) = \mu(\tau(z_0))f(z_0)f(x), \tag{46}$$

$$f(xz_0^2) = -f(z_0)f(x), \tag{47}$$

$$\mu(x)f(\tau(x)z_0) = f(xz_0), \tag{48}$$

$$f(x\tau(z_0)) = \mu(x)f(\tau(x)\tau(z_0)), \tag{49}$$

$$f(z_0^2) = f(z_0\tau(z_0)) = 0. \tag{50}$$

***Proof*** Equation (44): Let $f \neq 0$ be a non-zero solution of equation (12). We will derive (44) by contradiction. Assume that $f(z_0) = 0$. Putting $y = z_0$ in equation (12) we get

$$\mu(z_0)f(\tau(z_0)xz_0) - f(xz_0z_0) = 2f(x)f(z_0) = 0 \tag{51}$$

Replacing $y$ by $yz_0$ in (12) and using (51) and (12) we get

$$\mu(yz_0)f(\tau(y)xz_0\tau(z_0)) - f(xyz_0z_0) = 2f(x)f(yz_0)$$

$$= \mu(y)f(\tau(y)xz_0z_0) - f(xz_0yz_0)$$

$$= 2f(y)f(xz_0).$$

So, we deduce that $f(y)f(xz_0) = f(x)f(yz_0)$ for all $x, y \in S$. Since $f \neq 0$, then there exists $\alpha \in C$ such that $f(xz_0) = \alpha f(x)$ for all $x \in S$. Furthermore, $\alpha \neq 0$, because if $\alpha = 0$ we get $f(xz_0) = 0$ for all $x \in S$ and equation (12) implies that $f = 0$. This contradicts the assumption that $f \neq 0$.

Now, by substituting $f(xz_0) = \alpha f(x)$ into (12) we get

$$\mu(y)f(\tau(y)x) - f(xy) = \frac{2}{\alpha} f(x)f(y) \text{ for all } x, y \in S. \tag{52}$$

Switching $x$ and $y$ in (52) we get

$$-f(yx) + \mu(x)f(\tau(x)y) = \frac{2}{\alpha} f(x)f(y),\ x, y \in S. \tag{53}$$

If we replace $y$ by $\tau(y)$ in (52) and multiplying the result obtained by $\mu(y)$ we get

$$-\mu(y)f(x\tau(y)) + f(yx) = \frac{2}{\alpha} f(x)\mu(y)f(\tau(y)),\ x, y \in S. \tag{54}$$

By adding (54) and (53) we obtain

$$-\mu(y)f(x\tau(y)) + \mu(x)f(\tau(x)y) = \frac{2}{\alpha} f(x)[\mu(y)f(\tau(y)) + f(y)],\ x, y \in S. \tag{55}$$

By replacing $x$ by $\tau(x)$ in (55) and multiplying the result obtained by $\mu(x)$ we get

$$f(xy) - \mu(xy)f(\tau(x)\tau(y)) = \frac{2}{\alpha}\mu(x)f(\tau(x))[\mu(y)f(\tau(y)) + f(y)]. \tag{56}$$

By replacing $y$ by $\tau(y)$ in (55) and multiplying the result obtained by $\mu(y)$ we get

$$\mu(xy)f(\tau(x)\tau(y)) - f(xy) = \frac{2}{\alpha} f(x)[f(y) + \mu(y)f(\tau(y))]. \tag{57}$$

By adding (56) and (57) we obtain

$$[f(x) + \mu(x)f(\tau(x))][\mu(y)f(\tau(y)) + f(y)] = 0,\ x, y \in S. \tag{58}$$

So, $\mu(x)f(\tau(x)) = -f(x)$ for all $x \in S$. Now, we will discuss the following two cases.

(1) $\tau$ is an involutive anti-automorphism. By using $\mu(x)f(\tau(x)) = -f(x)$ for all $x \in S$ we get $f(\tau(y)x) = -\mu(\tau(y)x)f(\tau(x)y)$ for all $x, y \in S$. Substituting this in equation (52) we obtain

$$f(xy) + \mu(x)f(\tau(x)y) = 2\frac{-f(x)}{\alpha} f(y),\ x, y \in S. \tag{59}$$

By replacing $x$ by $\tau(x)$ in (59) and multiplying the result obtained by $\mu(x)$ we deduce that $f(x) = \mu(x)f(\tau(x))$ for all $x \in S$. So, we have $f(x) = -\mu(x)f(\tau(x)) = -f(x)$, which implies that $f = 0$. This contradicts the assumption that $f \neq 0$.

(2) $\tau$ is an involutive automorphism. Then from $\mu(x)f(\tau(x)) = -f(x)$ for all $x \in S$ we get $f(\tau(y)x) = -\mu(\tau(y)x)f(y\tau(x))$ for all $x, y \in S$. Substituting this in equation (52) we obtain

$$f(xy) + \mu(x)f(y\tau(x)) = 2\frac{-f(x)}{\alpha}f(y) \text{ for all } x, y \in S. \tag{60}$$

By replacing $x$ by $\tau(x)$ in (60) and multiplying the result obtained by $\mu(x)$ and using $\mu(x)f(\tau(x)) = -f(x)$ we get

$$h(yx) + \mu(x)h(\tau(x)y) = 2h(x)h(y) \text{ for all } x, y \in S.$$

where $h = \frac{f}{\alpha}$. So, from [11] $\mu(x)f(\tau(x)) = f(x)$ for all $x \in S$. Consequently, $\mu(x)f(\tau(x)) = f(x) = -f(x)$ for all $x \in S$, which implies that $f = 0$. This contradicts the assumption that $f \neq 0$ and this proves (44).

Equation (45): By replacing $y$ by $yz_0$ in (12) we get

$$\mu(yz_0)f(\tau(y)xz_0\tau(z_0)) - f(xyz_0z_0) = 2f(x)f(yz_0). \tag{61}$$

Replacing $x$ by $xz_0$ in (12) we get

$$\mu(y)f(\tau(y)xz_0z_0) - f(xyz_0z_0) = 2f(y)f(xz_0). \tag{62}$$

Subtracting these equations results in

$$\mu(yz_0)f(\tau(y)xz_0\tau(z_0)) - \mu(y)f(\tau(y)xz_0z_0) \tag{63}$$
$$= 2f(x)f(yz_0) - 2f(y)f(xz_0).$$

On the other hand, from (12) we have

$$\mu(yz_0)f(\tau(y)xz_0\tau(z_0)) - \mu(y)f(\tau(y)xz_0z_0)$$
$$= \mu(y)[\mu(z_0)f(\tau(z_0)\tau(y)xz_0) - f(\tau(y)xz_0z_0)]$$
$$= 2\mu(y)f(z_0)f(\tau(y)x).$$

This implies that

$$f(x)f(yz_0) - f(y)f(xz_0) = \mu(y)f(\tau(y)x)f(z_0) \tag{64}$$

for all $x, y \in S$. Since $f(x)f(yz_0)-f(y)f(xz_0) = -[f(y)f(xz_0)-f(x)f(yz_0)]$, then we deduce $\mu(y)f(\tau(y)x)f(z_0) = -\mu(x)f(\tau(x)y)f(z_0)$. Now, by using (44) we deduce (45).

Equation (49): By replacing $x$ by $x\tau(z_0)$ in (12) we get

$$\mu(y)f(\tau(y)x\tau(z_0)z_0) - f(xy\tau(z_0)z_0) \tag{65}$$

$$= 2f(y)f(x\tau(z_0)).$$

From (45) we have $\mu(\tau(x))f(xy\tau(z_0)z_0) = \mu(\tau(x))f(\tau(\tau(x))(y\tau(z_0)z_0)) = -\mu(y)f(\tau(y)\tau(x)\tau(z_0)z_0)$ and then equation (65) can be written as follows:

$$f(\tau(y)x\tau(z_0)z_0) + \mu(x)f(\tau(y)\tau(x)\tau(z_0)z_0) \tag{66}$$

$$= 2f(y)\mu(\tau(y))f(x\tau(z_0)).$$

By replacing $x$ by $\tau(x)$ in (66) and multiplying the result obtained by $\mu(x)$ and using $f \neq 0$ we get (49).

From equations (45) and (49) we have

$$\mu(\tau(x))f(xz_0) = -\mu(z_0)f(\tau(z_0)\tau(x))$$

$$= -\mu(z_0)\mu(\tau(x))f(x\tau(z_0)) = f(\tau(x)z_0).$$

This proves (48).

Equation (43): Replacing $x$ by $\tau(x)$ in (12) we get

$$\mu(y)f(\tau(y)\tau(x)z_0) - f(\tau(x)yz_0) = 2f(\tau(x))f(y), \ \ x, y \in S. \tag{67}$$

We will discuss the following two possibilities.

(1) $\tau$ is an involutive automorphism. From (48) we have

$$f(\tau(y)\tau(x)z_0) = f(\tau(yx)z_0) = \mu(\tau(yx))f(yxz_0)$$

and in view of (67) we obtain

$$\mu(\tau(x))f(yxz_0) - f(\tau(x)yz_0) = 2f(\tau(x))f(y), \ x, y \in S.$$

Since

$$\begin{aligned}\mu(\tau(x))f(yxz_0) - f(\tau(x)yz_0) &= -\mu(\tau(x))[\mu(x)f(\tau(x)yz_0) - f(yxz_0)]\\ &= -\mu(\tau(x))2f(y)f(x),\end{aligned}$$

then we deduce that

$$-2\mu(\tau(x))f(x)f(y) = 2f(\tau(x))f(y)$$

for all $x, y \in S$. Since $f \neq 0$ then we have (43).

(2) $\tau$ is an involutive anti-automorphism. Using (48) we have

$$f(\tau(y)\tau(x)z_0) = f(\tau(xy)z_0) = \mu(\tau(yx))f(xyz_0)$$

and $f(\tau(x)yz_0) = \mu(\tau(x)y)f(\tau(y)xz_0)$. Now, equation (67) can be written as follows:

$$\mu(\tau(x))f(xyz_0) - \mu(\tau(x)y)f(\tau(y)xz_0) = 2f(\tau(x))f(y)$$

$$= -\mu(\tau(x))[\mu(y)f(\tau(y)xz_0) - f(xyz_0)]$$

$$= -\mu(\tau(x))2f(x)f(y).$$

Since $f \neq 0$ then we obtain again (43).

Equation (46): Putting $x = \tau(z_0)$ in (12), using (43) we get

$$\mu(y)f(\tau(y)\tau(z_0)z_0) - f(\tau(z_0)yz_0) = 2f(y)f(\tau(z_0))$$

$$= -2f(y)\mu(\tau(z_0))f(z_0).$$

Since

$$\mu(y)f(\tau(y)\tau(z_0)z_0) = -\mu(\tau(z_0)z_0)f(yz_0\tau(z_0)) = -f(yz_0\tau(z_0))$$

then we obtain

$$f(\tau(z_0)yz_0) = \mu(\tau(z_0))f(y)f(z_0)$$

for all $y \in S$. We see that we deal with (46).

Equation (47): Replacing $y$ by $z_0$ in (12) and using (46) we get

$$\mu(z_0)f(\tau(z_0)xz_0) - f(xz_0z_0)$$

$$= 2f(x)f(z_0) = f(x)f(z_0) - f(xz_0z_0),$$

which proves (46).

Equation (50): By replacing $x$ by $z_0$ in (48) we get $\mu(z_0)f(\tau(z_0)z_0) = f(z_0^2)$. From (43) we have $f(\tau(z_0)z_0) = -f(\tau(z_0)z_0)$, then we conclude that

$$f(\tau(z_0)z_0) = f(z_0^2) = 0,$$

which proves (50). This completes the proof.

**Lemma 6** *Let $f: S \longrightarrow C$ be a non-zero solution of equation (12). Then*

*(1) The function defined by*

$$g(x) \; := \frac{f(xz_0)}{f(z_0)} \; \text{for } x \in S$$

*is a non-zero solution of the variant of d'Alembert's functional equation (15).*

*(2) The function $g$ from (1) has the form $g = \frac{\chi + \mu\chi\circ\tau}{2}$, where $\chi : S \longrightarrow C$, $\chi \neq 0$, is a multiplicative function.*

***Proof***

(1) From (46), (47), (12) and the definition of $g$ we have

$$(f(z_0))^2[g(xy) + \mu(y)g(\tau(y)x)] = f(z_0)\mu(y)f(\tau(y)xz_0) + f(z_0)f(xyz_0)$$

$$= \mu(y)\mu(z_0)f(\tau(y)xz_0\tau(z_0)z_0) - f(xyz_0z_0^2)$$

$$= \mu(yz_0)f(\tau(yz_0)(xz_0)z_0) - f((xz_0)(yz_0)z_0)$$

$$= 2f(xz_0)f(yz_0).$$

Dividing the last equation by $(f(z_0))^2$ we get $g$ satisfies the variant of d'Alembert's functional equation (15). In view of (47) and the definition of $g$ we get

$$g(z_0^2) = \frac{f(z_0z_0^2)}{f(z_0)}$$

$$= \frac{-f(z_0)f(z_0)}{f(z_0)} = -f(z_0) \neq 0.$$

Then $g$ is a non-zero solution of equation (15).

(2) By replacing $x$ by $xz_0$ in (12) we get

$$\mu(y)f(\tau(y)xz_0^2) - f(xyz_0^2) = 2f(y)f(xz_0). \tag{68}$$

By using (47), equation (68) can be written as follows:

$$-\mu(y)f(\tau(y)x) + f(xy) = 2f(y)g(x), \; x, y \in S, \tag{69}$$

where $g$ is the function defined above. If we replace $y$ by $yz_0$ in (12) we get

$$\mu(yz_0)f(\tau(y)x\tau(z_0)z_0) - f(xyz_0z_0) = 2f(x)f(yz_0). \tag{70}$$

By using (46), (47) we obtain

$$\mu(y)f(\tau(y)x) + f(xy) = 2f(x)g(y),\ \ x, y \in S. \tag{71}$$

By adding (71) and (69) we get that the pair $f$, $g$ satisfies the sine addition law

$$f(xy) = f(x)g(y) + f(y)g(x) \text{ for all } \ x, y \in S.$$

Now, in view of [13, Lemma 3.4.] $g$ is abelian. Since $g$ is a non-zero solution of d'Alembert's functional equation (15) then from [3, Proposition 9.31] there exists a non-zero multiplicative function $\chi\colon S \longrightarrow C$ such that $g = \frac{\chi+\mu\chi\circ\tau}{2}$. This completes the proof.

The following theorem is the main result of this section.

**Theorem 4** *The non-zero solutions $f : S \longrightarrow C$ of the functional equation (12) are the functions of the form*

$$f = \frac{\mu\chi\circ\tau - \chi}{2}\chi(z_0), \tag{72}$$

*where $\chi : S \longrightarrow C$ is a multiplicative function such that $\chi(z_0) \neq 0$ and $\mu(z_0)\chi(\tau(z_0)) = -\chi(z_0)$.*

*If $S$ is a topological semigroup and that $\tau : S \longrightarrow S$ and $\mu\colon S \longrightarrow C$ are continuous, then the non-zero solution $f$ of equation (12) is continuous if and only if $\chi$ is continuous.*

***Proof*** Simple computations show that $f$ defined by (72) is a solution of (12). Conversely, let $f : S \longrightarrow C$ be a non-zero solution of the functional equation (12). By putting $y = z_0$ in (12) we get

$$f(x) = \frac{\mu(z_0)f(\tau(z_0)xz_0) - f(xz_0z_0)}{2f(z_0)} \tag{73}$$

$$= \frac{1}{2}(\mu(z_0)g(\tau(z_0)x) - g(xz_0)),$$

where g is the function defined by $g(x) = \frac{f(xz_0)}{f(z_0)}$ and that from Lemma 6 has the form $g = \frac{\chi+\mu\chi\circ\tau}{2}$, where $\chi : S \longrightarrow C$, $\chi \neq 0$ is a multiplicative function. Substituting this into (73) we find that $f$ has the form

$$f = \frac{\chi(z_0) - \mu(z_0)\chi(\tau(z_0))}{2}\,\frac{\mu\chi\circ\tau - \chi}{2}. \tag{74}$$

Furthermore, from (48) $f$ satisfies $\mu(x)f(\tau(x)z_0) = f(xz_0)$ for all $x \in S$. By applying the last expression of $f$ in (48) we get after computations that

$$[\mu(z_0)\chi(\tau(z_0)) + \chi(z_0)][\chi - \mu\chi \circ \tau] = 0.$$

Since $\chi \neq \mu\chi \circ \tau$, we obtain $\mu(z_0)\chi(\tau(z_0)) + \chi(z_0) = 0$ and then from (74) we have

$$f = \frac{\mu\chi \circ \tau - \chi}{2}\chi(z_0).$$

For the topological statement we use [3, Theorem 3.18(d)]. This completes the proof.

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