Some Different Type Integral Inequalities and Their Applications

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Abstract In this article, we first present some integral inequalities for Gauss-Jacobi type quadrature formula involving generalized-$\mathbf{m}$-$((h_1^p, h_2^q); (\eta_1, \eta_2))$-convex mappings. Secondly, an identity pertaining twice differentiable mappings defined on $\mathbf{m}$-invex set is used. By using the notion of generalized-$\mathbf{m}$-$((h_1^p, h_2^q); (\eta_1, \eta_2))$-convexity and the obtained identity as an auxiliary result, some new estimates with respect to Hermite-Hadamard, Ostrowski, and Simpson type inequalities via fractional integrals are established. It is pointed out that some new special cases can be deduced from main results. At the end, some applications to special means for different positive real numbers are provided as well.

1 Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$. For any subset $K \subseteq \mathbb{R}^n$, K° is the interior of K. The set of integrable functions on the interval $[a, b]$ is denoted by $L[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1 *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \qquad (1)$$

This inequality (1) is also known as trapezium inequality.

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The trapezium type inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. For other recent results which generalize, improve, and extend the inequality (1) through various classes of convex functions interested readers are referred to [1–47].

Also the following result is known in the literature as the Ostrowski inequality [33], which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a, b]$.

Theorem 2 *Let $f : I \longrightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I° of I, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a, b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right], \quad \forall x \in [a, b]. \tag{2}$$

The following inequality is well known in the literature as Simpson's inequality:

Theorem 3 *Let $f : [a, b] \longrightarrow \mathbb{R}$ be four time differentiable on the interval (a, b) and having the fourth derivative bounded on (a, b), that is $\|f^{(4)}\|_\infty = \sup_{x\in(a,b)} |f^{(4)}|$
$< \infty$. Then, we have*

$$\left| \int_a^b f(t)dt - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{1}{2880}\|f^{(4)}\|_\infty (b-a)^5. \tag{3}$$

Inequality (3) gives an error bound for the classical Simpson quadrature formula, which is one of the most used quadrature formulae in practical applications.

In recent years, various generalizations, extensions, and variants of such inequalities have been obtained. For other recent results concerning Ostrowski type inequalities, see [21, 33]. For other recent results concerning Simpson type inequalities, see [32, 38].

Gauss-Jacobi type quadrature formula [40] is defined as follows:

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^\star |f|, \tag{4}$$

for certain $B_{m,k}$, γ_k and rest $R_m^\star |f|$. In [31], Liu obtained integral inequalities for P-function related to the left-hand side of (4), and in [48], Özdemir et al. also presented several integral inequalities concerning the left-hand side of (4) via some kinds of convexity.

Let us recall some special functions and evoke some basic definitions as follows:

Definition 1 The Euler beta function is defined for $a, b > 0$ as

$$\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt. \tag{5}$$

Definition 2 ([34]) Let $f \in L[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u}u^{\alpha-1}du$. Here $J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x)$.
Note that $\alpha = 1$, the fractional integral reduces to the classical integral.

Definition 3 ([49]) A set $S \subseteq \mathbb{R}^n$ is said to be invex set with respect to the mapping $\eta : S \times S \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in S$ for every $x, y \in S$ and $t \in [0,1]$.

The invex set S is also termed an η-connected set.

Definition 4 ([50]) Let $h : [0,1] \longrightarrow \mathbb{R}$ be a non-negative function and $h \neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if

$$f\big(x + t\eta(y,x)\big) \leq h(1-t)f(x) + h(t)f(y) \tag{6}$$

for each $x, y \in K$ and $t \in [0,1]$ where $f(\cdot) > 0$.

Clearly, when putting $h(t) = t$ in Definition 4, f becomes a preinvex function [51]. If the mapping $\eta(y,x) = y - x$ in Definition 4, then the non-negative function f reduces to h-convex mappings [52].

Definition 5 ([53]) Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \longrightarrow \mathbb{R}^n$. A function $f : S \longrightarrow [0,+\infty)$ is said to be s-preinvex (or s-Breckner-preinvex) with respect to η and $s \in (0,1]$, if for every $x, y \in S$ and $t \in [0,1]$,

$$f\big(x + t\eta(y,x)\big) \leq (1-t)^s f(x) + t^s f(y). \tag{7}$$

Definition 6 ([54]) A function $f : K \longrightarrow \mathbb{R}$ is said to be s-Godunova-Levin-Dragomir-preinvex of second kind, if

$$f\big(x + t\eta(y,x)\big) \leq (1-t)^{-s} f(x) + t^{-s} f(y), \tag{8}$$

for each $x, y \in K$, $t \in (0,1)$ and $s \in (0,1]$.

Definition 7 ([55]) A non-negative function $f : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be tgs-convex on K if the inequality

$$f\big((1-t)x + ty\big) \leq t(1-t)[f(x) + f(y)] \tag{9}$$

grips for all $x, y \in K$ and $t \in (0, 1)$.

Definition 8 ([33]) A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to MT-convex functions, if it is non-negative and $\forall\, x, y \in I$ and $t \in (0, 1)$ satisfies the subsequent inequality

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{10}$$

Definition 9 ([38]) Let $K \subseteq \mathbb{R}$ be an open m-invex set respecting $\eta : K \times K \longrightarrow \mathbb{R}$ and $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$. A function $f : K \longrightarrow \mathbb{R}$ is said to be generalized (m, h_1, h_2)-preinvex, if

$$f\big(mx + t\eta(y, mx)\big) \leq mh_1(t) f(x) + h_2(t) f(y) \tag{11}$$

is valid for all $x, y \in K$ and $t \in [0, 1]$, for some fixed $m \in (0, 1]$.

The concept of η-convex functions (at the beginning was named by φ-convex functions), considered in [14], has been introduced as the following.

Definition 10 Consider a convex set $I \subseteq \mathbb{R}$ and a bifunction $\eta : f(I) \times f(I) \longrightarrow \mathbb{R}$. A function $f : I \longrightarrow \mathbb{R}$ is called convex with respect to η (briefly η-convex), if

$$f\big(\lambda x + (1-\lambda)y\big) \leq f(y) + \lambda\eta(f(x), f(y)), \tag{12}$$

is valid for all $x, y \in I$ and $\lambda \in [0, 1]$.

Geometrically it says that if a function is η-convex on I, then for any $x, y \in I$, its graph is on or under the path starting from $(y, f(y))$ and ending at $(x, f(y) + \eta(f(x), f(y)))$. If $f(x)$ should be the end point of the path for every $x, y \in I$, then we have $\eta(x, y) = x - y$ and the function reduces to a convex one. For more results about η-convex functions, see [7, 8, 13, 14].

Definition 11 ([1]) Let $I \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \longrightarrow \mathbb{R}$. Consider $f : I \longrightarrow \mathbb{R}$ and $\eta_2 : f(I) \times f(I) \longrightarrow \mathbb{R}$. The function f is said to be (η_1, η_2)-convex if

$$f\big(x + \lambda\eta_1(y, x)\big) \leq f(x) + \lambda\eta_2(f(y), f(x)), \tag{13}$$

is valid for all $x, y \in I$ and $\lambda \in [0, 1]$.

Motivated by the above literatures, the main objective of this article is to establish in Section 2 integral inequalities using two lemmas as auxiliary results for the left-hand side of Gauss-Jacobi type quadrature formula and some new estimates on

Hermite-Hadamard, Ostrowski, and Simpson type inequalities via fractional integrals associated with generalized-$\mathbf{m}$-$((h_1^p, h_2^q); (\eta_1, \eta_2))$-convex mappings. Also, some new special cases will be deduced. In Section 3, some applications to special means for different positive real numbers will be given as well. In Section 4, some conclusion and future research are given.

2 Main Results

The following definitions will be used in this section.

Definition 12 Let $\mathbf{m} : [0, 1] \longrightarrow (0, 1]$ be a function. A set $K \subseteq \mathbb{R}^n$ is named as $\mathbf{m}$-invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $\mathbf{m}(t)x + \xi\eta(y, \mathbf{m}(t)x) \in K$ holds for each $x, y \in K$ and any $t, \xi \in [0, 1]$.

Remark 1 In Definition 12, under certain conditions, the mapping $\eta(y, \mathbf{m}(t)x)$ for any $t, \xi \in [0, 1]$ could reduce to $\eta(y, mx)$. For example, when $\mathbf{m}(t) = m$ for all $t \in [0, 1]$, then the $\mathbf{m}$-invex set degenerates an m-invex set on K.

We next introduce the concept of generalized-$\mathbf{m}$-$((h_1^p, h_2^q); (\eta_1, \eta_2))$-convex mappings.

Definition 13 Let $K \subseteq \mathbb{R}$ be an open $\mathbf{m}$-invex set with respect to the mapping $\eta_1 : K \times K \longrightarrow \mathbb{R}$ and $\mathbf{m} : [0, 1] \longrightarrow (0, 1]$. Suppose $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$ and $\varphi : I \longrightarrow K$ are continuous. Consider $f : K \longrightarrow (0, +\infty)$ and $\eta_2 : f(K) \times f(K) \longrightarrow \mathbb{R}$. The mapping f is said to be generalized-$\mathbf{m}$-$((h_1^p, h_2^q); (\eta_1, \eta_2))$-convex if

$$f\big(\mathbf{m}(t)\varphi(x) + \xi\eta_1(\varphi(y), \mathbf{m}(t)\varphi(x))\big)$$
$$\leq \left[\mathbf{m}(\xi)h_1^p(\xi)f^r(x) + h_2^q(\xi)\eta_2(f^r(y), f^r(x))\right]^{\frac{1}{r}}, \tag{14}$$

holds for all $x, y \in I$, $r \neq 0$, $t, \xi \in [0, 1]$ and any fixed $p, q > -1$.

Remark 2 In Definition 13, if we choose $\mathbf{m} = p = q = r = 1$ and $\varphi(x) = x$, then we get Definition 11.

Remark 3 In Definition 13, if we choose $\mathbf{m} = p = q = r = 1$, $h_1(t) = 1$, $h_2(t) = t$, $\eta_1(\varphi(y), \mathbf{m}(t)\varphi(x)) = \varphi(y) - \mathbf{m}(t)\varphi(x)$, $\eta_2(f^r(y), f^r(x)) = \eta(f^r(y), f^r(x))$ and $\varphi(x) = x$, $\forall x \in I$, then we get Definition 10. Also, in Definition 13, if we choose $\mathbf{m} = p = q = r = 1$, $h_1(t) = 1$, $h_2(t) = t$ and $\varphi(x) = x$, $\forall x \in I$, then we get Definition 11. Under some suitable choices as we have done above, we can get also the Definitions 5 and 6.

Remark 4 Let us discuss some special cases in Definition 13 as follows:

(I) If taking $h_1(t) = h(1-t)$ and $h_2(t) = h(t)$, then we get generalized-**m**-$((h^p(1-t), h^q(t)); (\eta_1, \eta_2))$-convex mappings.
(II) If taking $h_1(t) = (1-t)^s$ and $h_2(t) = t^s$ for $s \in (0,1]$, then we get generalized-**m**-$(((1-t)^{sp}, t^{sq}); (\eta_1, \eta_2))$-Breckner-convex mappings.
(III) If taking $h_1(t) = (1-t)^{-s}$ and $h_2(t) = t^{-s}$ for $s \in (0,1]$, then we get generalized-**m**-$(((1-t)^{-sp}, t^{-sq}); (\eta_1, \eta_2))$-Godunova-Levin-Dragomir-convex mappings.
(IV) If taking $h_1(t) = h_2(t) = t(1-t)$, then we get generalized-**m**-$((t(1-t))^{sp}, (t(1-t))^{sq}); (\eta_1, \eta_2))$-convex mappings.
(V) If taking $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$ and $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized-**m**-$\left(\left(\left(\frac{\sqrt{1-t}}{2\sqrt{t}}\right)^p, \left(\frac{\sqrt{t}}{2\sqrt{1-t}}\right)^q\right); (\eta_1, \eta_2)\right)$-convex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.
Let us see the following example of a generalized-**m**-$((h_1^p, h_2^q); (\eta_1, \eta_2))$-convex mapping which is not convex.

Example 1 Let us take $\mathbf{m} = r = \frac{1}{2}$, $h_1(t) = t^l$, $h_2(t) = t^s$ for all $l, s \in [0,1]$, any fixed $p, q \geq 1$ and φ an identity function. Consider the function $f : [0,+\infty) \longrightarrow [0,+\infty)$ by

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1; \\ 2, & x > 1. \end{cases}$$

Define two bifunctions $\eta_1 : [0,+\infty) \times [0,+\infty) \longrightarrow \mathbb{R}$ and $\eta_2 : [0,+\infty) \times [0,+\infty) \longrightarrow [0,+\infty)$ by

$$\eta_1(x,y) = \begin{cases} -y, & 0 \leq y \leq 1; \\ x+y, & y > 1, \end{cases}$$

and

$$\eta_2(x,y) = \begin{cases} x+y, & x \leq y; \\ 2(x+y), & x > y. \end{cases}$$

Then f is generalized $\frac{1}{2}$-$((t^{lp}, t^{sq}); (\eta_1, \eta_2))$-convex mapping. But f is not preinvex with respect to η_1 and also it is not convex (consider $x = 0, y = 2$ and $t \in (0,1]$).

We claim the following integral identity.

Lemma 1 *Let* $\varphi : I \longrightarrow K$ *be a continuous function and* $\boldsymbol{m} : [0,1] \longrightarrow (0,1]$. *Assume that* $f : K = [\boldsymbol{m}(t)\varphi(a), \boldsymbol{m}(t)\varphi(a) + \eta(\varphi(b), \boldsymbol{m}(t)\varphi(a))] \longrightarrow$

$\mathbb{R}$ is a continuous function on K° with respect to $\eta : K \times K \longrightarrow \mathbb{R}$ for $\eta(\varphi(b), \boldsymbol{m}(t)\varphi(a)) > 0$ and $\forall\, t \in [0, 1]$. Then for any fixed $p, q > 0$, we have

$$\int_{\boldsymbol{m}(t)\varphi(a)}^{\boldsymbol{m}(t)\varphi(a)+\eta(\varphi(b),\boldsymbol{m}(t)\varphi(a))} (x - \boldsymbol{m}(t)\varphi(a))^p \times (\boldsymbol{m}(t)\varphi(a) + \eta(\varphi(b), \boldsymbol{m}(t)\varphi(a)) - x)^q f(x)dx$$
$$= \eta^{p+q+1}(\varphi(b), \boldsymbol{m}(t)\varphi(a)) \int_0^1 \xi^p (1-\xi)^q f(\boldsymbol{m}(t)\varphi(a)+\xi\eta(\varphi(b), \boldsymbol{m}(t)\varphi(a)))d\xi.$$

We denote

$$T_f^{p,q}(\eta, \varphi, \boldsymbol{m}; a, b) := \int_{\boldsymbol{m}(t)\varphi(a)}^{\boldsymbol{m}(t)\varphi(a)+\eta(\varphi(b),\boldsymbol{m}(t)\varphi(a))} (x - \boldsymbol{m}(t)\varphi(a))^p \times (\boldsymbol{m}(t)\varphi(a) + \eta(\varphi(b), \boldsymbol{m}(t)\varphi(a)) - x)^q f(x)dx. \quad (15)$$

Proof We observe that

$$T_f^{p,q}(\eta, \varphi, \mathbf{m}; a, b)$$
$$= \eta(\varphi(b), \mathbf{m}(t)\varphi(a)) \int_0^1 (\mathbf{m}(t)\varphi(a) + \xi\eta(\varphi(b), \mathbf{m}(t)\varphi(a)) - \mathbf{m}(t)\varphi(a))^p$$
$$\times (\mathbf{m}(t)\varphi(a) + \eta(\varphi(b), \mathbf{m}(t)\varphi(a)) - \mathbf{m}(t)\varphi(a) - \xi\eta(\varphi(b), \mathbf{m}(t)\varphi(a)))^q$$
$$\times f(\mathbf{m}(t)\theta(a) + \xi\Lambda(\theta(b), \mathbf{m}(t)\theta(a)))d\xi$$
$$= \eta^{p+q+1}(\varphi(b), \mathbf{m}(t)\varphi(a)) \int_0^1 \xi^p (1-\xi)^q f(\mathbf{m}(t)\varphi(a)+\xi\eta(\varphi(b), \mathbf{m}(t)\varphi(a)))d\xi.$$

This completes the proof of the lemma.

Remark 5 In Lemma 1, if we choose $\mathbf{m}(t) \equiv 1$ for any $t \in [0, 1]$, $\eta(\varphi(b), \mathbf{m}(t)\varphi(a)) = \varphi(b) - \mathbf{m}(t)\varphi(a)$ and $\varphi(x) = x$ for all $x \in I$, then we get the left-hand side of (4).

With the help of Lemma 1, we have the following results.

Theorem 4 *Let $k > 1$, $0 < r \le 1$ and $p_1, p_2 > -1$. Suppose $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$, $\varphi : I \longrightarrow K$ are continuous functions and $\boldsymbol{m} : [0, 1] \longrightarrow (0, 1]$. Assume that $f : K = [\boldsymbol{m}(t)\varphi(a), \boldsymbol{m}(t)\varphi(a) + \eta_1(\varphi(b), \boldsymbol{m}(t)\varphi(a))] \longrightarrow (0, +\infty)$ is a continuous mapping on K° with respect to $\eta_1 : K \times K \longrightarrow \mathbb{R}$ for $\eta_1(\varphi(b), \boldsymbol{m}(t)\varphi(a)) > 0$ for all $t \in [0, 1]$ and $\eta_2 : f(K) \times f(K) \longrightarrow \mathbb{R}$. If $f^{\frac{k}{k-1}}$ is generalized-$\boldsymbol{m}$-$((h_1^{p_1}, h_2^{p_2}); (\eta_1, \eta_2))$-convex mapping on an open $\boldsymbol{m}$-invex set K, then for any fixed $p, q > 0$, we have*

$$\left|T_f^{p,q}(\eta_1, \varphi, \boldsymbol{m}; a, b)\right| \leq \eta_1^{p+q+1}(\varphi(b), \boldsymbol{m}(t)\varphi(a))\sqrt[k]{\beta(kp+1, kq+1)} \tag{16}$$

$$\times\Big[f^{\frac{rk}{k-1}}(a)I^r(h_1(\xi), \boldsymbol{m}(\xi); p_1, r) + \eta_2\left(f^{\frac{rk}{k-1}}(b), f^{\frac{rk}{k-1}}(a)\right)I^r(h_2(\xi); p_2, r)\Big]^{\frac{k-1}{rk}},$$

where

$$I(h_1(\xi), \boldsymbol{m}(\xi); p_1, r) := \int_0^1 \boldsymbol{m}^{\frac{1}{r}}(\xi)h_1^{\frac{p_1}{r}}(\xi)d\xi, \quad I(h_2(\xi); p_2, r) := \int_0^1 h_2^{\frac{p_2}{r}}(\xi)d\xi.$$

Proof Since $f^{\frac{k}{k-1}}$ is generalized-**m**-$((h_1^{p_1}, h_2^{p_2}); (\eta_1, \eta_2))$-convex mapping on K, combining with Lemma 1, Hölder inequality, Minkowski inequality, and properties of the modulus, we get

$$\begin{aligned}
\left|T_f^{p,q}(\eta_1, \varphi, \mathbf{m}; a, b)\right| &\leq |\eta_1(\varphi(b), \mathbf{m}(t)\varphi(a))|^{p+q+1}\left[\int_0^1 \xi^{kp}(1-\xi)^{kq}d\xi\right]^{\frac{1}{k}} \\
&\times\left[\int_0^1 \left|f(\mathbf{m}(t)\varphi(a) + \xi\eta_1(\varphi(b), \mathbf{m}(t)\varphi(a)))\right|^{\frac{k}{k-1}}d\xi\right]^{\frac{k-1}{k}} \\
&\leq \eta_1^{p+q+1}(\varphi(b), \mathbf{m}(t)\varphi(a))\sqrt[k]{\beta(kp+1, kq+1)} \\
&\times\left[\int_0^1 \left[\mathbf{m}(\xi)h_1^{p_1}(\xi)f^{\frac{rk}{k-1}}(a) + h_2^{p_2}(\xi)\eta_2\left(f^{\frac{rk}{k-1}}(b), f^{\frac{rk}{k-1}}(a)\right)\right]^{\frac{1}{r}}d\xi\right]^{\frac{k-1}{k}} \\
&\leq \eta_1^{p+q+1}(\varphi(b), \mathbf{m}(t)\varphi(a))\sqrt[k]{\beta(kp+1, kq+1)} \\
&\times\left\{\left(\int_0^1 \mathbf{m}^{\frac{1}{r}}(\xi)h_1^{\frac{p_1}{r}}(\xi)f^{\frac{k}{k-1}}(a)d\xi\right)^r\right. \\
&\left.+\left(\int_0^1 \eta_2^{\frac{1}{r}}\left(f^{\frac{rk}{k-1}}(b), f^{\frac{rk}{k-1}}(a)\right)h_2^{\frac{p_2}{r}}(\xi)d\xi\right)^r\right\}^{\frac{k-1}{rk}} \\
&= \eta_1^{p+q+1}(\varphi(b), \mathbf{m}(t)\varphi(a))\sqrt[k]{\beta(kp+1, kq+1)} \\
&\times\Big[f^{\frac{rk}{k-1}}(a)I^r(h_1(\xi), \mathbf{m}(\xi); p_1, r) + \eta_2\left(f^{\frac{rk}{k-1}}(b), f^{\frac{rk}{k-1}}(a)\right)I^r(h_2(\xi); p_2, r)\Big]^{\frac{k-1}{rk}}.
\end{aligned}$$

So, the proof of this theorem is completed.

We point out some special cases of Theorem 4.

Corollary 1 *In Theorem 4 for $k = 2$, we get*

$$\left|T_f^{p,q}(\eta_1, \varphi, \boldsymbol{m}; a, b)\right| \leq \eta_1^{p+q+1}(\varphi(b), \boldsymbol{m}(t)\varphi(a))\sqrt{\beta(2p+1, 2q+1)} \tag{17}$$

$$\times \sqrt[2r]{f^{2r}(a) I^r(h_1(\xi), \boldsymbol{m}(\xi); p_1, r) + \eta_2\left(f^{2r}(b), f^{2r}(a)\right) I^r(h_2(\xi); p_2, r)}.$$

Corollary 2 *In Theorem 4 for $h_1(t) = h(1-t)$, $h_2(t) = h(t)$ and $\boldsymbol{m}(t) = m \in (0, 1]$ for all $t \in [0, 1]$, we get*

$$\left|T_f^{p,q}(\eta_1, \varphi, m; a, b)\right| \leq \eta_1^{p+q+1}(\varphi(b), m\varphi(a))\sqrt[k]{\beta(kp+1, kq+1)} \tag{18}$$

$$\times\left[m f^{\frac{rk}{k-1}}(a) I^r(h(1-\xi); p_1, r) + \eta_2\left(f^{\frac{rk}{k-1}}(b), f^{\frac{rk}{k-1}}(a)\right) I^r(h(\xi); p_2, r)\right]^{\frac{k-1}{rk}}.$$

Corollary 3 *In Corollary 2 for $h_1(t) = (1-t)^s$ and $h_2(t) = t^s$, we get*

$$\left|T_f^{p,q}(\eta_1, \varphi, m; a, b)\right| \leq \eta_1^{p+q+1}(\varphi(b), m\varphi(a))\sqrt[k]{\beta(kp+1, kq+1)} \tag{19}$$

$$\times\left[m f^{\frac{rk}{k-1}}(a)\left(\frac{r}{r+sp_1}\right)^r + \eta_2\left(f^{\frac{rk}{k-1}}(b), f^{\frac{rk}{k-1}}(a)\right)\left(\frac{r}{r+sp_2}\right)^r\right]^{\frac{k-1}{rk}}.$$

Corollary 4 *In Corollary 2 for $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ and $r > s \cdot \max\{p_1, p_2\}$, we get*

$$\left|T_f^{p,q}(\eta_1, \varphi, m; a, b)\right| \leq \eta_1^{p+q+1}(\varphi(b), m\varphi(a))\sqrt[k]{\beta(kp+1, kq+1)} \tag{20}$$

$$\times\left[m f^{\frac{rk}{k-1}}(a)\left(\frac{r}{r-sp_1}\right)^r + \eta_2\left(f^{\frac{rk}{k-1}}(b), f^{\frac{rk}{k-1}}(a)\right)\left(\frac{r}{r-sp_2}\right)^r\right]^{\frac{k-1}{rk}}.$$

Corollary 5 *In Theorem 4 for $h_1(t) = h_2(t) = t(1-t)$ and $\boldsymbol{m}(t) = m \in (0, 1]$ for all $t \in [0, 1]$, we get*

$$\left|T_f^{p,q}(\eta_1, \varphi, m; a, b)\right| \leq \eta_1^{p+q+1}(\varphi(b), m\varphi(a))\sqrt[k]{\beta(kp+1, kq+1)} \tag{21}$$

$$\times\left[m f^{\frac{rk}{k-1}}(a)\beta^r\left(1+\frac{p_1}{r}, 1+\frac{p_1}{r}\right) + \eta_2\left(f^{\frac{rk}{k-1}}(b), f^{\frac{rk}{k-1}}(a)\right)\beta^r\left(1+\frac{p_2}{r}, 1+\frac{p_2}{r}\right)\right]^{\frac{k-1}{rk}}.$$

Corollary 6 *In Corollary 2 for $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$ and $r > \frac{1}{2} \cdot \max\{p_1, p_2\}$, we get*

$$\left|T_f^{p,q}(\eta_1,\varphi,m;a,b)\right| \leq \eta_1^{p+q+1}(\varphi(b),m\varphi(a))\sqrt[k]{\beta(kp+1,kq+1)} \tag{22}$$

$$\times\left[mf^{\frac{rk}{k-1}}(a)\beta^r\left(1-\frac{p_1}{2r},1+\frac{p_1}{2r}\right)+\eta_2\left(f^{\frac{rk}{k-1}}(b),f^{\frac{rk}{k-1}}(a)\right)\beta^r\left(1-\frac{p_2}{2r},1+\frac{p_2}{2r}\right)\right]^{\frac{k-1}{rk}}.$$

Theorem 5 *Let $l \geq 1$, $0 < r \leq 1$ and $p_1, p_2 > -1$. Suppose $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$, $\varphi : I \longrightarrow K$ are continuous functions and $\boldsymbol{m} : [0,1] \longrightarrow (0,1]$. Assume that $f : K = [\boldsymbol{m}(t)\varphi(a), \boldsymbol{m}(t)\varphi(a)+\eta_1(\varphi(b),\boldsymbol{m}(t)\varphi(a))] \longrightarrow (0,+\infty)$ is a continuous mapping on K° with respect to $\eta_1 : K \times K \longrightarrow \mathbb{R}$ for $\eta_1(\varphi(b),\boldsymbol{m}(t)\varphi(a)) > 0$ for all $t \in [0,1]$ and $\eta_2 : f(K)\times f(K) \longrightarrow \mathbb{R}$. If f^l is generalized-$\boldsymbol{m}$-$((h_1^{p_1},h_2^{p_2});(\eta_1,\eta_2))$-convex mapping on an open $\boldsymbol{m}$-invex set K, then for any fixed $p, q > 0$, we have*

$$\left|T_f^{p,q}(\eta_1,\varphi,\boldsymbol{m};a,b)\right| \leq \eta_1^{p+q+1}(\varphi(b),\boldsymbol{m}(t)\varphi(a))\beta^{\frac{l-1}{l}}(p+1,q+1) \tag{23}$$

$$\times\sqrt[rl]{f^{rl}(a)J^r(h_1(\xi),\boldsymbol{m}(\xi);p,q,p_1,r)+\eta_2\left(f^{rl}(b),f^{rl}(a)\right)J^r(h_2(\xi);p,q,p_2,r)},$$

where

$$J(h_1(\xi),\boldsymbol{m}(\xi);p,q,p_1,r) := \int_0^1 \boldsymbol{m}^{\frac{1}{r}}(\xi)\xi^p(1-\xi)^q h_1^{\frac{p_1}{r}}(\xi)d\xi;$$

$$J(h_2(\xi);p,q,p_2,r) := \int_0^1 \xi^p(1-\xi)^q h_2^{\frac{p_2}{r}}(\xi)d\xi.$$

Proof Since f^l is generalized-$\mathbf{m}$-$((h_1^{p_1},h_2^{p_2});(\eta_1,\eta_2))$-convex mapping on K, combining with Lemma 1, the well-known power mean inequality, Minkowski inequality, and properties of the modulus, we get

$$\left|T_f^{p,q}(\eta_1,\varphi,\mathbf{m};a,b)\right| = \left|\eta_1^{p+q+1}(\varphi(b),\mathbf{m}(t)\varphi(a))\right|$$

$$\times\left|\int_0^1\left[\xi^p(1-\xi)^q\right]^{\frac{l-1}{l}}\left[\xi^p(1-\xi)^q\right]^{\frac{1}{l}}f(\mathbf{m}(t)\varphi(a)+\xi\eta_1(\varphi(b),\mathbf{m}(t)\varphi(a)))d\xi\right|$$

$$\leq |\eta_1(\varphi(b),\mathbf{m}(t)\varphi(a))|^{p+q+1}\left[\int_0^1\xi^p(1-\xi)^q d\xi\right]^{\frac{l-1}{l}}$$

$$\times\left[\int_0^1\xi^p(1-\xi)^q\left|f(\mathbf{m}(t)\varphi(a)+\xi\eta_1(\varphi(b),\mathbf{m}(t)\varphi(a)))\right|^l d\xi\right]^{\frac{1}{l}}$$

$$\leq \eta_1^{p+q+1}(\varphi(b),\mathbf{m}(t)\varphi(a))\beta^{\frac{l-1}{l}}(p+1,q+1)$$

$$\times\left[\int_0^1 \xi^p(1-\xi)^q\Big[\mathbf{m}(\xi)h_1^{p_1}(\xi)f^{rl}(a)+h_2^{p_2}(\xi)\eta_2\left(f^{rl}(b),f^{rl}(a)\right)\Big]^{\frac{1}{r}}d\xi\right]^{\frac{1}{l}}$$

$$\leq \eta_1^{p+q+1}(\varphi(b),\mathbf{m}(t)\varphi(a))\beta^{\frac{l-1}{l}}(p+1,q+1)$$

$$\times\left\{\left(\int_0^1 \mathbf{m}^{\frac{1}{r}}(\xi)\xi^p(1-\xi)^q h_1^{\frac{p_1}{r}}(\xi)f^l(a)d\xi\right)^r\right.$$

$$\left.+\left(\int_0^1 \xi^p(1-\xi)^q h_2^{\frac{p_2}{r}}(\xi)\eta_2^{\frac{1}{r}}\left(f^{rl}(b),f^{rl}(a)\right)d\xi\right)^r\right\}^{\frac{1}{rl}}$$

$$=\eta_1^{p+q+1}(\varphi(b),\mathbf{m}(t)\varphi(a))\beta^{\frac{l-1}{l}}(p+1,q+1)$$

$$\times\sqrt[rl]{f^{rl}(a)J^r(h_1(\xi),\mathbf{m}(\xi);p,q,p_1,r)+\eta_2\left(f^{rl}(b),f^{rl}(a)\right)J^r(h_2(\xi);p,q,p_2,r)}.$$

So, the proof of this theorem is completed.

Let us discuss some special cases of Theorem 5.

Corollary 7 *In Theorem 5 for* $l=1$, *we get*

$$\left|T_f^{p,q}(\eta_1,\varphi,\boldsymbol{m};a,b)\right|\leq \eta_1^{p+q+1}(\varphi(b),\boldsymbol{m}(t)\varphi(a)) \tag{24}$$

$$\times\sqrt[r]{f^r(a)J^r(h_1(\xi),\boldsymbol{m}(\xi);p,q,p_1,r)+\eta_2\left(f^r(b),f^r(a)\right)J^r(h_2(\xi);p,q,p_2,r)}.$$

Corollary 8 *In Theorem 5 for* $h_1(t)=h(1-t)$, $h_2(t)=h(t)$ *and* $\boldsymbol{m}(t)=m\in(0,1]$ *for all* $t\in[0,1]$, *we get*

$$\left|T_f^{p,q}(\eta_1,\varphi,m;a,b)\right|\leq \eta_1^{p+q+1}(\varphi(b),m\varphi(a))\beta^{\frac{l-1}{l}}(p+1,q+1) \tag{25}$$

$$\times\sqrt[rl]{mf^{rl}(a)J^r(h(1-\xi);p,q,p_1,r)+\eta_2\left(f^{rl}(b),f^{rl}(a)\right)J^r(h(\xi);p,q,p_2,r)}.$$

Corollary 9 *In Corollary 8 for* $h_1(t)=(1-t)^s$, $h_2(t)=t^s$ *and* $0<s\leq r$, *we get*

$$\left|T_f^{p,q}(\eta_1,\varphi,m;a,b)\right|\leq \eta_1^{p+q+1}(\varphi(b),m\varphi(a))\beta^{\frac{l-1}{l}}(p+1,q+1) \tag{26}$$

$$\times\sqrt[rl]{mf^{rl}(a)\beta^r\left(p+1,q+\frac{sp_1}{r}+1\right)+\eta_2\left(f^{rl}(b),f^{rl}(a)\right)\beta^r\left(q+1,p+\frac{sp_2}{r}+1\right)}.$$

Corollary 10 *In Corollary 8 for* $h_1(t)=(1-t)^{-s}$ *and* $h_2(t)=t^{-s}$, *we get*

$$\left|T_f^{p,q}(\eta_1,\varphi,m;a,b)\right|\leq \eta_1^{p+q+1}(\varphi(b),m\varphi(a))\beta^{\frac{l-1}{l}}(p+1,q+1) \tag{27}$$

$$\times \sqrt[rl]{mf^{rl}(a)\beta^{r}\left(p+1, q-\frac{sp_1}{r}+1\right)+\eta_2\left(f^{rl}(b), f^{rl}(a)\right)\beta^{r}\left(q+1, p-\frac{sp_2}{r}+1\right)}.$$

Corollary 11 *In Theorem 5 for* $h_1(t) = h_2(t) = t(1-t)$ *and* $\boldsymbol{m}(t) = m \in (0, 1]$ *for all* $t \in [0, 1]$, *we get*

$$\left|T_f^{p,q}(\eta_1, \varphi, m; a, b)\right| \leq \eta_1^{p+q+1}(\varphi(b), m\varphi(a))\beta^{\frac{l-1}{l}}(p+1, q+1) \tag{28}$$
$$\times\left[mf^{rl}(a)\beta^{r}\left(p+\frac{p_1}{r}+1, q+\frac{p_1}{r}+1\right)\right.$$
$$\left.+\eta_2\left(f^{rl}(b), f^{rl}(a)\right)\beta^{r}\left(p+\frac{p_2}{r}+1, q+\frac{p_2}{r}+1\right)\right]^{\frac{1}{rq}}.$$

Corollary 12 *In Corollary 8 for* $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$ *and* $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, *we get*

$$\left|T_f^{p,q}(\eta_1, \varphi, m; a, b)\right| \leq \eta_1^{p+q+1}(\varphi(b), m\varphi(a))\beta^{\frac{l-1}{l}}(p+1, q+1) \tag{29}$$
$$\times\left[mf^{rl}(a)\beta^{r}\left(p-\frac{p_1}{2r}+1, q+\frac{p_1}{2r}+1\right)\right.$$
$$\left.+\eta_2\left(f^{rl}(b), f^{rl}(a)\right)\beta^{r}\left(p+\frac{p_2}{2r}+1, q-\frac{p_2}{2r}+1\right)\right]^{\frac{1}{rq}}.$$

For establishing our second main results regarding generalizations of Hermite-Hadamard, Ostrowski, and Simpson type inequalities associated with generalized-$\mathbf{m}$-$((h_1^{p_1}, h_2^{p_2}); (\eta_1, \eta_2))$-convexity via fractional integrals, we need the following lemma.

Lemma 2 *Let* $\varphi : I \longrightarrow K$ *be a continuous function and* $\boldsymbol{m} : [0, 1] \longrightarrow (0, 1]$. *Suppose* $K = [\boldsymbol{m}(t)\varphi(a), \boldsymbol{m}(t)\varphi(a) + \eta(\varphi(b), \boldsymbol{m}(t)\varphi(a))] \subseteq \mathbb{R}$ *be an open* $\boldsymbol{m}$*-invex subset with respect to* $\eta : K \times K \longrightarrow \mathbb{R}$ *and let* $\eta(\varphi(b), \boldsymbol{m}(t)\varphi(a)) > 0$ *for all* $t \in [0, 1]$. *Assume that* $f : K \longrightarrow \mathbb{R}$ *be a twice differentiable mapping on* K° *and* $f'' \in L(K)$. *Then for any* $\lambda \in [0, 1]$ *and* $\alpha > 0$, *the following identity holds:*

$$\frac{\lambda-1}{\eta(\varphi(b), \boldsymbol{m}(t)\varphi(a))}\left\{\eta^{\alpha+1}(\varphi(x), \boldsymbol{m}(t)\varphi(a))f'(\boldsymbol{m}(t)\varphi(a)+\eta(\varphi(x), \boldsymbol{m}(t)\varphi(a)))\right.$$
$$\left.+\eta^{\alpha+1}(\varphi(x), \boldsymbol{m}(t)\varphi(b))f'(\boldsymbol{m}(t)\varphi(b)+\eta(\varphi(x), \boldsymbol{m}(t)\varphi(b)))\right\}$$
$$+\frac{1+\alpha-\lambda}{\eta(\varphi(b), \boldsymbol{m}(t)\varphi(a))}\left\{\eta^{\alpha}(\varphi(x), \boldsymbol{m}(t)\varphi(a))f(\boldsymbol{m}(t)\varphi(a)+\eta(\varphi(x), \boldsymbol{m}(t)\varphi(a)))\right.$$
$$\left.+\eta^{\alpha}(\varphi(x), \boldsymbol{m}(t)\varphi(b))f(\boldsymbol{m}(t)\varphi(b)+\eta(\varphi(x), \boldsymbol{m}(t)\varphi(b)))\right\}$$

$$+\frac{\lambda}{\eta(\varphi(b),\boldsymbol{m}(t)\varphi(a))}$$

$$\times\Big\{\eta^{\alpha}(\varphi(x),\boldsymbol{m}(t)\varphi(a))f(\boldsymbol{m}(t)\varphi(a))+\eta^{\alpha}(\varphi(x),\boldsymbol{m}(t)\varphi(b))f(\boldsymbol{m}(t)\varphi(b))\Big\}$$

$$-\frac{\Gamma(\alpha+2)}{\eta(\varphi(b),\boldsymbol{m}(t)\varphi(a))}\times\Big[J^{\alpha}_{(\boldsymbol{m}(t)\varphi(a)+\eta(\varphi(x),\boldsymbol{m}(t)\varphi(a)))^-}f(\boldsymbol{m}(t)\varphi(a))$$

$$+J^{\alpha}_{(\boldsymbol{m}(t)\varphi(b)+\eta(\varphi(x),\boldsymbol{m}(t)\varphi(b)))^-}f(\boldsymbol{m}(t)\varphi(b))\Big]$$

$$=\frac{\eta^{\alpha+2}(\varphi(x),\boldsymbol{m}(t)\varphi(a))}{\eta(\varphi(b),\boldsymbol{m}(t)\varphi(a))} \tag{30}$$

$$\times\int_0^1\xi(\lambda-\xi^{\alpha})f''(\boldsymbol{m}(t)\varphi(a)+\xi\eta(\varphi(x),\boldsymbol{m}(t)\varphi(a)))d\xi$$

$$+\frac{\eta^{\alpha+2}(\varphi(x),\boldsymbol{m}(t)\varphi(b))}{\eta(\varphi(b),\boldsymbol{m}(t)\varphi(a))}$$

$$\times\int_0^1\xi(\lambda-\xi^{\alpha})f''(\boldsymbol{m}(t)\varphi(b)+\xi\eta(\varphi(x),\boldsymbol{m}(t)\varphi(b)))d\xi.$$

We denote

$$\Delta^{\alpha}_{f}(\eta,\varphi,\boldsymbol{m};\lambda,x,a,b):=\frac{\eta^{\alpha+2}(\varphi(x),\boldsymbol{m}(t)\varphi(a))}{\eta(\varphi(b),\boldsymbol{m}(t)\varphi(a))} \tag{31}$$

$$\times\int_0^1\xi(\lambda-\xi^{\alpha})f''(\boldsymbol{m}(t)\varphi(a)+\xi\eta(\varphi(x),\boldsymbol{m}(t)\varphi(a)))d\xi$$

$$+\frac{\eta^{\alpha+2}(\varphi(x),\boldsymbol{m}(t)\varphi(b))}{\eta(\varphi(b),\boldsymbol{m}(t)\varphi(a))}\int_0^1\xi(\lambda-\xi^{\alpha})f''(\boldsymbol{m}(t)\varphi(b)+\xi\eta(\varphi(x),\boldsymbol{m}(t)\varphi(b)))d\xi.$$

Proof A simple proof of the equality (30) can be done by performing two integration by parts in the integrals above and changing the variables. The details are left to the interested reader. This completes the proof of our lemma.

Using Lemma 2, we now state the following theorems for the corresponding version for power of second derivative.

Theorem 6 *Let* $0<r\leq 1$ *and* $p_1,p_2>-1$. *Suppose* $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$, $\varphi:I\longrightarrow K$ *are continuous functions and* $\boldsymbol{m}:[0,1]\longrightarrow(0,1]$. *Let* $K=[\boldsymbol{m}(t)\varphi(a),\boldsymbol{m}(t)\varphi(a)+\eta_1(\varphi(b),\boldsymbol{m}(t)\varphi(a))]\subseteq\mathbb{R}$ *be an open* $\boldsymbol{m}$*-invex subset with respect to* $\eta_1:K\times K\longrightarrow\mathbb{R}$ *and let* $\eta_1(\varphi(b),\boldsymbol{m}(t)\varphi(a))>0$ *for all* $t\in[0,1]$ *and* $\eta_2:f(K)\times f(K)\longrightarrow\mathbb{R}$. *Assume that* $f:K\longrightarrow(0,+\infty)$ *be a* *twice differentiable mapping on* K°. *If* f''^q *is generalized-*$\boldsymbol{m}$*-*$((h_1^{p_1}, h_2^{p_2}); (\eta_1, \eta_2))$*-convex mapping on* K, $q > 1$, $p^{-1} + q^{-1} = 1$, *then for any* $\lambda \in [0, 1]$ *and* $\alpha > 0$, *the following inequality for fractional integrals holds:*

$$\begin{aligned}\left|\Delta_f^\alpha(\eta_1, \varphi, \boldsymbol{m}; \lambda, x, a, b)\right| &\leq \frac{\sqrt[p]{D(\alpha, \lambda, p)}}{\eta_1(\varphi(b), \boldsymbol{m}(t)\varphi(a))} \\ &\times \Big\{ |\eta_1(\varphi(x), \boldsymbol{m}(t)\varphi(a))|^{\alpha+2} \\ &\times \sqrt[rq]{(f''(a))^{rq} I^r(h_1(\xi), \boldsymbol{m}(\xi); p_1, r) + \eta_2\left((f''(x))^{rq}, (f''(a))^{rq}\right) I^r(h_2(\xi); p_2, r)} \\ &+ |\eta_1(\varphi(x), \boldsymbol{m}(t)\varphi(b))|^{\alpha+2} \\ &\times \sqrt[rq]{(f''(b))^{rq} I^r(h_1(\xi), \boldsymbol{m}(\xi); p_1, r) + \eta_2\left((f''(x))^{rq}, (f''(b))^{rq}\right) I^r(h_2(\xi); p_2, r)} \Big\},\end{aligned} \tag{32}$$

where

$$D(\alpha, \lambda, p) := \int_0^1 |\xi(\lambda - \xi^\alpha)|^p d\xi$$

and $I(h_1(\xi), \boldsymbol{m}(\xi); p_1, r)$, $I(h_2(\xi); p_2, r)$ *are defined as in Theorem 4.*

Proof Using relation (31), generalized-$\mathbf{m}$-$((h_1^{p_1}, h_2^{p_2}); (\eta_1, \eta_2))$-convexity of f''^q, Hölder inequality, Minkowski inequality, and properties of the modulus, we have

$$\begin{aligned}\left|\Delta_f^\alpha(\eta_1, \varphi, \mathbf{m}; \lambda, x, a, b)\right| &\leq \frac{|\eta_1(\varphi(x), \mathbf{m}(t)\varphi(a))|^{\alpha+2}}{|\eta_1(\varphi(b), \mathbf{m}(t)\varphi(a))|} \\ &\times \int_0^1 |\xi(\lambda - \xi^\alpha)||f''(\mathbf{m}(t)\varphi(a) + \xi\eta_1(\varphi(x), \mathbf{m}(t)\varphi(a)))|d\xi \\ &+ \frac{|\eta_1(\varphi(x), \mathbf{m}(t)\varphi(b))|^{\alpha+2}}{|\eta_1(\varphi(b), \mathbf{m}(t)\varphi(a))|} \\ &\times \int_0^1 |\xi(\lambda - \xi^\alpha)||f''(\mathbf{m}(t)\varphi(b) + \xi\eta_1(\varphi(x), \mathbf{m}(t)\varphi(b)))|d\xi \\ &\leq \frac{|\eta_1(\varphi(x), \mathbf{m}(t)\varphi(a))|^{\alpha+2}}{\eta_1(\varphi(b), \mathbf{m}(t)\varphi(a))} \left(\int_0^1 |\xi(\lambda - \xi^\alpha)|^p d\xi\right)^{\frac{1}{p}} \\ &\times \left(\int_0^1 \left(f''(\mathbf{m}(t)\varphi(a) + \xi\eta_1(\varphi(x), \mathbf{m}(t)\varphi(a)))\right)^q d\xi\right)^{\frac{1}{q}} \\ &+ \frac{|\eta_1(\varphi(x), \mathbf{m}(t)\varphi(b))|^{\alpha+2}}{\eta_1(\varphi(b), \mathbf{m}(t)\varphi(a))} \left(\int_0^1 |\xi(\lambda - \xi^\alpha)|^p d\xi\right)^{\frac{1}{p}}\end{aligned}$$

$$\times\left(\int_0^1 \left(f''(\mathbf{m}(t)\varphi(b)+\xi\eta_1(\varphi(x),\mathbf{m}(t)\varphi(b)))\right)^q d\xi\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta_1(\varphi(x),\mathbf{m}(t)\varphi(a))|^{\alpha+2}}{\eta_1(\varphi(b),\mathbf{m}(t)\varphi(a))}\sqrt[p]{D(\alpha,\lambda,p)}$$

$$\times\left(\int_0^1 \left[\mathbf{m}(\xi)h_1^{p_1}(\xi)(f''(a))^{rq}+h_2^{p_2}(\xi)\eta_2\left((f''(x))^{rq},(f''(a))^{rq}\right)\right]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}$$

$$+\frac{|\eta_1(\varphi(x),\mathbf{m}(t)\varphi(b))|^{\alpha+2}}{\eta_1(\varphi(b),\mathbf{m}(t)\varphi(a))}\sqrt[p]{D(\alpha,\lambda,p)}$$

$$\times\left(\int_0^1 \left[\mathbf{m}(\xi)h_1^{p_1}(\xi)(f''(b))^{rq}+h_2^{p_2}(\xi)\eta_2\left((f''(x))^{rq},(f''(b))^{rq}\right)\right]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta_1(\varphi(x),\mathbf{m}(t)\varphi(a))|^{\alpha+2}}{\eta_1(\varphi(b),\mathbf{m}(t)\varphi(a))}\sqrt[p]{D(\alpha,\lambda,p)}$$

$$\times\left\{\left(\int_0^1 \mathbf{m}^{\frac{1}{r}}(\xi)h_1^{\frac{p_1}{r}}(\xi)(f''(a))^q d\xi\right)^r\right.$$

$$\left.+\left(\int_0^1 h_2^{\frac{p_2}{r}}(\xi)\eta_2^{\frac{1}{r}}\left((f''(x))^{rq},(f''(a))^{rq}\right)d\xi\right)^r\right\}^{\frac{1}{rq}}$$

$$+\frac{|\eta_1(\varphi(x),\mathbf{m}(t)\varphi(b))|^{\alpha+2}}{\eta_1(\varphi(b),\mathbf{m}(t)\varphi(a))}\sqrt[p]{D(\alpha,\lambda,p)}$$

$$\times\left\{\left(\int_0^1 \mathbf{m}^{\frac{1}{r}}(\xi)h_1^{\frac{p_1}{r}}(\xi)(f''(b))^q d\xi\right)^r\right.$$

$$\left.+\left(\int_0^1 h_2^{\frac{p_2}{r}}(\xi)\eta_2^{\frac{1}{r}}\left((f''(x))^{rq},(f''(b))^{rq}\right)d\xi\right)^r\right\}^{\frac{1}{rq}}$$

$$=\frac{\sqrt[p]{D(\alpha,\lambda,p)}}{\eta_1(\varphi(b),\mathbf{m}(t)\varphi(a))}$$

$$\times\left\{|\eta_1(\varphi(x),\mathbf{m}(t)\varphi(a))|^{\alpha+2}\right.$$

$$\times\sqrt[rq]{(f''(a))^{rq}I^r(h_1(\xi),\mathbf{m}(\xi);p_1,r)+\eta_2\left((f''(x))^{rq},(f''(a))^{rq}\right)I^r(h_2(\xi);p_2,r)}$$

$$+|\eta_1(\varphi(x),\mathbf{m}(t)\varphi(b))|^{\alpha+2}$$

$$\left.\times\sqrt[rq]{(f''(b))^{rq}I^r(h_1(\xi),\mathbf{m}(\xi);p_1,r)+\eta_2\left((f''(x))^{rq},(f''(b))^{rq}\right)I^r(h_2(\xi);p_2,r)}\right\}.$$

So, the proof of this theorem is completed.

Let us discuss some special cases of Theorem 6.

Corollary 13 *In Theorem 6 for $p = q = 2$, we get*

$$\left|\Delta_f^{\alpha}(\eta_1, \varphi, \boldsymbol{m}; \lambda, x, a, b)\right| \le \frac{1}{\eta_1(\varphi(b), \boldsymbol{m}(t)\varphi(a))}\sqrt{\frac{\lambda^2}{3} + \frac{1}{2\alpha+3} - \frac{2\lambda}{\alpha+3}}$$
$$\times \Big\{ |\eta_1(\varphi(x), \boldsymbol{m}(t)\varphi(a))|^{\alpha+2}$$
$$\times \sqrt[2r]{(f''(a))^{2r} I^r(h_1(\xi), \boldsymbol{m}(\xi); p_1, r) + \eta_2\left((f''(x))^{2r}, (f''(a))^{2r}\right) I^r(h_2(\xi); p_2, r)}$$
$$+ |\eta_1(\varphi(x), \boldsymbol{m}(t)\varphi(b))|^{\alpha+2}$$
$$\times \sqrt[2r]{(f''(b))^{2r} I^r(h_1(\xi), \boldsymbol{m}(\xi); p_1, r) + \eta_2\left((f''(x))^{2r}, (f''(b))^{2r}\right) I^r(h_2(\xi); p_2, r)}\Big\}. \quad (33)$$

Corollary 14 *In Theorem 6, if we choose $\eta_1(\varphi(y), \boldsymbol{m}(t)\varphi(x)) = \varphi(y) - \boldsymbol{m}(t)\varphi(x)$ and $\lambda = \boldsymbol{m}(t) \equiv 1, \forall t \in [0, 1]$, we get the following generalized Hermite-Hadamard type inequality for fractional integrals:*

$$\left|\Delta_f^{\alpha}(\varphi; 1, x, a, b)\right| = \left| \frac{\alpha}{(\varphi(b) - \varphi(a))} \left\{ \left((\varphi(x) - \varphi(a))^{\alpha} + (\varphi(b) - \varphi(x))^{\alpha}\right) f(\varphi(x)) \right\}\right.$$
$$+ \frac{1}{(\varphi(b) - \varphi(a))} \left\{ (\varphi(x) - \varphi(a))^{\alpha} f(\varphi(a)) + (\varphi(b) - \varphi(x))^{\alpha} f(\varphi(b)) \right\}$$
$$\left. - \frac{\Gamma(\alpha+2)}{(\varphi(b) - \varphi(a))} \times \left[J^{\alpha}_{(\varphi(x))^-} f(\varphi(a)) + J^{\alpha}_{(\varphi(x))^-} f(\varphi(b)) \right] \right|$$

$$\le \frac{1}{(\varphi(b) - \varphi(a))} \sqrt[p]{\frac{\alpha}{2(\alpha+2)}}$$
$$\times \Big\{ (\varphi(x) - \varphi(a))^{\alpha+2}$$
$$\times \sqrt[rq]{(f''(a))^{rq} I^r(h_1(\xi); p_1, r) + \eta_2\left((f''(x))^{rq}, (f''(a))^{rq}\right) I^r(h_2(\xi); p_2, r)}$$
$$+ (\varphi(b) - \varphi(x))^{\alpha+2}$$
$$\times \sqrt[rq]{(f''(b))^{rq} I^r(h_1(\xi); p_1, r) + \eta_2\left((f''(x))^{rq}, (f''(b))^{rq}\right) I^r(h_2(\xi); p_2, r)}\Big\}. \quad (34)$$

Corollary 15 *In Theorem 6, if we choose $\eta_1(\varphi(y), \boldsymbol{m}(t)\varphi(x)) = \varphi(y) - \boldsymbol{m}(t)\varphi(x)$, $\lambda = 0$ and $\boldsymbol{m}(t) \equiv 1, \forall t \in [0, 1]$, we get the following generalized Ostrowski type inequality for fractional integrals:*

$$\left|\Delta_f^{\alpha}(\varphi; 0, x, a, b)\right|$$

$$= \left| \frac{1}{(\varphi(a)-\varphi(b))} \left\{ \left((\varphi(x)-\varphi(a))^{\alpha+1} + (\varphi(b)-\varphi(x))^{\alpha+1} \right) f'(\varphi(x)) \right\} \right.$$

$$+ \frac{1+\alpha}{(\varphi(b)-\varphi(a))} \left\{ \left((\varphi(x)-\varphi(a))^{\alpha} + (\varphi(b)-\varphi(x))^{\alpha} \right) f(\varphi(x)) \right\}$$

$$\left. - \frac{\Gamma(\alpha+2)}{(\varphi(b)-\varphi(a))} \times \left[J^{\alpha}_{(\varphi(x))^-} f(\varphi(a)) + J^{\alpha}_{(\varphi(x))^-} f(\varphi(b)) \right] \right|$$

$$\leq \frac{1}{(\varphi(b)-\varphi(a))} \sqrt[p]{\frac{1}{p(\alpha+1)+1}}$$

$$\times \left\{ (\varphi(x)-\varphi(a))^{\alpha+2} \right.$$

$$\times \sqrt[rq]{(f''(a))^{rq} I^r(h_1(\xi); p_1, r) + \eta_2\left((f''(x))^{rq}, (f''(a))^{rq}\right) I^r(h_2(\xi); p_2, r)}$$

$$+ (\varphi(b)-\varphi(x))^{\alpha+2}$$

$$\left. \times \sqrt[rq]{(f''(b))^{rq} I^r(h_1(\xi); p_1, r) + \eta_2\left((f''(x))^{rq}, (f''(b))^{rq}\right) I^r(h_2(\xi); p_2, r)} \right\}. \tag{35}$$

Corollary 16 *In Theorem 6, if we choose* $\eta_1(\varphi(y), \boldsymbol{m}(t)\varphi(x)) = \varphi(y) - \boldsymbol{m}(t)\varphi(x)$, $x = \frac{a+b}{2}$ *and* $\boldsymbol{m}(t) \equiv 1, \forall t \in [0,1]$, *we get the following generalized Simpson type inequality for fractional integrals:*

$$\left| \Delta^{\alpha}_{f}\left(\varphi; \lambda, \frac{a+b}{2}, a, b\right) \right| = \left| \frac{\lambda - 1}{(\varphi(b)-\varphi(a))} \right.$$

$$\times \left\{ \left(\left(\varphi\left(\frac{a+b}{2}\right) - \varphi(a) \right)^{\alpha+1} + \left(\varphi(b) - \varphi\left(\frac{a+b}{2}\right) \right)^{\alpha+1} \right) f'\left(\varphi\left(\frac{a+b}{2}\right)\right) \right\}$$

$$+ \frac{1+\alpha-\lambda}{(\varphi(b)-\varphi(a))}$$

$$\times \left\{ \left(\left(\varphi\left(\frac{a+b}{2}\right) - \varphi(a) \right)^{\alpha} + \left(\varphi(b) - \varphi\left(\frac{a+b}{2}\right) \right)^{\alpha} \right) f\left(\varphi\left(\frac{a+b}{2}\right)\right) \right\}$$

$$+ \frac{\lambda}{(\varphi(b)-\varphi(a))}$$

$$\times\left\{\left(\varphi\left(\frac{a+b}{2}\right)-\varphi(a)\right)^{\alpha} f(\varphi(a))+\left(\varphi(b)-\varphi\left(\frac{a+b}{2}\right)\right)^{\alpha} f(\varphi(b))\right\}$$

$$-\frac{\Gamma(\alpha+2)}{(\varphi(b)-\varphi(a))}\times\left[J^{\alpha}_{\left(\varphi\left(\frac{a+b}{2}\right)\right)^-} f(\varphi(a))+J^{\alpha}_{\left(\varphi\left(\frac{a+b}{2}\right)\right)^-} f(\varphi(b))\right]\Bigg|$$

$$\le \frac{\sqrt[p]{D(\alpha,\lambda,p)}}{(\varphi(b)-\varphi(a))}$$

$$\times\left\{\left(\varphi\left(\frac{a+b}{2}\right)-\varphi(a)\right)^{\alpha+2}\Bigg[(f''(a))^{rq} I^r(h_1(\xi);p_1,r)\right.$$

$$+\eta_2\left(\left(f''\left(\frac{a+b}{2}\right)\right)^{rq},(f''(a))^{rq}\right) I^r(h_2(\xi);p_2,r)\Bigg]^{\frac{1}{rq}} \quad (36)$$

$$+\left(\varphi(b)-\varphi\left(\frac{a+b}{2}\right)\right)^{\alpha+2}\Bigg[(f''(b))^{rq} I^r(h_1(\xi);p_1,r)$$

$$\left.+\eta_2\left(\left(f''\left(\frac{a+b}{2}\right)\right)^{rq},(f''(b))^{rq}\right) I^r(h_2(\xi);p_2,r)\Bigg]^{\frac{1}{rq}}\right\}.$$

Corollary 17 *In Theorem 6 for $h_1(t)=h(1-t)$, $h_2(t)=h(t)$ and $\boldsymbol{m}(t)=m\in(0,1]$ for all $t\in[0,1]$, we get the following inequality for generalized-m-$((h^{p_1}(1-t),h^{p_2}(t));(\eta_1,\eta_2))$-convex mappings:*

$$\left|\Delta^{\alpha}_f(\eta_1,\varphi,m;\lambda,x,a,b)\right|\le\frac{\sqrt[p]{D(\alpha,\lambda,p)}}{\eta_1(\varphi(b),m\varphi(a))}$$

$$\times\Bigg\{|\eta_1(\varphi(x),m\varphi(a))|^{\alpha+2}$$

$$\times\sqrt[rq]{m(f''(a))^{rq}I^r(h(1-\xi);p_1,r)+\eta_2\left((f''(x))^{rq},(f''(a))^{rq}\right)I^r(h(\xi);p_2,r)}$$

$$+|\eta_1(\varphi(x),m\varphi(b))|^{\alpha+2}$$

$$\times\sqrt[rq]{m(f''(b))^{rq}I^r(h(1-\xi);p_1,r)+\eta_2\left((f''(x))^{rq},(f''(b))^{rq}\right)I^r(h(\xi);p_2,r)}\Bigg\}. \quad (37)$$

Corollary 18 *In Corollary 17 for $h_1(t)=(1-t)^s$ and $h_2(t)=t^s$, we get the following inequality for generalized-m-$(((1-t)^{sp_1},t^{sp_2});(\eta_1,\eta_2))$-Breckner-convex mappings:*

$$\left|\Delta_f^{\alpha}(\eta_1, \varphi, m; \lambda, x, a, b)\right| \leq \frac{\sqrt[p]{D(\alpha, \lambda, p)}}{\eta_1(\varphi(b), m\varphi(a))}$$
$$\times \Bigg\{ |\eta_1(\varphi(x), m\varphi(a))|^{\alpha+2}$$
$$\times \sqrt[rq]{m(f''(a))^{rq} \left(\frac{r}{r+sp_1}\right)^r + \eta_2\left((f''(x))^{rq}, (f''(a))^{rq}\right) \left(\frac{r}{r+sp_2}\right)^r} \tag{38}$$
$$+ |\eta_1(\varphi(x), m\varphi(b))|^{\alpha+2}$$
$$\times \sqrt[rq]{m(f''(b))^{rq} \left(\frac{r}{r+sp_1}\right)^r + \eta_2\left((f''(x))^{rq}, (f''(b))^{rq}\right) \left(\frac{r}{r+sp_2}\right)^r} \Bigg\}.$$

Corollary 19 *In Corollary 17 for $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ and $r > s \cdot \max\{p_1, p_2\}$, we get the following inequality for generalized-m-$(((1-t)^{-sp_1}, t^{-sp_2}); (\eta_1, \eta_2))$-Godunova-Levin-Dragomir-convex mappings:*

$$\left|\Delta_f^{\alpha}(\eta_1, \varphi, m; \lambda, x, a, b)\right| \leq \frac{\sqrt[p]{D(\alpha, \lambda, p)}}{\eta_1(\varphi(b), m\varphi(a))}$$
$$\times \Bigg\{ |\eta_1(\varphi(x), m\varphi(a))|^{\alpha+2}$$
$$\times \sqrt[rq]{m(f''(a))^{rq} \left(\frac{r}{r-sp_1}\right)^r + \eta_2\left((f''(x))^{rq}, (f''(a))^{rq}\right) \left(\frac{r}{r-sp_2}\right)^r} \tag{39}$$
$$+ |\eta_1(\varphi(x), m\varphi(b))|^{\alpha+2}$$
$$\times \sqrt[rq]{m(f''(b))^{rq} \left(\frac{r}{r-sp_1}\right)^r + \eta_2\left((f''(x))^{rq}, (f''(b))^{rq}\right) \left(\frac{r}{r-sp_2}\right)^r} \Bigg\}.$$

Corollary 20 *In Theorem 6 for $h_1(t) = h_2(t) = t(1-t)$ and $\boldsymbol{m}(t) = m \in (0, 1]$ for all $t \in [0, 1]$, we get the following inequality for generalized-m-$((t(1-t))^{sp_1}, (t(1-t))^{sp_2}); (\eta_1, \eta_2))$-convex mappings:*

$$\left|\Delta_f^{\alpha}(\eta_1, \varphi, m; \lambda, x, a, b)\right| \leq \frac{\sqrt[p]{D(\alpha, \lambda, p)}}{\eta_1(\varphi(b), m\varphi(a))}$$
$$\times \Bigg\{ |\eta_1(\varphi(x), m\varphi(a))|^{\alpha+2} \Big[m(f''(a))^{rq} \beta^r \left(1 + \frac{p_1}{r}, 1 + \frac{p_1}{r}\right) \tag{40}$$
$$+ \eta_2\left((f''(x))^{rq}, (f''(a))^{rq}\right) \beta^r \left(1 + \frac{p_2}{r}, 1 + \frac{p_2}{r}\right) \Big]^{\frac{1}{rq}}$$

$$+ |\eta_1(\varphi(x), m\varphi(b))|^{\alpha+2}\Big[m(f''(b))^{rq}\beta^r\left(1+\frac{p_1}{r}, 1+\frac{p_1}{r}\right)$$
$$+ \eta_2\left((f''(x))^{rq}, (f''(b))^{rq}\right)\beta^r\left(1+\frac{p_2}{r}, 1+\frac{p_2}{r}\right)\Big]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 21 *In Corollary 17 for* $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$ *and* $r > \frac{1}{2} \cdot \max\{p_1, p_2\}$, *we get the following inequality for generalized-m* -$\left(\left(\left(\frac{\sqrt{1-t}}{2\sqrt{t}}\right)^p, \left(\frac{\sqrt{t}}{2\sqrt{1-t}}\right)^q\right); (\eta_1, \eta_2)\right)$*-convex mappings:*

$$\left|\Delta_f^{\alpha}(\eta_1, \varphi, m; \lambda, x, a, b)\right| \leq \frac{\sqrt[p]{D(\alpha, \lambda, p)}}{\eta_1(\varphi(b), m\varphi(a))}$$
$$\times\Bigg\{|\eta_1(\varphi(x), m\varphi(a))|^{\alpha+2}\Big[m(f''(a))^{rq}\beta^r\left(1-\frac{p_1}{2r}, 1+\frac{p_1}{2r}\right)$$
$$+ \eta_2\left((f''(x))^{rq}, (f''(a))^{rq}\right)\beta^r\left(1-\frac{p_2}{2r}, 1+\frac{p_2}{2r}\right)\Big]^{\frac{1}{rq}} \tag{41}$$
$$+ |\eta_1(\varphi(x), m\varphi(b))|^{\alpha+2}\Big[m(f''(b))^{rq}\beta^r\left(1-\frac{p_1}{2r}, 1+\frac{p_1}{2r}\right)$$
$$+ +\eta_2\left((f''(x))^{rq}, (f''(b))^{rq}\right)\beta^r\left(1-\frac{p_2}{2r}, 1+\frac{p_2}{2r}\right)\Big]^{\frac{1}{rq}}\Bigg\}.$$

Theorem 7 *Let* $0 < r \leq 1$ *and* $p_1, p_2 > -1$. *Suppose* $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$, $\varphi : I \longrightarrow K$ *are continuous functions and* $\boldsymbol{m} : [0, 1] \longrightarrow (0, 1]$. *Let* $K = [\boldsymbol{m}(t)\varphi(a), \boldsymbol{m}(t)\varphi(a) + \eta_1(\varphi(b), \boldsymbol{m}(t)\varphi(a))] \subseteq \mathbb{R}$ *be an open* $\boldsymbol{m}$*-invex subset with respect to* $\eta_1 : K \times K \longrightarrow \mathbb{R}$ *and let* $\eta_1(\varphi(b), \boldsymbol{m}(t)\varphi(a)) > 0$ *for all* $t \in [0, 1]$ *and* $\eta_2 : f(K) \times f(K) \longrightarrow \mathbb{R}$. *Assume that* $f : K \longrightarrow (0, +\infty)$ *be a twice differentiable mapping on* K°. *If* f''^q *is generalized-*$\boldsymbol{m}$-$((h_1^{p_1}, h_2^{p_2}); (\eta_1, \eta_2))$*-convex mapping on* K *and* $q \geq 1$, *then for any* $\lambda \in [0, 1]$ *and* $\alpha > 0$, *the following inequality for fractional integrals holds:*

$$\left|\Delta_f^{\alpha}(\eta_1, \varphi, \boldsymbol{m}; \lambda, x, a, b)\right| \leq \frac{C^{1-\frac{1}{q}}(\alpha, \lambda)}{\eta_1(\varphi(b), \boldsymbol{m}(t)\varphi(a))}$$
$$\times\Bigg\{|\eta_1(\varphi(x), \boldsymbol{m}(t)\varphi(a))|^{\alpha+2}\Big[(f''(a))^{rq}F^r(h_1(\xi), \boldsymbol{m}(\xi); \lambda, \alpha, p_1, r)$$
$$+ \eta_2\left((f''(x))^{rq}, (f''(a))^{rq}\right)F^r(h_2(\xi); \lambda, \alpha, p_2, r)\Big]^{\frac{1}{rq}} \tag{42}$$
$$+ |\eta_1(\varphi(x), \boldsymbol{m}(t)\varphi(b))|^{\alpha+2}\Big[(f''(b))^{rq}F^r(h_1(\xi), \boldsymbol{m}(\xi); \lambda, \alpha, p_1, r)$$
$$+ \eta_2\left((f''(x))^{rq}, (f''(b))^{rq}\right)F^r(h_2(\xi); \lambda, \alpha, p_2, r)\Big]^{\frac{1}{rq}}\Bigg\},$$

where

$$C(\alpha,\lambda):=\frac{\alpha\lambda^{1+\frac{2}{\alpha}}+1}{\alpha+2}-\frac{\lambda}{2};$$

$$F(h_1(\xi),\boldsymbol{m}(\xi);\lambda,\alpha,p_1,r):=\int_0^1 \boldsymbol{m}^{\frac{1}{r}}(\xi)|\xi(\lambda-\xi^{\alpha})|h_1^{\frac{p_1}{r}}(\xi)d\xi;$$

$$F(h_2(\xi);\lambda,\alpha,p_2,r):=\int_0^1 |\xi(\lambda-\xi^{\alpha})|h_2^{\frac{p_2}{r}}(\xi)d\xi.$$

Proof Using relation (31), generalized-**m**-$((h_1^{p_1},h_2^{p_2});(\eta_1,\eta_2))$-convexity of f''^q, the well-known power mean inequality, Minkowski inequality, and properties of the modulus, we have

$$\left|\Delta_f^{\alpha}(\eta_1,\varphi,\mathbf{m};\lambda,x,a,b)\right|\le\frac{|\eta_1(\varphi(x),\mathbf{m}(t)\varphi(a))|^{\alpha+2}}{|\eta_1(\varphi(b),\mathbf{m}(t)\varphi(a))|}$$

$$\times\int_0^1|\xi(\lambda-\xi^{\alpha})||f''(\mathbf{m}(t)\varphi(a)+\xi\eta_1(\varphi(x),\mathbf{m}(t)\varphi(a)))|d\xi$$

$$+\frac{|\eta_1(\varphi(x),\mathbf{m}(t)\varphi(b))|^{\alpha+2}}{|\eta_1(\varphi(b),\mathbf{m}(t)\varphi(a))|}$$

$$\times\int_0^1|\xi(\lambda-\xi^{\alpha})||f''(\mathbf{m}(t)\varphi(b)+\xi\eta_1(\varphi(x),\mathbf{m}(t)\varphi(b)))|d\xi$$

$$\le\frac{|\eta_1(\varphi(x),\mathbf{m}(t)\varphi(a))|^{\alpha+2}}{\eta_1(\varphi(b),\mathbf{m}(t)\varphi(a))}\left(\int_0^1|\xi(\lambda-\xi^{\alpha})|d\xi\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1|\xi(\lambda-\xi^{\alpha})|\left(f''(\mathbf{m}(t)\varphi(a)+\xi\eta_1(\varphi(x),\mathbf{m}(t)\varphi(a)))\right)^q d\xi\right)^{\frac{1}{q}}$$

$$+\frac{|\eta_1(\varphi(x),\mathbf{m}(t)\varphi(b))|^{\alpha+2}}{\eta_1(\varphi(b),\mathbf{m}(t)\varphi(a))}\left(\int_0^1|\xi(\lambda-\xi^{\alpha})|d\xi\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1|\xi(\lambda-\xi^{\alpha})|\left(f''(\mathbf{m}(t)\varphi(b)+\xi\eta_1(\varphi(x),\mathbf{m}(t)\varphi(b)))\right)^q d\xi\right)^{\frac{1}{q}}$$

$$\le\frac{|\eta_1(\varphi(x),\mathbf{m}(t)\varphi(a))|^{\alpha+2}}{\eta_1(\varphi(b),\mathbf{m}(t)\varphi(a))}C^{1-\frac{1}{q}}(\alpha,\lambda)$$

$$\times\left(\int_0^1|\xi(\lambda-\xi^{\alpha})|\Big[\mathbf{m}(\xi)h_1^{p_1}(\xi)(f''(a))^{rq}+h_2^{p_2}(\xi)\eta_2\left((f''(x))^{rq},(f''(a))^{rq}\right)\Big]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}$$

$$+\frac{|\eta_1(\varphi(x),\mathbf{m}(t)\varphi(b))|^{\alpha+2}}{\eta_1(\varphi(b),\mathbf{m}(t)\varphi(a))}C^{1-\frac{1}{q}}(\alpha,\lambda)$$

$$\times\left(\int_0^1|\xi(\lambda-\xi^{\alpha})|\Big[\mathbf{m}(\xi)h_1^{p_1}(\xi)(f''(b))^{rq}+h_2^{p_2}(\xi)\eta_2\left((f''(x))^{rq},(f''(b))^{rq}\right)\Big]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}$$

$$\leq\frac{|\eta_1(\varphi(x),\mathbf{m}(t)\varphi(a))|^{\alpha+2}}{\eta_1(\varphi(b),\mathbf{m}(t)\varphi(a))}C^{1-\frac{1}{q}}(\alpha,\lambda)$$

$$\times\left\{\left(\int_0^1\mathbf{m}^{\frac{1}{r}}(\xi)|\xi(\lambda-\xi^{\alpha})|h_1^{\frac{p_1}{r}}(\xi)(f''(a))^q d\xi\right)^r\right.$$

$$\left.+\left(\int_0^1|\xi(\lambda-\xi^{\alpha})|h_2^{\frac{p_2}{r}}(\xi)\eta_2^{\frac{1}{r}}\left((f''(x))^{rq},(f''(a))^{rq}\right)d\xi\right)^r\right\}^{\frac{1}{rq}}$$

$$+\frac{|\eta_1(\varphi(x),\mathbf{m}(t)\varphi(b))|^{\alpha+2}}{\eta_1(\varphi(b),\mathbf{m}(t)\varphi(a))}C^{1-\frac{1}{q}}(\alpha,\lambda)$$

$$\times\left\{\left(\int_0^1\mathbf{m}^{\frac{1}{r}}(\xi)|\xi(\lambda-\xi^{\alpha})|h_1^{\frac{p_1}{r}}(\xi)(f''(b))^q d\xi\right)^r\right.$$

$$\left.+\left(\int_0^1|\xi(\lambda-\xi^{\alpha})|h_2^{\frac{p_2}{r}}(\xi)\eta_2^{\frac{1}{r}}\left((f''(x))^{rq},(f''(b))^{rq}\right)d\xi\right)^r\right\}^{\frac{1}{rq}}$$

$$=\frac{C^{1-\frac{1}{q}}(\alpha,\lambda)}{\eta_1(\varphi(b),\mathbf{m}(t)\varphi(a))}$$

$$\times\Big\{|\eta_1(\varphi(x),\mathbf{m}(t)\varphi(a))|^{\alpha+2}\Big[(f''(a))^{rq}F^r(h_1(\xi),\mathbf{m}(\xi);\lambda,\alpha,p_1,r)$$

$$+\eta_2\left((f''(x))^{rq},(f''(a))^{rq}\right)F^r(h_2(\xi);\lambda,\alpha,p_2,r)\Big]^{\frac{1}{rq}}$$

$$+|\eta_1(\varphi(x),\mathbf{m}(t)\varphi(b))|^{\alpha+2}\Big[(f''(b))^{rq}F^r(h_1(\xi),\mathbf{m}(\xi);\lambda,\alpha,p_1,r)$$

$$+\eta_2\left((f''(x))^{rq},(f''(b))^{rq}\right)F^r(h_2(\xi);\lambda,\alpha,p_2,r)\Big]^{\frac{1}{rq}}\Big\}.$$

So, the proof of this theorem is completed.

Let us discuss some special cases of Theorem 7.

Corollary 22 *In Theorem 7 for $q = 1$, we get*

$$\left|\Delta_f^{\alpha}(\eta_1, \varphi, \boldsymbol{m}; \lambda, x, a, b)\right| \leq \frac{1}{\eta_1(\varphi(b), \boldsymbol{m}(t)\varphi(a))}$$
$$\times \Bigg\{ |\eta_1(\varphi(x), \boldsymbol{m}(t)\varphi(a))|^{\alpha+2} \Big[(f''(a))^r F^r(h_1(\xi), \boldsymbol{m}(\xi); \lambda, \alpha, p_1, r)$$
$$+ \eta_2\left((f''(x))^r, (f''(a))^r\right) F^r(h_2(\xi); \lambda, \alpha, p_2, r)\Big]^{\frac{1}{r}} \tag{43}$$
$$+ |\eta_1(\varphi(x), \boldsymbol{m}(t)\varphi(b))|^{\alpha+2} \Big[(f''(b))^r F^r(h_1(\xi), \boldsymbol{m}(\xi); \lambda, \alpha, p_1, r)$$
$$+ \eta_2\left((f''(x))^r, (f''(b))^r\right) F^r(h_2(\xi); \lambda, \alpha, p_2, r)\Big]^{\frac{1}{r}} \Bigg\}.$$

Corollary 23 *In Theorem 7, if we choose $\eta_1(\varphi(y), \boldsymbol{m}(t)\varphi(x)) = \varphi(y) - \boldsymbol{m}(t)\varphi(x)$ and $\lambda = \boldsymbol{m}(t) \equiv 1, \forall t \in [0, 1]$, we get the following generalized Hermite-Hadamard type inequality for fractional integrals:*

$$\left|\Delta_f^{\alpha}(\varphi; 1, x, a, b)\right| \leq \left(\frac{\alpha}{2(\alpha+2)}\right)^{1-\frac{1}{q}} \frac{1}{(\varphi(b) - \varphi(a))}$$
$$\times \Bigg\{ (\varphi(x) - \varphi(a))^{\alpha+2} \Big[(f''(a))^{rq} F^r(h_1(\xi); 1, \alpha, p_1, r)$$
$$+ \eta_2\left((f''(x))^{rq}, (f''(a))^{rq}\right) F^r(h_2(\xi); 1, \alpha, p_2, r)\Big]^{\frac{1}{rq}} \tag{44}$$
$$+ (\varphi(b) - \varphi(x))^{\alpha+2} \Big[(f''(b))^{rq} F^r(h_1(\xi); 1, \alpha, p_1, r)$$
$$+ \eta_2\left((f''(x))^{rq}, (f''(b))^{rq}\right) F^r(h_2(\xi); 1, \alpha, p_2, r)\Big]^{\frac{1}{rq}} \Bigg\}.$$

Corollary 24 *In Theorem 7, if we choose $\eta_1(\varphi(y), \boldsymbol{m}(t)\varphi(x)) = \varphi(y) - \boldsymbol{m}(t)\varphi(x)$, $\lambda = 0$ and $\boldsymbol{m}(t) \equiv 1, \forall t \in [0, 1]$, we get the following generalized Ostrowski type inequality for fractional integrals:*

$$\left|\Delta_f^{\alpha}(\varphi; 0, x, a, b)\right| \leq \left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}} \frac{1}{(\varphi(b) - \varphi(a))}$$
$$\times \Bigg\{ (\varphi(x) - \varphi(a))^{\alpha+2} \Big[(f''(a))^{rq} F^r(h_1(\xi); 0, \alpha, p_1, r) \tag{45}$$
$$+ \eta_2\left((f''(x))^{rq}, (f''(a))^{rq}\right) F^r(h_2(\xi); 0, \alpha, p_2, r)\Big]^{\frac{1}{rq}}$$

$$+ (\varphi(b) - \varphi(x))^{\alpha+2}\Big[(f''(b))^{rq} F^r(h_1(\xi); 0, \alpha, p_1, r)$$

$$+ \eta_2\left((f''(x))^{rq}, (f''(b))^{rq}\right) F^r(h_2(\xi); 0, \alpha, p_2, r)\Big]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 25 *In Theorem 7, if we choose* $\eta_1(\varphi(y), \boldsymbol{m}(t)\varphi(x)) = \varphi(y) - \boldsymbol{m}(t)\varphi(x)$, $x = \dfrac{a+b}{2}$ *and* $\boldsymbol{m}(t) \equiv 1, \forall\, t \in [0, 1]$, *we get the following generalized Simpson type inequality for fractional integrals:*

$$\left|\Delta_f^{\alpha}\left(\varphi; \lambda, \frac{a+b}{2}, a, b\right)\right| \leq \frac{C^{1-\frac{1}{q}}(\alpha, \lambda)}{(\varphi(b) - \varphi(a))}$$

$$\times \left\{ \left(\varphi\left(\frac{a+b}{2}\right) - \varphi(a)\right)^{\alpha+2} \left[(f''(a))^{rq} F^r(h_1(\xi); \lambda, \alpha, p_1, r)\right.\right.$$

$$+ \eta_2\left(\left(f''\left(\frac{a+b}{2}\right)\right)^{rq}, (f''(a))^{rq}\right) F^r(h_1(\xi); \lambda, \alpha, p_2, r)\Big]^{\frac{1}{rq}} \tag{46}$$

$$+ \left(\varphi(b) - \varphi\left(\frac{a+b}{2}\right)\right)^{\alpha+2} \Big[(f''(b))^{rq} F^r(h_1(\xi); \lambda, \alpha, p_1, r)$$

$$+ \eta_2\left(\left(f''\left(\frac{a+b}{2}\right)\right)^{rq}, (f''(b))^{rq}\right) F^r(h_1(\xi); \lambda, \alpha, p_2, r)\Big]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 26 *In Theorem 7 for* $h_1(t) = h(1-t)$, $h_2(t) = h(t)$ *and* $\boldsymbol{m}(t) = m \in (0, 1]$ *for all* $t \in [0, 1]$, *we get the following inequality for generalized-m-*$((h^{p_1}(1-t), h^{p_2}(t)); (\eta_1, \eta_2))$*-convex mappings:*

$$\left|\Delta_f^{\alpha}(\eta_1, \varphi, m; \lambda, x, a, b)\right| \leq \frac{C^{1-\frac{1}{q}}(\alpha, \lambda)}{\eta_1(\varphi(b), m\varphi(a))}$$

$$\times \Big\{|\eta_1(\varphi(x), m\varphi(a))|^{\alpha+2}\Big[m(f''(a))^{rq} F^r(h(1-\xi); \lambda, \alpha, p_1, r)$$

$$+ \eta_2\left((f''(x))^{rq}, (f''(a))^{rq}\right) F^r(h(\xi); \lambda, \alpha, p_2, r)\Big]^{\frac{1}{rq}} \tag{47}$$

$$+ |\eta_1(\varphi(x), m\varphi(b))|^{\alpha+2}\Big[m(f''(b))^{rq} F^r(h(1-\xi); \lambda, \alpha, p_1, r)$$

$$+ \eta_2\left((f''(x))^{rq}, (f''(b))^{rq}\right) F^r(h(\xi); \lambda, \alpha, p_2, r)\Big]^{\frac{1}{rq}}\Big\}.$$

Corollary 27 *In Corollary 26 for $h_1(t) = (1-t)^s$ and $h_2(t) = t^s$, we get the following inequality for generalized-m-$(((1-t)^{sp_1}, t^{sp_2}); (\eta_1, \eta_2))$-Breckner-convex mappings:*

$$\begin{aligned}
\left|\Delta_f^{\alpha}(\eta_1, \varphi, m; \lambda, x, a, b)\right| &\leq \frac{C^{1-\frac{1}{q}}(\alpha, \lambda)}{\eta_1(\varphi(b), m\varphi(a))} \\
&\times \Big\{|\eta_1(\varphi(x), m\varphi(a))|^{\alpha+2}\Big[m(f''(a))^{rq} F^r((1-\xi)^s; \lambda, \alpha, p_1, r) \\
&+ \eta_2\left((f''(x))^{rq}, (f''(a))^{rq}\right) F^r(\xi^s; \lambda, \alpha, p_2, r)\Big]^{\frac{1}{rq}} \\
&+ |\eta_1(\varphi(x), m\varphi(b))|^{\alpha+2}\Big[m(f''(b))^{rq} F^r((1-\xi)^s; \lambda, \alpha, p_1, r) \\
&+ \eta_2\left((f''(x))^{rq}, (f''(b))^{rq}\right) F^r(\xi^s; \lambda, \alpha, p_2, r)\Big]^{\frac{1}{rq}}\Big\}.
\end{aligned} \tag{48}$$

Corollary 28 *In Corollary 26 for $h_1(t) = (1-t)^{-s}$ and $h_2(t) = t^{-s}$, we get the following inequality for generalized-m-$(((1-t)^{-sp_1}, t^{-sp_2}); (\eta_1, \eta_2))$-Godunova-Levin-Dragomir-convex mappings:*

$$\begin{aligned}
\left|\Delta_f^{\alpha}(\eta_1, \varphi, m; \lambda, x, a, b)\right| &\leq \frac{C^{1-\frac{1}{q}}(\alpha, \lambda)}{\eta_1(\varphi(b), m\varphi(a))} \\
&\times \Big\{|\eta_1(\varphi(x), m\varphi(a))|^{\alpha+2}\Big[m(f''(a))^{rq} F^r((1-\xi)^{-s}; \lambda, \alpha, p_1, r) \\
&+ \eta_2\left((f''(x))^{rq}, (f''(a))^{rq}\right) F^r(\xi^{-s}; \lambda, \alpha, p_2, r)\Big]^{\frac{1}{rq}} \\
&+ |\eta_1(\varphi(x), m\varphi(b))|^{\alpha+2}\Big[m(f''(b))^{rq} F^r((1-\xi)^{-s}; \lambda, \alpha, p_1, r) \\
&+ \eta_2\left((f''(x))^{rq}, (f''(b))^{rq}\right) F^r(\xi^{-s}; \lambda, \alpha, p_2, r)\Big]^{\frac{1}{rq}}\Big\}.
\end{aligned} \tag{49}$$

Corollary 29 *In Theorem 7 for $h_1(t) = h_2(t) = t(1-t)$ and $\mathbf{m}(t) = m \in (0, 1]$ for all $t \in [0, 1]$, we get the following inequality for generalized-m-$((t(1-t))^{sp_1}, (t(1-t))^{sp_2}); (\eta_1, \eta_2))$-convex mappings:*

$$\begin{aligned}
\left|\Delta_f^{\alpha}(\eta_1, \varphi, m; \lambda, x, a, b)\right| &\leq \frac{C^{1-\frac{1}{q}}(\alpha, \lambda)}{\eta_1(\varphi(b), m\varphi(a))} \\
&\times \Big\{|\eta_1(\varphi(x), m\varphi(a))|^{\alpha+2}\Big[m(f''(a))^{rq} F^r(\xi(1-\xi); \lambda, \alpha, p_1, r)
\end{aligned}$$

$$+\eta_2\left((f''(x))^{rq},(f''(a))^{rq}\right)F^r(\xi(1-\xi);\lambda,\alpha,p_2,r)\Big]^{\frac{1}{rq}}$$
$$+|\eta_1(\varphi(x),m\varphi(b))|^{\alpha+2}\Big[m(f''(b))^{rq}F^r(\xi(1-\xi);\lambda,\alpha,p_1,r) \quad (50)$$
$$+\eta_2\left((f''(x))^{rq},(f''(b))^{rq}\right)F^r(\xi(1-\xi);\lambda,\alpha,p_2,r)\Big]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 30 *In Corollary 26 for* $h_1(t)=\dfrac{\sqrt{1-t}}{2\sqrt{t}}$ *and* $h_2(t)=\dfrac{\sqrt{t}}{2\sqrt{1-t}}$, *we get the following inequality for generalized-m-*$\left(\left(\left(\frac{\sqrt{1-t}}{2\sqrt{t}}\right)^p,\left(\frac{\sqrt{t}}{2\sqrt{1-t}}\right)^q\right);(\eta_1,\eta_2)\right)$*-convex mappings:*

$$\left|\Delta_f^{\alpha}(\eta_1,\varphi,m;\lambda,x,a,b)\right|\le\frac{C^{1-\frac{1}{q}}(\alpha,\lambda)}{\eta_1(\varphi(b),m\varphi(a))}$$
$$\times\Bigg\{|\eta_1(\varphi(x),m\varphi(a))|^{\alpha+2}\Bigg[m(f''(a))^{rq}F^r\left(\left(\frac{\sqrt{1-\xi}}{2\sqrt{\xi}}\right);\lambda,\alpha,p_1,r\right)$$
$$+\eta_2\left((f''(x))^{rq},(f''(a))^{rq}\right)F^r\left(\left(\frac{\sqrt{\xi}}{2\sqrt{1-\xi}}\right);\lambda,\alpha,p_2,r\right)\Bigg]^{\frac{1}{rq}} \quad (51)$$
$$+|\eta_1(\varphi(x),m\varphi(b))|^{\alpha+2}\Bigg[m(f''(b))^{rq}F^r\left(\left(\frac{\sqrt{1-\xi}}{2\sqrt{\xi}}\right);\lambda,\alpha,p_1,r\right)$$
$$+\eta_2\left((f''(x))^{rq},(f''(b))^{rq}\right)F^r\left(\left(\frac{\sqrt{\xi}}{2\sqrt{1-\xi}}\right);\lambda,\alpha,p_2,r\right)\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

Remark 6 For $\alpha=1$, by our Theorems 6 and 7, we can get some new special Hermite-Hadamard, Ostrowski, and Simpson type inequalities for classical integrals associated with generalized-**m**-$((h_1^{p_1},h_2^{p_2});(\eta_1,\eta_2))$-convex mappings.

Remark 7 Also, applying our Theorems 6 and 7, for different values of $\lambda\in(0,1)$, for different values of $p_1,p_2>-1$, for different choices of function $\mathbf{m}(t)$ and if $0<f''(x)\le L$ for all $x\in I$, we can get some new special Hermite-Hadamard, Ostrowski, and Simpson type inequalities for fractional integrals associated with generalized-**m**-$((h_1^{p},h_2^{p_2});(\eta_1,\eta_2))$-convex mappings.

3 Applications to Special Means

Definition 14 A function $M:\mathbb{R}_+^2\longrightarrow\mathbb{R}_+$ is called a Mean function if it has the following properties:

1. Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,
2. Symmetry: $M(x, y) = M(y, x)$,
3. Reflexivity: $M(x, x) = x$,
4. Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x, y) \leq M(x', y')$,
5. Internality: $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

We consider some means for different positive real numbers α, β.

1. The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}.$$

2. The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}.$$

3. The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}.$$

4. The power mean:

$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1.$$

5. The identric mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

6. The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln\beta - \ln\alpha}.$$

7. The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}; \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$. Let us consider continuous functions $\varphi : I \longrightarrow K$, $\eta_1 : K \times K \longrightarrow \mathbb{R}$, $\eta_2 : f(K) \times f(K) \longrightarrow \mathbb{R}$ and $\overline{M} := M(\varphi(a), \varphi(b)) : [\varphi(a), \varphi(a) + \eta_1(\varphi(b), \varphi(a))] \times [\varphi(a), \varphi(a) + \eta_1(\varphi(b), \varphi(a))] \longrightarrow \mathbb{R}_+$, which is one of the above-mentioned means. Therefore one can obtain various inequalities using the results of Section 2 for these means as follows. If we take $\mathbf{m}(t) \equiv 1$, $\forall t \in [0, 1]$ and replace $\eta_1(\varphi(x), \mathbf{m}(t)\varphi(y)) = M(\varphi(x), \varphi(y))$ for all $x, y \in I$ for value $m = 1$, in (32) and (42), one can obtain the following interesting inequalities involving means:

$$\begin{aligned}\left|\Delta_f^{\alpha}(\overline{M}, \varphi; \lambda, x, a, b)\right| &\le \frac{\sqrt[p]{D(\alpha, \lambda, p)}}{\overline{M}}\\ &\times \Big\{ M^{\alpha+2}(\varphi(x), \varphi(a))\\ &\times \sqrt[rq]{(f''(a))^{rq} I^r(h_1(\xi); p_1, r) + \eta_2\left((f''(x))^{rq}, (f''(a))^{rq}\right) I^r(h_2(\xi); p_2, r)}\\ &+ M^{\alpha+2}(\varphi(x), \varphi(b))\\ &\times \sqrt[rq]{(f''(b))^{rq} I^r(h_1(\xi); p_1, r) + \eta_2\left((f''(x))^{rq}, (f''(b))^{rq}\right) I^r(h_2(\xi); p_2, r)} \Big\},\end{aligned} \tag{52}$$

$$\begin{aligned}\left|\Delta_f^{\alpha}(\overline{M}, \varphi; \lambda, x, a, b)\right| &\le \frac{C^{1-\frac{1}{q}}(\alpha, \lambda)}{\overline{M}}\\ &\times \Big\{ M^{\alpha+2}(\varphi(x), \varphi(a)) \Big[(f''(a))^{rq} F^r(h_1(\xi); \lambda, \alpha, p_1, r)\\ &+ \eta_2\left((f''(x))^{rq}, (f''(a))^{rq}\right) F^r(h_2(\xi); \lambda, \alpha, p_2, r) \Big]^{\frac{1}{rq}}\\ &+ M^{\alpha+2}(\varphi(x), \varphi(b)) \Big[(f''(b))^{rq} F^r(h_1(\xi); \lambda, \alpha, p_1, r)\\ &+ \eta_2\left((f''(x))^{rq}, (f''(b))^{rq}\right) F^r(h_2(\xi); \lambda, \alpha, p_2, r) \Big]^{\frac{1}{rq}} \Big\}.\end{aligned} \tag{53}$$

Letting $\overline{M} := A, G, H, P_r, I, L, L_p$ in (52) and (53), we get the inequalities involving means for a particular choices of f''^q that are generalized-1-$((h_1^{p_1}, h_2^{p_2}); (\eta_1, \eta_2))$-convex mappings.

Remark 8 Also, applying our Theorems 6 and 7 for appropriate choices of functions h_1 and h_2 (see Remark 4) such that f''^q to be generalized-1-$((h_1^{p_1}, h_2^{p_2}); (\eta_1, \eta_2))$-convex mappings (for example $f(x) = x^{\alpha}$, where $\alpha > 1, \forall x > 0$; $f(x) = \frac{1}{x}$, $\forall x > 0$ etc.), we can deduce some new inequalities using above special means. The details are left to the interested reader.

4 Conclusion

In this article, we first presented some integral inequalities for Gauss-Jacobi type quadrature formula involving generalized-**m**-$((h_1^p, h_2^q); (\eta_1, \eta_2))$-convex mappings. Secondly, an identity pertaining twice differentiable mappings defined on **m**-invex set is used for derived some new estimates with respect to Hermite-Hadamard, Ostrowski, and Simpson type inequalities via fractional integrals associated with generalized-**m**-$((h_1^{p_1}, h_2^{p_2}); (\eta_1, \eta_2))$-convex mappings. Also, some new special cases are given. At the end, some applications to special means for different positive real numbers are provided as well. Motivated by this interesting class we can indeed see to be vital for fellow researchers and scientists working in the same domain. We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard, Ostrowski, and Simpson type integral inequalities for various kinds of convex and preinvex functions involving local fractional integrals, fractional integral operators, Caputo k-fractional derivatives, q-calculus, (p, q)-calculus, time scale calculus, and conformable fractional integrals.

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