# Exact Solution to Systems of Linear First-Order Integro-Differential Equations with Multipoint and Integral Conditions

**M. M. Baiburin and E. Providas**

**Abstract** This paper is devoted to the study of nonhomogeneous systems of linear first-order ordinary integro-differential equations of Fredholm type with multipoint and integral boundary constraints. Sufficient conditions for the solvability and correctness of the problem are established and the unique solution is provided in closed-form. The approach followed is based on the extension theory of operators.

## 1 Introduction

Mathematical modeling in the theory of automatic control, the theory of oscillation, mathematical physics, biology, applied mathematics, and economics, very often, leads to the study of multipoint boundary value problems for differential, functional-differential, and integro-differential equations. These types of boundary value problems and their solutions have been investigated by many researchers, for example, [1, 2, 5, 8, 9, 21]. Of special interest are the multipoint boundary value problems for a system of differential equations (DEs) and integro-differential equations (IDEs), see, for example, [4, 6, 11, 12, 22]. It should be noted that obtaining exact solutions even to multipoint boundary value problems for a differential, or an integro-differential equation, is a difficult task. Therefore, usually numerical methods are employed as in [3, 7, 15] and elsewhere.

Recently, in [6] the solution to a class of boundary value problems for a system of linear first-order DEs coupled with multipoint and integral conditions has been obtained in closed-form. Here, we continue this study to systems of linear first-order

M. M. Baiburin
Department of Fundamental Mathematics, L.N. Gumilyov Eurasian National University, Astana, Republic of Kazakhstan
e-mail: merkhasyl@mail.ru

E. Providas (✉)
University of Thessaly, Larissa, Greece
e-mail: providas@uth.gr



T. M. Rassias, P. M. Pardalos (eds.), *Mathematical Analysis and Applications*, Springer Optimization and Its Applications 154,
https://doi.org/10.1007/978-3-030-31339-5_1

ordinary IDEs of Fredholm type with multipoint and integral boundary constraints. The method proposed is based on the extension theory of linear operators in a Banach space, as it has been developed in terms of inverse operators [5, 13] and in terms of direct operators [14], and has been used to investigate the correctness properties to some extensions of operators [10, 16, 20] and more recently to solve exactly initial and two-point boundary value problems for integro-differential equations [17–19].

We first examine the solvability conditions and then obtain the exact solution of the following system of IDEs subject to multipoint and integral boundary conditions:

$$
\begin{gathered}
y'(x) - Ay(x) - \sum_{i=0}^{m} G_i(x) \int_0^1 H_i(t) y(t) dt = f(x), \quad x \in [0,1], \\
\sum_{i=0}^{m} A_i y(x_i) + \sum_{j=0}^{s} B_j \int_{\xi_j}^{\xi_{j+1}} C_j(t) y(t) dt = \mathbf{0},
\end{gathered} \tag{1}
$$

where $A,\ A_i,\ B_j$ are $n \times n$ constant matrices, $G_i(x),\ H_i(x),\ C_j(x)$ are variable $n \times n$ matrices, whose elements are continuous functions on $[0, 1]$, $f(x)$ is a vector of $n$ continuous functions on $[0, 1]$, and $y(x)$ is a vector of $n$ sought continuous functions with continuous derivatives on $[0, 1]$; the points $x_i, \xi_j$ satisfy the conditions $0 = x_0 < x_1 < \cdots < x_{m-1} < x_m = 1$, $0 = \xi_0 < \xi_1 < \cdots < \xi_s < \xi_{s+1} = 1$. The problem (1) may be obtained as a perturbation of a corresponding boundary value problem for a system of first-order DEs, specifically

$$
\begin{gathered}
y'(x) - Ay(x) = f(x), \\
\sum_{i=0}^{m} A_i y(x_i) + \sum_{j=0}^{s} B_j \int_{\xi_j}^{\xi_{j+1}} C_j(t) y(t) dt = \mathbf{0},
\end{gathered} \tag{2}
$$

whose solvability and the construction of the exact solution were investigated in [6].

The rest of the paper is organized as follows. In Section 2 some necessary definitions are given and preliminary results are derived. In Section 3 the two main theorems for the existence and the construction of the exact solution are presented. Lastly, some conclusions are drawn in Section 4.

## 2 Definitions and Preliminary Results

Let $X, Y$ be complex Banach spaces. Let $P : X \to Y$ denote a linear operator and $D(P)$ and $R(P)$ its domain and the range, respectively. An operator $P$ is called an *extension* of the operator $P_0 : X \to Y$ if $D(P_0) \subseteq D(P)$ and $Pu = P_0 u$, for all $u \in D(P_0)$. An operator $P : X \to Y$ is called *correct* if $R(P) = Y$ and the inverse operator $P^{-1}$ exists and is continuous on $Y$.

We say that the problem $Pu = f$, $f \in Y$, is correct if the operator $P$ is correct. The problem $Pu = f$ with a linear operator $P$ is uniquely solvable on $R(P)$ if the corresponding homogeneous problem $Pu = 0$ has only a zero solution, i.e. if $\ker P = \{0\}$. The problem $Pu = f$ is said to be everywhere solvable on $Y$ if it admits a solution for any $f \in Y$.

Throughout this paper, we use lowercase letters and brackets to designate vectors and capital letters and square brackets to symbolize matrices. The unit and zero matrices are denoted by $\mathbf{I}$ and [0], respectively, and the zero column vector by $\mathbf{0}$.

The set of all complex numbers is specified by $\mathbf{C}$. If $c_i \in \mathbf{C}$, $i = 1, \ldots, n$, then we write $c = (c_1, \ldots, c_n) \in \mathbf{C}^n$. By $C_n[0, 1]$, we mean the space of continuous vector functions $f = f(x) = (f_1(x), \ldots, f_n(x))$ with norm

$$\|f\|_{C_n} = \|f_1(x)\| + \|f_2(x)\| + \cdots + \|f_n(x)\|, \quad \|f(x)\| = \max_{x \in [0,1]} |f(x)|. \tag{3}$$

Let $f = f(x) = col\,(f_1(x), \ldots, f_n(x)) \in C_n[0, 1]$. Further, let the operators $L, K, H : C_n[0, 1] \to C_n[0, 1]$ be defined by the matrices

$$L(x) = \begin{bmatrix} l_{11}(x) & \cdots & l_{1n}(x) \\ \vdots & \ldots & \vdots \\ l_{n1}(x) & \cdots & l_{nn}(x) \end{bmatrix}, \quad K(x) = \begin{bmatrix} k_{11}(x) & \cdots & k_{1n}(x) \\ \vdots & \ldots & \vdots \\ k_{n1}(x) & \cdots & k_{nn}(x) \end{bmatrix},$$

$$H(x) = \begin{bmatrix} h_{11}(x) & \cdots & h_{1n}(x) \\ \vdots & \ldots & \vdots \\ h_{n1}(x) & \cdots & h_{nn}(x) \end{bmatrix},$$

where $l_{ij},\ k_{ij},\ h_{ij} \in C[0, 1]$. Let $l_0 = \max |l_{ij}|,\ k_0 = \max |k_{ij}|,\ h_0 = \max |h_{ij}|$, $i, j = 1, \ldots, n$. Finally, consider the points $\xi_j,\ j = 0, \ldots, s + 1$ satisfying the conditions $0 = \xi_0 < \xi_1 < \cdots < \xi_s < \xi_{s+1} = 1$.

We now prove the next lemma which is used several times in the sequel.

**Lemma 1** *The next estimates are true*

$$\|Lf\|_{C_n} \leq l_0 n \|f\|_{C_n}, \tag{4}$$

$$\|\int_0^x L(t) f(t) dt\|_{C_n} \leq l_0 n \|f\|_{C_n}, \quad x \in [0, 1], \tag{5}$$

$$\|K(x) \int_0^x L(t) f(t) dt\|_{C_n} \leq k_0 l_0 n^2 \|f\|_{C_n}, \quad x \in [0, 1], \tag{6}$$

$$\|\int_0^1 K(x) \int_0^x L(t) f(t) dt dx\|_{C_n} \leq k_0 l_0 n^2 \|f\|_{C_n}, \tag{7}$$

$$\|\int_{\xi_j}^{\xi_{j+1}} K(x) \int_0^x L(t) f(t) dt dx\|_{C_n} \leq k_0 l_0 n^2 \|f\|_{C_n}, \tag{8}$$

$$\|H(x)\int_0^1 K(\xi)\int_0^{\xi} L(t)f(t)dtd\xi\|_{C_n} \le h_0k_0l_0n^3\|f\|_{C_n}, \quad x\in[0,1]. \tag{9}$$

***Proof*** The properties (4)–(6), (8) have been proved in [6]. We prove (7) and (9). Let $\phi(x) = col(\phi_1(x), \dots, \phi_n(x)) = \int_0^x L(t)f(t)dt$. Then, from (5) follows that

$$\begin{aligned}\|\int_0^1 K(x)\int_0^x L(t)f(t)dtdx\|_{C_n} &= \|\int_0^1 K(x)\phi(x)dx\|_{C_n}\\ &\le k_0n\|\phi(x)\|_{C_n}\\ &= k_0n\|\int_0^x L(t)f(t)dt\|_{C_n}\\ &\le k_0l_0n^2\|f\|_{C_n}.\end{aligned}$$

We now prove (9). Let $\phi = col\left(\phi_1, \dots, \phi_n\right) = \int_0^1 K(\xi)\int_0^{\xi} L(t)f(t)dtd\xi$. Then, from (4) and (7) follows that

$$\begin{aligned}\|H(x)\int_0^1 K(\xi)\int_0^{\xi} L(t)f(t)dtd\xi\|_{C_n} &= \|H(x)\phi\|_{C_n}\\ &\le h_0n\|\phi\|_{C_n}\\ &= h_0n\|\int_0^1 K(\xi)\int_0^{\xi} L(t)f(t)dtd\xi\|_{C_n}\\ &\le h_0k_0l_0n^3\|f\|_{C_n}.\end{aligned}$$

The lemma is proved. □

## 3 Main Results

Let the operator $P$ associated with problem (1) be defined as

$$\begin{aligned}&Py = y'(x) - Ay(x) - \sum_{i=0}^{m} G_i(x)\int_0^1 H_i(t)y(t)dt,\\ &D(P) = \left\{y(x)\in C_n^1[0,1] : \sum_{i=0}^{m} A_i y(x_i) + \sum_{j=0}^{s} B_j\int_{\xi_j}^{\xi_{j+1}} C_j(t)y(t)dt = \mathbf{0}\right\},\end{aligned} \tag{10}$$

where $A,\ A_i,\ B_j$ are $n\times n$ constant matrices and $G_i(x),\ H_i(x),\ C_j(x)$ are variable $n\times n$ matrices with elements continuous functions on [0, 1]; the points $x_i, \xi_j$ satisfy the conditions $0 = x_0 < x_1 < \cdots < x_{m-1} < x_m = 1,\ 0 = \xi_0 < \xi_1 < \cdots < \xi_s < \xi_{s+1} = 1$. Note that the operator $P$ is an extension of the minimal operator $P_0$ defined by

$$\begin{aligned} P_0 y &= y'(x) - Ay(x), \\ D(P_0) &= \Big\{ y(x) \in C_n^1[0,1] : y(x_i) = \mathbf{0},\ \int_{\xi_j}^{\xi_{j+1}} C_j(t)y(t)dt = \mathbf{0}, \\ &\qquad \int_0^1 H_i(t)y(t)dt = \mathbf{0},\ i = 0,\ldots,m,\ j = 0,\ldots,s \Big\}. \end{aligned} \tag{11}$$

Moreover, we may write the operator $P$ compactly as

$$\begin{aligned} Py &= y'(x) - Ay(x) - \mathbf{G}z(y), \\ D(P) &= \left\{ y(x) \in C_n^1[0,1] : \mathbf{A}y(\mathbf{x}) + \mathbf{B}\psi(y) = \mathbf{0} \right\}, \end{aligned} \tag{12}$$

where the composite matrices

$$\mathbf{G} = \left[\, G_0\ G_1\ \ldots\ G_m \,\right], \quad \mathbf{A} = \left[\, A_0\ A_1\ \ldots\ A_m \,\right], \quad \mathbf{B} = \left[\, B_0\ B_1\ \ldots\ B_s \,\right],$$

the compound column vectors

$$\begin{aligned} z(y) &= col\left(z_0(y), z_1(y), \ldots, z_m(y)\right), \\ y(\mathbf{x}) &= col\left(y(x_0), y(x_1), \ldots, y(x_m)\right), \\ \psi(y) &= col\left(\psi_0(y), \psi_1(y), \ldots, \psi_s(y)\right), \end{aligned}$$

and the $n \times 1$ vectors

$$\begin{aligned} z_i(y) &= \int_0^1 H_i(t)y(t)dt, \quad i = 0,\ldots,m, \\ \psi_j(y) &= \int_{\xi_j}^{\xi_{j+1}} C_j(t)y(t)dt, \quad j = 0,\ldots,s. \end{aligned}$$

By using (10) or (12), we can express the system (1) equivalently in the elegant operator form

$$Py = f(x), \qquad f(x) \in C_n[0,1]. \tag{13}$$

Theorem 1 below provides the criteria for the existence of a unique solution to the problem (13).

We first consider the $n \times n$ matrix $e^{xA}$ and define the following $n \times n$ matrices

$$L_i = \int_0^1 H_i(t)e^{tA}dt,$$

$$\Lambda_j = \int_{\xi_j}^{\xi_{j+1}} C_j(t)e^{tA}dt,$$

$$\Delta_{ik}(G) = \int_0^{x_i} e^{(x_i-t)A}G_k(t)dt,$$

$$V_{ik}(G) = \int_0^1 H_i(x)\int_0^x e^{(x-t)A}G_k(t)dtdx,$$

$$W_{jk}(G) = \int_{\xi_j}^{\xi_{j+1}} C_j(x)\int_0^x e^{(x-t)A}G_k(t)dtdx,$$

where $i, k = 0, \dots, m,\ \ j = 0, \dots, s$, and the compound matrices

$$e^{\mathbf{x}A} = col\left[e^{x_0A}\ \ e^{x_1A}\ \ \dots\ \ e^{x_mA}\right],$$
$$L = col\,[L_0\ \ L_1\ \cdots\ L_m], \quad \Lambda = col\,[\Lambda_0\ \ \Lambda_1\ \cdots\ \Lambda_s],$$
$$\Delta_G = [\Delta_{ik}(G)], \quad V_G = [V_{ik}(G)], \quad W_G = \left[W_{jk}(G)\right]. \tag{14}$$

**Theorem 1** *The problem (13) is uniquely solvable on* $C_n[0, 1]$ *if*

$$\det \mathbf{T} = \det \begin{bmatrix} \mathbf{A}\Delta_G + \mathbf{B}W_G & \mathbf{A}e^{\mathbf{x}A} + \mathbf{B}\Lambda \\ V_G - I & L \end{bmatrix} \neq 0. \tag{15}$$

***Proof*** It suffices to show that $\ker P = \{0\}$ if $\det T \neq 0$. Assume that $\det T \neq 0$. Consider the homogeneous problem $Py = \mathbf{0}$ consisting of the homogeneous equation

$$y'(x) - Ay(x) - \mathbf{G}z(y) = \mathbf{0}, \tag{16}$$

and the boundary conditions

$$\mathbf{A}y(\mathbf{x}) + \mathbf{B}\psi(y) = \mathbf{0}. \tag{17}$$

Let the auxiliary integro-functional equation

$$y(x) = e^{xA}\mathbf{d} + e^{xA}\sum_{i=0}^{m}\int_0^x e^{-tA}G_i(t)dtz_i(y), \tag{18}$$

or in a compact form

$$y(x) = e^{xA}\mathbf{d} + e^{xA}\int_0^x e^{-tA}\mathbf{G}(t)dtz(y), \tag{19}$$

where $e^{xA}$ is a fundamental $n \times n$ matrix to the homogeneous differential equation $y'(x) - Ay(x) = \mathbf{0}$ and $\mathbf{d}$ is an arbitrary column vector with constant coefficients. It is easy to verify that from (19) follows the homogeneous equation (16). Therefore every solution of (19) is also a solution of (16). From (18), we have

$$y(x_i) = e^{x_i A}\mathbf{d} + e^{x_i A}\sum_{k=0}^{m}\int_0^{x_i} e^{-tA}G_k(t)dtz_k(y), \tag{20}$$

$$H_i(x)y(x) = H_i(x)e^{xA}\mathbf{d} + H_i(x)e^{xA}\sum_{k=0}^{m}\int_0^{x} e^{-tA}G_k(t)dtz_k(y), \tag{21}$$

$$C_j(x)y(x) = C_j(x)e^{xA}\mathbf{d} + C_j(x)e^{xA}\sum_{k=0}^{m}\int_0^{x} e^{-tA}G_k(t)dtz_k(y), \tag{22}$$

for $i = 0, \ldots, m,\ \ j = 0, \ldots, s$. By integrating (21) and (22), we get

$$\begin{aligned}\int_0^1 H_i(x)y(x)dx &= \int_0^1 H_i(x)e^{xA}dx\mathbf{d} \\ &\quad + \sum_{k=0}^{m}\int_0^1 H_i(x)e^{xA}\int_0^x e^{-tA}G_k(t)dtdxz_k(y),\end{aligned} \tag{23}$$

$$\begin{aligned}\int_{\xi_j}^{\xi_{j+1}} C_j(x)y(x)dx &= \int_{\xi_j}^{\xi_{j+1}} C_j(x)e^{xA}dx\mathbf{d} \\ &\quad + \sum_{k=0}^{m}\int_{\xi_j}^{\xi_{j+1}} C_j(x)e^{xA}\int_0^x e^{-tA}G_k(t)dtdxz_k(y).\end{aligned} \tag{24}$$

We rewrite (20), (23), (24) in the compact matrix form

$$y(\mathbf{x}) = e^{\mathbf{x}A}\mathbf{d} + \Delta_G z(y), \tag{25}$$

$$z(y) = L\mathbf{d} + V_G z(y), \tag{26}$$

$$\psi(y) = \Lambda\mathbf{d} + W_G z(y), \tag{27}$$

where the matrices $e^{\mathbf{x}A},\ L,\ \Lambda,\ \Delta_G,\ V_G,\ W_G$ are defined in (14). By utilizing (25) and (27), the boundary conditions in (17) are written as

$$\mathbf{A}\left(e^{\mathbf{x}A}\mathbf{d} + \Delta_G z(y)\right) + \mathbf{B}\left(\Lambda\mathbf{d} + W_G z(y)\right) = \mathbf{0}. \tag{28}$$

From (26) and (28) we obtain the system

$$\begin{bmatrix} \mathbf{A}\Delta_G + \mathbf{B}W_G & \mathbf{A}e^{\mathbf{x}A} + \mathbf{B}\Lambda \\ V_G - I & L \end{bmatrix} \begin{pmatrix} z(y) \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \tag{29}$$

or

$$\mathbf{T}\begin{pmatrix} z(y) \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \tag{30}$$

The assumption that $\det \mathbf{T} \neq 0$ implies $z(y) = \mathbf{0},\ \mathbf{d} = \mathbf{0}$. Substituting these values into (19), we obtain $y(x) = \mathbf{0}$. Hence $\ker P = \{0\}$ and the operator $P$ is uniquely solvable. The theorem is proved. □

*Remark 1* Note that the system of integro-differential equations (1) for $G_i \equiv [0],\ i = 0, \ldots, m$, degenerates to the system of differential equations (2). By setting $G_i = [0],\ i = 0, \ldots, m$, into (15), we obtain

$$\begin{aligned} \det \mathbf{T} &= \det \begin{bmatrix} \mathbf{A}\Delta_G + \mathbf{B}W_G & \mathbf{A}e^{\mathbf{x}A} + \mathbf{B}\Lambda \\ V_G - I & L \end{bmatrix} \\ &\overset{G_i=[0]}{=} \det \begin{bmatrix} [0] & \mathbf{A}e^{\mathbf{x}A} + \mathbf{B}\Lambda \\ -I & L \end{bmatrix} \\ &= \det \left[\mathbf{A}e^{\mathbf{x}A} + \mathbf{B}\Lambda\right] \\ &= \det \left[\sum_{i=0}^{m} A_i e^{x_i A} + \sum_{j=0}^{s} B_j \int_{\xi_j}^{\xi_{j+1}} C_j(x) e^{xA} dx\right] \neq 0, \end{aligned} \tag{31}$$

which is the sufficient solvability condition for the differential problem (2) derived in [6].

We introduce now the $\mathbf{C}^n$ vectors

$$\begin{aligned} \phi_i(f) &= \int_0^{x_i} e^{(x_i - t)A} f(t)dt, \\ \nu_i(f) &= \int_0^1 H_i(x) \int_0^x e^{(x-t)A} f(t)dtdx, \\ \omega_j(f) &= \int_{\xi_j}^{\xi_{j+1}} C_j(x) \int_0^x e^{(x-t)A} f(t)dtdx, \end{aligned}$$

for $i = 0, \ldots, m,\ j = 0, \ldots, s$, and the combined vectors

$$\phi_f = col(\phi_0(f), \phi_1(f), \ldots, \phi_m(f)), \quad \nu_f = col(\nu_0(f), \ldots, \nu_m(f)),$$
$$\omega_f = col(\omega_0(f), \ldots, \omega_s(f)). \tag{32}$$

**Theorem 2** *Let (15) hold true. Then the problem (13) is correct on* $C_n[0, 1]$ *and its unique solution is given by*

$$y(x) = e^{xA}\int_0^x e^{-tA} f(t)dt - \left[e^{xA}\int_0^x e^{-tA}\mathbf{G}(t)dt \;\; e^{xA}\right]\mathbf{T}^{-1}\begin{pmatrix}\mathbf{A}\phi_f + \mathbf{B}\omega_f \\ \nu_f\end{pmatrix}. \tag{33}$$

***Proof*** The problem (13) encompasses the nonhomogeneous system of integro-differential equations

$$y'(x) - Ay(x) - \mathbf{G}z(y) = f(x), \tag{34}$$

and the boundary conditions

$$\mathbf{A}y(\mathbf{x}) + \mathbf{B}\psi(y) = \mathbf{0}. \tag{35}$$

Take the auxiliary integro-functional equation

$$y(x) = e^{xA}\mathbf{d} + e^{xA}\sum_{i=0}^{m}\int_0^x e^{-tA}G_i(t)dt z_i(y) + e^{xA}\int_0^x e^{-tA} f(t)dt, \tag{36}$$

or in the compact matrix form

$$y(x) = \left[e^{xA}\int_0^x e^{-tA}\mathbf{G}(t)dt \;\; e^{xA}\right]\begin{pmatrix} z(y) \\ \mathbf{d}\end{pmatrix} + e^{xA}\int_0^x e^{-tA} f(t)dt, \tag{37}$$

for every $f(x) \in C_n[0, 1]$; $e^{xA}$ is a fundamental $n \times n$ matrix to the homogeneous differential equation $y'(x) - Ay(x) = \mathbf{0}$ and $\mathbf{d}$ is an arbitrary column vector with constant elements. Observe that differentiation of (37) yields (34). Hence, a solution of (37) is also a solution of (34). From (36), we get

$$y(x_i) = e^{x_i A}\mathbf{d} + e^{x_i A}\sum_{k=0}^{m}\int_0^{x_i} e^{-tA}G_k(t)dt z_k(y) + e^{x_i A}\int_0^{x_i} e^{-tA} f(t)dt, \tag{38}$$

$$H_i(x)y(x) = H_i(x)e^{xA}\mathbf{d} + H_i(x)e^{xA}\sum_{k=0}^{m}\int_0^x e^{-tA}G_k(t)dt z_k(y)$$
$$+H_i(x)e^{xA}\int_0^x e^{-tA} f(t)dt, \tag{39}$$

$$C_j(x)y(x) = C_j(x)e^{xA}\mathbf{d} + C_j(x)e^{xA}\sum_{k=0}^{m}\int_0^x e^{-tA}G_k(t)dtz_k(y)$$
$$+C_j(x)e^{xA}\int_0^x e^{-tA}f(t)dt, \tag{40}$$

for $i = 0, \dots, m,\ j = 0, \dots, s$. By integrating (39), (40), we obtain

$$\begin{aligned}\int_0^1 H_i(x)y(x)dx = &\int_0^1 H_i(x)e^{xA}dx\mathbf{d}\\ &+\sum_{k=0}^{m}\int_0^1 H_i(x)e^{xA}\int_0^x e^{-tA}G_k(t)dtdxz_k(y)\\ &+\int_0^1 H_i(x)e^{xA}\int_0^x e^{-tA}f(t)dtdx,\end{aligned} \tag{41}$$

$$\begin{aligned}\int_{\xi_j}^{\xi_{j+1}} C_j(x)y(x)dx = &\int_{\xi_j}^{\xi_{j+1}} C_j(x)e^{xA}dx\mathbf{d}\\ &+\sum_{k=0}^{m}\int_{\xi_j}^{\xi_{j+1}} C_j(x)e^{xA}\int_0^x e^{-tA}G_k(t)dtdxz_k(y)\\ &+\int_{\xi_j}^{\xi_{j+1}} C_j(x)e^{xA}\int_0^x e^{-tA}f(t)dtdx,\end{aligned} \tag{42}$$

for $i = 0, \dots, m,\ j = 0, \dots, s$. We rewrite (38), (41), (42) in the compact matrix form

$$y(\mathbf{x}) = e^{\mathbf{x}A}\mathbf{d} + \Delta_G z(y) + \phi_f, \tag{43}$$
$$z(y) = L\mathbf{d} + V_G z(y) + \nu_f, \tag{44}$$
$$\psi(y) = \Lambda\mathbf{d} + W_G z(y) + \omega_f, \tag{45}$$

where the matrices $e^{\mathbf{x}A},\ L,\ \Lambda,\ \Delta_G,\ V_G,\ W_G$ are defined in (14) and the vectors $\phi_f,\ \nu_f,\ \omega_f$ are given in (32). By utilizing (43) and (45), the boundary conditions in (35) are recast as

$$\mathbf{A}\left(e^{\mathbf{x}A}\mathbf{d} + \Delta_G z(y) + \phi_f\right) + \mathbf{B}\left(\Lambda\mathbf{d} + W_G z(y) + \omega_f\right) = \mathbf{0}. \tag{46}$$

From (44) and (46), we obtain the system

$$\begin{bmatrix}\mathbf{A}\Delta_G + \mathbf{B}W_G & \mathbf{A}e^{\mathbf{x}A} + \mathbf{B}\Lambda\\ V_G - I & L\end{bmatrix}\begin{pmatrix}z(y)\\ \mathbf{d}\end{pmatrix} = -\begin{pmatrix}\mathbf{A}\phi_f + \mathbf{B}\omega_f\\ \nu_f\end{pmatrix}, \tag{47}$$

or

$$\mathbf{T}\begin{pmatrix} z(y) \\ \mathbf{d} \end{pmatrix} = -\begin{pmatrix} \mathbf{A}\phi_f + \mathbf{B}\omega_f \\ \nu_f \end{pmatrix}. \tag{48}$$

Since $\det \mathbf{T} \neq 0$ by hypothesis, we have

$$\begin{pmatrix} z(y) \\ \mathbf{d} \end{pmatrix} = -\mathbf{T}^{-1}\begin{pmatrix} \mathbf{A}\phi_f + \mathbf{B}\omega_f \\ \nu_f \end{pmatrix}. \tag{49}$$

Substitution of (49) into (37) yields the solution (33) to the problem (34)–(35). Since this solution holds for all $f(x) \in C_n[0, 1]$, then the system (34)–(35) is everywhere solvable. Thus, (33) is the unique solution to the nonhomogeneous problem (13) which can be denoted conveniently as $y(x) = P^{-1}f(x)$. To prove the correctness of the problem (13) it remains to show that the inverse operator $P^{-1}$ is bounded.

Let $r = m + 1$ and write the matrix $\mathbf{T}$ conveniently as

$$\mathbf{T} = \begin{bmatrix} \mathbf{A}\Delta_G + \mathbf{B}W_G & \mathbf{A}e^{\mathbf{x}A} + \mathbf{B}\Lambda \\ V_G - I & L \end{bmatrix} = \begin{bmatrix} \mathbf{T}^{1r} & \mathbf{T}^{11} \\ \mathbf{T}^{rr} & \mathbf{T}^{r1} \end{bmatrix}, \tag{50}$$

where $\mathbf{T}^{1r} = \mathbf{A}\Delta_G + \mathbf{B}W_G$, $\mathbf{T}^{11} = \mathbf{A}e^{\mathbf{x}A} + \mathbf{B}\Lambda$, $\mathbf{T}^{rr} = V_G - I$ and $\mathbf{T}^{r1} = L$. Let also the analogously partitioned matrix

$$\Pi = \mathbf{T}^{-1} = \begin{bmatrix} \Pi^{r1} & \Pi^{rr} \\ \Pi^{11} & \Pi^{1r} \end{bmatrix}, \tag{51}$$

where

$$\Pi^{1r} = \begin{bmatrix} \Pi_0^{1r} \cdots \Pi_m^{1r} \end{bmatrix}, \quad \Pi^{r1} = \begin{bmatrix} \Pi_0^{r1} \\ \vdots \\ \Pi_m^{r1} \end{bmatrix}, \quad \Pi^{rr} = \begin{bmatrix} \Pi_{00}^{rr} & \cdots & \Pi_{0m}^{rr} \\ \vdots & \cdots & \vdots \\ \Pi_{m0}^{rr} & \cdots & \Pi_{mm}^{rr} \end{bmatrix}, \tag{52}$$

and $\Pi^{11}$, $\Pi_i^{1r}$, $\Pi_i^{r1}$, $\Pi_{ik}^{rr}$, $i, k = 0, \ldots, m$ are $n \times n$ matrices. Then,

$$\Pi\begin{pmatrix} \mathbf{A}\phi_f + \mathbf{B}\omega_f \\ \nu_f \end{pmatrix} = \begin{pmatrix} \Pi^{r1}\left(\mathbf{A}\phi_f + \mathbf{B}\omega_f\right) + \Pi^{rr}\nu_f \\ \Pi^{11}\left(\mathbf{A}\phi_f + \mathbf{B}\omega_f\right) + \Pi^{1r}\nu_f \end{pmatrix}. \tag{53}$$

Substitution of (53) into solution (33) yields

$$y(x) = \int_0^x e^{(x-t)A}f(t)dt$$
$$- \left[\int_0^x e^{(x-t)A}\mathbf{G}(t)dt \quad e^{xA}\right]\begin{pmatrix} \Pi^{r1}(\mathbf{A}\phi_f + \mathbf{B}\omega_f) + \Pi^{rr}\nu_f \\ \Pi^{11}(\mathbf{A}\phi_f + \mathbf{B}\omega_f) + \Pi^{1r}\nu_f \end{pmatrix}$$

$$
\begin{aligned}
&= \int_0^x e^{(x-t)A} f(t)dt \\
&\quad - \int_0^x e^{(x-t)A}\mathbf{G}(t)dt\Pi^{r1}(\mathbf{A}\phi_f + \mathbf{B}\omega_f) - \int_0^x e^{(x-t)A}\mathbf{G}(t)dt\Pi^{rr}\nu_f \\
&\quad - e^{xA}\Pi^{11}(\mathbf{A}\phi_f + \mathbf{B}\omega_f) - e^{xA}\Pi^{1r}\nu_f. \qquad (54)
\end{aligned}
$$

Let the maxima absolute elements (ae) for each of the following $n \times n$ matrices be denoted by

$$
\begin{aligned}
&k^{(0)} = \max_{ae}\left[e^{xA}\right], \quad k_j = \max_{ae}\left[B_j C_j(x) e^{xA}\right], \quad l^{(0)} = \max_{ae}\left[e^{-xA}\right], \\
&l^{(1)} = \max_{ae}\left[\int_0^x \sum_{i=0}^m e^{(x-t)A} G_i(t)\Pi_i^{r1} dt\right], \quad l_i^{(2)} = \max_{ae}\left[A_i e^{(x_i-t)A}\right], \\
&l_k^{(3)} = \max_{ae}\left[\sum_{i=0}^m \int_0^x e^{(x-t)A} G_i(t)\Pi_{ik}^{rr} dt\right], \quad l^{(4)} = \max_{ae}\left[e^{xA}\Pi^{11}\right], \\
&\hat{h}_k = \max_{ae}\left[H_k(x)e^{xA}\right], \quad \tilde{h}_i = \max_{ae}\left[e^{xA}\Pi_i^{1r}\right]. \qquad (55)
\end{aligned}
$$

Notice that the elements of the above matrices are continuous functions on [0, 1] since the elements of the fundamental matrix $e^{xA}$ and the inverse matrix $e^{-xA}$ are continuous functions.

We now find some estimates for the terms appearing in (54). First, note that $\mathbf{A}\phi_f + \mathbf{B}\omega_f \in \mathbf{C}^n$, since both $A_i\phi_i(f)$ and $B_j\omega_j(f) \in \mathbf{C}^n$, and by the triangle inequality and properties (5), (8), we have

$$
\begin{aligned}
\|\mathbf{A}\phi_f + \mathbf{B}\omega_f\|_{C_n} &\le \|\mathbf{A}\phi_f\|_{C_n} + \|\mathbf{B}\omega_f\|_{C_n} \\
&= \|\sum_{i=0}^m A_i\phi_i(f)\|_{C_n} + \|\sum_{j=0}^s B_j\omega_j(f)\|_{C_n} \\
&\le \sum_{i=0}^m \|\int_0^{x_i} A_i e^{(x_i-t)A} f(t)dt\|_{C_n} \\
&\quad + \sum_{j=0}^s \|\int_{\xi_j}^{\xi_{j+1}} B_j C_j(x) e^{xA}\int_0^x e^{-tA} f(t)dtdx\|_{C_n} \\
&\le \sum_{i=0}^m l_i^{(2)} n\|f\|_{C_n} + \sum_{j=0}^s k_j l^{(0)} n^2\|f\|_{C_n}
\end{aligned}
$$

$$= n\left(\sum_{i=0}^{m} l_i^{(2)} + l^{(0)} n \sum_{j=0}^{s} k_j\right)\|f\|_{C_n}. \quad (56)$$

By means of (6), we obtain

$$\|\int_0^x e^{(x-t)A} f(t)dt\|_{C_n} = \|e^{xA}\int_0^x e^{-tA} f(t)dt\|_{C_n} \le k^{(0)} l^{(0)} n^2 \|f\|_{C_n}. \quad (57)$$

Utilization of (4) and the relation (56) produces

$$\begin{aligned}
&\|\int_0^x e^{(x-t)A}\mathbf{G}(t)dt\Pi^{r1}(\mathbf{A}\phi_f + \mathbf{B}\omega_f)\|_{C_n} \\
&= \|\int_0^x \sum_{i=0}^{m} e^{(x-t)A} G_i(t)\Pi_i^{r1} dt(\mathbf{A}\phi_f + \mathbf{B}\omega_f)\|_{C_n} \\
&\le l^{(1)} n\|\mathbf{A}\phi_f + \mathbf{B}\omega_f\|_{C_n} \\
&\le l_1 n^2\left(\sum_{i=0}^{m} l_i^{(2)} + l^{(0)} n \sum_{j=0}^{s} k_j\right)\|f\|_{C_n}. \quad (58)
\end{aligned}$$

From (4) and (7) follows that

$$\begin{aligned}
&\|\int_0^x e^{(x-t)A}\mathbf{G}(t)dt\Pi^{rr}\nu_f\|_{C_n} \\
&= \|\int_0^x e^{(x-t)A}\left(\sum_{i=0}^{m} G_i(t)\Pi_{i0}^{rr}, \ldots, \sum_{i=0}^{m} G_i(t)\Pi_{im}^{rr}\right) dt col\,(\nu_0(f), \ldots, \nu_m(f))\|_{C_n} \\
&\le \sum_{k=0}^{m}\|\sum_{i=0}^{m}\int_0^x e^{(x-t)A} G_i(t)\Pi_{ik}^{rr} dt\nu_k(f)\|_{C_n} \\
&\le \sum_{k=0}^{m} l_k^{(3)} n\|\nu_k(f)\|_{C_n} \\
&= \sum_{k=0}^{m} l_k^{(3)} n\|\int_0^1 H_k(x)e^{xA}\int_0^x e^{-tA} f(t)dtdx\|_{C_n} \\
&\le \sum_{k=0}^{m} l_k^{(3)} n\hat{h}_k n l^{(0)} n\|f\|_{C_n} \\
&= l^{(0)} n^3 \sum_{k=0}^{m} l_k^{(3)}\hat{h}_k\|f\|_{C_n}. \quad (59)
\end{aligned}$$

Further, by using property (4) and the relation (56), we acquire

$$
\begin{aligned}
\|e^{xA}\Pi^{11}(\mathbf{A}\phi_f+\mathbf{B}\omega_f)\|_{C_n} &\leq l^{(4)}n\|\mathbf{A}\phi_f+\mathbf{B}\omega_F\|_{C_n} \\
&\leq l^{(4)}n^2\left(\sum_{i=0}^{m} l_i^{(2)}+l^{(0)}n\sum_{j=0}^{s}k_j\right)\|f\|_{C_n}. \quad (60)
\end{aligned}
$$

Finally, by employing (9), we get

$$
\begin{aligned}
\|e^{xA}\Pi^{1r}\nu_f\|_{C_n} &= \|e^{xA}\sum_{i=0}^{m}\Pi_i^{1r}\nu_i(f)\|_{C_n} \\
&\leq \sum_{i=0}^{m}\|e^{xA}\Pi_i^{1r}\nu_i(f)\|_{C_n} \\
&= \sum_{i=0}^{m}\|e^{xA}\Pi_i^{1r}\int_0^1 H_i(\xi)e^{\xi A}\int_0^{\xi} e^{-tA}f(t)dtd\xi\|_{C_n} \\
&\leq l^{(0)}n^3\sum_{i=0}^{m}\hat{h}_i\tilde{h}_i\|f\|_{C_n}. \quad (61)
\end{aligned}
$$

From (54) and (57)–(60), follows that

$$
\begin{aligned}
\|y(x)\|_{C_n} \leq & \left[k^{(0)}l^{(0)}n^2+l_1n^2\left(\sum_{i=0}^{m}l_i^{(2)}+l^{(0)}n\sum_{j=0}^{s}k_j\right)+l^{(0)}n^3\sum_{k=0}^{m}l_k^{(3)}\hat{h}_k\right. \\
& \left. +l^{(4)}n^2\left(\sum_{i=0}^{m}l_i^{(2)}+l^{(0)}n\sum_{j=0}^{s}k_j\right)+l^{(0)}n^3\sum_{i=0}^{m}\hat{h}_i\tilde{h}_i\right]\|f\|_{C_n} \\
\leq & \ \gamma\|f\|_{C_n}. \quad (62)
\end{aligned}
$$

where $\gamma > 0$. The last inequality proves the boundedness and correctness of the operator $P$ and problem (13). The theorem is proved. □

## 4 Conclusions

We have studied a class of nonhomogeneous systems of $n$ linear first-order ordinary Fredholm type integro-differential equations subject to general multipoint and integral boundary constraints. We have established sufficient solvability and uniqueness criteria and we have derived a ready to use exact solution formula. The method proposed requires the knowledge of a fundamental matrix of the corresponding homogeneous system of first-order differential equations. The solution process can be easily implemented to any computer algebra system.

## References


1. A.R. Abdullaev, E.A. Skachkova, On a multipoint boundary-value problem for a second-order differential equation. Vestn. Perm. Univ. Ser. Math. Mech. Inf. **2**(25), 5–9 (2014)
2. A.R. Abdullaev, E.A. Skachkova, On one class of multipoint boundary-value problems for a second-order linear functional-differential equation. J. Math. Sci. **230**, 647–650 (2018). https://doi.org/10.1007/s10958-018-3761-9
3. R.P. Agarwal, The numerical solutions of multipoint boundary value problems. J. Comput. Appl. Math. **5**, 17–24 (1979)
4. M.T. Ashordiya, A Criterion for the solvability of a multipoint boundary value problem for a system of generalized ordinary differential equations. Differ. Uravn. **32**, 1303–1311 (1996)
5. M.M. Baiburin, Multipoint value problems for second order differential operator. Vestn. KAZgu. Math. **1**(37), 36–40 (2005) [in Russian]
6. M.M. Baiburin, On multi-point boundary value problems for first order linear differential equations system. Appl. Math. Control Prob. **3**, 16–30 (2018) [in Russian]
7. D.S. Dzhumabaev, On one approach to solve the linear boundary value problems for Fredholm integro-differential equations. J. Comput. Appl. Math. **294**, 342–357 (2016). https://doi.org/10.1016/j.cam.2015.08.023
8. C.P. Gupta, A generalized multi-point boundary value problem for second order ordinary differential equations. Appl. Math. Comput. **89**, 133–146 (1998). https://doi.org/10.1016/S0096-3003(97)81653-0
9. V.A. Il'in, E.I. Moiseev, Nonlocal boundary value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects. Differ. Equ. **23**, 803–810 (1987)
10. A.K. Iskakova, On the correctness of a single operator in $C_n[0, 1]$. Vestn. KAZgu. Math. Mech. Inf. **5**(19), 68–72 (1999) [in Russian]
11. I.T. Kiguradze, Boundary-value problems for systems of ordinary differential equations. J. Math. Sci. **43**, 2259–2339 (1988). https://doi.org/10.1007/BF01100360
12. V.S. Klimov, A multipoint boundary value problem for a system of differential equations. Differ. Uravn. **5**, 1532–1534 (1969) [in Russian]
13. B.K. Kokebaev, M. Otelbaev, A.N. Shynybekov, Questions on extension and restriction of operators. Dokl. Akad. Nauk SSSR **271**, 24–26 (1983)
14. R.O. Oinarov, I.N. Parasidis, Correctly solvable extensions of operators with finite deficiencies in Banach space. Izv. Akad. Kaz. SSR. **5**, 42–46 (1988) [in Russian]
15. T. Ojika, W. Wayne, A numerical method for the solution of multi-point problems for ordinary differential equations with integral constraints. J. Math. Anal. Appl. **72**, 500–511 (1979). https://doi.org/10.1016/0022-247X(79)90243-9
16. M.O. Otelbaev, A.K. Iskakova, On the multipoint value problem for the operator $Ly = y'(t)$ in $C_1[0, 1]$. Vestn. KAZgu. Math. Mech. Inf. **5**(19), 111–115 (1999) [in Russian]
17. I.N. Parasidis, E. Providas, Resolvent operators for some classes of integro-differential equations, in *Mathematical Analysis, Approximation Theory and Their Applications*, ed. by T. Rassias, V. Gupta. Springer Optimization and Its Applications, vol. 111 (Springer, Cham, 2016), pp. 535–558. https://doi.org/10.1007/978-3-319-31281-1
18. I.N. Parasidis, E. Providas, Extension operator method for the exact solution of integro-differential equations, in *Contributions in Mathematics and Engineering*, ed. by P. Pardalos, T. Rassias (Springer, Cham, 2016), pp. 473–496. https://doi.org/10.1007/978-3-319-31317-7

19. I.N. Parasidis, E. Providas, On the exact solution of nonlinear integro-differential equations, in *Applications of Nonlinear Analysis*, ed. by T. Rassias. Springer Optimization and Its Applications, vol. 134 (Springer, Cham, 2018), pp. 591–609. https://doi.org/10.1007/978-3-319-89815-5
20. I. Parassidis, P. Tsekrekos, Correct selfadjoint and positive extensions of nondensely defined symmetric operators. Abstr. Appl. Anal. **2005**(7), 767–790 (2005)
21. I.V. Parkhimovich, Multipoint boundary value problems for linear integro-differential equations in a class of smooth functions. Differ. Uravn. **8**, 549–552 (1972) [in Russian]
22. M.N. Yakovlev, Estimates of the solutions systems of integro-differential equations subject to many-points and integral boundary conditions. Zap. Nauchn. Sem. LOMI **124**, 131–139 (1983) [in Russian]