Philippe de la Hire: Was He Desargues' Schüler?

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Abstract Philippe de la Hire (1640–1718) was the third of the seventeenth century pioneers of projective geometry, after Girard Desargues (1591–1661) and Blaise Pascal (1623–1662). We know very little about La Hire beyond what he tells us in his various published works and what Bernard de Fontenelle reported in his *Eloge*, issued soon after La Hire's death. That vacuum of information has been filled with misinformation and speculation. Beyond the annoying falsehoods, there is the very real issue of the degree of influence of Desargues in La Hire's geometry, especially his *Nouvelle méthode en Géométrie pour les Sections des Superficies coniques et Cylindriques* of 1673. La Hire's originality has been questioned, beginning, apparently, soon after the 1673 publication, reviving in the late nineteenth century after publication of Desargues' long lost work of 1639 and, again, in the writing of eminent scholar René Taton around 1950. Much of the discussion of originality has centered on the availability of Desargues' booklet to La Hire, rather than an examination of the work itself. The claim of this study is that comparison of the work of Desargues and La Hire shows that Desargues' influence was minimal.

1 Introduction

Richard Westfall ends the short entry for Philippe de la Hire on the Galileo website Westfall (2018):

> This extensive bibliography is misleading. It records my effort to find something about him beyond the small budget of information in Fontenelle's Éloge, which is apparently the source of every biographical treatment of La Hire. There is an extraordinary dearth of material on this important man.

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This vacuum on the life and work of Philippe de la Hire (1640–1718) has been filled with speculation, much improbable. Although we have no evidence that La Hire did any mathematics before the year of Desargues' death, 1661, or that the two ever met, W.W. Rouse Ball wrote that La Hire was Desargues' "favorite pupil" (Ball 1960, p. 317). And Wikipedia, 2018, has

> Upon his return to Paris [in 1664], [La Hire] became a disciple of Girard Desargues from whom he learned geometrical perspective. Wikipedia (2018)

Beyond unfounded statements, as illustrated above, there is an important question about the originality of La Hire's first major work on the conic sections, the *Nouvelle méthode* (La Hire 1673) of 1673. Girard Desargues had produced a bold work (Desargues 1639) on conic sections, employing projective methods, in 1639. It was a short book in 50 copies, which all seem to have disappeared within 20 years. La Hire also employed projective methods. Further, both Desargues and Philippe de la Hire had collaborated with the engraver Abraham Bosse (1604–1676), and Philippe's father was a friend of both Bosse and Desargues. Further still, in the nineteenth century, a handwritten transcript, dated 1679, by La Hire, of Desargues (1639) was found. La Hire claimed to have read Desargues' work only in 1679. But did La Hire study Desargues' work before 1673? Scholar René Taton (1915–2004) thought the answer was "yes." Among various articles in which he questioned La Hire's originality is the entry on La Hire for the *Dictionary of Scientific Biography* (Taton 1970). His judgment has been widely followed in online information. The MacTutor biography of La Hire (MacTutor 2018) simply quotes Taton's article. On the other hand, the one broad study of La Hire's mathematics, an unpublished thesis by Zbynek Sir, from 2002, does not support Taton (Zbynek 2002, p. 227).

We argue that the goals and methods of La Hire's *Nouvelle méthode*, of 1673, are very different from those of Desargues' 1639 (Desargues 1639). Our argument centers on a detailed comparison of the handling of the pole and polar concept, perhaps the most central topic of projective geometry. We conclude that La Hire's claim to not have seen Desargues' mathematical work before 1679 is plausible.

Section 2 provides background, starting with the *Conics* of Apollonius, on the pole/polar concept and the related concept of harmonic conjugates. Sections 3, 4, and 5 provide background on Girard Desargues and Philippe de la Hire and their work. Section 6, the longest, is the detailed comparison of development of the pole/polar concept by La Hire and Desargues. Then we are ready to turn, again, in Sect. 7, to a history of arguments, over the centuries, for Desargues' influence on La Hire.

2 A Little Background from Apollonius

The *Conics* of Apollonius was written, about 200 BC, in the definition–proposition–proof form found in Euclid's *Elements*. As with Euclid, the initial propositions of the *Conics* were already well established. Due to the high quality of the work, and good fortune, the *Conics* survived into the European renaissance, while other ancient works on conic sections were lost.

Apollonius defined a *conic surface* as that traced out by a line, extending indefinitely in both directions, turning on a point, the *vertex*, A, and traveling along a *base circle* (not coplanar with the vertex). The intersection, or section, of a plane with this surface was called a *conic section*, although planes on the vertex or parallel to the base circle were excluded.

When the slicing plane meets the base plane (plane of the base circle) in line FG, and BC is the diameter of the base circle perpendicular to FG, then the line ED, which is the intersection of the slicing plane and the plane ABC, is a *diameter* of the conic in that all the chords of the conic section which are parallel to FG are bisected by ED. See Fig. 1 Left. These chords are the *ordinates* corresponding to diameter ED. E and D, the intersections of the slicing plane with AB and AC, are *vertices* of the conic section corresponding to diameter ED, and the midpoint of ED is the *center* of the conic section. (When the slicing plane is parallel to AB or AC, there is only one vertex and no center, and the conic is called a *parabola*.)

Apollonius spent much of Book 1 showing that any other chord on the center was also a diameter; in the case of the parabola, any parallel to the diameter found as above is another diameter.

Book 1 covers the relation of a point, A, and a line, LK, now called the *polar* of A with respect to a given conic. In the simplest case, let A lie outside the conic, and let the tangents from A meet the conic in K and L. See Fig. 1 Right. Then KL is the *polar* of A, and A is the *pole* of KL. (Note that these names are from the early nineteenth century.) When $KL \cap ED = M$, then Apollonius showed (Apollonius of Perga 1696, Book 1 Prop 34, 36; Book 3 Prop 37).

$$\frac{AE}{AD} = \frac{ME}{MD}, \quad \text{abbreviated} \quad H(AM, ED).$$

Further, if a line on A meets the conic in H and N, and meets LK in G, then $H(AG, HN)$.

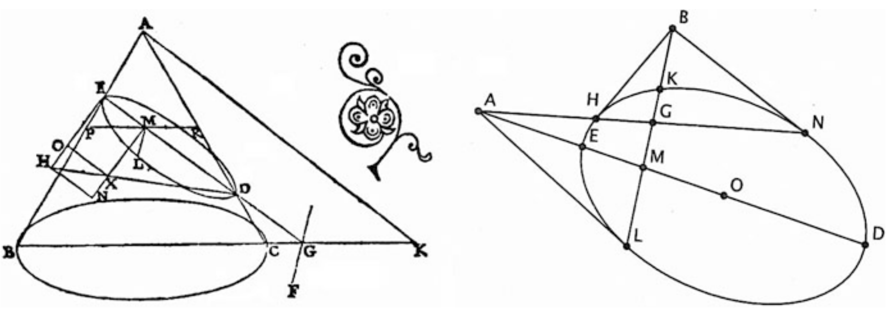

Fig. 1 Left: from *Conics* of Apollonius, 1696 ed. Right: Pole and polar property

Definition (La Hire, 1673) Let A, E, M, D be collinear points with exactly one of A and M between E and D. Then A and M are *harmonic conjugates* of E and D if

$$\frac{AE}{AD} = \frac{ME}{MD} \quad \text{abbreviated} \quad H(AM, ED).$$

When the order is clear, we refer to $\{A, E, M, D\}$ as a *harmonic set*.

3 To 1639

The first four books of the *Conics* of Apollonius had been edited, translated into Latin, and published by the late sixteenth century. (Books 5–7 were later recovered from Arabic sources, while Book 8 remains lost.) This provoked several attempts to simplify the study, including short works by Werner (1522) and Maurolico (1575), and Mydorge (1631/1639) of 1631 and 1639.

In 1639, a short work by Girard Desargues (1591–1661) was published in 50 copies, with the title *Brouillon project d'une atteinte aux événements des rencontres d'un cône avec un plan*, henceforth referred to as *Brouillon project* (Desargues 1639). The title could be translated as *Draft study of the intersections of a plane with a cone*. More than a simplification, this innovative work intended to put the study of the conics on a new foundation, which we would call projective.

The mathematicians and scientists of Paris were in close communication, facilitated by the Minim priest Marin Mersenne. This "circle," or informal "academy," included Mydorge, Desargues, Roberval, Étienne Pascal and his son Blaise. By correspondence, Mersenne had ties to Descartes, Fermat, and the other major scientists of Europe. The painter Laurent de la Hyre (1606–1656), the father of Philippe, is reported to have been a good friend of Desargues, and his attention to perspective is attributed to this friendship (Sorensen 2009). Laurent would have known Father Mersenne if for no other reason than he made a series of 18 paintings to decorate the refectory of the Paris convent of the Minim fathers. This association of scientists did not survive Mersenne's death in 1648, and these participants were dead by 1662; when Philippe de la Hire worked on conic sections in the late 1660s and the 1670s, he seems to have done so in isolation.

All copies of the *Brouillon project* apparently disappeared soon after its publication, a mark of its difficulty and unusual character. Desargues and his ideas did influence at least one in Mersenne's circle, the young Blaise Pascal. Pascal wrote in his *Essay pour les coniques* (Pascal 1640) of 1640, "I have tried as far as I could to imitate [Desargues'] method of approaching this material" (Field and Gray 1987, p. 183). However, a copy of the *Brouillon projet* came into the hands of Philippe de la Hire by 1679 and he made a transcription, which, in turn, disappeared until 1845. The existence of Desargues' work was never in question, since it was discussed in various letters which survived, and was at the center of a vitriolic dispute in the 1640s (Taton 1951a).

4 On the Life of La Hire

Philippe de la Hire trained to be a painter, the profession of his father, but moved to mathematics. The evidence of this change is limited.

In his short *Éloge* of 1718 (Fontenelle 1718), Fontenelle wrote that the teenage Philippe was interested in geometric aspects of painting, including perspective. In Italy, from 1660 until 1664, he developed his art, but also found a love for Greek geometry, especially the *Conics* of Apollonius.

Fontenelle reported that, after 1664, La Hire continued his geometric studies, going deeper into the subject, and La Hire himself tells us in the initial *Epistre* to La Hire (1673) of 1673 that he had applied himself to the study of geometry for *plusieurs années*. He must also have continued in painting, for he was admitted to the painters' guild, *l'Académie de Saint-Luc*, in 1670 (Bénézit 1960, Vol 3, p. 137).

We know nothing specific until 1672, the year of his first publication in mathematics, a short work entitled *Observations de Ph. de la Hire sur les points d'attouchement de trois Lignes droits qui touchent la Section d'un Cone..., et sur le centre de la mesme Section* (La Hire 1772). This short work comprises seven propositions with corollaries. It was brought out by Abraham Bosse (1604–1676). The problem addressed is that of constructing an *arc rampant*, and that problem seems to have been the focus of a controversy between Bosse [See Bosse 1672] and (Nicolas)-François Blondel (1618–1686), where the disagreement centered on the value of approximate methods.

The tracing of the *arc rampant* is the second problem of Blondel's 1673 *Résolution des quatres principaux problèmes d'architecture*:

> to find a Conic Section tangent to three given straight lines, in one plane, at a given point on two of these lines: in other words, to describe geometrically the *arcs rampant* of all types of foot segments (*pieds droits*) and heights. (Blondel 1673)

Bosse had worked with Desargues. In 1643 he brought out the first part of a treatise on stone cutting (Bosse 1643), incorporating rules found by Desargues. In that work of 1643 [p. 50], Bosse wrote that he hoped to produce a second part. In his preamble to La Hire (1772), of 1672, he wrote that, working with diagrams for this second part, with a "very capable person"—identified by Taton as master mason M. Rouget (Taton 1953, p. 94), he realized the need for a mathematical collaborator on construction of the *arc rampant*. Bosse turned to Philippe, the son of his friend. The result was La Hire (1772) of 1672. (The preamble is in Taton 1953, p. 97; Rouget was mentioned by La Hire on p. 97 of La Hire 1673.)

Both La Hire and Blondel solved the problem; when the *pieds droits* are EC and DB, the point of tangency to be found on BC is A, in Fig. 2 Left, the harmonic conjugate of S, where CB and ED meet at S. That Bosse did not solve the problem shows his limited understanding of the *Conics* of Apollonius. La Hire and Blondel independently introduced the term *harmonic* in their works of 1673.

La Hire's work of 1672 employs classic Euclidean geometry to specify the construction, in three-dimensions, of a cone, where a section of that cone gives the required *arc rampant*. Use of this geometry is puzzling, since La Hire seems to

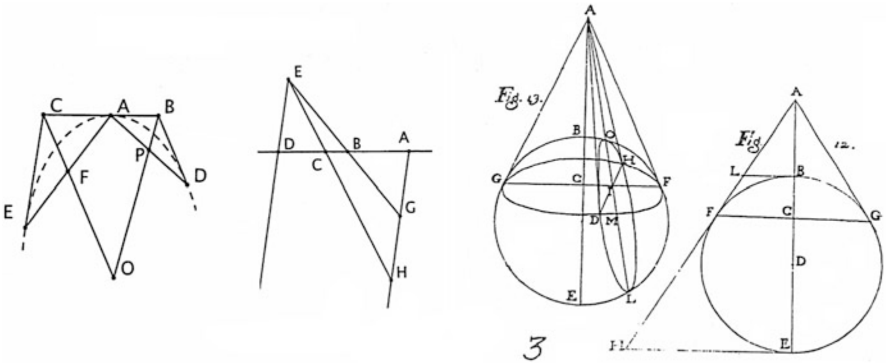

Fig. 2 Left: *arc rampant*, based on La Hire 1672 Prop. 1. Center Left: La Hire's Lemma 3, 1673. Right: La Hire's Lemma's 8 and 9, 1673

have already been reworking the geometry of conic sections, which would appear in 1673, as the *Nouvelle Methode*. The construction of the *arc rampant* is a simple consequence of the projective theory of pole and polar as set out in *Nouvelle Methode*. We will turn to that treatment in Sect. 6.

5 Desargues

Girard Desargues came from a wealthy Lyon family. In the 1630s he lived in Paris and became a member of Father Mersenne's circle. He acquired a thorough knowledge of the Greek heritage in geometry, and his works show an interest in applications of geometry, including a 12-page pamphlet on perspective drawing, *Exemple de l'une des manières universelles du S.G.D.L. touchant la pratique de la perspective*, of 1636, followed by a work on sundials and, with Abraham Bosse, the work on stone cutting. The theorem on perspective triangles which bears his name appeared in a work published by Bosse in 1648, but even that was little known. We know he was involved in several construction projects as architect, but it would be difficult to label his profession.

6 A Comparison: Pole and Polar in La Hire and Desargues

La Hire

We have already noted the pole — polar pairing of a point and a line, with respect to a given conic section, as found in the *Conics* of Apollonius. Both Desargues and La Hire approached the topic projectively, by handling the topic for a circle and then showing that the relation found for the base circle was preserved in projection to the conic section in the slicing plane. Beyond this similarity, they proceeded in very different ways.

We start with La Hire. Although later than Desargues, his method is simpler. He began his 1673 *Nouvelle méthode* with the definition, as we have seen, of *harmonic*:

> I call the straight line AD cut in 3 parts harmonically when the rectangle contained by all AD and the middle part BC is equal to the rectangle contained by the two extreme parts AB, CD.

Note that when collinear points A, B, C are given, there is exactly one point D so $H(AB, CD)$.

La Hire immediately entered, in his 1673 work, into a sequence of lemmas. Lemmas 2 through 6 showed that in a line-to-line projection from a point in a plane, a harmonic set is projected to a harmonic set. He needed numerous lemmas because he did not treat parallel lines as concurrent (at infinity) as Desargues had. It is instructive to look at one of the proofs. Here is his Lemma 3, with proof. In modern terms, it is the claim that the midpoint of a segment, AH, and the collinear point at infinity are harmonic conjugates of A and H, but La Hire avoided any reference to infinity. La Hire's proof is based on similar triangles and properties of proportion.

Theorem 1 (La Hire's Lemma 3, 1673) *Suppose $H(AC, BD)$ on line AD and AD is projected from a point E onto line AH that is parallel to ED, B to G and C to H. Then G is the midpoint of AH.*

Proof See Fig. 2 Center Left. By similar triangles,

$$\frac{AG}{DE} = \frac{AB}{BD} \text{ and } \frac{AH}{DE} = \frac{AC}{CD}.$$

Solving for DE in each equation gives

$$\frac{BD \cdot AG}{AB} = \frac{CD \cdot AH}{AC}. \text{ So } \frac{AH}{AG} = \frac{BD \cdot AC}{AB \cdot CD}.$$

The numerator is $(CB + CD)(AB + CB) = AB \cdot CD + CB(AB + BC + CD) = AB \cdot CD + CB \cdot AD$ so division by $AB \cdot CD = AD \cdot CB$ shows that $AH \div AG = 2$. □

Lemma 7 follows, the claim that when two lines meet at B so $H(BG, FH)$ and $H(BD, CE)$ for points on those lines, then the lines on corresponding points, GD, FC, and HE, are concurrent (or parallel).

In Lemmas 8 and 9, we consider a point A outside a given circle, where tangents from A meet the circle in points F and G. See La Hire's *Fig.* 12 and 13 of 1673, in our Fig. 2, when A is on the diameter on B. In La Hire's *Fig.* 12, we have $H(AC, BE)$, shown by similar triangles ALB and AHE, the tangent property:

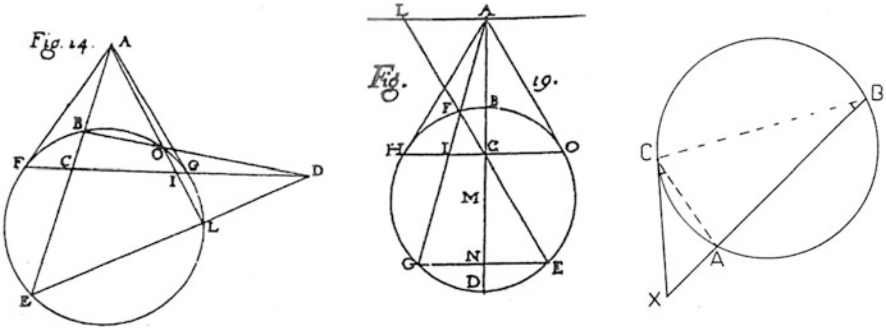

Fig. 3 Left: La Hire's Lemma 10, 1673. Center La Hire's Lemmas 13 and 14, 1673. Right: Euclid Book 3 Prop 36. For any line XAB on X, $XA \cdot XB$ is constant = XC^2

$LB = LF$ and $HF = HE$, and a parallel projection from AH to AE. This is Lemma 8.

In Lemma 9, the result of Lemma 8 is extended to any line on A meeting the circle in O and L, and meeting the polar in I. [La Hire's *Fig.* 13.] The sphere for which the given circle is a great circle is cut perpendicular to the plane AGE, giving a circle for which OL is a diameter and DH the polar of A. So by Lemma 8, $H(AI, OL)$.

For Lemma 10, we consider two lines on A that cut the circle. See La Hire's *Fig.* 14, in our Fig. 3, where FG is the polar of A. We know $H(BE, AC)$ and $H(OL, AI)$. By Lemma 7, chords BO and EL meet, at D, on the polar of A. [Note that BL and EO will also meet on the polar of A.] Then La Hire observed that when point B coalesces with O, and, likewise, point E coalesces with L, then BO and EL become tangents to the circle at O and at L, still meeting on line FG. It follows that when a point D is on the *polar* of A and outside the circle, then the *pole* of D lies on A.

What happens when a point, C, is inside a circle? La Hire, in Lemmas 13 and 14, drew a tentative polar AL — see La Hire's *Fig.* 19 — perpendicular, at A, to the diameter on C that meets the circle at B and D, with $H(AC, BD)$. We start with the secant on L and C, meeting the circle at E and F, then with GE drawn parallel to HO, La Hire showed line EF meets HO at C and, when continued to line LA, that $H(LC, EF)$.

In modern language, La Hire had proved the *Pole-Polar Theorem*:

Theorem 2 *Given a circle, let x be the polar of X and y the polar of Y. Then Y is on x exactly when X is on y.*

(Note that cases when X or Y is on the circle or at the center of the circle were not considered.)

What about the conic sections? The *Pole-Polar Theorem* applies just as well to the conic sections: "all that follows is a simple application of these lemmas ... in all the conic and cylindrical sections"(p. 15). In particular (p. 35): Suppose that, in the base plane,

> line eg will be cut at points e, f, p, g in three harmonic parts. But the lines drawn joining these points of division to the vertex A [of the cone] will meet line EG on the slicing plane in points E, F, P, G and by Lemma 5 or 6 it will be cut in these points E, F, P, G harmonically in 3 parts... .

Desargues

How did Desargues handle that material? For him, the key figure was a quadrilateral inscribed in a conic, and he began with the relation that he called *involution*, on collinear points.

Definition (Desargues) Collinear pairs $L, M;\ I, K;\ H, G$ are points *in involution* when there is a collinear point Q — called the *souche* — so $QL \cdot QM = QI \cdot QK = QH \cdot QG$, with Q separating either all or none of the pairs.

In Fig. 4 Right, for example, pairs $P, Q;\ L, M;\ H, G;\ I, K$ are in involution, where the *souche* would be between Q and L.

Desargues characterized in the following lemma three pairs of points in involution, in terms of what is now called a cross-ratio (Field and Gray 1987, p. 87).

Lemma 1 *Let $B, H;\ C, G;\ D, F$ be three pairs in involution with* souche *A. Then*

$$\frac{GB \cdot GH}{CB \cdot CH} = \frac{GD \cdot GF}{CD \cdot CF} = \frac{AG}{AC}.$$

This is the same as $CR(GC, BD) = CR(GC, HF)$ in absolute value, where $CR(XY, WZ)$ denotes $\frac{XW \cdot YZ}{XZ \cdot YW}$. [Corresponding equations hold for the other pairs.]

Proof By the definition of involution, $AB \cdot AH = AC \cdot AG = AD \cdot AF$ with A separating all pairs or none. Since $\frac{AG}{AF} = \frac{AD}{AC}$ then by adding or subtracting in both

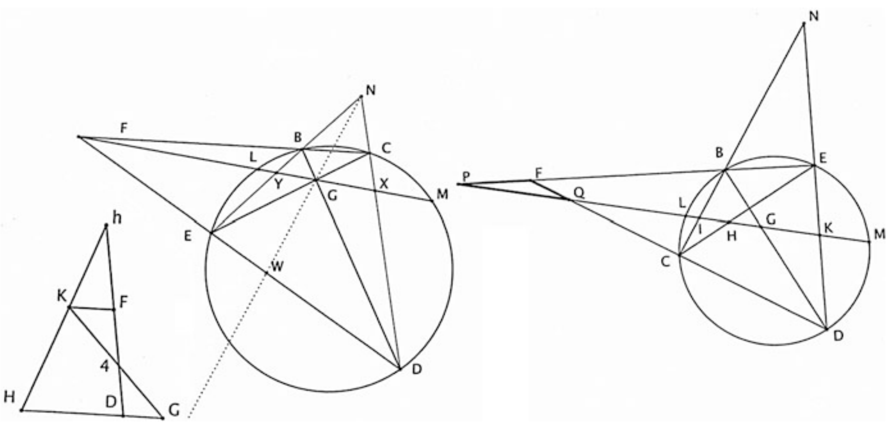

Fig. 4 Left: Menelaus' Theorem, based on La Hire's *Fig.* 10 for Desargues 1639 (Taton 1951a, p. 126). Center: NG is polar of F. Right: based on La Hire's *Fig.* 14 for Desargues 1639 (Taton 1951a, p. 142)

the numerator and denominator, this fraction equals $\frac{GD}{FC}$. Likewise $\frac{AF}{AC} = \frac{AG}{AD}$ gives the equal fraction $\frac{GF}{CD}$. Thus,

$$\frac{GD \cdot GF}{FC \cdot CD} = \frac{AD}{AC} \cdot \frac{AG}{AD} = \frac{AG}{AC}. \text{ In a similar way, } \frac{GB \cdot GH}{CB \cdot CH} = \frac{AG}{AC}.$$

(One can retrace steps in the proof to show the converse, that $\frac{GB \cdot GH}{CB \cdot CH} = \frac{GD \cdot GF}{CD \cdot CF}$ implies that the pairs $B, H;\ C, G;\ D, F$ are in involution with a *souche* A.)

What for La Hire was a "line divided harmonically," is found in Desargues's work as *four points in involution*: When H, B; F, D; C, G are three pairs in involution, and points D and F coalesce into point F, and points B and H coalesce into point H, then pairs H, F; C, G are *four points in involution* . Since

$$\frac{GH}{CH} = \frac{GF}{CF},$$

we have a harmonic set with $H(GC, HF)$ (Field and Gray 1987, pp. 74–84).

After the initial material on involution and an introduction to conic sections as sections of a *roll* or conic surface (*rouleau*) Desargues introduced the *traversale*, a line, p, where for a given figure—which need not be a conic—and a point, F, every line on F that meets the figure in points B and C must meet p in point O so $H(FO, BC)$. So the point and *traversale* are pole and polar.

We consider Fig. 4 Center, with quadrilateral $BCDE$ inscribed in a circle. Initially, only $BCDE$ matters, whose diagonals meet at G and opposite sides at N and F. FG meets opposite sides EB and DC at Y and X, respectively. Desargues showed $H(FG, XY)$. And by projection from N, we get the same result for any line on F, making NG the *traversale* of F with respect to quadrilateral $BCDE$.

And if Desargues did not need the circle, then how did he prove it? He used Menelaus's Theorem, applied four times to $\triangle XYN$ (Field and Gray 1987, p. 115).

Desargues' proved what is now call Menelaus's Theorem, and used it repeatedly, as in his proof that the involution relation among points is preserved in a line-to-line projection (Field and Gray 1987, p. 92). It follows that the harmonic relation would also be preserved. Desargues attributed the theorem to Ptolemy, who had proved a spherical case in his *Almageste* [I.13] (Field and Gray 1987, p. 91). La Hire did not use this theorem.

Here is Menelaus's Theorem: If collinear points D, H, G lie, respectively, on sides (extended) $4h$, hK, and $K4$ of triangle $Kh4$, then

$$\frac{Dh}{D4} = \frac{Hh}{HK} \cdot \frac{GK}{G4}.$$

The proof is by triangle similarity in Fig. 4 Left, where KF is drawn parallel to line HDG.

We return to the proof that $H(FG, XY)$. When $\triangle XYN$ is

$$cut\ by\ line\ BGD,\ then\ \frac{GX}{GY} = \frac{DX \cdot BN}{DN \cdot BY},$$

$$cut\ by\ line\ CGE,\ then\ \frac{GX}{GY} = \frac{CX \cdot EN}{CN \cdot EY},$$

$$cut\ by\ line\ FED,\ then\ \frac{FX}{FY} = \frac{DX \cdot EN}{DN \cdot EY},$$

$$cut\ by\ line\ FBC,\ then\ \frac{FX}{FY} = \frac{CX \cdot BN}{CN \cdot BY}.$$

It follows that

$$(\frac{GX}{GY})^2 = (\frac{FX}{FY})^2,\ \text{so}\ \frac{GX}{GY} = \frac{FX}{FY},\ \text{so}\ H(GF, XY).$$

(We note that La Hire did give essentially this result in Prop 20 of Book 1 of La Hire (1685), his most complete and best known work on conic sections. There we have a complete quadrilateral, like $BNCG$, producing a harmonic set on a line like FD.)

What else did Desargues prove? First, the points of intersection of a line with the six sides of a complete quadrilateral are points in involution. Menelaus's Theorem is applied four times to produce the required equality of cross-ratios (Field and Gray 1987, p. 107).

Next is Desargues's Involution Theorem. Now the circle is involved. La Hire has nothing like this. In fact, it would not be seen again until the middle of the nineteenth century. In proving the Involution Theorem, Desargues first proved it for a circle.

Theorem 3 *Let $BCDE$ be inscribed in a conic, where BE meets CD at F. Let a line meet the conic at L and M, meet CD at Q, BE at P, CB at I, ED at K, BD at G, and CE at H. Then pairs Q, P; L, M; I, K; H, G are points in involution. [See Figure 4 Right.] (Field and Gray 1987, p. 108).*

Proof By the Euclid Book 3 Prop 36 (See Fig. 3 Right),

$$PL \cdot PM = PB \cdot PE,\ QL \cdot QM = QC \cdot QD,\ FB \cdot FE = FC \cdot FD.$$

With, additionally, four applications of Menelaus's Theorem to $\triangle PQF$, with lines EHC, BGD, BIC, and EKD, we reach both

$$\frac{GQ \cdot HQ}{GP \cdot HP} = \frac{QL \cdot QM}{PL \cdot PM}\ \text{and}\ \frac{IQ \cdot KQ}{IP \cdot KP} = \frac{QL \cdot QM}{PL \cdot PM}.$$

This means that $L, M;\ P, Q;\ G, H$ are points in involution, as are $L, M;\ P, Q;\ I, K$. We conclude that $L, M;\ Q, P;\ G, H;\ I, K$ are in involution, for both involution relations would involve the same *souche*. (Field and Gray 1987, pp. 108–110).

Desargues then considered the *traversale* when the figure in question is a conic. He simply observed that in the three dimensional cone, when points in involution on a line which cuts the circle in the base plane are joined by lines to the vertex of the cone, then those lines are concurrent at the vertex and so they meet the corresponding line in the plane of the section in, also, points in involution (Field and Gray 1987, p. 110). This, a generation before La Hire, was the revolutionary claim that those properties of a circle which are preserved under projection must hold for conic sections.

Only at this point did Desargues handle the question of tangents to a conic. When F is the *pole*, or *but*, corresponding to a certain *polar*, or *traversale*, the tangents from F to the conic section meet it at the points on the *traversale*. The argument is a quick one: as a line on F moves across the conic section, the two points at which that line meets the conic coalesce into a single point, which must necessarily be a point of tangency, and must be a point where the *traversale* meets the conic. Where La Hire began with tangents, for Desargues they were close to an afterthought.

7 Was La Hire Desargues' *Schüler*?

The similarity in approach of Desargues and La Hire apparently aroused suspicion by some contemporaries. We can read this concern in La Hire's letter (Poudre 1864, p. 231) accompanying his 1679 transcription of *Brouillon project*. La Hire wrote, with typical politeness, that if he had known the work of Desargues then he would not have discovered the method used in 1673, for he would not have believed it possible to improve on something so simple and general. La Hire defended his independence, pointing out that Desargues had used long compositions of ratios which he, La Hire, had not, and for this reason it would not be wrong to judge his work superior. Further, Apollonius had already employed the harmonic division of lines associated with conics, and both he, La Hire, and Desargues would have learned from this same teacher.

We have some information, from G. W. Leibniz, about general awareness in the 1670s of Desargues' work and its possible influence on La Hire's 1673 (La Hire 1673). Henry Oldenburg wrote to Leibniz, in April 1673, about a 1670 criticism of Desargues by Grégoire Huret. It is clear that both Huret and Oldenburg had only second-hand knowledge of Desargues' work, but that they had some sense of the projective character of the work: "Mr. [John] Collins feels that if we correctly gauge the mind and aim of the author, then the doctrine [of Desargues] merits praise rather than condemnation; the plan of this work was to treat the conic sections as projections of small circles which lie on the surface of a sphere... ." (Leibniz 1884, in Latin, p. 40–41). In letters from the years 1674 or 1676, Leibniz noted with approval Desargues' universal treatment of the conics as one genre, where line constructions to resolve problems apply to all the conics, and that Desargues treated parallel lines as concurrent. Later (Le Goff 1994, p. 189) Oldenburg told Leibniz that he was unable to procure a copy of Desargues' work.

In a letter to Etienne Périer, nephew of Blaise Pascal, Leibniz wrote, in a crossed out section, (Echeverría 1994, August 1676, p. 287) "not long ago there appeared a new *Methods des sections Coniques* whose author was a friend of Monsieur de Boss and *disciple* of Monsieur des Argues (who was a great friend of Mons. Pascal) and spoke also of the properties of lines cut harmonically and of their application to conics in a manner strongly approaching that of [Desargues]." This is evidence that La Hire's 1673 work was known of in mathematical circles, and, despite the disappearance of Desargues' work, the similarity of La Hire's work to that of Desargues was already commented on.

When La Hire's transcription of the *Brouillon Projet* was published by Poudra in 1864, the conic sections were part of a projective geometry that had been transformed by Poncelet and others. Particularly in this light, La Hire showed not the boldness, not the inventiveness, not the depth of understanding that could be read—with some effort—in the work of Desargues. La Hire surely had some exposure to the ideas of Desargues, for he had worked with Bosse, his father's friend, and Bosse had worked with Desargues. How independent would he have been in his 1673 work?

In short commentary, without any detailed examination, scholars Charles Taylor and Ernst Lehmann, in passages recognizing the achievement of La Hire, referred to him, respectively, as a "disciple" (Taylor 1881, p. lxiv) and a "Schüler" (Lehmann 1888, p. 1) of Desargues.

The issue returned about 1950 when when an original copy of *Brouillon Projet* was found in the French Bibliothèeque Nationale. Soon after, René Taton produced a study of the mathematical work of Desargues Taton (1951a), and several related articles.

While praising La Hire's clarity of presentation, he wrote of La Hire's treatises of 1673 and 1685 (Taton 1951b, p. 16) "their originality seems very contestable despite the contrary affirmation of Ph. de La Hire himself." And in the *Dictionary of Scientific Biography*, "The 'Nouvelle méthode' clearly displayed Desargues' influence, even though La Hire, in a note written in 1679 . . ., affirmed that he did not become aware of the latter's work until after publication of his own. Yet what we know about La Hire's training seems to contradict this assertion. Furthermore, the resemblance of their projective descriptions is too obvious for La Hire's not to appear to have been an adaption of Desargues'." Taton (1970).

Le Goff (1994, p. 204) wrote, in 1994, that La Hire "seems to revert to the orthodoxy of the ancients, with his 'line cut harmonically,' which only retains involution in the case of harmonic division." But after noting this difference with Desargues, he continued, "It is probable that Abraham Bosse, if not Laurent de La Hire, would have had a copy of Brouillon in their library. It is hard to believe the words of Phillipe de la Hire, according to which he was only acquainted with this text in 1679, when he transcribed a copy."

Our comparison of the development of the pole-polar concept at the hands of Desargues and La Hire shows, at least in this case, that Desargues's work had minimal influence on La Hire. La Hire, as had Apollonius, began with tangents to the conic, by which the polar was characterized, and by which the harmonic division of a secant was shown; Desargues defined the polar by the harmonic division of a secant, and his concept of harmonic division began with involution — not found in La Hire. Further comparison of the *Brouillon Projet* with La Hire's 1673 work finds essentially nothing in common, in either detail or broad plan. Desargues used Menelaus's Theorem repeatedly; La Hire never did. La Hire did not treat parallel lines as concurrent, adding much to the length of his work. La Hire is detailed in showing myriad resulting properties of the three conics, each treated separately, in the tradition of those, like Mydorge, who wished to simplify what Apollonius had done. Desargues, on the other hand, aimed to unify the treatment of the conics in developing common properties, as he wrote to Mersenne in a letter of 1638 (Taton 1951a, p. 83, 84). With the topic of the foci, La Hire in 1673 offered little that was not already covered by Apollonius; as interpreted by Hogendijk (1991), Desargues, in some of the most puzzling parts of his work, was revamping in a completely new, projective, way the place of foci.

If we look at La Hire's 1685 work (La Hire 1685), we see specific borrowing from Desargues. Most prominent is the harmonic pencil of concurrent lines, which Desargues spoke of as *rameau correspondants entr'eaux*, branches (lines) corresponding among themselves (Field and Gray 1987, p. 96). This is a set of four concurrent lines where any other line intersecting the four does so in a harmonic set. La Hire did not have this concept in 1673, but he did in 1685 (La Hire 1673, Book 1 Prop XI), calling such concurrent lines *harmonicales*, and he made crucial use of the concept in, for example, his Book 8, on foci. One does not find any comparable borrowing in 1673.

On the one hand, yes, many mathematicians in the 1670s knew that Desargues treated parallel lines as concurrent, at a point at infinity, and they had some general idea that he used projections in work on conic sections. And La Hire, even in relative isolation, could not have avoided this. But those, like Taton, who question the originality and honesty of La Hire, base their opinion primarily on the availability of Desargues' 1639 work to La Hire. We see that Le Goff, in 1994, could point to the significant difference in goals and style of Desargues and La Hire while at the same time claiming that La Hire could not have been independent since people he knew were likely to have had a copy of Desargues' 1639 booklet. Zbynek Sir, in 2002, came to another conclusion: that the only influence of Desargues would be in the *méthode projective spaciale*, but of the other methods attributed to La Hire, "nous ne trouvons pas de trace chez Desargues." (Zbynek 2002, p. 219).

On the other hand, notions of projective methods were in the air in the seventeenth century. Apollonius, once he developed the abscissa-ordinate equations of the conic sections, did not return to the three-dimensional cone except for a few problems, as at the end of Book 1, to describe the cone from which a given conic section could originate. Remember that there was no science of perspective when Apollonius wrote. However, Mydorge (1631/1639, Book 1 Prop 36) and Werner

(1522, Prop 15, 16) did resort in places to the three-dimensional context of a cone. They brought in a tangent plane to a cone, in which the tangent line to a conic section lies. Mydorge (1631/1639, p. 14) has a diagram of a cone sliced in an ellipse identical to that 1696 Apollonius diagram in Fig. 1 Left. How would an artist, conscious of techniques of perspective drawing, not see the circle and ellipse as projections of each other from the vertex? Recall that Desargues wrote a pamphlet on perspective drawing and La Hire was himself an artist.

It is unlikely that we will know anything definitive on the degree of influence of Desargues on La Hire's 1673 geometry. But we can conclude that the details have the character of an independent work.

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