Chapter 10
Representativeness

Abstract A referendum paradox occurs when a collective decision by a majority in a representative body contradicts the majority opinion in the electorate at large. We discuss this paradox as an introduction to the problems of constructing optimally representative committees. Two important studies are reviewed and the notion of a Condorcet committee introduced. We also deal with power distribution in committees *vis-à-vis* the electorate at large.

10.1 The Referendum Paradox

Since direct democracy is for several reasons impossible in contemporary political systems, some principle of representation has to be resorted in those systems that call themselves democratic. Rightly or wrongly, most current systems of governance of political entities declare themselves democratic. Yet, a wide variety of principles are being used in composing the representative bodies making decisions on behalf and in the name of the populations at large. In business environments, the highest decision making bodies are typically not envisaged to be democratic in the same sense as in political contexts, but quite often we encounter the representation problem when various working groups or task forces are being set up. The population or electorate in those contexts typically consists of boards or councils or plenary assemblies. In what follows we draw upon some results from political science to shed light to the process of rational composition of representative bodies. We start with the description of a puzzling – but at the same time quite understandable – phenomenon sometimes called the referendum paradox.

There are 10 single-member districts, each with 100 voters. The districts are not necessarily geographical entities, but may constitute partitioning of the electorate on other – age, occupation, ethnicity, income, stock share – grounds. In any case, it is here assumed that each district, however defined, sends one representative to the task force, parliament or working group to be elected. The representative body is then expected to vote on a dichotomous issue, i.e. one that results either in a 'yes' or 'no' stand depending on which position has more votes in the body. An illustrative distribution of voters over positions and districts is exhibited in Table 10.1.

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Table 10.1 Referendum paradox

Opinion	District 1	…	District 9	District 10	Total
Yes	45	…	45	100	505
No	55	…	55	0	495

The paradox consists of the fact that direct and representative majority voting lead to opposite outcomes with a clear margin. To wit, if we assume that each representative votes according to what s/he believes (correctly) to be the opinion of the majority of voters in his district, the outcome in the representative body is 9 to 1 in favour of 'no', but if the voters vote directly on the issue, the winner is 'yes' with a 505 to 495 margin. The observers of the U.S. presidential elections are, of course, familiar with this paradox in a slightly different disguise.

The referendum paradox is just another way of saying that moving from direct to indirect (representative) decision making comes with a price. Even if the representatives aim at faithfully representing their electors by reflecting the majority opinion of the latter, the outcome in the representative body may contradict the view of the majority of the electorate. This possibility is unavoidable as long as the simple majority principle is being utilized. It becomes, however, less likely in homogeneous electorates. There is another reason for not being overly worried about the referendum paradox, viz. it deals with a single issue, while one could perhaps expect that in the long run, i.e. with a long sequence of issues, the majorities in the representative body and in the electorate at large are more likely to coincide on the average. This is, course, a conjecture.

Another conjecture suggests that when choosing representatives the voters are not primarily interested in having their own views reflected in the final outcomes, since most of the time they simply do not have them. One could argue that in modern democracies very few if any voters have an opinion on every issue on the legislative agenda of the parliament. The same presumably holds for many other representative bodies. Hence, the voter's main interest is in the qualities – personal or professional – of the candidates competing for representative positions.

10.2 Optimal Committees

A pioneering paper on optimal committees from the social choice perspective is the article of Chamberlin and Courant (1983). It starts from assuming that each individual has a preference ranking over the candidates competing for available committee (working group, council, board) membership. The optimality of the ensuing committee is defined in terms of these preferences. The second crucial assumption is that for each possible committee and each voter, the latter is represented in the former by one committee member, viz. that member which has the highest position in the voter's ranking. Thus, for example, if the committee consists of three members and voter i ranks them in third, fifth and tenth position, then i is represented by the member whom s/he ranks third. This highest ranked member will be called i's representative in the committee. Now, the desirability of any given committee to any voter, is defined in a natural way: the higher the representative is positioned in the voter's ranking, the better s/he is represented. Or, stated in another way, the degree of misrepresentation of the voter is the smaller, the higher his/her representative in this committee is ranked. An obvious way of measuring the degree of misrepresentation of a committee for a voter is count the number of candidates that are ranked higher than the voter's representative in the committee. E.g. if the voter's representative is ranked first by him/her, the degree of misrepresentation of this committee is 0 for this voter, if his/her representative is ranked third, the degree of misrepresentation is 2 and so on. The determination of the best or optimal committee in this sense is basically straight-forward: one generates all possible committees and sums up the voters' degrees of misrepresentation for each committee. The committee with the smallest sum is the optimal one. This way of computing optimal committees is linearly related to the Borda scores of each committee member.

The procedure outlined above differs from extant methods of proportional representation, but captures important features that underlie the concept of proportionality. At the same time it is subject to the criticism that in the optimal committee the members may represent voters in very different ways since the average Borda score sum of the members allows for a wide variation in the scores of individual members. Stated in a another way, the optimal committees in the Chamberlin-Courant procedure typically consists of members with widely varying constituencies or support sizes. The constituency $N_j(c)$ of a candidate j in a committee c is defined as the number of voters who rank j higher than any other candidate in c. That is,

$$N_j(c) = |\{i \in N | r_{ij} \leq r_{is}, \forall s \in c\}|$$

Here N is the set of voters and r_{ij} denotes the rank assigned to member j by voter i.

To see how the Chamberlin-Courant procedure works, consider the following example involving nine voters, five candidates and the committee of three members. The voter preferences over the candidates are presented in Table 10.2. There are 10 possible ways of selecting a 3-member committee out of the five candidates. One of them is ABC. Its associated degree of misrepresentation is computed as follows: there are three voters each of whom assigns A the degree of misrepresentation of 4 and two voters assigning A the degree 3, while four voters assign A a zero degree of misrepresentation. Similarly, B is assigned 1 degree by four voters and 2 degrees by two voters. C, in turn, is associated with the degree 2 by seven voters. Hence the degree of misrepresentation associated with the committee ABC is 40. The degrees of the other nine committees can be computed in the same manner. It turns out that the committee BCD has the smallest degree of misrepresentation, viz. 39.

The example shows that the constituencies of the committee members in the optimal committee are far from being equal-sized: seven voters are associated with

Table 10.2 The Chamberlin-Courant procedure illustrated

4 voters	3 voters	2 voters
A	B	C
B	D	D
C	C	B
D	E	A
E	A	E

B, two with C and none with D. On the other hand, the candidate ranked first by more voters than any other candidate, viz. A, is not present in the optimal committee. It is difficult to say which is more counterintuitive: either an optimal committee where the candidate ranked first by more voters than any other candidate is not present or an optimal committee that includes a member with an empty constituency. Both these counterintuitive features are results of the fact that in defining the optimal committee Chamberlin and Courant resort to the Borda count, while in defining the assignment of voters to various committee members, another positional criterion is being applied.

Viewed from the representation point of view the Chamberlin-Courant method has a flaw that is related to what was just said, viz. it does not guarantee that each representative stands for equally many voters. To rectify this, Monroe devised a technique that guarantees that each representative represents the same number of voters Monroe (1995).

The fundamental idea is to divide the electorate of size n into segments of equal size, that is, each segment consists of n/k voters where k is the size of the committee. Here each segment represents the same number of voters. The way to achieve a fully proportional representation in Monroe's sense is to start from the voters' preference profile over all candidates and to construct every possible k-member committee where each voter is assigned to the member that minimizes his/her misrepresentation under the constraint that the number of voters assigned to each member is the same, viz. n/k. For a given committee this is done by first assigning every voter to the candidate that best represents him/her. Suppose that voter i assigns candidate j the rank r_{ij}. Monroe suggests the misrepresentation measure of Chamberlin and Courant, viz. $\mu_{ij} = r_{ij} - 1$. In other words, the misrepresentation related to a candidate is simply his/her rank minus unity. Thus, for any given committee one first assigns each voter to the committee member for whom his/her misrepresentation is smallest (ties are broken randomly). Since this does not in general lead to a uniform distribution of voters over committee members, one proceeds by transferring voters from one candidate to another to achieve a situation where each member represents n/k voters. The criterion for transfer is the following: of any two voters, say i and l, the one that suffers less from the transfer is re-assigned. This means that those voters who are indifferent or nearly indifferent between the candidates they are assigned to before and after re-assignment are transferred first.

Once the transfers required to make each constituency of equal size have been performed, the degree of misrepresentation associated with the committee is computed as the sum of the individual misrepresentations of the voters. The committee with the smallest value of misrepresentation is the fully proportional committee. This description of the construction process is basically one that was presented in Monroe's article. For large candidate sets it is very tedious. Fortunately, Potthoff and Brams (1998) have devised an integer programming algorithm for computing fully proportional committees (Brams 2008).

Let there be m candidates. Define the value of the variable x_j to be 1 if candidate j is a winner (i.e. belongs to the fully proportional committee) and $x_j = 0$, otherwise. The sum over all candidates of x_j's thus indicates the number of candidates in the committee. Now define $x_{ij} = 1$, if candidate i is assigned to voter j, and $x_{ij} = 0$, otherwise. For a fixed value of the variable i the sum over j indicates the number of voters associated with candidate i. The objective function to be minimized is the sum of misrepresentation values:

$$z = \sum_i \sum_j \mu_{ij}$$

The constraints under which the minimization is to be done are

$$\sum_i x_i = k$$

$$\sum_i x_{ij} = 1, \text{ for each } \quad j = 1, \ldots, n$$

$$-Lx_i + \sum_j x_{ij} \geq 0, \text{ for each } \quad i = 1, \ldots, m$$

$$-Ux_i + \sum jx_{ij} \leq 0, \text{ for each } \quad i = 1, \ldots, m$$

$$x_i \text{ is an integer less than or equal to 1, for each } \quad i$$

$$x_{ij} \text{ is an integer less than or equal to 1, for each } \quad i \quad \text{and } j$$

Here L is the lower bound of the number of voters assigned to each candidate, while U is the upper bound of this number. Since the constituencies in fully proportional committees are of equal size, L is equal to the largest integer smaller than n/k, while U is the smallest integer larger than n/k. Should n/k be an integer the constraints involving L and U can be replaced by a single constraint:

$$-\frac{n}{k}x_i + \sum_j x_{ij} = 0, \text{ for each } \ i$$

Monroe's idea of full proportionality and its precise formulation and solution through integer programming as suggested by Brams and Potthoff allow for various kinds of extensions. To wit, the measure of misrepresentation, μ_{ij}, can be defined in ways that differ from the one described above. Moreover, the voters can be assigned to several candidates instead of just one. Some of these variations can be easily accommodated by the integer programming approach.

10.3 Topics for Further Reflection

1. Is the referendum paradox possible in the presidential election system of the United States? If it is, can you suggest a way of avoiding it? If it isn't, explain why.
2. The Chamberlin-Courant procedure for determining optimal committees uses all rank positions to determine the degree of misrepresentation associated with a candidate. Suppose a different count were used; to wit, suppose all first ranks were associated with 0 degree of misrepresentation while all lower positions had the same misrepresentation value, viz. 1. Compute the optimal 3-member committee in the Table 10.2 profile using this alternative count.
3. A strong Condorcet committee is a committee that would defeat all other committees in pairwise majority voting contests, i.e. in each comparison this committee would be supported by more voters than its opponent. Suppose that the voter preferences over committees are induced by the respective misrepresentation values: the smaller the value the more preferable the committee. Focusing on three-member committees in Table 10.2, is the optimal BCD committee a Condorcet one?

10.4 Suggestions for Reading

The determination of the nearly optimal and/or nearly fully proportional committees has been shown to be computationally tractable by Skowron et al. (2015). Similar computational results pertaining to Condorcet committees – i.e. committees that are undefeated by any others in pairwise comparisons with a majority of individual votes – are presented by Darmann (2013). Methods based on dichotomous preferences and approval voting are discussed by Brams et al. (2006) and Kilgour (2010).

Answers to Selected Problems

1. It is possible and has, in fact, occurred a few times, most recently in the 2016 election where Donald J. Trump won, but his competitor Hillary Clinton received more popular votes.

2. The degree of misrepresentation for committee ABC is $5 + 6 + 7 = 18$. Any two-member committee that includes D or E can be improved by replacing this member by A, B or C thereby decreasing the degree of misrepresentation by 4, 6 or 7. Similarly, any one-member committee that includes D or E can be improved by replacing these by A, B or C. ABC cannot be improved upon by replacing any member with either D or E. Thus it is the optimal one under the modified count.
3. Yes, it is. For example, when compared with the committee ABC, each of the first four voters assigns ABC the misrepresentation value $1 + 2 = 3$, each of the next three voters assign ABC the misrepresentation value $4 + 2 = 6$ and each of the last two voters assigns ABC the misrepresentation value $3 + 2 = 5$. The corresponding values for BCD are $1 + 2 + 3 = 6$, $2 + 1 = 3$ and $2 + 1 = 3$. So, while four voters deem ABC preferable to BCD, five others have the opposite preference. Hence, BCD beats ABC by a majority. By studying all 10 committees of three, it turns out that BCD defeats all others and is thus the strong Condorcet committee.

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