Synthesis of Structurally Restricted b-bounded Petri Nets: Complexity Results

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Abstract. Let $b \in \mathbb{N}^+$. A b-bounded Petri net (b-net) *solves* a transition system (TS) if its reachability graph and the TS are isomorphic. Synthesis (of b-nets) is the problem of finding for a TS A a b-net N that solves it. This paper investigates the computational complexity of synthesis, where the searched net is structurally restricted in advance. The restrictions relate to the cardinality of the preset and the postset of N's transitions and places. For example, N is *choice-free* (CF) if the postset-cardinality of its places do not exceed one. If additionally the preset-cardinality of N's transitions is at most one then it is fork-attribution. This paper shows that deciding if A is solvable by a pure or test-free b-net N which is *choice-free*, *fork-attribution*, *free-choice*, *extended free-choice* or *asymmetric-choice*, respectively, is NP-complete. Moreover, we show that deciding if A is solvable by a b-bounded weighted (m, n)-*T-systems*, $m, n \in \mathbb{N}$, is NP-complete if m, n belong to the input. On the contrary, synthesis for this class becomes tractable if $m, n \in \mathbb{N}$ are chosen *a priori*. We contrast this result with the fact that synthesis for weighted (m, n)-*S-systems*, being the T-systems's dual class, is NP-complete for any fixed $m, n \geq 2$.

1 Introduction

Examining the behaviour of a system and deducing its behavioral properties is the task of system *analyses*. Its counterpart, *synthesis*, is the task to find for a given behavioral specification an implementing system. A valid synthesis procedure computes systems which are correct by design. However, the chances for obtaining an (efficient) algorithm for both analyses and synthesis, depend drastically on the given specification and the searched system: In [8] it has been shown that deciding liveness (the behavioral property) is EXPSPACE-hard for bounded Petri nets (the system), while it is NP-complete for free-choice Petri nets and polynomial for 1-safe free-choice nets. Similarly, the reachability problem is EXPSPACE-hard for bounded Petri nets, PSPACE-complete for free-choice 1-safe nets, NP-complete for acyclic 1-safe and conflict-free nets and polynomial for 1-safe conflict-free nets [8,10].

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In [12] it has been shown that it is impossible to decide if a *modal* transition system (the specification) can be implemented by a bounded Petri net, while synthesis of bounded Petri nets can be done in polynomial time if the specification is a transition system (TS, for short) [1]. An even better procedure for synthesis from TS is possible if the searched bounded Petri net is to be choice-free or a marked graph [4,7]. Moreover, restricting the searched system to b-bounded Petri nets makes synthesis from modal TSs decidable for every fixed integer b [13].

In this paper, we investigate the following instance of synthesis: The specification is a TS A and the searched system is a b-bounded Petri net N (b-net, for short). We demand that N implements A up to isomorphism, that is, N's reachability graph and A are isomorphic. Recently, in [15] we have shown that deciding the existence of N is NP-complete for every fixed $b \geq 1$. However, the former examples provide several results where restricting the system makes the corresponding analyses and synthesis problems easier. Encouraged by these results, we continue our work of [15] in this paper and address whether structurally restricting a searched b-net N influences positively the computational complexity of synthesis. The restrictions relate to the preset- and postset-cardinality of N's transitions and places and correspond to well-known subclasses of Petri nets [3,6,9,14]. Surprisingly, it turns out that almost all applied net restrictions do not bring the synthesis down to polynomial time. More exactly, we show that synthesis remains intractable if N is pure or test-free and satisfies one of the following properties: *choice-free* [6,14], *fork-attribution* [14], *free-choice*, *extended free-choice* or *asymmetric-choice* [3]. Moreover, we adapt the classes of *(weighted) T-systems* and *(weighted) marked graphs* [9] for b-nets and introduce for $m, n \in \mathbb{N}$ their extension of *weighted* (m, n)*-T-systems* restricting the cardinality of the preset and the postset of N's places by m and n, respectively. We show that synthesis of weighted (m, n)-T-systems is hard if m, n are part of the input and becomes tractable for every fixed m, n. In particular, synthesis of b-bounded weighted T-systems is polynomial which answers partly a question from [5, p.144]. Furthermore, we introduce their dual class of *weighted* (m, n)*-S-systems* which restricts the cardinality of the preset and postset of N's transitions by m and n, respectively. In contrast to the result of its dual class, deciding if A is implementable by a pure or test-free b-net, being a weighted (m, n)-S-system, is NP-complete for every fixed $m, n \geq 2$. We get all intractability results by a reduction of the cubic monotone one-in-three-3-sat-problem and partly apply our methods from [15]. However, the reductions here are extremely specialized and tailored to synthesis of restricted nets.

The next Sect. 2 introduces all necessary preliminary notions, Sect. 3 presents our main result and Sect. 4 closes the paper.

2 Preliminaries

This section introduces all necessary preliminary notions and Fig. 1 gives corresponding examples. In the remainder of this paper, if not stated explicitly otherwise then $b \in \mathbb{N}^+$ is assumed to be arbitrary but fixed.

Transition Systems. An *initialized transition system* (TS, for short) $A = (S, E, \delta, s_0)$ consists of a finite disjoint set S of states, E of events, a partial *transition function* $\delta : S \times E \to S$ and an *initial state* $s_0 \in S$. A can be interpreted as edge-labeled directed graph where every triple $\delta(s, e) = s'$ is an e-labeled edge $s \xrightarrow{e} s'$, called *transition*. An event e *occurs* at state s, denoted by $s \xrightarrow{e}$, if $\delta(s, e) = s'$ for some state s'. This notation is extended to words $w' = wa$, $w \in E^*, a \in E$, by inductively defining $s \xrightarrow{\varepsilon} s$ for all $s \in S$ and $s \xrightarrow{w'} s''$ if and only if there is a state $s' \in S$ satisfying $s \xrightarrow{w} s'$ and $s' \xrightarrow{a} s''$. If $w \in E^*$ then $s \xrightarrow{w}$ denotes that there is a state $s' \in S$ such that $s \xrightarrow{w} s'$. We assume all TSs to be *reachable*: $\forall s \in S, \exists w \in E^* : s_0 \xrightarrow{w} s$.

***b*-bounded Petri Nets.** A *b-bounded Petri net* (b-net, for short) $N = (P, T, f, M_0)$ consists of finite and disjoint sets of *places* P and *transitions* T, a (total) *flow function* $f : P \times T \to \{0, \dots, b\}^2$ and an *initial marking* $M_0 : P \to \{0, \dots, b\}$. If $f(p, t) = (m, n)$ then $f^-(p, t) = m$ defines the *consuming effect* of t on p. Similarly, $f^+(p, t) = n$ defines t's *producing effect* on p. The *preset* of a place p is defined by ${}^\bullet p = \{t \in T \mid f^+(p, t) > 0\}$, the set of transitions that produce on p. Accordingly, p's *postset* is defined by $p^\bullet = \{t \in T \mid f^-(p, t) > 0\}$ and contains the transitions that consume from p. Similarly, the *preset* ${}^\bullet t = \{p \in P \mid f^-(p, t) > 0\}$ of a transition t is defined by the places from which t consumes and its *postset* $t^\bullet = \{p \in P \mid f^+(p, t) > 0\}$ by the places on which t produces. Notice that neither ${}^\bullet p \cap p^\bullet$ nor ${}^\bullet t \cap t^\bullet$ is necessarily empty. A transition $t \in T$ can *fire* or *occur* in a marking $M : P \to \{0, \dots, b\}$, denoted by $M \xrightarrow{t}$, if $M(p) \geq f^-(p, t)$ and $M(p) - f^-(p, t) + f^+(p, t) \leq b$ for all places $p \in P$. The firing of t in marking M leads to the marking $M'(p) = M(p) - f^-(p, t) + f^+(p, t)$ for $p \in P$, denoted by $M \xrightarrow{t} M'$. Again, this notation extends to sequences $\sigma \in T^*$ and the *reachability set* $RS(N) = \{M \mid \exists \sigma \in T^* : M_0 \xrightarrow{\sigma} M\}$ contains all of N's reachable markings. The firing rule preserves the *b-boundedness* of N by definition: $M(p) \leq b$ for all places p and all $M \in RS(N)$. The *reachability graph* of N is the TS $A_N = (RS(N), T, \delta, M_0)$, where for every reachable marking M of N and transition $t \in T$ with $M \xrightarrow{t} M'$ the transition function δ of A_N is defined by $\delta(M, t) = M'$.

Structurally Restricted Subclasses of *b*-nets. A b-net N is *pure* if $\forall (p, t) \in P \times T : f^-(p, t) = 0$ or $f^+(p, t) = 0$, that is, $\forall p \in P : {}^\bullet p \cap p^\bullet = \emptyset$; *test-free* if $\forall (p, t) \in P \times T : f(p, t) \neq (0, 0) \Rightarrow f^-(p, t) \neq f^+(p, t)$; *choice-free* (CF) or *place-output-nonbranching* if $\forall p \in P : |p^\bullet| \leq 1$; *fork-attribution* (FA) if it is CF and, additionally, $\forall t \in T : |{}^\bullet t| \leq 1$; *free-choice* (FC) if $\forall p, \tilde{p} \in P : p^\bullet \cap \tilde{p}^\bullet \neq \emptyset \Rightarrow |p^\bullet| = |\tilde{p}^\bullet| = 1$; *extended-free-choice* (EFC) if $\forall p, \tilde{p} \in P : p^\bullet \cap \tilde{p}^\bullet \neq \emptyset \Rightarrow p^\bullet = \tilde{p}^\bullet$; *asymmetric-choice* (AC) if $\forall p, \tilde{p} \in P : p^\bullet \cap \tilde{p}^\bullet \neq \emptyset \Rightarrow (p^\bullet \subseteq \tilde{p}^\bullet$ or $\tilde{p}^\bullet \subseteq p^\bullet)$; for $m, n \in \mathbb{N}$ a *weighted (m, n)-T-system* if $\forall p \in P : |{}^\bullet p| \leq m, |p^\bullet| \leq n$; for $m, n \in \mathbb{N}$ a *weighted (m, n)-S-system* if $\forall t \in T : |{}^\bullet t| \leq m, |t^\bullet| \leq n$.

***b*-bounded Regions.** For the purpose of finding a b-net N implementing a TS A, we want to synthesize N's components purely from the input A. Demanding A

$$s_0 \xrightarrow{k} s_1 \xrightarrow{k} s_2 \xrightarrow{z} s_3 \xrightarrow{o} s_4 \xrightarrow{k} s_5 \xrightarrow{k} s_6$$

sup	s_0	s_1	s_2	s_3	s_4	s_5	s_6	sig	k	z	o
sup_1	0	1	2	2	0	1	2	sig_1	$(0,1)$	$(0,0)$	$(2,0)$
sup_2	2	1	0	2	2	1	0	sig_2	$(1,0)$	$(0,2)$	$(0,0)$
sup_3	2	2	2	0	2	2	2	sig_3	$(0,0)$	$(2,0)$	$(0,2)$
sup_4	0	0	0	2	2	2	2	sig_4	$(0,0)$	$(0,2)$	$(0,0)$
sup_5	0	1	2	1	0	1	2	sig_5	$(0,1)$	$(1,0)$	$(1,0)$

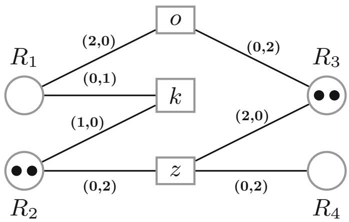

Fig. 1. Top: Input TS A. Middle: For $i \in \{1,2,3,4,5\}$ pure 2-regions $R_i = (sup_i, sig_i)$ of A, where $R_1, \dots, R_4$ already solve all of A's (E)SSP atoms. For example, the region R_1 solves (k, s_i), $\forall i \in \{2,3,6\}$ and (o, s_i), $\forall i \in \{0,1,4,5\}$. Bottom: Pure 2-net $N_A^{\mathcal{R}}$, built by $\mathcal{R} = \{R_1, R_2, R_3, R_4\}$, where, for example, ${}^\bullet R_3 = \{o\}$ and $R_3^\bullet = \{z\}$ and ${}^\bullet o = \{R_1\}$ and $o^\bullet = \{R_3\}$. Moreover, $N_A^{\mathcal{R}}$ is FA because of $|R^\bullet| \leq 1$ and $|{}^\bullet e_{\mathcal{R}}| \leq 1$ for all $R \in \mathcal{R}$ and $e \in E(A)$. The net $N_A^{\mathcal{R}}$ origins from $N_A^{\mathcal{R}'}$, where $\mathcal{R}' = \mathcal{R} \cup \{\ R_5\ \}$, by removing R_5. Both $\mathcal{R}$ and $\mathcal{R}'$ are b-admissible sets. Thus, the reachability graphs of their synthesized nets are both isomorphic to A. However, because $z \in R_3^\bullet \cap R_5^\bullet$ and $R_5^\bullet = \{z, o\}$, the net $N_A^{\mathcal{R}'}$ is not even free-choice.

and A_N to be isomorphic suggests that A's events correspond to N's transitions. However, the notion of a *place* is not known for TSs. A *b-bounded region* R (b-region, for short) of a TS $A = (S, E, \delta, s_0)$ is a pair $R = (sup, sig)$ of *support* $sup : S \to \{0, \dots, b\}$ and *signature* $sig : E \to \{0, \dots, b\}^2$, where $sig^-(e) = m$ and $sig^+(e) = n$ for $sig(e) = (m, n)$, such that for every edge $s \xrightarrow{e} s'$ of A holds $sup(s) \geq sig^-(e)$ and $sup(s') = sup(s) - sig^-(e) + sig^+(e)$. A region (sup, sig) models a place p and the corresponding part of the flow function f: $sig^+(e)$ models $f^+(e)$, $sig^-(e)$ models $f^-(e)$ and $sup(s)$ models $M(p)$ in the marking $M \in RS(N)$ corresponding to $s \in S(A)$. Accordingly, a region R is *test-free* if $sig(e) \neq (0,0)$ implies $sig^-(e) \neq sig^+(e)$. The *preset* of R is defined by ${}^\bullet R = \{e \in E \mid sig^+(e) > 0\}$ and its *postset* by $R^\bullet = \{e \in E \mid sig^-(e) > 0\}$. The Region R is *pure* if ${}^\bullet R \cap R^\bullet = \emptyset$. For a set $\mathcal{R}$ of b-regions and $e \in E$ we define by ${}^\bullet e_{\mathcal{R}} = \{(sup, sig) \in \mathcal{R} \mid sig^-(e) > 0\}$ the *preset* and by $e^\bullet_{\mathcal{R}} = \{(sup, sig) \in \mathcal{R} \mid sig^+(e) > 0\}$ the *postset* of e (in accordance to $\mathcal{R}$). Every set $\mathcal{R}$ of b-regions of A defines the *synthesized b-net* $N_A^{\mathcal{R}} = (\mathcal{R}, E, f, M_0)$ with flow function $f((sup, sig), e) = sig(e)$ and initial marking $M_0((sup, sig)) = sup(s_0)$ for all $(sup, sig) \in \mathcal{R}, e \in E$. We emphasize once again that a *region* R of $\mathcal{R}$ becomes a *place* of $N_A^{\mathcal{R}}$ with the preset ${}^\bullet R$ and the postset $R^\bullet$. Moreover, every *event* $e \in E$ becomes a *transition* of $N_A^{\mathcal{R}}$ with preset ${}^\bullet e = {}^\bullet e_{\mathcal{R}}$ and postset $e^\bullet = e_{\mathcal{R}}{}^\bullet$. It is well known that $A_{N_A^{\mathcal{R}}}$ and A are isomorphic if and only if $\mathcal{R}$'s regions solve certain separation atoms [2], to be introduced next.

A pair (s, s') of distinct states of A define a *state separation atom* (SSP atom, for short). A b-region $R = (sup, sig)$ *solves* (s, s') if $sup(s) \neq sup(s')$. The meaning of R is to ensure that $N_A^{\mathcal{R}}$ contains at least one place R such that $M(R) \neq M'(R)$ for the markings M and M' corresponding to s and s', respectively. If there is a b-region that solves (s, s') then s and s' are called *b-solvable*. If every SSP atom of A is b-solvable then A has the *b-state separation property* (b-SSP, for short).

A pair (e, s) of event $e \in E$ and state $s \in S$ where e does not occur at s, that is $\neg s \xrightarrow{e}$, define an *event state separation atom* (ESSP atom, for short). A b-region $R = (sup, sig)$ *solves* (e, s) if $sig^-(e) > sup(s)$ or $sup(s) - sig^-(e) + sig^+(e) > b$. The meaning of R is to ensure that there is at least one place R in $N_A^{\mathcal{R}}$ such that $\neg M \xrightarrow{e}$ for the marking M corresponding to s. If there is a b-region that solves (e, s) then e and s are called *b-solvable*. If every ESSP atom of A is b-solvable then A has the *b-event state separation property* (b-ESSP, for short).

A set $\mathcal{R}$ of b-regions of A is called *b-admissible* if for every of A's (E)SSP atoms there is a b-region R in $\mathcal{R}$ that solves it. The following lemma, borrowed from [2, p.163], summarizes the already implied connection between the existence of b-admissible sets of A and (the solvability of) synthesis:

Lemma 1. ([2]). *A b-net N has a reachability graph isomorphic to a given TS A if and only if there is a b-admissible set* $\mathcal{R}$ *of A such that* $N = N_A^{\mathcal{R}}$.

We say a b-net N *solves* A if A_N and A are isomorphic. By Lemma 1, searching for a restricted b-net reduces to finding a b-admissible set of accordingly restricted regions. The following two examples illustrate this fact.

Example 1. If $\mathcal{R}$ is a b-admissible set of pure regions of A satisfying $\forall R \in \mathcal{R}$: $|R^\bullet| \leq 1$ and $\forall e \in E(A) : |{}^\bullet e_{\mathcal{R}}| \leq 1$ then $N_A^{\mathcal{R}}$ is a pure FA b-net solving A.

Example 2. If $\mathcal{R}$ is a b-admissible set of pure regions of A and $\forall e \in E(A)$: $|{}^\bullet e_{\mathcal{R}}| \leq 2, |e_{\mathcal{R}}{}^\bullet| \leq 2$ then $N_A^{\mathcal{R}}$ is a pure solving b-net, being a weighted $(2, 2)$-S-system.

3 Our Contribution

Theorem 1. *For a given TS A the following conditions are true:*

1. *If* $P \in \{CF, FA, FC, EFC, AC\}$ *then to decide if A is solvable by a pure or a test-free b-net which is P is NP-complete.*
2. *Given integers* $\ell, \ell' \in \mathbb{N}$*, deciding if A is solvable by a pure or a test-free b-net, being a weighted* (ℓ, ℓ')*-T-System, is NP-complete.*
3. *For any fixed* $\ell, \ell' \geq 2$*, deciding if A is solvable by a pure or a test-free b-net, being a weighted* (ℓ, ℓ')*-S-system, is NP-complete.*

4. *For any fixed $\ell, \ell' \in \mathbb{N}$, one can decide in polynomial time if A is solvable by a b-net, being a weighted (ℓ, ℓ')-T-System.*

To prove Theorem 1.1–Theorem 1.3 we show that the corresponding decision problems are in NP and NP-hard. Membership in NP can be seen as follows: By Lemma 1, if N is a b-net that solves A then there is a b-admissible set $\mathcal{R}'$ of A such that $N_A^{\mathcal{R}'} = N$. By definition, A has at most $|S|^2$ SSP atoms and at most $|E| \cdot |S|$ ESSP atoms. Thus, there is a b-admissible subset $\mathcal{R} \subseteq \mathcal{R}'$ with $|\mathcal{R}| \leq |S|^2 + |E| \cdot |S|$. In particular, $N_A^{\mathcal{R}}$ originates from $N_A^{\mathcal{R}'} = N$ by (possibly) removing places, which can not increase any preset- or postset cardinality. Consequently, removing places preserves property $P \in \{CA, FA, FC, EFC, AC\}$, the weighted (m, n)-T-system property and the weighted (m, n)-S-system property. This makes $N_A^{\mathcal{R}}$ a searched net. A non-deterministic Turing machine can guess in polynomial time a corresponding set $\mathcal{R}$, check its b-admissibility, build $N_A^{\mathcal{R}}$ and check its structural properties in accordance to the regarded decision problem.

To show hardness we use the NP-complete problem CUBIC MONOTONE ONE-IN-THREE-3-SAT (CM 1-IN-3 3SAT) from [11] which is defined as follows: The input for CM 1-IN-3 3SAT is a negation-free boolean expression $\varphi = \{\zeta_0, \dots, \zeta_{m-1}\}$ of three-clauses $\zeta_0, \dots, \zeta_{m-1}$ with set of variables $V(\varphi)$ where every variable occurs in exactly three clauses. Notice that this implies $|V(\varphi)| = m$. The question is whether there is a subset $M \subseteq V(\varphi)$ satisfying $|M \cap \zeta_i| = 1, \forall i \in \{0, \dots, m-1\}$.

For Theorem 1.(1–2) we reduce an input instance φ with m clauses (in polynomial time) to a TS A_φ^b satisfying the following condition:

Condition 1. *1. If a test-free b-net solves A_φ^b then φ is one-in-three satisfiable.*
2. If φ is one-in-three satisfiable then there is a b-admissible set $\mathcal{R}$ of pure regions of A_φ^b satisfying $\forall R \in \mathcal{R} : |R^\bullet| \leq 1 \wedge |^\bullet R| \leq 7m+4$ and $\forall e \in E(A) : |^\bullet e_{\mathcal{R}}| \leq 1$.

A reduction that satisfies Condition 1 proves Theorem 1.(1–2) as follows: By definition of test-freeness, every b-net of Theorem 1.(1–2) is *at least* test-free, although possibly further restricted. Hence, Condition 1.1 ensures that if A_φ^b is solvable by such a net then φ has a one-in-three model. Moreover, a b-admissible set $\mathcal{R}$ that satisfies Condition 1.2 implies that $N_{A_\varphi^b}^{\mathcal{R}}$ is a pure b-net that is FA and solves A, cf. Example 1. Every pure FA b-net is test-free (by $f^+(p,t) = 0$ or $f^-(p,t) = 0$) and CF (by definition). By $N_{A_\varphi^b}^{\mathcal{R}}$ being CF, all of its places p satisfy $|p^\bullet| \leq 1$. Thus, the net is also FC, EFC and AC. Finally, by $\ell = 7m+4$ and $\ell' = 1$, the net $N_{A_\varphi^b}^{\mathcal{R}}$ is a weighted (ℓ, ℓ')-T-system. Altogether, Condition 1 ensures that A_φ^b is solvable by a b-net of Theorem 1.(1–2) if and only if φ is one-in-three satisfiable.

For Theorem 1.3 we reduce φ to a TS B_φ^b that satisfies the following condition:

Condition 2. *1. If a test-free b-net solves B_φ^b then φ is one-in-three satisfiable.*
2. If φ is one-in-three satisfiable then there is a b-admissible set $\mathcal{R}$ of pure regions such that $|^\bullet e_{\mathcal{R}}| \leq 2$ and $|e_{\mathcal{R}}{}^\bullet| \leq 2$ for all $e \in E(A)$.

A reduction satisfying Condition 2 proves Theorem 1.3 as follows: By the definition of test-freeness and weighted (m, n)-S-systems, a pure weighted $(2, 2)$-S-system is a test-free weighted (m, n)-S-System for all $m, n \geq 2$. Moreover, a b-admissible set that satisfies Condition 2.2 implies that $N^{\mathcal{R}}_{B^b_\varphi}$ is a pure weighted $(2, 2)$-S-system solving B^b_φ, cf. Example 2. Thus, Condition 2 ensures that B^b_φ is solvable by a b-net of Theorem 1.3 if and only if φ is one-in-three satisfiable.

3.1 The Reduction and the Proof of Condition 1.1 and Condition 2.2

In accordance to Condition 1.1 and Condition 2.1, our goal is to combine the existence of a b-admissible set $\mathcal{R}$, the b-solvability of A^b_φ and B^b_φ, with the one-in-three satisfiability of φ. For this purpose, both TSs (among others) apply gadgets that represent φ's clauses and use their variables as events. Moreover, both A^b_φ and B^b_φ have a certain separation atom and the signature of a solving region (sup, sig) defines a one-in-three model of φ via the variable events. So far, this approach is like that of [15]. However, the main difference and the biggest challenge is to consider the restrictions of Condition 1.1 and Condition 2.2. To master this challenge, we apply refined, specialized and different gadgets. Particularly noteworthy in this context is the representation of φ's clauses by $\{0, \dots, b\}^3$-grids instead of simple sequences, as it has been done in [15].

We proceed by introducing the gadgets of A^b_φ and B^b_φ that represent φ's clauses. In particular, the clause-gadgets' functionality will serve as motivation for the remaining parts of A^b_φ and B^b_φ, which are presented afterwards.

Let $i \in \{0, \dots, m-1\}$. The TSs A^b_φ and B^b_φ have for the clause $C_i = \{X_{i,0}, X_{i,1}, X_{i,2}\}$ the $\{0, \dots, b\}^3$-grid C^b_i with transitions that use the variables of C_i as events. More exactly, the $\{0, \dots, b\}^3$-grid C^b_i is built by the following sequences $P^{i,0}_{\alpha,\beta}, P^{i,1}_{\alpha,\beta}, P^{i,2}_{\alpha,\beta}$, where $\alpha, \beta \in \{0, \dots, b\}$. Figure 2 shows C^2_i.

$$P^{i,0}_{\alpha,\beta} = \quad t^i_{0,\alpha,\beta} \xrightarrow{X_{i,0}} t^i_{1,\alpha,\beta} \xrightarrow{X_{i,0}} \cdots \xrightarrow{X_{i,0}} t^i_{b-1,\alpha,\beta} \xrightarrow{X_{i,0}} t^i_{b,\alpha,\beta}$$

$$P^{i,1}_{\alpha,\beta} = \quad t^i_{\alpha,\beta,0} \xrightarrow{X_{i,1}} t^i_{\alpha,\beta,1} \xrightarrow{X_{i,1}} \cdots \xrightarrow{X_{i,1}} t^i_{\alpha,\beta,b-1} \xrightarrow{X_{i,1}} t^i_{\alpha,\beta,b}$$

$$P^{i,2}_{\alpha,\beta} = \quad t^i_{\alpha,0,\beta} \xrightarrow{X_{i,2}} t^i_{\alpha,1,\beta} \xrightarrow{X_{i,2}} \cdots \xrightarrow{X_{i,2}} t^i_{\alpha,b-1,\beta} \xrightarrow{X_{i,2}} t^i_{\alpha,b,\beta}$$

Among others, C^b_i provides the following sequence P_i where each of $X_{i,0}, X_{i,1}$ and $X_{i,2}$ occur b times in a row:

$$P_i = t^i_{0,0,0} \xrightarrow{X_{i,0}} \dots \xrightarrow{X_{i,0}} t^i_{b,0,0} \xrightarrow{X_{i,1}} \dots \xrightarrow{X_{i,1}} t^i_{b,0,b} \xrightarrow{X_{i,2}} \dots \xrightarrow{X_{i,2}} t^i_{b,b,b}$$

Notice that, except for $t^i_{b,b,b}$, every variable of C_i occur at every state of C^b_i. This has the advantage that we never have to solve an ESSP atom (X, s) such that $X \in \{X_{i,0}, X_{i,1}, X_{i,2}\}$ and s occur in the same grid and s is a source of another variable event $Y \in \{X_{i,0}, X_{i,1}, X_{i,2}\} \setminus \{X\}$. This property is crucial to ensure Condition 1.2 and Condition 2.2. In particular, it prevents atoms like $(X_{i,1}, t^i_{b-1,0,0})$ which would be unsolvable for $b \geq 2$.

The TSs A^b_φ and B^b_φ use the grid C^b_i as follows: Both TSs have at least one separation atom such that a corresponding b-solving region (sup, sig) satisfies either $sup(t^i_{0,0,0}) = 0$ and $sup(t^i_{b,b,b}) = b$ or $sup(t^i_{0,0,0}) = b$ and $sup(t^i_{b,b,b}) = 0$. In the following, we assume $sup(t^i_{0,0,0}) = 0$ and $sup(t^i_{b,b,b}) = b$ and argue that this implies that there is exactly one $X \in \{X_{i,0}, X_{i,1}, X_{i,2}\}$ with $sig(X) \neq (0,0)$. If $X \in \{X_{i,0}, X_{i,1}, X_{i,2}\}$ then, by $sup(t^i_{0,0,0}) = 0$ and $t^i_{0,0,0}\xrightarrow{X}$, we have immediately $sig^-(X) = 0$ (no consuming is possible). Moreover, by the definition of regions, we have $sup(s') = sup(s) - sig^-(e) + sig^+(e)$ for every $s\xrightarrow{e}s' \in P_i$. We use all this together and obtain inductively that $b = sup(t^i_{b,b,b}) = b \cdot (sig^+(X_{i,0}) + sig^+(X_{i,1}) + sig^+(X_{i,2})) > 0 = sup(t^i_{0,0,0})$. It is easy to see that this expression is satisfied if and only if there is exactly one variable event with a positive value sig^+ (and this value equals 1). Thus, there is exactly one event $X \in \{X_{i,0}, X_{i,1}, X_{i,2}\}$ with $sig(X) \neq (0,0)$. By the arbitrariness of i this is simultaneously true for all grids $C^b_0, \dots, C^b_{m-1}$. Consequently, the set $M = \{X \in V(\varphi) \mid sig(X) \neq (0,0)\}$ selects exactly one element of every clause C_i which makes it a one-in-three model of φ. Similarly, if $sup(t^i_{0,0,0}) = b$ and $sup(t^i_{b,b,b}) = 0$ then M yields also a one-in-three model of φ.

With the just presented functionality of C^b_i in mind, in what follows, we introduce A^b_φ's and B^b_φ's remaining parts. In particular, we explain how they collaborate to ensure the existence of a region satisfying $sup(t^i_{0,0,0}) = 0$ and $sup(t^i_{b,b,b}) = b$ or $sup(t^i_{0,0,0}) = b$ and $sup(t^i_{b,b,b}) = 0$. Before we start, the following lemma provides a basic result, to be used in the sequel, and shows how to connect the signature of some events with the solvability of an ESSP atom.

Lemma 2. *Let* $q_0\xrightarrow{e_1}\dots\xrightarrow{e_1}q_b\xrightarrow{e_2}q_{b+1}\xrightarrow{e_3}q_{b+2}\xrightarrow{e_1}\dots\xrightarrow{e_1}q_{2b+2}$ *be a sequence of a TS* $A = (S, E, \delta, s_0)$*, where* e_1, e_2, e_3 *are pairwise distinct events, which starts and ends with* e_1 *b-times in a row. A test-free b-region solves the ESSP atom* (e_1, q_{b+1}) *if and only if* $sig(e_1) = (0,1)$*,* $sig^-(e_2) = sig^+(e_2)$ *and* $sig(e_3) = (b,0)$ *or* $sig(e_1) = (1,0)$*,* $sig^-(e_2) = sig^+(e_2)$ *and* $sig(e_3) = (0,b)$*.*

We start by introducing the parts of A^b_φ. Figure 2 sketches a snippet of A^2_φ. The initial state of A^b_φ is s. Firstly, the TS A^b_φ has the sequence Q^b:

$$Q^b = s \xrightarrow{a} q_0 \xrightarrow{k} \cdots \xrightarrow{k} q_b \xrightarrow{z} q_{b+1} \xrightarrow{o} q_{b+2} \xrightarrow{k} \cdots \xrightarrow{k} q_{2b+2}$$

The sequence Q^b provides the ESSP-atom (k, q_{b+1}). If A^b_φ is b-solvable then, by Lemma 1, there is a b-admissible set $\mathcal{R}$ of (test-free) regions such that $N = N^{\mathcal{R}}_{A^b_\varphi}$. As $\mathcal{R}$ is b-admissible, there is a test-free b-region $(sup, sig) \in \mathcal{R}$ that solves (k, q_{b+1}). By Lemma 2, we have either $sig^-(z) = sig^+(z)$ and $sig(o) = (b,0)$ or $sig^-(z) = sig^+(z)$ and $sig(o) = (0,b)$. Let's discuss the former case. The region R implies for transitions $s\xrightarrow{o}s'$ and $s''\xrightarrow{z}s'''$ (of A^b_φ) that $sup(s) = b$, $sup(s') = 0$ and $sup(s'') = sup(s''')$. The TS A^b_φ uses this to ensure a particular

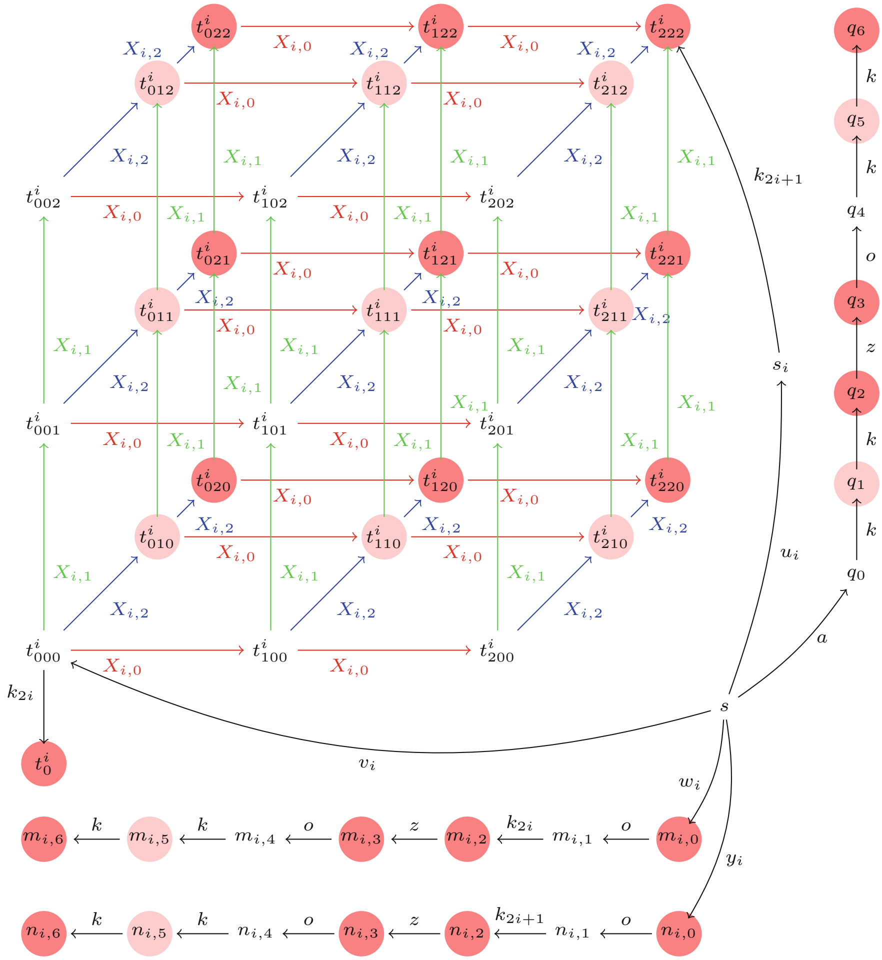

Fig. 2. A snippet of A^2_φ showing the sequences Q^2, M^2_i, N^2_i, the $\{0,1,2\}^3$-grid C^2_i for the clause $C_i = \{X_{i,0}, X_{i,1}, X_{i,2}\}$ and the paths $L_{i,0}$ and $L_{i,1}$. For clarity, edges labeled by the same variable event have the same color. The coloring of the states corresponds to the 2-region R_1 which is defined in Table 1 and where $X_{i,0} \in M$: Light (dark) red colored states are mapped to 1 (2) and the others are mapped to 0. (Color figure online)

signature of the events k_{2i}, k_{2i+1} that are provided by the following sequences N^b_i and M^b_i, for all $i \in \{0, \dots, m-1\}$:

$$M^b_i = s \xrightarrow{w_i} m_{i,0} \xrightarrow{o} m_{i,1} \xrightarrow{k_{2i}} m_{i,2} \xrightarrow{z} m_{i,3} \xrightarrow{o} m_{i,4} \xrightarrow{k} \dots \xrightarrow{k} m_{i,b+4}$$

$$N^b_i = s \xrightarrow{y_i} n_{i,0} \xrightarrow{o} n_{i,1} \xrightarrow{k_{2i+1}} n_{i,2} \xrightarrow{z} n_{i,3} \xrightarrow{o} n_{i,4} \xrightarrow{k} \dots \xrightarrow{k} n_{i,b+4}$$

The TS A^b_φ uses M^b_i, N^b_i, R and the occurrences of z and o for the announced goal as follows: By $sig(o) = (b, 0)$, we have $sup(m_{i,1}) = sup(n_{i,1}) = 0$ and $sup(m_{i,3}) = sup(n_{i,3}) = b$ which, by $sig^-(z) = sig^+(z)$, implies $sup(m_{i,2}) = sup(n_{i,2}) = b$. By $m_{i,1} \xrightarrow{k_{2i}} m_{i,2}$, $n_{i,1} \xrightarrow{k_{2i+1}} n_{i,2}$ this leads to $sig(k_{2i}) = sig(k_{2i+1}) = (0, b)$. In particular, for all edges $s \xrightarrow{k_{2i}} s'$ and $s'' \xrightarrow{k_{2i+1}} s'''$ of A^b_φ holds $sup(s) = sup(s'') = 0$ and $sup(s') = sup(s''') = b$. Finally, A^b_φ uses other occurrences of k_{2i} and k_{2i+1} to ensure $sup(t^i_{0,0,0}) = 0$ and $sup(t^i_{b,b,b}) = b$. More exactly, A^b_φ installs the paths $L_{i,0} = s \xrightarrow{v_i} t^i_{0,0,0} \xrightarrow{k_{2i}} t^i_0$ and $L_{i,1} = s \xrightarrow{u_i} s_i \xrightarrow{k_{2i+1}} t^i_{b,b,b}$. On the one hand, $L_{i,0}$ ensures reachability of A^b_φ. On the other hand, by $t^i_{0,0,0} \xrightarrow{k_{2i}} t^i_0$, $s_i \xrightarrow{k_{2i+1}} t^i_{b,b,b}$ and the discussion above, $L_{i,0}, L_{i,1}$ ensure that $sup_0(t^i_{0,0,0}) = 0$ and $sup_0(t^i_{b,b,b}) = b$.

Similarly, one argues that $sig(o) = (0, b)$ and $sig^-(z) = sig^+(z)$ yields $sig(k_{2i}) = sig(k_{2i+1}) = (b, 0)$, implying $sup_1(t^i_{0,0,0}) = b$ and $sup_1(t^i_{b,b,b}) = 0$. By the discussed functionality of the grids, this proves that A^b_φ satisfies Condition 1.1.

We proceed by presenting the remaining gadgets of B^b_φ. The TS B^b_φ has the initial state s and it has for every $i \in \{0, \ldots, m-1\}$ the following six sequences:

$$F^b_i = s \xrightarrow{b^i_{2m+5}} a^i_{2m+5} \;\cdots\; a^i_1 \xrightarrow{b^i_0} f^i_0 \xrightarrow{k} \cdots \xrightarrow{k} f^i_b \xrightarrow{z_{2i}} f^i_{b+1} \xrightarrow{o} f^i_{b+2} \xrightarrow{k} \cdots \xrightarrow{k} f^i_{2b+2}$$

$$G^b_i = s \xrightarrow{d^i_{2m+5}} c^i_{2m+5} \cdots c^i_1 \xrightarrow{d^i_0} g^i_0 \xrightarrow{k} \cdots \xrightarrow{k} g^i_b \xrightarrow{z_{2i+1}} g^i_{b+1} \xrightarrow{o} g^i_{b+2} \xrightarrow{k} \cdots \xrightarrow{k} g^i_{2b+2}$$

$$M^b_i = s \xrightarrow{w^i_{2m+5}} r^i_{2m+5} \cdots r^i_1 \xrightarrow{w^i_0} m^i_0 \xrightarrow{o} m^i_1 \xrightarrow{k_{2i}} m^i_2 \xrightarrow{z_{2i}} m^i_3 \xrightarrow{o} m^i_4 \xrightarrow{k} \cdots \xrightarrow{k} m^i_{2b+2}$$

$$N^b_i = s \xrightarrow{y^i_{2m+5}} s^i_{2m+5} \cdots s^i_1 \xrightarrow{y^i_0} n^i_0 \xrightarrow{o} n^i_1 \xrightarrow{k_{2i+1}} n^i_2 \xrightarrow{z_{2i+1}} n^i_3 \xrightarrow{o} n^i_4 \xrightarrow{k} \cdots \xrightarrow{k} n^i_{2b+2}$$

$$L_{i,0} = s \xrightarrow{v^i_{2m+5}} q^i_{2m+5} \cdots q^i_1 \xrightarrow{v^i_0} t^i_0 \xrightarrow{k_{2i}} t^i_0 \qquad L_{i,1} = s \xrightarrow{u^i_{2m+5}} p^i_{2m+5} \cdots p^i_2 \xrightarrow{u^i_1} p^i_1 \xrightarrow{k_{2i+1}} t^i_{b,b,b}$$

In terms of Condition 2.2, the gadgets $M^b_i, N^b_i, L_{i,0}$ and $L_{i,1}$ work similar to the corresponding ones of A^b_φ. However, Condition 2.2 requires to distribute the task of one event to multiple events. For example, the events $z_0, \ldots, z_{2m-1}$ of B^b_φ play the same role as z of A^b_φ. This is achieved by F^b_i and G^b_i. More exactly, if B^b_φ is b-solvable then, by Lemma 1, every atom (k, f^i_{b+1}) is too. By Lemma 2, if (sup, sig) is a solving test-free b-region then $sig(k) = (0, 1)$ and $sig(o) = (b, 0)$ or $sig(k) = (1, 0)$ and $sig(o) = (0, b)$. If $sig(k) = (0, 1)$ then, by $sup(f^i_b) = sup(g^i_b) = b \cdot sig^+(k) = b$ and $sup(f^i_{b+1}) = sup(f^i_{b+1}) = b$, we get $sig^+(z_i) = sig^-(z_i)$ and, thus, $sig(k_i) = (0, b)$, $\forall i \in \{0, \ldots, 2m-1\}$. Similarly, if $sig(k) = (1, 0)$ then $sig(k_i) = (b, 0)$, $\forall i \in \{0, \ldots, 2m-1\}$. Thus, by the grids' functionality, the set $M = \{X \in V(\varphi) \mid sig(X) \neq (0, 0)\}$ is a sought model.

3.2 The Proof of Condition 1.2 and Condition 2.2

In this section, we provide b-admissible sets of A^b_φ and B^b_φ in accordance to Condition 1.2 and Condition 2.2, respectively. For the sake of simplicity, we present for every region (sup, sig) only its signature sig and the value $sup(s)$ of the initial state s. Because A^b_φ and B^b_φ are reachable and $sup(s'') = sup(s') - sig^-(e) + sig^+(e)$ for every transition $s' \xrightarrow{e} s''$, this completely defines the region. In the remainder of this section, unless stated explicitly otherwise, let $i \in \{0, \dots, m-1\}$ and M be a one-in-three model of φ. Moreover, for $\alpha \in \{0,1,2\}$ let $\beta_\alpha, \gamma_\alpha \in \{0, \dots, m-1\} \setminus \{i\}$ be the distinct indices such that $X_{i,\alpha} \in C_i \cap C_{\beta_\alpha} \cap C_{\gamma_\alpha}$, that is, $\beta_\alpha, \gamma_\alpha$ choose the other two clauses of φ containing $X_{i,\alpha}$.

We start with Condition 1.2 and provide a b-admissible set $\mathcal{R}$ of pure regions of A^b_φ such that $|R^\bullet| \leq 1$ and $|{}^\bullet e_\mathcal{R}| \leq 1$ for all $R \in \mathcal{R}$ and $e \in E(A^b_\varphi)$. Moreover, because A^b_φ has exactly $7m+4$ events, every region R of A^b_φ satisfies $|{}^\bullet R| \leq 7m+4$. For abbreviation, we define $U = \{u_0, \dots, u_{m-1}\}, V = \{v_0, \dots, v_{m-1}\}, W = \{w_0, \dots, w_{m-1}\}, Y = \{y_0, \dots, y_{m-1}\}$ and $K = \{k_0, \dots, k_{2m-1}\}$. We solve all atoms concerning the events of $\{a\} \cup U \cup V \cup W \cup Y$ with the region $R = (sup, sig)$, defined by $sup(s) = 0$ and $sig(e) = (0, b)$ if $e \in \{a\} \cup U \cup V \cup W \cup Y$ and, otherwise, $sig(e) = (0,0)$. This region satisfies $|R^\bullet| = 0$ (no event consumes). Moreover, none of the subsequently presented regions of A^b_φ is in the preset of any of $\{a\} \cup U \cup V \cup W \cup Y$, thus, $|{}^\bullet e_\mathcal{R}| \leq 1$ for $e \in \{a\} \cup U \cup V \cup W \cup Y$. We proceed with presenting for every event $k, z, o, v, k_{2i}, k_{2i+1}$ and $X_{i,0}, X_{i,1}, X_{i,2}$ corresponding regions that solves it. Every row of Table 1 (below) defines a region $R = (sup_R, sig_R)$ with $sup_R(s) = 0$ as follows: For every $e \in E(A^b_\varphi)$ we have either $sig_R(e) = (0,0)$ or $sig_R(e) \in \{(1,0), (0,1), (b,0), (0,b)\}$. In the latter case, e occurs according to its signature in the corresponding column either as a single event or as member of the event set shown. For example, for R_1 we have $sig_{R_1}(k) = (0,1)$ and $sig_{R_1}(e) = (0,1)$ for $e \in M$.

Table 1. Pure regions of A^b_φ that solve $k, z, o, k_{2i}, k_{2i+1}$ and $X_{i,0}, X_{i,1}, X_{i,2}$.

R	$(1,0)$	$(0,1)$	$(b,0)$	$(0,b)$
R_1		k, M	o	W, Y, K
R_2	k			z, a
R_3			z	a, o, U, V
R_4				z, U, V
$R^z_{k_{2i}}$			k_{2i}	z, u_i, v_i, w_i
$R^z_{k_{2i+1}}$			k_{2i+1}	z, u_i, y_i
$R^\alpha_{k_{2i}}$ for $X_{i,\alpha} \notin M$		$X_{i,\alpha}$		$a, Y, \ell \in \{i, \beta_\alpha, \gamma_\alpha\} : u_\ell, k_{2\ell}$, $W \setminus \{w_\ell \mid \ell \in \{i, \beta_\alpha, \gamma_\alpha\}$
$R_{k_{2i+1}}$				$k_{2i+1}, a, W, V, U \setminus \{u_{2i+1}\}, Y \setminus \{y_{2i+1}\}$
$R_{X_{i,\alpha}}$	$X_{i,\alpha}$			$v_i, v_{\beta_\alpha}, v_{\gamma_\alpha}$

The regions of Table 1 solve the events $k, z, o, k_{2i}, k_{2i+1}$ and $X_{i,0}, X_{i,1}, X_{i,2}$ as follows. (k): R_1 solves k at the sinks of z and R_2 solves k at the remaining states. (z): R_2 solves z at the sources of k and R_3 solves z at o's sources and at s. $R^z_{k_{2i}}$ and $R^z_{k_{2i+1}}$, where $i \in \{0, \dots, m-1\}$, solve z at the sources of $k_0, \dots, k_{2m-1}$. Finally, R_4 solves z at the remaining states. (o): R_1 solves o at the sources of k, $k_0, \dots, k_{2m-1}$ and at s and R_3 solves o at the remaining states. (k_{2i}): R_1 solves k_{2i} at all sources of o and all sources of $X_{i,\alpha}$ in C^b_i, where $X_{i,\alpha} \in M$. $R^z_{k_{2i}}$ solves k_{2i} at all sources of k_j, where $2i \neq j \in \{0, \dots, 2m-1\}$ and at s. The remaining atoms are solved by (the two regions defined by) $R^\alpha_{k_{2i}}$, where $\alpha \in \{0, 1, 2\}$ such that $X_{i,\alpha} \notin M$. (k_{2i+1}): R_1 solve k_{2i+1} at $n_{i,0}$ and $R^z_{k_{2i+1}}$ at s and $R_{k_{2i+1}}$ at all remaining states. ($X_{i,\alpha}$): If $X_{i,\alpha} \in M$ then the region R_1 solves it at t^i_0, otherwise, $X_{i,\alpha}$ is solved at t^i_0 by $R^\alpha_{k_{2i}}$. The remaining atoms are solved by $R_{X_{i,\alpha}}$.

In the following we argue that A^b_φ has the SSP, too: To separate $S(Q_b)$ from $S(A^b_\varphi) \setminus S(Q_b)$ we use the region $R_Q = (sup_Q, sig_Q)$ where $sup_Q(s) = 0$, $sig_Q(a) = (0, b)$ and $sig_Q(e) = (0, 0)$ for the other events. Moreover, the states of Q_b are pairwise separated by R_1, R_2 and R_4. To separate the states $S(M^b_i)$ from $S(A^b_\varphi) \setminus S(M^b_i)$ we define the region $R_{M_i} = (sup_{M_i}, sig_{M_i})$ where $sup_{M_i}(s) = 0$, $sig_{M_i}(w_i) = (0, b)$ and $sig_{M_i}(e) = (0, 0)$ for the other events. The states of M^b_i are pairwise separated by R_1, R_2, R_3 and R_4. Similarly, the states $S(N^b_i)$ are separated by R_1, R_2, R_3, R_4 and $R_{N_i} = (sup_{N_i}, sig_{N_i})$ where $sup_{N_i}(s) = 0$, $sig_{N_i}(y_i) = (0, b)$ and $sig_{N_i}(e) = (0, 0)$ for the other events. To separate the states of $S(C^b_i) \cup \{t^i_0, s_i\}$ from all the other states we use the region $R_{C_i} = (sup_{C_i}, sig_{C_i})$ where $sup(C_i)(s) = 0$, $sig_{C_i}(u_i) = sig_{C_i}(v_i) = (0, b)$ and $sig_{C_i}(e) = (0, 0)$ for the other events. Moreover, the states of $S(C^b_i) \cup \{t^i_0, s_i\}$ are pairwise separated by $R_1, R_{k_{2i+1}}$ and $R^\alpha_{X_{i,\alpha}}$, where $X_{i,\alpha} \notin M$.

Altogether, the set $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4$ where $\mathcal{R}_1 = \{R_1, R_2, R_3, R_4\}$, $\mathcal{R}_2 = \{R^z_{k_{2i}}, R^z_{k_{2i+1}}, R^\alpha_{k_{2i}}, R_{k_{2i+1}} \mid i \in \{0, \dots, m-1\}, \alpha \in \{0, 1, 2\}, X_{i,\alpha} \notin M\}$, $\mathcal{R}_3 = \{R_{X_{i,\alpha}} \mid i \in \{0, \dots, m-1\}, \alpha \in \{0, 1, 2\}\}$ and $\mathcal{R}_3 = \{R_Q, R_{M_i}, R_{N_i}, R_{C_i} \mid i \in \{0, \dots, m-1\}\}$, is an admissible set of A^b_φ. We briefly argue that it is FA: It is easy to see that every presented region $R \in \mathcal{R}$ satisfy $|R^\bullet| \leq 1$. Moreover, $|{}^\bullet e_{\mathcal{R}}| \leq 1$ is also true for $e \in E(A^b_\varphi)$: The regions $R_1 \in {}^\bullet o$, $R_2 \in {}^\bullet k$, $R_3 \in {}^\bullet z$ and $R^z_{k_{2i}} \in {}^\bullet k_{2i}$ and $R^z_{k_{2i+1}} \in {}^\bullet k_{2i+1}$ are unique. Furthermore, if $X_{i,\alpha} = X_{j,\beta} = X_{\ell,\gamma}$ then $R_{X_{i,\alpha}} = R_{X_{j,\beta}} = R_{X_{\ell,\gamma}}$ where $i, j, \ell \in \{0, \dots, m-1\}, \alpha, \beta, \gamma \in \{0, 1, 2\}$. As $\mathcal{R}$ is a set, this region is the only element in ${}^\bullet X_{i,\alpha}$. No other region $(sup, sig) \in \mathcal{R}$ satisfies $sig^-(e) > 0$ for any $e \in E(A^b_\varphi)$. Thus, A^b_φ satisfies Condition 1.2.

To prove Condition 2.2 we provide a b-admissible set $\mathcal{R}$ of pure regions of B^b_φ such that $|e_{\mathcal{R}}{}^\bullet| \leq 2$ and $|{}^\bullet e_{\mathcal{R}}| \leq 2$ for all $e \in E(B^b_\varphi)$. For brevity, we define for $j \in \{0, \dots, m-1\}$ the following sets: $B_j = \{b^i_j \mid i \in \{0, \dots, m-1\}\}$, $D_j = \{d^i_j \mid i \in \{0, \dots, m-1\}\}$, $U_j = \{u^i_j \mid i \in \{0, \dots, m-1\}\}$, $V_j = \{v^i_j \mid i \in \{0, \dots, m-1\}\}$, $W_j = \{w^i_j \mid i \in \{0, \dots, m-1\}\}$, $Y_j = \{y^i_j \mid i \in \{0, \dots, m-1\}\}$, $K = \{k_i \mid i \in \{0, \dots, 2m-1\}\}$ and $Z = \{z_i \mid i \in \{0, \dots, 2m-1\}\}$. By a little abuse of notation, we let $\mathcal{C}_i = F^b_i \cup G^b_i \cup M^b_i \cup N^b_i \cup F^b_i \cup C^b_i \cup L_{i,0} \cup L_{i,1}$ and $\delta_i = 2m + 5 - i$. Table 2 (below) defines a regions R of B^b_φ with $sup_R(s) = 0$.

Table 2. Pure b-regions of B^b_φ that solve several separation atoms.

R	$(1,0)$	$(0,1)$	$(b,0)$	$(0,b)$
R_1		k, M	o	W_0, Y_0, K
R_2	k			Z, B_0, D_0
R_3			o	Z, W_3, Y_3
$R_{z_{2i}}$			z_{2i}	b^i_1, w^i_1
$R^0_{z_{2i+1}}$			z_{2i+1}	v^i_5, k_{2i+1}, d^i_1
$R^1_{z_{2i+1}}$			z_{2i+1}	d^i_0, y^i_1
$R_{k_{2i+1}}$			k_{2i+1}	b^i_1, w^i_1
R^2_{2i}			z_{2i}	$k_{2i}, b^i_0,\ (V_{\delta_i} \cup U_{\delta_i} \cup B_{\delta_i} \cup D_{\delta_i} \cup W_{\delta_i} \cup Y_{\delta_i}) \setminus E(\mathcal{C}_i)$

$(k), (o)$: The regions R_1 and R_2 solve k and the regions R_1 and R_3 solve o. $(z_{2i}), (z_{2i+1})$: The region R_2 solves z_{2i}, z_{2i+1} at k's sources and R_3 solves them at o's sources, at $s_{i,1}, s_{i,2}, s_{i,3}$ and at $r_{i,1}, r_{i,2}, r_{i,3}$. R^2_{2i} solves z_{2i} at the remaining states of $\mathcal{C}_i \setminus \{t^i_0\}$ and $R_{z_{2i}}$ solves z_{2i} at the remaining states of B^b_φ. $R^0_{z_{2i+1}}$ solves z_{2i+1} at n^i_0, n^i_1 and $s_{i,1}$ and $R^1_{z_{2i+1}}$ solves it at the remaining states.

(k_{2i}): For a correct referencing, we need the following definitions: If $j \in \{0,\dots,m-1\}$ then let $\alpha_j \in \{0,1,2\}$ be the index such that $X_{j,\alpha_j} \in M$ and let by $\beta_j < \gamma_j \in \{0,1,2\} \setminus \{\alpha_j\}$ the other variable events of C^b_j be chosen. Moreover, let $\ell \neq j \in \{0,\dots,m-1\}$ such that $X_{i,\beta_i} \in C_i \cap C_\ell, \cap C_j$ and let $\ell' \neq j' \in \{0,\dots,m-1\}$ such that $X_{i,\gamma_i} \in C_i \cap C_{\ell'}, \cap C_{j'}$. That is, ℓ, j and ℓ', j' choose the other two clauses where $X_{i,\beta_i}, X_{i,\gamma_i}$ occur. We use this to define the region $R^0_{2i} = (sup^0_{2i}, sig^0_{2i})$ where $sup^0_{2i}(s) = 0$, $sig(X_{i,\beta_i}) = (1,0)$ and for $\delta \in \{i, \ell, j\}$ it is $sig^0_{2i}(k_{2\delta}) = (b,0)$ and $sig^0_{2i}(w^\delta_0) = (0,b)$ if $X_{i,\beta_i} = X_{\delta,\beta_\delta}$ and $sig^0_{2i}(w^\delta_2) = (0,b)$ if $X_{i,\beta_i} = X_{\delta,\gamma_\delta}$. Similarly, we define the region $R^1_{2i} = (sup^1_{2i}, sig^1_{2i})$ by $sup^1_{2i}(s) = 0$, $sig(X_{i,\gamma_i}) = (1,0)$ and for $\delta \in \{i, \ell', j'\}$ it is $sig^1_{2i}(k_{2\delta}) = (b,0)$ and $sig^1_{2i}(w_{\delta,2}) = (0,b)$ if $X_{i,\gamma_i} = X_{\delta,\gamma_\delta}$ and $sig^1_{2i}(w_{\delta,0}) = (0,b)$ if if $X_{i,\gamma_i} = X_{\delta,\beta_\delta}$. Notice that if $X_{i,\beta_i} = X_{\delta,\gamma_\delta}$ then $R^0_{2i} = R^1_{2\delta}$ and if $X_{i,\gamma_i} = X_{\delta,\beta_\delta}$ then $R^1_{2i} = R^0_{2\delta}$. This is our way to correctly, restrict the postset of the events $w^{\cdots}_0$ and $w^{\cdots}_2$. The region R_1 solves k_{2i} at m^i_0 and the sinks of X_{i,α_i}. R^0_{2i} and R^1_{2i} solve k_{2i} at all states of $C^b_i \cup \{s\}$ and $\bigcup_{j=1}^{2m+5} \{q^\ell_j, p^\ell_j, a^\ell_j, c^\ell_j, r^\ell_j, s^\ell_j \mid \ell \in \{0,\dots,m-1\} \setminus \{i\}\}$. Finally, to solve k_{2i} at the remaining states we use the region R^2_{2i}defined as follows: If $\alpha = 2m+5-i$ then $R^2_{2i} = (sup^2_{2i}, sig^2_{2i})$ is defined by $sup^2_{2i}(s) = 0$, $sig^2_{2i}(k_{2i}) = sig^2_{2i}(b_{i,0}) = sig^2_{2i}(e)$, where $e \in \{v_{j,\alpha}, u_{j,\alpha}, b_{j,\alpha}, d_{j,\alpha}, w_{j,\alpha}, y_{j,\alpha} \mid j \in \{\ 0,\dots,m-1\} \setminus \{i\}\}$ and $sig^2_{2i}(z_{2i}) = (b,0)$.

(k_{2i+1}): R_1 and $R_{k_{2i+1}}$ solve k_{2i+1} at all states of B^b_φ.

$(X_{i,0}, X_{i,1}, X_{i,2})$: Let $\alpha_i, \beta_i, \gamma_i$ be defined as above. To separate $X_{i,\alpha_i} = X_{\ell,\alpha_\ell} = X_{j,\alpha_j}$, i, j, ℓ pairwise distinct, from $q^i_1, q^i_2, q^\ell_1, q^\ell_2, q^j_1, q^j_2$, respectively, we use the region $R^i_q = R^\ell_q = R^j_q$ that maps s to 0, X_{i,α_1} to $(0,b)$, v^i_0, v^ℓ_0, v^j_0 to $(b,0)$, v^i_2, v^ℓ_2, v^j_2 to $(0,b)$ and the other events to $(0,0)$. This region is necessary as the presets ${}^\bullet v^i_0, {}^\bullet v^\ell_0, {}^\bullet v^j_0$ have already two elements. To separate X_{i,α_i} from the remaining states, we use $R^i_{\alpha_i} = (sup^i_{\alpha_i}, sig^i_{\alpha_i})$, where $sup^i_{\alpha_i}(s) = 0$, $sig^i_{\alpha_i}(X_{i\alpha_i}) = (1, 0)$ $sig^i_{\alpha_i}(v^i_1) = sig^i_{\alpha_i}(v^\ell_1) = sig^i_{\alpha_i}(v^j_1) = (0, b)$ and $X_{i,\alpha_i} \in C_i \cap C_\ell \cap C_j$.

The regions $R^i_{\beta_i}$ for X_{i,β_i} and $R^i_{\gamma_i}$ for X_{i,γ_i} are defined accordingly, where we use $v_3^{\cdot\cdot}$ and $v_4^{\cdot\cdot}$ (without repetition or confusion) as preset events, respectively. Notice that, so far, $X_{i,\beta_i}, X_{i,\gamma_i}$ are already separated from $q_1, \dots, q_{2m+5}$ by R^0_{2i} and R^1_{2i}, respectively.

($u^i_j, v^i_j, b^i_j, d^i_j, w^i_j, y^i_j, j \in \{1, \dots, 2m-5\}$): So far, for all of these events e holds $|{}^\bullet e_{\mathcal{R}}| = 0$ and, even more, if $j \neq 1$ then $|e_{\mathcal{R}}{}^\bullet| \leq 1$. Hence, for $e, e' \in \{u^i_j, v^i_j, b^i_j, d^i_j, w^i_j, y^i_j, j \in \{1, \dots, 2m-4\}\}$ with $\xrightarrow{e'} x \xrightarrow{e} \in B^b_\varphi$ we use the region (sup_e, sig_e) where $sup_e(s) = 0$, $sig_e(e') = (0, b)$ and $sig_e(e) = (b, 0)$ and $sig_e(e'') = (0, 0)$ for $E(B^b_\varphi) \setminus \{e, e'\}$. Notice that e, e' are unique and that this region also separates x. For the $2m+5$-indexed events we use the region where all these (and only these) events are mapped to $(b, 0)$ and s is mapped to b.

So far, the presented regions justify B^b_φ's b-ESSP. It remains to justify its b-SSP: One verifies that all distinct states $s, s' \in \mathcal{C}_i$ are separated by the already presented regions. If $e \in \{u^i_j, v^i_j, b^i_j, d^i_j, w^i_j, y^i_j \mid i \in \{0, \dots, m-1\}, j \in \{1, \dots, 2m-5\}\}$ and $s \xrightarrow{e}$ then s is separated by the region defined for the separation of e. Moreover, so far, if $e \in \{u^i_j, v^i_j, b^i_j, d^i_j, w^i_j, y^i_j \mid i \in \{0, \dots, m-1\}, j \in \{m, \dots, 2m+6\}\}$ then $|e_{\mathcal{R}}{}^\bullet| = 1$. Hence, we choose for every $i \in \{0, \dots, m-1\}$ the region $R_{\mathcal{C}_i} = (sup_{\mathcal{C}_i}, sig_{\mathcal{C}_i})$ where $sup_{\mathcal{C}_i}(s) = 0$, $sig_{\mathcal{C}_i}(e) = (0, b)$ if $e \in \{u^i_j, v^i_j, b^i_j, d^i_j, w^i_j, y^i_j \mid j = 2m+5-i\}\}$ and, otherwise, $sig_{\mathcal{C}_i}(e) = (0, 0)$. Clearly, $R_{\mathcal{C}_i}$ separates the remaining states in question from $S(B^b_\varphi) \setminus \mathcal{C}_i$. Moreover, the regions $R_{\mathcal{C}_0}, \dots, R_{\mathcal{C}_{m-1}}$ preserve the $(2, 2)$-S-system property.

Altogether, the union of all introduced regions yields a b-admissible set $\mathcal{R}$ of pure regions that has the $(2, 2)$-S-system property.

3.3 The Proof of Theorem 1.4

By Lemma 1, a b-net N, being a weighted (m, n)-T-system, solves A if and only if there is a b-admissible set $\mathcal{R}$ with $N = N^{\mathcal{R}}_A$. By definition, every $R = (sup, sig) \in \mathcal{R}$ satisfies $|{}^\bullet R| = |\{e \in E(A) \mid sig^+(e) > 0\}| \leq m$ and $|R^\bullet| = |\{e \in E(A) \mid sig^-(e) > 0\}| \leq n$. The maximum set $\mathcal{R}$ of A's b-regions that satisfy the (m, n)-condition is computable in polynomial time: To define $R = (sup, sig) \in \mathcal{R}$ we have for $\ell \in \{1, \dots, m\}$ and $\ell' \in \{1, \dots, n\}$ at most $\binom{|E|}{\ell}$ and $\binom{|E|}{\ell'}$ events for ${}^\bullet R$ and $R^\bullet$, respectively. This makes at most $\binom{|E|}{\ell} \cdot \binom{|E|}{\ell'} \cdot (b+1)^{\ell+\ell'}$ possibilities for sig, each of it is to combine with the at most $b+1$ values for $sup(s_0)$. As b, m and n are not part of the input, altogether, there are at most $\mathcal{O}(|E|^{m+n})$ b-regions. Moreover, one can decide in polynomial time if $sup(s_0)$ and sig define actually a fitting b-region as follows: Firstly, compute a spanning tree A' of A, having at most $|S(A)|$ paths, in time $\mathcal{O}(|E(A)| \cdot |S(A)|^3)$ [16]. Secondly, use $sup(s_0)$ and sig to determine $sup(s_j)$ for all $s_j \in S(A)$ by the unique path $s_0 \xrightarrow{e_1} \dots \xrightarrow{e_j} s_j \in A'$. Thirdly, check for the at most $|S|^2 \cdot |E|$ edges $s \xrightarrow{e} s' \in A$ if both $sup(s) \geq sig^-(e)$ and $sup(s') = sup(s) + sig^-(e) + sig^+(e) \leq b$ are satisfied.

Having computed the (maximum) set $\mathcal{R}$, it remains to check (in polynomial time) whether the at most $|S|^2 + |S| \cdot |E|$ separation atoms of A are solved by $\mathcal{R}$.

4 Conclusion

This paper shows that deciding if a TS is solvable by a b-net which is CF, FA, FC, EFC or AC remains NP-complete. Moreover, our proof imply that synthesis is also hard if the searched net is to be *behaviorally free choice*, *behaviorally asymmetric choice* or *reducedly asymmetric choice* [3]. Furthermore, we show that synthesis of (m, n)-S-systems is NP-complete for every fixed $m, n \geq 2$. While synthesis of weighted (m, n)-T-systems, being dual to the S-systems, is also hard if m, n are part of the input, it becomes tractable for any fixed m, n. In particular, fixing m, n puts the problem into the complexity class XP. Consequently, for future work, it remains to be investigated whether the synthesis of weighted (m, n)-T-systems parameterized by $m + n$ is fixed parameter tractable.

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