Chapter 5
Surface Elasticity Models: Comparison Through the Condition of the Anti-plane Surface Wave Propagation

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Abstract In order to discuss the peculiarities of few models of surface elasticity we consider here the dispersion relations for anti-plane surface waves. We show that the dispersion curves are quite sensitive to the choice of the model. We consider here the linear Gurtin-Murdoch model, strain- and stress-gradient surface elasticity models.

Keywords: Surface elasticity · Anti-plane surface wave propagation · Dispersion curves · Gurtin-Murdoch model · Strain-gradient surface elasticity model · Stress-gradient surface elasticity models

5.1 Introduction

The interest to generalized models of continua grows recently with respect to appearance of new microstructured materials as well as in order to describe new phenomena observed at the micro- and nano-scale, see, e.g., Forest et al (2011); Liebold and Müller (2015); Aifantis (2016). In particular, the surface elasticity models found various applications in micro- and nano-mechanics, see, e.g., Duan et al (2008); Wang et al (2011); Javili et al (2013b,a); Eremeyev (2016) and the reference therein. Having origin in the landscape works by Laplace (1805, 1806); Young (1805); Poisson (1831) and Gibbs, see Longley and Van Name (1928), the rational continual model of the surface elasticity was developed by Gurtin and Murdoch (1975, 1978). Later it was generalized by Steigmann and Ogden (1997, 1999) in order to take into account bending surface stiffness. As surface mechanics should describe quite

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different phenomena, in the literature are known various extensions of surface-related mechanics, see, e.g., dell'Isola and Seppecher (1997); dell'Isola et al (2012b); Placidi et al (2014); Lurie et al (2016, 2009); Belov et al (2019); Eremeyev (2019b) and the references therein.

The presence of surface stresses influences the effective (apparent) properties of nanostructured materials, such as nano-composites (Kushch et al, 2013; Nazarenko et al, 2016, 2018; Zemlyanova and Mogilevskaya, 2018; Han et al, 2018) or nano-plates and shells (Altenbach and Eremeyev, 2011; Altenbach et al, 2010, 2012; Ru, 2016). In addition, surface energy may result in new phenomena as the appearance surface/interfacial waves considered within the Gurtin-Murdoch approach (Xu et al, 2015; Eremeyev et al, 2016) and for certain generalizations of the Gurtin-Murdoch model (Eremeyev, 2017, 2019b,a). Let us note that this class of waves exist also for another type of media with surface energy such as strain-gradient media, see Vardoulakis and Georgiadis (1997); Georgiadis et al (2000); Yerofeyev and Sheshenina (2005); dell'Isola et al (2012a); Rosi et al (2015); Li et al (2015); Gourgiotis and Georgiadis (2015). The comparison of the Gurtin-Murdoch model with the Toupin-Mindlin strain gradient elasticity was given by Eremeyev et al (2018b), whereas the similarities with the dynamics of a square lattice was discussed by Eremeyev and Sharma (2019).

The aim of this paper is to compare the dispersion relations and condition of existence of anti-plane surface waves in various media with surface energy. The key-point of the surface elasticity is the presence of surface stresses $\boldsymbol{\tau}$. For the latter we assume additional constitutive equation. Here we consider the classic Gurtin-Murdoch model as well two extensions such as surface strain and surface stress gradient elasticity.

The paper is organized as follows. First, in Sect. 5.2 we present the basic equations for an elastic half-space with surface stresses. Then in Sect. 5.3 we consider various constitutive equations for $\boldsymbol{\tau}$. Here we introduce both the integral and differential constitutive equations. In other words, we consider both strongly and weak nonlocal models of surface elasticity. Finally, we discuss the dispersion relations in Sect. 5.4.

5.2 Anti-plane Motions of an Elastic Half-Space

In what follows we restrict ourselves by isotropic materials undergoing infinitesimal deformations. So in the bulk we have the Hooke law

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{e} + \lambda\boldsymbol{I}\ \mathrm{tr}\,\boldsymbol{e}, \quad \boldsymbol{e} = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right), \tag{5.1}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{e}$ are the stress and strain tensors, respectively, λ and μ are Lamé elastic moduli, tr is the trace operator, the superscript T stands for the transpose operation, ∇ is the 3D nabla operator, and $\boldsymbol{I}$ is the 3D unit tensor. Hereinafter we use the direct (coordinate-free) tensor calculus as described in Simmonds (1994); Lebedev et al (2010); Eremeyev et al (2018a). As a result, the gradient of the displacement vector

$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ is given by

$$\nabla \boldsymbol{u} = \frac{\partial u_j}{\partial x_i} \boldsymbol{i}_i \otimes \boldsymbol{i}_j,$$

where $\otimes$ denotes the dyadic product, x_1, x_2, x_3 are Cartesian coordinates with corresponding base vectors $\boldsymbol{i}_k$, $k = 1, 2, 3$, $\boldsymbol{x} = x_i \boldsymbol{i}_i$ is the position vector, t is time, and Einstein's summation rule is utilized. The equation of the motion is given by

$$\nabla \cdot \boldsymbol{\sigma} = \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2}, \tag{5.2}$$

where ρ is the mass density and the dot stands for scalar product. For a free surface with surface stresses we get the generalized Young-Laplace equation as a boundary condition

$$\boldsymbol{n} \cdot \boldsymbol{\sigma} = \nabla_s \cdot \boldsymbol{\tau} - m \frac{\partial^2 \boldsymbol{u}}{\partial t^2}, \tag{5.3}$$

where $\boldsymbol{n}$ is the unit outward vector of normal to the boundary, $\nabla_s \equiv \boldsymbol{P} \cdot \nabla$ is the surface nabla operator, $\boldsymbol{P} \equiv \boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n}$ is the surface unit second-order tensor, and m is the surface mass density, see Gurtin and Murdoch (1978).

Let us consider anti-plane motions of an elastic half-space given by the inequality $-\infty \le x_3 \le 0$. The displacement vector takes the form

$$\boldsymbol{u} = u(x_2, x_3, t) \boldsymbol{i}_1, \tag{5.4}$$

see Achenbach (1973). In this case the equation of motion (5.2) is reduced to the wave equation

$$\mu \Delta u = \rho \partial_t^2 u, \tag{5.5}$$

where $\Delta = \partial_2^2 + \partial_3^2$ is the 2D Laplace operator. For brevity, in what follows we will denote partial derivatives as $\partial_k = \partial/\partial x_k$ and $\partial_t = \partial/\partial t$. For the anti-plane motion $\boldsymbol{\tau}$ takes the form

$$\boldsymbol{\tau} = \tau (\boldsymbol{i}_1 \otimes \boldsymbol{i}_2 + \boldsymbol{i}_2 \otimes \boldsymbol{i}_1), \quad \tau = \tau(x_2, x_3, t)$$

with only one surface stress $\tau(x_2, x_3, t)$. As a result, the generalized Young-Laplace equation (5.3) can be transformed into

$$\sigma_{31} = \partial_2 \tau - m \partial_t^2 u$$

or, considering Hooke's law (5.1), into

$$\mu \partial_3 u = \partial_2 \tau - m \partial_t^2 u. \tag{5.6}$$

Thus, to complete the boundary-value problem statement one needs in the constitutive relations for τ.

5.3 Constitutive Relations Within the Surface Elasticity

After Gurtin and Murdoch (1975) in addition to constitutive equations in the bulk one should independently introduce constitutive relations for surface stresses $\boldsymbol{\tau}$. Here we consider the simplified linear Gurtin-Murdoch model and some of its extensions.

5.3.1 Simplified Linear Gurtin-Murdoch Model

Within this model we get the following constitutive relation

$$\boldsymbol{\tau} = 2\mu_s \boldsymbol{\epsilon} + \lambda_s \boldsymbol{P}\ \mathrm{tr}\,\boldsymbol{\epsilon}, \tag{5.7}$$

where the surface strain tensor is defined by the formula

$$\boldsymbol{\epsilon} = \frac{1}{2}\left[\boldsymbol{P}\cdot(\nabla_s \boldsymbol{u}) + (\nabla_s \boldsymbol{u})^T \cdot \boldsymbol{P}\right],$$

and λ_s and μ_s are the surface Lamé moduli.

For anti-plane deformations we get that

$$\boldsymbol{\epsilon} = \epsilon(\boldsymbol{i}_1 \otimes \boldsymbol{i}_2 + \boldsymbol{i}_2 \otimes \boldsymbol{i}_1), \quad \epsilon = \frac{1}{2}\partial_2 u$$

and

$$\tau = \mu_s \partial_2 u. \tag{5.8}$$

Let us note that as the anti-plane motions constitute a very specific class of deformations, in this case τ takes form (5.8) also for linearized (non-simplified) Gurtin-Murdoch model, see also discussion by Ru (2010), as well as for the linear Steigmann-Ogden model.

5.3.2 Linear Stress-gradient Surface Elasticity

Motivated by long range surface interactions as described by de Gennes (1981); de Gennes et al (2004); Israelachvili (2011), we recently proposed the integral-type constitutive relations of Eringen's type (Eremeyev, 2019a)

$$\boldsymbol{\tau}(\boldsymbol{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha(\|\boldsymbol{x}-\boldsymbol{x}'\|)\,[2\mu_s \boldsymbol{\epsilon}(\boldsymbol{x}') + \lambda_s\,(\mathrm{tr}\,\boldsymbol{\epsilon}(\boldsymbol{x}'))\,\boldsymbol{P}]\,\mathrm{d}x_1'\mathrm{d}x_2', \tag{5.9}$$

where $\alpha(s)$ is a kernel function, which can be taken as a fundamental solution of an elliptic differential equation. For example, introducing an elliptic differential operator

$\mathcal{L}$ we define α as the normalized solution of

$$\mathcal{L}(\partial_1, \partial_2)\alpha = \delta(\boldsymbol{x}), \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha(\|\boldsymbol{x} - \boldsymbol{x}'\|)\mathrm{d}x_1'\mathrm{d}x_2' = 1, \tag{5.10}$$

where $\delta(\boldsymbol{x})$ is the Dirac delta-function. In this case we can transform (5.9) into differential form

$$\mathcal{L}(\partial_1, \partial_2)\boldsymbol{\tau} = 2\mu_s\boldsymbol{\epsilon} + \lambda_s\boldsymbol{P}\,\mathrm{tr}\,\boldsymbol{\epsilon}. \tag{5.11}$$

After Eringen (2002) we can consider

$$\mathcal{L} = -q^{-2}\Delta + 1,$$

where the parameter q is a reciprocal length, as an example of proper strongly non-local model. Here we have

$$\alpha(s) = \frac{1}{2\pi}K_0(qs),$$

where K_0 is a modified Bessel function of the second kind. So we get the following stress-gradient constitutive equation

$$-q^{-2}\Delta\boldsymbol{\tau} + \boldsymbol{\tau} = 2\mu_s\boldsymbol{\epsilon} + \lambda_s\boldsymbol{P}\,\mathrm{tr}\,\boldsymbol{\epsilon}. \tag{5.12}$$

Other choices of the kernel functions are also possible, see Eringen (2002). For example, if we take $\alpha = \delta(\boldsymbol{x})$ we get (5.7).

In the case of anti-plane motions, Eq. (5.9) can be transformed into one scalar integral equation

$$\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha(\|\boldsymbol{x} - \boldsymbol{x}'\|)\mu_s\partial_2 u(\boldsymbol{x}')\,\mathrm{d}x_1'\mathrm{d}x_2',$$

or into its differential counterpart

$$\mathcal{L}(0, \partial_2)\tau = \mu_s\partial_2 u.$$

For $\mathcal{L} = -q^{-2}\Delta + 1$, this becomes

$$-q^{-2}\partial_2^2\tau + \tau = \mu_s\partial_2 u. \tag{5.13}$$

Obviously, Eq. (5.13) transforms into (5.8) at $q \to \infty$. So for this limit we get the classic Gurtin-Murdoch model. The presented here model belongs to the class of strongly non-local materials according to Maugin's classification, see Maugin (2017) for the general framework and Eremeyev (2019a) for more detail.

5.3.3 Linear Strain-gradient Surface Elasticity

Another non-local generalization of the Gurtin-Murdoch model can be introduced considering higher order gradient terms in the surface energy density

$$W_s = W_s(\boldsymbol{\epsilon}, \nabla_s \nabla_s \boldsymbol{u}),$$

see Eremeyev (2017). Partially the model was motivated by consideration of hyperbolic metasurfaces, see Eremeyev (2019b) and the reference therein. As a result, we came to the constitutive relation

$$\boldsymbol{\tau} = \mu_s \boldsymbol{\epsilon} + \lambda_s \boldsymbol{P} \operatorname{tr} \boldsymbol{\epsilon} - \mu_2 \nabla_s \cdot (\nabla_s \nabla_s \boldsymbol{u}), \tag{5.14}$$

where μ_2 is an additional surface elastic modulus. Here in the model there also exist surface hyperstresses as in the 3D strain-gradient elasticity, given by the formula

$$\boldsymbol{\mu} = \mu_2 \nabla_s \nabla_s \boldsymbol{u}.$$

For anti-plane deformations, Eq. (5.14) takes the form

$$\tau = \mu_s \partial_2 u - \mu_2 \partial_2^3 u. \tag{5.15}$$

As a result, Eq. (5.6) becomes a forth-order differential equation with respect to the tangent derivative.

5.4 Dispersion Relations

Considering the models above, we came to the boundary-value problem in the half-space which consists of the wave equation (5.5) and the boundary condition (5.6) where τ was introduced within the Gurtin-Murdoch, stress- and strain-gradient models according to (5.8), (5.13), and (5.15), respectively. Assuming steady-state behaviour, we consider a solution of (5.5) in the form

$$u = U(x_2, x_3) \exp(-\mathrm{i}\omega t), \tag{5.16}$$

where U is an amplitude, ω is a circular frequency, and $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. With (5.16), Eq. (5.5) transforms into

$$\mu \Delta U = -\rho \omega^2 U, \tag{5.17}$$

which has decaying at $x_3 \to -\infty$ solution

$$U = U_0 \exp(\varkappa x_3) \exp(\mathrm{i} k x_2), \tag{5.18}$$

where k is a wavenumber, U_0 is a constant, and $\varkappa$ is given by

$$\varkappa = \varkappa(k,\omega) \equiv \sqrt{k^2 - \frac{\omega^2}{c_T^2}}, \quad c_T = \sqrt{\frac{\mu}{\rho}},$$

where c_T is the phase velocity of transverse waves in the bulk (Achenbach, 1973).

A nontrivial solution of (5.18), that is with $U_0 \neq 0$, exists if and only if it satisfies the boundary conditions at $x_3 = 0$. The latter will lead to a dispersion relation, i.e., an equation relating k and ω.

The displacement field $u(x_2, x_3, t)$ according to (5.16) and (5.18) leads to a surface stress in the form

$$\tau = T \exp(\mathrm{i}kx_2)\exp(-\mathrm{i}\omega t),$$

where T is a constant. For (5.8), (5.13), and (5.15), T is given by

$$T = \mathrm{i}k\mu_s U_0, \tag{5.19}$$

$$T = \frac{\mathrm{i}k}{1 + q^{-2}k^2}\mu_s U_0, \tag{5.20}$$

$$T = \mathrm{i}k(\mu_s + \mu_2 k^2)U_0, \tag{5.21}$$

respectively. Substituting these dependencies into (5.6) we get the dispersion relations

$$\mu\varkappa(k,\omega) = m\omega^2 - \mu_s k^2, \tag{5.22}$$

$$\mu\varkappa(k,\omega) = m\omega^2 - q^2\frac{\mu_s k^2}{k^2 + q^2}, \tag{5.23}$$

$$\mu\varkappa(k,\omega) = m\omega^2 - \mu_s k^2 - \mu_2 k^4. \tag{5.24}$$

Introducing the phase velocity $c = \omega/k$ and characteristic wavenumber $p = \rho/m$ we transform (5.22)-(5.24) into dimensionless forms

$$\frac{c^2}{c_T^2} = \frac{c_s^2}{c_T^2} + \frac{p}{|k|}\sqrt{1 - \frac{c^2}{c_T^2}}, \tag{5.25}$$

$$\frac{c^2}{c_T^2} = \frac{c_s^2}{c_T^2}\left(1 + \frac{k^2}{q^2}\right)^{-1} + \frac{p}{|k|}\sqrt{1 - \frac{c^2}{c_T^2}}, \tag{5.26}$$

$$\frac{c^2}{c_T^2} = \frac{c_s^2}{c_T^2} + \frac{\mathbb{K}}{p^4}k^4 + \frac{p}{|k|}\sqrt{1 - \frac{c^2}{c_T^2}}, \tag{5.27}$$

where $\mathbb{K} = \mu_2 p^4/(c_T^2 m)$ and $c_s = \sqrt{\mu_s/m}$ is the shear wave velocity in the thin film associated with the Gurtin-Murdoch model.

Typical dispersion curves for these models are shown in Fig. 5.1 for different values of parameters. Let us discuss some similarities in dispersion curves. All curves start from the point $(0, c_T)$ with a horizontal tangent. So for small k that is for long waves there is no significant difference in models as it should be. Indeed, surface nonlocality plays a role for short waves. Moreover, within a fixed range $0 \leq k \leq k_1$,

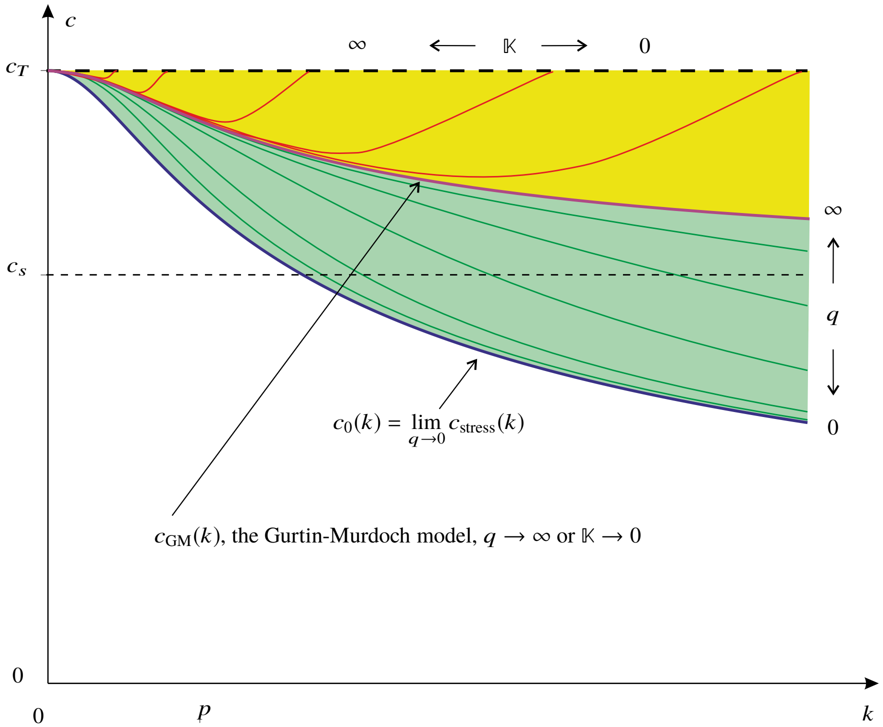

Fig. 5.1 Dispersion relations. c_{GM} curve corresponds to the Gurtin-Murdoch model. The dispersion curves c_{stress} for stress-gradient surface elasticity occupies the green area whereas dispersion curves c_{strain} for strain-gradient surface elasticity are in the yellow area. Here we assumed that $c_s = 3/4c_T$

the dispersion curve of the stress-gradient model for $q \to \infty$ will come arbitrarily close to the dispersion curve of the Gurtin-Murdoch model. The same behaviour demonstrate the dispersion curves for the strain-gradient model when $\mathbb{K} \to 0$. In what follows we assume the following notations: $c_{\text{GM}} = c_{\text{GM}}(k)$, $c_{\text{stress}} = c_{\text{stress}}(k)$, and $c_{\text{strain}} = c_{\text{strain}}(k)$ denote the phase velocity for the Gurtin-Murdoch, stress- and strain-gradient models.

For fixed q and $\mathbb{K}$ we have different behaviour of the dispersion curves for the stress- and strain gradient models at $k \to \infty$. For stress-gradient model we have that $c_{\text{stress}} \to 0$ when $k \to \infty$. Let us remind that the Gurtin-Murdoch dispersion curve tends to the finite velocity c_s at $k \to \infty$, see GM-curve in Fig. 5.1. For the strain-gradient model the dispersion curves approach the line $c_{\text{strain}} = c_T$ at $k = k_{\max}$, where $k_{\max}$ takes the value

$$k_{\max} = \sqrt{\frac{c_T^2 - c_s^2}{\mathbb{K}}}.$$

For the stress-gradient surface elasticity all dispersion curves are enclosed between the lower limiting curve for $q \to 0$, given by the formula

$$c_0^2 = \frac{c_T^2 p^2}{2k^2}\left(\sqrt{1+\frac{4k^2}{p^2}}-1\right), \tag{5.28}$$

and the dispersion curve of the Gurtin-Murdoch model, see Fig. 5.1. So we have the following bounds for c_{stress}

$$c_0(k) \le c_{\text{stress}}(k) \le c_{\text{GM}}(k). \tag{5.29}$$

Let us note that the dispersion curves for a square lattice lie also below the GM-curve, see Eremeyev and Sharma (2019). For the strain gradient model all dispersive curves are enclosed between the GM-curve and the line $c = c_T$, see Fig. 5.1,

$$c_{\text{GM}}(k) \le c_{\text{strain}}(k) \le c_T. \tag{5.30}$$

Thus, GM-curve separates dispersion curves for the stress- and stress-gradient model.

For all considered above models we consider the surface kinetic energy in the simplest form

$$K_s = \frac{1}{2} m \partial_t \boldsymbol{u} \cdot \partial_t \boldsymbol{u},$$

as was introduced by Gurtin and Murdoch (1978). Introduction of higher-order terms in the surface kinetic energy may significantly change the behaviour of the dispersion curves as in the case of the 3D models, see e.g. Askes and Aifantis (2011).

5.5 Conclusions

We have considered here the propagation of anti-plane surface waves in an elastic half-space with surfaces stresses within various models of surface elasticity. The linear Gurtin-Murdoch elasticity and strain- and stress-gradient surface elasticity models were compared. From the mathematical point of view the difference between the models consists of the boundary conditions at the half-space boundary. The analysis of dispersion relations was performed and the upper and lower bounds for the dispersion curves were found. In particular, it was shown that the dispersion curve for the Gurtin-Murdoch model separates the areas of dispersion curves for strain- and stress gradient surface elasticity.

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