Chapter 18
On Micropolar Theory with Inertia Production

Elena Vilchevskaya

Abstract This paper presents a new aspect in generalized continuum theory, namely micropolar media showing structural change. Initially the necessary theoretical framework for a micropolar continuum is presented. To this end the standard macroscopic equations for mass and linear and angular momentum are complemented by a recently proposed balance equation for the moment of inertia tensor containing a production term. The new balance and, in particular, the production is interpreted mesoscopically by taking the inner structure of micropolar media into account. Various of examples for the term are presented.

Keywords: Micropolar continua · Structural transformations · Inertia production

18.1 Introduction

Mechanics of Micropolar Continua is a theory with independent force and moment (couple) actions. That theory incorporates local rotations of points as well as translations assumed in classical elasticity. The idea of a couple stress can be traced to Voigt (1887) where the effects of couple stresses were investigated and a generalization of the classical theory of symmetric elasticity to a non-symmetric theory was made. The approach was further elaborated by the Cosserat brothers (Cosserat and Cosserat, 1909) who suggested to consider the rotational degrees of freedom of material particles as independent variables and so every particle contains six degrees of freedom: three displacements assigned to the macro-element, plus three rotations

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referring to the micro-structure. Thus, force and moment actions in the continuum were introduced independently, as it was done by Euler, and the angular momentum equation was explicitly used instead of being reduced to a symmetry statement of the stress tensor. This material model referred to as the Cosserat continuum provides the mathematical characterization of solid bodies with microstructure in which couple stresses, body couples, and local motions are included. These peculiarities of the Cosserat continuum model give a possibility to describe more complex media, for example, micro-inhomogeneous materials, particle assemblies, viscous fluids, fiber suspensions, liquid crystals, etc. Later Eringen and associates started to use the term micropolar to describe Cosserat media.

Although Hellinger (1914) paid tribute to the potential of the theory and obtained the general constitutive relations for stress and couple-stress, the ideas of the Cosserat continuum were not widely accepted and it was not until the 1960's that fully developed microstructure theories evolved. In fact, it was only after a paper by Ericksen and Truesdell (1957) that the ideas of the Cosserat brothers were revived. In this paper, the purely kinematical description of Cosserat continua emphasizing the one- and two-dimensional cases of rods and shells was developed. The original concept was modified in two ways. Firstly, the concept of directors defining the orientation of the material particle was introduced and secondly, these directors were also allowed to deform to describe a deformation of the material particle at the microscale. These are micromorphic continua in Eringen's classification (Eringen, 1999, 2001), which have nine degrees of freedom (three for microrotation and six for microdeformation). A particular case is that of continua with microstretch (Eringen, 1969), where the directors are orthogonal, but permit isotropic expansion or contraction in addition to rotation. This means, particles of microstretch continua have four additional degrees of freedom more than classical continua.

Later Günther (1958) developed a linear theory of the Cosserat continuum with an application to the continuum theory of dislocations, Grioli (1960) elaborated a theory of elasticity with a non-symmetric stress tensor, and Ericksen (1960c,a,b, 1961) developed a theory of anisotropic fluids and liquid crystal assuming that a fluid is a three-dimensional point continuum with one director at each point. Since the mid of 20th century a lot of publications devoted to the Cosserat continuum have appeared. Not even trying to give detailed information about various contributions we just refer to Mindlin and Tiersten (1962), Aero and Kuvshinskii (1961), Kröner (1963), Palmov (1964), Toupin (1962, 1964) as pioneers in the field. Later the micropolar elasticity was considered for example in Maugin (1998); Neff and Jeong (2009); Jeong and Neff (2010); Dyszłewicz (2004); Pietraszkiewicz and Eremeyev (2009); Ramezani and Naghdabadi (2007). There are also many publications on the plastic and visco-elastic Cosserat continuum, among them Lippmann (1969); de Borst (1993); Steinmann (1994); Forest and Sievert (2003, 2006); Grammenoudis and Tsakmakis (2005); Neff (2006). Variational problems in the micropolar continuum are investigated in Steinmann and Stein (1997); Nistor (2002). It is also worth mentioning recent collections (Maugin and Metrikine, 2010; Altenbach et al, 2011; Altenbach and Eremeyev, 2012; Sansour and Skatulla, 2012; Altenbach and Forest, 2016) and references therein where modern views on the micropolar media are presented.

The essential developments in the field of micropolar theory were made by Eringen and Suhubi (1964a,b); Eringen (1999, 2001); Eringen and Kafadar (1976); Eringen (1997). Unlike Ericksen and Truesdell (1957) and other early contributions, where the orientation of the material particle was defined by directors, they considered a field of orthogonal transformations (rotations) and not the directors themselves. In analogy to rigid body dynamics Eringen extended the Cosserat theory to include body microinertia effects and used the microinertia tensor $\boldsymbol{J}$ as the orientational descriptor. A truly new notion in his approach is an establishment of existence of a conservation law of micro-inertia. It is based on the concept of an indestructible "material particle" (polar particle) that is phenomenologically equivalent to a rigid body, see for example Eringen (1997); Truesdell and Toupin (1960); Mindlin (1964); Eringen and Kafadar (1976), where it is supposed that the inertia tensor changes only due to rotation of the material particle as a rigid body. So the inertia tensor in the current configuration can be written as follows:

$$\boldsymbol{J} = \boldsymbol{Q} \cdot \boldsymbol{J}_0 \cdot \boldsymbol{Q}^T, \tag{18.1}$$

where the inertia tensor in the reference configuration, $\boldsymbol{J}_0$, is a priory known constant tensor, $\boldsymbol{Q}$ is the tensor of microrotation.

Equation (18.1) can be rewritten in differential form:

$$\frac{\delta \boldsymbol{J}}{\delta t} = \boldsymbol{\omega} \times \boldsymbol{J} - \boldsymbol{J} \times \boldsymbol{\omega}, \tag{18.2}$$

where the Poisson equation

$$\frac{\delta \boldsymbol{Q}}{\delta t} = \boldsymbol{\omega} \times \boldsymbol{Q} \tag{18.3}$$

is taken into account. Here $\boldsymbol{\omega}$ is the angular velocity. We denote by

$$\frac{\delta(\cdot)}{\delta t} = \frac{d(\cdot)}{dt} + (\mathbf{v} - \mathbf{w}) \cdot \boldsymbol{\nabla}(\cdot) \tag{18.4}$$

the substantial derivative of a field quantity that characterizes the rate of change a property of the material point that was in the observation point at the certain moment of time, $\frac{d(\cdot)}{dt}$ is the total derivative that determines the rate of change of property in an observation point, $\mathbf{v}$ is the velocity of the material point and $\mathbf{w}$ is the mapping velocity of the observational point (see Ivanova et al (2016)).

Even if a micromorphic structure is considered, following Eringen (1976); Eringen and Kafadar (1976); Eringen (1999), many papers use the balance law for the conservation of inertia (e.g., see Oevel and Schröter, 1981; Chen, 2007). A different approach was suggested in Dłuzewski (1993), where it was assumed that the inertia of polar particles may change as the continuum deforms. Furthermore, in order to take the interaction of suspensions with viscous fluids surrounding the suspensions into account, Eringen (1984, 1985, 1991) proposed a modified balance law for microinertia:

$$\frac{\delta \boldsymbol{J}}{\delta t} = \boldsymbol{\omega} \times \boldsymbol{J} - \boldsymbol{J} \times \boldsymbol{\omega} - \boldsymbol{F}, \tag{18.5}$$

where the additional term $\boldsymbol{F}$ describes changes of the microinertia of rigid suspensions due to the fluid sticking to the suspensions.

This idea was further elaborated by Ivanova and Vilchevskaya (2016) who clearly stated that the tensor of inertia should be treated as an independent field. They considered the micropolar theory based on the spatial description. Within the spatial description, it is customarily to refer thermodynamic state quantities to an elementary volume V, fixed in space, containing an ensemble of micro-particles. In their approach, the tensor of inertia associated with the elementary volume was obtained as a result of averaging of the inertia tensors of micro-particles that constitute V. Because the elementary volume is an open system, its inertia tensor can change due to the inertia flux as micro-particles travel across the bounding surface. Moreover, to take into account internal structural transformations, such as combination or fragmentation of micro-particles, chemical reactions, or changes of anisotropy of the material, the authors in Ivanova and Vilchevskaya (2016) assumed that the inertia tensor in the reference configuration is no longer a constant tensor but an additional independent variable characterizing structural transformations in the media. As a result, they proposed a governing equation for the inertia tensor, which in contrast to former theories contains a production term. On the macroscopic continuum level, the production term must be considered as a new constitutive quantity for which an additional constitutive equation has to be formulated. The form of the constitutive equation depends on the problem under consideration and can be a function of many physical quantities, such as temperature, pressure, flow rate, etc.

For a better understanding of these new concepts, the authors in Ivanova and Vilchevskaya (2016) also presented a mesoscopic theory. The main idea was to connect information on a mesoscale by taking the intrinsic microstructure within the elementary volume into account with the balances of micropolar continua on the continuum level in combination with suitable constitutive equations. A similar approach for the case of material description was presented in the series of papers by Stojanović et al (1964) and Rivlin (1968), where the discrete structure of macro-particles constituting the medium was taken into account. It was assumed that each macro-particle consists of a number of micro-particles and characterizes by a position vector to the center of gravity of micro-particles and a number of directors. Later a transition from the dynamics of single particles to micropolar continuum was done by many researchers, for example, the homogenization approaches in Ehlers et al (2003) was based on the volume averages and in Mandadapu et al (2018) it was derived by means of the Irving-Kirkwood procedure. Various homogenization procedures also were used for the determination of the micropoloar moduli, see for example, van der Sluis et al (1999); Larsson and Diebels (2007); Larsson and Zhang (2007). However, the production term in the kinetic equation for the inertia tensor has never been considered from this point of view, to the best of our knowledge.

In this paper different examples of the production term introduction will be discussed and a potential of this approach for modeling materials with higher inner degrees of freedom by various example problems will be illustrated. It will be shown that the new term in the balance of inertia allows to model additional features of materials, namely processes inherent of considerable structural changes.

In what follows, we firstly consider the theoretical aspects of micropolar theory with the inertia production from the mesoscopic point of view, which results in an answer to the question of how to determine the inertial and kinematic characteristics of the polar particle within the spatial description. Then, because the balance for the inertia tensor field is extended by the production term, we also discuss possible forms of the production term on the continuum level in relation with properties of micro-particles which are located in the elementary volume.

18.2 Outline of the Theory

Within the spatial description, it is customary to refer thermodynamic state quantities to an elementary volume, fixed in space. If the length scale difference between an elementary or micro-particle (microscale) and the whole medium under consideration (macroscale) is sufficiently large (e.g., a sand grain in a sand heap), then the elementary volume containing sufficiently many micro-particles can be introduced in the sense of a representative volume element (RVE). It means that within that approach a continuum is understood as a manifold of RVEs (Fig. 18.1). The RVE, by turn, is constructed as a manifold of micro-particles and links the micro- and mesoscales. Note that the presence of a very large number of micro-particles in the RVE is required since establishing a continuous field theory would not be possible otherwise, and fluctuations would become dominant.

Let us consider a volume element containing $N(\mathbf{r}, t)$ micro-particles. The position vector $\mathbf{r}$ corresponds to the geometrical center of the volume and is independent of the motion of the medium. Generally the volume may be considered as moving but we will suppose that it is fixed in space. It means that the position vector is

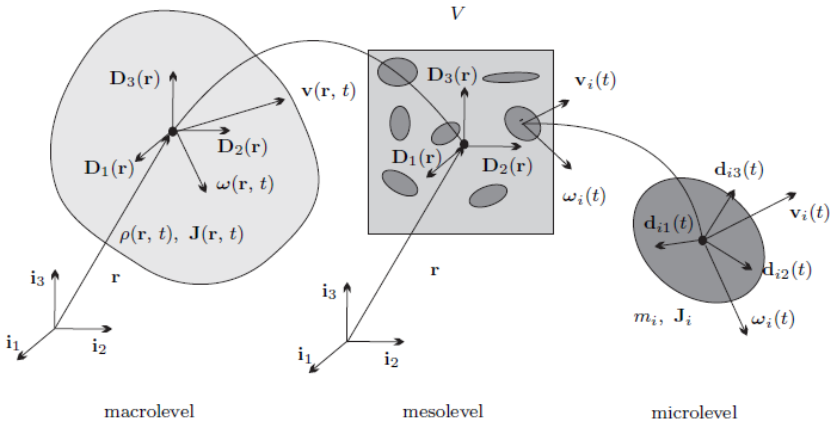

Fig. 18.1 Continuum as a manifold of elementary volumes

independent on time, t, and the velocity of the observational point in (18.4) equals to zero. Note that the volume is an open system since, as the medium moves, different micro-particles each having their own mass, inertia tensor, translational and angular velocities pass through the volume.

Since at different moments the volume consists of different micro-particles each having their own mass and inertia tensor, one has to introduce corresponding fields at the macrolevel as effective characteristics. The micro-particles within the RVE are assumed to be replaceable by an ensemble of identical average particles each having an average mass and an average tensor of inertia.

$$m(\mathbf{r}, t) = \frac{1}{N}\sum_{i=1}^{N} m_i, \qquad \hat{\boldsymbol{J}}(\mathbf{r}, t) = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{J}_i. \tag{18.6}$$

The second equation calls for a short explanation. The field $\hat{\boldsymbol{J}}(\mathbf{r}, t)$ characterizes the size, shape and orientation of the average particle rather than the mass distribution over the RVE. In fact, it is nothing more than an effective characteristic of rotational inertia that should not be associated with a real material particle. Note that if the micro-particles are randomly oriented within the RVE, then, due to symmetry consideration, the averaged tensor $\hat{\boldsymbol{J}}(\mathbf{r}, t)$ must be a spherical tensor.

If $n(\mathbf{r}, t) = N/V$ denotes the number of particles per unit volume, then the mass density, ρ, and the volumetric density of the moment of inertia are expressed as:

$$\rho = nm, \qquad \rho\boldsymbol{J} = n\hat{\boldsymbol{J}}, \tag{18.7}$$

where $\boldsymbol{J} = \hat{\boldsymbol{J}}/m$ is the average geometrical moment of inertia of a single particle. However, from the continuum point of view, $\boldsymbol{J}$ stands for the specific density of the moment of inertia of the elementary volume and $\rho\boldsymbol{J}$ refers to the volumetric density of the inertia tensor of the elementary volume. Thus, the inertial characteristics of the elementary volume are assumed to coincide with those of the average particle. Within this approach, the inertial properties of the medium are only weakly dependent on the size of the elementary volume.

The momentum and spin of the elementary volume consisting of the original micro-particles are required to equal those of the elementary volume consisting of average particles. The linear and angular velocities are obtained from these conditions

$$\frac{1}{N}\sum_{i=1}^{N} m_i\mathbf{v}_i = m\mathbf{v}, \qquad \text{or} \qquad \frac{1}{V}\sum_{i=1}^{N} m_i\mathbf{v}_i = \rho\mathbf{v}, \tag{18.8}$$

$$\frac{1}{N}\sum_{i=1}^{N} \boldsymbol{J}_i\cdot\boldsymbol{\omega}_i = \rho\boldsymbol{J}\cdot\boldsymbol{\omega}. \tag{18.9}$$

Similar definitions for the mass density and so-called barycentric velocity, $\mathbf{v}$, can be found in treatises on multi-component and porous media (Loret and Simões, 2005; Wilmanski, 2008).

It should be noted that with respect to the translational degrees of freedom, the spatial description only considers the current configuration. Nevertheless, the concept of a reference configuration for the rotational degrees of freedom should be introduced. To describe the average particle rotation, we choose some fixed state of the medium that may be taken at $t = 0$ or another fixed instant and call this state the reference configuration. In order to determine the orientations of particles, reference directors $\boldsymbol{D}_k(\mathbf{r})$, $(k = 1,2,3)$ must be locally introduced at each point of the space (Fig. 18.1a). These directors may coincide with the base vectors of the reference coordinate system or can be chosen independently and, say, coincide with the primary axes of $\boldsymbol{J}(\mathbf{r}, 0)$.

Note also that within the spatial description the translational and angular velocities are the primary quantities. The displacement, $\mathbf{u}$, and the microrotation tensor, $\boldsymbol{Q}$, have to be found as solutions from the corresponding differential equations:

$$\mathbf{v} = \frac{\delta \mathbf{u}}{\delta t}, \quad \frac{\delta \boldsymbol{Q}}{\delta t} = \boldsymbol{\omega} \times \boldsymbol{Q}, \tag{18.10}$$

provided $\mathbf{v}$ and $\boldsymbol{\omega}$ are known. Furthermore, the microrotational tensor is different from the rotation tensor of the elementary volume as a rigid body as well as from the rotation tensor obtained by averaging over all micro-particles found in a given volume element at a given time. In fact, it describes the change of directors from the reference to the current position. To this end, it suffices to postulate that the tensors of rotation of all macroparticles in the reference configuration are identity tensors $\boldsymbol{Q}(\mathbf{r}, 0) = \boldsymbol{I}$.

Now we assume that the tensor of inertia in spatial description has a representation similar to (18.1):

$$\boldsymbol{J}(\mathbf{r},t) = \boldsymbol{Q}(\mathbf{r},t) \cdot \boldsymbol{J}_0(\mathbf{r},t) \cdot \boldsymbol{Q}^T(\mathbf{r},t), \quad \boldsymbol{J}_0(\mathbf{r},0) = \tilde{\mathbf{J}}_0(\mathbf{r}). \tag{18.11}$$

The key point within this approach is that the inertia tensor in the reference configuration is no longer a known characteristic of the medium. Indeed, let us suppose that $\boldsymbol{\omega}(\mathbf{r}, t) = \mathbf{0}$, then $\boldsymbol{Q}(\mathbf{r}, t) = \boldsymbol{I}$ and Eqns. (18.6), (18.7) determine the reference inertia tensor $\boldsymbol{J}_0(\mathbf{r}, t)$. Being the averaged characteristic of the micro-particles within the elementary volume the inertia tensor in the reference configuration may change due to different reasons.

- In the case of an inhomogeneous distribution of micro-particles within the medium, the tensor of inertia associated with the elementary volume changes in a certain way as micro-particles travel across its boundary surface S. Mathematically, the balance of $\rho \boldsymbol{J}$ can be expressed as

$$\frac{d}{dt} \int_V \rho \mathbf{J}_0 \, dV = - \int_S (\mathbf{n} \cdot \mathbf{v}) \rho \mathbf{J}_0 \, dS, \tag{18.12}$$

  where $\mathbf{n}$ is the outward unit normal to S.
  Equation (18.12), after application of Gauss's theorem and pulling the differentiation under the integral, leads to the following local statement for the balance of inertia:

$$\frac{\delta \boldsymbol{J}_0}{\delta t} = \mathbf{0}, \quad \Longleftrightarrow \quad \frac{\partial \boldsymbol{J}_0}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} \boldsymbol{J}_0 = \mathbf{0}. \tag{18.13}$$

Here the local conservation of mass is taken into the account. The solution of equation (18.13) determines the inertia tensor change due to inertia flux, however, there is no inertia production.

- The size and shape of the micro-particles within the elementary volume may change due to phase transitions and chemical reactions or due to fragmentation or combination of the micro-particles. Thus, the size and shape of the average particle also change and leads to a change of $\boldsymbol{J}_0$.
- Nonspherical micro-particles may have a tendency to align with an external applied field or conversely to realign due to thermal motion. This describes a transition from the isotropic state to nonisotropic one or vice versa. It will reflect a change in the average tensor of inertia. Note that if the micro-particles within the elementary volume remain the same and only change their orientation in space, from (18.6), (18.7) follows that the spherical part of the inertia tensor is constant and the changes of anisotropy of the material are characterized only by the deviatoric part of $\boldsymbol{J}_0$.

In view of the above remarks, we may conclude that the inertia tensor in the reference configuration should be treated as a variable rather then a parameter. As a result the reference inertia tensor has to satisfy the following balance equation:

$$\frac{\delta \boldsymbol{J}_0}{\delta t} = \boldsymbol{\chi}_0, \tag{18.14}$$

or in explicit form

$$\frac{\partial \boldsymbol{J}_0}{\partial t} = -\mathbf{v} \cdot \boldsymbol{\nabla} \boldsymbol{J}_0 + \boldsymbol{\chi}_0, \tag{18.15}$$

where the first term on the right side describes the inertia flux and the production term $\boldsymbol{\chi}_0$ reflects structural transformations of the media. The form of the production term depends on the physical interpretation of microstructural changes. It can, for instance, depend on $\boldsymbol{J}_0$ as well as other characteristics of the medium, such as density, temperature, stresses, etc. It can also depend on external stimuli such as external electric or magnetic fields.

Then from (18.11), $(18.10)_2$ and (18.14) it follows that the tensor of inertia in the current configuration has the form:

$$\frac{\delta \boldsymbol{J}}{\delta t} = \boldsymbol{\omega} \times \boldsymbol{J} - \boldsymbol{J} \times \boldsymbol{\omega} + \boldsymbol{\chi}, \qquad \boldsymbol{\chi} = \boldsymbol{Q} \cdot \boldsymbol{\chi}_0 \cdot \boldsymbol{Q}^T, \tag{18.16}$$

where the first two terms describe the inertia tensor change due to rotation of the average particle and the last term is responsible for the inertia production due to internal structural transformations.

The inertia tensor production leads to the production of spin. Then the balance of moment of momentum is formulated as follows

$$\rho \frac{\delta (\boldsymbol{J} \cdot \boldsymbol{\omega})}{\delta t} = \boldsymbol{\nabla} \cdot \boldsymbol{\mu} + \boldsymbol{\sigma}_{\times} + \rho \mathbf{m} + \rho \boldsymbol{\chi} \cdot \boldsymbol{\omega}. \tag{18.17}$$

Here $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are the non-symmetric Cauchy and couple stress tensors, $(\mathbf{a}\otimes\mathbf{b})_\times = \mathbf{a}\times\mathbf{b}$ is the Gibbsian cross, and $\mathbf{m}$ is the specific couple density. The extra term in the balance equation, with $\boldsymbol{\chi}$, describes the moment of momentum production due to structural transformations.

However, by taking into account the balance equation for the inertia tensor in the current configuration (18.16) we obtain the spin balance equation in classical form:

$$\rho \boldsymbol{J}\cdot\frac{\delta\boldsymbol{\omega}}{\delta t} = -\boldsymbol{\omega}\times\boldsymbol{J}\cdot\boldsymbol{\omega} + \boldsymbol{\nabla}\cdot\boldsymbol{\mu} + \boldsymbol{\sigma}_\times + \rho\mathbf{m}. \tag{18.18}$$

Thus in the suggested model, the basic equations are essentially the same as in the classical approach except for replacement of the classical conservation law of micro-inertia by the inertia tensor balance equation containing the production term.

It should be also noted that traditionally the tensor of inertia of a continuum particle plays a role only in context with rotations. However, within this approach the balance equation for $\boldsymbol{J}$ and hence the production term in Eqn.(18.14) are physical meaningful by themselves, independent of the angular velocity and may serve as an indicator of the internal structural changes.

18.3 Special Cases for the Production Term

We shall now proceed and illustrate the theory by several examples. By the first example, it is intended to show what happens to the tensor of inertia if the number of micro-particles and their size within the elementary volume change, for example due to the presence of a crusher. By the second example, the impact of a changing moment of inertia onto rotational motion will be demonstrated. In particular, the change of the state of rotation of a isotropic thermoelastic continuum will be studied. The average particle will undergo a nonuniform change of external temperature affecting its moment of inertia. Note that within the classical framework of micropolar theory a change of temperature would not influence rotation. However, within the to-be-presented theory, changes in temperature will influence the inertia tensor and hence couple to angular velocity. The last example will describe changes of anisotropy of a material under an external electrical field.

18.3.1 Milling Matter in a Crusher

As the first very simple example let us consider a continuous flow of granular matter of height H moving on a conveyor belt in the x-direction at a constant, prescribed speed, $\mathbf{v}_0 = v_0\mathbf{e}_x$. On its way it enters a region $0 \le x \le L$, where it is continuously crushed by an external distributed force, $\mathbf{p}_0 = p_0\mathbf{e}_y$, applied at the top so that smaller and smaller particles will form. Note that in spite of the fact that the micro-particles have a very irregular shape the homogenized tensor of inertia on a continuum level is

isotropic due to the statistically random distribution of micro-particles of different size and shape, as illustrated in Fig. 18.2 in the left inset. During the milling process the mass and characteristic size of a micro-particle will decrease over time (right inset on the top of the figure), which, under the assumption that the distribution stays isotropic, leads to a decreasing moment of inertia on the macro-level. At the same time the mass density of the elementary volume remains the same. The isotropy means that the production term also has to be a spherical tensor, $\chi = \chi \boldsymbol{I}$, and Eq. (18.16) turns into a scalar one:

$$\frac{\delta J}{\delta t} = \chi. \tag{18.19}$$

Here the identity $\mathbf{a} \times \boldsymbol{I} = \boldsymbol{I} \times \mathbf{a}$ is taken into account.

We assume that the production term is given by the following expression:

$$\chi = -\alpha_0 \operatorname{tr} \boldsymbol{\sigma}\, (J - J_*)\big(H(x) - H(x - L)\big)\boldsymbol{I}, \tag{18.20}$$

where $\boldsymbol{\sigma}$ is the stress tensor, $H(x)$ is the Heaviside step function, J_* and α_0 are positive constants, which can be interpreted intuitively as being related to the minimum grain size the particles can be crushed to, and to the inverse of the particle toughness. Thus, being the characteristics of the material and not of the crusher, they may be considered as constitutive properties. At the same time $\operatorname{tr}\boldsymbol{\sigma}$ describes a conversion of the crusher action to a material response. In other words, it is related to the effectiveness of the crusher and transmission of its external forces into the material. Hence, in this case, the production term depends on the material properties, external action, and space.

Since the material in the crusher is under a significant pressure we will model it as a linear-elastic material. For linear elasticity the Cauchy stresses $\boldsymbol{\sigma}$ is related to the strain $\boldsymbol{\varepsilon}$ by:

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon}, \qquad \boldsymbol{C} = k(J)\boldsymbol{I}\boldsymbol{I} + 2\mu(J)\left({}^4\boldsymbol{I} - \frac{1}{3}\boldsymbol{I}\boldsymbol{I}\right), \tag{18.21}$$

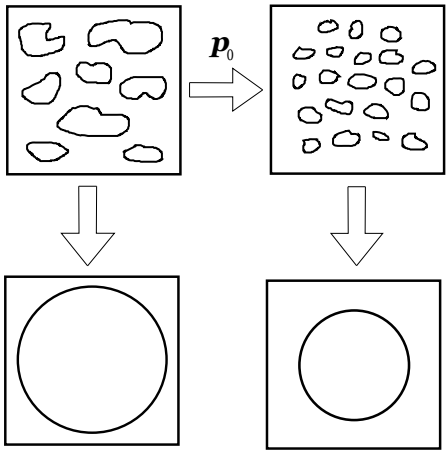

Fig. 18.2 Structural shape change and corresponding homogenization

where ${}^4\boldsymbol{I}_{ijkl} = (\delta_{ik}\delta_{lj} + \delta_{il}\delta_{kj})/2$ denotes the fourth rank identity tensor, δ_{ij} is the Kronecker symbol. It is assumed that the bulk and shear, moduli, k and μ, depend on the particle size:

$$k(J) = k_* f_k\,(J/J_*), \quad \mu(J) = \mu_* f_\mu\,(J/J_*), \quad f_k(1) = f_\mu(1) = 1. \tag{18.22}$$

Here k_* and μ_* are the bulk and shear moduli of the material consisting of the particles of the minimal size. The functions f_k and f_μ depend on the material and have to be obtained from experiments. For example, in Hamilton (1971) it was shown that the bulk modulus decreases and the shear modulus increases with decreasing grain size. Thus we assume that the elastic modules depend on the moment of inertia as follows:

$$k(J) = k_*\,(J/J_*)^2, \quad \mu(J) = \mu_*\,(J/J_*)^{-2}. \tag{18.23}$$

Furthermore it is assumed that the material moves freely along the x-axis while its movement in the y-direction is limited by the walls. Then in the absence of a body force the equilibrium condition $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = \mathbf{0}$, (18.19) and (18.20) yield:

$$\frac{\partial \bar{J}}{\partial \bar{t}} + \frac{\partial \bar{J}}{\partial \bar{x}} = -\xi\,(\bar{J} - \bar{J}_*)\big(H(\bar{x}) - H(\bar{x}-1)\big), \quad \xi = \frac{9\bar{\alpha}}{3 + \dfrac{\mu_*}{k_*}\left(\dfrac{\bar{J}_*}{\bar{J}}\right)^4}. \tag{18.24}$$

The bar on symbols refers to dimensionless quantities, namely,

$$\bar{x} = \frac{x}{L}, \qquad \bar{t} = \frac{v_0}{L}t, \qquad \bar{J} = \frac{J}{J^0}, \qquad \bar{J}_* = \frac{J_*}{J^0}, \qquad \bar{\alpha} = \frac{Lp_0\alpha_0}{v_0}, \tag{18.25}$$

where J^0 is the maximal moment of inertia.

Eqn. (18.24) has to be supplemented with initial and boundary conditions. In order to obtain a non-trivial solution, the initial distribution of the particles along the vertical axis at the left side of the crusher at $x = 0$ has to be inhomogeneous. For simplicity we will consider a linear distribution:

$$\bar{J}(0, \bar{z}, \bar{t}) = \left(\bar{J}_* - \bar{J}^0\right)\bar{z} + \bar{J}^0. \tag{18.26}$$

The numerical solutions of (18.24), (18.26) based on an explicit method of discrete integration are presented in Fig. 18.3, where the distributions of the moment of inertia for a vertical and a horizontal cut within the crusher area (stationary case) are visualized in dimensionless form. It is clearly visible that the moment of inertia decreases linearly from the bottom to the top of the crusher. At the same time, the distributions of the moment of inertia along the x-axis have an exponential shape.

During the computation the following parameters were used

$$\bar{\alpha} = 1.5, \qquad \frac{\mu_*}{k_*} = 0.1.$$

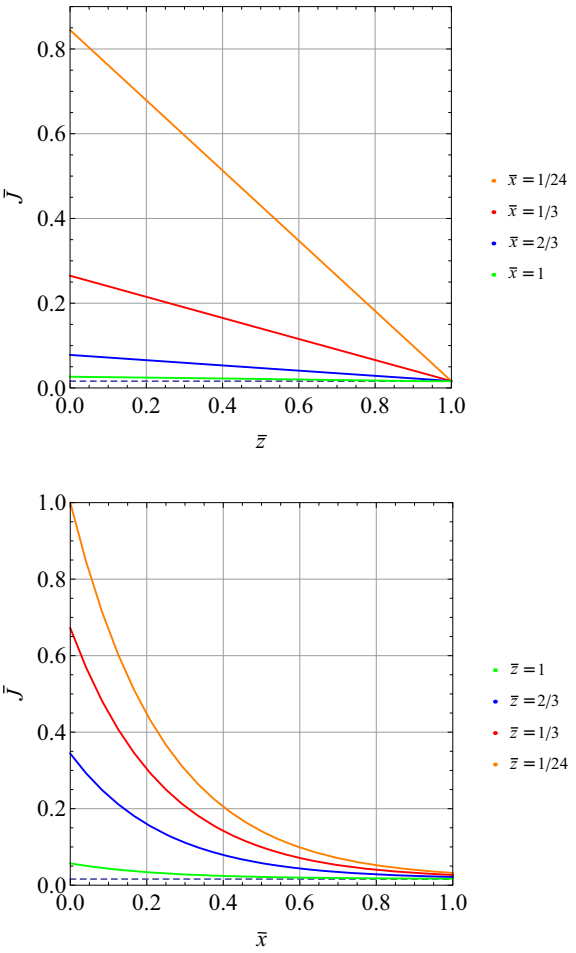

Fig. 18.3 Distribution of the moment of inertia (top inset – vertical cut, bottom inset – horizontal cut through crusher region)

The influence of the last parameter on the crushing process is shown in Fig. 18.4, where the stationary distributions of the moment of inertia along the crusher region for constant and for variable elastic moduli are presented. As can be expected the difference is more pronounced for large values of the moment of inertia. In Müller et al (2017) an analytical solution to a very simple one dimensional initial-boundary value problem for non-homogeneous crushing of particles was found based on the method of characteristics. The similar problem for viscous material was considered in Fomicheva et al (2019).

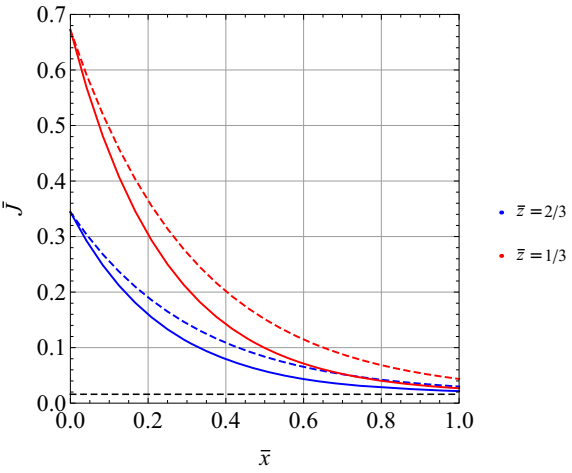

Fig. 18.4 Profiles of the moment of inertia for variable and constant elastic moduli (solid and dashed lines, respectively)

18.3.2 Turning Heat Conduction into Space-varying Rotational Motion

We next consider a medium consisting of thermoelastic micro-particles homogeneously distributed over a rectangular plate: $x \in [0, L]$, $y \in [-L_y, L_y]$, $z \in [-L_z, L_z]$. The medium represents the behavior of a homogeneous mix of micro-particles of arbitrary size and shape on the mesoscale, that results in the isotropic tensor of inertia on a continuum level, $\boldsymbol{J} = J\boldsymbol{I}$.

Initially the temperature of the media is also homogeneous and equal to T_0. By positioning the medium in between two reservoirs kept at temperatures T_0 and T_L and attached at positions $x = 0$ and $x = L$ of the region, respectively, the temperature of this medium will gradually change. The temperature development is described by the heat conduction equation after (Zhilin, 2012):

$$\rho\, c_v \frac{\delta T}{\delta t} = \boldsymbol{\sigma}_d : (\boldsymbol{\nabla} \otimes \mathbf{v} + \boldsymbol{I} \times \boldsymbol{\omega}) + \boldsymbol{\mu}_d : (\boldsymbol{\nabla} \otimes \boldsymbol{\omega}) + \rho q - \boldsymbol{\nabla} \cdot \mathbf{h}\,. \tag{18.27}$$

Here T is the absolute temperature, c_v is the specific heat capacity at constant volume, double convolution means $(\mathbf{a} \otimes \mathbf{b}) : (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})$, q is the specific heat source, $\mathbf{h}$ is the heat flux, and $\boldsymbol{\sigma}_d$ and $\boldsymbol{\mu}_d$ are the inelastic (dissipative) parts of the stress tensor and couple stress tensor:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_e + \boldsymbol{\sigma}_d, \quad \boldsymbol{\mu} = \boldsymbol{\mu}_e + \boldsymbol{\mu}_d, \tag{18.28}$$

where $\boldsymbol{\sigma}_e$ and $\boldsymbol{\mu}_e$ are the elastic (velocity independent) parts of the stress tensor and couple stress tensor.

For simplicity we suppose that the macro-particles have only rotational degrees of freedom and their translational velocities are equal to zero. Then, for an unconstrained

medium in absence of body forces the momentum balance equation is automatically fulfilled.

The system of equations (18.18), (18.19) and (18.27) has to be supplemented by constitutive equations. We suppose that the heat conduction is governed by the Fourier's law

$$\mathbf{h} = -\kappa \boldsymbol{\nabla} T,$$

with κ being the thermal conductivity, the elastic part of the couple stress tensor equals to zero, and write the following constitutive equation for its dissipative part according to Zhilin (2012):

$$\boldsymbol{\mu}_d = -\beta(\boldsymbol{\nabla} \times \boldsymbol{\omega}) \times \boldsymbol{I}, \tag{18.29}$$

where β has the meaning of a frictional coefficient. In order to formulate a constitutive equation for the production term we consider the free thermal expansion of the spherical particle under the assumption that the temperature increase is instantaneously assumed by the particle. Then the moment of inertia changes in accordance with the temperature field:

$$J(x,t) = J_0 \left[1 + \alpha(T(x,t) - T_0)\right]^2, \tag{18.30}$$

with J_0 being the initial moment of inertia, and α being the linear coefficient of thermal expansion. Therefore the production term can be written as:

$$\chi = \frac{\partial J}{\partial t} = 2J_0\alpha \left(1 + \alpha(T - T_0)\right) \frac{\partial T}{\partial t}. \tag{18.31}$$

As one can see the production depends on the material properties and vanishes at the constant temperature field.

To keep the problem one-dimensional we also assume that:

$$\boldsymbol{\omega}(x,t) = \omega(x,t)\mathbf{e}_z, \quad \mathbf{m}(x,t) = m_0\mathbf{e}_z. \tag{18.32}$$

Thus the development of temperature, the moment of inertia and angular velocity can be obtained as a result of solution of a coupled system of partial differential equations in dimensionless form:

$$\begin{aligned} \frac{\partial \bar{T}}{\partial \bar{t}} &= \delta \left(\frac{\partial \bar{\omega}}{\partial \bar{x}}\right)^2 + \frac{\partial^2 \bar{T}}{\partial \bar{x}^2}, \\ \frac{\partial \bar{J}}{\partial \bar{t}} &= 2\bar{\alpha}[1 + \bar{\alpha}(\bar{T} - 1)]\frac{\partial \bar{T}}{\partial \bar{t}}, \\ \bar{J}\frac{\partial \bar{\omega}}{\partial \bar{t}} &= \eta \frac{\partial^2 \bar{\omega}}{\partial \bar{x}^2} + \bar{m}, \end{aligned} \tag{18.33}$$

$$\bar{\alpha} = \alpha T_0, \quad \delta = \frac{\beta m_0}{\kappa T_0 J_0}, \quad \eta = \frac{\beta c_v}{\kappa J_0}, \quad \bar{m} = \omega_0 \frac{L^2}{D}, \quad \omega_0 = \sqrt{\frac{m_0}{J_0}}, \quad D = \frac{\kappa}{\rho c_v},$$

$$\bar{x} = \frac{x}{L}, \qquad \bar{t} = \frac{D}{L^2}t, \qquad \bar{T} = \frac{T}{T_0}, \qquad \bar{J} = \frac{J}{J_0}, \qquad \bar{\omega} = \frac{\omega}{\omega_0}.$$

with the following initial and boundary conditions:

$$\bar{T}(\bar{x}, \bar{t} = 0) = 1, \quad \bar{J}(\bar{x}, \bar{t} = 0) = 1, \quad \bar{\omega}(\bar{x}, \bar{t} = 0) = 0,$$
$$\bar{T}(\bar{x} = 0, \bar{t}) = 1, \quad \bar{T}(\bar{x} = 1, \bar{t}) = \frac{T_L}{T_0}, \quad \left.\frac{\partial \bar{\omega}}{\partial \bar{x}}\right|_{\bar{x}=0;1} = 0.$$

Note that the angular velocity related boundary conditions are necessary only if the viscosity is taken into the account. The proper choice of boundary conditions for the angular velocity is a complex issue. Generally, two types of boundary conditions are considered. The first type is so-called "strict adhesion", see, for example, Eringen (2001) ($\bar{\omega}(\bar{x} = 0; 1, \bar{t}) = 0$). The second one used here corresponds to an absent couple stress on the boundary.

The developments of angular velocity at three dimensionless times, $\bar{t} = 0.005$ (green), $\bar{t} = 0.01$ (blue), and $\bar{t} = 0.03$ (red) are depicted in Fig. 18.5. It is seen that the obtained profile of angular velocity is nonlinear in contrast to the classical approach where the angular velocity does not change over the sample length. The nonlinear behavior reflects the fact that distribution of the inertia moment over the sample mimics the temperature profile and as a result it follows from Eq. (18.33)

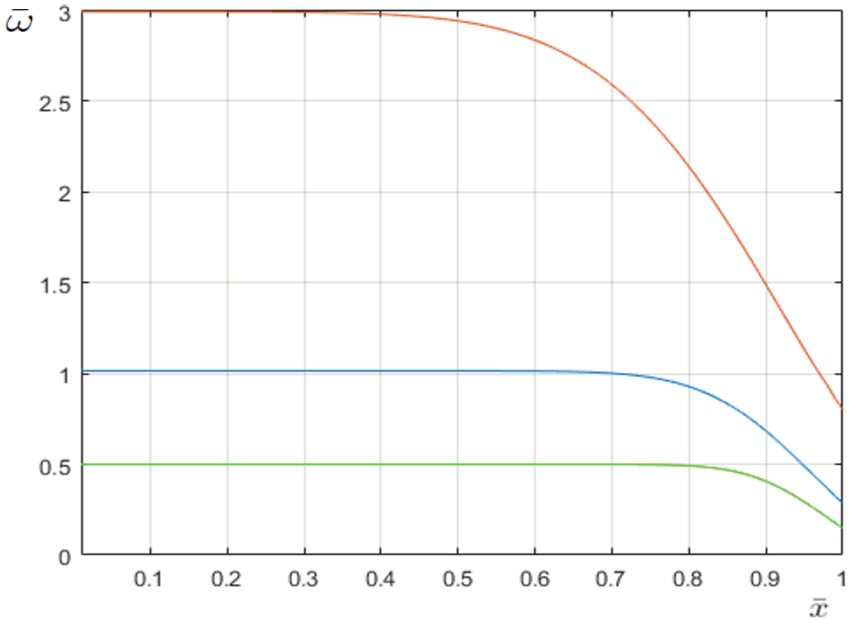

Fig. 18.5 Angular velocity distribution over the sample at different moments of time ($\bar{m} = 100$, $\bar{T}(\bar{x} = 1, \bar{t}) = 2, \eta = 1, \delta = 1$)

that the angular acceleration varies for particles with different temperature. Different boundary conditions and time-dependent the volume moment couple density was considered in Morozova et al (2019).

18.3.3 Dipolar Polarization

In order to describe anisotropic changes let us consider a material, which, on a mesoscale, consists of an assembly of dipoles. Due to thermal motion the dipoles are randomly oriented in the substance so that there is no overall charge in the material (Fig. 18.6, top left). The initial, homogenized macro-inertia tensor is then the isotropic spherical tensor.

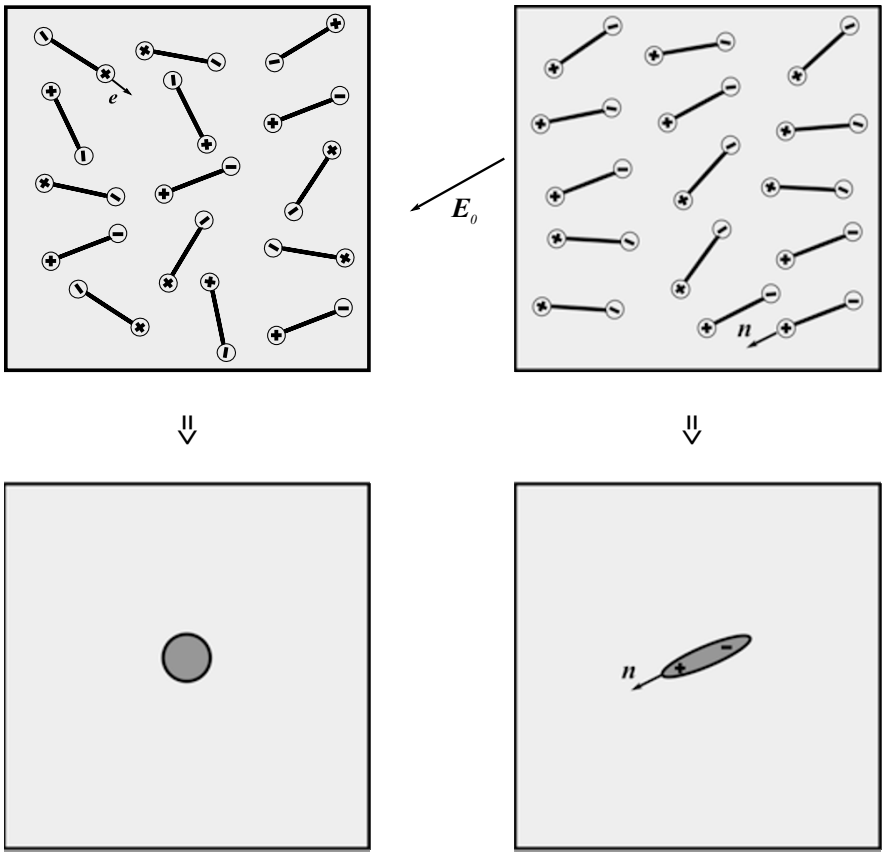

Fig. 18.6 Structural shape change and corresponding homogenization

When an external electric field $\boldsymbol{E} = E_0\mathbf{n}$ is applied, the dipoles tend to align with the applied field to lower their electrostatic energy, basically the positive end of the dipole would like to join the negative end of the applied field. Such behavior can be observed in dipolar polarization, for materials with build-in dipoles that are independent of each other, i.e. they don't interact and they can be rotated freely by an applied field (Kestelman et al, 2013). However, even in case of liquids or gases, where molecules are free to rotate, a complete alignment is impossible because the tendency of dipoles to orient along into the field direction will be counteracted by thermal motion. Thus, as a result of the combined action of the external and internal actions, a dominating orientation of the molecules along the direction of the electric field occurs and the transversally-isotropic state is achieved (Fig. 18.6, right).

Upon the applied electric field removal the thermal motion randomizes the alignment of the dipoles and returns the material into the initial isotropic state. The objective is now to describe the transition processes.

First note that switching back and forth between the isotropic and transversally-isotropic states is characterized only by the deviatoric part of the inertia tensor since the micro-particles within the elementary volume do not change. Thus, we have a purely deviatoric production. Second, since the micro-particles rotate in all possible directions during the orientation and disorientation processes, the macroscopic spin and, therefore, the macroscopic angular velocity is zero. Hence, the macrorotation tensor is an identity tensor. Under an assumption that the process occurs in the same manner at all points of space the balance of the inertia tensor simplifies and reads:

$$\boldsymbol{J}^{sh}(t) = J_0\boldsymbol{I} = \text{const}, \qquad \frac{\mathrm{d}\boldsymbol{J}^{\mathrm{dev}}(\mathrm{t})}{\mathrm{dt}} = \boldsymbol{\chi}, \tag{18.34}$$

where

$$\boldsymbol{J}^{dev}(t) = \frac{1}{15}\left(c(t)^2 - a(t)^2\right)(\boldsymbol{I} - 3\mathbf{nn})\,. \tag{18.35}$$

$a(t)$ is the semi-axis of the plane of isotropy and $c(t)$ is the semi-axis in the direction of $\mathbf{n}$.

The production term describing the microparticle alignment has to depend on the direction of the external field and on the time of particle orientation under the field action, τ_p, which defines the polarization setting time. The higher the magnitude of the electric field and lower the temperature the shorter is the time of dipolar polarization setting. For instance, $\tau_p = \alpha(T)/E_0$ can be taken as a simple example. Here the parameter α in units of $\mathrm{V}\cdot\mathrm{s/m}$ is an increasing function of temperature. Thus, we can postulate the following form of the production term:

$$\boldsymbol{\chi}(t) = \frac{J_\infty}{\tau_p}\exp(-t/\tau_p)\,(\boldsymbol{I} - 3\mathbf{nn})\,, \tag{18.36}$$

where $J_\infty = (c_\infty^2 - a_\infty^2)/15$ is the value reached at $t \to \infty$. Note that in that case, the production is explicitly time-dependent. However, since the exponent is a rapidly decreasing function one can assume that the production stops when $t > 5\tau_p$. It also worth mentioning that τ_p characterizes the combined effect of the electric field

magnitude and thermal motion. Nevertheless the direction of the electric field appears explicitly in the production term.

Integration of the Eq. (18.34) with zero initial condition gives a development of the inertia tensor in time:

$$\boldsymbol{J}(t) = J_0\boldsymbol{I} + \frac{1}{15}\left(c_\infty^2 - a_\infty^2\right)\left(1 - \exp(-t/\tau_p)\right)(\boldsymbol{I} - 3\mathbf{nn})\,. \tag{18.37}$$

A few comments are now in order. First, note that the inertia tensor (18.37) corresponds to a particle that is oriented in the direction of the electric fields. Then the moment couple on the macro level will be zero and therefore zero angular velocity satisfies the spin balance in the absence of stress tensors. Second, the parameter J_∞ cannot be defined easily. If all dipoles would be uniformly aligned in **n**-direction then the homogenized tensor of inertia would coincide with the inertia tensor of the dipole. But the perfect alignment is not reachable because of randomizing effect of the thermal motion. Hence J_∞ characterizes the "equilibrium" distribution of dipoles over orientations. It is tempting to interrelated J_∞ to the maximal polarization density **P** that can be reached in the material at the given electric field and temperature, but we leave it with this remark.

Now let us consider the reverse process. The production term associated with the thermal motion of the dipoles depends on the temperature and has to disappear as soon as the isotropic case is reached. Having that in mind, we choose the production of microinertia in the most simple form:

$$\boldsymbol{\chi}_{tm}(t) = -\frac{1}{\tau_r(T)}\boldsymbol{J}^{dev}(t), \tag{18.38}$$

where τ_r defines the relaxation time. The smaller it is, the faster the transition from order to disorder will be achieved. Being a quantity associated with the thermal motion the relaxation time has to be a decreasing function of the temperature.

Then it follows that the deviatoric part of the inertia tensor decreases exponentially in time and the inertia tensor turns eventually into a spherical tensor:

$$\boldsymbol{J}(t) = J_0\boldsymbol{I} + \frac{1}{15}\left(c_1^2 - a_1^2\right)\exp(-t/\tau_r)\,(\boldsymbol{I} - 3\mathbf{nn})\,. \tag{18.39}$$

Here c_1 and a_1 are the spheroid axes at the moment t_1, when the external field was removed.

18.4 Conclusions and Outlook

The intention of this paper is to draw attention on some recent activities in the field of micropolar media capable of structural change. One of its main feature is a new balance equation for the tensor of inertia containing a production term. The new balance and in particular the production are interpreted mesoscopically by taking

the inner structure of micropolar matter into account. In fact, it is an attempt to generalize the classical approach based on the concept of an indestructible material particle consisting of a statistically significant number of subunits on a mesoscopic scale. Within the classical approach, there should be no exchange of subunits between the material particles. Furthermore, the polar continuum particle assumed to be equivalent to a rigid body and can be neither destroyed nor generated. However, this means that within this framework certain processes and effects in materials can simply not be modeled.

The new approach emphasizes the idea that it may become necessary to abandon the concept of the rigid material particle if one wishes to describe micropolar matter in which structural changes or chemical reactions occur. The approach is based on the spatial description where a representative volume element is treated as a continuum polar particle. It does not impose strict constraints on the motion of micro-particles, rather it embraces the idea of an open system, allowing a priori for exchange of mass, momentum, energy, tensor of inertia, etc., between and within the representative volume elements. For a better understanding of this new concept an underlying mesoscopic theory is presented. The main idea is to connect information on a mesoscale by taking the intrinsic microstructure within RVE into account with the macroscopic world, i.e., with the balances of micropolar continua in combination with suitable constitutive equations. This new approach enables us to study the temporal development of rotational inertial characteristics. In this context, the tensor of inertia is an additional internal variable characterizing structural transformations of the media. Moreover, in contrast to the material description where all neighboring material particles have to remain in the neighborhood during their motion, the spatial description does not impose strict constraints on the motion of material points. As a result, the neighboring particles can separate and travel significant distances from one another as happens in soils, granular and powder-like materials.

The extended theory seems particularly promising in context with the description of materials with complex or variable structure, such as suspensions or liquid crystals where the structural state of the fluid matter is actively controlled by applying external electromagnetic fields and temperature changes. Also, the approach has a potential for modeling processes going on under an influence of different physical and thermo-mechanical factors or taking into account their mutual influence. For example, a dielectric polarization in an alternating electric and temperature fields or electret production where a dielectric is placed in strong electric field and subjected to additional physical action or mechanoelectrets where the electret state can occur from mechanical deformation without external electrical field.

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