Chapter 9
Electromagnetic Waves in an Inhomogeneous Medium

In the previous chapter, we considered the direct scattering problem for acoustic waves in an inhomogeneous medium. We now consider the case of electromagnetic waves. However, our aim is not to simply prove the electromagnetic analogue of each theorem in Chap. 8 but rather to select the basic ideas of the previous chapter, extend them when possible to the electromagnetic case, and then consider some themes that were not considered in Chap. 8, but ones that are particularly relevant to the case of electromagnetic waves. In particular, we shall consider two simple problems, one in which the electromagnetic field has no discontinuities across the boundary of the medium and the second where the medium is an imperfect conductor such that the electromagnetic field does not penetrate deeply into the body. This last problem is an approximation to the more complicated transmission problem for a piecewise constant medium and leads to what is called the exterior impedance problem for electromagnetic waves.

After a brief discussion of the physical background to electromagnetic wave propagation in an inhomogeneous medium, we show existence and uniqueness of a solution to the direct scattering problem for electromagnetic waves in an inhomogeneous medium. By means of a reciprocity relation for electromagnetic waves in an inhomogeneous medium, we then show that, for a conducting medium, the set of electric far field patterns corresponding to incident time-harmonic plane waves moving in arbitrary directions is complete in the space of square integrable tangential vector fields on the unit sphere. However, we show that this set of far field patterns is in general not complete for a dielectric medium. Finally, we establish the existence and uniqueness of a solution to the exterior impedance problem and show that the set of electric far field patterns is again complete in the space of square integrable tangential vector fields on the unit sphere. These results for the exterior impedance problem will be used in the next chapter when we discuss the inverse scattering problem for electromagnetic waves in an inhomogeneous medium. We note, as in the case of acoustic waves, that our ideas and methods can be extended to more complicated scattering problems involving discontinuous fields, piecewise

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continuous refractive indexes, etc. but, for the sake of clarity and brevity, we do not consider these more general problems in this book.

9.1 Physical Background

We consider electromagnetic wave propagation in an inhomogeneous isotropic medium in $\mathbb{R}^3$ with electric permittivity $\varepsilon = \varepsilon(x) > 0$, magnetic permeability $\mu = \mu_0$, and electric conductivity $\sigma = \sigma(x)$ where μ_0 is a positive constant. We assume that $\varepsilon(x) = \varepsilon_0$ and $\sigma(x) = 0$ for all x outside some sufficiently large ball where ε_0 is a constant. Then if J is the current density, the electric field $\mathcal{E}$ and magnetic field $\mathcal{H}$ satisfy the *Maxwell equations*, namely

$$\operatorname{curl}\mathcal{E} + \mu_0\,\frac{\partial\mathcal{H}}{\partial t} = 0, \quad \operatorname{curl}\mathcal{H} - \varepsilon(x)\,\frac{\partial\mathcal{E}}{\partial t} = J. \tag{9.1}$$

Furthermore, in an isotropic conductor, the current density is related to the electric field by *Ohm's law*

$$J = \sigma\mathcal{E}. \tag{9.2}$$

For most metals, σ is very large and hence it is often reasonable in many theoretical investigations to approximate a metal by a fictitious *perfect conductor* in which σ is taken to be infinite. However, in this chapter, we shall assume that the inhomogeneous medium is not a perfect conductor, i.e., σ is finite. If σ is nonzero, the medium is called a *conductor*, whereas if $\sigma = 0$ the medium is referred to as a *dielectric*.

We now assume that the electromagnetic field is time-harmonic, i.e., of the form

$$\mathcal{E}(x,t) = \frac{1}{\sqrt{\varepsilon_0}}\,E(x)\,e^{-i\omega t}, \quad \mathcal{H}(x,t) = \frac{1}{\sqrt{\mu_0}}\,H(x)\,e^{-i\omega t}$$

where ω is the frequency. Then from (9.1) and (9.2) we see that E and H satisfy the time-harmonic Maxwell equations

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikn(x)E = 0 \tag{9.3}$$

in $\mathbb{R}^3$ where the (positive) wave number k is defined by $k^2 = \varepsilon_0\mu_0\omega^2$ and the *refractive index* $n = n(x)$ is given by

$$n(x) := \frac{1}{\varepsilon_0}\left(\varepsilon(x) + i\,\frac{\sigma(x)}{\omega}\right).$$

In order to be able to formulate an integral equation of Lippmann–Schwinger type for the direct scattering problem we assume that $n \in C^{1,\alpha}(\mathbb{R}^3)$ for some $0 < \alpha < 1$ and, as usual, that $m := 1 - n$ has compact support. As in the previous chapter, we define $D := \{x \in \mathbb{R}^3 : m(x) \neq 0\}$. For an integral equation formulation of the direct scattering problem in the case when n is discontinuous across ∂D we refer the reader to [241].

We consider the following scattering problem for (9.3). Let $E^i, H^i \in C^1(\mathbb{R}^3)$ be a solution of the Maxwell equations for a homogeneous medium

$$\operatorname{curl} E^i - ikH^i = 0, \quad \operatorname{curl} H^i + ikE^i = 0 \tag{9.4}$$

in all of $\mathbb{R}^3$. We then want to find a solution $E, H \in C^1(\mathbb{R}^3)$ of (9.3) in $\mathbb{R}^3$ such that if

$$E = E^i + E^s, \quad H = H^i + H^s \tag{9.5}$$

the scattered field E^s, H^s satisfies the *Silver–Müller radiation condition*

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0 \tag{9.6}$$

uniformly for all directions $x/|x|$ where $r = |x|$.

For the next three sections of this chapter, we shall be concerned with the scattering problem (9.3)–(9.6). The existence and uniqueness of a solution to this problem were first given by Müller [332] for the more general case when $\mu = \mu(x)$. The proof simplifies considerably for the case we are considering, i.e., $\mu = \mu_0$, and we shall present this proof in the next section.

9.2 Existence and Uniqueness

Under the assumptions given in the previous section for the refractive index n, we shall show in this section that there exists a unique solution to the scattering problem (9.3)–(9.6). Our analysis follows that of Colton and Kress [99] and is based on reformulating (9.3)–(9.6) as an integral equation. We first prove the following theorem, where

$$\Phi(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x - y|}, \quad x \neq y,$$

as usual, denotes the fundamental solution to the Helmholtz equation and

$$m := 1 - n.$$

Theorem 9.1 *Let $E, H \in C^1(\mathbb{R}^3)$ be a solution of the scattering problem (9.3)–(9.6). Then E satisfies the integral equation*

$$E(x) = E^i(x) - k^2 \int_{\mathbb{R}^3} \Phi(x, y) m(y) E(y)\, dy + \operatorname{grad} \int_{\mathbb{R}^3} \frac{1}{n(y)} \operatorname{grad} n(y) \cdot E(y)\, \Phi(x, y)\, dy, \quad x \in \mathbb{R}^3. \tag{9.7}$$

Proof Let $x \in \mathbb{R}^3$ be an arbitrary point and choose an open ball B with unit outward normal ν such that B contains the support of m and $x \in B$. From the Stratton–Chu formula (6.5) applied to E, H, we have

$$\begin{aligned} E(x) = &- \operatorname{curl} \int_{\partial B} \nu(y) \times E(y)\, \Phi(x, y)\, ds(y) \\ &+ \operatorname{grad} \int_{\partial B} \nu(y) \cdot E(y)\, \Phi(x, y)\, ds(y) \\ &- ik \int_{\partial B} \nu(y) \times H(y)\, \Phi(x, y)\, ds(y) \\ &+ \operatorname{grad} \int_B \frac{1}{n(y)} \operatorname{grad} n(y) \cdot E(y)\, \Phi(x, y)\, dy \\ &- k^2 \int_B m(y) E(y)\, \Phi(x, y)\, dy \end{aligned} \tag{9.8}$$

since $\operatorname{curl} H + ikE = ikmE$ and $n \operatorname{div} E = -\operatorname{grad} n \cdot E$. Note that in the volume integrals over B we can integrate over all of $\mathbb{R}^3$ since m has support in B. The Stratton–Chu formula applied to E^i, H^i gives

$$\begin{aligned} E^i(x) = &- \operatorname{curl} \int_{\partial B} \nu(y) \times E^i(y)\, \Phi(x, y)\, ds(y) \\ &+ \operatorname{grad} \int_{\partial B} \nu(y) \cdot E^i(y)\, \Phi(x, y)\, ds(y) \\ &- ik \int_{\partial B} \nu(y) \times H^i(y)\, \Phi(x, y)\, ds(y). \end{aligned} \tag{9.9}$$

Finally, from the version of the Stratton–Chu formula corresponding to Theorem 6.7, we see that

$$- \operatorname{curl} \int_{\partial B} \nu(y) \times E^s(y)\, \Phi(x,y)\, ds(y)$$

$$+ \operatorname{grad} \int_{\partial B} \nu(y) \cdot E^s(y)\, \Phi(x,y)\, ds(y) \tag{9.10}$$

$$- ik \int_{\partial B} \nu(y) \times H^s(y)\, \Phi(x,y)\, ds(y) \;=\; 0.$$

With the aid of $E = E^i + E^s$, $H = H^i + H^s$ we can now combine (9.8)–(9.10) to conclude that (9.7) is satisfied. □

We now want to show that every solution of the integral equation (9.7) is also a solution to (9.3)–(9.6).

Theorem 9.2 *Let $E \in C(\mathbb{R}^3)$ be a solution of the integral equation (9.7). Then E and $H := \operatorname{curl} E/ik$ are a solution of (9.3)–(9.6).*

Proof Since m has compact support, from Theorem 8.1 we can conclude that if $E \in C(\mathbb{R}^3)$ is a solution of (9.7) then $E \in C^{1,\alpha}(\mathbb{R}^3)$. Hence, by the relation $\operatorname{grad}_x \Phi(x,y) = -\operatorname{grad}_y \Phi(x,y)$, Gauss' divergence theorem and Theorem 8.1, we have

$$\operatorname{div} \int_{\mathbb{R}^3} \Phi(x,y) m(y) E(y)\, dy = \int_{\mathbb{R}^3} \operatorname{div}\{m(y)E(y)\} \Phi(x,y)\, dy \tag{9.11}$$

and

$$(\Delta + k^2) \int_{\mathbb{R}^3} \frac{1}{n(y)} \operatorname{grad} n(y) \cdot E(y)\, \Phi(x,y)\, dy = -\frac{1}{n(x)} \operatorname{grad} n(x) \cdot E(x) \tag{9.12}$$

for $x \in \mathbb{R}^3$. Taking the divergence of (9.7) and using (9.11) and (9.12), we see that

$$u := \frac{1}{n} \operatorname{div}(nE)$$

satisfies the integral equation

$$u(x) + k^2 \int_{\mathbb{R}^3} \Phi(x,y) m(y) u(y)\, dy = 0, \quad x \in \mathbb{R}^3.$$

Hence, from Theorems 8.3 and 8.7 we can conclude that $u(x) = 0$ for $x \in \mathbb{R}^3$, that is,

$$\operatorname{div}(nE) = 0 \quad \text{in } \mathbb{R}^3. \tag{9.13}$$

Therefore, the integral equation (9.7) can be written in the form

$$
\begin{aligned}
E(x) = E^i(x) - k^2 & \int_{\mathbb{R}^3} \Phi(x,y) m(y) E(y)\, dy \\
& - \operatorname{grad} \int_{\mathbb{R}^3} \Phi(x,y) \operatorname{div} E(y)\, dy, \quad x \in \mathbb{R}^3,
\end{aligned}
\tag{9.14}
$$

and thus for $H := \operatorname{curl} E/ik$ we have

$$
H(x) = H^i(x) + ik \operatorname{curl} \int_{\mathbb{R}^3} \Phi(x,y) m(y) E(y)\, dy, \quad x \in \mathbb{R}^3. \tag{9.15}
$$

In particular, by Theorem 8.1 this implies $H \in C^{1,\alpha}(\mathbb{R}^3)$ since $E \in C^{1,\alpha}(\mathbb{R}^3)$. We now use the vector identity (6.4), the Maxwell equations (9.4), and (9.11), (9.13)–(9.15) to deduce that

$$
\begin{aligned}
\operatorname{curl} H(x) + ikE(x) &= ik(\operatorname{curl}\operatorname{curl} - k^2) \int_{\mathbb{R}^3} \Phi(x,y) m(y) E(y)\, dy \\
&\qquad - ik \operatorname{grad} \int_{\mathbb{R}^3} \Phi(x,y) \operatorname{div} E(y)\, dy \\
&= -ik(\Delta + k^2) \int_{\mathbb{R}^3} \Phi(x,y) m(y) E(y)\, dy \\
&\qquad - ik \operatorname{grad} \int_{\mathbb{R}^3} \operatorname{div}\{n(y)E(y)\} \Phi(x,y)\, dy \\
&= ikm(x)E(x)
\end{aligned}
$$

for $x \in \mathbb{R}^3$. Therefore E, H satisfy (9.3). Finally, the decomposition (9.5) and the radiation condition (9.6) follow readily from (9.7) and (9.15) with the aid of (2.15) and (6.26). □

We note that in (9.7) we can replace the region of integration by any domain G such that the support of m is contained in $\bar{G}$ and look for solutions in $C(\bar{G})$. Then for $x \in \mathbb{R}^3 \setminus \bar{G}$ we define $E(x)$ by the right-hand side of (9.7) and obviously obtain a continuous solution to (9.7) in all of $\mathbb{R}^3$.

In order to show that (9.7) is uniquely solvable we need to establish the following unique continuation principle for the Maxwell equations.

Theorem 9.3 *Let G be a domain in $\mathbb{R}^3$ and let $E, H \in C^1(G)$ be a solution of*

$$
\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikn(x)E = 0 \tag{9.16}
$$

in G such that $n \in C^{1,\alpha}(G)$. Suppose E, H vanishes in a neighborhood of some $x_0 \in G$. Then E, H is identically zero in G.

Proof From the representation formula (9.8) and Theorem 8.1, since by assumption $n \in C^{1,\alpha}(G)$, we first can conclude that $E \in C^{1,\alpha}(B)$ for any ball B with $\bar{B} \subset G$. Then, using $\operatorname{curl} E = ikH$ from (9.8) we have $H \in C^{2,\alpha}(B)$ whence, in particular, $H \in C^2(G)$ follows.

Using the vector identity (6.4), we deduce from (9.16) that

$$\Delta H + \frac{1}{n(x)} \operatorname{grad} n(x) \times \operatorname{curl} H + k^2 n(x) H = 0 \quad \text{in } G$$

and the proof is completed by applying Lemma 8.5 to the real and imaginary parts of the Cartesian components of H. □

Theorem 9.4 *The scattering problem (9.3)–(9.6) has at most one solution E, H in $C^1(\mathbb{R}^3)$.*

Proof Let E, H denote the difference between two solutions. Then E, H clearly satisfy the radiation condition (9.6) and the Maxwell equations for a homogeneous medium outside some ball B containing the support of m. From Gauss' divergence theorem and the Maxwell equations (9.3), denoting as usual by ν the exterior unit normal to B, we have that

$$\int_{\partial B} \nu \times E \cdot \bar{H}\, ds = \int_B (\operatorname{curl} E \cdot \bar{H} - E \cdot \operatorname{curl} \bar{H})\, dx = ik \int_B (|H|^2 - \bar{n}\, |E|^2)\, dx \tag{9.17}$$

and hence

$$\operatorname{Re} \int_{\partial B} \nu \times E \cdot \bar{H}\, ds = -k \int_B \operatorname{Im} n\, |E|^2 dx \leq 0.$$

Hence, by Theorem 6.11, we can conclude that $E(x) = H(x) = 0$ for $x \in \mathbb{R}^3 \setminus \bar{B}$. By Theorem 9.3 the proof is complete. □

We are now in a position to show that there exists a unique solution to the electromagnetic scattering problem.

Theorem 9.5 *The scattering problem (9.3)–(9.6) for an inhomogeneous medium has a unique solution and the solution E, H depends continuously on the incident field E^i, H^i with respect to the maximum norm.*

Proof By Theorems 9.2 and 9.4, it suffices to prove the existence of a solution $E \in C(\mathbb{R}^3)$ to (9.7). As in the proof of Theorem 8.7, it suffices to look for solutions of (9.7) in an open ball B containing the support of m. We define an electromagnetic operator $T_e : C(\bar{B}) \to C(\bar{B})$ on the Banach space of continuous vector fields in $\bar{B}$ by

$$
\begin{aligned}
(T_e E)(x) := -k^2 & \int_B \Phi(x,y) m(y) E(y)\, dy \\
& + \operatorname{grad} \int_B \frac{1}{n(y)} \operatorname{grad} n(y) \cdot E(y)\, \Phi(x,y)\, dy, \quad x \in \bar{B}.
\end{aligned}
\tag{9.18}
$$

Since T_e has a weakly singular kernel it is a compact operator. Hence, we can apply the Riesz–Fredholm theory and must show that the homogeneous equation corresponding to (9.7) has only the trivial solution. If this is done, Eq. (9.7) can be solved and the inverse operator $(I - T_e)^{-1}$ is bounded. From this it follows that E, H depend continuously on the incident field with respect to the maximum norm.

By Theorem 9.2, a continuous solution E of $E - T_e E = 0$ solves the homogeneous scattering problem (9.3)–(9.6) with $E^i = 0$ and hence, by Theorem 9.4, it follows that $E = 0$. The theorem is now proved. □

9.3 The Far Field Patterns

We now want to examine the far field patterns of the scattering problem (9.3)–(9.6) where the refractive index $n = n(x)$ again satisfies the assumptions of Sect. 9.1. As in Sect. 6.6 the incident electromagnetic field is given by the plane wave described by the matrices $E^i(x,d)$ and $H^i(x,d)$ defined by

$$
\begin{aligned}
E^i(x,d)p &= \frac{i}{k} \operatorname{curl}\operatorname{curl} p\, e^{ik\, x\cdot d} = ik\,(d \times p) \times d\, e^{ik\, x\cdot d}, \\
H^i(x,d)p &= \operatorname{curl} p\, e^{ik\, x\cdot d} = ik\, d \times p\, e^{ik\, x\cdot d},
\end{aligned}
\tag{9.19}
$$

where d is a unit vector giving the direction of propagation and $p \in \mathbb{R}^3$ is a constant vector giving the polarization. Because of the linearity of the direct scattering problem with respect to the incident field, we can also express the scattered waves by matrices. From Theorem 6.9, we see that

$$
\begin{aligned}
E^s(x,d)p &= \frac{e^{ik|x|}}{|x|}\, E_\infty(\hat{x},d)p + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty, \\
H^s(x,d)p &= \frac{e^{ik|x|}}{|x|}\, \hat{x} \times E_\infty(\hat{x},d)p + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty,
\end{aligned}
\tag{9.20}
$$

where E_∞ is the electric far field pattern. Furthermore, from (6.88) and Green's vector theorem (6.3), we can immediately deduce the following reciprocity relation.

Theorem 9.6 *Let E_∞ be the electric far field pattern of the scattering problem (9.3)–(9.6) and (9.19). Then for all vectors $\hat{x}, d \in \mathbb{S}^2$ we have*

$$E_\infty(\hat{x}, d) = [E_\infty(-d, -\hat{x})]^\top.$$

Motivated by our study of acoustic waves in Chap. 8, we now want to use this reciprocity relation to show the equivalence of the completeness of the set of electric far field patterns and the uniqueness of the solution to an electromagnetic interior transmission problem. In this chapter, we shall only be concerned with the homogeneous problem, defined as follows.

Homogeneous Electromagnetic Interior Transmission Problem *Find a solution* $E_0, E_1, H_0, H_1 \in C^1(D) \cap C(\bar{D})$ *of*

$$\begin{aligned} \operatorname{curl} E_1 - ikH_1 = 0, \quad \operatorname{curl} H_1 + ikn(x)E_1 = 0 \quad \text{in } D, \\ \operatorname{curl} E_0 - ikH_0 = 0, \quad \operatorname{curl} H_0 + ikE_0 = 0 \quad \text{in } D, \end{aligned} \tag{9.21}$$

satisfying the boundary condition

$$\nu \times (E_1 - E_0) = 0, \quad \nu \times (H_1 - H_0) = 0 \quad \text{on } \partial D, \tag{9.22}$$

where again $D := \{x \in \mathbb{R}^3 : m(x) \neq 0\}$ *and where we assume that D is connected with a connected* C^2 *boundary.*

In order to establish the connection between electric far field patterns and the electromagnetic interior transmission problem, we now recall the definition of the Hilbert space

$$L_t^2(\mathbb{S}^2) := \left\{ g : \mathbb{S}^2 \to \mathbb{C}^3 : g \in L^2(\mathbb{S}^2),\ \nu \cdot g = 0 \text{ on } \mathbb{S}^2 \right\}$$

of square integrable tangential fields on the unit sphere. Let $\{d_n : n = 1, 2, \dots\}$ be a countable dense set of unit vectors on $\mathbb{S}^2$ and consider the set $\mathcal{F}$ of electric far field patterns defined by

$$\mathcal{F} := \left\{ E_\infty(\cdot\,, d_n)e_j : n = 1, 2, \dots,\ j = 1, 2, 3 \right\}$$

where e_1, e_2, e_3 are the Cartesian unit coordinate vectors in $\mathbb{R}^3$. Recalling the definition of an electromagnetic Herglotz pair and Herglotz kernel given in Sect. 6.6, we can now prove the following theorem due to Colton and Päivärinta [122].

Theorem 9.7 *A tangential vector field g is in the orthogonal complement* $\mathcal{F}^\perp$ *of* $\mathcal{F}$ *if and only if there exists a solution of the homogeneous electromagnetic interior transmission problem such that* E_0, H_0 *is an electromagnetic Herglotz pair with Herglotz kernel* ikh *where* $h(d) = \overline{g(-d)}$.

Proof Suppose that $g \in L^2_t(\mathbb{S}^2)$ satisfies

$$\int_{\mathbb{S}^2} E_\infty(\hat{x}, d_n)e_j \cdot \overline{g(\hat{x})}\, ds(\hat{x}) = 0$$

for $n = 1, 2, \dots$ and $j = 1, 2, 3$. By the reciprocity relation, this is equivalent to

$$\int_{\mathbb{S}^2} E_\infty(-d, -\hat{x})\overline{g(\hat{x})}\, ds(\hat{x}) = 0$$

for all $d \in \mathbb{S}^2$, i.e.,

$$\int_{\mathbb{S}^2} E_\infty(\hat{x}, d)h(d)\, ds(d) = 0 \tag{9.23}$$

for all $\hat{x} \in \mathbb{S}^2$ where $h(d) = \overline{g(-d)}$. Analogous to Lemma 6.35, from the integral equation (9.7) it can be seen that the left-hand side of (9.23) represents the electric far field pattern of the scattered wave E^s_0, H^s_0 corresponding to the incident wave E^i_0, H^i_0 given by the electromagnetic Herglotz pair

$$E^i_0(x) = \int_{\mathbb{S}^2} E^i(x, d)h(d)\, ds(d) = ik \int_{\mathbb{S}^2} h(d)\, e^{ik\, x\cdot d}\, ds(d),$$

$$H^i_0(x) = \int_{\mathbb{S}^2} H^i(x, d)h(d)\, ds(d) = \operatorname{curl} \int_{\mathbb{S}^2} h(d)\, e^{ik\, x\cdot d}\, ds(d).$$

Hence, (9.23) is equivalent to a vanishing far field pattern of E^s_0, H^s_0 and thus, by Theorem 6.10, equivalent to $E^s_0 = H^s_0 = 0$ in $\mathbb{R}^3\setminus B$, i.e., with $E_0 := E^i_0,\ H_0 := H^i_0$ and $E_1 := E^i_0 + E^s_0,\ H_1 := H^i_0 + H^s_0$ we have solutions to (9.21) satisfying the boundary condition (9.22). □

In the case of a conducting medium, i.e., $\operatorname{Im} n \neq 0$, we can use Theorem 9.7 to deduce the following result [122].

Theorem 9.8 *In a conducting medium, the set $\mathcal{F}$ of electric far field patterns is complete in $L^2_t(\mathbb{S}^2)$.*

Proof Recalling that an electromagnetic Herglotz pair vanishes if and only if its Herglotz kernel vanishes (Theorem 3.27 and Definition 6.33), we see from Theorem 9.7 that it suffices to show that the only solution of the homogeneous electromagnetic interior transmission problem (9.21) and (9.22) is $E_0 = E_1 = H_0 = H_1 = 0$. However, analogous to (9.17), from Gauss' divergence theorem and the Maxwell equations (9.21) we have

$$\int_{\partial D} \nu \cdot E_1 \times \bar{H}_1\, ds = ik \int_D (|H_1|^2 - \bar{n}\, |E_1|^2)\, dx,$$

$$\int_{\partial D} \nu \cdot E_0 \times \bar{H}_0 \, ds = ik \int_D (|H_0|^2 - |E_0|^2)\, dx.$$

From these two equations, using the transmission conditions (9.22) we obtain

$$\int_D (|H_1|^2 - \bar{n}\,|E_1|^2)\, dx = \int_D (|H_0|^2 - |E_0|^2)\, dx$$

and taking the imaginary part of both sides gives

$$\int_D \operatorname{Im} n\, |E_1|^2 dx = 0.$$

From this, we conclude by unique continuation that $E_1 = H_1 = 0$ in D. From (9.22) we now have vanishing tangential components of E_0 and H_0 on the boundary ∂D whence $E_0 = H_0 = 0$ in D follows from the Stratton–Chu formulas (6.8) and (6.9). □

In contrast to Theorem 9.8, the set $\mathcal{F}$ of electric far field patterns is not in general complete for a dielectric medium. We shall show this for a spherically stratified medium in the next section.

We conclude this section with a short analysis of the far field operator $F : L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$ defined by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} E_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2, \tag{9.24}$$

and begin with an analog of Theorem 8.17.

Lemma 9.9 *Let E^i_g, H^i_g and E^i_h, H^i_h be electromagnetic Herglotz pairs with kernels $g, h \in L^2_t(\mathbb{S}^2)$, respectively, and let E_g, H_g and E_h, H_h be the solutions of (9.4)–(9.6) with E^i, H^i equal to E^i_g, H^i_g and E^i_h, H^i_h, respectively. Then*

$$k \int_D \operatorname{Im} n\, E_g \bar{E}_h \, dx = -2\pi(Fg, h) - 2\pi(g, Fh) - (Fg, Fh),$$

where $(\cdot\,,\cdot)$ denotes the inner product on $L^2_t(\mathbb{S}^2)$.

Proof Noting that

$$\bar{H}_h \cdot (\nu \times E_g) = -\frac{1}{ik}\, (\nu \times E_g) \cdot \operatorname{curl} \bar{E}_h$$

and

$$\bar{E}_h \cdot (\nu \times H_g) = \frac{1}{ik}\, \bar{E}_h \cdot (\nu \times \operatorname{curl} E_g) = -\frac{1}{ik}\, (\nu \times \bar{E}_h) \cdot \operatorname{curl} E_g$$

from Green's second vector integral theorem we obtain

$$\int_{\partial D} \left\{ \overline{H_h} \cdot (\nu \times E_g) - \overline{E_h} \cdot (\nu \times H_g) \right\} ds = 2k \int_D \operatorname{Im} n \, E_g \bar{E}_h \, dx.$$

Now the statement of the lemma follows from (6.103). □

Theorem 9.10 *Assume that* $\operatorname{Im} n = 0$. *Then the far field operator* F *is compact and normal, i.e.,* $FF^* = F^*F$, *and has an infinite number of eigenvalues.*

Proof Under the assumption $\operatorname{Im} n = 0$ from Lemma 9.9 we have that

$$2\pi(Fg, h) + 2\pi(g, Fh) + (Fg, Fh) = 0.$$

From this the normality of F follows analogous to the proof of Theorem 6.39 with the aid of the reciprocity relation in Theorem 9.6. By the uniqueness result contained in Theorem 9.7 the nullspace of F is either trivial or finite dimensional since by Remark 4.4 in [59] the transmission eigenvalues have finite multiplicity. From this the statement on the eigenvalues follows by the spectral theorem for compact normal operators (see [375]). □

Corollary 9.11 *Assume that* $\operatorname{Im} n = 0$. *Then the scattering operator* $S : L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$ *defined by*

$$S := I + \frac{1}{2\pi} F$$

is unitary.

Proof Analogous to the proof of Corollary 6.40. □

To conclude this section, with the aid of Lidski's Theorem 8.15 we will extend the statement of Theorem 9.10 on the eigenvalues of F to the case where $\operatorname{Im} n \neq 0$. For this we note that the argument for showing that the far field operator F is a trace class operator (see p. 324) carries over from the acoustic case to the electromagnetic case.

Theorem 9.12 *The far field operator* F *has an infinite number of eigenvalues.*

Proof In view of Theorem 9.10 we only need to consider the case where $\operatorname{Im} n \neq 0$. Recall that we assume that $n(x) \geq 0$ for all $x \in D$.

We first show that in this case F is injective. From $Fg = 0$, by Rellich's lemma and the unique continuation principle we conclude that the scattered wave for the solution E_g, H_g to (9.4)–(9.6) with E^i, H^i equal to the electromagnetic Herglotz pair E^i_g, H^i_g with kernel $g \in L^2_t(\mathbb{S}^2)$ vanishes in $\mathbb{R}^3$. Therefore $E_g = E^i_g$ and by Lemma 9.10 and again the unique continuation principle we obtain that $E^i_g = 0$ in $\mathbb{R}^3$, whence $g = 0$ follows by Theorem 3.27.

By Lemma 9.9 we have that

$$4\pi \operatorname{Im}(-iFg, g) = 2\pi \ [(Fg, g) + (g, Fg)] = k \int_D \operatorname{Im} n \, |E_g|^2 dx + \|Fg\|^2 \geq 0$$

for all $g \in L^2_t(\mathbb{S}^2)$. Therefore the operator $-iF$ satisfies the assumptions of Theorem 8.15 and the statement of the theorem follows. □

Corollary 9.13 *If* $\operatorname{Im} n \neq 0$ *the eigenvalues of the far field operator* F *lie in the disk*

$$|\lambda|^2 + 4\pi \operatorname{Re} \lambda < 0$$

whereas if $\operatorname{Im} n = 0$ *they lie on the circle*

$$|\lambda|^2 + 4\pi \ \operatorname{Re} \ \lambda = 0$$

in the complex plane.

Proof This follows from Lemma 9.9 by setting $g = h$ and $Fg = \lambda g$. □

9.4 The Spherically Stratified Dielectric Medium

In this section, we shall consider the class $\mathcal{F}$ of electric far field patterns for a spherically stratified dielectric medium. Our aim is to show that in this case there exist wave numbers k such that $\mathcal{F}$ is not complete in $L^2_t(\mathbb{S}^2)$. It suffices to show that when $n(x) = n(r)$, $r = |x|$, $\operatorname{Im} n = 0$ and, as a function of r, $n \in C^2$, there exist values of k such that there exists a nontrivial solution to the homogeneous electromagnetic interior transmission problem

$$\begin{aligned} &\operatorname{curl} E_1 - ikH_1 = 0, \quad \operatorname{curl} H_1 + ikn(r)E_1 = 0 \quad \text{in } B, \\ &\operatorname{curl} E_0 - ikH_0 = 0, \quad \operatorname{curl} H_0 + ikE_0 = 0 \quad \text{in } B, \end{aligned} \tag{9.25}$$

with the boundary condition

$$\nu \times (E_1 - E_0) = 0, \quad \nu \times (H_1 - H_0) = 0 \quad \text{on } \partial B, \tag{9.26}$$

where E_0, H_0 is an electromagnetic Herglotz pair, where now B is an open ball of radius a with exterior unit normal ν and where $\operatorname{Im} n = 0$. Analogous to the construction of the spherical vector wave functions in Theorem 6.26 from the scalar spherical wave functions, we will develop special solutions to the electromagnetic transmission problem (9.25) and (9.26) from solutions to the acoustic interior transmission problem

$$\Delta w + k^2 n(r) w = 0, \quad \Delta v + k^2 v = 0 \quad \text{in } B, \tag{9.27}$$

$$w - v = 0, \quad \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial B. \tag{9.28}$$

Assuming that the solutions w, v of (9.27) and (9.28) are three times continuously differentiable, we now define

$$\begin{aligned} E_1(x) &:= \operatorname{curl}\{x w(x)\}, \ H_1(x) := \frac{1}{ik} \operatorname{curl} E_1(x), \\ E_0(x) &:= \operatorname{curl}\{x v(x)\}, \ H_0(x) := \frac{1}{ik} \operatorname{curl} E_0(x). \end{aligned} \tag{9.29}$$

Then, from the identity (6.4) together with

$$\Delta\{x w(x)\} = x \Delta w(x) + 2 \operatorname{grad} w(x)$$

and (9.27) we have that

$$\begin{aligned} ik \operatorname{curl} H_1(x) &= \operatorname{curl}\operatorname{curl}\operatorname{curl}\{x w(x)\} = -\operatorname{curl} \Delta\{x w(x)\} \\ &= k^2 \operatorname{curl}\{x n(r) w(x)\} = k^2 n(r) \operatorname{curl}\{x w(x)\} = k^2 n(r) E_1(x), \end{aligned}$$

that is,

$$\operatorname{curl} H_1 + ikn(r) E_1 = 0,$$

and similarly

$$\operatorname{curl} H_0 + ik E_0 = 0.$$

Hence, E_1, H_1 and E_0, H_0 satisfy (9.25). From $w - v = 0$ on ∂B we have that

$$x \times \{E_1(x) - E_0(x)\} = x \times \{\operatorname{grad}[w(x) - v(x)] \times x\} = 0, \quad x \in \partial B,$$

that is,

$$\nu \times (E_1 - E_0) = 0 \quad \text{on } \partial B.$$

Finally, setting $u = w - v$ in the relation

$$\operatorname{curl}\operatorname{curl}\{xu(x)\} = -\Delta\{xu(x)\} + \operatorname{grad}\operatorname{div}\{xu(x)\}$$

$$= -x\Delta u(x) + \operatorname{grad}\left\{u(x) + r\,\frac{\partial u}{\partial r}\,(x)\right\}$$

and using the boundary condition (9.28), we deduce that

$$\nu \times (H_1 - H_0) = 0 \quad \text{on } \partial B$$

is also valid. Hence, from a three times continuously differentiable solution w, v to the scalar transmission problem (9.27) and (9.28), via (9.29) we obtain a solution E_1, H_1 and E_0, H_0 to the electromagnetic transmission problem (9.25) and (9.26). Note, however, that in order to obtain a nontrivial solution through (9.29) we have to insist that w and v are not spherically symmetric.

We proceed as in Sect. 8.4 and, after introducing spherical coordinates (r, θ, φ), look for solutions to (9.27) and (9.28) of the form

$$\begin{aligned} v(r,\theta) &= a_l\, j_l(kr)\, P_l(\cos\theta), \\ w(r,\theta) &= b_l\,\frac{y_l(r)}{r}\, P_l(\cos\theta), \end{aligned} \tag{9.30}$$

where P_l is Legendre's polynomial, j_l is a spherical Bessel function, a_l and b_l are constants to be determined, and the function y_l is a solution of

$$y_l'' + \left(k^2 n(r) - \frac{l(l+1)}{r^2}\right) y_l = 0 \tag{9.31}$$

for $r > 0$ such that y_l is continuous for $r \geq 0$. However, in contrast to the analysis of Sect. 8.4, we are only interested in solutions which are dependent on θ, i.e., in solutions for $l \geq 1$. In particular, the ordinary differential equation (9.31) now has singular coefficients. We shall show that if $n(r) > 1$ for $0 \leq r < a$ or $0 < n(r) < 1$ for $0 \leq r < a$, then for each $l \geq 1$ there exist an infinite set of values of k and constants $a_l = a_l(k)$, $b_l = b_l(k)$, such that (9.30) is a nontrivial solution of (9.27) and (9.28). From Sect. 6.6 we know that E_0, H_0, given by (9.29), is an electromagnetic Herglotz pair. Hence, by Theorem 9.7, for such values of k the set of electric far field patterns is not complete.

To show the existence of values of k such that (9.30) yields a nontrivial solution of (9.27) and (9.28), we need to examine the asymptotic behavior of solutions to (9.31). To this end, we use the Liouville transformation

$$\xi := \int_0^r [n(\rho)]^{1/2} d\rho, \quad z(\xi) := [n(r)]^{1/4} y_l(r) \tag{9.32}$$

to transform (9.31) to

$$z'' + [k^2 - p(\xi)]z = 0 \tag{9.33}$$

where

$$p(\xi) := \frac{n''(r)}{4\,[n(r)]^2} - \frac{5}{16}\,\frac{[n'(r)]^2}{[n(r)]^3} + \frac{l(l+1)}{r^2 n(r)}.$$

Note that since $n(r) > 0$ for $r \geq 0$ and n is in C^2, the transformation (9.32) is invertible and p is well defined and continuous for $r > 0$. In order to deduce the required asymptotic estimates, we rewrite (9.33) in the form

$$z'' + \left(k^2 - \frac{l(l+1)}{\xi^2} - g(\xi)\right) z = 0 \tag{9.34}$$

where

$$g(\xi) := \frac{l(l+1)}{r^2 n(r)} - \frac{l(l+1)}{\xi^2} + \frac{n''(r)}{4\,[n(r)]^2} - \frac{5}{16}\,\frac{[n'(r)]^2}{[n(r)]^3}\,, \qquad r = r(\xi), \tag{9.35}$$

and note that since $n(r) = 1$ for $r \geq a$ we have

$$\int_1^\infty |g(\xi)|\,d\xi < \infty \quad \text{and} \quad \int_0^1 \xi\,|g(\xi)|\,d\xi < \infty.$$

For $\lambda > 0$ we now define the functions E_λ and M_λ by

$$E_\lambda(\xi) := \begin{cases} \left[-\dfrac{Y_\lambda(\xi)}{J_\lambda(\xi)}\right]^{1/2}, & 0 < \xi < \xi_\lambda, \\[2ex] 1, & \xi_\lambda \leq \xi < \infty, \end{cases}$$

and

$$M_\lambda(\xi) := \begin{cases} [2\,|Y_\lambda(\xi)|\,J_\lambda(\xi)]^{1/2}, & 0 < \xi < \xi_\lambda, \\[2ex] \left[J_\lambda^2(\xi) + Y_\lambda^2(\xi)\right]^{1/2}, & \xi_\lambda \leq \xi < \infty, \end{cases}$$

where J_λ is the Bessel function, Y_λ the Neumann function, and ξ_λ is the smallest positive root of the equation

$$J_\lambda(\xi) + Y_\lambda(\xi) = 0.$$

Note that ξ_λ is less than the first positive zero of J_λ. For the necessary information on Bessel and Neumann functions of nonintegral order we refer the reader to [86, 293]. We further define G_λ by

$$G_\lambda(k,\xi) := \frac{\pi}{2}\int_0^\xi \rho M_\lambda^2(k\rho)\,|g(\rho)|\,d\rho$$

where g is given by (9.35). Noting that for $k > 0$ and $\lambda \geq 0$ we have that G_λ is finite when r is finite, we can now state the following result from Olver [343, p. 450].

Theorem 9.14 *Let $k > 0$ and $l \geq -1/2$. Then (9.34) has a solution z which, as a function of ξ, is continuous in $[0,\infty)$, twice continuously differentiable in $(0,\infty)$, and is given by*

$$z(\xi) = \sqrt{\frac{\pi\xi}{2k}}\;\{J_\lambda(k\xi) + \varepsilon_l(k,\xi)\} \tag{9.36}$$

where

$$\lambda = l + \frac{1}{2}$$

and

$$|\varepsilon_l(k,\xi)| \leq \frac{M_\lambda(k\xi)}{E_\lambda(k\xi)}\;\left\{e^{G_\lambda(k,\xi)} - 1\right\}.$$

In order to apply Theorem 9.14 to obtain an asymptotic estimate for a continuous solution y_l of (9.31), we fix $\xi > 0$ and let k be large. Then for $\lambda > 0$ we have that there exist constants C_1 and C_2, both independent of k, such that

$$\begin{aligned} |G_\lambda(k,\xi)| &\leq C\left\{\int_0^1 M_\lambda^2(k\rho)\,d\rho + \frac{1}{k}\int_1^\infty |g(\rho)|\,d\rho\right\} \\ &\leq C_1\left\{\frac{1}{k}\int_{1/k}^1 \frac{d\rho}{\rho} + \frac{1}{k}\right\} \;=\; C_1\left\{\frac{\ln k}{k} + \frac{1}{k}\right\}. \end{aligned} \tag{9.37}$$

Hence, for z defined by (9.36) we have from Theorem 9.14, (9.37) and the asymptotics for the Bessel function J_λ that

$$\begin{aligned} z(\xi) &= \sqrt{\frac{\pi\xi}{2k}}\;\left\{J_\lambda(k\xi) + O\left(\frac{\ln k}{k^{3/2}}\right)\right\} \\ &= \frac{1}{k}\,\cos\left(k\xi - \frac{\lambda\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{\ln k}{k^2}\right) \end{aligned} \tag{9.38}$$

for fixed $\xi > 0$ and λ as defined in Theorem 9.14. Furthermore, it can be shown that the asymptotic expansion (9.38) can be differentiated with respect to ξ, the error estimate being $O(\ln k/k)$. Hence, from (9.32) and (9.38) we can finally conclude that if y_l is defined by (9.32) then

$$y_l(r) = \frac{1}{k[n(r)]^{1/4}[n(0)]^{l/2+1/4}} \cos\left(k\int_0^r [n(\rho)]^{1/2} d\rho - \frac{\lambda\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{\ln k}{k^2}\right) \tag{9.39}$$

where the asymptotic expansion for $[n(r)]^{1/4} y_l(r)$ can be differentiated with respect to r, the error estimate being $O(\ln k/k)$.

We now note that, from the above estimates, w, as defined by (9.30), is a C^2 solution of $\Delta w + k^2 n(r) w = 0$ in $B \setminus \{0\}$ and is continuous in B. Hence, by the removable singularity theorem for elliptic differential equations (cf. [366],p. 104) we have that $w \in C^2(B)$. Since $n \in C^{1,\alpha}(\mathbb{R}^3)$, we can conclude from Green's formula (8.14) and Theorem 8.1 that $w \in C^3(B)$ and hence E_1 and H_1 are continuously differentiable in B.

We now return to the scalar interior transmission problem (9.27) and (9.28) and note that (9.30) will be a nontrivial solution provided there exists a nontrivial solution a_l, b_l of the homogeneous linear system

$$\begin{aligned} b_l \frac{y_l(a)}{a} \quad &- a_l j_l(ka) = 0 \\ b_l \frac{d}{dr}\left(\frac{y_l(r)}{r}\right)_{r=a} &- a_l k j_l'(ka) = 0. \end{aligned} \tag{9.40}$$

The system (9.40) will have a nontrivial solution provided the determinant of the coefficients vanishes, that is,

$$d := \det \begin{pmatrix} \dfrac{y_l(a)}{a} & -j_l(ka) \\ \dfrac{d}{dr}\left(\dfrac{y_l(r)}{r}\right)_{r=a} & -k j_l'(ka) \end{pmatrix} = 0. \tag{9.41}$$

Recalling the asymptotic expansions (2.42) for the spherical Bessel functions, i.e.,

$$\begin{aligned} j_l(kr) &= \frac{1}{kr}\cos\left(kr - \frac{l\pi}{2} - \frac{\pi}{2}\right) + O\left(\frac{1}{k^2}\right), \quad k \to \infty, \\ j_l'(kr) &= \frac{1}{kr}\sin\left(kr - \frac{l\pi}{2} + \frac{\pi}{2}\right) + O\left(\frac{1}{k^2}\right), \quad k \to \infty, \end{aligned} \tag{9.42}$$

we see from (9.39) and (9.42) and the addition formula for the sine function that

$$d = \frac{1}{a^2 k\,[n(0)]^{l/2+1/4}} \left\{ \sin\left(k\int_0^a [n(r)]^{1/2} dr - ka\right) + O\left(\frac{\ln k}{k}\right) \right\}.$$

Therefore, a sufficient condition for (9.41) to be valid for a discrete set of values of k is that either $n(r) > 1$ for $0 \le r < a$ or $n(r) < 1$ for $0 \le r < a$. Hence we have the following theorem [122].

Theorem 9.15 *Assume that* $\operatorname{Im} n = 0$ *and that* $n(x) = n(r)$ *is spherically stratified,* $n(r) = 1$ *for* $r \geq a$, $n(r) > 1$ *or* $0 < n(r) < 1$ *for* $0 \leq r < a$ *and, as a function of* r, $n \in C^2$. *Then there exists an infinite set of wave numbers* k *such that the set* $\mathcal{F}$ *of electric far field patterns is not complete in* $L_t^2(\mathbb{S}^2)$.

9.5 The Exterior Impedance Boundary Value Problem

The mathematical treatment of the scattering of time harmonic electromagnetic waves by a body which is not perfectly conducting but which does not allow the electric and magnetic field to penetrate deeply into the body leads to what is called an exterior impedance boundary value problem for electromagnetic waves (cf. [223, p. 511] and [411, p. 304]). In particular, such a model is sometimes used for coated media instead of the more complicated transmission problem. In addition to being an appropriate theme for this chapter, we shall also need to make use of the mathematical theory of the exterior impedance boundary value problem in our later treatment of the inverse scattering problem for electromagnetic waves. The first rigorous proof of the existence of a unique solution to the exterior impedance boundary value problem for electromagnetic waves was given by Colton and Kress in [96]. Here we shall provide a simpler proof of this result by basing our ideas on those developed for a perfect conductor in Chap. 6. We first define the problem under consideration where for the rest of this section D is a bounded domain in $\mathbb{R}^3$ with connected C^2 boundary ∂D with unit outward normal ν.

Exterior Impedance Problem *Given a Hölder continuous tangential field* c *on* ∂D *and a positive constant* λ, *find a solution* $E, H \in C^1(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ *of the Maxwell equations*

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikE = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D} \tag{9.43}$$

satisfying the impedance boundary condition

$$\nu \times \operatorname{curl} E - i\lambda\, (\nu \times E) \times \nu = c \quad \text{on } \partial D \tag{9.44}$$

and the Silver–Müller radiation condition

$$\lim_{r \to \infty} (H \times x - rE) = 0 \tag{9.45}$$

uniformly for all directions $\hat{x} = x/|x|$.

The uniqueness of a solution to (9.43)–(9.45) is easy to prove.

Theorem 9.16 *The exterior impedance problem has at most one solution provided* $\lambda > 0$.

Proof If $c = 0$, then from (9.44) and the fact that $\lambda > 0$ we have that

$$\operatorname{Re} k \int_{\partial D} \nu \times E \cdot \bar{H}\, ds = -\lambda \int_{\partial D} |\nu \times E|^2 ds \leq 0.$$

We can now conclude from Theorem 6.11 that $E = H = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. □

We now turn to the existence of a solution to the exterior impedance problem, always assuming that $\lambda > 0$. To this end, we recall the definition of the space $C^{0,\alpha}(\partial D)$ of Hölder continuous functions defined on ∂D from Sect. 3.1 and the space $C_t^{0,\alpha}(\partial D)$ of Hölder continuous tangential fields defined on ∂D from Sect. 6.3. We also recall from Theorems 3.2, 3.4, and 6.14 that the single-layer operator $S : C^{0,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ defined by

$$(S\varphi)(x) := 2 \int_{\partial D} \Phi(x, y)\varphi(y)\, ds(y), \quad x \in \partial D,$$

the double-layer operator $K : C^{0,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ defined by

$$(K\varphi)(x) := 2 \int \frac{\partial \Phi(x, y)}{\partial \nu(y)}\, \varphi(y)\, ds(y), \quad x \in \partial D,$$

and the magnetic dipole operator $M : C_t^{0,\alpha}(\partial D) \to C_t^{0,\alpha}(\partial D)$ defined by

$$(Ma)(x) := 2 \int_{\partial D} \nu(x) \times \operatorname{curl}_x \{a(y)\Phi(x, y)\}\, ds(y), \quad x \in \partial D$$

are all compact. Furthermore, with the spaces

$$C^{0,\alpha}(\operatorname{Div}, \partial D) = \left\{ a \in C_t^{0,\alpha}(\partial D) : \operatorname{Div} a \in C^{0,\alpha}(\partial D) \right\}$$

and

$$C^{0,\alpha}(\operatorname{Curl}, \partial D) = \left\{ b \in C_t^{0,\alpha}(\partial D) : \operatorname{Curl} b \in C^{0,\alpha}(\partial D) \right\}$$

which were also introduced in Sect. 6.3, the electric dipole operator $N : C^{0,\alpha}(\operatorname{Curl}, \partial D) \to C^{0,\alpha}(\operatorname{Div}, \partial D)$ defined by

$$(Na)(x) := 2\, \nu(x) \times \operatorname{curl}\operatorname{curl} \int_{\partial D} \Phi(x, y)\, \nu(y) \times a(y)\, ds(y), \quad x \in \partial D,$$

is bounded by Theorem 6.19.

With these definitions and facts recalled, following Hähner [170], we now look for a solution of the exterior impedance problem in the form

$$E(x) = \int_{\partial D} \Phi(x, y) b(y)\, ds(y) + i\lambda \operatorname{curl} \int_{\partial D} \Phi(x, y)\, \nu(y) \times (S_0^2 b)(y)\, ds(y)$$

$$+ \operatorname{grad} \int_{\partial D} \Phi(x, y)\varphi(y)\, ds(y) + i\lambda \int_{\partial D} \Phi(x, y)\nu(y)\varphi(y)\, ds(y),$$

$$H(x) = \frac{1}{ik} \operatorname{curl} E(x), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{9.46}$$

where S_0 is the single-layer operator in the potential theoretic limit $k = 0$ and the densities $b \in C_t^{0,\alpha}(\partial D)$ and $\varphi \in C^{0,\alpha}(\partial D)$ are to be determined. The vector field E clearly satisfies the vector Helmholtz equation and its Cartesian components satisfy the (scalar) Sommerfeld radiation condition. Hence, if we insist that $\operatorname{div} E = 0$ in $\mathbb{R}^3 \setminus \bar{D}$, then by Theorems 6.4 and 6.8 we have that E, H satisfy the Maxwell equations and the Silver–Müller radiation condition. Since $\operatorname{div} E$ satisfies the scalar Helmholtz equation and the Sommerfeld radiation condition, by the uniqueness for the exterior Dirichlet problem it suffices to impose $\operatorname{div} E = 0$ only on the boundary ∂D. From the jump and regularity conditions of Theorems 3.1, 3.3, 6.12, and 6.13, we can now conclude that (9.46) for $b \in C_t^{0,\alpha}(\partial D)$ and $\varphi \in C^{0,\alpha}(\partial D)$ ensures the regularity $E, H \in C^{0,\alpha}(\mathbb{R}^3 \setminus D)$ up to the boundary and that it solves the exterior impedance problem provided b and φ satisfy the integral equations

$$\begin{aligned} b + M_{11} b + M_{12} \varphi &= 2c \\ -i\lambda\varphi + M_{21} b + M_{22} \varphi &= 0, \end{aligned} \tag{9.47}$$

where

$$M_{11} b := Mb + i\lambda N P S_0^2 b - i\lambda P S b + \lambda^2 \{M(\nu \times S_0^2 b)\} \times \nu + \lambda^2 P S_0^2 b,$$

$$(M_{12}\varphi)(x) := 2i\lambda\, \nu(x) \times \int_{\partial D} \operatorname{grad}_x \Phi(x, y) \times \{\nu(y) - \nu(x)\}\varphi(y)\, ds(y)$$

$$+\lambda^2 (P S \nu\varphi)(x), \quad x \in \partial D,$$

$$(M_{21} b)(x) := -2 \int_{\partial D} \operatorname{grad}_x \Phi(x, y) \cdot b(y)\, ds(y), \quad x \in \partial D,$$

$$M_{22}\varphi := k^2 S\varphi + i\lambda K\varphi,$$

and where P stands for the orthogonal projection of a vector field defined on ∂D onto the tangent plane, that is, $Pa := (\nu \times a) \times \nu$. Noting the smoothing property $S_0 : C^{0,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$ from Theorem 3.4, as in the proof of Theorem 6.21 it is not difficult to verify that $M_{11} : C_t^{0,\alpha}(\partial D) \to C_t^{0,\alpha}(\partial D)$ is compact. Compactness of the operator $M_{12} : C^{0,\alpha}(\partial D) \to C_t^{0,\alpha}(\partial D)$ follows by applying Corollary 2.9 from [104] to the first term in the definition of M_{12}. Loosely speaking, compactness of M_{12} rests on the fact that the factor $\nu(x) - \nu(y)$ makes the kernel weakly singular. Finally, $M_{22} : C^{0,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ is compact, whereas $M_{21} : C_t^{0,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ is merely bounded. Writing the system (9.47) in the form

$$\begin{pmatrix} I & 0 \\ M_{21} & -i\lambda I \end{pmatrix} \begin{pmatrix} b \\ \varphi \end{pmatrix} + \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix} \begin{pmatrix} b \\ \varphi \end{pmatrix} = \begin{pmatrix} 2c \\ 0 \end{pmatrix},$$

we now see that the first of the two matrix operators has a bounded inverse because of its triangular form and the second is compact. Hence, we can apply the Riesz–Fredholm theory to (9.47).

For this purpose, suppose b and φ are a solution to the homogeneous equation corresponding to (9.47) (i.e., $c = 0$). Then the field E, H defined by (9.46) satisfies the homogeneous exterior impedance problem in $\mathbb{R}^3 \setminus \bar{D}$. Since $\lambda > 0$, we can conclude from Theorem 9.16 that $E = H = 0$ in $\mathbb{R}^3 \setminus D$. Viewing (9.46) as defining a solution of the vector Helmholtz equation in D, from the jump relations of Theorems 3.1, 3.3, 6.12, and 6.13 we see that

$$-\nu \times E_- = i\lambda \nu \times S_0^2 b, \quad -\nu \times \operatorname{curl} E_- = b \quad \text{on } \partial D, \tag{9.48}$$

$$-\operatorname{div} E_- = -i\lambda\varphi, \quad -\nu \cdot E_- = -\varphi \quad \text{on } \partial D. \tag{9.49}$$

Hence, with the aid of Green's vector theorem (6.2), we derive from (9.48) and (9.49) that

$$\int_D \left\{ |\operatorname{curl} E|^2 + |\operatorname{div} E|^2 - k^2 |E|^2 \right\} dx = i\lambda \int_{\partial D} \left\{ |S_0 b|^2 + |\varphi|^2 \right\} ds.$$

Taking the imaginary part of the last equation and recalling that $\lambda > 0$ now shows that $S_0 b = 0$ and $\varphi = 0$ on ∂D. Since S_0 is injective (see the proof of Theorem 3.12), we have that $b = 0$ on ∂D. The Riesz–Fredholm theory now implies the following theorem. The statement on the boundedness of the operator $\mathcal{A}$ follows from the fact that by the Riesz–Fredholm theory the inverse operator for (9.47) is bounded from $C_t^{0,\alpha}(\partial D) \times C^{0,\alpha}(\partial D)$ into itself and by applying the mapping properties of Theorems 3.3 and 6.13 to the solution (9.46).

Theorem 9.17 *Suppose $\lambda > 0$. Then for each $c \in C_t^{0,\alpha}(\partial D)$ there exists a unique solution to the exterior impedance problem. The operator $\mathcal{A}$ mapping the boundary data c onto the tangential component $\nu \times E$ of the solution is a bounded operator $\mathcal{A} : C_t^{0,\alpha}(\partial D) \to C^{0,\alpha}(\mathrm{Div}, \partial D)$.*

For technical reasons, we shall need in Chap. 11 sufficient conditions for the invertibility of the operator

$$NR - i\lambda R(I+M) : C^{0,\alpha}(\text{Div}, \partial D) \to C_t^{0,\alpha}(\partial D)$$

where the operator $R : C_t^{0,\alpha}(\partial D) \to C_t^{0,\alpha}(\partial D)$ is given by

$$Ra := a \times \nu.$$

To this end, we first try to express the solution of the exterior impedance problem in the form

$$E(x) = \text{curl} \int_{\partial D} a(y)\Phi(x,y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D},$$

where $a \in C^{0,\alpha}(\text{Div}, \partial D)$. From the jump conditions of Theorems 6.12 and 6.13, this leads to the integral equation

$$NRa - i\lambda RMa - i\lambda Ra = 2c \tag{9.50}$$

for the unknown density a. However, we can interpret the solution of the exterior impedance problem as the solution of the exterior Maxwell problem with boundary condition

$$\nu \times E = \mathcal{A}c \quad \text{on } \partial D,$$

and hence a also is required to satisfy the integral equation

$$a + Ma = 2\mathcal{A}c.$$

The last equation turns out to be a special case of Eq. (6.56) with $\eta = 0$ (and a different right-hand side). From the proof of Theorem 6.21, it can be seen that if k is not a Maxwell eigenvalue for D then $I + M$ has a trivial nullspace. Hence, since by Theorem 6.17 the operator $M : C^{0,\alpha}(\text{Div}, \partial D) \to C^{0,\alpha}(\text{Div}, \partial D)$ is compact, by the Riesz–Fredholm theory $(I+M)^{-1} : C^{0,\alpha}(\text{Div}, \partial D) \to C^{0,\alpha}(\text{Div}, \partial D)$ exists and is bounded. Hence, $(I+M)^{-1}\mathcal{A} : C_t^{0,\alpha}(\partial D) \to C^{0,\alpha}(\text{Div}, \partial D)$ is the bounded inverse of $NR - i\lambda R(I+M)$ and we have proven the following theorem.

Theorem 9.18 *Assume that $\lambda > 0$ and that k is not a Maxwell eigenvalue for D. Then the operator $NR - i\lambda R(I+M) : C^{0,\alpha}(\text{Div}, \partial D) \to C_t^{0,\alpha}(\partial D)$ has a bounded inverse.*

We shall now conclude this chapter by briefly considering the electric far field patterns corresponding to the exterior impedance problem (9.43)–(9.45) with c given by

$$c := -\nu \times \text{curl}\, E^i + i\lambda\, (\nu \times E^i) \times \nu \quad \text{on } \partial D$$

where E^i and H^i are given by (9.19). This corresponds to the scattering of the incident field (9.19) by the imperfectly conducting obstacle D where the total electric field $E = E^i + E^s$ satisfies the impedance boundary condition

$$\nu \times \operatorname{curl} E - i\lambda\,(\nu \times E) \times \nu = 0 \quad \text{on } \partial D \tag{9.51}$$

and E^s is the scattered electric field. From Theorem 6.9 we see that E^s has the asymptotic behavior

$$E^s(x,d)p = \frac{e^{ik|x|}}{|x|}\, E^\lambda_\infty(\hat{x},d)p + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty,$$

where E^λ_∞ is the electric far field pattern. From (6.88) and (9.51) we can easily deduce the following reciprocity relation [12].

Theorem 9.19 *For all vectors $\hat{x}, d \in \mathbb{S}^2$ we have*

$$E^\lambda_\infty(\hat{x},d) = [E^\lambda_\infty(-d,-\hat{x})]^\top.$$

We are now in a position to prove the analogue of Theorem 9.8 for the exterior impedance problem. In particular, recall the Hilbert space $L^2_t(\mathbb{S}^2)$ of tangential L^2 vector fields on the unit sphere, let $\{d_n : n = 1, 2, \dots\}$ be a countable dense set of unit vectors on $\mathbb{S}^2$ and denote by e_1, e_2, e_3 the Cartesian unit coordinate vectors in $\mathbb{R}^3$. For the electric far field patterns we now have the following theorem due to Angell, Colton, and Kress [12].

Theorem 9.20 *Assume $\lambda > 0$. Then the set*

$$\mathcal{F}_\lambda = \left\{E^\lambda_\infty(\cdot\,, d_n)e_j : n = 1, 2, \dots,\ j = 1, 2, 3\right\}$$

of electric far field patterns for the exterior impedance problem is complete in $L^2_t(\mathbb{S}^2)$.

Proof Suppose that $g \in L^2_t(\mathbb{S}^2)$ satisfies

$$\int_{\mathbb{S}^2} E^\lambda_\infty(\hat{x}, d_n)e_j \cdot g(\hat{x})\, ds(\hat{x}) = 0$$

for $n = 1, 2, \dots$ and $j = 1, 2, 3$. We must show that $g = 0$. As in the proof of Theorem 9.7, by the reciprocity Theorem 9.19, we have

$$\int_{\mathbb{S}^2} E^\lambda_\infty(-d, -\hat{x})g(\hat{x})\, ds(\hat{x}) = 0$$

for all $d \in \mathbb{S}^2$, i.e.,

$$\int_{\mathbb{S}^2} E^\lambda_\infty(\hat{x}, d)h(d)\,ds(d) = 0 \tag{9.52}$$

for all $\hat{x} \in \mathbb{S}^2$ where $h(d) = g(-d)$.

Now define the electromagnetic Herglotz pair E^i_0, H^i_0 by

$$E^i_0(x) = \int_{\mathbb{S}^2} E^i(x, d)h(d)\,ds(d) = ik \int_{\mathbb{S}^2} h(d)\,e^{ik\,x\cdot d}\,ds(d),$$

$$H^i_0(x) = \int_{\mathbb{S}^2} H^i(x, d)h(d)\,ds(d) = \operatorname{curl} \int_{\mathbb{S}^2} h(d)\,e^{ik\,x\cdot d}\,ds(d).$$

Analogous to Lemma 6.35 it can be seen that the left-hand side of (9.52) represents the electric far field pattern of the scattered field E^s_0, H^s_0 corresponding to the incident field E^i_0, H^i_0. Then from (9.52) we see that the electric far field pattern of E^s_0 vanishes and hence, from Theorem 6.10, both E^s_0 and H^s_0 are identically zero in $\mathbb{R}^3 \setminus D$. We can now conclude that E^i_0, H^i_0 satisfies the impedance boundary condition

$$\nu \times \operatorname{curl} E^i_0 - i\lambda\,(\nu \times E^i_0) \times \nu = 0 \quad \text{on } \partial D. \tag{9.53}$$

Gauss' theorem and the Maxwell equations (compare (9.17)) now imply that

$$\int_{\partial D} \nu \times E^i_0 \cdot \bar{H}^i_0\,ds = ik \int_D \left\{ |H^i_0|^2 - |E^i_0|^2 \right\} dx$$

and hence from (9.53) we have that

$$\lambda \int_{\partial D} |\nu \times E^i_0|^2 ds = ik^2 \int_D \left\{ |E^i_0|^2 - |H^i_0|^2 \right\} dx$$

whence $\nu \times E^i_0 = 0$ on ∂D follows since $\lambda > 0$. From (9.53) we now see that $\nu \times H^i_0 = 0$ on ∂D and hence from the Stratton–Chu formulas (6.8) and (6.9) we have that $E^i_0 = H^i_0 = 0$ in D and by analyticity (Theorem 6.3) $E^i_0 = H^i_0 = 0$ in $\mathbb{R}^3$. But now from Theorem 3.27 we conclude that $h = 0$ and consequently $g = 0$. □