Chapter 7
Inverse Electromagnetic Obstacle Scattering

This last chapter on obstacle scattering is concerned with the extension of the results from Chap. 5 on inverse acoustic scattering to inverse electromagnetic scattering. In order to avoid repeating ourselves, we keep this chapter short by referring back to the corresponding parts of Chap. 5 when appropriate. In particular, for notations and for the motivation of our analysis we urge the reader to get reacquainted with the corresponding analysis in Chap. 5 on acoustics. We again follow the general guideline of our book and consider only one of the many possible inverse electromagnetic obstacle problems: given the electric far field pattern for one or several incident plane electromagnetic waves and knowing that the scattering obstacle is perfectly conducting, find the shape of the scatterer.

We begin the chapter with a uniqueness result. Due to the lack of an appropriate selection theorem, we do not follow Schiffer's proof as in acoustics. Instead of this, we prove a uniqueness result following Isakov's approach and, in addition, we use a method based on differentiation with respect to the wave number. We also include the electromagnetic version of Karp's theorem.

We then proceed to establish a continuous dependence result on the boundary based on the integral equation approach. As an alternative for establishing Fréchet differentiability with respect to the boundary we present the electromagnetic version of an approach proposed by Kress and Päivärinta [271]. The following three sections then will present extensions of some of the iterative methods, decomposition methods, and sampling methods considered in Chap. 5 from acoustics to electromagnetics. In particular we will present the electromagnetic versions of the iterative method due to Johansson and Sleeman, the decomposition methods of Kirsch and Kress and of Colton and Monk, and conclude with a discussion of the linear sampling method in electromagnetic obstacle scattering.

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7.1 Uniqueness

For the investigation of uniqueness in inverse electromagnetic obstacle scattering, as in the case of the Neumann and the impedance boundary condition in acoustics, Schiffer's method of Theorem 5.1 cannot be applied since the appropriate selection theorem in electromagnetics requires the boundary to be sufficiently smooth (see [297]). However, the methods used in Theorem 5.6 for inverse acoustic scattering can be extended to the case of inverse electromagnetic scattering from perfect and impedance conductors. We consider boundary conditions of the form $BE = 0$ on ∂D, where $BE = \nu \times E$ for a perfect conductor and $BE = \nu \times \operatorname{curl} E - i\lambda\,(\nu \times E) \times \nu$ for the impedance boundary condition. In the latter case, the real-valued function λ is assumed to be continuous and positive to ensure well-posedness of the direct scattering problem as proven in Theorem 9.17.

Theorem 7.1 *Assume that D_1 and D_2 are two scatterers with boundary conditions B_1 and B_2 such that for a fixed wave number the electric far field patterns for both scatterers coincide for all incident directions and all polarizations. Then $D_1 = D_2$ and $B_1 = B_2$.*

Proof The proof is completely analogous to that of Theorem 5.6 for the acoustic case which was based on the reciprocity relations from Theorems 3.24 and 3.25 and Holmgren's Theorem 2.3. In the electromagnetic case we have to use the reciprocity relations from Theorems 6.31 and 6.32 and Holmgren's Theorem 6.5 and instead of point sources $\Phi(\cdot\,, z)$ as incident fields we use electric dipoles $\operatorname{curl}\operatorname{curl} p\Phi(\cdot\,, z)$.

□

A corresponding uniqueness result for the inverse electromagnetic transmission problem has been proven by Hähner [174].

For diversity, we now prove a uniqueness theorem for fixed direction and polarization.

Theorem 7.2 *Assume that D_1 and D_2 are two perfect conductors such that for one fixed incident direction and polarization the electric far field patterns of both scatterers coincide for all wave numbers contained in some interval $0 < k_1 < k < k_2 < \infty$. Then $D_1 = D_2$.*

Proof We will use the fact that the scattered wave depends analytically on the wave number k. Deviating from our usual notation, we indicate the dependence on the wave number by writing $E^i(x; k)$, $E^s(x; k)$, and $E(x; k)$. Since the fundamental solution to the Helmholtz equation depends analytically on k, the integral operator $I + M + iNPS_0^2$ in the integral equation (6.56) is also analytic in k. (For the reader who is not familiar with analytic operators, we refer to Sect. 8.5.) From the fact that for each $k > 0$ the inverse operator of $I + M + iNPS_0^2$ exists, by using a Neumann series argument it can be deduced that the inverse $(I + M + iNPS_0^2)^{-1}$ is also analytic in k. Therefore, the analytic dependence of the right-hand side $c = 2\,E^i(\cdot\,; k) \times \nu$ of (6.56) for the scattering problem implies that the solution a also depends analytically on k and consequently from the representation (6.55) it can be seen that the scattered field $E^s(\cdot\,;k)$ also depends analytically on k. In addition, from (6.55) it also follows that the derivatives of E^s with respect to the space variables and with respect to the wave number can be interchanged. Therefore, from the vector Helmholtz equation $\Delta E + k^2 E = 0$ for the total field $E = E^i + E^s$ we derive the inhomogeneous vector Helmholtz equation

$$\Delta F + k^2 F = -2kE$$

for the derivative

$$F := \frac{\partial E}{\partial k}\,.$$

Let k_0 be an accumulation point of the wave numbers for the incident waves and assume that $D_1 \neq D_2$. By Theorem 6.10, the electric far field pattern uniquely determines the scattered field. Hence, for any incident wave E^i the scattered wave E^s for both obstacles coincide in the unbounded component G of the complement of $D_1 \cup D_2$. Without loss of generality, we assume that $(\mathbb{R}^3 \setminus G) \setminus \bar{D}_2$ is a nonempty open set and denote by D^* a connected component of $(\mathbb{R}^3 \setminus G) \setminus \bar{D}_2$. Then E is defined in D^* since it describes the total wave for D_2, that is, E satisfies the vector Helmholtz equation in D^* and fulfills homogeneous boundary conditions $\nu \times E = 0$ and $\operatorname{div} E = 0$ on ∂D^* for each k with $k_1 < k < k_2$. By differentiation with respect to k, it follows that $F(\cdot\,;k_0)$ satisfies the same homogeneous boundary conditions. Therefore, from Green's vector theorem (6.3) applied to $E(\cdot\,;k_0)$ and $F(\cdot\,;k_0)$ we find that

$$2k_0 \int_{D^*} |E|^2 dx = \int_{D^*} \left\{ \bar{F} \Delta E - E \Delta \bar{F} \right\} dx = 0,$$

whence $E = 0$ first in D^* and then by analyticity everywhere outside $D_1 \cup D_2$. This implies that E^i satisfies the radiation condition whence $E^i = 0$ in $\mathbb{R}^3$ follows (cf. p. 231). This is a contradiction. □

Concerning uniqueness for one incident wave under a priori assumptions on the shape of the scatterer we note that analogous to Theorem 5.4 using the explicit solution (6.85) it can be shown that a perfectly conducting ball is uniquely determined by the far field pattern for plane wave incidence with one direction d and polarization p. In the context of Theorem 5.5 it has been shown by Liu, Yamamoto, and Zou [306] that a perfectly conducting polyhedron is uniquely determined by the far field pattern for plane wave incidence with one direction d and two polarizations p_1 and p_2. We note that our proof of Theorem 5.5 for a convex polyhedron can be carried over to the perfect conductor case.

We include in this section on uniqueness the electromagnetic counterpart of Karp's theorem for acoustics. If the perfect conductor D is a ball centered at the origin, it is obvious from symmetry considerations that the electric far field pattern for incoming plane waves of the form (6.86) satisfies

$$E_\infty(Q\hat{x}, Qd)Qp = QE_\infty(\hat{x}, d)p \tag{7.1}$$

for all $\hat{x}, d \in \mathbb{S}^2$, all $p \in \mathbb{R}^3$, and all rotations Q, i.e., for all real orthogonal matrices Q with $\det Q = 1$. As shown by Colton and Kress [98], the converse of this statement is also true. We include a simplified version of the original proof.

The vectors $\hat{x}$, $p \times \hat{x}$ and $\hat{x} \times (p \times \hat{x})$ form a basis in $\mathbb{R}^3$ provided $p \times \hat{x} \neq 0$. Hence, since the electric far field pattern is orthogonal to $\hat{x}$, we can write

$$E_\infty(\hat{x}, d)p = [e_1(\hat{x}, d)p]\ p \times \hat{x} + [e_2(\hat{x}, d)p]\ \hat{x} \times (p \times \hat{x})$$

where

$$[e_1(\hat{x}, d)p] = [p \times \hat{x}] \cdot E_\infty(\hat{x}, d)p$$

and

$$[e_2(\hat{x}, d)p] = [\hat{x} \times (p \times \hat{x})] \cdot E_\infty(\hat{x}, d)p$$

and the condition (7.1) is equivalent to

$$e_j(Q\hat{x}, Qd)Qp = e_j(\hat{x}, d)p, \quad j = 1, 2.$$

This implies that

$$\int_{\mathbb{S}^2} e_j(\hat{x}, d)p\, ds(d) = \int_{\mathbb{S}^2} e_j(Q\hat{x}, d)Qp\, ds(d)$$

and therefore

$$\int_{\mathbb{S}^2} E_\infty(\hat{x}, d)p\, ds(d) = c_1(\theta)\ p \times \hat{x} + c_2(\theta)\ \hat{x} \times (p \times \hat{x}) \tag{7.2}$$

for all $\hat{x} \in \mathbb{S}^2$ and all $p \in \mathbb{R}^3$ with $p \times \hat{x} \neq 0$ where c_1 and c_2 are functions depending only on the angle θ between $\hat{x}$ and p. Given $p \in \mathbb{R}^3$ such that $0 < \theta < \pi/2$, we also consider the vector

$$q := 2\,\hat{x} \cdot p\,\hat{x} - p$$

which clearly makes the same angle with $\hat{x}$ as p. From the linearity of the electric far field pattern with respect to polarization, we have

$$E_\infty(\hat{x}, d)(\lambda p + \mu q) = \lambda E_\infty(\hat{x}, d)p + \mu E_\infty(\hat{x}, d)q \tag{7.3}$$

for all $\lambda, \mu \in \mathbb{R}$. Since $q \times \hat{x} = -p \times \hat{x}$, from (7.2) and (7.3) we can conclude that

$$c_j(\theta_{\lambda\mu}) = c_j(\theta), \quad j = 1, 2,$$

for all $\lambda, \mu \in \mathbb{R}$ with $\lambda \neq \mu$ where $\theta_{\lambda\mu}$ is the angle between $\hat{x}$ and $\lambda p + \mu q$. This now implies that both functions c_1 and c_2 are constants since, by choosing λ and μ appropriately, we can make $\theta_{\lambda\mu}$ to be any angle between 0 and π. With these constants, by continuity, (7.2) is valid for all $\hat{x} \in \mathbb{S}^2$ and all $p \in \mathbb{R}^3$.

Choosing a fixed but arbitrary vector $p \in \mathbb{R}^3$ and using the Funk–Hecke formula (2.45), we consider the superposition of incident plane waves given by

$$\widetilde{E}^i(x) = \frac{i}{k} \operatorname{curl}\operatorname{curl} p \int_{\mathbb{S}^2} e^{ik\,x\cdot d} ds(d) = \frac{4\pi i}{k^2} \operatorname{curl}\operatorname{curl} p \,\frac{\sin k|x|}{|x|}. \tag{7.4}$$

Then, by Lemma 6.35 and (7.2), the corresponding scattered wave $\widetilde{E}^s$ has the electric far field pattern

$$\widetilde{E}_\infty(\hat{x}) = c_1\, p \times \hat{x} + c_2\, \hat{x} \times (p \times \hat{x}).$$

From this, with the aid of (6.26) and (6.27), we conclude that

$$\widetilde{E}^s(x) = \frac{ic_1}{k} \operatorname{curl} p \,\frac{e^{ik|x|}}{|x|} + \frac{c_2}{k^2} \operatorname{curl}\operatorname{curl} p \,\frac{e^{ik|x|}}{|x|}. \tag{7.5}$$

Using (7.4) and (7.5) and setting $r = |x|$, the boundary condition $\nu \times (\widetilde{E}^i + \widetilde{E}^s) = 0$ on ∂D can be brought into the form

$$\nu(x) \times \{g_1(r)\, p + g_2(r)\, p \times x + g_3(r)\,(p\cdot x)\, x\} = 0, \quad x \in \partial D, \tag{7.6}$$

for some functions g_1, g_2, g_3. In particular,

$$g_1(r) = \frac{4\pi i}{k^2 r} \left\{ \frac{d}{dr} \frac{\sin kr}{r} + k^2 \sin kr + \frac{c_2}{4\pi i} \frac{d}{dr} \frac{e^{ikr}}{r} + \frac{c_2 k^2}{4\pi i}\, e^{ikr} \right\}.$$

For a fixed, but arbitrary $x \in \partial D$ with $x \neq 0$ we choose p to be orthogonal to x and take the scalar product of (7.6) with $p \times x$ to obtain

$$g_1(r)\, x \cdot \nu(x) = 0, \quad x \in \partial D.$$

Assume that $g_1(r) \neq 0$. Then $x \cdot \nu(x) = 0$ and inserting $p = x \times \nu(x)$ into (7.6) we arrive at the contradiction $g_1(r)\, x = 0$. Hence, since $x \in \partial D$ can be chosen arbitrarily, we have that $g_1(r) = 0$ for all $x \in \partial D$ with $x \neq 0$. Since g_1 does not vanish identically and is analytic, it can have only discrete zeros. Therefore, $r = |x|$ must be constant for all $x \in \partial D$, i.e., D is a ball with center at the origin.

7.2 Continuity and Differentiability of the Far Field Mapping

In this section, as in the case of acoustic obstacle scattering, we wish to study some of the properties of the far field mapping

$$\mathcal{F} : \partial D \mapsto E_\infty$$

which for a fixed incident plane wave E^i maps the boundary ∂D of the perfect conductor D onto the electric far field pattern E_∞ of the scattered wave.

We first briefly wish to indicate why the weak solution methods used in the proof of Theorems 5.8 and 5.9 have no immediate counterpart for the electromagnetic case. Recall the electric to magnetic boundary component map $\mathcal{A}$ from Theorem 6.22 that for radiating solutions to the Maxwell equations transforms the tangential trace of the electric field onto the tangential trace of the magnetic field. In the remark after Theorem 6.23 we noted that $\mathcal{A}$ is a bijective bounded operator from $H^{-1/2}(\mathrm{Div}, \partial D)$ onto $H^{-1/2}(\mathrm{Div}, \partial D)$ with a bounded inverse.

Now let S_R denote the sphere of radius R centered at the origin and recall the definition (6.60) of the vector spherical harmonics U_n^m and V_n^m. Then for the tangential field

$$a = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left\{ a_n^m \, U_n^m + b_n^m \, V_n^m \right\}$$

with Fourier coefficients a_n^m and b_n^m the norm on $H^{-1/2}(S_R)$ can be written as

$$\|a\|^2_{H^{-1/2}(S_R)} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=-n}^{n} \left\{ |a_n^m|^2 + |b_n^m|^2 \right\}.$$

Since $\mathrm{Div}\, U_n^m = -\sqrt{n(n+1)}\, Y_n^m$ and $\mathrm{Div}\, V_n^m = 0$, the norm on the Sobolev space $H^{-1/2}(\mathrm{Div}, S_R)$ is equivalent to

$$\|a\|^2_{H^{-1/2}(\mathrm{Div}, S_R)} = \sum_{n=1}^{\infty} \left\{ n \sum_{m=-n}^{n} |a_n^m|^2 + \frac{1}{n} \sum_{m=-n}^{n} |b_n^m|^2 \right\}.$$

From the expansion (6.73) for radiating solutions to the Maxwell equations, we see that $\mathcal{A}$ maps the tangential field a with Fourier coefficients a_n^m and b_n^m onto

$$\mathcal{A}a = \frac{1}{ik} \sum_{n=1}^{\infty} \left\{ \delta_n \sum_{m=-n}^{n} a_n^m \, V_n^m + \frac{k^2}{\delta_n} \sum_{m=-n}^{n} b_n^m \, U_n^m \right\} \qquad (7.7)$$

where

$$\delta_n := \frac{k h_n^{(1)\prime}(kR)}{h_n^{(1)}(kR)} + \frac{1}{R}, \quad n = 1, 2, \dots .$$

Comparing this with (5.22), we note that

$$\delta_n = \gamma_n + \frac{1}{R},$$

that is, we can use the results from the proof of Theorem 5.8 on the coefficients γ_n. There does not exist a positive t such that $h_n^{(1)}(t) = 0$ or $h_n^{(1)}(t) + t h_n^{(1)\prime}(t) = 0$ since the Wronskian (2.37) does not vanish. Therefore, we have confirmed that the operator $\mathcal{A}$ in the special case of a ball indeed is bijective. Furthermore, from

$$c_1 n \le |\delta_n| \le c_2 n$$

which is valid for all n and some constants $0 < c_1 < c_2$, it is confirmed that $\mathcal{A}$ maps $H^{-1/2}(\text{Div}, S_R)$ boundedly onto itself.

However, different from the acoustic case, due to the factor k^2 in the second term of the expansion (7.7) the operator $ik\mathcal{A}$ in the limiting case $k = 0$ no longer remains bijective. This reflects the fact that for $k = 0$ the Maxwell equations decouple. Therefore, there is no obvious way of splitting $\mathcal{A}$ into a strictly coercive and a compact operator as was done for the Dirichlet to Neumann map in the proof of Theorem 5.8.

Hence, for the continuous dependence on the boundary in electromagnetic obstacle scattering, we rely on the integral equation approach. For this, we describe a modification of the boundary integral equations used for proving existence of a solution to the exterior Maxwell problem in Theorem 6.21 which was introduced by Werner [423] and simplified by Hähner [171, 173]. In addition to surface potentials, it also contains volume potentials which makes it less satisfactory from a numerical point of view. However, it will make the investigation of the continuous dependence on the boundary easier since it avoids dealing with the more complicated second term in the approach (6.55) containing the double curl. We recall the notations introduced in Sects. 6.3 and 6.4. After choosing an open ball B such that $\bar{B} \subset D$, we try to find the solution to the exterior Maxwell problem in the form

$$\begin{aligned} E(x) = {} & \operatorname{curl} \int_{\partial D} \Phi(x, y) a(y)\, ds(y) \\ & - \int_{\partial D} \Phi(x, y) \varphi(y) \nu(y)\, ds(y) - \int_B \Phi(x, y) b(y)\, dy, \end{aligned} \tag{7.8}$$

$$H(x) = \frac{1}{ik} \operatorname{curl} E(x), \quad x \in \mathbb{R}^3 \setminus \partial D.$$

We assume that the densities $a \in C^{0,\alpha}(\mathrm{Div}, \partial D)$, $\varphi \in C^{0,\alpha}(\partial D)$ and $b \in C^{0,\alpha}(B)$ satisfy the three integral equations

$$\begin{aligned} a + M_{11}a + M_{12}\varphi + M_{13}b &= 2c \\ \varphi + M_{22}\varphi + M_{23}b &= 0 \\ b + M_{31}a + M_{32}\varphi + M_{33}b &= 0 \end{aligned} \tag{7.9}$$

where the operators are given by $M_{11} := M$, $M_{22} := K$, $M_{12}\varphi := -\nu \times S(\nu\varphi)$, and

$$(M_{13}b)(x) := -2\,\nu(x) \times \int_B \Phi(x, y)b(y)\,dy, \quad x \in \partial D,$$

$$(M_{23}b)(x) := -2 \int_B b(y) \cdot \mathrm{grad}_x\, \Phi(x, y)\,dy, \quad x \in \partial D,$$

$$(M_{31}a)(x) := i\eta(x)\,\mathrm{curl} \int_{\partial D} \Phi(x, y)a(y)\,ds(y), \quad x \in B,$$

$$(M_{32}\varphi)(x) := -i\eta(x) \int_{\partial D} \Phi(x, y)\varphi(y)\nu(y)\,ds(y), \quad x \in B,$$

$$(M_{33}b)(x) := -i\eta(x) \int_B \Phi(x, y)b(y)\,dy, \quad x \in B,$$

and where $\eta \in C^{0,\alpha}(\mathbb{R}^3)$ is a function with $\eta > 0$ in B and $\mathrm{supp}\,\eta = \bar{B}$.

First assume that we have a solution to these integral equations. Then clearly $\mathrm{div}\,E$ is a radiating solution to the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}$ and, by the jump relations, the second integral equation implies $\mathrm{div}\,E = 0$ on ∂D. Hence, $\mathrm{div}\,E = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ because of the uniqueness for the exterior Dirichlet problem. Now, with the aid of Theorems 6.4 and 6.8, we conclude that E, H is a radiating solution to the Maxwell equations in $\mathbb{R}^3 \setminus \bar{D}$. By the jump relations, the first integral equation ensures the boundary condition $\nu \times E = c$ on ∂D is satisfied.

We now establish that the system (7.9) of integral equations is uniquely solvable. For this, we first observe that all the integral operators M_{ij} are compact. The compactness of $M_{11} = M$ and $M_{22} = K$ is stated in Theorems 6.17 and 3.4 and the compactness of M_{33} follows from the fact that the volume potential operator maps $C(\bar{B})$ boundedly into $C^{1,\alpha}(\bar{B})$ (see Theorem 8.1) and the imbedding Theorem 3.2. The compactness of $M_{12} : C^{0,\alpha}(\partial D) \to C^{0,\alpha}(\mathrm{Div}, \partial D)$ follows from Theorem 3.4 and the representation

$$(\mathrm{Div}\, M_{12}\varphi)(x) = 2\,\nu(x) \cdot \int_{\partial D} \varphi(y)\{\nu(x) - \nu(y)\} \times \mathrm{grad}_x\, \Phi(x, y)\,ds(y), \quad x \in \partial D,$$

which can be derived with the help of (6.43). The term $\nu(x)-\nu(y)$ makes the kernel weakly singular in a way such that Corollary 2.9 from [104] can be applied. For the other terms, compactness is obvious since the kernels are sufficiently smooth. Hence, by the Riesz–Fredholm theory it suffices to show that the homogeneous system only allows the trivial solution.

Assume that a, φ, b solve the homogeneous form of (7.9) and define E, H by (7.8). Then, by the above analysis, we already know that E, H solve the homogeneous exterior Maxwell problem whence $E = H = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ follows. The jump relations then imply that

$$\nu \times \operatorname{curl} E_- = 0, \quad \nu \cdot E_- = 0 \quad \text{on } \partial D. \tag{7.10}$$

From the third integral equation and the conditions on η, we observe that we may view b as a field in $C^{0,\alpha}(\mathbb{R}^3)$ with support in $\bar{B}$. Therefore, by the jump relations for volume potentials (see Theorem 8.1), we have $E \in C^2(D)$ and, in view of the third integral equation,

$$\Delta E + k^2 E = b = -i\eta E \quad \text{in } D. \tag{7.11}$$

From (7.10) and (7.11) with the aid of Green's vector theorem (6.2), we now derive

$$\int_D \left\{ |\operatorname{curl} E|^2 + |\operatorname{div} E|^2 - (k^2 + i\eta)|E|^2 \right\} dx = 0,$$

whence, taking the imaginary part,

$$\int_B \eta |E|^2 dx = 0$$

follows. This implies $E = 0$ in B and from (7.11) we obtain $b = \Delta E + k^2 E = 0$ in D. Since solutions to the Helmholtz equation are analytic, from $E = 0$ in B we obtain $E = 0$ in D. The jump relations now finally yield $a = \nu \times E_+ - \nu \times E_- = 0$ and $\varphi = \operatorname{div} E_+ - \operatorname{div} E_- = 0$. Thus, we have established unique solvability for the system (7.9).

We are now ready to outline the proof of the electromagnetic analogue to the continuous dependence result of Theorem 5.17. We again consider surfaces Λ which are starlike with respect to the origin and represented in the form (5.20) with a positive function $r \in C^{1,\alpha}(\mathbb{S}^2)$ with $0 < \alpha < 1$. We recall from Sect. 5.3 what we mean by convergence of surfaces $\Lambda_n \to \Lambda$, $n \to \infty$, and L^2 convergence of functions f_n from $L^2(\Lambda_n)$ to a function f in $L^2(\Lambda)$. (For consistency with the rest of our book, we again choose C^2 surfaces Λ_n and Λ instead of $C^{1,\alpha}$ surfaces.)

Theorem 7.3 *Let (Λ_n) be a sequence of starlike C^2 surfaces which converges with respect to the $C^{1,\alpha}$ norm to a C^2 surface Λ as $n \to \infty$ and let E_n, H_n and E, H be radiating solutions to the Maxwell equations in the exterior of Λ_n and*

Λ, respectively. Assume that the continuous tangential components of E_n on Λ_n are L^2 convergent to the tangential components of E on Λ. Then the sequence (E_n), together with all its derivatives, converges to E uniformly on compact subsets of the open exterior of Λ.

Proof As in the proof of Theorem 5.17, we transform the boundary integral equations in (7.9) onto a fixed reference surface by substituting $x = r(\hat{x})\,\hat{x}$ to obtain integral equations over the unit sphere for the surface densities

$$\tilde{a}(\hat{x}) := \hat{x} \times a(r(\hat{x})\,\hat{x}), \quad \tilde{\varphi}(\hat{x}) := \varphi(r(\hat{x})\,\hat{x}).$$

Since the weak singularities of the operators M_{11}, M_{22} and M_{12} are similar in structure to those of K and S which enter into the combined double- and single-layer approach to the exterior Dirichlet problem, proceeding as in Theorem 5.17 it is possible to establish an estimate of the form (5.55) for the boundary integral terms in the transformed equations corresponding to (7.9). For the mixed terms like M_{31} and M_{13}, estimates of the type (5.55) follow trivially from Taylor's formula and the smoothness of the kernels. Finally, the volume integral term corresponding to M_{33} does not depend on the boundary at all. Based on these estimates, the proof is now completely analogous to that of Theorem 5.17. □

Without entering into details we wish to mention that the above approach can also be used to show Fréchet differentiability with respect to the boundary analogously to Theorem 5.15 (see [356]).

In an alternate approach for establishing Fréchet differentiability, we extend a technique due to Kress and Päivärinta [271] from acoustic to electromagnetic scattering. For this, in a slightly more general setting, we consider a family of scatterers D_h with boundaries represented in the form

$$\partial D_h = \{x + h(x) : x \in \partial D\} \tag{7.12}$$

where $h : \partial D \to \mathbb{R}^3$ is of class C^2 and sufficiently small in the C^2 norm on ∂D. Then we may consider the operator $\mathcal{F}$ as a mapping from a ball

$$V := \{h \in C^2(\partial D) : \|h\|_{C^2} < \delta\} \subset C^2(\partial D)$$

with sufficiently small radius $\delta > 0$ into $L^2_t(\mathbb{S}^2)$.

From Theorem 6.44 we recall the bounded linear operator $A : L^2_t(\partial D) \to L^2_t(\mathbb{S}^2)$ which maps the electric tangential components on ∂D of radiating solutions to the Maxwell equations in $\mathbb{R}^3 \setminus \bar{D}$ onto the electric far field pattern. Further we denote by A_h the operator A with ∂D replaced by ∂D_h and define the integral operator $G_h : L^2_t(\partial D_h) \to L^2_t(\partial D_h)$ by

$$(G_h a)(x) := \nu(x) \times \int_{\partial D_h} E_e(x\,, y)[\nu(y) \times a(y)]\, ds(y), \quad x \in \partial D,$$

in terms of the total electric field E_e for scattering of the electric dipole field E_e^i, H_e^i given by (6.89) from D.

Lemma 7.4 *Assume that $\bar{D} \subset D_h$. Then for the far fields E_∞ and $E_{h,\infty}$ for scattering of an incident field E^i, H^i from D and D_h, respectively, we have the factorization*

$$E_{h,\infty} - E_\infty = A_h G_h \left(\nu \times H_h|_{\partial D_h}\right) \tag{7.13}$$

where H_h denotes the total magnetic field for scattering from D_h.

Proof As indicated in the formulation of the lemma, we distinguish the solution to the scattering problem for the domain D_h by the subscript h, that is, $E_h = E^i + E_h^s$ and $H_h = H^i + H_h^s$. By Huygens' principle, i.e., Theorem 6.24, the scattered field can be written as

$$E^s(x) = \int_{\partial D} [E_e^i(\cdot, x)]^\top [\nu \times H]\, ds, \quad x \in \mathbb{R}^3 \setminus \bar{D}_h. \tag{7.14}$$

From this we obtain that

$$\begin{aligned}
-E^s(x) &= \int_{\partial D} \left\{[E_e^s(\cdot, x)]^\top [\nu \times H] + [H_e^s(\cdot, x)]^\top [\nu \times E]\right\} ds \\
&= \int_{\partial D_h} \left\{[E_e^s(\cdot, x)]^\top [\nu \times H] + [H_e^s(\cdot, x)]^\top [\nu \times E]\right\} ds \\
&= \int_{\partial D_h} \left\{[E_e^s(\cdot, x)]^\top [\nu \times H^i] + [H_e^s(\cdot, x)]^\top [\nu \times E^i]\right\} ds \\
&= \int_{\partial D_h} \left\{[E_e^s(\cdot, x)]^\top [\nu \times H^i] - [H_e^s(\cdot, x)]^\top [\nu \times E_h^s]\right\} ds \\
&= \int_{\partial D_h} [E_e^s(\cdot, x)]^\top [\nu \times H_h]\, ds, \quad x \in \mathbb{R}^3 \setminus \bar{D}_h,
\end{aligned}$$

where we have used the perfect conductor boundary condition, the vector Green's theorem (applied component-wise), and the radiation condition.

On the other hand, the representation (7.14) applied to D_h yields

$$E_h^s(x) = \int_{\partial D_h} [E_e^i(\cdot, x)]^\top [\nu \times H_h\, ds, \quad x \in \mathbb{R}^3 \setminus \bar{D}_h,$$

and (7.13) follows by adding the last two equations and passing to the far field. □

Theorem 7.5 *The boundary to far field mapping* $\mathcal{F} : \partial D_h \mapsto E_\infty$ *is Fréchet differentiable. The derivative is given by*

$$\mathcal{F}'(\partial D) : h \mapsto \mathcal{E}_{h,\infty},$$

where $\mathcal{E}_{h,\infty}$ *is the electric far field pattern of the uniquely determined radiating solution* $\mathcal{E}_h, \mathcal{H}_h$ *to the Maxwell equations in* $\mathbb{R}^3 \setminus \bar{D}$ *satisfying the boundary condition*

$$\nu \times \mathcal{E}_h = -ik\, \nu \times (H \times \nu)\, \nu \cdot h - \nu \times \operatorname{Grad}\{(\nu \cdot h)\,(\nu \cdot E)\} \quad \text{on } \partial D \tag{7.15}$$

in terms of the total field $E = E^i + E^s$, $H = H^i + H^s$.

Proof We use the notations introduced in connection with Lemma 7.4. For simplicity we assume that ∂D is analytic. Then, by the regularity results on elliptic boundary value problems, the fields E, H and E_e, H_e can be extended as solutions to the Maxwell equations across the boundary ∂D. (This follows from Sects. 6.1 and 6.6 in [325] by considering the boundary value problem for the Maxwell equations equivalently as a boundary value problem for the vector Helmholtz equation with boundary condition for the tangential components and the divergence.) Hence (7.13) remains valid also if $\bar{D} \not\subset D_h$ provided that h is sufficiently small.

For simplicity we confine ourselves to the case where k is not a Maxwell eigenvalue. As in the proof of Theorem 6.43, from Huygen's principle (6.58) we obtain the integral equation

$$b + M'b = 2\{\nu \times H^i\} \times \nu \tag{7.16}$$

for the tangential component $b = \{\nu \times H\} \times \nu$ of the magnetic field. Because of our assumption on k the operator $I + M' : L^2_t(\partial D) \to L^2_t(\partial D)$ has a trivial nullspace and consequently a bounded inverse. By M'_h we denote the operator M' with ∂D replaced by ∂D_h and interpret it as an operator $M'_h : C_t(\partial D) \to C_t(\partial D)$ by substituting $x = \xi + h(\xi)$ and $y = \eta + h(\eta)$. With the aid of the decomposition (6.32) of the kernel of M, proceeding as in the proof of Theorem 5.14 it can be shown that

$$\|M'_h - M'\|_\infty \le c\|h\|_{C^2(\partial D)}$$

for some constant c depending on ∂D. Hence, by a Neumann series argument, from (7.16) it can be deduced that we have continuity

$$|\nu_h(y + h(y)) \times H_h(y + h(y)) - \nu(y) \times H(y)| \to 0, \quad \|h\|_{C^2(\partial D)} \to 0,$$

uniformly for all $y \in \partial D$. From this, in view of the continuity of H, it follows that

$$\int_{\partial D_h} E_e(x, \cdot)[\nu \times H_h]\, ds = \int_{\partial D_h} E_e(x, \cdot)[\nu \times H]\, ds + o\left(\|h\|_{C^2(\partial D)}\right)$$

uniformly for all sufficiently large $|x|$.

Using the symmetry relation (6.93) and the boundary condition $\nu \times E_e(x, \cdot) = 0$ on ∂D, from Gauss' divergence theorem we obtain

$$\int_{\partial D_h} E_e(x, \cdot)[\nu \times H]\, ds - ik \int_{D_h^*} \left\{ [E_e(\cdot\,, x)]^\top E + [H_e(\cdot\,, x)]^\top H \right\} \chi\, dy,$$

where

$$D_h^* := \{y \in D_h : y \notin D\} \cup \{y \in D : y \notin D_h\}$$

and $\chi(y) = 1$ if $y \in D_h$ and $y \notin D$ and $\chi(y) = -1$ if $y \in D$ and $y \notin D_h$. With the aid of the boundary condition $\nu \times E = 0$ on ∂D it can be shown that the volume integral over D_h^* can be approximated by a surface integral over ∂D through

$$\int_{D_h^*} \left\{ [E_e(\cdot\,, x)]^\top E + [H_e(\cdot\,, x)]^\top H \right\} \chi\, dy$$

$$= \int_{\partial D} \left\{ [E_e(\cdot\,, x)]^\top [\nu\,(E \cdot \nu)] + [H_e(\cdot\,, x)]^\top [\nu \times (H \times \nu)] \right\} \nu \cdot h\, ds + o\left(\|h\|_{C^1(\partial D)}\right)$$

uniformly for all sufficiently large $|x|$. Note that $\nu \times E = 0$ on ∂D implies that $\nu \cdot H = 0$ on ∂D as consequence of the Maxwell equations and the identity (6.43). Also as a consequence of the latter identity, with the help of the surface divergence theorem we can deduce that

$$k \int_{\partial D} \nu{\cdot}h\, [E_e(\cdot\,, x)]^\top [\nu\,(E{\cdot}\nu)]\, ds = -i \int_{\partial D} [H_e(\cdot\,, x)]^\top [\nu {\times} \operatorname{Grad}\{(\nu{\cdot}h)\,(\nu{\cdot}E)\}]\, ds.$$

Hence, putting the preceding four equations together and using the boundary condition (7.15) we find that

$$\int_{\partial D_h} E_e(x, \cdot)[\nu \times H_h]\, ds = \int_{\partial D} [H_e(\cdot\,, x)]^\top [\nu \times \mathcal{E}]\, ds + o\left(\|h\|_{C^2(\partial D)}\right). \tag{7.17}$$

On the other hand, from the Stratton–Chu formula (6.16) applied to $\mathcal{E}_h, \mathcal{H}_h$, the radiation condition and the boundary condition $\nu \times E_e = 0$ on ∂D we conclude that

$$\mathcal{E}_h(x) = \int_{\partial D} [H_e(\cdot\,, x)]^\top [\nu \times \mathcal{E}]\, ds, \quad x \in \mathbb{R}^3 \setminus \bar{D}.$$

Therefore, we can rewrite (7.17) as

$$\int_{\partial D_h} E_e(x, \cdot)[\nu \times H_h]\, ds = \mathcal{E}_h(x) + o\left(\|h\|_{C^2(\partial D)}\right)$$

and passing to the far field, with the aid of the identity (7.13), it follows that

$$E_{h,\infty} - E_\infty = \mathcal{E}_{h,\infty} + o\left(\|h\|_{C^2(\partial D)}\right).$$

This completes the proof. □

This approach to proving Fréchet differentiability has been extended to the impedance boundary condition by Haddar and Kress [169].

7.3 Iterative Solution Methods

All the iterative methods for solving the inverse obstacle problem in acoustics described in Sect. 5.4, in principle, have extensions to electromagnetic inverse obstacle scattering.

Here, in order to avoid repetitions, we only present an electromagnetic version of the method due to Johansson and Sleeman as suggested by Pieper [350]. We recall Huygens' principle from Theorem 6.24 and to circumvent the use of the hypersingular operator N we start from the representation

$$H(x) = H^i(x) + \operatorname{curl} \int_{\partial D} \nu(y) \times H(y)\, \Phi(x, y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{7.18}$$

for the total magnetic field H in terms of the incident field H^i and the representation

$$H_\infty(\hat{x}) = \frac{ik}{4\pi}\, \hat{x} \times \int_{\partial D} \nu(y) \times H(y)\, e^{-ik\,\hat{x}\cdot y}\, ds(y), \quad \hat{x} \in \mathbb{S}^2, \tag{7.19}$$

for the magnetic far field pattern H_∞. From (7.18), as in the proof of Theorem 6.43, from the jump relations we find that the tangential component

$$a := \nu \times H \quad \text{on } \partial D$$

satisfies

$$a(x) - 2\int_{\partial D} \nu(x) \times \{\operatorname{curl}_x \Phi(x, y) a(y)\}\, ds(y) = 2\, \nu(x) \times H^i(x), \quad x \in \partial D, \tag{7.20}$$

and (7.19) can be written as

$$\frac{ik}{4\pi}\, \hat{x} \times \int_{\partial D} a(y) e^{-ik\,\hat{x}\cdot y}\, ds(y) = H_\infty(\hat{x}), \quad \hat{x} \in \mathbb{S}^2. \tag{7.21}$$

We call (7.20) the field equation and (7.21) the data equation and interpret them as two integral equations for the unknown boundary and the unknown tangential component a of the total magnetic field on the boundary. Both equations are linear with respect to a and nonlinear with respect to ∂D. Equation (7.21) is severely ill-posed whereas (7.20) is well-posed provided k is not a Maxwell eigenvalue for D.

As in Sect. 5.4 there are three possible options for an iterative solution of the system (7.20)– (7.21). Here, from these we only briefly discuss the case where, given an approximation for the boundary ∂D, we solve the well-posed equation of the second kind (7.20) for a. Since the perfect conductor boundary condition $\nu \times E = 0$ on ∂D by the identity (6.43) implies that $\nu \cdot H = 0$ on ∂D, the full three-dimensional field H on ∂D is available via $H = a \times \nu$. Then, keeping H fixed, Eq. (7.21) is linearized with respect to ∂D to update the boundary approximation.

To describe this linearization in more detail, using the parameterization (5.20) for starlike ∂D and recalling the notation (5.32), we introduce the parameterized far field operator

$$A_\infty : C^2(\mathbb{S}^2) \times L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$$

by

$$A_\infty(r, b)(\hat{x}) := \frac{ik}{4\pi}\, \hat{x} \times \int_{\mathbb{S}^2} \nu_r(\hat{y}) \times b(\hat{y}) e^{-ikr(\hat{y})\, \hat{x}\cdot\hat{y}} ds(\hat{y}), \quad \hat{x} \in \mathbb{S}^2. \tag{7.22}$$

Here ν_r denotes the transformed normal vector as given by (5.33) in terms of the transformation $p_r : \mathbb{S}^2 \to \partial D$. Now the data equation (7.21) can be written in the operator form

$$A_\infty(r, b) = H_\infty \tag{7.23}$$

where we have set

$$b := J_r\, (a \circ p_r) \times \nu_r \tag{7.24}$$

with the Jacobian J_r of p_r. To update the boundary, the linearization

$$A'_\infty(r, b)q = H_\infty - A_\infty(r, b)$$

of (7.23) needs to be solved for q. The derivative A'_∞ is given by

$$\left(A'_\infty(r, \psi)q\right)(\hat{x}) = \frac{k^2}{4\pi}\, \hat{x} \times \int_{\mathbb{S}^2} \nu_r(\hat{y}) \times b(\hat{y}) e^{-ikr(\hat{y})\, \hat{x}\cdot\hat{y}}\, \hat{x}\cdot\hat{y}\, q(\hat{y}) ds(\hat{y})$$

$$+ \frac{ik}{4\pi}\, \hat{x} \times \int_{\mathbb{S}^2} (\nu'_r q)(\hat{y}) \times b(\hat{y}) e^{-ikr(\hat{y})\, \hat{x}\cdot\hat{y}} ds(\hat{y}), \quad \hat{x} \in \mathbb{S}^2,$$

where

$$(\nu_r)'q = \frac{p_q - \operatorname{Grad} q}{\sqrt{r^2 + |\operatorname{Grad} r|^2}} - \frac{rq + \operatorname{Grad} r \cdot \operatorname{Grad} q}{r^2 + |\operatorname{Grad} r|^2}\, \nu_r$$

denotes the derivative of ν_r, see (5.33) and (5.37).

We present two examples for reconstructions that were provided to us by Olha Ivanyshyn. The synthetic data were obtained by applying the spectral method of Sect. 3.7 to the integral equation (6.56) for $\eta = k$. For this, the unknown tangential vector field was represented in terms of its three Cartesian components and (6.56) was interpreted as a system of three scalar integral equations and the variant (3.146) of Wienert's method was applied. In both examples the synthetic data consisted of 242 values of the far field.

Correspondingly, for the reconstruction the number of collocation points on $\mathbb{S}^2$ for the data equation (7.21) also was chosen as 242. For the field equation (7.20) again Wienert's spectral method (3.146) was applied with 242 collocation points and 338 quadrature points corresponding to $N = 10$ and $N' = 12$ in (3.148). For the approximation space for the radial function representing the boundary of the scatterer, spherical harmonics up to order six were chosen.

The wave number was $k = 1$ and the incident direction $d = (0, 0, -1)$ and the polarization $p = (1, 0, 0)$ are indicated in the figures by a solid and a dashed arrow, respectively. The iterations were started with a ball of radius $3.5Y_0^0 = 0.9873$ centered at the origin. For the surface update H^1 penalization was applied with the regularization parameter selected by trial and error as $\alpha_n = \alpha\gamma^n$ depending on the iteration number n with $\alpha = 0.5$ and $\gamma = 2/3$.

Both to the real and imaginary part of the far field data 1% of normally distributed noise was added, i.e.,

$$\frac{\|H_\infty - H_\infty^\delta\|_{L^2(\mathbb{S}^2)}}{\|H_\infty\|_{L^2(\mathbb{S}^2}} \le 0.01.$$

In terms of the relative data error

$$\varepsilon_r := \frac{\|H_\infty - H_{r,\infty}\|_{L^2(\mathbb{S}^2)}}{\|H_\infty\|_{L^2(\mathbb{S}^2)}}$$

with the given far field data H_∞ and the far field $H_{r,\infty}$ corresponding to the radial function r, a stopping criterion was incorporated such that the iteration was carried on as long as $\varepsilon_r > 0.05$ or $\varepsilon_r > \varepsilon_{r+q}$. The figures show the exact shape on the left and the reconstruction on the right.

The first example is a cushion shaped scatterer with radial function

$$r(\theta, \varphi) = \sqrt{0.8 + 0.5(\cos 2\varphi - 1)(\cos 4\theta - 1)}, \quad \theta \in [0, \pi],\ \varphi \in [0, 2\pi].$$

Figure 7.1 shows the reconstruction after 19 iteration steps with the final data error $\varepsilon_r = 0.026$.

The second example is a pinched ball with radial function

$$r(\theta, \varphi) = \sqrt{1.44 + 0.5 \cos 2\varphi(\cos 2\theta - 1)}, \quad \theta \in [0, \pi], \ \varphi \in [0, 2\pi].$$

Figure 7.2 shows the reconstruction after nine iteration steps with data error $\varepsilon_r = 0.012$.

In passing we note that, in principle, instead of (7.24) one also could substitute $b := (a \circ p_r) \times \nu_r$, i.e., linearize also with respect to the surface element. However, numerical examples indicate that this variant is less stable.

For an implementation of regularized Newton iterations for the simultaneous linearization of the field equation (7.20) and the data equation (7.21) using spectral methods in the spirit of [145, 146] we refer to [217, 298, 299] where both the theoretical and numerical analysis were made more efficient by the use of the Piola transformation from continuum mechanics.

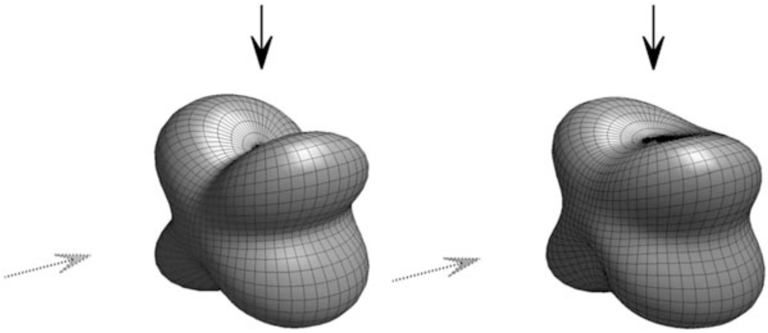

Fig. 7.1 Reconstruction of a cushion from noisy data

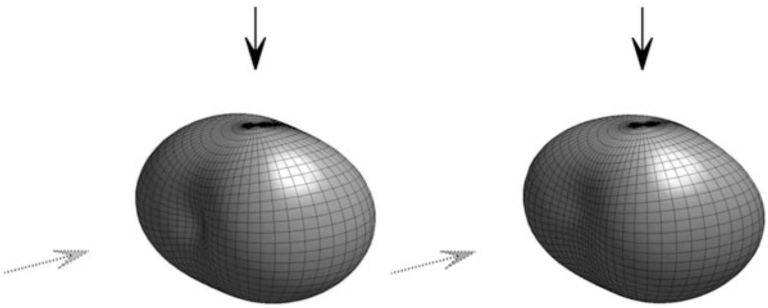

Fig. 7.2 Reconstruction of a pinched ball from noisy data

7.4 Decomposition Methods

We begin this section by describing the electromagnetic version of the decomposition method proposed by Kirsch and Kress for inverse acoustic obstacle scattering. We confine our analysis to inverse scattering from a perfectly conducting obstacle. Extensions to other boundary conditions are also possible.

We again first construct the scattered wave E^s from a knowledge of its electric far field pattern E_∞. To this end, we choose an auxiliary closed C^2 surface Γ with unit outward normal ν contained in the unknown scatterer D such that k is not a Maxwell eigenvalue for the interior of Γ. For example, we can choose Γ to be a sphere of radius R such that $j_n(kR) \neq 0$ and $j_n(kR)+kRj_n'(kR) \neq 0$ for $n = 0, 1, \dots$. Given the internal surface Γ, we try to represent the scattered field as the electromagnetic field

$$E^s(x) = \operatorname{curl} \int_\Gamma \Phi(x, y)a(y)\,ds(y), \quad H^s(x) = \frac{1}{ik}\operatorname{curl} E^s(x) \tag{7.25}$$

of a magnetic dipole distribution a from the space $L_t^2(\Gamma)$ of tangential L^2 fields on Γ. From (6.26) we see that the electric far field pattern of E^s is given by

$$E_\infty(\hat{x}) = \frac{ik}{4\pi}\,\hat{x} \times \int_\Gamma e^{-ik\,\hat{x}\cdot y}a(y)\,ds(y), \quad \hat{x} \in \mathbb{S}^2.$$

Hence, given the (measured) electric far field pattern E_∞, we have to solve the ill-posed integral equation of the first kind

$$M_\infty a = E_\infty \tag{7.26}$$

for the density a where the integral operator $M_\infty : L_t^2(\Gamma) \to L_t^2(\mathbb{S}^2)$ is defined by

$$(M_\infty a)(\hat{x}) := \frac{ik}{4\pi}\,\hat{x} \times \int_\Gamma e^{-ik\,\hat{x}\cdot y}a(y)\,ds(y), \quad \hat{x} \in \mathbb{S}^2. \tag{7.27}$$

As for the corresponding operator (5.70) in acoustics, the operator (7.27) has an analytic kernel and therefore the integral equation (7.26) is severely ill-posed. We now establish some properties of M_∞.

Theorem 7.6 *The far field operator M_∞ defined by (7.27) is injective and has dense range provided k is not a Maxwell eigenvalue for the interior of Γ.*

Proof Let $M_\infty a = 0$ and define an electromagnetic field by (7.25). Then E^s has vanishing electric far field pattern $E_\infty = 0$, whence $E^s = 0$ in the exterior of Γ follows by Theorem 6.10. After introducing, analogous to (6.33), the magnetic dipole operator $M : L^2(\Gamma) \to L^2(\Gamma)$, by the L^2 jump relation (6.53) we find that

$$a + Ma = 0.$$

Employing the argument used in the proof of Theorem 6.23, by the Fredholm alternative we see that the nullspaces of $I + M$ in $L^2(\Gamma)$ and in $C(\Gamma)$ coincide. Therefore, a is continuous and, by the jump relations of Theorem 6.12 for continuous densities, $H^s, -E^s$ represents a solution to the Maxwell equations in the interior of Γ satisfying the homogeneous boundary condition $\nu \times H^s = 0$ on Γ. Hence, by our assumption on the choice of Γ we have $H^s = E^s = 0$ everywhere in $\mathbb{R}^3$. The jump relations now yield $a = 0$ on Γ, whence M_∞ is injective.

The adjoint operator $M_\infty^* : L_t^2(\mathbb{S}^2) \to L_t^2(\Gamma)$ of M_∞ is given by

$$(M_\infty^* g)(y) = \left(\nu(y) \times \frac{ik}{4\pi} \int_{\mathbb{S}^2} e^{ik\,\hat{x}\cdot y}\, \hat{x} \times g(\hat{x})\, ds(\hat{x}) \right) \times \nu(y), \quad y \in \Gamma.$$

Let $M_\infty^* g = 0$. Then

$$E(y) := \int_{\mathbb{S}^2} e^{ik\,\hat{x}\cdot y}\, \hat{x} \times g(\hat{x})\, ds(\hat{x}), \quad H(y) := \frac{1}{ik} \operatorname{curl} E(y), \quad y \in \mathbb{R}^3,$$

defines an electromagnetic Herglotz pair satisfying $\nu \times E = 0$ on Γ. Hence, $E = H = 0$ in the interior of Γ by our assumption on the choice of Γ. Since E and H are analytic in $\mathbb{R}^3$, it follows that $E = H = 0$ everywhere. Theorem 3.27 now yields $g = 0$ on Γ, whence M_∞^* is also injective and by Theorem 4.6 the range of M_∞ is dense in $L_t^2(\mathbb{S}^2)$. □

We now define a magnetic dipole operator $\widetilde{M} : L_t^2(\Gamma) \to L_t^2(\Lambda)$ by

$$(\widetilde{M}a)(x) := \nu(x) \times \operatorname{curl} \int_\Gamma \Phi(x, y) a(y)\, ds(y), \quad x \in \Lambda, \tag{7.28}$$

where Λ denotes a closed C^2 surface with unit outward normal ν containing Γ in its interior. The proof of the following theorem is similar to that of Theorem 7.6.

Theorem 7.7 *The operator $\widetilde{M}$ defined by (7.28) is injective and has dense range provided k is not a Maxwell eigenvalue for the interior of Γ.*

Now we know that by our choice of Γ the integral equation of the first kind (7.26) has at most one solution. Analogous to the acoustic case, its solvability is related to the question of whether or not the scattered wave can be analytically extended as a solution to the Maxwell equations across the boundary ∂D.

For the same reasons as in the acoustic case, we combine a Tikhonov regularization for the integral equation (7.26) and a defect minimization for the boundary search into one cost functional. We proceed analogously to Definition 5.23 and choose a compact (with respect to the $C^{1,\beta}$ norm, $0 < \beta < 1$,) subset U of the set of all starlike closed C^2 surfaces described by

$$\Lambda = \left\{ r(\hat{x})\, \hat{x} : \hat{x} \in \mathbb{S}^2 \right\}, \quad r \in C^2(\mathbb{S}^2),$$

satisfying the a priori assumption

$$0 < r_i(\hat{x}) \leq r(\hat{x}) \leq r_e(\hat{x}), \quad \hat{x} \in \mathbb{S}^2,$$

with given functions r_i and r_e representing surfaces Λ_i and Λ_e such that the internal auxiliary surface Γ is contained in the interior of Λ_i and that the boundary ∂D of the unknown scatterer D is contained in the annulus between Λ_i and Λ_e. We now introduce the cost functional

$$\mu(a, \Lambda; \alpha) := \|M_\infty a - E_\infty\|^2_{L^2_t(\mathbb{S}^2)} + \alpha\|a\|^2_{L^2_t(\Gamma)} + \gamma\|\nu \times E^i + \widetilde{M}a\|^2_{L^2_t(\Lambda)}. \tag{7.29}$$

Again, $\alpha > 0$ denotes the regularization parameter for the Tikhonov regularization of (7.26) and $\gamma > 0$ denotes a suitable coupling parameter which for theoretical purposes we always assume equal to one.

Definition 7.8 Given the incident field E^i, a (measured) electric far field pattern $E_\infty \in L^2_t(\mathbb{S}^2)$, and a regularization parameter $\alpha > 0$, a surface Λ_0 from the compact set U is called *optimal* if there exists $a_0 \in L^2_t(\Gamma)$ such that a_0 and Λ_0 minimize the cost functional (7.29) simultaneously over all $a \in L^2_t(\Gamma)$ and $\Lambda \in U$, that is, we have

$$\mu(a_0, \Lambda_0; \alpha) = m(\alpha),$$

where

$$m(\alpha) := \inf_{a \in L^2_t(\Gamma),\ \Lambda \in U} \mu(a, \Lambda; \alpha).$$

For this reformulation of the electromagnetic inverse obstacle problem as a nonlinear optimization problem, we can state the following counterparts of Theorems 5.24–5.26. We omit the proofs since, except for minor adjustments, they literally coincide with those for the acoustic case. The use of Theorems 5.17 and 5.21, of course, has to be replaced by the corresponding electromagnetic versions given in Theorems 7.3 and 7.7.

Theorem 7.9 *For each $\alpha > 0$ there exists an optimal surface $\Lambda \in U$.*

Theorem 7.10 *Let E_∞ be the exact electric far field pattern of a domain D such that ∂D belongs to U. Then we have convergence of the cost functional* $\lim_{\alpha \to 0} m(\alpha) = 0$.

Theorem 7.11 *Let (α_n) be a null sequence and let (Λ_n) be a corresponding sequence of optimal surfaces for the regularization parameter α_n. Then there exists a convergent subsequence of (Λ_n). Assume that E_∞ is the exact electric far field pattern of a domain D such that ∂D is contained in U. Then every limit point Λ^* of (Λ_n) represents a surface on which the total field vanishes.*

Variants of these results were first established by Blöhbaum [34]. We will not repeat all the possible modifications mentioned in Sect. 5.5 for acoustic waves such as using more than one incoming wave, the limited aperture problem or using near field data. It is also straightforward to replace the magnetic dipole distribution on the internal surface for the approximation of the scattered field by an electric dipole distribution.

The above method for the electromagnetic inverse obstacle problem has been numerically implemented and tested by Haas, Rieger, Rucker, and Lehner [163].

We now proceed with briefly describing the extension of Potthast's point source method to electromagnetic obstacle scattering where again we start from Huygens' principle. By Theorem 6.24 the scattered field is given by

$$E^s(x) = \frac{i}{k}\,\operatorname{curl}\operatorname{curl}\int_{\partial D} \nu(y) \times H(y)\,\Phi(x,y)\,ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{7.30}$$

and the far field pattern by

$$E_\infty(\hat{x}) = \frac{ik}{4\pi}\,\hat{x} \times \int_{\partial D} [\nu(y) \times H(y)] \times \hat{x}\, e^{-ik\,\hat{x}\cdot y}\,ds(y), \quad \hat{x} \in \mathbb{S}^2, \tag{7.31}$$

in terms of the total magnetic field H. We choose an auxiliary closed C^2 surface Λ such that the scatterer D is contained in the interior of Λ and approximate the point source $\Phi(x,\cdot)$ for x in the exterior of Λ by a Herglotz wave function such that

$$\Phi(x,y) \approx \frac{1}{4\pi}\int_{\mathbb{S}^2} e^{ik\,y\cdot d} g_x(d)\,ds(d) \tag{7.32}$$

for all y in the interior of Λ and some scalar kernel function $g_x \in L^2(\mathbb{S}^2)$. In Sect. 5.5 we have described how such an approximation can be achieved uniformly with respect to y up to derivatives of second order on compact subsets of the interior of Λ by solving the ill-posed linear integral equation (5.93). With the aid of (6.4) and $\operatorname{grad}_x \Phi(x,y) = -\operatorname{grad}_y \Phi(x,y)$, we first transform (7.30) into

$$\begin{aligned} E^s(x) = {} & \frac{i}{k}\int_{\partial D} \big([\nu(y)) \times H(y)] \cdot \operatorname{grad}_y\big)\operatorname{grad}_y \Phi(x,y)\,ds(y) \\ & + ik\int_{\partial D} \nu(y) \times H(y)\,\Phi(x,y)\,ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}. \end{aligned} \tag{7.33}$$

With the aid of

$$\big(a(y)\cdot\operatorname{grad}_y\big)\operatorname{grad}_y e^{ik\,y\cdot d} + k^2 a(y)\,e^{ik\,y\cdot d} = k^2 d \times [a(y) \times d]\,e^{ik\,y\cdot d}$$

for $a = \nu \times H$ we now insert (7.32) into (7.33) and use (7.31) to obtain

$$E^s(x) \approx \int_{\mathbb{S}^2} g_x(d) E_\infty(-d)\, ds(d) \tag{7.34}$$

as an approximation for the scattered wave E^s. Knowing an approximation for the scattered wave, in principle the boundary ∂D can be found from the boundary condition $\nu \times (E^i + E^s) = 0$ on ∂D. For further details we refer to [29].

We conclude this section on decomposition methods with a short presentation of the electromagnetic version of the method of Colton and Monk. Again we confine ourselves to scattering from a perfect conductor and note that there are straightforward extensions to other boundary conditions.

As in the acoustic case, we try to find a superposition of incident plane electromagnetic fields with different directions and polarizations which lead to simple scattered fields and far field patterns. Starting from incident plane waves of the form (6.86), we consider as incident wave a superposition of the form

$$\widetilde{E}^i(x) = \int_{\mathbb{S}^2} ik\, e^{ik\,x\cdot d} g(d)\, ds(d), \quad \widetilde{H}^i(x) = \frac{1}{ik} \operatorname{curl} \widetilde{E}^i(x), \quad x \in \mathbb{R}^3, \tag{7.35}$$

with a tangential field $g \in L^2_t(\mathbb{S}^2)$, i.e., the incident wave is an electromagnetic Herglotz pair. By Lemma 6.35, the corresponding electric far field pattern

$$\widetilde{E}_\infty(\hat{x}) = \int_{\mathbb{S}^2} E_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2$$

is obtained by superposing the far field patterns $E_\infty(\cdot, d)g(d)$ for the incoming directions d with polarization $g(d)$. We note that by the Reciprocity Theorem 6.30 we may consider (7.35) also as a superposition with respect to the observation directions instead of the incident directions and in this case the method we are considering is sometimes referred to as dual space method.

If we want the scattered wave to become a prescribed radiating solution $\widetilde{E}^s, \widetilde{H}^s$ with explicitly known electric far field pattern $\widetilde{E}_\infty$, given the (measured) far field patterns for all incident directions and polarizations, we need to solve the linear integral equation of the first kind

$$Fg = \widetilde{E}_\infty \tag{7.36}$$

with the far field operator $F : L^2_t(\mathbb{S}^2) \to L^2_t(\mathbb{S}^2)$ defined by (6.98).

We need to assume the prescribed field $\widetilde{E}^s, \widetilde{H}^s$ is defined in the exterior of the unknown scatterer. For example, if we have the a priori information that the origin is contained in D, then for actual computations obvious choices for the prescribed scattered field would be the electromagnetic field

$$\widetilde{E}^s(x) = \operatorname{curl} a\Phi(x,0), \quad \widetilde{H}^s(x) = \frac{1}{ik}\operatorname{curl}\widetilde{E}^s(x)$$

of a magnetic dipole at the origin with electric far field pattern

$$\widetilde{E}_\infty(\hat{x}) = \frac{ik}{4\pi}\,\hat{x}\times a,$$

or the electromagnetic field

$$\widetilde{E}^s(x) = \operatorname{curl}\operatorname{curl} a\Phi(x,0), \quad \widetilde{H}^s(x) = \frac{1}{ik}\operatorname{curl}\widetilde{E}^s(x)$$

of an electric dipole with far field pattern

$$\widetilde{E}_\infty(\hat{x}) = \frac{k^2}{4\pi}\,\hat{x}\times(a\times\hat{x}).$$

Another more general possibility is to choose the radiating vector wave functions of Sect. 6.5 with the far field patterns given in terms of vector spherical harmonics (see Theorem 6.28).

We have already investigated the far field operator F. From Corollary 6.37, we know that F is injective and has dense range if and only if there does not exist an electromagnetic Herglotz pair which satisfies the homogeneous perfect conductor boundary condition on ∂D. Therefore, for the sequel we will make the assumption that k is not a Maxwell eigenvalue for D. This then implies that the inhomogeneous interior Maxwell problem for D is uniquely solvable. The classical approach to solve this boundary value problem is to seek the solution in the form of the electromagnetic field of a magnetic dipole distribution

$$E(x) = \operatorname{curl}\int_{\partial D} a(y)\Phi(x,y)\,ds(y), \quad H(x) = \frac{1}{ik}\operatorname{curl} E(x), \quad x\in D,$$

with a tangential field $a \in C^{0,\alpha}(\text{Div}, \partial D)$. Then, given $c \in C^{0,\alpha}(\text{Div}, \partial D)$, by the jump relations of Theorem 6.12 the electric field E satisfies the boundary condition $\nu \times E = c$ on ∂D if the density a solves the integral equation

$$a - Ma = -2c \tag{7.37}$$

with the magnetic dipole operator M defined by (6.33). The assumption that there exists no nontrivial solution to the homogeneous interior Maxwell problem in D now can be utilized to show with the aid of the jump relations that $I - M$ has a trivial nullspace in $C^{0,\alpha}(\text{Div}, \partial D)$ (for the details see [104]). Hence, by the Riesz–Fredholm theory $I - M$ has a bounded inverse $(I - M)^{-1}$ from $C^{0,\alpha}(\text{Div}, \partial D)$ into $C^{0,\alpha}(\text{Div}, \partial D)$. This implies solvability and well-posedness of the interior Maxwell problem. The proof of the following theorem is now completely analogous to that of Theorem 5.27.

Theorem 7.12 *Assume that k is not a Maxwell eigenvalue for D. Let (E_n, H_n) be a sequence of $C^1(D) \cap C(\bar{D})$ solutions to the Maxwell equations in D such that the boundary data $c_n = \nu \times E_n$ on ∂D are weakly convergent in $L^2_t(\partial D)$. Then the sequence (E_n, H_n) converges uniformly on compact subsets of D to a solution E, H to the Maxwell equations.*

From now on, we assume that $\mathbb{R}^3 \setminus D$ is contained in the domain of definition of $\widetilde{E}^s, \widetilde{H}^s$, that is, for the case of the above examples for $\widetilde{E}^s, \widetilde{H}^s$ with singularities at $x = 0$ we assume the origin to be contained in D. We associate the following uniquely solvable interior Maxwell problem

$$\operatorname{curl} \widetilde{E}^i - ik\widetilde{H}^i = 0, \quad \operatorname{curl} \widetilde{H}^i + ik\widetilde{E}^i = 0 \quad \text{in } D, \tag{7.38}$$

$$\nu \times (\widetilde{E}^i + \widetilde{E}^s) = 0 \quad \text{on } \partial D \tag{7.39}$$

to the inverse scattering problem. From Theorem 6.41 we know that the solvability of the integral equation (7.36) is connected to this interior boundary value problem, i.e., (7.36) is solvable for $g \in L^2_t(\mathbb{S}^2)$ if and only if the solution $\widetilde{E}^i, \widetilde{H}^i$ to (7.38) and (7.39) is an electromagnetic Herglotz pair with kernel ikg.

The Herglotz integral operator $H : L^2_t(\mathbb{S}^2) \to L^2_t(\Lambda)$ defined by

$$(Hg)(x) := ik\, \nu(x) \times \int_{\mathbb{S}^2} e^{ik\, x\cdot d} g(d)\, ds(d), \quad x \in \Lambda, \tag{7.40}$$

where ν denotes the unit outward normal to the surface and Λ represents the tangential component of the electric field on Λ for the Herglotz pair with kernel ikg.

Theorem 7.13 *The Herglotz operator H defined by (7.40) is injective and has dense range provided k is not a Maxwell eigenvalue for the interior of Λ.*

Proof The operator H is related to the adjoint of the far field integral operator given by (7.27). Therefore, the statement of the theorem is equivalent to Theorem 7.6. □

We are now ready to reformulate the inverse scattering problem as a nonlinear optimization problem analogous to Definition 5.28 in the acoustic case. We recall the description of the set U of admissible surfaces from p. 291 and pick a closed C^2 surface Γ_e such that Λ_e is contained in the interior of Γ_e where we assume that k is not a Maxwell eigenvalue for the interior of Γ_e. We now introduce a cost functional by

$$\mu(g, \Lambda; \alpha) := \|Fg - \widetilde{E}_\infty\|^2_{L^2_t(\mathbb{S}^2)} + \alpha\|Hg\|^2_{L^2_t(\Gamma_e)} + \gamma\|Hg + \nu\times\widetilde{E}^s\|^2_{L^2_t(\Lambda)}. \tag{7.41}$$

Definition 7.14 Given the (measured) electric far field $E_\infty \in L_t^2(\mathbb{S}^2 \times \mathbb{S}^2)$ for all incident and observation directions and all polarizations and a regularization parameter $\alpha > 0$, a surface Λ_0 from the compact set U is called *optimal* if

$$\inf_{g \in L_t^2(\mathbb{S}^2)} \mu(g, \Lambda_0; \alpha) = m(\alpha)$$

where

$$m(\alpha) := \inf_{g \in L_t^2(\mathbb{S}^2),\ \Lambda \in U} \mu(\varphi, \Lambda; \alpha).$$

For this electromagnetic optimization problem, we can state the following counterparts to Theorems 5.29–5.31. Variants of these results were first established by Blöhbaum [34].

Theorem 7.15 *For each $\alpha > 0$, there exists an optimal surface $\Lambda \in U$.*

Proof The proof is analogous to that of Theorem 5.29 with the use of Theorem 7.12 instead of Theorem 5.27. □

Theorem 7.16 *For all incident and observation directions and all polarizations let E_∞ be the exact electric far field pattern of a domain D such that ∂D belongs to U. Then we have convergence of the cost functional* $\lim_{\alpha \to 0} m(\alpha) = 0$.

Proof The proof is analogous to that of Theorem 5.30. Instead of Theorem 5.22 we use Theorem 7.13 and instead of (5.100) we use the corresponding relation

$$\widetilde{E}_\infty - Fg = A(Hg + \nu \times \widetilde{E}^s) \tag{7.42}$$

where $A : L_t^2(\partial D) \to L_t^2(\mathbb{S}^2)$ is the bounded injective operator introduced in Theorem 6.44 that maps the electric tangential component of radiating solutions to the Maxwell equations in $\mathbb{R}^3 \setminus D$ onto the electric far field pattern. □

Theorem 7.17 *Let (α_n) be a null sequence and let (Λ_n) be a corresponding sequence of optimal surfaces for the regularization parameter α_n. Then there exists a convergent subsequence of (Λ_n). Assume that for all incident and observation directions and all polarizations E_∞ is the exact electric far field pattern of a domain D such that ∂D belongs to U. Assume further that the solution $\widetilde{E}^i, \widetilde{H}^i$ to the associated interior Maxwell problem (7.38) and (7.39) can be extended as a solution to the Maxwell equations across the boundary ∂D into the interior of Γ_e with continuous boundary values on Γ_e. Then every limit point Λ^* of (Λ_n) represents a surface on which the boundary condition (7.39) is satisfied, i.e., $\nu \times (\widetilde{E}^i + \widetilde{E}^s) = 0$ on Λ^*.*

Proof The proof is analogous to that of Theorem 5.31 with the use of Theorems 7.12 and 7.13 instead of Theorems 5.27 and 5.22 and of (7.42) instead of (5.100). □

Using the completeness result of Theorem 6.42, it is possible to design a variant of the above method for which one does not have to assume that k is not a Maxwell eigenvalue for D.

As in acoustics, the decomposition method of Colton and Monk is closely related to the linear sampling method that we are going to discuss in the next section. For numerical examples using the latter method to solve three-dimensional electromagnetic inverse scattering problems we refer the reader to [59].

7.5 Sampling Methods

Analogous to Sect. 5.6, based on the far field operator F which in the case of electromagnetic waves is defined in Theorem 6.37, i.e.,

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} E_\infty(\hat{x}, d)g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2, \tag{7.43}$$

a factorization method can be considered in terms of the ill-posed linear operator equation

$$(F^*F)^{1/4} g_z = E_{e,\infty}(\cdot, z)p. \tag{7.44}$$

Here the right-hand side $E_{e,\infty}(\cdot, z)p$ is the far field pattern of an electric dipole with source z and polarization p. However, at the time this is being written, this method has not yet been justified, for example, by proving an analogue of Corollary 5.41 although the far field operator is also compact and normal in the electromagnetic case (see Theorem 6.39). In addition a factorization of the far field operator also is available in the form

$$F = \frac{2\pi i}{k}\, AN^*A^*$$

in terms of the tangential component to far field operator A of Theorem 6.44 and the hypersingular boundary integral operator N defined in (6.48). However, for establishing an obvious analogue of Lemma 5.38 coercivity of N_i (the operator N with k replaced by i) remains open. Nevertheless, for the case of a ball the above factorization method has been justified by Collino, Fares, and Haddar [85].

In Sect. 5.6 we described the linear sampling method for solving the inverse scattering problem for a sound-soft obstacle. Our analysis was based on first presenting the factorization method for solving this inverse scattering problem and then deriving Corollary 5.43 as the final result on the linear sampling method as a consequence of the factorization method. As just pointed out, the factorization method has not been established for the case of a perfect conductor and hence we can develop the linear sampling method for electromagnetic obstacle scattering only up to the analogue of Theorem 5.35 (cf. [51, 59, 88]). Although we shall not do so here, the inverse scattering problem with limited aperture data can also be handled [49, 59].

Analogous to the scalar case, our analysis is based on an examination of the equation

$$Fg = E_{e,\infty}(\cdot, z)p, \tag{7.45}$$

where now the far field operator F is given by (7.43) and

$$E_{e,\infty}(\hat{x}, z)p = \frac{ik}{4\pi}\,(\hat{x} \times p) \times \hat{x}\, e^{-ik\,\hat{x}\cdot z}$$

is the far field pattern of an electric dipole with source z and polarization p (we could also have considered the right-hand side of (7.45) to be the far field pattern of a magnetic dipole). Equation (7.45) is known as the *far field equation*. If $z \in D$, it is seen that if g_z is a solution of the far field equation, then by Theorem 6.41 the scattered field E_g^s due to the vector Herglotz wave function ikE_g as incident field and the electric dipole $E_e(\cdot, z)p$ coincide in $\mathbb{R}^3 \setminus \bar{D}$. Hence, by the trace theorem, the tangential traces $\nu \times E_g^s = -ik\,\nu \times E_g$ and $\nu \times E_e(\cdot, z)p$ coincide on ∂D. As $z \in D$ tends to ∂D we have that

$$\|\nu \times E_e(\cdot, z)p\|_{H^{-1/2}(\mathrm{Div},\partial D)} \to \infty,$$

and hence $\|\nu \times E_g\|_{H^{-1/2}(\mathrm{Div},\partial D)} \to \infty$ also. Thus $\|g\|_{L_t^2(\mathbb{S}^2)} \to \infty$ and this behavior determines ∂D. Unfortunately, the above argument is only heuristic since it is based on the assumption that g satisfies the far field equation for $z \in D$, and in general the far field equation has no solution for $z \in D$. This follows from the fact that by Theorem 6.41 if g satisfies the far field equation, then the Herglotz pair $E = ikE_g$, $H = \operatorname{curl} E_g$ is the solution of the interior Maxwell problem

$$\operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikE = 0 \quad \text{in } D, \tag{7.46}$$

and

$$\nu \times [E + E_e(\cdot, z)p] = 0 \quad \text{on } \partial D, \tag{7.47}$$

which in general is not possible. However, using denseness properties of Herglotz pairs the following foundation of the linear sampling method can be established.

To achieve this, we first present modified versions of the denseness results of Theorems 6.44 and 7.13.

Corollary 7.18 *The operator* $A : H^{-1/2}(\mathrm{Div}, \partial D) \to L_t^2(\mathbb{S}^2)$ *which maps the electric tangential component of radiating solutions* $E, H \in H_{\mathrm{loc}}(\mathrm{curl}, \mathbb{R}^3 \setminus \bar{D})$ *to the Maxwell equations onto the electric far field pattern* E_∞ *is bounded, injective, and has dense range.*

Proof Injectivity of A is a consequence of Rellich's lemma and the trace estimate (6.52). Boundedness of A follows from the representation (6.115) via duality pairing in view of the continuous dependence of the solution to the scattering problem on the incident direction d of the plane waves. From (6.115) we also observe that the dual operator $A^\top : L^2_t(\mathbb{S}^2) \to H^{-1/2}(\text{Curl}, \partial D)$ of A is given by

$$A^\top g = \overline{A^* \overline{g}}, \quad g \in L^2_t(\mathbb{S}^2),$$

in terms of the L^2 adjoint A^*. From the proof of Theorem 6.44 we know that A^* is injective. Consequently $A^\top$ is injective and therefore A has dense range by the Hahn–Banach theorem. □

Corollary 7.19 *The Herglotz operator $H : L^2_t(\mathbb{S}^2) \to H^{-1/2}(\text{Div}, \partial D)$ defined by*

$$(Hg)(x) := ik\,\nu(x) \times \int_{\mathbb{S}^2} e^{ik\,x\cdot d} g(d)\,ds(d), \quad x \in \partial D, \tag{7.48}$$

is injective and has dense range provided k is not a Maxwell eigenvalue for D.

Proof In view of Theorem 7.13 we only need to be concerned with the denseness of $H(L^2_t(\mathbb{S}^2))$ in $H^{-1/2}(\text{Div}, \partial D)$. From (7.48), in view of the duality pairing for $H^{-1/2}(\text{Div}, \partial D)$ and its dual space $H^{-1/2}(\text{Curl}, \partial D)$, interchanging the order of integration we observe that the dual operator $H^\top : H^{-1/2}(\text{Curl}, \partial D) \to L^2_t(\mathbb{S}^2)$ of H is given by

$$H^\top a = \frac{2\pi}{ik}\,\overline{AN\overline{a}}, \quad a \in H^{-1/2}(\text{Curl}, \partial D), \tag{7.49}$$

in terms of the boundary data to far field operator $A : H^{-1/2}(\text{Div}, \partial D) \to L^2_t(\mathbb{S}^2)$ and the electric dipole operator $N : H^{-1/2}(\text{Curl}, \partial D) \to H^{-1/2}(\text{Div}, \partial D)$. Since A and N are bounded, (7.49) represents the dual operator on $H^{-1/2}(\text{Curl}, \partial D)$. Both A and N are injective, the latter because of our assumption on k. Hence $H^\top$ is injective and the dense range of H follows by the Hahn–Banach theorem. □

When k is not a Maxwell eigenvalue, well-posedness of the interior Maxwell problem (7.46)–(7.47) in $H(\text{curl}, D)$ with the tangential trace of the electric dipole replaced by an arbitrary $c \in H^{-1/2}(\text{Div}, \partial D)$ can be established by solving the integral equation (7.37) in $H^{-1/2}(\text{Div}, \partial D)$. Now Corollary 7.19 can be interpreted as denseness of Herglotz pairs in the space of solutions to the Maxwell equations in D with respect to $H(\text{curl}, D)$. For an alternate proof we refer to [102].

Lemma 7.20 $E_{e,\infty}(\cdot\,, z)p \in A\left(H^{-1/2}(\text{Div}, \partial D)\right)$ *if and only if $z \in D$.*

Proof If $z \in D$, then clearly $A(-\nu \times E_e(\cdot, z)p) = E_{e,\infty}(\cdot\,, z)p$. Now let $z \in \mathbb{R}^3 \setminus \bar{D}$ and assume that there is a tangential vector field $c \in H^{-1/2}(\text{Div}, \partial D)$ such that $Ac = E_{e,\infty}(\cdot, z)p$. Then by Theorem 6.11 the radiating field E^s corresponding to the boundary data c and the electric dipole $E_e(\cdot, z)p$ coincide in $(\mathbb{R}^3 \setminus \bar{D}) \setminus \{z\}$. But this is a contradiction since $E^s \in H_{\text{loc}}(\text{curl}, \mathbb{R}^3 \setminus \bar{D})$ but $E_e(\cdot, z)p$ is not. □

Now we are ready to establish our main result on the linear sampling method in inverse electromagnetic obstacle scattering.

Theorem 7.21 *Assume that k is not a Maxwell eigenvalue for D and let F be the far field operator (7.43) for scattering from a perfect conductor. Then the following hold:*

1. *For $z \in D$ and a given $\varepsilon > 0$ there exists a $g_z^\varepsilon \in L_t^2(\mathbb{S}^2)$ such that*

$$\|Fg_z^\varepsilon - E_{e,\infty}(\cdot, z)p\|_{L_t^2(\mathbb{S}^2)} < \varepsilon \tag{7.50}$$

   and the Herglotz wave field $E_{g_z^\varepsilon}$ with kernel g_z^ε converges to the solution of (7.46) and (7.47) in $H(\text{curl}, D)$ as $\varepsilon \to 0$
2. *For $z \notin D$ every $g_z^\varepsilon \in L_t^2(\mathbb{S}^2)$ that satisfies (7.50) for a given $\varepsilon > 0$ is such that*

$$\lim_{\varepsilon \to 0} \|H_{g_z^\varepsilon}\|_{H(\text{curl}, D)} = \infty.$$

Proof As pointed out above, under the assumption on k we have well-posedness of the interior Maxwell problem in the $H(\text{curl}, D)$ setting. Given $\varepsilon > 0$, by Corollary 7.19 we can choose $g_z \in L_t^2(\mathbb{S}^2)$ such that

$$\|Hg_z^\varepsilon + \nu \times E_e(\cdot, z)p\|_{H^{-1/2}(\text{Div}, \partial D)} < \frac{\varepsilon}{\|A\|},$$

where A denotes the boundary component to far field operator from Corollary 7.18. Then (7.50) follows from observing that

$$F = -AH.$$

Now if $z \in D$, then by the well-posedness of the interior Maxwell problem the convergence $Hg_z^\varepsilon + \nu \times E_e(\cdot, z)p \to 0$ as $\varepsilon \to 0$ in $H^{-1/2}(\text{Div}, \partial D)$ implies convergence $E_{g_z^\varepsilon} \to E$ as $\varepsilon \to 0$ in $H(\text{curl}, D)$ where E is the solution to (7.46) and (7.47). Hence, the first statement is proven.

In order to prove the second statement, for $z \notin D$ assume to the contrary that there exists a null sequence (ε_n) and corresponding Herglotz wave functions E_n with kernels $g_n = g_z^{\varepsilon_n}$ such that $\|E_n\|_{H(\text{curl}, D)}$ remains bounded. Then without loss of generality we may assume weak convergence $E_n \rightharpoonup E \in H(\text{curl}, D)$ as $n \to \infty$. Denote by $E^s, H^s \in H_{\text{loc}}(\text{curl}\, \mathbb{R}^3 \setminus \bar{D})$ the solution to the exterior Maxwell problem with $\nu \times E^s = \nu \times E$ on ∂D and denote its electric far field pattern by E_∞. Since Fg_n is the far field pattern of the scattered wave for the incident field $-E_n$ from (7.50) we conclude that $E_\infty = -E_{e,\infty}(\cdot, z)p$ and therefore $E_{e,\infty}(\cdot, z)p$ in $A(H^{-1/2}(\text{Div}, \partial D))$. But this contradicts Lemma 7.20. □

In particular we expect from the above theorem that $\|g_z^\varepsilon\|_{L_t^2(\mathbb{S}^2)}$ will be larger for $z \in \mathbb{R}^3 \setminus \bar{D}$ than it is for $z \in D$. We note that the assumption that k is not a Maxwell eigenvalue can be removed if the far field operator F is replaced by the combined far field operator (cf. Theorem 6.42)

$$(Fg)(\hat{x}) = \lambda \int_{L_t^2(\mathbb{S}^2)} E_\infty(\hat{x}, d) g(d)\, ds(d) + \mu \int_{L_t^2(\mathbb{S}^2)} H_\infty(\hat{x}, d)[g(d) \times d] ds(d), \quad \hat{x} \in \mathbb{S}^2,$$

where H_∞ is the magnetic far field pattern [49]. We also observe that in contrast to the scalar case of the linear sampling method an open question in the present case is how to obtain numerically the ε-approximate solution g_z^ε of the far field equation given by Theorem 7.21. In all numerical experiments implemented to date, Tikhonov regularization combined with the Morozov discrepancy principle is used to solve the far field equation and this procedure leads to a solution that exhibits the same behavior as g_z^ε (cf. [59]).