Chapter 3
Direct Acoustic Obstacle Scattering

This chapter is devoted to the solution of the direct obstacle scattering problem for acoustic waves. As in [104], we choose the method of integral equations for solving the boundary value problems. However, we decided to leave out some of the details in the analysis. In particular, we assume that the reader is familiar with the Riesz–Fredholm theory for operator equations of the second kind in dual systems as described in [104, 268]. We also do not repeat the technical proofs for the jump relations and regularity properties for single- and double-layer potentials. Leaving aside these two restrictions, however, we will present a rather complete analysis of the forward scattering problem. For the reader interested in a more comprehensive treatment of the direct problem, we suggest consulting our previous book [104] on this subject.

We begin by listing the jump and regularity properties of surface potentials in the classical setting of continuous and Hölder continuous functions and later present their extensions to the case of Sobolev spaces. We then proceed to establish the existence of the solution to the exterior Dirichlet problem via boundary integral equations and also describe some results on the regularity of the solution. In particular, we will establish the well-posedness of the Dirichlet-to-Neumann map in the Hölder and Sobolev space settings. We then proceed with a section on classical and generalized impedance problems to provide some material to be used later on in the book. Coming back to the far field pattern, we prove reciprocity relations that will be of importance in the study of the inverse scattering problem. We then use one of the reciprocity relations to derive some completeness results on the set of far field patterns corresponding to the scattering of incident plane waves propagating in different directions. For this we need to introduce and examine Herglotz wave functions and the far field operator which will both be of central importance later on for the inverse scattering problem.

Our presentation is in $\mathbb{R}^3$. For the sake of completeness, we include a section where we list the necessary modifications for the two-dimensional theory. We also add a section advertising a Nyström method for the numerical solution of

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the boundary integral equations in two dimensions by a spectral method based on approximations via trigonometric polynomials. Finally, we present the main ideas of a spectral method based on approximations via spherical harmonics for the numerical solution of the boundary integral equations in three dimensions that was developed and investigated by Wienert [427] and by Ganesh, Graham, and Sloan [143, 153].

3.1 Single- and Double-Layer Potentials

In this chapter, if not stated otherwise, we always will assume that the bounded set D is the open complement of an unbounded domain of class C^2, that is, we include scattering from more than one obstacle in our analysis noting that the C^2 smoothness implies that D has only a finite number of components.

We first briefly review the basic jump relations and regularity properties of acoustic single- and double-layer potentials. Given an integrable function φ, the integrals

$$u(x) := \int_{\partial D} \varphi(y)\Phi(x, y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D,$$

and

$$v(x) := \int_{\partial D} \varphi(y)\, \frac{\partial \Phi(x, y)}{\partial \nu(y)}\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D,$$

are called, respectively, *acoustic single-layer* and *acoustic double-layer potentials* with density φ. They are solutions to the Helmholtz equation in D and in $\mathbb{R}^3 \setminus \bar{D}$ and satisfy the Sommerfeld radiation condition. Green's formulas (2.5) and (2.9) show that any solution to the Helmholtz equation can be represented as a combination of single- and double-layer potentials. For continuous densities, the behavior of the surface potentials at the boundary is described by the following *jump relations*. By $\|\cdot\|_\infty = \|\cdot\|_{\infty,G}$ we denote the usual supremum norm of real or complex valued functions defined on a set $G \subset \mathbb{R}^3$.

Theorem 3.1 *Let ∂D be of class C^2 and let φ be continuous. Then the single-layer potential u with density φ is continuous throughout $\mathbb{R}^3$ and*

$$\|u\|_{\infty,\mathbb{R}^3} \le C\|\varphi\|_{\infty,\partial D}$$

for some constant C depending on ∂D. On the boundary we have

$$u(x) = \int_{\partial D} \varphi(y)\Phi(x, y)\, ds(y), \quad x \in \partial D, \tag{3.1}$$

$$\frac{\partial u_\pm}{\partial \nu}(x) = \int_{\partial D} \varphi(y) \frac{\partial \Phi(x,y)}{\partial \nu(x)} \, ds(y) \mp \frac{1}{2} \varphi(x), \quad x \in \partial D, \tag{3.2}$$

where

$$\frac{\partial u_\pm}{\partial \nu}(x) := \lim_{h \to +0} \nu(x) \cdot \operatorname{grad} u(x \pm h\nu(x))$$

is to be understood in the sense of uniform convergence on ∂D and where the integrals exist as improper integrals. The double-layer potential v with density φ can be continuously extended from D to $\bar{D}$ and from $\mathbb{R}^3 \setminus \bar{D}$ to $\mathbb{R}^3 \setminus D$ with limiting values

$$v_\pm(x) = \int_{\partial D} \varphi(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} \, ds(y) \pm \frac{1}{2} \varphi(x), \quad x \in \partial D, \tag{3.3}$$

where

$$v_\pm(x) := \lim_{h \to +0} v(x \pm h\nu(x))$$

and where the integral exists as an improper integral. Furthermore,

$$\|v\|_{\infty,\overline{D}} \le C\|\varphi\|_{\infty,\partial D}, \quad \|v\|_{\infty,\mathbb{R}^3 \setminus D} \le C\|\varphi\|_{\infty,\partial D}$$

for some constant C depending on ∂D and

$$\lim_{h \to +0} \left\{ \frac{\partial v}{\partial \nu}(x + h\nu(x)) - \frac{\partial v}{\partial \nu}(x - h\nu(x)) \right\} = 0, \quad x \in \partial D, \tag{3.4}$$

uniformly on ∂D.

Proof For a proof, we refer to Theorems 2.12, 2.13, 2.19, and 2.21 in [104]. Note that the estimates on the double-layer potential follow from Theorem 2.13 in [104] by using the maximum–minimum principle for harmonic functions in the limiting case $k = 0$ and Theorems 2.7 and 2.15 in [104]. □

An appropriate framework for formulating additional regularity properties of these surface potentials is provided by the concept of Hölder spaces. A real or complex valued function φ defined on a set $G \subset \mathbb{R}^3$ is called *uniformly Hölder continuous* with *Hölder exponent* $0 < \alpha \le 1$ if there is a constant C such that

$$|\varphi(x) - \varphi(y)| \le C|x - y|^\alpha \tag{3.5}$$

for all $x, y \in G$. We define the *Hölder space* $C^{0,\alpha}(G)$ to be the linear space of all functions defined on G which are bounded and uniformly Hölder continuous with exponent α. It is a Banach space with the norm

$$\|\varphi\|_\alpha := \|\varphi\|_{\alpha,G} := \sup_{x\in G} |\varphi(x)| + \sup_{\substack{x,y\in G\\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha}\,. \tag{3.6}$$

Clearly, for $\alpha < \beta$ each function $\varphi \in C^{0,\beta}(G)$ is also contained in $C^{0,\alpha}(G)$. For this imbedding, from the Arzelà–Ascoli theorem, we have the following compactness property (for a proof we refer to [104, p. 38] or [268, p. 105]).

Theorem 3.2 *Let* $0 < \alpha < \beta \leq 1$ *and let* G *be compact. Then the* imbedding operators

$$I^\beta : C^{0,\beta}(G) \to C(G), \quad I^{\alpha,\beta} : C^{0,\beta}(G) \to C^{0,\alpha}(G)$$

are compact.

For a vector field, Hölder continuity and the Hölder norm are defined analogously by replacing absolute values in (3.5) and (3.6) by Euclidean norms. We can then introduce the Hölder space $C^{1,\alpha}(G)$ of uniformly Hölder continuously differentiable functions as the space of differentiable functions φ for which $\operatorname{grad}\varphi$ (or the surface gradient $\operatorname{Grad}\varphi$ in the case $G = \partial D$) belongs to $C^{0,\alpha}(G)$. With the norm

$$\|\varphi\|_{1,\alpha} := \|\varphi\|_{1,\alpha,G} := \|\varphi\|_\infty + \|\operatorname{grad}\varphi\|_{0,\alpha}$$

the Hölder space $C^{1,\alpha}(G)$ is again a Banach space and we also have an imbedding theorem corresponding to Theorem 3.2.

Extending Theorem 3.1, we can now formulate the following regularity properties of single- and double-layer potentials in terms of Hölder continuity.

Theorem 3.3 *Let* ∂D *be of class* C^2 *and let* $0 < \alpha < 1$. *Then the single-layer potential* u *with density* $\varphi \in C(\partial D)$ *is uniformly Hölder continuous throughout* $\mathbb{R}^3$ *and*

$$\|u\|_{\alpha,\mathbb{R}^3} \leq C_\alpha \, \|\varphi\|_{\infty,\partial D}.$$

The first derivatives of the single-layer potential u *with density* $\varphi \in C^{0,\alpha}(\partial D)$ *can be uniformly Hölder continuously extended from* D *to* $\bar{D}$ *and from* $\mathbb{R}^3 \setminus \bar{D}$ *to* $\mathbb{R}^3 \setminus D$ *with boundary values*

$$\operatorname{grad} u_\pm(x) = \int_{\partial D} \varphi(y) \operatorname{grad}_x \Phi(x,y)\, ds(y) \mp \frac{1}{2}\,\varphi(x)\nu(x), \quad x \in \partial D, \tag{3.7}$$

where

$$\operatorname{grad} u_\pm(x) := \lim_{h\to+0} \operatorname{grad} u(x \pm h\nu(x))$$

and we have

$$\|\operatorname{grad} u\|_{\alpha,\overline{D}} \le C_\alpha \|\varphi\|_{\alpha,\partial D}, \quad \|\operatorname{grad} u\|_{\alpha,\mathbb{R}^3\setminus D} \le C_\alpha \|\varphi\|_{\alpha,\partial D}.$$

The double-layer potential v with density $\varphi \in C^{0,\alpha}(\partial D)$ can be uniformly Hölder continuously extended from D to $\bar{D}$ and from $\mathbb{R}^3 \setminus \bar{D}$ to $\mathbb{R}^3 \setminus D$ such that

$$\|v\|_{\alpha,\overline{D}} \le C_\alpha \|\varphi\|_{\alpha,\partial D}, \quad \|v\|_{\alpha,\mathbb{R}^3\setminus D} \le C_\alpha \|\varphi\|_{\alpha,\partial D}.$$

The first derivatives of the double-layer potential v with density $\varphi \in C^{1,\alpha}(\partial D)$ can be uniformly Hölder continuously extended from D to $\bar{D}$ and from $\mathbb{R}^3 \setminus \bar{D}$ to $\mathbb{R}^3 \setminus D$ such that

$$\|\operatorname{grad} v\|_{\alpha,\overline{D}} \le C_\alpha \|\varphi\|_{1,\alpha,\partial D}, \quad \|\operatorname{grad} v\|_{\alpha,\mathbb{R}^3\setminus D} \le C_\alpha \|\varphi\|_{1,\alpha,\partial D}.$$

In all inequalities, C_α denotes some constant depending on ∂D and α.

Proof For a proof, we refer to the Theorems 2.12, 2.16, 2.17, and 2.23 in [104]. □

For the direct values of the single- and double-layer potentials on the boundary ∂D, we have more regularity. This can be conveniently expressed in terms of the mapping properties of the single- and double-layer operators S and K, given by

$$(S\varphi)(x) := 2\int_{\partial D} \Phi(x, y)\varphi(y)\, ds(y), \quad x \in \partial D, \tag{3.8}$$

$$(K\varphi)(x) := 2\int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(y)}\, \varphi(y)\, ds(y), \quad x \in \partial D, \tag{3.9}$$

and the normal derivative operators K' and T, given by

$$(K'\varphi)(x) := 2\int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(x)}\, \varphi(y)\, ds(y), \quad x \in \partial D, \tag{3.10}$$

$$(T\varphi)(x) := 2\, \frac{\partial}{\partial \nu(x)} \int_{\partial D} \frac{\partial \Phi(x, y)}{\partial \nu(y)}\, \varphi(y)\, ds(y), \quad x \in \partial D. \tag{3.11}$$

Theorem 3.4 *Let ∂D be of class C^2. Then the operators S, K and K' are bounded operators from $C(\partial D)$ into $C^{0,\alpha}(\partial D)$, the operators S and K are also bounded from $C^{0,\alpha}(\partial D)$ into $C^{1,\alpha}(\partial D)$, and the operator T is bounded from $C^{1,\alpha}(\partial D)$ into $C^{0,\alpha}(\partial D)$.*

Proof The statements on S and T are contained in the preceding theorem and proofs for the operators K and K' can be found in Theorems 2.15, 2.22, and 2.30 of [104]. □

We wish to point out that all these jump and regularity properties essentially are deduced from the corresponding results for the classical single- and double-layer potentials for the Laplace equation by smoothness arguments on the difference between the fundamental solutions for the Helmholtz and the Laplace equation.

Clearly, by interchanging the order of integration, we see that S is self-adjoint and K and K' are adjoint with respect to the bilinear form

$$\langle \varphi, \psi \rangle := \int_{\partial D} \varphi \psi \, ds,$$

that is,

$$\langle S\varphi, \psi \rangle = \langle \varphi, S\psi \rangle \quad \text{and} \quad \langle K\varphi, \psi \rangle = \langle \varphi, K'\psi \rangle$$

for all $\varphi, \psi \in C(\partial D)$. To derive further properties of the boundary integral operators, let u and v denote the double-layer potentials with densities φ and ψ in $C^{1,\alpha}(\partial D)$, respectively. Then by the jump relations of Theorem 3.1, Green's theorem (2.3) and the radiation condition we find that

$$\int_{\partial D} T\varphi \, \psi \, ds = 2\int_{\partial D} \frac{\partial u}{\partial \nu} \, (v_+ - v_-) \, ds = 2\int_{\partial D} (u_+ - u_-) \, \frac{\partial v}{\partial \nu} \, ds = \int_{\partial D} \varphi T \psi \, ds,$$

that is, T also is self-adjoint. Now, in addition, let w denote the single-layer potential with density $\varphi \in C(\partial D)$. Then

$$\int_{\partial D} S\varphi \, T\psi \, ds = 4\int_{\partial D} w \, \frac{\partial v}{\partial \nu} \, ds = 4\int_{\partial D} v_- \, \frac{\partial w_-}{\partial \nu} \, ds = \int_{\partial D} (K - I)\psi (K' + I)\varphi \, ds,$$

whence

$$\int_{\partial D} \varphi \, ST\psi \, ds = \int_{\partial D} \varphi (K^2 - I)\psi \, ds$$

follows for all $\varphi \in C(\partial D)$ and $\psi \in C^{1,\alpha}(\partial D)$. Thus, we have proven the relation

$$ST = K^2 - I \tag{3.12}$$

and similarly it can be shown that the adjoint relation

$$TS = K'^2 - I \tag{3.13}$$

is also valid. Throughout the book I stands for the identity operator.

Looking at the regularity and mapping properties of surface potentials, we think it is natural to start with the classical Hölder space case. As worked out in detail by Kirsch [234], the corresponding results in the Sobolev space setting can be deduced from these classical results through the use of a functional analytic tool provided by Lax [290], that is, the classical results are stronger. Since we shall be referring to Lax's theorem several times in the sequel, we prove it here.

Theorem 3.5 *Let X and Y be normed spaces both of which are equipped with a scalar product $(\cdot\,,\cdot)$ and assume that there exists a positive constant c such that*

$$|(\varphi, \psi)| \leq c\|\varphi\|\,\|\psi\| \tag{3.14}$$

for all $\varphi, \psi \in X$. Let $U \subset X$ be a subspace and let $A : U \to Y$ and $B : Y \to X$ be bounded linear operators satisfying

$$(A\varphi, \psi) = (\varphi, B\psi) \tag{3.15}$$

for all $\varphi \in U$ and $\psi \in Y$. Then $A : U \to Y$ is bounded with respect to the norms induced by the scalar products.

Proof We denote the norms induced by the scalar products by $\|\cdot\|_s$. Consider the bounded operator $M : U \to X$ given by $M := BA$ with $\|M\| \leq \|B\|\,\|A\|$. Then, as a consequence of (3.15), M is self-adjoint, that is, $(M\varphi, \psi) = (\varphi, M\psi)$ for all $\varphi, \psi \in U$. Therefore, using the Cauchy–Schwarz inequality, we obtain

$$\|M^n\varphi\|_s^2 = (M^n\varphi, M^n\varphi) = (\varphi, M^{2n}\varphi) \leq \|M^{2n}\varphi\|_s$$

for all $\varphi \in U$ with $\|\varphi\|_s \leq 1$ and all $n \in \mathbb{N}$. From this, by induction, it follows that

$$\|M\varphi\|_s \leq \|M^{2^n}\varphi\|_s^{2^{-n}}.$$

By (3.14) we have $\|\varphi\|_s \leq \sqrt{c}\,\|\varphi\|$ for all $\varphi \in X$. Hence,

$$\|M\varphi\|_s \leq \left\{\sqrt{c}\,\|M^{2^n}\varphi\|\right\}^{2^{-n}} \leq \left\{\sqrt{c}\,\|\varphi\|\,\|M\|^{2^n}\right\}^{2^{-n}} = \left\{\sqrt{c}\,\|\varphi\|\right\}^{2^{-n}}\|M\|.$$

Passing to the limit $n \to \infty$ now yields

$$\|M\varphi\|_s \leq \|M\|$$

for all $\varphi \in U$ with $\|\varphi\|_s \leq 1$. Finally, for all $\varphi \in U$ with $\|\varphi\|_s \leq 1$, we again have from the Cauchy–Schwarz inequality that

$$\|A\varphi\|_s^2 = (A\varphi, A\varphi) = (\varphi, M\varphi) \leq \|M\varphi\|_s \leq \|M\|.$$

From this the statement follows. □

We now use Lax's Theorem 3.5 to prove the mapping properties of surface potentials in Sobolev spaces. For an introduction into the classical Sobolev spaces $H^1(D)$ and $H^1_{\text{loc}}(\mathbb{R}^3 \setminus \bar{D})$ for domains and the Sobolev spaces $H^p(\partial D)$, $p \in \mathbb{R}$, on the boundary ∂D we refer to Adams [2], Kirsch and Hettlich [244], and McLean [315]. We note that $H^1_{\text{loc}}(\mathbb{R}^3 \setminus \bar{D})$ is the space of all functions $u : \mathbb{R}^3 \setminus \bar{D} \to \mathbb{C}$ such that $u \in H^1((\mathbb{R}^3 \setminus \bar{D}) \cap B)$ for all open balls B containing $\bar{D}$. For an introduction of the spaces $H^p(\partial D)$ in two dimensions using a Fourier series approach we also refer to [268].

Theorem 3.6 *Let ∂D be of class C^2 and let $H^1(\partial D)$ denote the usual Sobolev space. Then the operator S is bounded from $L^2(\partial D)$ into $H^1(\partial D)$. Assume further that ∂D belongs to $C^{2,\alpha}$. Then the operators K and K' are bounded from $L^2(\partial D)$ into $H^1(\partial D)$ and the operator T is bounded from $H^1(\partial D)$ into $L^2(\partial D)$.*

Proof We prove the boundedness of $S : L^2(\partial D) \to H^1(\partial D)$. Let $X = C^{0,\alpha}(\partial D)$ and $Y = C^{1,\alpha}(\partial D)$ be equipped with the usual Hölder norms and introduce scalar products on X by the L^2 scalar product and on Y by the H^1 scalar product

$$(u, v)_{H^1(\partial D)} := \int_{\partial D} \left\{\varphi\bar{\psi} + \operatorname{Grad}\varphi \cdot \operatorname{Grad}\bar{\psi}\right\} ds.$$

By interchanging the order of integration, we have

$$\int_{\partial D} S\varphi\, \psi\, ds = \int_{\partial D} \varphi\, S\psi\, ds \tag{3.16}$$

for all $\varphi, \psi \in C(\partial D)$. For $\varphi \in C^{0,\alpha}(\partial D)$ and $\psi \in C^2(\partial D)$, by Gauss' surface divergence theorem and (3.16) we have

$$\int_{\partial D} \operatorname{Grad} S\varphi \cdot \operatorname{Grad}\psi\, ds = -\int_{\partial D} \varphi\, S(\operatorname{Div}\operatorname{Grad}\psi)\, ds. \tag{3.17}$$

(For the reader who is not familiar with vector analysis on surfaces, we refer to Sect. 6.3.) Using again Gauss' surface divergence theorem and the relation $\operatorname{grad}_x \Phi(x, y) = -\operatorname{grad}_y \Phi(x, y)$, we find that

$$\int_{\partial D} \Phi(x, y) \operatorname{Div}\operatorname{Grad}\psi(y)\, ds(y) = \operatorname{div}\int_{\partial D} \Phi(x, y) \operatorname{Grad}\psi(y)\, ds(y), \quad x \notin \partial D.$$

Hence, with the aid of the jump relations of Theorem 3.1 and 3.3 (see also Theorem 6.13), for $\psi \in C^2(\partial D)$ we obtain

$$S(\operatorname{Div}\operatorname{Grad}\psi) = \tilde{S}(\operatorname{Grad}\psi)$$

where the bounded operator $\tilde{S} : C^{0,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ is given by

$$(\tilde{S}a)(x) := 2\operatorname{div}\int_{\partial D} \Phi(x, y) a(y)\, ds(y), \quad x \in \partial D,$$

for Hölder continuous tangential fields a on ∂D. Therefore, from (3.17) we have

$$\int_{\partial D} \operatorname{Grad} S\varphi \cdot \operatorname{Grad}\psi\, ds = -\int_{\partial D} \varphi\, \tilde{S}(\operatorname{Grad}\psi)\, ds \tag{3.18}$$

for all $\varphi \in C^{0,\alpha}(\partial D)$ and $\psi \in C^2(\partial D)$. Since, for fixed φ, both sides of (3.18) represent bounded linear functionals on $C^{1,\alpha}(\partial D)$, (3.18) is also true for all $\varphi \in C^{0,\alpha}(\partial D)$ and $\psi \in C^{1,\alpha}(\partial D)$. Hence, from (3.16) and (3.18) we have that the operators $S : C^{0,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$ and $S^* : C^{1,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ given by

$$S^*\psi := \overline{S\bar{\psi}} - \overline{\tilde{S}\,\mathrm{Grad}\,\bar{\psi}}$$

are adjoint, i.e.,

$$(S\varphi, \psi)_{H^1(\partial D)} = (\varphi, S^*\psi)_{L^2(\partial D)}$$

for all $\varphi \in C^{0,\alpha}(\partial D)$ and $\psi \in C^{1,\alpha}(\partial D)$. By Theorem 3.3, both S and S^* are bounded with respect to the Hölder norms. Hence, from Lax's Theorem 3.5 we see that there exists a positive constant C such that

$$\|S\varphi\|_{H^1(\partial D)} \le C\|\varphi\|_{L^2(\partial D)}$$

for all $\varphi \in C^{0,\alpha}(\partial D)$. The proof of the boundedness of $S : L^2(\partial D) \to H^1(\partial D)$ is now finished by observing that $C^{0,\alpha}(\partial D)$ is dense in $L^2(\partial D)$.

The proofs of the assertions on K, K', and T are similar in structure and for details we refer the reader to [234]. □

Corollary 3.7 *If ∂D is of class C^2, then the operator S is bounded from $H^{-1/2}(\partial D)$ into $H^{1/2}(\partial D)$. Assume further that ∂D belongs to $C^{2,\alpha}$. Then the operators K and K' are bounded from $H^{-1/2}(\partial D)$ into $H^{1/2}(\partial D)$ and the operator T is bounded from $H^{1/2}(\partial D)$ into $H^{-1/2}(\partial D)$.*

Proof We prove the statement on S. The L^2 adjoint S^* of S has kernel $2\overline{\Phi(x, y)}$ and therefore also is bounded from $L^2(\partial D)$ into $H^1(\partial D)$. By duality, this implies that S is bounded from $H^{-1}(\partial D)$ into $L^2(\partial D)$. Now the boundedness of $S : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ follows by the interpolation property of the Sobolev spaces $H^{1/2}(\partial D)$ (see Theorem 8.13 in [268]). The proofs of the assertions on K, K', and T are analogous. □

In view of the compactness of the *imbedding operators* $I^{p,q}$ from $H^p(\partial D)$ into $H^q(\partial D)$ for $p > q$, from Corollary 3.7 we observe that the operators S, K, and K' are compact from $H^{-1/2}(\partial D)$ into $H^{-1/2}(\partial D)$ and from $H^{1/2}(\partial D)$ into $H^{1/2}(\partial D)$. For the following corollary we make use of the *trace theorem* which states that the restriction of a function $u \in C^2(D) \cap C(\bar{D})$ to its boundary values $u|_{\partial D}$ can be uniquely extended via a bounded linear operator $\sigma : H^1(D) \to H^{1/2}(\partial D)$ known as *trace operator*, i.e.,

$$\|\sigma u\|_{H^{1/2}(\partial D)} \le C\,\|u\|_{H^1(D)} \tag{3.19}$$

for all $u \in H^1(D)$ and some positive constant C and $\sigma u = u|_{\partial D}$ (see [315]). In the following corollary we say that a linear operator mapping $H^{-1/2}(\partial D)$ or $H^{1/2}(\partial D)$ into $H^1_{\rm loc}(\mathbb{R}^3 \setminus \bar{D})$ is bounded if it is a bounded operator into $H^1((\mathbb{R}^3 \setminus \bar{D}) \cap B)$ for all open balls B containing $\bar{D}$.

Corollary 3.8 *Let ∂D be of class $C^{2,\alpha}$. The single-layer potential defines bounded linear operators from $H^{-1/2}(\partial D)$ into $H^1(D)$ and into $H^1_{\rm loc}(\mathbb{R}^3 \setminus \bar{D})$. The double-layer potential defines bounded linear operators from $H^{1/2}(\partial D)$ into $H^1(D)$ and into $H^1_{\rm loc}(\mathbb{R}^3 \setminus \bar{D})$.*

Proof Let u be the single-layer potential with density $\varphi \in C^{0,\alpha}(\partial D)$. Then, by Green's theorem and the jump relations of Theorem 3.3, we have

$$\int_D \left\{|\operatorname{grad} u|^2 - k^2|u|^2\right\} dx = \int_{\partial D} \bar{u}\, \frac{\partial u}{\partial \nu}\, ds = \frac{1}{4} \int_{\partial D} \overline{S\varphi}\, (\varphi + K'\varphi)\, ds.$$

Therefore, by the preceding Corollary 3.7, we can estimate

$$\|\operatorname{grad} u\|^2_{L^2(D)} - k^2 \|u\|^2_{L^2(D)} \le \frac{1}{4} \|S\varphi\|_{H^{1/2}(\partial D)} \|\varphi + K'\varphi\|_{H^{-1/2}(\partial D)} \le c_1\, \|\varphi\|^2_{H^{-1/2}(\partial D)}$$

for some positive constant c_1. In terms of the volume potential operator V as introduced in Theorem 8.2, interchanging orders of integration we have

$$(u, u)_{L^2(D)} = (\varphi, \overline{V\,\bar{u}})_{L^2(\partial D)}$$

and estimating with the aid of the trace theorem and the mapping property of Theorem 8.2 for the volume potential operator V yields

$$\|u\|^2_{L^2(D)} \le C \|\varphi\|_{H^{-1/2}(\partial D)} \|\overline{V\,\bar{u}}\|_{H^1(D)} \le c_2 \|\varphi\|_{H^{-1/2}(\partial D)} \|u\|_{L^2(D)}$$

for some positive constant c_2. Now the statement on the single-layer potential for the interior domain D follows by combining the last two inequalities and using the denseness of $C^{0,\alpha}(\partial D)$ in $H^{-1/2}(\partial D)$. The proof carries over to the exterior domain $\mathbb{R}^3 \setminus \bar{D}$ by considering the product χu for some smooth cut-off function χ with compact support.

The case of the double-layer potential v with density φ is treated analogously through using

$$\int_D \left\{|\operatorname{grad} v|^2 - k^2|v|^2\right\} dx = \frac{1}{4} \int_{\partial D} T\bar{\varphi}\, (K\varphi - \varphi)\, ds,$$

which follows from Green's theorem and the jump relations, and

$$(v, v)_{L^2(D)} = \left(\varphi, \overline{\frac{\partial}{\partial \nu} V \bar{v}}\right)_{L^2(\partial D)}$$

which is obtained by interchanging orders of integration. □

In addition to the boundary trace operator for H^1 solutions to the Helmholtz equation as described by (3.19) we also need to clarify the meaning of the normal derivative in this case. For this we define $H^1_\Delta(D) := \{u \in H^1(D) : \Delta u \in L^2(D)\}$ with norm

$$\|u\|^2_{H^1_\Delta(D)} := \|u\|^2_{H^1(D)} + \|\Delta u\|^2_{L^2(D)}$$

where Δu must be interpreted as a distributional derivative. For $u \in C^2(\bar{D})$ and $w \in H^1(D)$, by Green's integral theorem we have

$$\int_{\partial D} w \, \frac{\partial u}{\partial \nu} \, ds = \int_D (w \Delta u + \operatorname{grad} w \cdot \operatorname{grad} u) \, dx.$$

In view of this, for $u \in C^2(\bar{D})$ we define the normal derivative trace τu by the duality pairing

$$\langle \tau u, \varphi \rangle_{H^{-1/2}(\partial D), H^{1/2}(\partial D)} := \int_D (w \Delta u + \operatorname{grad} w \cdot \operatorname{grad} u) \, dx \tag{3.20}$$

where $w \in H^1(D)$ and $\varphi \in H^{1/2}(\partial D)$ are such that $w = \varphi$ on ∂D in the sense of the trace operator in (3.19). Clearly, the right-hand side of (3.20) has the same value for all $w \in H^1(D)$ with boundary trace $w = \varphi$ on ∂D. The well-posedness of the weak Dirichlet problem for harmonic functions (see [199, 268]) implies that for $w \in H^1(D)$ with $\Delta w = 0$ in D and $w = \varphi$ on ∂D we have that

$$\|w\|_{H^1(D)} \le c \|\varphi\|_{H^{1/2}(\partial D)}$$

with some positive constant c independent of φ. Consequently

$$\left|\langle \tau u, \varphi \rangle_{H^{-1/2}(\partial D), H^{1/2}(\partial D)}\right| \le C \|u\|_{H^1_\Delta(D)} \|\varphi\|_{H^{1/2}(D)} \tag{3.21}$$

for all $\varphi \in H^{1/2}(\partial D)$ and some positive constant C. Thus for each $u \in C^2(\bar{D})$ by (3.20) we have defined a bounded linear functional τu on $H^{1/2}(\partial D)$ with

$$\|\tau u\|_{H^{-1/2}(\partial D)} \le C \|u\|_{H^1_\Delta(D)},$$

i.e., $\tau : C^2(\bar{D}) \to H^{-1/2}(\partial D)$ is a bounded linear operator with respect to $\|\cdot\|_{H^1_\Delta(D)}$. By denseness we can extend τ as a bounded linear operator $\tau : H^1_\Delta(D) \to H^{-1/2}(\partial D)$ and can understand τu as normal derivative $\partial_\nu u$ for $u \in H^1_\Delta(D)$. The operator τ is known as normal derivative trace operator. Note that solutions to the Helmholtz equation in D clearly belong to $H^1_\Delta(D)$. From the definition (3.20) it is obvious that Green's integral theorem remains valid for functions $u \in H^1_\Delta(D)$ and $w \in H^1(D)$. These ideas carry over to the exterior domain $\mathbb{R}^3 \setminus \bar{D}$ by considering products χu for some smooth cut-off function χ with compact support.

Finally we note that the above analysis by denseness arguments also implies that the jump relations for the boundary trace and the normal derivative trace of the single- and double-layer potential remain valid in the Sobolev space setting. For a different approach to proving Theorem 3.6 and its two corollaries we refer to McLean [315] and to Nédélec [337].

The jump relations of Theorem 3.1 can also be extended through the use of Lax's theorem from the case of continuous densities to L^2 densities. This was done by Kersten [231]. In the L^2 setting, the jump relations (3.1)–(3.4) have to be replaced by

$$\lim_{h\to+0}\int_{\partial D}|2u(x \pm h\nu(x)) - (S\varphi)(x)|^2 ds(x) = 0, \tag{3.22}$$

$$\lim_{h\to+0}\int_{\partial D}\left|2\,\frac{\partial u}{\partial\nu}\,(x \pm h\nu(x)) - (K'\varphi)(x) \pm \varphi(x)\right|^2 ds(x) = 0 \tag{3.23}$$

for the single-layer potential u with density $\varphi \in L^2(\partial D)$ and

$$\lim_{h\to+0}\int_{\partial D}|2v(x \pm h\nu(x)) - (K\varphi)(x) \mp \varphi(x)|^2 ds(x) = 0, \tag{3.24}$$

$$\lim_{h\to+0}\int_{\partial D}\left|\frac{\partial v}{\partial\nu}\,(x + h\nu(x)) - \frac{\partial v}{\partial\nu}\,(x - h\nu(x))\right|^2 ds(x) = 0 \tag{3.25}$$

for the double-layer potential v with density $\varphi \in L^2(\partial D)$. Using Lax's theorem, Hähner [171] has also established that

$$\lim_{h\to+0}\int_{\partial D}\left|\operatorname{grad} u(\cdot \pm h\nu) - \int_{\partial D}\operatorname{grad}_x \Phi(\cdot\,,y)\varphi(y)\,ds(y) \pm \frac{1}{2}\,\varphi\nu\right|^2 ds = 0 \tag{3.26}$$

for single-layer potentials u with $L^2(\partial D)$ density φ, extending the jump relation (3.7).

3.2 Scattering from a Sound-Soft Obstacle

The scattering of time-harmonic acoustic waves by sound-soft obstacles leads to the following problem.

Direct Acoustic Obstacle Scattering Problem *Given an entire solution u^i to the Helmholtz equation representing an incident field, find a solution*

$$u = u^i + u^s$$

to the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}$ such that the scattered field u^s satisfies the Sommerfeld radiation condition and the total field u satisfies the boundary condition

$$u = 0 \quad on\ \partial D.$$

Clearly, after renaming the unknown functions, this direct scattering problem is a special case of the following Dirichlet problem.

Exterior Dirichlet Problem *Given a continuous function f on ∂D, find a radiating solution $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ to the Helmholtz equation*

$$\Delta u + k^2 u = 0 \quad in\ \mathbb{R}^3 \setminus \bar{D},$$

which satisfies the boundary condition

$$u = f \quad on\ \partial D.$$

We briefly sketch uniqueness, existence, and well-posedness for this boundary value problem.

Theorem 3.9 *The exterior Dirichlet problem has at most one solution.*

Proof We have to show that solutions to the homogeneous boundary value problem $u = 0$ on ∂D vanish identically. If u had a normal derivative in the sense of uniform convergence, we could immediately apply Theorem 2.13 to obtain $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. However, in our formulation of the exterior Dirichlet problem we require u only to be continuous up to the boundary which is the natural assumption for posing the Dirichlet boundary condition. There are two possibilities to overcome this difficulty: either we can use the fact that the solution to the Dirichlet problem belongs to $C^{1,\alpha}(\mathbb{R}^3 \setminus D)$ provided the given boundary data is in $C^{1,\alpha}(\partial D)$ (cf. [104] or [303]), or we can justify the application of Green's theorem by a more direct argument using convergence theorems for Lebesgue integration. Despite the fact that later we will also need the result on the smoothness of solutions to the exterior Dirichlet problem up to the boundary, we briefly sketch a variant of the second alternative based on an approximation idea due to Heinz (see [148] and also [419] and [225, p. 144]). It is more satisfactory since it does not rely on techniques used in the existence results. Thus, we state and prove the following lemma which then justifies the application of Theorem 2.13. Note that this uniqueness result for the Dirichlet problem requires no regularity assumptions on the boundary ∂D. □

Lemma 3.10 *Let $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ be a solution to the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}$ which satisfies the homogeneous boundary condition $u = 0$ on ∂D. Define $D_R := \{y \in \mathbb{R}^3 \setminus \bar{D} : |y| < R\}$ and $S_R := \{y \in \mathbb{R}^3 : |y| = R\}$ for sufficiently large R. Then* $\operatorname{grad} u \in L^2(D_R)$ *and*

$$\int_{D_R} |\operatorname{grad} u|^2\, dx - k^2 \int_{D_R} |u|^2 dx = \int_{S_R} u\, \frac{\partial \bar{u}}{\partial \nu}\, ds. \tag{3.27}$$

Proof We first assume that u is real valued. We choose an odd function $\psi \in C^1(\mathbb{R})$ such that $\psi(t) = 0$ for $0 \le t \le 1$, $\psi(t) = t$ for $t \ge 2$ and $\psi'(t) \ge 0$ for all t, and set $u_n := \psi(nu)/n$. We then have uniform convergence $\|u - u_n\|_\infty \to 0,\ n \to \infty$. Since $u = 0$ on the boundary ∂D, the functions u_n vanish in a neighborhood of ∂D and we can apply Green's theorem (2.2) to obtain

$$\int_{D_R} \operatorname{grad} u_n \cdot \operatorname{grad} u\, dx = k^2 \int_{D_R} u_n u\, dx + \int_{S_R} u_n\, \frac{\partial u}{\partial \nu}\, ds.$$

It can be easily seen that

$$0 \le \operatorname{grad} u_n(x) \cdot \operatorname{grad} u(x) = \psi'(nu(x))\, |\operatorname{grad} u(x)|^2 \to |\operatorname{grad} u(x)|^2, \quad n \to \infty,$$

for all x not contained in $\{x \in D_R : u(x) = 0,\ \operatorname{grad} u(x) \neq 0\}$. Since as a consequence of the implicit function theorem the latter set has Lebesgue measure zero, Fatou's lemma tells us that $\operatorname{grad} u \in L^2(D_R)$.

Now assume $u = v + i\, w$ with real functions v and w. Then, since v and w also satisfy the assumptions of our lemma, we have $\operatorname{grad} v,\ \operatorname{grad} w \in L^2(D_R)$. From

$$\operatorname{grad} v_n + i\ \operatorname{grad} w_n = \psi'(nv)\ \operatorname{grad} v + i\ \psi'(nw)\ \operatorname{grad} w$$

we can estimate

$$|(\operatorname{grad} v_n + i\ \operatorname{grad} w_n) \cdot \operatorname{grad} \bar{u}| \le 2\|\psi'\|_\infty \left\{ |\operatorname{grad} v|^2 + |\operatorname{grad} w|^2 \right\}.$$

Hence, by the Lebesgue dominated convergence theorem, we can pass to the limit $n \to \infty$ in Green's theorem

$$\int_{D_R} \{(\operatorname{grad} v_n + i\ \operatorname{grad} w_n) \cdot \operatorname{grad} \bar{u} + (v_n + i w_n)\Delta \bar{u}\}\, dx = \int_{S_R} (v_n + i\, w_n)\, \frac{\partial \bar{u}}{\partial \nu}\, ds$$

to obtain (3.27). □

The existence of a solution to the exterior Dirichlet problem can be based on boundary integral equations. In the so-called *layer approach*, we seek the solution in the form of acoustic surface potentials. Here, we choose an approach in the form of a combined acoustic double- and single-layer potential

$$u(x) = \int_{\partial D} \left\{ \frac{\partial \Phi(x,y)}{\partial \nu(y)} - i\eta \Phi(x,y) \right\} \varphi(y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial D, \tag{3.28}$$

with a density $\varphi \in C(\partial D)$ and a real coupling parameter $\eta \neq 0$. Then from the jump relations of Theorem 3.1 we see that the potential u given by (3.28) in $\mathbb{R}^3 \setminus \bar{D}$ solves the exterior Dirichlet problem provided the density is a solution of the integral equation

$$\varphi + K\varphi - i\eta S\varphi = 2f. \tag{3.29}$$

Combining Theorems 3.2 and 3.4, the operators $S, K : C(\partial D) \to C(\partial D)$ are seen to be compact. Therefore, the existence of a solution to (3.29) can be established by the Riesz–Fredholm theory for equations of the second kind with a compact operator.

Let φ be a continuous solution to the homogeneous form of (3.29). Then the potential u given by (3.28) satisfies the homogeneous boundary condition $u_+ = 0$ on ∂D whence by the uniqueness for the exterior Dirichlet problem $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ follows. The jump relations (3.1)–(3.4) now yield

$$-u_- = \varphi, \quad -\frac{\partial u_-}{\partial \nu} = i\eta\varphi \quad \text{on } \partial D. \tag{3.30}$$

Hence, using Green's theorem (2.2), we obtain

$$i\eta \int_{\partial D} |\varphi|^2 ds = \int_{\partial D} \bar{u}_- \frac{\partial u_-}{\partial \nu}\, ds = \int_D \left\{ |\operatorname{grad} u|^2 - k^2 |u|^2 \right\} dx. \tag{3.31}$$

Taking the imaginary part of the last equation shows that $\varphi = 0$. Thus, we have established uniqueness for the integral equation (3.29), i.e., injectivity of the operator $I + K - i\eta S : C(\partial D) \to C(\partial D)$. Therefore, by the Riesz–Fredholm theory, $I + K - i\eta S$ is bijective and the inverse $(I + K - i\eta S)^{-1} : C(\partial D) \to C(\partial D)$ is bounded. Hence, the inhomogeneous equation (3.29) possesses a solution and this solution depends continuously on f in the maximum norm. From the representation (3.28) of the solution as a combined double- and single-layer potential, with the aid of the regularity estimates in Theorem 3.1, the continuous dependence of the density φ on the boundary data f shows that the exterior Dirichlet problem is well-posed, i.e., small deviations in f in the maximum norm ensure small deviations in u in the maximum norm on $\mathbb{R}^3 \setminus D$ and small deviations of all its derivatives in the maximum norm on closed subsets of $\mathbb{R}^3 \setminus \bar{D}$. We summarize these results in the following theorem.

Theorem 3.11 *The exterior Dirichlet problem has a unique solution and the solution depends continuously on the boundary data with respect to uniform convergence of the solution on* $\mathbb{R}^3 \setminus D$ *and all its derivatives on closed subsets of* $\mathbb{R}^3 \setminus \bar{D}$.

Note that for $\eta = 0$ the integral equation (3.29) becomes nonunique if k is a so-called irregular wave number or internal resonance, i.e., if there exist nontrivial solutions u to the Helmholtz equation in the interior domain D satisfying homogeneous Neumann boundary conditions $\partial u/\partial\nu = 0$ on ∂D. The approach (3.28) was introduced independently by Brakhage and Werner [44], Leis [296], and Panich [346] in order to remedy this nonuniqueness deficiency of the classical double-layer approach due to Vekua [414] and Weyl [426]. For an investigation on the proper choice of the coupling parameter η with respect to the condition of the integral equation (3.29), we refer to Kress [259] and Chandler-Wilde, Graham, Langdon, and Lindner [77].

In the literature, a variety of other devices has been designed for overcoming the nonuniqueness difficulties of the double-layer integral equation. The combined single- and double-layer approach seems to be the most attractive method from a theoretical point of view since its analysis is straightforward as well as from a numerical point of view since it never fails and can be implemented without additional computational cost as compared with the double-layer approach.

In order to be able to use Green's representation formula for the solution of the exterior Dirichlet problem, we need its normal derivative. However, assuming the given boundary values to be merely continuous means that in general the normal derivative will not exist. Hence, we need to impose some additional smoothness condition on the boundary data.

From Theorems 3.2 and 3.4 we also have compactness of the operators $S, K : C^{1,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$. Hence, by the Riesz–Fredholm theory, the injective operator $I + K - i\eta S : C^{1,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$ again has a bounded inverse $(I + K - i\eta S)^{-1} : C^{1,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$. Therefore, given a right-hand side f in $C^{1,\alpha}(\partial D)$, the solution φ of the integral equation (3.29) belongs to $C^{1,\alpha}(\partial D)$ and depends continuously on f in the $\|\cdot\|_{1,\alpha}$ norm. Using the regularity results of Theorem 3.3 for the derivatives of single- and double-layer potentials, from (3.28) we now find that u belongs to $C^{1,\alpha}(\mathbb{R}^3 \setminus D)$ and depends continuously on f. In particular, the normal derivative $\partial u/\partial\nu$ of the solution u exists and belongs to $C^{0,\alpha}(\partial D)$ if $f \in C^{1,\alpha}(\partial D)$ and is given by

$$\frac{\partial u}{\partial \nu} = \mathcal{A} f$$

where

$$\mathcal{A} := (i\eta I - i\eta K' + T)(I + K - i\eta S)^{-1} : C^{1,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$$

is bounded. The operator $\mathcal{A}$ transfers the boundary values, i.e., the Dirichlet data, into the normal derivative, i.e., the Neumann data, and therefore it is called the *Dirichlet-to-Neumann map*.

For the sake of completeness, we wish to show that $\mathcal{A}$ is bijective and has a bounded inverse. This is equivalent to showing that

$$i\eta I - i\eta K' + T : C^{1,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$$

is bijective and has a bounded inverse. Since T is not compact, the Riesz–Fredholm theory cannot be employed in a straightforward manner. In order to regularize the operator, we first examine the exterior Neumann problem.

Exterior Neumann Problem *Given a continuous function g on ∂D, find a radiating solution $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ to the Helmholtz equation*

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D}$$

which satisfies the boundary condition

$$\frac{\partial u}{\partial \nu} = g \quad \text{on } \partial D$$

in the sense of uniform convergence on ∂D.

The exterior Neumann problem describes acoustic scattering from sound-hard obstacles. Uniqueness for the Neumann problem follows from Theorem 2.13. To prove existence we again use a combined single- and double-layer approach. We overcome the problem that the normal derivative of the double-layer potential in general does not exist if the density is merely continuous by incorporating a smoothing operator, that is, we seek the solution in the form

$$u(x) = \int_{\partial D} \left\{ \Phi(x, y)\, \varphi(y) + i\eta\, \frac{\partial \Phi(x, y)}{\partial \nu(y)}\, (S_0^2 \varphi)(y) \right\} ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{3.32}$$

with continuous density φ and a real coupling parameter $\eta \neq 0$. By S_0 we denote the single-layer operator (3.8) in the potential theoretic limit case $k = 0$. Note that by Theorem 3.4 the density $S_0^2 \varphi$ of the double-layer potential belongs to $C^{1,\alpha}(\partial D)$. The idea of using a smoothing operator as in (3.32) was first suggested by Panich [346]. From Theorem 3.1 we see that (3.32) solves the exterior Neumann problem provided the density is a solution of the integral equation

$$\varphi - K'\varphi - i\eta T S_0^2 \varphi = -2g. \tag{3.33}$$

By Theorems 3.2 and 3.4 both $K' + i\eta T S_0^2 : C(\partial D) \to C(\partial D)$ and $K' + i\eta T S_0^2 : C^{0,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ are compact. Hence, the Riesz–Fredholm theory is available in both spaces.

Let φ be a continuous solution to the homogeneous form of (3.33). Then the potential u given by (3.32) satisfies the homogeneous Neumann boundary condition $\partial u_+ / \partial \nu = 0$ on ∂D whence by the uniqueness for the exterior Neumann problem $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ follows. The jump relations (3.1)–(3.4) now yield

$$-u_- = i\eta S_0^2 \varphi, \quad -\frac{\partial u_-}{\partial \nu} = -\varphi \quad \text{on } \partial D \tag{3.34}$$

and, by interchanging the order of integration and using Green's integral theorem as above in the proof for the Dirichlet problem, we obtain

$$i\eta \int_{\partial D} |S_0\varphi|^2 ds = i\eta \int_{\partial D} \varphi S_0^2 \bar{\varphi}\, ds = \int_{\partial D} \bar{u}_- \frac{\partial u_-}{\partial \nu}\, ds = \int_D \left\{ |\operatorname{grad} u|^2 - k^2 |u|^2 \right\} dx$$

whence $S_0\varphi = 0$ on ∂D follows. The single-layer potential w with density φ and wave number $k = 0$ is continuous throughout $\mathbb{R}^3$, harmonic in D and in $\mathbb{R}^3 \setminus \bar{D}$ and vanishes on ∂D and at infinity. Therefore, by the maximum–minimum principle for harmonic functions, we have $w = 0$ in $\mathbb{R}^3$ and the jump relation (3.2) yields $\varphi = 0$. Thus, we have established injectivity of the operator $I - K' - i\eta T S_0^2$ and, by the Riesz–Fredholm theory, $(I - K' - i\eta T S_0^2)^{-1}$ exists and is bounded in both $C(\partial D)$ and $C^{0,\alpha}(\partial D)$. From this we conclude the existence of the solution to the Neumann problem for continuous boundary data g and the continuous dependence of the solution on the boundary data.

Theorem 3.12 *The exterior Neumann problem has a unique solution and the solution depends continuously on the boundary data with respect to uniform convergence of the solution on* $\mathbb{R}^3 \setminus D$ *and all its derivatives on closed subsets of* $\mathbb{R}^3 \setminus \bar{D}$.

In the case when $g \in C^{0,\alpha}(\partial D)$, the solution φ to the integral equation (3.33) belongs to $C^{0,\alpha}(\partial D)$ and depends continuously on g in the norm of $C^{0,\alpha}(\partial D)$. Using the regularity results of Theorem 3.3 for the single- and double-layer potentials, from (3.32) we now find that u belongs to $C^{1,\alpha}(\mathbb{R}^3 \setminus D)$. In particular, the boundary values u on ∂D are given by

$$u = \mathcal{B} g,$$

where

$$\mathcal{B} = (i\eta S_0^2 + i\eta K S_0^2 + S)(K' - I + i\eta T S_0^2)^{-1} : C^{0,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$$

is bounded. Clearly, the operator $\mathcal{B}$ is the inverse of $\mathcal{A}$. Thus, we can summarize our regularity analysis as follows.

Theorem 3.13 *The Dirichlet-to-Neumann map* $\mathcal{A}$ *which transfers the boundary values of a radiating solution to the Helmholtz equation into its normal derivative is a bijective bounded operator from* $C^{1,\alpha}(\partial D)$ *onto* $C^{0,\alpha}(\partial D)$ *with bounded inverse. The solution to the exterior Dirichlet problem belongs to* $C^{1,\alpha}(\mathbb{R}^3 \setminus D)$ *if the boundary values are in* $C^{1,\alpha}(\partial D)$ *and the mapping of the boundary data into the solution is continuous from* $C^{1,\alpha}(\partial D)$ *into* $C^{1,\alpha}(\mathbb{R}^3 \setminus D)$.

Instead of looking for classical solutions in the spaces of continuous or Hölder continuous functions one can also pose and solve the boundary value problems for the Helmholtz equation in a weak formulation for the boundary condition either in an L^2 sense or in a Sobolev space setting. This then leads to existence results under weaker regularity assumptions on the given boundary data and to continuous dependence in different norms. The latter, in particular, can be useful in the error analysis for approximate solution methods.

In the Sobolev space setting, the solution to the exterior Dirichlet problem is required to belong to the energy space $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ and the boundary condition $u = f$ on ∂D for a given $f \in H^{1/2}(\partial D)$ has to be understood in the sense of the trace operator. This simplifies the uniqueness issue since the identity (3.27) is obvious for functions in $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$. The existence analysis via the combined double- and single-layer potential (3.28) with a density $\varphi \in H^{1/2}(\partial D)$ and the integral equation (3.29) can be carried over in a straightforward manner. For the exterior Neumann problem the boundary condition $\partial u/\partial \nu = g$ on ∂D for $g \in H^{-1/2}(\partial D)$ has to be understood in the sense of the normal derivative trace operator. Again the existence analysis via the combined single- and double-layer potential (3.32) with a density $\varphi \in H^{-1/2}(\partial D)$ and the integral equation (3.33) carries over. Corollary 3.7 implies well-posedness in the sense that the mapping from the boundary values $f \in H^{1/2}(\partial D)$ onto the solution $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ is continuous. Further, we note that analogous to Theorem 3.13 the Dirichlet-to-Neumann map $\mathcal{A}$ is a bijective bounded operator from $H^{1/2}(\partial D)$ onto $H^{-1/2}(\partial D)$ with a bounded inverse.

Boundary integral equations for obstacle scattering problems can also be obtained from Green's representation theorem. The basis of this so-called direct method can be formulated by the following theorem which follows immediately from Theorem 2.5 and the jump relations of Theorem 3.1.

Theorem 3.14 *Let $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C^{1,\alpha}(\mathbb{R}^3 \setminus D)$ be a radiating solution to the Helmholtz equation. Then the boundary values and the normal derivative satisfy*

$$\begin{pmatrix} u \\ \partial u/\partial \nu \end{pmatrix} = \begin{pmatrix} K & -S \\ T & -K' \end{pmatrix} \begin{pmatrix} u \\ \partial u/\partial \nu \end{pmatrix}, \tag{3.35}$$

i.e., the operator in (3.35) is a projection operator in the product space of the boundary values and the normal derivatives of radiating solutions to the Helmholtz equation. This projection operator is known as the Calderón projection.

Obviously, given the Dirichlet data f, any linear combination of the two equations in (3.35) will lead to an integral equation for the unknown Neumann data g such as for example

$$g + K'g - i\eta Sg = Tf + i\eta(f - Kf) \tag{3.36}$$

as an integral equation of the second kind for the unknown g. The operator in (3.36) is the adjoint of the operator in the equation (3.29). Therefore, by the Riesz–Fredholm theory (3.36) is uniquely solvable. We refrain from writing down further examples. However we note that (3.36), in principle, as consequence of Green's representation theorem provides only a necessary condition for the unknown Neumann data. Therefore, for a complete existence analysis based only on (3.36) one still has to show that solutions of (3.36) indeed lead to solutions of the exterior Dirichlet problem.

We note that Theorem 3.14 remains valid for H^1 solutions to the Helmholtz equation since Green's integral theorem and consequently also Green's representation formula remain valid for H^1 solutions.

A major drawback of the integral equation approach to constructively proving existence of solutions for scattering problems is the relatively strong regularity assumption on the boundary to be of class C^2. It is possible to slightly weaken the regularity and allow *Lyapunov boundaries* instead of C^2 boundaries and still remain within the framework of compact operators in the spaces of Hölder continuous functions. The boundary is said to satisfy a Lyapunov condition if at each point $x \in \partial D$ the normal vector ν to the surface exists and if there are positive constants L and α such that for the angle $\theta(x, y)$ between the normal vectors at x and y there holds $\theta(x, y) \leq L|x - y|^\alpha$ for all $x, y \in \partial D$. For the treatment of the Dirichlet problem for Lyapunov boundaries, which does not differ essentially from that for C^2 boundaries, we refer to Mikhlin [317].

However, the situation changes considerably if the boundary is allowed to have edges and corners since this affects the compactness of the double-layer integral operator in the space of continuous functions. Here, under suitable assumptions on the nature of the edges and corners, the double-layer integral operator can be decomposed into the sum of a compact operator and a bounded operator with norm less than one reflecting the behavior at the edges and corners, and then the Riesz–Fredholm theory still can be employed. For details, we refer to Sect. 3.6 for the two-dimensional case. Resorting to single-layer potentials in the Sobolev space setting as introduced above is another efficient option to handle edges and corners (see Hsiao and Wendland [199] and McLean [315]).

Explicit solutions for the direct scattering problem are only available for special geometries and special incoming fields. In general, to construct a solution one must resort to numerical methods, for example, the numerical solution of the boundary integral equations. An introduction into numerical approximation for integral equations of the second kind by the Nyström method, collocation method, and Galerkin method is contained in [268]. We will describe in some detail Nyström methods for the two- and three-dimensional case at the end of this chapter.

For future reference, we present the solution for the scattering of a plane wave

$$u^i(x) = e^{ik\,x\cdot d}$$

by a sound-soft ball of radius R with center at the origin. The unit vector d describes the direction of propagation of the incoming wave. In view of the Jacobi–Anger expansion (2.46) and the boundary condition $u^i + u^s = 0$, we expect the scattered wave to be given by

$$u^s(x) = -\sum_{n=0}^{\infty} i^n (2n+1)\,\frac{j_n(kR)}{h_n^{(1)}(kR)}\,h_n^{(1)}(k|x|)\,P_n(\cos\theta), \tag{3.37}$$

where θ denotes the angle between x and d. By the asymptotic behavior (2.38) and (2.39) of the spherical Bessel and Hankel functions for large n, we have

$$\frac{j_n(kR)}{h_n^{(1)}(kR)}\, h_n^{(1)}(k|x|) = O\left(\frac{n!\,(2kR)^n}{(2n+1)!}\,\frac{R^n}{|x|^n}\right), \quad n \to \infty,$$

uniformly on compact subsets of $\mathbb{R}^3 \setminus \{0\}$. Therefore, the series (3.37) is uniformly convergent on compact subsets of $\mathbb{R}^3 \setminus \{0\}$. Hence, by Theorem 2.15 the series represents a radiating field in $\mathbb{R}^3 \setminus \{0\}$, and therefore indeed solves the scattering problem for the sound-soft ball.

For the far field pattern, we see by Theorem 2.16 that

$$u_\infty(\hat{x}) = \frac{i}{k}\sum_{n=0}^{\infty}(2n+1)\,\frac{j_n(kR)}{h_n^{(1)}(kR)}\,P_n(\cos\theta). \tag{3.38}$$

Clearly, as we expect from symmetry reasons, it depends only on the angle θ between the observation direction $\hat{x}$ and the incident direction d.

In general, for the scattering problem the boundary values are as smooth as the boundary since they are given by the restriction of the analytic function u^i to ∂D. In particular, for domains D of class C^2 our regularity analysis shows that the scattered field u^s is in $C^{1,\alpha}(\mathbb{R}^3 \setminus D)$. Therefore, we may apply Green's formula (2.9) with the result

$$u^s(x) = \int_{\partial D}\left\{u^s(y)\,\frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u^s}{\partial \nu}(y)\,\Phi(x,y)\right\} ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}.$$

Green's theorem (2.3), applied to the entire solution u^i and $\Phi(x,\cdot)$, gives

$$0 = \int_{\partial D}\left\{u^i(y)\,\frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u^i}{\partial \nu}(y)\,\Phi(x,y)\right\} ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}.$$

Adding these two equations and using the boundary condition $u^i + u^s = 0$ on ∂D gives the following theorem. The representation for the far field pattern is obtained with the aid of (2.15).

Theorem 3.15 *For the scattering of an entire field u^i from a sound-soft obstacle D we have*

$$u(x) = u^i(x) - \int_{\partial D}\frac{\partial u}{\partial \nu}(y)\,\Phi(x,y)\,ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{3.39}$$

and the far field pattern of the scattered field u^s is given by

$$u_\infty(\hat{x}) = -\frac{1}{4\pi}\int_{\partial D}\frac{\partial u}{\partial \nu}(y)\,e^{-ik\,\hat{x}\cdot y}\,ds(y), \quad \hat{x} \in \mathbb{S}^2. \tag{3.40}$$

In physics, the representation (3.39) for the scattered field through the so-called *secondary sources* on the boundary is known as *Huygens' principle*.

We conclude this section by briefly giving the motivation for the *Kirchhoff* or *physical optics approximation* which is frequently used in applications as a physically intuitive procedure to simplify the direct scattering problem. The solution for the scattering of a plane wave with incident direction d at a plane $\Gamma := \{x \in \mathbb{R}^3 : x \cdot \nu = 0\}$ through the origin with normal vector ν is described by

$$u(x) = u^i(x) + u^s(x) = e^{ik\,x\cdot d} - e^{ik\,x\cdot\tilde{d}},$$

where $\tilde{d} = d - 2\,\nu\cdot d\,\nu$ denotes the reflection of d at the plane Γ. Clearly, $u^i + u^s = 0$ is satisfied on Γ and we evaluate

$$\frac{\partial u}{\partial \nu} = ik\{\nu \cdot d\,u^i + \nu \cdot \tilde{d}\,u^s\} = 2ik\,\nu \cdot d\,u^i = 2\,\frac{\partial u^i}{\partial \nu}\,.$$

For large wave numbers k, i.e., for small wavelengths, in a first approximation a convex object D locally may be considered at each point x of ∂D as a plane with normal $\nu(x)$. This leads to setting

$$\frac{\partial u}{\partial \nu} = 2\,\frac{\partial u^i}{\partial \nu}$$

on the region $\partial D_- := \{x \in \partial D : \nu(x) \cdot d < 0\}$ which is illuminated by the plane wave with incident direction d, and

$$\frac{\partial u}{\partial \nu} = 0$$

in the shadow region $\partial D_+ := \{x \in \partial D : \nu(x) \cdot d \geq 0\}$. Thus, in the Kirchhoff approximation for the scattering of a plane wave with incident direction d at a convex sound-soft obstacle the total wave is approximated by

$$u(x) \approx e^{ik\,x\cdot d} - 2\int_{\partial D_-} \frac{\partial e^{ik\,y\cdot d}}{\partial \nu(y)}\,\Phi(x,y)\,ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{3.41}$$

and the far field pattern of the scattered field is approximated by

$$u_\infty(\hat{x}) \approx -\frac{1}{2\pi}\int_{\partial D_-} \frac{\partial e^{ik\,y\cdot d}}{\partial \nu(y)}\,e^{-ik\,\hat{x}\cdot y}\,ds(y), \quad \hat{x} \in \mathbb{S}^2. \tag{3.42}$$

In this book, the Kirchhoff approximation does not play an important role since we are mainly interested in scattering at low and intermediate values of the wave number.

3.3 Impedance Boundary Conditions

In addition to the two standard boundary conditions for sound-soft and sound-hard obstacles, the so-called *impedance boundary conditions* were introduced to model scattering problems for penetrable obstacles approximately by scattering problems for impenetrable obstacles. The classical impedance condition for the total wave $u = u^i + u^s$, also known as the Leontovich condition, is given by

$$\frac{\partial u}{\partial \nu} + ik\lambda u = 0 \quad \text{on} \, , \partial D \tag{3.43}$$

where $\lambda \in C(\partial D)$ is a given complex valued function with nonnegative real part. On occasion we will also call the impedance boundary condition a *Robin condition*. The *generalized impedance boundary condition* is described by

$$\frac{\partial u}{\partial \nu} + ik\,(\lambda u - \operatorname{Div} \mu \operatorname{Grad} u) = 0 \quad \text{on } \partial D, \tag{3.44}$$

where $\lambda \in C(\partial D)$ and $\mu \in C^1(\partial D)$ are given complex valued functions with nonnegative real parts. Grad and Div denote the surface gradient and surface divergence on ∂D. For their definition and basic properties we refer the reader to Sect. 6.3. As compared with the Leontovich condition, the wider class of impedance conditions (3.44) can provide more accurate models, for example, for imperfectly conducting or coated obstacles (see [135, 168, 389]).

As in the previous section, after renaming the unknowns we consider the scattering problems as special cases of exterior boundary value problems and begin with the classical impedance condition. Given a function $g \in C(\partial D)$ the exterior impedance boundary value problem consists of finding a radiating solution $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ to the Helmholtz equation satisfying the boundary condition

$$\frac{\partial u}{\partial \nu} + ik\lambda u = g \quad \text{on } \partial D, \tag{3.45}$$

where, as for the exterior Neumann problem, the normal derivative is understood in the sense of uniform convergence on ∂D. The uniqueness of a solution follows from Theorem 2.13. Existence can be shown by seeking a solution in the form of the modified single- and double-layer potential (3.32) and imitating the proof of Theorem 3.12. For convenience we formulate the following theorem leaving the details of the proof to the reader.

Theorem 3.16 *The exterior impedance problem has a unique solution and the solution depends continuously on the boundary data with respect to uniform convergence of the solution on* $\mathbb{R}^3 \setminus D$ *and all its derivatives on closed subsets of* $\mathbb{R}^3 \setminus \bar{D}$.

We now consider the exterior boundary value problem with the generalized impedance boundary condition: Given a function g on ∂D find a radiating solution to the Helmholtz equation in $\mathbb{R}^3 \setminus \bar{D}$ which satisfies the boundary condition

$$\frac{\partial u}{\partial \nu} + ik\,(\lambda u - \operatorname{Div}\mu \operatorname{Grad} u) = g \quad \text{on } \partial D. \tag{3.46}$$

Deviating from the prevailing practice in this book to treat the boundary integral equations in the classical spaces of continuous and Hölder continuous functions, in order to deal with the differential operator in the boundary condition (3.46) we have chosen to work in a Sobolev space setting. As in [43], where existence of a solution is established by a variational approach, for a given $g \in L^2(\partial D)$, we seek a solution u in

$$H^{1,1}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D}) := \left\{ u \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D}) : u|_{\partial D} \in H^1(\partial D) \right\}.$$

Note that the boundary trace of functions in $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$, in general, only belongs to $H^{1/2}(\partial D)$, see the trace theorem (3.19). Then the surface gradient $\operatorname{Grad} u$ is in $L^2(\partial D)$ and the surface divergence $\operatorname{Div}\mu \operatorname{Grad} u$ has to be understood in the weak sense of Definition 6.15, i.e., the boundary condition (3.46) means that u has to satisfy

$$\int_{\partial D} \left(\psi\, \frac{\partial u}{\partial \nu} + ik\lambda\psi u + ik\mu \operatorname{Grad}\psi \cdot \operatorname{Grad} u \right) ds = \int_{\partial D} \psi g\, ds \tag{3.47}$$

for all $\psi \in H^1(\partial D)$. The normal derivative in (3.47) has to be understood in the sense of duality as defined by (3.20).

For a solution u for the homogeneous problem, inserting $\psi = \bar{u}|_{\partial D}$ in the weak form (3.47) of the boundary condition for $g = 0$ we obtain that

$$\int_{\partial D} \bar{u}\, \frac{\partial u}{\partial \nu}\, ds = -ik \int_{\partial D} \left\{ \lambda |u|^2 + \mu\, |\operatorname{Grad} u|^2 \right\} ds.$$

Hence in view of our assumption $\operatorname{Re}\lambda \geq 0$ and $\operatorname{Re}\mu \geq 0$, we can conclude that

$$\operatorname{Im} \int_{\partial D} \bar{u}\, \frac{\partial u}{\partial \nu}\, ds \leq 0$$

and from this $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ follows again by Theorem 2.13. We note that Theorem 2.13 remains valid for H^1 solutions to the Helmholtz equation since its proof is based on Green's integral theorem and Green's representation formula which also hold for H^1 solutions as pointed out earlier. Therefore the exterior boundary value problem with generalized impedance boundary condition has at most one solution.

We seek the solution u in the form of a combined double- and single-layer potential of the form (3.28) with a density $\varphi \in H^1(\partial D)$. Then, by Theorem 3.6 and Corollary 3.8 we have that $u \in H^{1,1}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$. From the jump relations (3.22) and (3.23) for L^2 densities we observe that the boundary condition (3.46) is satisfied provided φ solves the integro-differential equation

$$A\varphi = 2g \tag{3.48}$$

with the operator A given by

$$A := T + i\eta(I - K') + ik\,(\lambda I - \operatorname{Div}\mu\operatorname{Grad})\,(I + K - i\eta S). \tag{3.49}$$

For the differential operator in this equation we provide the following lemma.

Lemma 3.17 *The modified Laplace–Beltrami operator given by*

$$L\varphi := -\operatorname{Div}\operatorname{Grad}\varphi + \varphi$$

is an isomorphism from $H^1(\partial D)$ onto $H^{-1}(\partial D)$.

Proof From the definition (6.39) of the weak surface divergence we have that

$$(L\varphi, \psi) = (\operatorname{Grad}\varphi, \operatorname{Grad}\psi) + (\varphi, \psi)$$

for $\varphi, \psi \in H^1(\partial D)$ and consequently

$$\|L\varphi\|_{H^{-1}(\partial D)} = \sup_{\|\psi\|_{H^1(\partial D)}=1} |(L\varphi, \psi)| \le C_1\|\varphi\|_{H_1(\partial D)} \tag{3.50}$$

and

$$|(L\varphi, \varphi)| \ge C_2\|\varphi\|^2_{H_1(\partial D)} \tag{3.51}$$

for all $\varphi \in H^1(\partial D)$ and some positive constants C_1 and C_2. From (3.50) we have that $L : H^1(\partial D) \to H^{-1}(\partial D)$ is bounded and from (3.51) we can conclude that it is injective and has closed range. Assuming that it is not surjective implies the existence of some $\chi \neq 0$ in the dual space $\left(H^{-1}(\partial D)\right)^* = H^1(\partial D)$ that vanishes on $L\left(H^1(\partial D)\right)$, i.e., $(L\varphi, \chi) = 0$ for all $\varphi \in H^1(\partial D)$. Choosing $\varphi = \chi$ yields $(L\chi, \chi) = 0$ and from (3.51) we obtain the contradiction $\chi = 0$. Hence $L : H^1(\partial D) \to H^{-1}(\partial D)$ is bijective and by Banach's open mapping theorem it is an isomorphism. □

Lemma 3.18 *Assume that ∂D is of class $C^{3,\alpha}$. Then the operator*

$$A + ik\mu L : H^1(\partial D) \to H^{-1}(\partial D)$$

is compact.

Proof By Theorem 3.6 the operators T and K' are bounded from $H^1(\partial D)$ into $L^2(D)$ and, in extension of Theorem 3.6, it is known that S and K map $H^1(\partial D)$ into $H^2(\partial D)$ and are bounded provided ∂D is of class $C^{3,\alpha}$ (see [233, 315]). Clearly, L is bounded from $H^2(\partial D)$ into $L^2(\partial D)$. Therefore, in view of our assumptions $\mu \in C^1(\partial D)$ and $\lambda \in C(\partial D)$, all terms in the sum defining the operator A are bounded from $H^1(\partial D)$ into $L^2(\partial D)$ except the term $\varphi \mapsto ik \operatorname{Div} \mu \operatorname{Grad} \varphi$. Then, observing that in the decomposition

$$\operatorname{Div} \mu \operatorname{Grad} \varphi = \mu \operatorname{Div} \operatorname{Grad} \varphi + \operatorname{Grad} \mu \cdot \operatorname{Grad} \varphi$$

the second term is also bounded from $H^1(\partial D)$ into $L^2(\partial D)$, we can conclude that the operator $A + ik\mu L : H^1(\partial D) \to L^2(\partial D)$ is bounded. Hence the statement of the lemma follows from the compact embedding of $L^2(\partial D)$ into $H^{-1}(\partial D)$. □

Lemma 3.19 *Assume that $\mu(x) \neq 0$ for all $x \in \partial D$ and that ∂D is of class $C^{3,\alpha}$. Then for each $g \in L^2(\partial D)$ the Eq. (3.48) has a unique solution $\varphi \in H^1(\partial D)$ and this solution depends continuously on g.*

Proof Because of Lemmas 3.17 and 3.18 by the Riesz–Fredholm theory it suffices to show that the operator A is injective. Assume that $\varphi \in H^1(\partial D)$ satisfies $A\varphi = 0$. Then, the combined double- and single-layer potential (3.28) satisfies the homogeneous generalized impedance condition on ∂D and the above uniqueness result implies $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. Taking the boundary trace implies $\varphi + K\varphi - i\eta S\varphi = 0$. From the existence proof for the exterior Dirichlet problem in Theorem 3.11 we know that $I + K - i\eta S$ is injective in $C(\partial D)$. The proof for this remains valid in $H^1(\partial D)$. Hence, finally we conclude that $\varphi = 0$ and our proof is finished. □

Now we can summarize our existence analysis in the following theorem. The continuous dependence follows from Theorem 3.6 and Corollary 3.8.

Theorem 3.20 *Assume that $\mu(x) \neq 0$ for all $x \in \partial D$ and that ∂D is of class $C^{3,\alpha}$. Then for each $g \in L^2(\partial D)$ the exterior generalized impedance problem has a unique solution u and the solution depends continuously on g with respect to both the norm on $H^1(\partial D)$ and the norm on $H^1((\mathbb{R}^3 \setminus \bar{D}) \cap B)$ for all open balls B containing $\bar{D}$.*

The above impedance conditions (3.43) and (3.46) are local conditions whereas we now also will briefly discuss *nonlocal impedance conditions* of the form

$$A\,u + B\,\frac{\partial u}{\partial \nu} = g \quad \text{on } \partial D \tag{3.52}$$

for solutions u to the Helmholtz equation defined either in $\mathbb{R}^3 \setminus \bar{D}$ or in D for a given function g on ∂D. Here, one of the two operators A and B will contain integral operators, or more general pseudo differential operators defined in Sobolev spaces on ∂D. In general, these nonlocal impedance conditions have no immediate physical interpretation and only will serve us as an analytic tool for the investigation of mathematical problems related to direct and inverse obstacle scattering.

We have already met a nonlocal impedance condition in the proof of Theorem 3.12. The Cauchy data (3.34) for the modified single- and double-layer potential u correspond to the nonlocal impedance condition

$$u_- + i\eta S_0^2 \frac{\partial u_-}{\partial \nu} = 0 \quad \text{on } \partial D$$

whereas (3.30) corresponds to the classical impedance condition

$$\frac{\partial u_-}{\partial \nu} - i\eta u_- = 0 \quad \text{on } \partial D.$$

We recall from Sect. 2.1 the *transmission problem* to find the scattered field u^s as a radiating solution to the Helmholtz equation with wave number k in $\mathbb{R}^3 \setminus \bar{D}$ and the transmitted field v as a solution to the Helmholtz equation with wave number k_D in D such that the total field $u = u^i + u^s$ and v satisfy the transmission conditions

$$u = v, \quad \frac{1}{\rho}\frac{\partial u}{\partial \nu} = \frac{1}{\rho_D}\frac{\partial v}{\partial \nu} \quad \text{on } \partial D. \tag{3.53}$$

For the sake of simplicity, we only consider the case where $\rho_D = \rho$. The extension of the following analysis to the case $\rho_D \neq \rho$ is straightforward. We also want to allow absorption, i.e., complex wave numbers k_D with nonnegative real and imaginary part. For an incident field $u^i = 0$, by Green's integral theorem we find that

$$\operatorname{Im} \int_{\partial D} u^s \frac{\partial \bar{u}^s}{\partial \nu}\, ds = \operatorname{Im} \int_{\partial D} v \frac{\partial \bar{v}}{\partial \nu}\, ds = 2 \operatorname{Re} k_D \operatorname{Im} k_D \int_D |v|^2\, dx \geq 0.$$

By Theorem 2.13 it follows that $u^s = 0$ in $\mathbb{R}^3 \setminus \bar{D}$, that is, we have uniqueness for the solution.

Usually this transmission problem is reduced to a two-by-two system of boundary integral equations over the interface ∂D for a pair of unknowns, see among others [97, 272]. This can be done either by the direct method combining the Calderón projections from Theorem 3.14 and its counterpart for the domain D or by a potential approach. For a survey on methods for solving the transmission problem using only a single integral equation over ∂D we refer to [253]. As an addition to the selection of available single integral equations for the transmission problem, we will reduce the transmission problem to a scattering problem in $\mathbb{R}^3 \setminus \bar{D}$ with a nonlocal impedance boundary condition in terms of the Dirichlet-to-Neumann operator for the domain D which then can be solved via one integral equation. To prepare for this we establish the following theorem on the invertibility of the single-layer boundary integral operator.

Theorem 3.21 *Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D and that ∂D is of class C^2. Then the single-layer potential operator $S : C^{0,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$ is bijective with a bounded inverse.*

Proof Let $\varphi \in C^{0,\alpha}(\partial D)$ satisfy $S\varphi = 0$. Then the single-layer potential u with density φ has boundary values $u = 0$ on ∂D. By the uniqueness for the exterior Dirichlet problem u vanishes in $\mathbb{R}^3 \setminus \bar{D}$, and by the assumption on k it also vanishes in D. Now the jump relations for the normal derivative of single-layer potentials imply $\varphi = 0$. Hence S is injective.

To prove surjectivity, we choose a second wave number $k_0 > 0$ such that k_0^2 is not a Neumann eigenvalue for $-\Delta$ in D and distinguish the fundamental solutions and the boundary integral operators for the two different wave numbers by the indices k and k_0. Let $\psi \in C^{1,\alpha}(\partial D)$ satisfy $T_{k_0}\psi = 0$. Then the double-layer potential v with density ψ has normal derivative $\partial_\nu v = 0$ on both sides of ∂D. By the uniqueness for the exterior Neumann problem v vanishes in $\mathbb{R}^3 \setminus \bar{D}$, and by the assumption on k_0 it also vanishes in D. Therefore the jump relations imply $\psi = 0$. Hence $T_{k_0} : C^{1,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ is injective.

Then, given $f \in C^{1,\alpha}(\partial D)$ the two equations $S_k\varphi = f$ and $T_{k_0}S_k\varphi = T_{k_0}f$ for $\varphi \in C^{0,\alpha}(\partial D)$ are equivalent. In view of (3.13) we have $T_{k_0}S_k = C - I$ where

$$C := K_{k_0}'^2 + T_{k_0}(S_k - S_{k_0}).$$

From the first term in the series

$$4\pi \operatorname{Grad}_x[\Phi_k(x, y) - \Phi_{k_0}(x, y)] = \sum_{m=2}^{\infty} \frac{i^m}{m!}(k^m - k_0^m) \operatorname{Grad}_x |x - y|^{m-1}$$

we observe that $\operatorname{Grad}(S_k - S_{k_0})$ has the same mapping properties as the single-layer potential operator. This implies that $S_k - S_{k_0}$ is bounded from $C(\partial D)$ into $C^{1,\alpha}(\partial D)$ and therefore compact from $C^{0,\alpha}(\partial D)$ into $C^{1,\alpha}(\partial D)$. Then Theorem 3.4 implies that $C : C^{0,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ is compact. Therefore, the Riesz–Fredholm theory can be applied and injectivity of $T_{k_0}S_k$ implies solvability of $T_{k_0}S_k\varphi = T_{k_0}f$ and consequently also of $S_k\varphi = f$. Hence, we have bijectivity of S_k and the Banach open mapping theorem implies the boundedness of the inverse $S_k^{-1} : C^{1,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$. □

Returning to the transmission problem, we assume that k_D^2 is not a Dirichlet eigenvalue for $-\Delta$ in D. Then the Dirichlet-to-Neumann operator

$$\mathcal{A}_{k_D} : C^{1,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$$

is well defined by the mapping taking $f \in C^{1,\alpha}(\partial D)$ into the normal derivative $\mathcal{A}_{k_D} f := \partial_\nu v$ of the unique solution $v \in C^{1,\alpha}(\bar{D})$ of $\Delta v + k_D^2 v = 0$ satisfying the Dirichlet condition $v = f$ on ∂D. Using the preceding Theorem 3.21 and a single-layer approach for the interior Dirichlet problem the representation

$$\mathcal{A}_{k_D} = (I + K'_{k_D})S_{k_D}^{-1} \tag{3.54}$$

can be obtained. In particular, this implies that $\mathcal{A}_{k_D} : C^{1,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ is bounded. Then, for $\rho_D = \rho$, the transmission problem (3.53) can be seen to be equivalent to the scattering problem for $u = u^i + u^s$ in $\mathbb{R}^3 \setminus \bar{D}$ with the nonlocal impedance condition

$$\frac{\partial u^s}{\partial \nu} - \mathcal{A}_{k_D} u^s = -\frac{\partial u^i}{\partial \nu} + \mathcal{A}_{k_D} u^i \quad \text{on } \partial D. \tag{3.55}$$

Once we have determined the scattered wave u^s in $\mathbb{R}^3 \setminus \bar{D}$ from (3.55), the transmitted wave v in D can be obtained via Green's representation theorem from its Cauchy data $v = u$ and $\partial_\nu v = A_{k_D} v = A_{k_D} u = \partial_\nu u$ on ∂D.

The single-layer potential u^s with density $\varphi \in C^{0,\alpha}(\partial D)$ and wave number k satisfies the boundary condition (3.55) provided

$$-\varphi + K'_k\varphi - \mathcal{A}_{k_D} S_k\varphi = -2\,\frac{\partial u^i}{\partial \nu} + 2\mathcal{A}_{k_D} u^i. \tag{3.56}$$

From the uniqueness for the solution of the transmission problem, and consequently also for the solution of its equivalent reformulation (3.55), it follows that for a solution φ of the homogeneous form of (3.56) the corresponding potential vanishes $u^s = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. Taking the boundary trace we obtain that $S_k\varphi = 0$. If we assume that in addition to k_D^2 also k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D we have injectivity of S_k and therefore $\varphi = 0$. Hence the operator $-I + K'_k - \mathcal{A}_{k_D} S_k$ is injective.

With the aid of (3.54) we rewrite

$$-I + K'_k - \mathcal{A}_{k_D} S_k = -I + K'_k - I - K'_{k_D} - \mathcal{A}_{k_D}(S_k - S_{k_D}).$$

From the proof of the preceding Theorem 3.21 we know already that $S_k - S_{k_D} : C^{0,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$ is compact. Since the Dirichlet-to-Neumann operator is bounded from $C^{1,\alpha}(\partial D)$ into $C^{0,\alpha}(\partial D)$ in addition to $K'_k, K'_{k_D} : C^{0,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ also $\mathcal{A}_{k_D}(S_k - S_{k_D}) : C^{0,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ is compact. Thus, finally the Riesz–Fredholm theory applies to equation (3.56) and we can summarize in the following theorem.

Theorem 3.22 *Under the assumption that both k^2 and k_D^2 are not Dirichlet eigenvalues for the negative Laplacian in D the equation (3.56) is uniquely solvable.*

To avoid the restriction on k_D, instead of using the Dirichlet-to-Neumann operator, we propose using the Robin-to-Neumann operator $R_{k_D} : C^{1,\alpha}(\partial D) \to C^{0,\alpha}(\partial D)$ defined by the mapping taking $f \in C^{1,\alpha}(\partial D)$ into the normal derivative $R_{k_D} f = \partial_\nu v$ of the unique solution $v \in C^{1,\alpha}(D)$ of $\Delta v + k_D^2 v = 0$ satisfying the Robin condition

$$v + i\,\frac{\partial v}{\partial \nu} = f \quad \text{on } \partial D. \tag{3.57}$$

Uniqueness for the solution follows by inserting the homogeneous form of the boundary condition (3.57) into Green's integral theorem applied to v and $\bar{v}$ and taking the imaginary part. From the single-layer approach for the solution of (3.57) we observe that

$$R_{k_D} = (I + K'_{k_D})[S_{k_D} + i(I + K'_{k_D})]^{-1}. \tag{3.58}$$

The corresponding nonlocal impedance condition now becomes

$$\frac{\partial u^s}{\partial \nu} - R_{k_D}\left[u^s + i\,\frac{\partial u^s}{\partial \nu}\right] = -\frac{\partial u^i}{\partial \nu} + R_{k_D}\left[u^i + i\,\frac{\partial u^i}{\partial \nu}\right] \quad \text{on } \partial D. \tag{3.59}$$

For all wave numbers k and k_D we are allowing, this impedance problem can be dealt with via a uniquely solvable integral equation as derived from the modified single- and double-layer approach (3.32). We omit working out the details.

In Sect. 10.2 in our analysis of transmission eigenvalues we again will treat a transmission problem by transforming it equivalently into a boundary value problem with a nonlocal impedance condition in a Sobolev space setting.

3.4 Herglotz Wave Functions and the Far Field Operator

In the sequel, for an incident plane wave $u^i(x) = u^i(x, d) = e^{ik\,x\cdot d}$ we will indicate the dependence of the scattered field, of the total field, and of the far field pattern on the incident direction d by writing, respectively, $u^s(x, d)$, $u(x, d)$, and $u_\infty(\hat{x}, d)$.

Theorem 3.23 *The far field pattern for sound-soft obstacle scattering satisfies the* reciprocity relation

$$u_\infty(\hat{x}, d) = u_\infty(-d, -\hat{x}), \quad \hat{x}, d \in \mathbb{S}^2. \tag{3.60}$$

Proof By Green's theorem (2.3), the Helmholtz equation for the incident and the scattered wave and the radiation condition for the scattered wave we find

$$\int_{\partial D} \left\{ u^i(\cdot\,, d)\,\frac{\partial}{\partial \nu}\,u^i(\cdot\,, -\hat{x}) - u^i(\cdot\,, -\hat{x})\,\frac{\partial}{\partial \nu}\,u^i(\cdot\,, d) \right\} ds = 0$$

and

$$\int_{\partial D} \left\{ u^s(\cdot\,, d)\,\frac{\partial}{\partial \nu}\,u^s(\cdot\,, -\hat{x}) - u^s(\cdot\,, -\hat{x})\,\frac{\partial}{\partial \nu}\,u^s(\cdot\,, d) \right\} ds = 0.$$

From (2.14) we deduce that

$$4\pi u_\infty(\hat{x}, d) = \int_{\partial D} \left\{ u^s(\cdot\,, d)\, \frac{\partial}{\partial \nu}\, u^i(\cdot\,, -\hat{x}) - u^i(\cdot\,, -\hat{x})\, \frac{\partial}{\partial \nu}\, u^s(\cdot\,, d) \right\} ds$$

and, interchanging the roles of $\hat{x}$ and d,

$$4\pi u_\infty(-d, -\hat{x}) = \int_{\partial D} \left\{ u^s(\cdot\,, -\hat{x})\, \frac{\partial}{\partial \nu}\, u^i(\cdot\,, d) - u^i(\cdot\,, d)\, \frac{\partial}{\partial \nu}\, u^s(\cdot\,, -\hat{x}) \right\} ds.$$

We now subtract the last equation from the sum of the three preceding equations to obtain

$$\begin{aligned} &4\pi \{u_\infty(\hat{x}, d) - u_\infty(-d, -\hat{x})\} \\ &= \int_{\partial D} \left\{ u(\cdot\,, d)\, \frac{\partial}{\partial \nu}\, u(\cdot\,, -\hat{x}) - u(\cdot\,, -\hat{x})\, \frac{\partial}{\partial \nu}\, u(\cdot\,, d) \right\} ds \end{aligned} \tag{3.61}$$

whence (3.60) follows by using the boundary condition $u(\cdot\,, d) = u(\cdot\,, -\hat{x}) = 0$ on ∂D. □

In the derivation of (3.61), we only used the Helmholtz equation for the incident field in $\mathbb{R}^3$ and for the scattered field in $\mathbb{R}^3 \setminus \bar{D}$ and the radiation condition. Therefore, we can conclude that the reciprocity relation (3.60) is also valid for the sound-hard, Leontovich, generalized impedance, and transmission boundary conditions. It states that the far field pattern is unchanged if the direction of the incident field and the observation directions are interchanged.

For the scattering of a point source $w^i(x, z) = \Phi(x, z)$ located at $z \in \mathbb{R}^3 \setminus \bar{D}$ we denote the scattered field by $w^s(x, z)$, the total field by $w(x, z)$, and the far field pattern of the scattered wave by $w^s_\infty(\hat{x}, z)$.

Theorem 3.24 *For obstacle scattering of point sources and plane waves we have the* mixed reciprocity relation

$$4\pi\, w^s_\infty(-d, z) = u^s(z, d), \quad z \in \mathbb{R}^3 \setminus \bar{D},\ d \in \mathbb{S}^2. \tag{3.62}$$

Proof The statement follows by combining Green's theorems

$$\int_{\partial D} \left\{ w^i(\cdot\,, z)\, \frac{\partial}{\partial \nu}\, u^i(\cdot\,, d) - u^i(\cdot\,, d)\, \frac{\partial}{\partial \nu}\, w^i(\cdot\,, z) \right\} ds = 0$$

and

$$\int_{\partial D} \left\{ w^s(\cdot\,, z)\, \frac{\partial}{\partial \nu}\, u^s(\cdot\,, d) - u^s(\cdot\,, d)\, \frac{\partial}{\partial \nu}\, u^s(\cdot\,, z) \right\} ds = 0$$

and the representations

$$\int_{\partial D} \left\{ w^s(\cdot\,,z)\,\frac{\partial}{\partial\nu}\,u^i(\cdot\,,d) - u^i(\cdot\,,d)\,\frac{\partial}{\partial\nu}\,w^s(\cdot\,,z) \right\} ds = 4\pi\, w^s_\infty(-d,z)$$

and

$$\int_{\partial D} \left\{ u^s(\cdot\,,d)\,\frac{\partial}{\partial\nu}\,w^i(\cdot\,,z) - w^i(\cdot\,,z)\,\frac{\partial}{\partial\nu}\,u^s(\cdot\,,d) \right\} ds = u^s(z,d)$$

as in the proof of Theorem 3.23. □

Again the statement of Theorem 3.24 is valid for all boundary conditions. Since the far field pattern Φ_∞ of the incident field Φ is given by

$$\Phi_\infty(d,z) = \frac{1}{4\pi}\, e^{-ik\,d\cdot z}, \tag{3.63}$$

from (3.62) we conclude that

$$w_\infty(d,z) = \frac{1}{4\pi}\, u(z,-d) \tag{3.64}$$

for the far field pattern w_∞ of the total field w.

The proof of the following theorem is analogous to that of the two preceding theorems.

Theorem 3.25 *For obstacle scattering of point sources we have the* symmetry relation

$$w^s(x,y) = w^s(y,x), \quad x,y \in \mathbb{R}^3 \setminus \bar{D}. \tag{3.65}$$

We now ask the question if the far field patterns for a fixed sound-soft obstacle D and all incident plane waves are complete in $L^2(\mathbb{S}^2)$. We call a subset U of a Hilbert space X *complete* if the linear combinations of elements from U are dense in X, that is, if $X = \overline{\operatorname{span} U}$. Recall that U is complete in the Hilbert space X if and only if $(u,\varphi) = 0$ for all $u \in U$ implies that $\varphi = 0$ (see [130]).

Definition 3.26 A *Herglotz wave function* is a function of the form

$$v(x) = \int_{\mathbb{S}^2} e^{ik\,x\cdot d} g(d)\, ds(d), \quad x \in \mathbb{R}^3, \tag{3.66}$$

where $g \in L^2(\mathbb{S}^2)$. The function g is called the Herglotz kernel of v.

Herglotz wave functions are clearly entire solutions to the Helmholtz equation. We note that for a given $g \in L^2(\mathbb{S}^2)$ the function

$$v(x) = \int_{\mathbb{S}^2} e^{-ik\,x\cdot d} g(d)\,ds(d), \quad x \in \mathbb{R}^3,$$

also defines a Herglotz wave function. The following theorem establishes a one-to-one correspondence between Herglotz wave functions and their kernels.

Theorem 3.27 *Assume that the Herglotz wave function v with kernel g vanishes in all of $\mathbb{R}^3$. Then $g = 0$.*

Proof From $v(x) = 0$ for all $x \in \mathbb{R}^3$ and the Funk–Hecke formula (2.45), we see that $\int_{\mathbb{S}^2} g\, Y_n\, ds = 0$ for all spherical harmonics Y_n of order $n = 0, 1, \dots$. Now $g = 0$ follows from the completeness of the spherical harmonics (Theorem 2.8). □

Lemma 3.28 *For a given function $g \in L^2(\mathbb{S}^2)$ the solution to the scattering problem for the incident wave*

$$v^i(x) = \int_{\mathbb{S}^2} e^{ik\,x\cdot d} g(d)\,ds(d), \quad x \in \mathbb{R}^3$$

is given by

$$v^s(x) = \int_{\mathbb{S}^2} u^s(x, d) g(d)\,ds(d), \quad x \in \mathbb{R}^3 \setminus \bar{D},$$

and has the far field pattern

$$v_\infty(\hat{x}) = \int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(d)\,ds(d), \quad \hat{x} \in \mathbb{S}^2.$$

Proof Multiply (3.28) and (3.29) by g, integrate with respect to d over $\mathbb{S}^2$, and interchange orders of integration. □

Now, the rather surprising answer to our completeness question, due to Colton and Kirsch [90], will be that the far field patterns are complete in $L^2(\mathbb{S}^2)$ if and only if there does not exist a nontrivial Herglotz wave function v that vanishes on ∂D. A nontrivial Herglotz wave function that vanishes on ∂D, of course, is a Dirichlet eigenfunction, i.e., a solution to the Dirichlet problem in D with zero boundary condition, and this is peculiar since from physical considerations the eigenfunctions corresponding to the *Dirichlet eigenvalues* of the negative Laplacian in D should have nothing to do with the exterior scattering problem at all.

Theorem 3.29 *Let (d_n) be a sequence of unit vectors that is dense on $\mathbb{S}^2$ and define the set $\mathcal{F}$ of far field patterns by $\mathcal{F} := \{u_\infty(\cdot\,, d_n) : n = 1, 2, \dots\}$. Then $\mathcal{F}$ is complete in $L^2(\mathbb{S}^2)$ if and only if there does not exist a Dirichlet eigenfunction for D which is a Herglotz wave function.*

Proof Deviating from the original proof by Colton and Kirsch [90], we make use of the reciprocity relation. By the continuity of u_∞ as a function of d and

Theorem 3.23, the completeness condition

$$\int_{\mathbb{S}^2} u_\infty(\hat{x}, d_n) h(\hat{x})\, ds(\hat{x}) = 0, \quad n = 1, 2, \dots,$$

for a function $h \in L^2(\mathbb{S}^2)$ is equivalent to the condition

$$\int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(d)\, ds(d) = 0, \quad \hat{x} \in \mathbb{S}^2, \tag{3.67}$$

for $g \in L^2(\mathbb{S}^2)$ with $g(d) = h(-d)$. By Theorem 3.27 and Lemma 3.28, the existence of a nontrivial function g satisfying (3.67) is equivalent to the existence of a nontrivial Herglotz wave function v^i (with kernel g) for which the far field pattern of the corresponding scattered wave v^s is $v_\infty = 0$. By Theorem 2.14, the vanishing far field $v_\infty = 0$ on $\mathbb{S}^2$ is equivalent to $v^s = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. By the boundary condition $v^i + v^s = 0$ on ∂D and the uniqueness for the exterior Dirichlet problem, this is equivalent to $v^i = 0$ on ∂D and the proof is finished. □

Clearly, by the Funk–Hecke formula (2.45), the spherical wave functions

$$u_n(x) = j_n(k|x|)\, Y_n\left(\frac{x}{|x|}\right)$$

provide examples of Herglotz wave functions. The spherical wave functions also describe Dirichlet eigenfunctions for a ball of radius R centered at the origin with the eigenvalues k^2 given in terms of the zeros $j_n(kR) = 0$ of the spherical Bessel functions. By arguments similar to those used in the proof of Rellich's Lemma 2.12, an expansion with respect to spherical harmonics shows that all the eigenfunctions for a ball are indeed spherical wave functions. Therefore, the eigenfunctions for balls are always Herglotz wave functions and by Theorem 3.29 the far field patterns for plane waves are not complete for a ball D when k^2 is a Dirichlet eigenvalue.

The corresponding completeness results for the transmission problem were given by Kirsch [232] and for the resistive boundary condition by Hettlich [186]. For extensions to Sobolev and Hölder norms we refer to Kirsch [233].

We can also express the result of Theorem 3.29 in terms of a far field operator.

Theorem 3.30 *The far field operator $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ defined by*

$$(Fg)(\hat{x}) := \int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(d)\, ds(d), \quad \hat{x} \in \mathbb{S}^2, \tag{3.68}$$

is injective and has dense range if and only if there does not exist a Dirichlet eigenfunction for D which is a Herglotz wave function.

Proof For the L^2 adjoint $F^* : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ the reciprocity relation (3.60) implies that

$$F^*g = \overline{RFR\bar{g}}, \tag{3.69}$$

where $R : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ is defined by

$$(Rg)(d) := g(-d). \tag{3.70}$$

Hence, the operator F is injective if and only if its adjoint F^* is injective. Observing that in a Hilbert space we have $N(F^*)^\perp = \overline{F(L^2(\mathbb{S}^2))}$ for bounded operators F (see Theorem 4.6), the statement of the corollary is indeed seen to be a reformulation of the preceding theorem. □

The far field operator F will play an essential role in our investigations of the inverse scattering problem in Chap. 5. For the preparation of this analysis we proceed by presenting some of its main properties.

Lemma 3.31 *The far field operator satisfies*

$$2\pi\,\{(Fg, h) - (g, Fh)\} = ik(Fg, Fh), \quad g, h \in L^2(\mathbb{S}^2), \tag{3.71}$$

where $(\cdot\,,\cdot)$ *denotes the inner product in* $L^2(\mathbb{S}^2)$.

Proof If v^s and w^s are radiating solutions of the Helmholtz equation with far field patterns v_∞ and w_∞, then from the far field asymptotics and Green's second integral theorem we deduce that

$$\int_{\partial D} \left(v^s\, \frac{\partial \overline{w^s}}{\partial \nu} - \overline{w_s}\, \frac{\partial v^s}{\partial \nu} \right) ds = -2ik \int_{\mathbb{S}^2} v_\infty \overline{w_\infty}\, ds. \tag{3.72}$$

From the far field representation of Theorem 2.6 we see that if w_h^i is a Herglotz wave function with kernel h, then

$$\int_{\partial D} \left(v^s(x)\, \frac{\partial \overline{w_h^i}}{\partial \nu}(x) - \overline{w_h^i}(x)\, \frac{\partial v^s}{\partial \nu}(x) \right) ds(x)$$

$$= \int_{\mathbb{S}^2} \overline{h(d)} \int_{\partial D} \left(v^s(x)\, \frac{\partial e^{-ik\,x\cdot d}}{\partial \nu(x)} - e^{-ik\,x\cdot d}\, \frac{\partial v^s}{\partial \nu}(x) \right) ds(x)\, ds(d)$$

$$= 4\pi \int_{\mathbb{S}^2} \overline{h(d)}\, v_\infty(d)\, ds(d).$$

Now let v_g^i and v_h^i be the Herglotz wave functions with kernels $g, h \in L^2(\mathbb{S}^2)$, respectively, and let v_g and v_h be the solutions to the obstacle scattering problem with incident fields v_g^i and v_h^i, respectively. We denote by $v_{g,\infty}$ and $v_{h,\infty}$ the far field patterns corresponding to v_g and v_h, respectively. Then we can combine the two previous equations to obtain

$$\begin{aligned} &-2ik\,(Fg, Fh) + 4\pi(Fg, h) - 4\pi(g, Fh) \\ &= -2ik \int_{\mathbb{S}^2} v_{g,\infty}\, \overline{v_{h,\infty}}\, ds + 4\pi \int_{\mathbb{S}^2} v_{g,\infty}\, \bar{h}\, ds - 4\pi \int_{\mathbb{S}^2} g\, \overline{v_{h,\infty}}\, ds \\ &= \int_{\partial D} \left(v_g\, \frac{\partial \overline{v_h}}{\partial \nu} - \overline{v_h}\, \frac{\partial v_g}{\partial \nu} \right) ds. \end{aligned} \tag{3.73}$$

From this the statement follows in view of the boundary condition. □

Theorem 3.32 *The far field operator F is compact and normal, i.e., $FF^* = F^*F$, and has an infinite number of eigenvalues.*

Proof Since F is an integral operator with continuous kernel, it is compact. From (3.71) we obtain that

$$(g, ikF^*Fh) = 2\pi \left\{ (g, Fh) - (g, F^*h) \right\}$$

for all $g, h \in L^2(\mathbb{S}^2)$ and therefore

$$ikF^*F = 2\pi(F - F^*). \tag{3.74}$$

Using (3.69) we can deduce that $(F^*g, F^*h) = (FR\bar{h}, FR\bar{g})$ and hence, from (3.71), it follows that

$$ik(F^*g, F^*h) = 2\pi \left\{ (g, F^*h) - (F^*g, h) \right\}$$

for all $g, h \in L^2(\mathbb{S}^2)$. If we now proceed as in the derivation of (3.74), we find that

$$ikFF^* = 2\pi(F - F^*) \tag{3.75}$$

and the proof for normality of F is completed. By the spectral theorem for compact normal operators (see [375]) there exists a countable complete set of orthonormal eigenelements of F. By Theorem 3.30 the nullspace of F is finite dimensional and therefore F has an infinite number of eigenvalues. □

Corollary 3.33 *The scattering operator $S : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ defined by*

$$S := I + \frac{ik}{2\pi}\, F \tag{3.76}$$

is unitary.

Proof From (3.74) and (3.75) we see that $SS^* = S^*S = I$. □

In view of (3.76), the unitarity of S implies that the eigenvalues of F lie on the circle with center at $(0, 2\pi/k)$ on the positive imaginary axis and radius $2\pi/k$.

The question of when we can find a superposition of incident plane waves such that the resulting far field pattern coincides with a prescribed far field is answered in terms of a solvability condition for an integral equation of the first kind in the following theorem.

Theorem 3.34 *Let v^s be a radiating solution to the Helmholtz equation with far field pattern v_∞. Then the integral equation of the first kind*

$$\int_{\mathbb{S}^2} u_\infty(\hat{x}, d)g(d)\,ds(d) = v_\infty(\hat{x}), \quad \hat{x} \in \mathbb{S}^2 \tag{3.77}$$

possesses a solution $g \in L^2(\mathbb{S}^2)$ if and only if v^s is defined in $\mathbb{R}^3 \setminus \bar{D}$, is continuous in $\mathbb{R}^3 \setminus D$ and the interior Dirichlet problem for the Helmholtz equation

$$\Delta v^i + k^2 v^i = 0 \quad \text{in } D \tag{3.78}$$

and

$$v^i + v^s = 0 \quad \text{on } \partial D \tag{3.79}$$

is solvable with a solution v^i being a Herglotz wave function.

Proof By Theorem 3.27 and Lemma 3.28, the solvability of the integral equation (3.77) for g is equivalent to the existence of a Herglotz wave function v^i (with kernel g) for which the far field pattern for the scattering by the obstacle D coincides with the given v_∞, i.e., the scattered wave coincides with the given v^s. This completes the proof. □

Special cases of Theorem 3.34 include the radiating spherical wave function

$$v^s(x) = h_n^{(1)}(k|x|)Y_n\left(\frac{x}{|x|}\right)$$

of order n with far field pattern

$$v_\infty = \frac{1}{ki^{n+1}}\,Y_n.$$

Here, for solvability of (3.77) it is necessary that the origin is contained in D.

The integral equation (3.77) will play a role in our analysis of the inverse scattering problem in Sect. 5.6. By reciprocity, the solvability of (3.77) is equivalent to the solvability of

$$\int_{\mathbb{S}^2} u_\infty(\hat{x}, d)h(\hat{x})\,ds(\hat{x}) = v_\infty(-d), \quad d \in \mathbb{S}^2, \tag{3.80}$$

where $h(\hat{x}) = g(-\hat{x})$. Since the Dirichlet problem (3.78) and (3.79) is solvable provided k^2 is not a Dirichlet eigenvalue, the crucial condition in Theorem 3.34 is the property of the solution to be a Herglotz wave function, that is, a strong regularity condition. In the special case $v_\infty = 1$, the connection between the solution to the integral equation (3.80) and the interior Dirichlet problem (3.78) and (3.79) as described in Theorem 3.34 was first established by Colton and Monk [111] without, however, making use of the reciprocity Theorem 3.23.

The original proof for Theorem 3.29 by Colton and Kirsch [90] is based on the following completeness result which we include for its own interest.

Theorem 3.35 *Let (d_n) be a sequence of unit vectors that is dense on $\mathbb{S}^2$. Then the normal derivatives of the total fields*

$$\left\{ \frac{\partial}{\partial \nu}\, u(\cdot\,, d_n) : n = 1, 2, \dots \right\}$$

corresponding to incident plane waves with directions (d_n) are complete in $L^2(\partial D)$.

Proof The weakly singular operators $K - iS$ and $K' - iS$ are both compact from $C(\partial D)$ into $C(\partial D)$ and from $L^2(\partial D)$ into $L^2(\partial D)$ and they are adjoint with respect to the L^2 bilinear form, i.e.,

$$\int_{\partial D} (K - iS)\varphi\, \psi\, ds = \int_{\partial D} \varphi\, (K' - iS)\psi\, ds$$

for all $\varphi, \psi \in L^2(\partial D)$. From the proof of Theorem 3.11, we know that the operator $I + K - iS$ has a trivial nullspace in $C(\partial D)$. Therefore, by the Fredholm alternative applied in the dual system $\langle C(\partial D), L^2(\partial D)\rangle$ with the L^2 bilinear form, the adjoint operator $I + K' - iS$ has a trivial nullspace in $L^2(\partial D)$. Again by the Fredholm alternative, but now applied in the dual system $\langle L^2(\partial D), L^2(\partial D)\rangle$ with the L^2 bilinear form, the operator $I + K - iS$ also has a trivial nullspace in $L^2(\partial D)$. Hence, by the Riesz–Fredholm theory for compact operators, both the operators $I + K - iS : L^2(\partial D) \to L^2(\partial D)$ and $I + K' - iS : L^2(\partial D) \to L^2(\partial D)$ are bijective and have a bounded inverse. This idea to employ the Fredholm alternative in two different dual systems for showing that the dimensions of the nullspaces for weakly singular integral operators of the second kind in the space of continuous functions and in the L^2 space coincide is due to Hähner [172].

From the representation (3.39), the boundary condition $u = 0$ on ∂D, and the jump relations of Theorem 3.1 we deduce that

$$\frac{\partial u}{\partial \nu} + K' \frac{\partial u}{\partial \nu} - iS \frac{\partial u}{\partial \nu} = 2\, \frac{\partial u^i}{\partial \nu} - 2iu^i.$$

Now let $g \in L^2(\partial D)$ satisfy

$$\int_{\partial D} g\, \frac{\partial u(\cdot\,, d_n)}{\partial \nu}\, ds = 0, \quad n = 1, 2, \dots.$$

This, by the continuity of the Dirichlet-to-Neumann map (Theorem 3.13), implies

$$\int_{\partial D} g \, \frac{\partial u(\cdot\,, d)}{\partial \nu} \, ds = 0$$

for all $d \in \mathbb{S}^2$. Then from

$$\frac{\partial u}{\partial \nu} = 2(I + K' - iS)^{-1} \left\{ \frac{\partial u^i}{\partial \nu} - iu^i \right\}$$

we obtain

$$\int_{\partial D} g \, (I + K' - iS)^{-1} \left\{ \frac{\partial}{\partial \nu} \, u^i(\cdot\,, d) - iu^i(\cdot\,, d) \right\} ds = 0$$

for all $d \in \mathbb{S}^2$, and consequently

$$\int_{\partial D} \varphi(y) \left\{ \frac{\partial}{\partial \nu(y)} \, e^{ik\, y \cdot d} - i e^{ik\, y \cdot d} \right\} ds(y) = 0$$

for all $d \in \mathbb{S}^2$ where we have set

$$\varphi := (I + K - iS)^{-1} g.$$

Therefore, since $I + K - iS$ is bijective, our proof will be finished by showing that $\varphi = 0$. To this end, by (2.15) and (2.16), we deduce from the last equation that the combined single- and double-layer potential

$$v(x) := \int_{\partial D} \varphi(y) \left\{ \frac{\partial \Phi(x, y)}{\partial \nu(y)} - i\Phi(x, y) \right\} ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D},$$

has far field pattern

$$v_\infty(\hat{x}) = \frac{1}{4\pi} \int_{\partial D} \varphi(y) \left\{ \frac{\partial}{\partial \nu(y)} \, e^{-ik\, y \cdot \hat{x}} - i e^{-ik\, y \cdot \hat{x}} \right\} ds(y) = 0, \quad \hat{x} \in \mathbb{S}^2.$$

By Theorem 2.14, this implies $v = 0$ in $\mathbb{R}^3 \setminus \bar{D}$, and letting x tend to the boundary ∂D with the help of the L^2 jump relations (3.22) and (3.24) yields $\varphi + K\varphi - iS\varphi = 0$, whence $\varphi = 0$ follows. □

With the tools involved in the proof of Theorem 3.35, we can establish the following result which we shall also need in our analysis of the inverse problem in Chap. 5.

Theorem 3.36 *The operator $A : C(\partial D) \to L^2(\mathbb{S}^2)$ which maps the boundary values of radiating solutions $w \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ to the Helmholtz* *equation onto the far field pattern w_∞ can be extended to an injective bounded linear operator $A : L^2(\partial D) \to L^2(\mathbb{S}^2)$ with dense range.*

Proof From the solution (3.28) to the exterior Dirichlet problem, for $\hat{x} \in \mathbb{S}^2$ we derive

$$w_\infty(\hat{x}) = \frac{1}{2\pi} \int_{\partial D} \left\{ \frac{\partial}{\partial \nu(y)} e^{-ik\, y\cdot\hat{x}} - ie^{-ik\, y\cdot\hat{x}} \right\} \Big((I + K - iS)^{-1} f \Big)(y)\, ds(y)$$

with the boundary values $w = f$ on ∂D. From this, given the boundedness of the operator $(I + K - iS)^{-1} : L^2(\partial D) \to L^2(\partial D)$ from the proof of Theorem 3.35, it is obvious that A is bounded from $L^2(\partial D) \to L^2(\mathbb{S}^2)$. The injectivity of A is also immediate from the proof of Theorem 3.35.

In order to show that A has dense range we rewrite it as an integral operator. To this end we note that in terms of the plane waves $u^i(x, d) = e^{ik\, x\cdot d}$ the far field representation (2.14) for a radiating solution w of the Helmholtz equation can be written in the form

$$w_\infty(\hat{x}) = \frac{1}{4\pi} \int_{\partial D} \left\{ \frac{\partial u^i(y, -\hat{x})}{\partial \nu(y)}\, w(y) - u^i(y, -\hat{x})\, \frac{\partial w}{\partial \nu}(y) \right\} ds(y), \quad \hat{x} \in \mathbb{S}^2.$$

(See also the proof of Theorem 3.24.) From this, with the aid of Green's integral theorem and the radiation condition, using the sound-soft boundary condition for the total wave $u = u^i + u^s$ on ∂D we conclude that

$$w_\infty(\hat{x}) = \frac{1}{4\pi} \int_{\partial D} \frac{\partial u(y, -\hat{x})}{\partial \nu(y)}\, w(y)\, ds(y), \quad \hat{x} \in \mathbb{S}^2,$$

that is,

$$(Af)(d) = \frac{1}{4\pi} \int_{\partial D} \frac{\partial u(y, -d)}{\partial \nu(y)}\, f(y)\, ds(y), \quad d \in \mathbb{S}^2. \tag{3.81}$$

Consequently the adjoint operator $A^* : L^2(\mathbb{S}^2) \to L^2(\partial D)$ can be expressed as the integral operator

$$(A^* g)(x) = \frac{1}{4\pi} \int_{\mathbb{S}^2} \frac{\partial \overline{u(x, -d)}}{\partial \nu(x)}\, g(d)\, ds(d), \quad x \in \partial D. \tag{3.82}$$

If for $g \in L^2(\mathbb{S}^2)$ we define the Herglotz wave function v_g^i in the form

$$v_g^i(x) = \int_{\mathbb{S}^2} e^{-ik\, x\cdot d}\, \overline{g(d)}\, ds(d) = \int_{\mathbb{S}^2} u^i(x, -d) \overline{g(d)}\, ds(d), \quad x \in \mathbb{R}^3,$$

then from Lemma 3.28 we have that

$$v_g(x) = \int_{\mathbb{S}^2} u(x, -d)\overline{g(d)}\, ds(d), \quad x \in \mathbb{R}^3,$$

is the total wave for scattering of v_g^i from D. Hence,

$$\overline{A^* g} = \frac{1}{4\pi} \frac{\partial v_g}{\partial \nu} = \frac{1}{4\pi} \left\{ \frac{\partial v_g^i}{\partial \nu} - \mathcal{A} v_g^i|_{\partial D} \right\} \tag{3.83}$$

with the Dirichlet-to-Neumann operator $\mathcal{A}$. Now let g satisfy $A^* g = 0$. Then (3.82) implies that $\partial v_g/\partial \nu = 0$ on ∂D. By definition we also have $v_g = 0$ on ∂D and therefore, by Holmgren's Theorem 2.3, it follows that $v_g = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. Thus the entire solution v_g^i satisfies the radiation condition and therefore must vanish identically. Thus $g = 0$, i.e., A^* is injective. Hence A has dense range by Theorem 4.6. □

Theorem 3.37 *For the far field operator F we have the factorization*

$$F = -2\pi A S^* A^*. \tag{3.84}$$

Proof For convenience we introduce the Herglotz operator $H : L^2(\mathbb{S}^2) \to L^2(\partial D)$ by

$$(Hg)(x) := \int_{\mathbb{S}^2} e^{ik\, x\cdot d} g(d)\, ds(d), \quad x \in \partial D. \tag{3.85}$$

Since Fg represents the far field pattern of the scattered wave corresponding to Hg as incident field, we clearly have

$$F = -AH. \tag{3.86}$$

The L^2 adjoint $H^* : L^2(\partial D) \to L^2(\mathbb{S}^2)$ is given by

$$(H^*\varphi)(\hat{x}) = \int_{\partial D} e^{-ik\, \hat{x}\cdot y} \varphi(y)\, ds(y), \quad \hat{x} \in \mathbb{S}^2,$$

and represents the far field pattern of the single-layer potential with density $4\pi\varphi$. Therefore

$$H^* = 2\pi A S \tag{3.87}$$

and consequently

$$H = 2\pi S^* A^*. \tag{3.88}$$

Now the statement follows by combining (3.86) and (3.88). □

We now wish to study Herglotz wave functions more closely. The concept of the growth condition in the following theorem for solutions to the Helmholtz equation was introduced by Herglotz in a lecture in 1945 in Göttingen and was studied further by Magnus [309] and Müller [329]. The equivalence stated in the theorem was shown by Hartman and Wilcox [184].

Theorem 3.38 *An entire solution v to the Helmholtz equation possesses the growth property*

$$\sup_{R>0} \frac{1}{R} \int_{|x|\le R} |v(x)|^2 dx < \infty \tag{3.89}$$

if and only if it is a Herglotz wave function, i.e., if and only if there exists a function $g \in L^2(\mathbb{S}^2)$ such that v can be represented in the form (3.66).

Proof Before we can prove this result, we need to note two properties for integrals containing spherical Bessel functions. From the asymptotic behavior (2.42), that is, from

$$j_n(t) = \frac{1}{t} \cos\left(t - \frac{n\pi}{2} - \frac{\pi}{2}\right) \left\{1 + O\left(\frac{1}{t}\right)\right\}, \quad t \to \infty,$$

we readily find that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T t^2 [j_n(t)]^2 dt = \frac{1}{2}, \quad n = 0, 1, 2, \dots. \tag{3.90}$$

We now want to establish that the integrals in (3.90) are uniformly bounded with respect to T and n. This does not follow immediately since the asymptotic behavior for the spherical Bessel functions is not uniformly valid with respect to the order n. If we multiply the differential formula (2.35) rewritten in the form

$$j_{n+1}(t) = -\frac{1}{\sqrt{t}} \frac{d}{dt} \sqrt{t}\, j_n(t) + \left(n + \frac{1}{2}\right) \frac{1}{t}\, j_n(t)$$

by two and subtract it from the recurrence relation (2.34), that is, from

$$j_{n-1}(t) + j_{n+1}(t) = \frac{2n+1}{t}\, j_n(t),$$

we obtain

$$j_{n-1}(t) - j_{n+1}(t) = \frac{2}{\sqrt{t}} \frac{d}{dt} \sqrt{t}\, j_n(t).$$

Hence, from the last two equations we get

$$\int_0^T t^2 \left\{[j_{n-1}(t)]^2 - [j_{n+1}(t)]^2\right\} dt = (2n+1)T\,[j_n(T)]^2$$

for $n = 1, 2, \dots$ and all $T > 0$. From this monotonicity, together with (3.90) for $n = 0$ and $n = 1$, it is now obvious that

$$\sup_{\substack{T>0 \\ n=0,1,2,\dots}} \frac{1}{T} \int_0^T t^2 [j_n(t)]^2 dt < \infty. \tag{3.91}$$

For the proof of the theorem, we first observe that any entire solution v of the Helmholtz equation can be expanded in a series

$$v(x) = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n\, a_n^m\, j_n(k|x|)\, Y_n^m\left(\frac{x}{|x|}\right) \tag{3.92}$$

and the series converges uniformly on compact subsets of $\mathbb{R}^3$. This follows from Green's representation formula (2.5) for v in a ball with radius R and center at the origin and inserting the addition theorem (2.43) with the roles of x and y interchanged, that is,

$$\Phi(x, y) = ik \sum_{n=0}^{\infty} \sum_{m=-n}^{n} j_n(k|x|)\, Y_n^m\left(\frac{x}{|x|}\right) h_n^{(1)}(k|y|)\, \overline{Y_n^m\left(\frac{y}{|y|}\right)}, \quad |x| < |y|.$$

Since the expansion derived for two different radii represents the same function in the ball with the smaller radius, the coefficients a_n^m do not depend on the radius R. Because of the uniform convergence, we can integrate term by term and use the orthonormality of the Y_n^m to find that

$$\frac{1}{R} \int_{|x| \le R} |v(x)|^2 dx = \frac{16\pi^2}{R} \sum_{n=0}^{\infty} \int_0^R r^2 [j_n(kr)]^2 dr \sum_{m=-n}^{n} \left|a_n^m\right|^2. \tag{3.93}$$

Now assume that v satisfies

$$\frac{1}{R} \int_{|x| \le R} |v(x)|^2 dx \le C$$

for all $R > 0$ and some constant $C > 0$. This, by (3.93), implies that

$$\frac{16\pi^2}{R} \sum_{n=0}^{N} \int_0^R r^2 [j_n(kr)]^2 dr \sum_{m=-n}^{n} \left|a_n^m\right|^2 \le C$$

for all $R > 0$ and all $N \in \mathbb{N}$. Hence, by first passing to the limit $R \to \infty$ with the aid of (3.90) and then letting $N \to \infty$ we obtain

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left|a_n^m\right|^2 \leq \frac{k^2 C}{8\pi^2}\,.$$

Therefore,

$$g := \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m \, Y_n^m$$

defines a function $g \in L^2(\mathbb{S}^2)$. From the Jacobi–Anger expansion (2.46) and the addition theorem (2.30), that is, from

$$e^{ik\,x\cdot d} = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n \, j_n(k|x|) \, Y_n^m\left(\frac{x}{|x|}\right) \overline{Y_n^m(d)}$$

we now derive

$$\int_{\mathbb{S}^2} g(d) e^{ik\,x\cdot d} ds(d) = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n \, a_n^m \, j_n(k|x|) \, Y_n^m\left(\frac{x}{|x|}\right) = v(x)$$

for all $x \in \mathbb{R}^3$, that is, we have shown that v can be represented in the form (3.66).

Conversely, for a given $g \in L^2(\mathbb{S}^2)$ we have an expansion

$$g = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m \, Y_n^m,$$

where, by Parseval's equality, the coefficients satisfy

$$\|g\|^2_{L^2(\mathbb{S}^2)} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left|a_n^m\right|^2 < \infty. \tag{3.94}$$

Then for the entire solution v to the Helmholtz equation defined by

$$v(x) := \int_{\mathbb{S}^2} e^{ik\,x\cdot d} g(d)\, ds(d), \quad x \in \mathbb{R}^3,$$

we again see by the Jacobi–Anger expansion that

$$v(x) = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^n \, a_n^m \, j_n(k|x|) \, Y_n^m\left(\frac{x}{|x|}\right) \tag{3.95}$$

and from (3.91), (3.93), and (3.95) we conclude that the growth condition (3.89) is fulfilled for v. The proof is now complete. □

With the help of (3.91), we observe that the series (3.93) has a convergent majorant independent of R. Hence, it is uniformly convergent for all $R > 0$ and we may interchange the limit $R \to \infty$ with the series and use (3.90) and (3.94) to obtain that for the Herglotz wave function v with kernel g we have

$$\lim_{R\to\infty} \frac{1}{R} \int_{|x|\le R} |v(x)|^2 dx = \frac{8\pi^2}{k^2} \, \|g\|^2_{L^2(\mathbb{S}^2)}.$$

3.5 The Two-Dimensional Case

The scattering from infinitely long cylindrical obstacles leads to exterior boundary value problems for the Helmholtz equation in $\mathbb{R}^2$. The two-dimensional case can be used as an approximation for the scattering from finitely long cylinders, and more important, it can serve as a model case for testing numerical approximation schemes in direct and inverse scattering. Without giving much of the details, we would like to show how all the results of this chapter remain valid in two dimensions after appropriate modifications of the fundamental solution, the radiation condition, and the spherical wave functions.

We note that in two dimensions there exist two linearly independent spherical harmonics of order n which can be represented by $e^{\pm in\varphi}$. Correspondingly, looking for solutions to the Helmholtz equation of the form

$$u(x) = f(kr)\, e^{\pm in\varphi}$$

in polar coordinates (r, φ) leads to the *Bessel differential equation*

$$t^2 f''(t) + tf'(t) + [t^2 - n^2]f(t) = 0 \tag{3.96}$$

with integer order $n = 0, 1, \dots$. The analysis of the Bessel equation which is required for the study of the two-dimensional Helmholtz equation, in particular the asymptotics of the solutions for large argument, is more involved than the corresponding analysis for the spherical Bessel equation (2.31). Therefore, here we will list only the relevant results without proofs. For a concise treatment of the Bessel equation for the purpose of scattering theory, we refer to Colton [86] or Lebedev [293].

By direct calculations and the ratio test, we can easily verify that for $n = 0, 1, 2, \dots$ the functions

$$J_n(t) := \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\,(n+p)!} \left(\frac{t}{2}\right)^{n+2p} \tag{3.97}$$

represent solutions to Bessel's equation which are analytic for all $t \in \mathbb{R}$ and these are known as *Bessel functions* of order n. As opposed to the spherical Bessel equation, here it is more complicated to construct a second linearly independent solution. Patient, but still straightforward, calculations together with the ratio test show that

$$\begin{aligned} Y_n(t) := \frac{2}{\pi} \left\{ \ln \frac{t}{2} + C \right\} J_n(t) - \frac{1}{\pi} \sum_{p=0}^{n-1} \frac{(n-1-p)!}{p!} \left(\frac{2}{t} \right)^{n-2p} \\ - \frac{1}{\pi} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\,(n+p)!} \left(\frac{t}{2} \right)^{n+2p} \{\psi(p+n) + \psi(p)\} \end{aligned} \tag{3.98}$$

for $n = 0, 1, 2, \dots$ provide solutions to Bessel's equation which are analytic for all $t \in (0, \infty)$. Here, we define $\psi(0) := 0$,

$$\psi(p) := \sum_{m=1}^{p} \frac{1}{m}, \quad p = 1, 2, \dots,$$

let

$$C := \lim_{p \to \infty} \left\{ \sum_{m=1}^{p} \frac{1}{m} - \ln p \right\}$$

denote Euler's constant, and if $n = 0$ the finite sum in (3.98) is set equal to zero. The functions Y_n are called *Neumann functions* of order n and the linear combinations

$$H_n^{(1,2)} := J_n \pm i Y_n$$

are called *Hankel functions* of the first and second kind of order n respectively.

From the series representation (3.97) and (3.98), by equating powers of t, it is readily verified that both $f_n = J_n$ and $f_n = Y_n$ satisfy the recurrence relation

$$f_{n+1}(t) + f_{n-1}(t) = \frac{2n}{t} f_n(t), \quad n = 1, 2, \dots. \tag{3.99}$$

Straightforward differentiation of the series (3.97) and (3.98) shows that both $f_n = J_n$ and $f_n = Y_n$ satisfy the differentiation formulas

$$f_{n+1}(t) = -t^n \frac{d}{dt} \left\{ t^{-n} f_n(t) \right\}, \quad n = 0, 1, 2, \dots, \tag{3.100}$$

and

$$t^n f_{n-1}(t) = \frac{d}{dt} \left\{t^n f_n(t)\right\}, \quad n = 1, 2, \dots . \tag{3.101}$$

The Wronskian

$$W(J_n(t), Y_n(t)) := J_n(t)Y_n'(t) - Y_n(t)J_n'(t)$$

satisfies

$$W' + \frac{1}{t} W = 0.$$

Therefore, $W(J_n(t), Y_n(t)) = C/t$ for some constant C and by passing to the limit $t \to 0$ it follows that

$$J_n(t)Y_n'(t) - J_n'(t)Y_n(t) = \frac{2}{\pi t} . \tag{3.102}$$

From the series representation of the Bessel and Neumann functions, it is obvious that

$$J_n(t) = \frac{t^n}{2^n\, n!}\left(1 + O\left(\frac{1}{n}\right)\right), \quad n \to \infty, \tag{3.103}$$

uniformly on compact subsets of $\mathbb{R}$ and

$$H_n^{(1)}(t) = \frac{2^n(n-1)!}{\pi i t^n}\left(1 + O\left(\frac{1}{n}\right)\right), \quad n \to \infty, \tag{3.104}$$

uniformly on compact subsets of $(0, \infty)$.

For large arguments, we have the following asymptotic behavior of the Hankel functions

$$\begin{aligned} H_n^{(1,2)}(t) &= \sqrt{\frac{2}{\pi t}}\, e^{\pm i\left(t - \frac{n\pi}{2} - \frac{\pi}{4}\right)} \left\{1 + O\left(\frac{1}{t}\right)\right\}, \quad t \to \infty, \\ H_n^{(1,2)\prime}(t) &= \sqrt{\frac{2}{\pi t}}\, e^{\pm i\left(t - \frac{n\pi}{2} + \frac{\pi}{4}\right)} \left\{1 + O\left(\frac{1}{t}\right)\right\}, \quad t \to \infty. \end{aligned} \tag{3.105}$$

For a proof, we refer to Lebedev [293]. Taking the real and the imaginary part of (3.105) we also have asymptotic formulas for the Bessel and Neumann functions.

Now we have listed all the necessary tools for carrying over the analysis of Chaps. 2 and 3 for the Helmholtz equation from three to two dimensions. The fundamental solution to the Helmholtz equation in two dimensions is given by

$$\Phi(x, y) := \frac{i}{4}\, H_0^{(1)}(k|x-y|), \quad x \neq y. \tag{3.106}$$

For fixed $y \in \mathbb{R}^2$, it satisfies the Helmholtz equation in $\mathbb{R}^2 \setminus \{y\}$. From the expansions (3.97) and (3.98), we deduce that

$$\Phi(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|} + \frac{i}{4} - \frac{1}{2\pi} \ln \frac{k}{2} - \frac{C}{2\pi} + O\left(|x-y|^2 \ln \frac{1}{|x-y|}\right) \tag{3.107}$$

for $|x - y| \to 0$. Therefore, the fundamental solution to the Helmholtz equation in two dimensions has the same singular behavior as the fundamental solution of Laplace's equation. As a consequence, Green's formula (2.5) and the jump relations and regularity results on single- and double-layer potentials of Theorems 3.1 and 3.3 can be carried over to two dimensions. From (3.107) we note that, in contrast to three dimensions, the fundamental solution does not converge for $k \to 0$ to the fundamental solution for the Laplace equation. This leads to some difficulties in the investigation of the convergence of the solution to the exterior Dirichlet problem as $k \to 0$ (see Werner [424] and Kress [261]).

In $\mathbb{R}^2$ the Sommerfeld radiation condition has to be replaced by

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0, \quad r = |x|, \tag{3.108}$$

uniformly for all directions $x/|x|$. From (3.105) it is obvious that the fundamental solution satisfies the radiation condition uniformly with respect to y on compact sets. Therefore, Green's representation formula (2.9) can be shown to be valid for two-dimensional radiating solutions. According to the form (3.108) of the radiation condition, the definition of the far field pattern (2.13) has to be replaced by

$$u(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} \left\{ u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty, \tag{3.109}$$

and, due to (3.105), the representation (2.14) has to be replaced by

$$u_\infty(\hat{x}) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{8\pi k}} \int_{\partial D} \left\{ u(y)\, \frac{\partial e^{-ik\,\hat{x}\cdot y}}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y)\, e^{-ik\,\hat{x}\cdot y} \right\} ds(y) \tag{3.110}$$

for $|\hat{x}| = x/|x|$. We explicitly write out the addition theorem

$$H_0^{(1)}(k|x-y|) = H_0^{(1)}(k|x|)\, J_0(k|y|) + 2\sum_{n=1}^{\infty} H_n^{(1)}(k|x|)\, J_n(k|y|) \cos n\theta \tag{3.111}$$

which is valid for $|x| > |y|$ in the sense of Theorem 2.11 and where θ denotes the angle between x and y. The proof is analogous to that of Theorem 2.11. We note that the entire spherical wave functions in $\mathbb{R}^2$ are given by $J_n(kr)e^{\pm in\varphi}$ and the radiating spherical wave functions by $H_n^{(1)}(kr)e^{\pm in\varphi}$. Similarly, the Jacobi–Anger expansion (2.46) assumes the form

$$e^{ik\,x\cdot d} = J_0(k|x|) + 2\sum_{n=1}^{\infty} i^n\, J_n(k|x|)\cos n\theta, \quad x \in \mathbb{R}^2. \tag{3.112}$$

With all these prerequisites, it is left as an exercise to establish that, with minor adjustments in the proofs, all the results of Sects. 2.5, 3.2, and 3.4 remain valid in two dimensions.

3.6 On the Numerical Solution in $\mathbb{R}^2$

We would like to include in our presentation an advertisement for what we think is the most efficient method for the numerical solution of the boundary integral equations for two-dimensional problems. Since it seems to be safe to state that the boundary curves in most practical applications are either analytic or piecewise analytic with corners, we restrict our attention to approximation schemes which are the most appropriate under these regularity assumptions. We begin with the analytic case where we recommend the Nyström method based on weighted trigonometric interpolation quadratures on an equidistant mesh. To support our preference for using trigonometric polynomial approximations we quote from Atkinson [20]: *…the most efficient numerical methods for solving boundary integral equations on smooth planar boundaries are those based on trigonometric polynomial approximations, and such methods are sometimes called spectral methods. When calculations using piecewise polynomial approximations are compared with those using trigonometric polynomial approximations, the latter are almost always the more efficient.*

We first describe the necessary parametrization of the integral equation (3.29) in the two-dimensional case. We assume that the boundary curve ∂D possesses a regular analytic and 2π-periodic parametric representation of the form

$$x(t) = (x_1(t), x_2(t)), \quad 0 \le t \le 2\pi, \tag{3.113}$$

in counterclockwise orientation satisfying $|x'(t)|^2 > 0$ for all t. Then, by straightforward calculations using $H_1^{(1)} = -H_0^{(1)\prime}$, we transform (3.29) into the parametric form

$$\psi(t) - \int_0^{2\pi} \{L(t,\tau) + i\eta M(t,\tau)\}\,\psi(\tau)\,d\tau = g(t), \quad 0 \le t \le 2\pi,$$

where we have set $\psi(t) := \varphi(x(t))$, $g(t) := 2f(x(t))$, and the kernels are given by

$$L(t,\tau) := \frac{ik}{2}\,\{x_2'(\tau)[x_1(\tau)-x_1(t)]-x_1'(\tau)[x_2(\tau)-x_2(t)]\}\,\frac{H_1^{(1)}(k|x(t)-x(\tau)|)}{|x(t)-x(\tau)|},$$

$$M(t,\tau) := \frac{i}{2}\,H_0^{(1)}(k|x(t)-x(\tau)|)\,|x'(\tau)|$$

for $t \neq \tau$. From the expansion (3.98) for the Neumann functions, we see that the kernels L and M have logarithmic singularities at $t = \tau$. Hence, for their proper numerical treatment, following Martensen [310] and Kussmaul [284], we split the kernels into

$$L(t,\tau) = L_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + L_2(t,\tau),$$

$$M(t,\tau) = M_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + M_2(t,\tau),$$

where

$$L_1(t,\tau) := \frac{k}{2\pi}\,\{x_2'(\tau)[x_1(t)-x_1(\tau)]-x_1'(\tau)[x_2(t)-x_2(\tau)]\}\,\frac{J_1(k|x(t)-x(\tau)|)}{|x(t)-x(\tau)|},$$

$$L_2(t,\tau) := L(t,\tau) - L_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right),$$

$$M_1(t,\tau) := -\frac{1}{2\pi}\,J_0(k|x(t)-x(\tau)|)\,|x'(\tau)|,$$

$$M_2(t,\tau) := M(t,\tau) - M_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right).$$

The kernels L_1, L_2, M_1, and M_2 turn out to be analytic. In particular, using the expansions (3.97) and (3.98) we can deduce the diagonal terms

$$L_2(t,t) = L(t,t) = \frac{1}{2\pi}\,\frac{x_1'(t)x_2''(t) - x_2'(t)x_1''(t)}{|x'(t)|^2}$$

and

$$M_2(t,t) = \left\{\frac{i}{2} - \frac{C}{\pi} - \frac{1}{\pi}\ln\left(\frac{k}{2}\,|x'(t)|\right)\right\}|x'(t)|$$

for $0 \leq t \leq 2\pi$. We note that despite the continuity of the kernel L, for numerical accuracy it is advantageous to separate the logarithmic part of L since the derivatives of L fail to be continuous at $t = \tau$.

Hence, we have to numerically solve an integral equation of the form

$$\psi(t) - \int_0^{2\pi} K(t,\tau)\psi(\tau)\,d\tau = g(t), \quad 0 \le t \le 2\pi, \tag{3.114}$$

where the kernel can be written in the form

$$K(t,\tau) = K_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + K_2(t,\tau) \tag{3.115}$$

with analytic functions K_1 and K_2 and with an analytic right-hand side g. Here we wish to point out that it is essential to split off the logarithmic singularity in a fashion which preserves the 2π-periodicity for the kernels K_1 and K_2.

For the numerical solution of integral equations of the second kind, in principle, there are three basic methods available, the *Nyström method*, the *collocation method*, and the *Galerkin method*. In the case of one-dimensional integral equations, the Nyström method is more practical than the collocation and Galerkin methods since it requires the least computational effort. In each of the three methods, the approximation requires the solution of a finite dimensional linear system. In the Nyström method, for the evaluation of each of the matrix elements of this linear system only an evaluation of the kernel function is needed, whereas in the collocation and Galerkin methods the matrix elements are single or double integrals demanding numerical quadratures. In addition, the Nyström method is generically stable in the sense that it preserves the condition of the integral equation whereas in the collocation and Galerkin methods the condition can be disturbed by a poor choice of the basis (see [268]).

In the case of integral equations for periodic analytic functions, using global approximations via trigonometric polynomials is superior to using local approximations via low order polynomial splines since the trigonometric approximations yield much better convergence. By choosing the appropriate basis, the computational effort for the global approximation is comparable to that for local approximations.

The Nyström method consists in the straightforward approximation of the integrals by quadrature formulas. In our case, for the 2π-periodic integrands, we choose an equidistant set of knots $t_j := \pi j/n,\ \ j = 0,\dots,2n-1$, and use the quadrature rule

$$\int_0^{2\pi} \ln\left(4\sin^2\frac{t-\tau}{2}\right) f(\tau)d\tau \approx \sum_{j=0}^{2n-1} R_j^{(n)}(t) f(t_j), \quad 0 \le t \le 2\pi, \tag{3.116}$$

with the quadrature weights given by

$$R_j^{(n)}(t) := -\frac{2\pi}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos m(t-t_j) - \frac{\pi}{n^2}\cos n(t-t_j), \quad j = 0,\dots,2n-1,$$

and the trapezoidal rule

$$\int_0^{2\pi} f(\tau)d\tau \approx \frac{\pi}{n}\sum_{j=0}^{2n-1} f(t_j). \tag{3.117}$$

Both these numerical integration formulas are obtained by replacing the integrand f by its trigonometric interpolation polynomial and then integrating exactly. The quadrature formula (3.116) was first used by Martensen [310] and Kussmaul [284]. Provided f is analytic, according to derivative-free error estimates for the remainder term in trigonometric interpolation for periodic analytic functions (see [258, 268]), the errors for the quadrature rules (3.116) and (3.117) decrease at least exponentially when the number $2n$ of knots is increased. More precisely, the error is of order $O(\exp(-n\sigma))$ where σ denotes half of the width of a parallel strip in the complex plane into which the real analytic function f can be holomorphically extended.

Of course, it is also possible to use quadrature rules different from (3.116) and (3.117) obtained from other approximations for the integrand f. However, due to their simplicity and high approximation order we strongly recommend the application of (3.116) and (3.117).

In the Nyström method, the integral equation (3.114) is replaced by the approximating equation

$$\psi^{(n)}(t) - \sum_{j=0}^{2n-1}\left\{R_j^{(n)}(t)K_1(t,t_j) + \frac{\pi}{n}K_2(t,t_j)\right\}\psi^{(n)}(t_j) = g(t) \tag{3.118}$$

for $0 \le t \le 2\pi$. Equation (3.118) is obtained from (3.114) by applying the quadrature rule (3.116) to $f = K_1(t,.)\psi$ and (3.117) to $f = K_2(t,.)\psi$. The solution of (3.118) reduces to solving a finite dimensional linear system. In particular, for any solution of (3.118) the values $\psi_i^{(n)} = \psi^{(n)}(t_i)$, $i = 0,\dots,2n-1$, at the quadrature points trivially satisfy the linear system

$$\psi_i^{(n)} - \sum_{j=0}^{2n-1}\left\{R_{|i-j|}^{(n)}K_1(t_i,t_j) + \frac{\pi}{n}\,K_2(t_i,t_j)\right\}\psi_j^{(n)} = g(t_i), \quad i = 0,\dots,2n-1, \tag{3.119}$$

where

$$R_j^{(n)} := R_j^{(n)}(0) = -\frac{2\pi}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos\frac{mj\pi}{n} - \frac{(-1)^j\pi}{n^2}\,, \quad j = 0,\dots,2n-1.$$

Conversely, given a solution $\psi_i^{(n)}$, $i = 0,\dots,2n-1$, of the system (3.119), the function $\psi^{(n)}$ defined by

$$\psi^{(n)}(t) := \sum_{j=0}^{2n-1} \left\{ R_j^{(n)}(t) K_1(t, t_j) + \frac{\pi}{n} \, K_2(t, t_j) \right\} \psi_j^{(n)} + g(t), \quad 0 \le t \le 2\pi, \tag{3.120}$$

is readily seen to satisfy the approximating equation (3.118). The formula (3.120) may be viewed as a natural interpolation of the values $\psi_i^{(n)}$, $i = 0, \dots, 2n-1$, at the quadrature points to obtain the approximating function $\psi^{(n)}$ and goes back to Nyström.

For the solution of the large linear system (3.119), we recommend the use of the fast iterative two-grid or multi-grid methods as described in [268] or, in more detail, in [164].

Provided the integral equation (3.114) itself is uniquely solvable and the kernels K_1 and K_2 and the right-hand side g are continuous, a rather involved error analysis (for the details we refer to [263, 268]) shows that

1. the approximating linear system (3.119), i.e., the approximating equation (3.118), is uniquely solvable for all sufficiently large n;
2. as $n \to \infty$ the approximate solutions $\psi^{(n)}$ converge uniformly to the solution ψ of the integral equation;
3. the convergence order of the quadrature errors for (3.116) and (3.117) carries over to the error $\psi^{(n)} - \psi$.

The latter, in particular, means that in the case of analytic kernels K_1 and K_2 and analytic right-hand sides g the approximation error decreases exponentially, i.e., there exist positive constants C and σ such that

$$|\psi^{(n)}(t) - \psi(t)| \le C\, e^{-n\sigma}, \quad 0 \le t \le 2\pi, \tag{3.121}$$

for all n. In principle, the constants in (3.121) are computable but usually they are difficult to evaluate. In most practical cases, it is sufficient to judge the accuracy of the computed solution by doubling the number $2n$ of knots and then comparing the results for the coarse and the fine grid with the aid of the exponential convergence order, i.e., by the fact that doubling the number $2n$ of knots will double the number of correct digits in the approximate solution.

For a numerical example, we consider the scattering of a plane wave by a cylinder with a non-convex kite-shaped cross section with boundary ∂D illustrated in Fig. 3.1 and described by the parametric representation

$$x(t) = (\cos t + 0.65 \cos 2t - 0.65, \; 1.5 \sin t), \quad 0 \le t \le 2\pi.$$

From the asymptotics (3.105) for the Hankel functions, analogous to (3.110) it can be deduced that the far field pattern of the combined potential (3.28) in two dimensions is given by

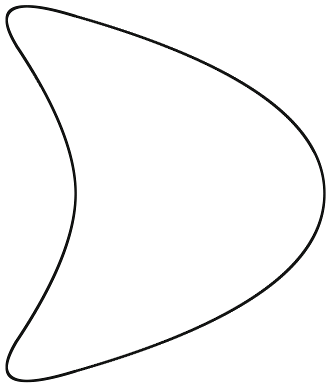

Fig. 3.1 Kite-shaped domain for numerical example

Table 3.1 Numerical results for Nyström's method

	n	$\text{Re}\, u_\infty(d)$	$\text{Im}\, u_\infty(d)$	$\text{Re}\, u_\infty(-d)$	$\text{Im}\, u_\infty(-d)$
$k = 1$	8	−1.62642413	0.60292714	1.39015283	0.09425130
	16	−1.62745909	0.60222343	1.39696610	0.09499454
	32	−1.62745750	0.60222591	1.39694488	0.09499635
	64	−1.62745750	0.60222591	1.39694488	0.09499635
$k = 5$	8	−2.30969119	1.52696566	−0.30941096	0.11503232
	16	−2.46524869	1.67777368	−0.19932343	0.06213859
	32	−2.47554379	1.68747937	−0.19945788	0.06015893
	64	−2.47554380	1.68747937	−0.19945787	0.06015893

$$u_\infty(\hat{x}) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{8\pi k}} \int_{\partial D} \{k\, \nu(y) \cdot \hat{x} + \eta\} e^{-ik\,\hat{x}\cdot y} \varphi(y)\, ds(y), \quad |\hat{x}| = 1, \tag{3.122}$$

which can be evaluated again by the trapezoidal rule after solving the integral equation for φ. Table 3.1 gives some approximate values for the far field pattern $u_\infty(d)$ and $u_\infty(-d)$ in the forward direction d and the backward direction $-d$. The direction d of the incident wave is $d = (1, 0)$ and, as recommended in [259], the coupling parameter is $\eta = k$. Note that the exponential convergence is clearly exhibited.

The corresponding quadrature method including its error and convergence analysis for the Neumann boundary condition has been described by Kress [264].

For domains D with corners, a uniform mesh yields only poor convergence and therefore has to be replaced by a graded mesh. We suggest to base this grading upon the idea of substituting an appropriate new variable and then using the Nyström method as described above for the transformed integral equation. With a suitable choice for the substitution, this will lead to high order convergence.

Without loss of generality, we confine our presentation to a boundary curve ∂D with one corner at the point x_0 and assume $\partial D \setminus \{x_0\}$ to be C^2 and piecewise analytic.

We do not allow cusps in our analysis, i.e., the angle γ at the corner is assumed to satisfy $0 < \gamma < 2\pi$.

Using the fundamental solution

$$\Phi_0(x, y) := \frac{1}{2\pi} \ln \frac{1}{|x-y|}, \quad x \neq y,$$

to the Laplace equation in $\mathbb{R}^2$ to subtract a vanishing term, we rewrite the combined double- and single-layer potential (3.28) in the form

$$u(x) = \int_{\partial D} \left[\left\{ \frac{\partial \Phi(x, y)}{\partial \nu(y)} - i\eta \Phi(x, y) \right\} \varphi(y) - \frac{\partial \Phi_0(x, y)}{\partial \nu(y)} \varphi(x_0) \right] ds(y)$$

for $x \in \mathbb{R}^2 \setminus \bar{D}$. This modification is notationally advantageous for the corner case and it makes the error analysis for the Nyström method work. The integral equation (3.29) now becomes

$$\begin{aligned} \varphi(x) - \varphi(x_0) + 2 \int_{\partial D} \left\{ \frac{\partial \Phi(x, y)}{\partial \nu(y)} - i\eta \Phi(x, y) \right\} \varphi(y)\, ds(y) \\ - 2 \int_{\partial D} \frac{\partial \Phi_0(x, y)}{\partial \nu(y)} \varphi(x_0)\, ds(y) = 2f(x), \quad x \in \partial D. \end{aligned} \tag{3.123}$$

Despite the corner at x_0, there is no change in the residual term in the jump relations since the density $\varphi - \varphi(x_0)$ of the leading term in the singularity vanishes at the corner. However, the kernel of the integral equation (3.123) at the corner no longer remains weakly singular. For a C^2 boundary, the weak singularity of the kernel of the double-layer operator rests on the inequality

$$|\nu(y) \cdot (x - y)| \leq L|x - y|^2, \quad x, y \in \partial D, \tag{3.124}$$

for some positive constant L. This inequality expresses the fact that the vector $x - y$ for x close to y is almost orthogonal to the normal vector $\nu(y)$. For a proof, we refer to [104]. However, in the vicinity of a corner (3.124) does not remain valid.

After splitting off the operator $K_0 : C(\partial D) \to C(\partial D)$ defined by

$$(K_0\varphi)(x) := 2 \int_{\partial D} \frac{\partial \Phi_0(x, y)}{\partial \nu(y)} [\varphi(y) - \varphi(x_0)]\, ds(y), \quad x \in \partial D,$$

from (3.107) we see that the remaining integral operator in (3.123) has a weakly singular kernel and therefore is compact. For the further investigation of the non-compact part K_0, we choose a sufficiently small positive number r and denote the two arcs of the boundary ∂D contained in the disk of radius r and center at the corner x_0 by A and B (see Fig. 3.2). These arcs intersect at x_0 with an angle γ and without loss of generality we restrict our presentation to the case where $\gamma < \pi$. By

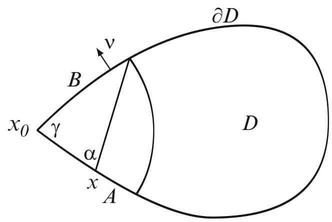

Fig. 3.2 Domain with a corner

elementary geometry and continuity, we can assume that r is chosen such that both A and B have length less than $2r$ and for the angle $\alpha(x, B)$ between the two straight lines connecting the points $x \in A \setminus \{x_0\}$ with the two endpoints of the arc B we have

$$0 < \alpha(x, B) \leq \pi - \frac{1}{2}\,\gamma, \quad x \in A \setminus x_0,$$

and analogously with the roles of A and B interchanged. For the sake of brevity, we confine ourselves to the case where the boundary ∂D in a neighborhood of the corner x_0 consists of two straight lines intersecting at x_0. Then we can assume that r is chosen such that the function $(x, y) \mapsto \nu(y) \cdot (y - x)$ does not change its sign for all $(x, y) \in A \times B$ and all $(x, y) \in B \times A$. Finally, for the two C^2 arcs A and B, there exists a constant L independent of r such that the estimate (3.124) holds for all $(x, y) \in A \times A$ and all $(x, y) \in B \times B$.

We now choose a continuous cut-off function $\psi : \mathbb{R}^2 \to [0, 1]$ such that $\psi(x) = 1$ for $0 \leq |x - x_0| \leq r/2$, $\psi(x) = 0$ for $r \leq |x - x_0| < \infty$ and define $K_{0,r} : C(\partial D) \to C(\partial D)$ by

$$K_{0,r}\varphi := \psi\, K_0(\psi\varphi).$$

Then, the kernel of $K_0 - K_{0,r}$ vanishes in a neighborhood of (x_0, x_0) and therefore is weakly singular.

We introduce the norm

$$\|\varphi\|_{\infty,0} := \max_{x \in \partial D} |\varphi(x) - \varphi(x_0)| + |\varphi(x_0)|,$$

which obviously is equivalent to the maximum norm. We now show that r can be chosen such that $\|K_{0,r}\|_{\infty,0} < 1$. Then, by the Neumann series, the operator $I + K_{0,r}$ has a bounded inverse and the results of the Riesz–Fredholm theory are available for the corner integral equation (3.123).

By our assumptions on the choice of r, we can estimate

$$|(K_{0,r}\varphi)(x_0)| \leq \frac{4Lr}{\pi}\, \|\varphi\|_{\infty,0} \tag{3.125}$$

since (3.124) holds for $x = x_0$ and all $y \in A \cup B$. For $x \in A \setminus \{x_0\}$ we split the integral into the parts over A and over B and evaluate the second one by using Green's integral theorem and our assumptions on the geometry to obtain

$$2 \int_B \left| \frac{\partial \Phi_0(x, y)}{\partial \nu(y)} \right| ds(y) = 2 \left| \int_B \frac{\partial \Phi_0(x, y)}{\partial \nu(y)} \, ds(y) \right| = \frac{\alpha(x, B)}{\pi}, \quad x \in A \setminus \{x_0\},$$

and consequently

$$|(K_{0,r}\varphi)(x)| \leq \left\{ \frac{2Lr}{\pi} + 1 - \frac{\gamma}{2\pi} \right\} \|\varphi\|_{\infty,0}, \tag{3.126}$$

which by symmetry is valid for all $x \in A \cup B \setminus \{x_0\}$. Summarizing, from the inequalities (3.125) and (3.126) we deduce that we can choose r small enough such that $\|K_{0,r}\|_{\infty,0} < 1$. For an analysis for more general domains with corners we refer to Ruland [380] and the literature therein.

The above analysis establishes the existence of a continuous solution to the integral equation (3.123). However, due to the singularities of elliptic boundary value problems in domains with corners (see [155]), this solution will have singularities in the derivatives at the corner. To take proper care of this corner singularity, we replace our equidistant mesh by a graded mesh through substituting a new variable in such a way that the derivatives of the new integrand vanish up to a certain order at the endpoints and then use the quadrature rules (3.116) and (3.117) for the transformed integrals.

We describe this numerical quadrature rule for the integral $\int_0^{2\pi} f(t)\,dt$ where the integrand f is analytic in $(0, 2\pi)$ but has singularities at the endpoints $t = 0$ and $t = 2\pi$. Let the function $w : [0, 2\pi] \to [0, 2\pi]$ be one-to-one, strictly monotonically increasing and infinitely differentiable. We assume that the derivatives of w at the endpoints $t = 0$ and $t = 2\pi$ vanish up to an order $p \in \mathbb{N}$. We then substitute $t = w(s)$ to obtain

$$\int_0^{2\pi} f(t)\,dt = \int_0^{2\pi} w'(s)\, f(w(s))\,ds.$$

Applying the trapezoidal rule to the transformed integral now yields the quadrature formula

$$\int_0^{2\pi} f(t)\,dt \approx \frac{\pi}{n} \sum_{j=1}^{2n-1} a_j\, f(s_j) \tag{3.127}$$

with the weights and mesh points given by

$$a_j = w'\left(\frac{j\pi}{n}\right), \quad s_j = w\left(\frac{j\pi}{n}\right), \quad j = 1, \ldots, 2n-1.$$

A typical example for such a substitution is given by

$$w(s) = 2\pi \, \frac{[v(s)]^p}{[v(s)]^p + [v(2\pi - s)]^p}\,, \qquad 0 \le s \le 2\pi, \tag{3.128}$$

where

$$v(s) = \left(\frac{1}{p} - \frac{1}{2}\right)\left(\frac{\pi - s}{\pi}\right)^3 + \frac{1}{p}\,\frac{s-\pi}{\pi} + \frac{1}{2}$$

and $p \ge 2$. Note that the cubic polynomial v is chosen such that $v(0) = 0$, $v(2\pi) = 1$ and $w'(\pi) = 2$. The latter property ensures, roughly speaking, that one half of the grid points is equally distributed over the total interval, whereas the other half is accumulated towards the two end points.

For an error analysis for the quadrature rule (3.127) with substitutions of the form described above and using the Euler–MacLaurin expansion, we refer to Kress [262]. Assume f is $2q+1$-times continuously differentiable on $(0, 2\pi)$ such that for some $0 < \alpha < 1$ with $\alpha p \ge 2q + 1$ the integrals

$$\int_0^{2\pi} \left[\sin\frac{t}{2}\right]^{m-\alpha} |f^{(m)}(t)|\,dt$$

exist for $m = 0, 1, \dots, 2q+1$. The error $E^{(n)}(f)$ in the quadrature (3.127) can then be estimated by

$$|E^{(n)}(f)| \le \frac{C}{n^{2q+1}} \tag{3.129}$$

with some constant C. Thus, by choosing p large enough, we can obtain almost exponential convergence behavior.

For the numerical solution of the corner integral equation (3.123), we choose a parametric representation of the form (3.113) such that the corner x_0 corresponds to the parameter $t = 0$ and rewrite (3.123) in the parameterized form

$$\begin{aligned} \psi(t) - \psi(0) - \int_0^{2\pi} K(t,\tau)\psi(\tau)\,d\tau \\ - \int_0^{2\pi} H(t,\tau)\psi(0)\,d\tau = g(t), \quad 0 \le t \le 2\pi, \end{aligned} \tag{3.130}$$

where K is given as above in the analytic case and where

$$H(t,\tau) = \begin{cases} \dfrac{1}{\pi}\, \dfrac{x_2'(\tau)[x_1(t) - x_1(\tau)] - x_1'(\tau)[x_2(t) - x_2(\tau)]}{|x(t) - x(\tau)|^2}\,, & t \ne \tau, \\[2ex] \dfrac{1}{\pi}\, \dfrac{x_2'(t)x_1''(t) - x_1'(t)x_2''(t)}{|x'(t)|^2}\,, & t = \tau,\ t \ne 0, 2\pi, \end{cases}$$

corresponds to the additional term in (3.123). For the numerical solution of the integral equation (3.130) by Nyström's method on the graded mesh, we also have to take into account the logarithmic singularity. We set $t = w(s)$ and $\tau = w(\sigma)$ to obtain

$$\int_0^{2\pi} K(t,\tau)\,\psi(\tau)\,d\tau = \int_0^{2\pi} K(w(s), w(\sigma))\; w'(\sigma)\psi(w(\sigma))\,d\sigma$$

and then write

$$K(w(s), w(\sigma)) = \tilde{K}_1(s,\sigma)\ln\left(4\sin^2\frac{s-\sigma}{2}\right) + \tilde{K}_2(s,\sigma).$$

This decomposition is related to (3.115) by

$$\tilde{K}_1(s,\sigma) = K_1(w(s), w(\sigma))$$

and

$$\tilde{K}_2(s,\sigma) = K(w(s), w(\sigma)) - \tilde{K}_1(s,\sigma)\ln\left(4\sin^2\frac{s-\sigma}{2}\right), \quad s \neq \sigma.$$

From

$$K_2(s,s) = \lim_{\sigma\to s}\left[K(s,\sigma) - K_1(s,\sigma)\ln\left(4\sin^2\frac{s-\sigma}{2}\right)\right],$$

we deduce the diagonal term

$$\tilde{K}_2(s,s) = K_2(w(s), w(s)) + 2\ln w'(s)\, K_1(w(s), w(s)).$$

Now, proceeding as in the derivation of (3.119), for the approximate values $\psi_i^{(n)} = \psi^{(n)}(s_i)$ at the quadrature points s_i for $i = 1,\dots,2n-1$ and $\psi_0^{(n)} = \psi^{(n)}(0)$ at the corner $s_0 = 0$ we arrive at the linear system

$$\psi_i^{(n)} - \psi_0^{(n)} - \sum_{j=1}^{2n-1}\left\{R_{|i-j|}^{(n)}\tilde{K}_1(s_i,s_j) + \frac{\pi}{n}\,\tilde{K}_2(s_i,s_j)\right\} a_j\,\psi_j^{(n)}$$
$$- \sum_{j=1}^{2n-1}\frac{\pi}{n}\,H(s_i,s_j)\,a_j\,\psi_0^{(n)} = g(s_i), \quad i = 0,\dots,2n-1. \tag{3.131}$$

A rigorous error analysis carrying over the error behavior (3.129) to the approximate solution of the integral equation obtained from (3.131) for the potential theoretic

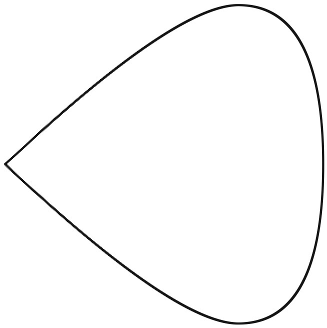

Fig. 3.3 Drop-shaped domain for numerical example

Table 3.2 Nyström's method for a domain with corner

	n	$\operatorname{Re} u_\infty(d)$	$\operatorname{Im} u_\infty(d)$	$\operatorname{Re} u_\infty(-d)$	$\operatorname{Im} u_\infty(-d)$
$k = 1$	16	−1.28558226	0.30687170	−0.53002440	−0.41033666
	32	−1.28549613	0.30686638	−0.53020518	−0.41094518
	64	−1.28549358	0.30686628	−0.53021014	−0.41096324
	128	−1.28549353	0.30686627	−0.53021025	−0.41096364
$k = 5$	16	−1.73779647	1.07776749	−0.18112826	−0.20507986
	32	−1.74656264	1.07565703	−0.19429063	−0.19451172
	64	−1.74656303	1.07565736	−0.19429654	−0.19453324
	128	−1.74656304	1.07565737	−0.19429667	−0.19453372

case $k = 0$ has been worked out by Kress [262]. Related substitution methods have been considered by Jeon [218] and by Elliott and Prössdorf [136, 137].

For a numerical example, we used the substitution (3.128) with order $p = 8$. We consider a drop-shaped domain with the boundary curve ∂D illustrated by Fig. 3.3 and given by the parametric representation

$$x(t) = (2\sin\frac{t}{2}\,,\ -\sin t), \quad 0 \le t \le 2\pi.$$

It has a corner at $t = 0$ with interior angle $\gamma = \pi/2$. The direction d of the incoming plane wave and the coupling parameter η are chosen as in our previous example. Table 3.2 clearly exhibits the fast convergence of the method.

3.7 On the Numerical Solution in $\mathbb{R}^3$

In three dimensions, for the numerical solution of the boundary integral equation (3.29) the Nyström, collocation, and Galerkin methods are still available. However, for surface integral equations we have to modify our statements on comparing the efficiency of the three methods. Firstly, there is no straightforward simple quadrature rule analogous to (3.116) available that deals appropriately with the singularity of the three-dimensional fundamental solution. Hence, the Nyström method loses some of its attraction. Secondly, for the surface integral equations there is no immediate choice for global approximations like the trigonometric polynomials in the one-dimensional periodic case. Therefore, local approximations by low order polynomial splines have been more widely used and the collocation method is the most important numerical approximation method. To implement the collocation method, the boundary surface is first subdivided into a finite number of segments, like curved triangles and squares. The approximation space is then chosen to consist of low order polynomial splines with respect to these surface elements. The simplest choices are piecewise constants or piecewise linear functions. Within each segment, depending on the degree of freedom in the chosen splines, a number of collocation points is selected. Then, the integrals for the matrix elements in the collocation system are evaluated using numerical integration. Due to the weak singularity of the kernels, the calculation of the improper integrals for the diagonal elements of the matrix, where the collocation points and the surface elements coincide, needs special attention. For a detailed description of this so-called *boundary element method* we refer to Rjasanow and Steinbach [376] and to Sauter and Schwab [385].

Besides these local approximations via boundary elements there are also global approaches available in the sense of spectral methods. For surfaces which can be mapped onto spheres, Atkinson [19] has developed a Galerkin method for the Laplace equation using spherical harmonics as the counterpart of the trigonometric polynomials. This method has been extended to the Helmholtz equation by Lin [304]. Based on spherical harmonics and transforming the boundary surface to a sphere as in Atkinson's method, Wienert [427] has developed a Nyström type method for the boundary integral equations for three-dimensional Helmholtz problems which exhibits exponential convergence for analytic boundary surfaces. Wienert's method has been further developed into a fully discrete Galerkin type method through the work of Ganesh, Graham, and Sloan [143, 153]. We conclude this chapter by introducing the main ideas of this method.

We begin by describing a numerical quadrature scheme for the integration of analytic functions over closed analytic surfaces Γ in $\mathbb{R}^3$ which are homeomorphic to the unit sphere $\mathbb{S}^2$ and then we proceed to corresponding quadratures for acoustic single- and double-layer potentials. To this end, we first introduce a suitable projection operator Q_N onto the linear space H_{N-1} of all spherical harmonics of order less than N. We denote by $-1 < t_1 < t_2 < \cdots < t_N < 1$ the zeros of the Legendre polynomial P_N (the existence of N distinct zeros of P_N in the interval $(-1, 1)$ is a consequence of the orthogonality relation (2.25), see [130, p. 236]) and by

$$\alpha_j := \frac{2(1-t_j^2)}{[N P_{N-1}(t_j)]^2}\,, \quad j = 1, \ldots, N,$$

the weights of the Gauss–Legendre quadrature rule which are uniquely determined by the property

$$\int_{-1}^{1} p(t)\,dt = \sum_{j=1}^{N} \alpha_j\, p(t_j) \tag{3.132}$$

for all polynomials p of degree less than or equal to $2N-1$ (see [131, p. 89]). We then choose a set of points x_{jk} on the unit sphere $\mathbb{S}^2$ given in polar coordinates by

$$x_{jk} := (\sin\theta_j \cos\varphi_k, \sin\theta_j \sin\varphi_k, \cos\theta_j)$$

for $j = 1, \dots, N$ and $k = 0, \dots, 2N-1$ where $\theta_j := \arccos t_j$ and $\varphi_k = \pi k/N$ and define $Q_N : C(\mathbb{S}^2) \to H_{N-1}$ by

$$Q_N f := \frac{\pi}{N} \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \alpha_j f(x_{jk}) \sum_{n=0}^{N-1} \sum_{m=-n}^{n} Y_n^{-m}(x_{jk})\, Y_n^m \tag{3.133}$$

where the spherical harmonics Y_n^m are given by (2.28). By orthogonality we clearly have

$$\int_{\mathbb{S}^2} Q_N f\, Y_n^{-m}\, ds = \frac{\pi}{N} \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \alpha_j f(x_{jk}) Y_n^{-m}(x_{jk}) \tag{3.134}$$

for $|m| \le n < N$. Since the trapezoidal rule with $2N$ knots integrates trigonometric polynomials of degree less than N exactly, we have

$$\frac{\pi}{N} \sum_{k=0}^{2N-1} Y_n^m(x_{jk}) Y_{n'}^{-m'}(x_{jk}) = \int_0^{2\pi} Y_n^m(\theta_j, \varphi)\, Y_{n'}^{-m'}(\theta_j, \varphi)\, d\varphi$$

for $|m|, |m'| \le n < N$ and these integrals, in view of (2.28), vanish if $m \ne m'$. For $m = m'$, by (2.27) and (2.28), $Y_n^m\, Y_{n'}^{-m}$ is a polynomial of degree less than $2N$ in $\cos\theta$. Hence, by the property (3.132) of the Gauss–Legendre quadrature rule, summing the previous equation we find

$$\frac{\pi}{N} \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \alpha_j Y_n^m(x_{jk})\, Y_{n'}^{-m'}(x_{jk}) = \int_{\mathbb{S}^2} Y_n^m\, Y_{n'}^{-m'}\, ds,$$

that is, $Q_N Y_n^m = Y_n^m$ for $|m| \le n < N$ and therefore Q_N is indeed a projection operator onto H_{N-1}. We note that Q_N is not an interpolation operator since by Theorem 2.7 we have $\dim H_{N-1} = N^2$ whereas we have $2N^2$ points x_{jk}. Therefore, it is also called a *hyperinterpolation operator*. With the aid of (2.22), the addition theorem (2.30) and (3.132) we can estimate

$$\|Q_N f\|_\infty \le \frac{1}{4N} \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \alpha_j \sum_{n=0}^{N-1} (2n+1)\, \|f\|_\infty = N^2 \|f\|_\infty$$

whence

$$\|Q_N\|_\infty \le N^2 \tag{3.135}$$

follows. However, this straightforward estimate is suboptimal and can be improved into

$$c_1\, N^{1/2} \le \|Q_N\|_\infty \le c_2\, N^{1/2} \tag{3.136}$$

with positive constants $c_1 < c_2$ (see [153, 395]). For analytic functions $f : \mathbb{S}^2 \to \mathbb{C}$, Wienert [427] has shown that the approximation error $f - Q_N f$ decreases exponentially, that is, there exist positive constants C and σ depending on f such that

$$\|f - Q_N f\|_{\infty,\mathbb{S}^2} \le C e^{-N\sigma} \tag{3.137}$$

for all $N \in \mathbb{N}$.

Integrating the approximation $Q_N f$ instead of f we obtain the so-called Gauss trapezoidal product rule

$$\int_{\mathbb{S}^2} f\, ds \approx \frac{\pi}{N} \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \alpha_j f(x_{jk}) \tag{3.138}$$

for the numerical integration over the unit sphere. For analytic surfaces Γ which can be mapped bijectively through an analytic function $q : \mathbb{S}^2 \to \Gamma$ onto the unit sphere, (3.138) can also be used after the substitution

$$\int_\Gamma g(\xi)\, ds(\xi) = \int_{\mathbb{S}^2} g(q(x))\, J_q(x)\, ds(x)$$

where J_q stands for the Jacobian of the mapping q. For analytic functions, the exponential convergence (3.137) carries over to the quadrature (3.138).

By passing to the limit $k \to 0$ in (2.44), with the help of (2.32) and (2.33), we find

$$\int_{\mathbb{S}^2} \frac{Y_n(y)}{|x-y|}\, ds(y) = \frac{4\pi}{2n+1}\, Y_n(x), \quad x \in \mathbb{S}^2,$$

for spherical harmonics Y_n of order n. This can be used together with the addition formula (2.30) to obtain the approximation

$$\int_{\mathbb{S}^2} \frac{f(y)}{|x-y|}\, ds(y) \approx \frac{\pi}{N} \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \alpha_j f(x_{jk}) \sum_{n=0}^{N-1} P_n(x_{jk}\cdot x), \quad x \in \mathbb{S}^2,$$

which again is based on replacing f by $Q_N f$. In particular, for the north pole $x_0 = (0, 0, 1)$ this reads

$$\int_{\mathbb{S}^2} \frac{f(y)}{|x_0-y|}\, ds(y) \approx \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \beta_j f(x_{jk}) \tag{3.139}$$

where

$$\beta_j := \frac{\pi \alpha_j}{N} \sum_{n=0}^{N-1} P_n(t_j), \quad j = 1, \ldots, N.$$

The exponential convergence for analytic densities $f : \mathbb{S}^2 \to \mathbb{C}$ again carries over from (3.137) to the numerical quadrature (3.139) of the harmonic single-layer potential.

For the extension of this quadrature scheme to more general surfaces Γ, we need to allow more general densities and we can do this without losing the rapid convergence order. Denote by $\tilde{\mathbb{S}}^2$ the cylinder

$$\tilde{\mathbb{S}}^2 := \{(\cos\varphi, \sin\varphi, \theta) : 0 \le \theta \le \pi,\ 0 \le \varphi \le 2\pi\}.$$

Then we can identify functions defined on $\tilde{\mathbb{S}}^2$ with functions on $\mathbb{S}^2$ through the mapping

$$(\cos\varphi, \sin\varphi, \theta) \mapsto (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

and, loosely speaking, in the sequel we refer to functions on $\tilde{\mathbb{S}}^2$ as functions on $\mathbb{S}^2$ depending on the azimuth φ at the poles. As Wienert [427] has shown, the exponential convergence is still true for the application of (3.139) to analytic functions $f : \tilde{\mathbb{S}}^2 \to \mathbb{C}$.

For the general surface Γ as above, we write

$$\int_{\Gamma} \frac{g(\eta)}{|q(x)-\eta|}\, ds(\eta) = \int_{\mathbb{S}^2} \frac{F(x,y) f(y)}{|x-y|}\, ds(y),$$

where we have set $f(y) := g(q(y)) J_q(y)$ and

$$F(x, y) := \frac{|x-y|}{|q(x)-q(y)|}, \quad x \ne y. \tag{3.140}$$

Unfortunately, as can be seen from simple examples, the function F in general cannot be extended as a continuous function on $\mathbb{S}^2 \times \mathbb{S}^2$. However, since on the unit sphere we have $|x - y|^2 = 2(1 - x \cdot y)$ from the estimate (see the proof of Theorem 2.2 in [104])

$$c_1|x - y|^2 \leq |q(x) - q(y)|^2 \leq c_2|x - y|^2$$

which is valid for all $x, y \in \mathbb{S}^2$ and some constants $0 < c_1 < c_2$ it can be seen that $F(x_0, \cdot)$ is analytic on $\tilde{\mathbb{S}}^2$.

For $\psi \in \mathbb{R}$, we define the orthogonal transformations

$$D_P(\psi) := \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D_T(\psi) := \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}.$$

Then for $x = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \in \mathbb{S}^2$ the orthogonal transformation

$$T_x := D_P(\varphi)D_T(\theta)D_P(-\varphi)$$

has the property $T_x x = (0, 0, 1)$ for $x \in \mathbb{S}^2$. Therefore

$$\int_{\mathbb{S}^2} \frac{F(x, y)f(y)}{|x - y|}\, ds(y) \approx \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \beta_j F(x, T_x^{-1}x_{jk}) f(T_x^{-1}x_{jk}) \tag{3.141}$$

is exponentially convergent for analytic densities f in the sense of (3.137) since x is the north pole for the set of quadrature points $T_x^{-1}x_{jk}$. It can be shown that the exponential convergence is uniform with respect to $x \in \mathbb{S}^2$.

By decomposing

$$\frac{e^{ik|x-y|}}{|x - y|} = \frac{\cos k|x - y|}{|x - y|} + i\,\frac{\sin k|x - y|}{|x - y|}\,,$$

we see that the integral equation (3.29) for the exterior Dirichlet problem is of the form

$$g(\xi) - \int_\Gamma \left\{ \frac{h_1(\xi, \eta)}{|\xi - \eta|} + \frac{\nu(\eta) \cdot (\xi - \eta)}{|\xi - \eta|^2}\, h_2(\xi, \eta) + h_3(\xi, \eta) \right\} g(\eta)\, ds(\eta) = w(\xi)$$

for $\xi \in \Gamma$ with analytic kernels h_1, h_2 and h_3. For our purpose of exposition, it suffices to consider only the singular part, that is, the case when $h_2 = h_3 = 0$. Using the substitution $\xi = q(x)$ and $\eta = q(y)$, the integral equation over Γ can be transformed into an integral equation over $\mathbb{S}^2$ of the form

$$f(x) - \int_{\mathbb{S}^2} \frac{k(x, y)F(x, y)}{|x - y|}\, f(y)\, ds(y) = v(x), \quad x \in \mathbb{S}^2, \tag{3.142}$$

with the functions f, k and v appropriately defined through g, h_1 and w and with F given as in (3.140). We write $A : C(\mathbb{S}^2) \to C(\mathbb{S}^2)$ for the weakly singular integral operator

$$(Af)(x) := \int_{\mathbb{S}^2} \frac{k(x, y)F(x, y)}{|x - y|} f(y)\, ds(y), \quad x \in \mathbb{S}^2,$$

occurring in (3.142). By using the quadrature rule (3.141), we arrive at an approximating quadrature operator $A_N : C(\mathbb{S}^2) \to C(\tilde{\mathbb{S}}^2)$ given by

$$(A_N f)(x) := \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \beta_j k(x, T_x^{-1} x_{jk}) F(x, T_x^{-1} x_{jk}) f(T_x^{-1} x_{jk}), \quad x \in \mathbb{S}^2. \tag{3.143}$$

We observe that the quadrature points $T_x^{-1} x_{jk}$ depend on x. Therefore, we cannot reduce the solution of the approximating equation

$$\tilde{f}_N - A_N \tilde{f}_N = v \tag{3.144}$$

to a linear system in the usual fashion of Nyström interpolation. A possible remedy for this difficulty is to apply the projection operator Q_N a second time. For this, two variants have been proposed. Wienert [427] suggested

$$f_N^w - A_N Q_N f_N^w = v \tag{3.145}$$

as the final approximating equation for the solution of (3.144). Analogous to the presentation in the first edition of this book, Graham and Sloan [153] considered solving (3.144) through the projection method with the final approximating equation of the form

$$f_N - Q_N A_N f_N = Q_N v. \tag{3.146}$$

As observed in [153] there is an immediate one-to-one correspondence between the solutions of (3.145) and (3.146) via $f_N = Q_N f_N^w$ and $f_N^w = v + A_N f_N$. Therefore, we restrict our outline on the numerical implementation to the second variant (3.146). Representing

$$f_N := \sum_{n=0}^{N-1} \sum_{m=-n}^{n} a_n^m Y_n^m$$

and using (3.134) and (3.143) we find that solving (3.146) is equivalent to solving the linear system

$$a_n^m - \sum_{n'=0}^{N-1} \sum_{m'=-n'}^{n'} R_{nn'}^{mm'} a_{n'}^{m'} = \frac{\pi}{N} \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \alpha_j v(x_{jk}) Y_n^{-m}(x_{jk}) \tag{3.147}$$

for $n = 0, \dots, N-1,\ m = -n, \dots, n$, where

$$R_{nn'}^{mm'} := \frac{\pi}{N} \sum_{j_1=1}^{N} \sum_{k_1=0}^{2N-1} \sum_{j_2=1}^{N} \sum_{k_2=0}^{2N-1} \alpha_{j_1} \beta_{j_2} K(x_{j_1 k_1}, x_{j_2 k_2})\, Y_n^{-m}(x_{j_1 k_1})\, Y_{n'}^{m'}(T_{x_{j_1 k_1}}^{-1} x_{j_2 k_2})$$

and

$$K(x, y) := k(x, T_x^{-1} y) F(x, T_x^{-1} y).$$

Since orthogonal transformations map spherical harmonics of order n into spherical harmonics of order n, we have

$$Y_{n'}^{m'}(D_T(-\theta)y) = \sum_{\mu=-n'}^{n'} Z_0(n', m', \mu, \theta) Y_{n'}^{\mu}(y)$$

with

$$Z_0(n', m', \mu, \theta) = \int_{\mathbb{S}^2} Y_{n'}^{m'}(D_T(-\theta)y)\, Y_{n'}^{-\mu}(y)\, ds(y)$$

and from (2.28) we clearly have

$$Y_{n'}^{m'}(D_P(-\varphi)y) = e^{-im'\varphi} Y_{n'}^{m'}(y).$$

From this we find that the coefficients in (3.147) can be evaluated recursively through the scheme

$$Z_1(j_1, k_1, j_2, \mu) := \sum_{k_2=0}^{2N-1} \beta_{j_2} e^{i\mu(\varphi_{k_2} - \varphi_{k_1})} K(x_{j_1 k_1}, x_{j_2 k_2}),$$

$$Z_2(j_1, k_1, n', \mu) := \sum_{j_2=1}^{N} Y_{n'}^{\mu}(x_{j_2,0}) Z_1(j_1, k_1, j_2, \mu),$$

$$Z_3(j_1, k_1, n', m') := \sum_{\mu=-n'}^{n'} Z_0(n', m', \mu, \theta_{j_1}) Z_2(j_1, k_1, n', \mu) e^{im'\varphi_{k_1}},$$

$$Z_4(j_1, m, n', m') := \sum_{k_1=0}^{2N-1} e^{-im\varphi_{k_1}} Z_3(j_1, k_1, n', m'),$$

$$R_{nn'}^{mm'} := \frac{\pi}{N} \sum_{j_1=1}^{N} \alpha_{j_1} Y_n^{-m}(x_{j_1,0}) Z_4(j_1, m, n', m')$$

by $O(N^5)$ multiplications provided the numbers $Z_0(n', m', \mu, \theta_{j_1})$ (which do not depend on the surface) are precalculated. The latter calculations can be based on

$$Z_0(n', m', \mu, \theta) = \int_{\mathbb{S}^2} (Q_N(Y_{n'}^{m'} \circ D_T(-\theta)))(y)\, Y_{n'}^{-\mu}(y)\, ds(y)$$
$$= \frac{\pi}{N} \sum_{j=1}^{N} \sum_{k=0}^{2N-1} \alpha_j Y_{n'}^{m'}(D_T(-\theta)x_{jk})\, Y_{n'}^{-\mu}(x_{jk}).$$

For further details we refer to [143, 427]. To obtain a convergence result, a further modification of (3.145) and (3.146) was required by using different orders N and N' for the projection operator Q_N and the approximation operator $A_{N'}$ such that

$$N' = \kappa N \tag{3.148}$$

for some $\kappa > 1$. Under this assumption Graham and Sloan [153] established superalgebraic convergence. We note that for the proof it is crucial that the exponent in the estimate (3.136) is less than one.

Table 3.3 gives approximate values for the far field pattern in the forward and backward direction for scattering of a plane wave with incident direction $d = (1, 0, 0)$ from a pinched ball with representation

$$r(\theta, \varphi) = \sqrt{1.44 + 0.5 \cos 2\varphi(\cos 2\theta - 1)}, \quad \theta \in [0, \pi],\ \varphi \in [0, 2\pi]$$

in polar coordinates. (For the shape of the pinched ball see Fig. 5.3.) The results were provided to us by Olha Ivanyshyn and obtained by using the combined double- and single-layer potential integral equation (3.29) with coupling parameter $\eta = k$ and applying the Graham and Sloan variant (3.146) of Wienert's method with $N' = 2N$. The rapid convergence behavior is clearly exhibited. For further numerical examples we refer to [143].

Table 3.3 Numerical results for Wienert's method

	N	$\mathrm{Re}\, u_\infty(d)$	$\mathrm{Im}\, u_\infty(d)$	$\mathrm{Re}\, u_\infty(-d)$	$\mathrm{Im}\, u_\infty(-d)$
$k = 1$	8	−1.43201720	1.40315084	0.30954060	0.93110842
	16	−1.43218545	1.40328665	0.30945849	0.93112270
	32	−1.43218759	1.40328836	0.30945756	0.93112274
	64	−1.43218759	1.40328836	0.30945756	0.93112274
$k = 5$	8	−1.73274564	5.80039242	1.86060183	0.92743363
	16	−2.10055735	5.86052809	1.56336545	1.07513529
	32	−2.10058191	5.86053941	1.56328188	1.07513840
	64	−2.10058191	5.86053942	1.56328188	1.07513841