Chapter 2
The Helmholtz Equation

Studying an inverse problem always requires a solid knowledge of the theory for the corresponding direct problem. Therefore, the following two chapters of our book are devoted to presenting the foundations of obstacle scattering problems for time-harmonic acoustic waves, i.e., to exterior boundary value problems for the scalar Helmholtz equation. Our aim is to develop the analysis for the direct problems to an extent which is needed in the subsequent chapters on inverse problems.

In this chapter we begin with a brief discussion of the physical background to scattering problems. We will then derive the basic Green representation theorems for solutions to the Helmholtz equation. Discussing the concept of the Sommerfeld radiation condition will already enable us to introduce the idea of the far field pattern which is of central importance in our book. For a deeper understanding of these ideas, we require sufficient information on spherical wave functions. Therefore, we present in two sections those basic properties of spherical harmonics and spherical Bessel functions that are relevant in scattering theory. We will then be able to derive uniqueness results and expansion theorems for solutions to the Helmholtz equation with respect to spherical wave functions. We also will gain a first insight into the ill-posedness of the inverse problem by examining the smoothness properties of the far field pattern. The study of the boundary value problems will be the subject of the next chapter.

2.1 Acoustic Waves

Consider the propagation of sound waves of small amplitude in a homogeneous isotropic medium in $\mathbb{R}^3$ viewed as an inviscid fluid. Let $v = v(x, t)$ be the velocity field and let $p = p(x, t)$, $\rho = \rho(x, t)$ and $S = S(x, t)$ denote the pressure, density, and specific entropy, respectively, of the fluid. The motion is then governed by *Euler's equation*

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$$\frac{\partial v}{\partial t} + (v \cdot \operatorname{grad})\, v + \frac{1}{\rho}\; \operatorname{grad} p = 0,$$

the *equation of continuity*

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0,$$

the *state equation*

$$p = f(\rho, S),$$

and the *adiabatic hypothesis*

$$\frac{\partial S}{\partial t} + v \cdot \operatorname{grad} S = 0,$$

where f is a function depending on the nature of the fluid. We assume that v, p, ρ, and S are small perturbations of the static state $v_0 = 0$, $p_0 = \text{constant}$, $\rho_0 = \text{constant}$, and $S_0 = \text{constant}$ and linearize to obtain the linearized Euler equation

$$\frac{\partial v}{\partial t} + \frac{1}{\rho_0}\; \operatorname{grad} p = 0,$$

the linearized equation of continuity

$$\frac{\partial \rho}{\partial t} + \rho_0 \operatorname{div} v = 0,$$

and the linearized state equation

$$\frac{\partial p}{\partial t} = \frac{\partial f}{\partial \rho}\,(\rho_0, S_0)\; \frac{\partial \rho}{\partial t}\; .$$

From this we obtain the *wave equation*

$$\frac{1}{c^2}\,\frac{\partial^2 p}{\partial t^2} = \Delta p$$

where the *speed of sound* c is defined by

$$c^2 = \frac{\partial f}{\partial \rho}\,(\rho_0, S_0).$$

From the linearized Euler equation, we observe that there exists a velocity potential $U = U(x, t)$ such that

$$v = \frac{1}{\rho_0}\,\operatorname{grad} U$$

and

$$p = -\frac{\partial U}{\partial t}\,.$$

Clearly, the velocity potential also satisfies the wave equation

$$\frac{1}{c^2}\,\frac{\partial^2 U}{\partial t^2} = \Delta U.$$

For time-harmonic acoustic waves of the form

$$U(x,t) = \operatorname{Re}\left\{u(x)\,e^{-i\omega t}\right\}$$

with frequency $\omega > 0$, we deduce that the complex valued space dependent part u satisfies the *reduced wave equation* or *Helmholtz equation*

$$\Delta u + k^2 u = 0$$

where the *wave number* k is given by the positive constant $k = \omega/c$. This equation carries the name of the physicist Hermann Ludwig Ferdinand von Helmholtz (1821–1894) for his contributions to mathematical acoustics and electromagnetics.

In the first part of this book we will be concerned with the scattering of time-harmonic waves by obstacles surrounded by a homogeneous medium, i.e., with exterior boundary value problems for the Helmholtz equation. However, studying the Helmholtz equation in some detail is also required for the second part of our book where we consider wave scattering from an inhomogeneous medium since we always will assume that the medium is homogeneous outside some sufficiently large sphere.

In obstacle scattering we must distinguish between the two cases of impenetrable and penetrable objects. For a *sound-soft* obstacle the pressure of the total wave vanishes on the boundary. Consider the scattering of a given incoming wave u^i by a sound-soft obstacle D. Then the total wave $u = u^i + u^s$, where u^s denotes the scattered wave, must satisfy the wave equation in the exterior $\mathbb{R}^3 \setminus \bar{D}$ of D and a Dirichlet boundary condition $u = 0$ on ∂D. Similarly, the scattering from *sound-hard* obstacles leads to a Neumann boundary condition $\partial u/\partial \nu = 0$ on ∂D where ν is the unit outward normal to ∂D since here the normal velocity of the acoustic wave vanishes on the boundary. More generally, allowing obstacles for which the normal velocity on the boundary is proportional to the excess pressure on the boundary leads to an *impedance boundary condition* of the form

$$\frac{\partial u}{\partial \nu} + ik\lambda u = 0 \quad \text{on } \partial D$$

with a positive constant λ.

The scattering by a penetrable obstacle D with constant density ρ_D and speed of sound c_D differing from the density ρ and speed of sound c in the surrounding medium $\mathbb{R}^3 \setminus \bar{D}$ leads to a transmission problem. Here, in addition to the superposition $u = u^i + u^s$ of the incoming wave u^i and the scattered wave u^s in $\mathbb{R}^3 \setminus \bar{D}$ satisfying the Helmholtz equation with wave number $k = \omega/c$, we also have a transmitted wave v in D satisfying the Helmholtz equation with wave number $k_D = \omega/c_D \neq k$. The continuity of the pressure and of the normal velocity across the interface leads to the *transmission conditions*

$$u = v, \quad \frac{1}{\rho}\frac{\partial u}{\partial \nu} = \frac{1}{\rho_D}\frac{\partial v}{\partial \nu} \quad \text{on } \partial D.$$

In addition to the transmission conditions, more general *resistive boundary conditions* have been introduced and applied. For their description and treatment we refer to [14].

In order to avoid repeating ourselves by considering all possible types of boundary conditions, we have decided to confine ourselves to working out the basic ideas only for the case of a sound-soft obstacle. On occasion, we will mention modifications and extensions to the other cases.

For the scattered wave u^s, the radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \quad r = |x|,$$

introduced by Sommerfeld [397] in 1912 will ensure uniqueness for the solutions to the scattering problems. From the two possible spherically symmetric solutions

$$\frac{e^{ik|x|}}{|x|} \quad \text{and} \quad \frac{e^{-ik|x|}}{|x|}$$

to the Helmholtz equation, only the first one satisfies the radiation condition. Since via

$$\text{Re}\left\{\frac{e^{ik|x|-i\omega t}}{|x|}\right\} = \frac{\cos(k|x| - \omega t)}{|x|}$$

this corresponds to an outgoing spherical wave, we observe that physically speaking the *Sommerfeld radiation condition* characterizes outgoing waves. Throughout the book by $|x|$ we denote the Euclidean norm of a point x in $\mathbb{R}^3$.

For more details on the physical background of linear acoustic waves, we refer to the article by Morse and Ingard [326] in the Encyclopedia of Physics and to Jones [224] and Werner [421].

2.2 Green's Theorem and Formula

We begin by giving a brief outline of some basic properties of solutions to the Helmholtz equation $\Delta u + k^2 u = 0$ with positive wave number k. Most of these can be deduced from the *fundamental solution*

$$\Phi(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y. \tag{2.1}$$

Straightforward differentiation shows that for fixed $y \in \mathbb{R}^3$ the fundamental solution satisfies the Helmholtz equation in $\mathbb{R}^3 \setminus \{y\}$.

A domain $D \subset \mathbb{R}^3$, i.e., an open and connected set, is said to be of *class* C^k, $k \in \mathbb{N}$, if for each point z of the boundary ∂D there exists a neighborhood V_z of z with the following properties: the intersection $V_z \cap \bar{D}$ can be mapped bijectively onto the half ball $\{x \in \mathbb{R}^3 : |x| < 1, \ x_3 \geq 0\}$, this mapping and its inverse are k-times continuously differentiable and the intersection $V_z \cap \partial D$ is mapped onto the disk $\{x \in \mathbb{R}^3 : |x| < 1, \ x_3 = 0\}$. On occasion, we will express the property of a domain D to be of class C^k also by saying that its boundary ∂D is of class C^k. By $C^k(D)$ we denote the linear space of real or complex valued functions defined on the domain D which are k-times continuously differentiable. By $C^k(\bar{D})$ we denote the subspace of all functions in $C^k(D)$ which together with all their derivatives up to order k can be extended continuously from D into the closure $\bar{D}$.

One of the basic tools in studying the Helmholtz equation is provided by Green's integral theorems. Let D be a bounded domain of class C^1 and let ν denote the unit normal vector to the boundary ∂D directed into the exterior of D. Then, for $u \in C^1(\bar{D})$ and $v \in C^2(\bar{D})$ we have *Green's first theorem*

$$\int_D (u \Delta v + \operatorname{grad} u \cdot \operatorname{grad} v)\, dx = \int_{\partial D} u \frac{\partial v}{\partial \nu}\, ds, \tag{2.2}$$

and for $u, v \in C^2(\bar{D})$ we have *Green's second theorem*

$$\int_D (u \Delta v - v \Delta u)\, dx = \int_{\partial D} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) ds. \tag{2.3}$$

For two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ or $\mathbb{C}^3$ we will denote by $a \cdot b := a_1 b_1 + a_2 b_2 + a_3 b_3$ the bilinear scalar product and by $|a| := \sqrt{a \cdot \bar{a}}$ the Euclidean norm. For complex numbers or vectors the bar indicates the complex conjugate. Note that our regularity assumptions on D are sufficient conditions for the validity of Green's theorems and can be weakened (see Kellogg [230]).

Theorem 2.1 *Let D be a bounded domain of class C^2 and let ν denote the unit normal vector to the boundary ∂D directed into the exterior of D. Assume that $u \in C^2(D) \cap C(\bar{D})$ is a function which possesses a normal derivative on the boundary in the sense that the limit*

$$\frac{\partial u}{\partial \nu}(x) = \lim_{h \to +0} \nu(x) \cdot \operatorname{grad} u(x - h\nu(x)), \quad x \in \partial D,$$

exists uniformly on ∂D. *Then we have* Green's formula

$$\begin{aligned} u(x) = &\int_{\partial D} \left\{ \frac{\partial u}{\partial \nu}(y)\, \Phi(x, y) - u(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y) \\ &- \int_D \left\{ \Delta u(y) + k^2 u(y) \right\} \Phi(x, y)\, dy, \quad x \in D, \end{aligned} \tag{2.4}$$

where the volume integral exists as improper integral. In particular, if u *is a solution to the Helmholtz equation*

$$\Delta u + k^2 u = 0 \quad \text{in } D,$$

then

$$u(x) = \int_{\partial D} \left\{ \frac{\partial u}{\partial \nu}(y)\, \Phi(x, y) - u(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y), \quad x \in D. \tag{2.5}$$

Proof First, we assume that $u \in C^2(\bar{D})$. We circumscribe the arbitrary fixed point $x \in D$ with a sphere $S(x; \rho) := \{y \in \mathbb{R}^3 : |x - y| = \rho\}$ contained in D and direct the unit normal ν to $S(x; \rho)$ into the interior of $S(x; \rho)$. We now apply Green's theorem (2.3) to the functions u and $\Phi(x, \cdot)$ in the domain $D_\rho := \{y \in D : |x - y| > \rho\}$ to obtain

$$\begin{aligned} &\int_{\partial D \cup S(x;\rho)} \left\{ \frac{\partial u}{\partial \nu}(y)\, \Phi(x, y) - u(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y) \\ &= \int_{D_\rho} \left\{ \Delta u(y) + k^2 u(y) \right\} \Phi(x, y)\, dy. \end{aligned} \tag{2.6}$$

Since on $S(x; \rho)$ we have

$$\Phi(x, y) = \frac{e^{ik\rho}}{4\pi\rho}$$

and

$$\operatorname{grad}_y \Phi(x, y) = \left(\frac{1}{\rho} - ik \right) \frac{e^{ik\rho}}{4\pi\rho}\, \nu(y),$$

a straightforward calculation, using the mean value theorem, shows that

$$\lim_{\rho\to 0}\int_{S(x;\rho)}\left\{u(y)\,\frac{\partial\Phi(x,y)}{\partial\nu(y)}-\frac{\partial u}{\partial\nu}\,(y)\,\Phi(x,y)\right\}ds(y)=u(x),$$

whence (2.4) follows by passing to the limit $\rho\to 0$ in (2.6). The existence of the volume integral as an improper integral is a consequence of the fact that its integrand is weakly singular.

The case where u belongs only to $C^2(D)\cap C(\bar{D})$ and has a normal derivative in the sense of uniform convergence is treated by first integrating over parallel surfaces to the boundary of D and then passing to the limit ∂D. For the concept of parallel surfaces, we refer to [104, 268, 311]. We note that the parallel surfaces for $\partial D\in C^2$ belong to C^1. □

In the literature, Green's formula (2.5) is also known as the *Helmholtz representation*. Obviously, Theorem 2.1 remains valid for complex values of k.

Theorem 2.2 *If u is a two times continuously differentiable solution to the Helmholtz equation in a domain D, then u is analytic.*

Proof Let $x\in D$ and choose a closed ball contained in D with center x. Then Theorem 2.1 can be applied in this ball and the statement follows from the analyticity of the fundamental solution for $x\neq y$. □

As a consequence of Theorem 2.2, a solution to the Helmholtz equation that vanishes in an open subset of its domain of definition must vanish everywhere.

In the sequel, by saying u is a solution to the Helmholtz equation we tacitly imply that u is twice continuously differentiable, and hence analytic, in the interior of its domain of definition.

The following theorem is a special case of a more general result for partial differential equations known as *Holmgren's theorem*.

Theorem 2.3 *Let D be as in Theorem 2.1 and let $u\in C^2(D)\cap C^1(\bar{D})$ be a solution to the Helmholtz equation in D such that*

$$u=\frac{\partial u}{\partial\nu}=0\quad\text{on }\Gamma \tag{2.7}$$

for some open subset $\Gamma\subset\partial D$. Then u vanishes identically in D.

Proof In view of (2.7), we use Green's representation formula (2.5) to extend the definition of u by setting

$$u(x):=\int_{\partial D\setminus\Gamma}\left\{\frac{\partial u}{\partial\nu}\,(y)\,\Phi(x,y)-u(y)\,\frac{\partial\Phi(x,y)}{\partial\nu(y)}\right\}ds(y)$$

for $x\in(\mathbb{R}^3\setminus\bar{D})\cup\Gamma$. Then, by Green's second integral theorem (2.3), applied to u and $\Phi(x,\cdot)$, we have $u=0$ in $\mathbb{R}^3\setminus\bar{D}$. By G we denote a component of $\mathbb{R}^3\setminus\bar{D}$ with $\Gamma\cap\partial G\neq\emptyset$. Clearly u solves the Helmholtz equation in $(\mathbb{R}^3\setminus\partial D)\cup\Gamma$ and therefore $u=0$ in D, since D and G are connected through the gap Γ in ∂D. □

Definition 2.4 A solution u to the Helmholtz equation whose domain of definition contains the exterior of some sphere is called *radiating* if it satisfies the *Sommerfeld radiation condition*

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0 \tag{2.8}$$

where $r = |x|$ and the limit is assumed to hold uniformly in all directions $x/|x|$.

Theorem 2.5 *Assume the bounded set D is the open complement of an unbounded domain of class C^2 and let ν denote the unit normal vector to the boundary ∂D directed into the exterior of D. Let $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ be a radiating solution to the Helmholtz equation*

$$\Delta u + k^2 u = 0 \quad in\ \mathbb{R}^3 \setminus \bar{D},$$

which possesses a normal derivative on the boundary in the sense that the limit

$$\frac{\partial u}{\partial \nu}(x) = \lim_{h\to+0} \nu(x)\cdot \operatorname{grad} u(x + h\nu(x)), \quad x \in \partial D,$$

exists uniformly on ∂D. Then we have Green's formula

$$u(x) = \int_{\partial D} \left\{ u(y)\,\frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y)\,\Phi(x,y)\right\} ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}. \tag{2.9}$$

Proof We first show that

$$\int_{S_r} |u|^2 ds = O(1), \quad r \to \infty, \tag{2.10}$$

where S_r denotes the sphere of radius r and center at the origin. To accomplish this, we observe that from the radiation condition (2.8) it follows that

$$\int_{S_r} \left\{ \left|\frac{\partial u}{\partial \nu}\right|^2 + k^2|u|^2 + 2k\ \operatorname{Im}\left(u\,\frac{\partial \bar{u}}{\partial \nu}\right)\right\} ds = \int_{S_r} \left|\frac{\partial u}{\partial \nu} - iku\right|^2 ds \to 0, \quad r \to \infty,$$

where ν is the unit outward normal to S_r. We take r large enough such that D is contained in S_r and apply Green's theorem (2.2) in $D_r := \{y \in \mathbb{R}^3 \setminus \bar{D} : |y| < r\}$ to obtain

$$\int_{S_r} u\,\frac{\partial \bar{u}}{\partial \nu}\,ds = \int_{\partial D} u\,\frac{\partial \bar{u}}{\partial \nu}\,ds - k^2 \int_{D_r} |u|^2 dy + \int_{D_r} |\operatorname{grad} u|^2 dy.$$

We now insert the imaginary part of the last equation into the previous equation and find that

$$\lim_{r\to\infty} \int_{S_r} \left\{ \left| \frac{\partial u}{\partial \nu} \right|^2 + k^2 |u|^2 \right\} ds = -2k \operatorname{Im} \int_{\partial D} u \frac{\partial \bar{u}}{\partial \nu} ds. \tag{2.11}$$

Both terms on the left-hand side of (2.11) are nonnegative. Hence, they must be individually bounded as $r \to \infty$ since their sum tends to a finite limit. Therefore, (2.10) is proven.

Now from (2.10) and the radiation condition

$$\frac{\partial \Phi(x, y)}{\partial \nu(y)} - ik\Phi(x, y) = O\left(\frac{1}{r^2}\right), \quad r \to \infty,$$

which is valid uniformly for $y \in S_r$, by the Cauchy–Schwarz inequality we see that

$$I_1 := \int_{S_r} u(y) \left\{ \frac{\partial \Phi(x, y)}{\partial \nu(y)} - ik\Phi(x, y) \right\} ds(y) \to 0, \quad r \to \infty,$$

and the radiation condition (2.8) for u and $\Phi(x, y) = O(1/r)$ for $y \in S_r$ yield

$$I_2 := \int_{S_r} \Phi(x, y) \left\{ \frac{\partial u}{\partial \nu}(y) - iku(y) \right\} ds(y) \to 0, \quad r \to \infty.$$

Hence,

$$\int_{S_r} \left\{ u(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y)\, \Phi(x, y) \right\} ds(y) = I_1 - I_2 \to 0, \quad r \to \infty.$$

The proof is now completed by applying Theorem 2.1 in the bounded domain D_r and passing to the limit $r \to \infty$. □

From Theorem 2.5 we deduce that radiating solutions u to the Helmholtz equation automatically satisfy Sommerfeld's finiteness condition

$$u(x) = O\left(\frac{1}{|x|}\right), \quad |x| \to \infty, \tag{2.12}$$

uniformly for all directions and that the validity of the Sommerfeld radiation condition (2.8) is invariant under translations of the origin. Wilcox [428] first established that the representation formula (2.9) can be derived without the additional condition (2.12) of finiteness. Our proof of Theorem 2.5 has followed Wilcox's proof. It also shows that (2.8) can be replaced by the weaker formulation

$$\int_{S_r} \left| \frac{\partial u}{\partial r} - iku \right|^2 ds \to 0, \quad r \to \infty,$$

with (2.9) still being valid. Of course, (2.9) then implies that (2.8) also holds.

Solutions to the Helmholtz equation which are defined in all of $\mathbb{R}^3$ are called *entire solutions*. An entire solution to the Helmholtz equation satisfying the radiation condition must vanish identically. This follows immediately from combining Green's formula (2.9) and Green's theorem (2.3).

We are now in a position to introduce the definition of the *far field pattern* or the *scattering amplitude* which plays a central role in this book.

Theorem 2.6 *Every radiating solution u to the Helmholtz equation has the asymptotic behavior of an outgoing spherical wave*

$$u(x) = \frac{e^{ik|x|}}{|x|}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty, \tag{2.13}$$

uniformly in all directions $\hat{x} = x/|x|$ *where the function* u_∞ *defined on the unit sphere* $\mathbb{S}^2$ *is known as the* far field pattern *of u. Under the assumptions of Theorem 2.5 we have*

$$u_\infty(\hat{x}) = \frac{1}{4\pi}\int_{\partial D}\left\{u(y)\,\frac{\partial e^{-ik\,\hat{x}\cdot y}}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y)\,e^{-ik\,\hat{x}\cdot y}\right\}ds(y), \quad \hat{x} \in \mathbb{S}^2. \tag{2.14}$$

Proof From

$$|x-y| = \sqrt{|x|^2 - 2\,x\cdot y + |y|^2} = |x| - \hat{x}\cdot y + O\left(\frac{1}{|x|}\right),$$

we derive

$$\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|x|}}{|x|}\left\{e^{-ik\,\hat{x}\cdot y} + O\left(\frac{1}{|x|}\right)\right\}, \tag{2.15}$$

and

$$\frac{\partial}{\partial \nu(y)}\,\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|x|}}{|x|}\left\{\frac{\partial e^{-ik\,\hat{x}\cdot y}}{\partial \nu(y)} + O\left(\frac{1}{|x|}\right)\right\} \tag{2.16}$$

uniformly for all $y \in \partial D$. Inserting this into Green's formula (2.9), the theorem follows. □

One of the main themes of our book will be to recover radiating solutions of the Helmholtz equation from a knowledge of their far field patterns. In terms of the mapping $A : u \mapsto u_\infty$ transferring the radiating solution u into its far field pattern u_∞, we want to solve the equation $Au = u_\infty$ for a given u_∞. In order to establish uniqueness for determining u from its far field pattern u_∞ and to understand the strong ill-posedness of the equation $Au = u_\infty$, we need to develop some facts on spherical wave functions. This will be the subject of the next two sections. We already can point out that the mapping A is extremely smoothing since from (2.14) we see that the far field pattern is an analytic function on the unit sphere.

2.3 Spherical Harmonics

For convenience and to introduce notations, we summarize some of the basic properties of spherical harmonics which are relevant in scattering theory and briefly indicate their proofs. For a more detailed study we refer to Lebedev [293].

Recall that solutions u to the Laplace equation $\Delta u = 0$ are called harmonic functions. The restriction of a homogeneous harmonic polynomial of degree n to the unit sphere $\mathbb{S}^2$ is called a *spherical harmonic* of order n.

Theorem 2.7 *There exist exactly* $2n + 1$ *linearly independent spherical harmonics of order* n*.*

Proof By the maximum–minimum principle for harmonic functions it suffices to show that there exist exactly $2n + 1$ linearly independent homogeneous harmonic polynomials H_n of degree n. We can write

$$H_n(x_1, x_2, x_3) = \sum_{k=0}^{n} a_{n-k}(x_1, x_2)\, x_3^k,$$

where the a_k are homogeneous polynomials of degree k in the two variables x_1 and x_2. Then, straightforward calculations show that H_n is harmonic if and only if the coefficients satisfy

$$a_{n-k} = -\frac{\Delta a_{n-k+2}}{k(k-1)}\,, \quad k = 2, \dots, n.$$

Therefore, choosing the two coefficients a_n and a_{n-1} uniquely determines H_n, and by setting

$$a_n(x_1, x_2) = x_1^{n-j} x_2^j, \quad a_{n-1}(x_1, x_2) = 0, \quad j = 0, \dots, n,$$

$$a_n(x_1, x_2) = 0, \quad a_{n-1}(x_1, x_2) = x_1^{n-1-j} x_2^j, \quad j = 0, \dots, n-1,$$

clearly we obtain $2n + 1$ linearly independent homogeneous harmonic polynomials of degree n. □

In principle, the proof of the preceding theorem allows a construction of all spherical harmonics. However, it is more convenient and appropriate to use polar coordinates for the representation of spherical harmonics. In polar coordinates (r, θ, φ), homogeneous polynomials clearly are of the form

$$H_n = r^n Y_n(\theta, \varphi),$$

and $\Delta H_n = 0$ is readily seen to be satisfied if

$$\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\,\sin\theta\,\frac{\partial Y_n}{\partial\theta}+\frac{1}{\sin^2\theta}\,\frac{\partial^2 Y_n}{\partial\varphi^2}+n(n+1)Y_n=0. \tag{2.17}$$

From Green's theorem (2.3), applied to two homogeneous harmonic polynomials H_n and $H_{n'}$, we have

$$0=\int_{\mathbb{S}^2}\left\{\overline{H}_{n'}\,\frac{\partial H_n}{\partial r}-H_n\,\frac{\partial \overline{H}_{n'}}{\partial r}\right\}ds=(n-n')\int_{\mathbb{S}^2}Y_n\overline{Y}_{n'}\,ds.$$

Therefore spherical harmonics satisfy the orthogonality relation

$$\int_{\mathbb{S}^2}Y_n\overline{Y}_{n'}\,ds=0,\quad n\neq n'. \tag{2.18}$$

We first construct spherical harmonics which only depend on the polar angle θ. Choose points x and y with $r=|x|<|y|=1$, denote the angle between x and y by θ, and set $t=\cos\theta$. Consider the function

$$\frac{1}{|x-y|}=\frac{1}{\sqrt{1-2tr+r^2}} \tag{2.19}$$

which for fixed y is a solution to Laplace's equation with respect to x. Since for fixed t with $-1\le t\le 1$ the right-hand side is an analytic function in r, we have the Taylor series

$$\frac{1}{\sqrt{1-2tr+r^2}}=\sum_{n=0}^{\infty}P_n(t)r^n. \tag{2.20}$$

The coefficients P_n in this expansion are called *Legendre polynomials* and the function on the left-hand side consequently is known as the *generating function* for the Legendre polynomials. For each $0<r_0<1$ the Taylor series

$$\frac{1}{\sqrt{1-r\exp(\pm i\theta)}}=1+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\,r^n\,e^{\pm in\theta}, \tag{2.21}$$

and all its term by term derivatives with respect to r and θ are absolutely and uniformly convergent for all $0\le r\le r_0$ and all $0\le\theta\le\pi$. Hence, by multiplying the Eq. (2.21) for the plus and the minus sign, we note that the series (2.20) and all its term by term derivatives with respect to r and θ are absolutely and uniformly convergent for all $0\le r\le r_0$ and all $-1\le t=\cos\theta\le 1$. Setting $\theta=0$ in (2.21) obviously provides a majorant for the series for all θ. Therefore, the geometric series is a majorant for the series in (2.20) and we obtain the inequality

$$|P_n(t)| \le 1, \quad -1 \le t \le 1, \quad n = 0, 1, 2, \dots. \tag{2.22}$$

Differentiating (2.20) with respect to r, multiplying by $1 - 2tr + r^2$, inserting (2.20) on the left-hand side, and then equating powers of r shows that the P_n satisfy the recursion formula

$$(n+1)P_{n+1}(t) - (2n+1)tP_n(t) + nP_{n-1}(t) = 0, \quad n = 1, 2, \dots. \tag{2.23}$$

Since, as easily seen from (2.20), we have $P_0(t) = 1$ and $P_1(t) = t$, the recursion formula shows that P_n indeed is a polynomial of degree n and that P_n is an even function if n is even and an odd function if n is odd.

Since for fixed y the function (2.19) is harmonic, differentiating (2.20) term by term, we obtain that

$$\sum_{n=0}^{\infty} \left\{ \frac{1}{\sin\theta} \frac{d}{d\theta} \sin\theta \frac{dP_n(\cos\theta)}{d\theta} + n(n+1)P_n(\cos\theta) \right\} r^{n-2} = 0.$$

Equating powers of r shows that the Legendre polynomials satisfy the *Legendre differential equation*

$$(1-t^2)P_n''(t) - 2tP_n'(t) + n(n+1)P_n(t) = 0, \quad n = 0, 1, 2, \dots, \tag{2.24}$$

and that the homogeneous polynomial $r^n P_n(\cos\theta)$ of degree n is harmonic. Therefore, $P_n(\cos\theta)$ represents a spherical harmonic of order n. The orthogonality (2.18) implies that

$$\int_{-1}^{1} P_n(t)P_{n'}(t)\, dt = 0, \quad n \ne n'.$$

Since we have uniform convergence, we may integrate the square of the generating function (2.20) term by term and use the preceding orthogonality to arrive at

$$\int_{-1}^{1} \frac{dt}{1 - 2tr + r^2} = \sum_{n=0}^{\infty} \int_{-1}^{1} [P_n(t)]^2\, dt\, r^{2n}.$$

On the other hand, we have

$$\int_{-1}^{1} \frac{dt}{1 - 2tr + r^2} = \frac{1}{r} \ln \frac{1+r}{1-r} = \sum_{n=0}^{\infty} \frac{2}{2n+1} r^{2n}.$$

Thus, we have proven the orthonormality relation

$$\int_{-1}^{1} P_n(t)P_m(t)\, dt = \frac{2}{2n+1}\, \delta_{nm}, \quad n, m = 0, 1, 2, \dots, \tag{2.25}$$

with the usual meaning for the Kronecker symbol δ_{nm}. Since $\text{span}\{P_0, \ldots, P_n\} = \text{span}\{1, \ldots, t^n\}$ the Legendre polynomials P_n, $n = 0, 1, \ldots,$ form a complete orthogonal system in $L^2[-1, 1]$.

We now look for spherical harmonics of the form

$$Y_n^m(\theta, \varphi) = f(\cos\theta)\, e^{im\varphi}.$$

Then (2.17) is satisfied provided f is a solution of the *associated Legendre differential equation*

$$(1 - t^2) f''(t) - 2t f'(t) + \left\{ n(n+1) - \frac{m^2}{1 - t^2} \right\} f(t) = 0. \tag{2.26}$$

Differentiating the Legendre differential equation (2.24) m-times shows that $g = P_n^{(m)}$ satisfies

$$(1 - t^2) g''(t) - 2(m+1) t g'(t) + (n - m)(n + m + 1) g(t) = 0.$$

From this it can be deduced that the *associated Legendre functions*

$$P_n^m(t) := (1 - t^2)^{m/2} \frac{d^m P_n(t)}{dt^m}, \quad m = 0, 1, \ldots, n, \tag{2.27}$$

solve the associated Legendre equation for $n = 0, 1, 2, \ldots$. In order to make sure that the functions $Y_n^m(\theta, \varphi) = P_n^m(\cos\theta)\, e^{im\varphi}$ are spherical harmonics, we have to prove that the harmonic functions $r^n Y_n^m(\theta, \varphi) = r^n P_n^m(\cos\theta)\, e^{im\varphi}$ are homogeneous polynomials of degree n. From the recursion formula (2.23) for the P_n and the definition (2.27) for the P_n^m, we first observe that

$$P_n^m(\cos\theta) = \sin^m\theta\, u_n^m(\cos\theta)$$

where u_n^m is a polynomial of degree $n - m$ which is even if $n - m$ is even and odd if $n - m$ is odd. Since in polar coordinates we have

$$r^m \sin^m\theta\, e^{im\varphi} = (x_1 + i x_2)^m,$$

it follows that

$$r^n\, Y_n^m(\theta, \varphi) = (x_1 + i x_2)^m\, r^{n-m}\, u_n^m(\cos\theta).$$

For $n - m$ even we can write

$$r^{n-m}\, u_n^m(\cos\theta) = r^{n-m} \sum_{k=0}^{\frac{1}{2}(n-m)} a_k \cos^{2k}\theta = \sum_{k=0}^{\frac{1}{2}(n-m)} a_k\, x_3^{2k}\, (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}(n-m)-k}$$

which is a homogeneous polynomial of degree $n-m$ and this is also true for $n-m$ odd. Putting everything together, we see that the $r^n\, Y_n^m(\theta,\varphi)$ are homogeneous polynomials of degree n.

Theorem 2.8 *The spherical harmonics*

$$Y_n^m(\theta,\varphi) := \sqrt{\frac{2n+1}{4\pi}\,\frac{(n-|m|)!}{(n+|m|)!}}\; P_n^{|m|}(\cos\theta)\, e^{im\varphi} \tag{2.28}$$

for $m = -n,\dots,n,\quad n = 0,1,2,\dots,$ *form a complete orthonormal system in* $L^2(\mathbb{S}^2)$.

Proof Because of (2.18) and the orthogonality of the $e^{im\varphi}$, the Y_n^m given by (2.28) are orthogonal. For $m > 0$ we evaluate

$$A_n^m := \int_0^\pi [P_n^m(\cos\theta)]^2 \sin\theta\, d\theta$$

by m partial integrations to get

$$A_n^m = \int_{-1}^1 (1-t^2)^m \left[\frac{d^m P_n(t)}{dt^m}\right]^2 dt = \int_{-1}^1 P_n(t)\,\frac{d^m}{dt^m}\, g_n^m(t)\, dt,$$

where

$$g_n^m(t) = (t^2-1)^m\, \frac{d^m P_n(t)}{dt^m}\ .$$

Hence

$$\frac{d^m}{dt^m}\, g_n^m(t) = \frac{(n+m)!}{(n-m)!}\, a_n t^n + \cdots$$

is a polynomial of degree n with a_n the leading coefficient in $P_n(t) = a_n t^n + \cdots$. Therefore, by the orthogonality (2.25) of the Legendre polynomials we derive

$$\frac{(n-m)!}{(n+m)!}\, A_n^m = \int_{-1}^1 a_n t^n\, P_n(t)\, dt = \int_{-1}^1 [P_n(t)]^2\, dt = \frac{2}{2n+1}\ ,$$

and the proof of the orthonormality of the Y_n^m is finished.

For fixed m the associated Legendre functions P_n^m for $n = m, m+1, \dots$ are orthogonal and they are complete in $L^2[-1,1]$ since we have

$$\operatorname{span}\left\{P_m^m,\dots,P_{m+n}^m\right\} = (1-t^2)^{m/2}\ \operatorname{span}\left\{1,\dots,t^n\right\}.$$

Writing $Y := \operatorname{span}\left\{Y_n^m : m = -n, \ldots, n,\ n = 0, 1, 2, \ldots\right\}$, it remains to show that Y is dense in $L^2(\mathbb{S}^2)$. Let $g \in C(\mathbb{S}^2)$. For fixed θ we then have Parseval's equality

$$2\pi \sum_{m=-\infty}^{\infty} |g_m(\theta)|^2 = \int_0^{2\pi} |g(\theta, \varphi)|^2 \, d\varphi \tag{2.29}$$

for the Fourier coefficients

$$g_m(\theta) = \frac{1}{2\pi} \int_0^{2\pi} g(\theta, \varphi) e^{-im\varphi} \, d\varphi$$

with respect to φ. Since the g_m and the right-hand side of (2.29) are continuous in θ, by Dini's theorem the convergence in (2.29) is uniform with respect to θ. Therefore, given $\varepsilon > 0$ there exists $M = M(\varepsilon) \in \mathbb{N}$ such that

$$\int_0^{2\pi} \left| g(\theta, \varphi) - \sum_{m=-M}^{M} g_m(\theta) e^{im\varphi} \right|^2 d\varphi = \int_0^{2\pi} |g(\theta, \varphi)|^2 \, d\varphi - 2\pi \sum_{m=-M}^{M} |g_m(\theta)|^2 < \frac{\varepsilon}{4\pi}$$

for all $0 \le \theta \le \pi$. The finite number of functions g_m, $m = -M, \ldots, M$, can now be simultaneously approximated by the associated Legendre functions, i.e., there exist $N = N(\varepsilon)$ and coefficients a_n^m such that

$$\int_0^{\pi} \left| g_m(\theta) - \sum_{n=|m|}^{N} a_n^m \, P_n^{|m|}(\cos\theta) \right|^2 \sin\theta \, d\theta < \frac{\varepsilon}{8\pi(2M+1)^2}$$

for all $m = -M, \ldots, M$. Then, combining the last two inequalities with the help of the Cauchy–Schwarz inequality, we find

$$\int_0^{\pi} \int_0^{2\pi} \left| g(\theta, \varphi) - \sum_{m=-M}^{M} \sum_{n=|m|}^{N} a_n^m \, P_n^{|m|}(\cos\theta) \, e^{im\varphi} \right|^2 \sin\theta \, d\varphi d\theta < \varepsilon.$$

Therefore, Y is dense in $C(\mathbb{S}^2)$ with respect to the L^2 norm and this completes the proof since $C(\mathbb{S}^2)$ is dense in $L^2(\mathbb{S}^2)$. □

We conclude our brief survey of spherical harmonics by proving the important *addition theorem.*

Theorem 2.9 *Let Y_n^m, $m = -n, \ldots, n$, be any system of $2n + 1$ orthonormal spherical harmonics of order n. Then for all $\hat{x}, \hat{y} \in \mathbb{S}^2$ we have*

$$\sum_{m=-n}^{n} Y_n^m(\hat{x})\,\overline{Y_n^m(\hat{y})} = \frac{2n+1}{4\pi}\,P_n(\cos\theta), \tag{2.30}$$

where θ denotes the angle between $\hat{x}$ and $\hat{y}$.

Proof We abbreviate the left-hand side of (2.30) by $Y(\hat{x}, \hat{y})$ and first show that Y only depends on the angle θ. Each orthogonal matrix Q in $\mathbb{R}^3$ transforms homogeneous harmonic polynomials of degree n again into homogeneous harmonic polynomials of degree n. Hence, we can write

$$Y_n^m(Q\hat{x}) = \sum_{k=-n}^{n} a_{mk} Y_n^k(\hat{x}), \quad m = -n, \dots, n.$$

Since Q is orthogonal and the Y_n^m are orthonormal, we have

$$\int_{\mathbb{S}^2} Y_n^m(Q\hat{x})\,\overline{Y_n^{m'}(Q\hat{x})}\,ds = \int_{\mathbb{S}^2} Y_n^m(\hat{x})\,\overline{Y_n^{m'}(\hat{x})}\,ds = \delta_{mm'}.$$

From this it can be seen that the matrix $A = (a_{mk})$ also is orthogonal and we obtain

$$Y(Q\hat{x}, Q\hat{y}) = \sum_{m=-n}^{n}\sum_{k=-n}^{n} a_{mk} Y_n^k(\hat{x}) \sum_{l=-n}^{n} \overline{a_{ml} Y_n^l(\hat{y})} = \sum_{k=-n}^{n} Y_n^k(\hat{x})\,\overline{Y_n^k(\hat{y})} = Y(\hat{x}, \hat{y})$$

whence $Y(\hat{x}, \hat{y}) = f(\cos\theta)$ follows. Since for fixed $\hat{y}$ the function Y is a spherical harmonic, by introducing polar coordinates with the polar axis given by $\hat{y}$ we see that $f = a_n P_n$ with some constant a_n. Hence, we have

$$\sum_{m=-n}^{n} Y_n^m(\hat{x})\,\overline{Y_n^m(\hat{y})} = a_n P_n(\cos\theta).$$

Setting $\hat{y} = \hat{x}$ and using $P_n(1) = 1$ (this follows from the generating function (2.20)) we obtain

$$a_n = \sum_{m=-n}^{n} |Y_n^m(\hat{x})|^2.$$

Since the Y_n^m are normalized, integrating the last equation over $\mathbb{S}^2$ we finally arrive at $4\pi a_n = 2n + 1$ and the proof is complete. □

2.4 Spherical Bessel Functions

We continue our study of spherical wave functions by introducing the basic properties of spherical Bessel functions. For a more detailed analysis we again refer to Lebedev [293].

We look for solutions to the Helmholtz equation of the form

$$u(x) = f(k|x|)\, Y_n\left(\frac{x}{|x|}\right),$$

where Y_n is a spherical harmonic of order n. From the differential equation (2.17) for the spherical harmonics, it follows that u solves the Helmholtz equation provided f is a solution of the *spherical Bessel differential equation*

$$t^2 f''(t) + 2tf'(t) + [t^2 - n(n+1)]f(t) = 0. \tag{2.31}$$

We note that for any solution f to the spherical Bessel differential equation (2.31) the function $g(t) := \sqrt{t}\, f(t)$ solves the Bessel differential equation with half integer order $n + 1/2$ and vice versa. By direct calculations, we see that for $n = 0, 1, \dots$ the functions

$$j_n(t) := \sum_{p=0}^{\infty} \frac{(-1)^p t^{n+2p}}{2^p p!\, 1 \cdot 3 \cdots (2n + 2p + 1)} \tag{2.32}$$

and

$$y_n(t) := -\frac{(2n)!}{2^n n!} \sum_{p=0}^{\infty} \frac{(-1)^p t^{2p-n-1}}{2^p p!(-2n+1)(-2n+3)\cdots(-2n+2p-1)} \tag{2.33}$$

represent solutions to the spherical Bessel differential equation (the first coefficient in the series (2.33) has to be set equal to one). By the ratio test, the function j_n is seen to be analytic for all $t \in \mathbb{R}$ whereas y_n is analytic for all $t \in (0, \infty)$. The functions j_n and y_n are called *spherical Bessel functions* and *spherical Neumann functions* of order n, respectively, and the linear combinations

$$h_n^{(1,2)} := j_n \pm i y_n$$

are known as *spherical Hankel functions* of the first and second kind of order n.

From the series representation (2.32) and (2.33), by equating powers of t, it is readily verified that both $f_n = j_n$ and $f_n = y_n$ satisfy the recurrence relation

$$f_{n+1}(t) + f_{n-1}(t) = \frac{2n+1}{t}\, f_n(t), \quad n = 1, 2, \dots. \tag{2.34}$$

Straightforward differentiation of the series (2.32) and (2.33) shows that both $f_n = j_n$ and $f_n = y_n$ satisfy the differentiation formulas

$$f_{n+1}(t) = -t^n \frac{d}{dt} \left\{t^{-n} f_n(t)\right\}, \quad n = 0, 1, 2, \dots, \tag{2.35}$$

and

$$t^{n+1} f_{n-1}(t) = \frac{d}{dt} \left\{t^{n+1} f_n(t)\right\}, \quad n = 1, 2, \dots. \tag{2.36}$$

Finally, from (2.31), the Wronskian

$$W(j_n(t), y_n(t)) := j_n(t)y_n'(t) - y_n(t)j_n'(t)$$

is readily seen to satisfy

$$W' + \frac{2}{t} W = 0,$$

whence $W(j_n(t), y_n(t)) = C/t^2$ for some constant C. This constant can be evaluated by passing to the limit $t \to 0$ with the result

$$j_n(t)y_n'(t) - j_n'(t)y_n(t) = \frac{1}{t^2}. \tag{2.37}$$

From the series representation of the spherical Bessel and Neumann functions, it is obvious that

$$j_n(t) = \frac{t^n}{1 \cdot 3 \cdots (2n+1)} \left(1 + O\left(\frac{1}{n}\right)\right), \quad n \to \infty, \tag{2.38}$$

uniformly on compact subsets of $\mathbb{R}$ and

$$h_n^{(1)}(t) = \frac{1 \cdot 3 \cdots (2n-1)}{i t^{n+1}} \left(1 + O\left(\frac{1}{n}\right)\right), \quad n \to \infty, \tag{2.39}$$

uniformly on compact subsets of $(0, \infty)$. With the aid of Stirling's formula $n! = \sqrt{2\pi n}\,(n/e)^n\,(1 + o(1))$, $n \to \infty$, which implies that

$$\frac{(2n)!}{n!} = 2^{2n+\frac{1}{2}} \left(\frac{n}{e}\right)^n (1 + o(1)), \quad n \to \infty,$$

from (2.39) we obtain

$$h_n^{(1)}(t) = O\left(\frac{2n}{et}\right)^n, \quad n \to \infty, \tag{2.40}$$

uniformly on compact subsets of $(0, \infty)$.

The spherical Bessel and Neumann functions can be expressed in terms of trigonometric functions. Setting $n = 0$ in the series (2.32) and (2.33) we have that

$$j_0(t) = \frac{\sin t}{t}, \quad y_0(t) = -\frac{\cos t}{t}$$

and consequently

$$h_0^{(1,2)}(t) = \frac{e^{\pm it}}{\pm it}. \tag{2.41}$$

Hence, by induction, from (2.41) and (2.35) it follows that the spherical Hankel functions are of the form

$$h_n^{(1)}(t) = (-i)^n \, \frac{e^{it}}{it} \left\{ 1 + \sum_{p=1}^{n} \frac{a_{pn}}{t^p} \right\}$$

and

$$h_n^{(2)}(t) = i^n \, \frac{e^{-it}}{-it} \left\{ 1 + \sum_{p=1}^{n} \frac{\bar{a}_{pn}}{t^p} \right\}$$

with complex coefficients $a_{1n}, \dots, a_{nn}$. From this we readily obtain the following asymptotic behavior of the spherical Hankel functions for large argument

$$\begin{aligned} h_n^{(1,2)}(t) &= \frac{1}{t} \, e^{\pm i\left(t - \frac{n\pi}{2} - \frac{\pi}{2}\right)} \left\{ 1 + O\left(\frac{1}{t}\right) \right\}, \quad t \to \infty, \\ h_n^{(1,2)\prime}(t) &= \frac{1}{t} \, e^{\pm i\left(t - \frac{n\pi}{2}\right)} \left\{ 1 + O\left(\frac{1}{t}\right) \right\}, \quad t \to \infty. \end{aligned} \tag{2.42}$$

Taking the real and the imaginary part of (2.42) we also have asymptotic formulas for the spherical Bessel and Neumann functions.

For solutions to the Helmholtz equation in polar coordinates, we can now state the following theorem on *spherical wave functions*.

Theorem 2.10 *Let Y_n be a spherical harmonic of order n. Then*

$$u_n(x) = j_n(k|x|) \, Y_n\left(\frac{x}{|x|}\right)$$

is an entire solution to the Helmholtz equation and

$$v_n(x) = h_n^{(1)}(k|x|)\, Y_n\left(\frac{x}{|x|}\right)$$

is a radiating solution to the Helmholtz equation in $\mathbb{R}^3 \setminus \{0\}$.

Proof Since we can write $j_n(kr) = k^n r^n w_n(r^2)$ with an analytic function $w_n : \mathbb{R} \to \mathbb{R}$ and since $r^n Y_n(\hat{x})$ is a homogeneous polynomial in x_1, x_2, x_3, the product $j_n(kr)\, Y_n(\hat{x})$ for $\hat{x} = x/|x|$ is regular at $x = 0$, i.e., u_n also satisfies the Helmholtz equation at the origin. That the radiation condition is satisfied for v_n follows from the asymptotic behavior (2.42) of the spherical Hankel functions of the first kind.

□

We conclude our brief discussion of spherical wave functions by the following *addition theorem* for the fundamental solution.

Theorem 2.11 *Let* Y_n^m, $m = -n, \dots, n$, $n = 0, 1, \dots,$ *be a set of orthonormal spherical harmonics. Then for* $|x| > |y|$ *we have*

$$\frac{e^{ik|x-y|}}{4\pi|x-y|} = ik \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_n^{(1)}(k|x|)\, Y_n^m\left(\frac{x}{|x|}\right) j_n(k|y|)\, \overline{Y_n^m\left(\frac{y}{|y|}\right)}. \tag{2.43}$$

The series and its term by term first derivatives with respect to $|x|$ *and* $|y|$ *are absolutely and uniformly convergent on compact subsets of* $|x| > |y|$.

Proof We abbreviate $\hat{x} = x/|x|$ and $\hat{y} = y/|y|$. From Green's theorem (2.3) applied to $u_n^m(z) = j_n(k|z|)\, Y_n^m(\hat{z})$ with $\hat{z} = z/|z|$ and $\Phi(x, z)$, we have

$$\int_{|z|=r} \left\{ u_n^m(z)\, \frac{\partial \Phi(x,z)}{\partial \nu(z)} - \frac{\partial u_n^m}{\partial \nu}(z)\, \Phi(x,z) \right\} ds(z) = 0, \quad |x| > r,$$

and from Green's formula (2.9), applied to $v_n^m(z) = h_n^{(1)}(k|z|)\, Y_n^m(\hat{z})$, we have

$$\int_{|z|=r} \left\{ v_n^m(z)\, \frac{\partial \Phi(x,z)}{\partial \nu(z)} - \frac{\partial v_n^m}{\partial \nu}(z)\, \Phi(x,z) \right\} ds(z) = v_n^m(x), \quad |x| > r.$$

From the last two equations, noting that on $|z| = r$ we have

$$u_n^m(z) = j_n(kr)\, Y_n^m(\hat{z}), \qquad \frac{\partial u_n^m}{\partial \nu}(z) = k j_n'(kr)\, Y_n^m(\hat{z})$$

and

$$v_n^m(z) = h_n^{(1)}(kr)\, Y_n^m(\hat{z}), \qquad \frac{\partial v_n^m}{\partial \nu}(z) = k h_n^{(1)\prime}(kr)\, Y_n^m(\hat{z})$$

and using the Wronskian (2.37), we see that

$$\frac{1}{ikr^2}\int_{|z|=r} Y_n^m(\hat{z})\,\Phi(x,z)\,ds(z) = j_n(kr)\,h_n^{(1)}(k|x|)\,Y_n^m(\hat{x}), \quad |x|>r,$$

and by transforming the integral into one over the unit sphere we get

$$\int_{\mathbb{S}^2} Y_n^m(\hat{z})\,\Phi(x,r\hat{z})\,ds(\hat{z}) = ik\,j_n(kr)\,h_n^{(1)}(k|x|)\,Y_n^m(\hat{x}), \quad |x|>r. \tag{2.44}$$

We can now apply Theorem 2.8 to obtain from the orthogonal expansion

$$\Phi(x,y) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\int_{\mathbb{S}^2} Y_n^m(\hat{z})\,\Phi(x,r\hat{z})\,ds(\hat{z})\,\overline{Y_n^m(\hat{y})}$$

and (2.44) that the series (2.43) is valid for fixed x with $|x|>r$ and with respect to y in the L^2 sense on the sphere $|y|=r$ for arbitrary r. With the aid of the Cauchy–Schwarz inequality, the Addition Theorem 2.9 for the spherical harmonics and the inequalities (2.22), (2.38), and (2.39) we can estimate

$$\sum_{m=-n}^{n}\left|h_n^{(1)}(k|x|)\,Y_n^m(\hat{x})\,j_n(k|y|)\,\overline{Y_n^m(\hat{y})}\right|$$

$$\leq \frac{2n+1}{4\pi}\,|\,h_n^{(1)}(k|x|)\,j_n(k|y|)\,| \;=\; O\left(\frac{|y|^n}{|x|^n}\right), \quad n\to\infty,$$

uniformly on compact subsets of $|x|>|y|$. Hence, we have a majorant implying absolute and uniform convergence of the series (2.43). The absolute and uniform convergence of the derivatives with respect to $|x|$ and $|y|$ can be established analogously with the help of estimates for the derivatives j_n' and $h_n^{(1)\prime}$ corresponding to (2.38) and (2.39) which follow readily from (2.35). □

Passing to the limit $|x|\to\infty$ in (2.44) with the aid of (2.15) and (2.42), we arrive at the *Funk–Hecke formula*

$$\int_{\mathbb{S}^2} e^{-ikr\,\hat{x}\cdot\hat{z}}\,Y_n(\hat{z})\,ds(\hat{z}) = \frac{4\pi}{i^n}\,j_n(kr)\,Y_n(\hat{x}), \quad \hat{x}\in\mathbb{S}^2, \quad r>0,$$

for spherical harmonics Y_n of order n. Obviously, this may be rewritten in the form

$$\int_{\mathbb{S}^2} e^{-ik\,x\cdot\hat{z}}\,Y_n(\hat{z})\,ds(\hat{z}) = \frac{4\pi}{i^n}\,j_n(k|x|)\,Y_n\left(\frac{x}{|x|}\right), \quad x\in\mathbb{R}^3. \tag{2.45}$$

Proceeding as in the proof of the previous theorem, from (2.45) and Theorem 2.9 we can derive the *Jacobi–Anger expansion*

$$e^{ik\,x\cdot d} = \sum_{n=0}^{\infty} i^n (2n+1)\, j_n(k|x|)\, P_n(\cos\theta), \quad x \in \mathbb{R}^3, \tag{2.46}$$

where d is a unit vector, θ denotes the angle between x and d and the convergence is uniform on compact subsets of $\mathbb{R}^3$.

2.5 The Far Field Pattern

In this section we first establish the one-to-one correspondence between radiating solutions to the Helmholtz equation and their far field patterns.

Lemma 2.12 (Rellich) *Assume the bounded set D is the open complement of an unbounded domain and let $u \in C^2(\mathbb{R}^3 \setminus \bar{D})$ be a solution to the Helmholtz equation satisfying*

$$\lim_{r\to\infty} \int_{|x|=r} |u(x)|^2 ds = 0. \tag{2.47}$$

Then $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$.

Proof For sufficiently large $|x|$, by Theorem 2.8 we have a Fourier expansion

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m(|x|)\, Y_n^m(\hat{x})$$

with respect to spherical harmonics where $\hat{x} = x/|x|$. The coefficients are given by

$$a_n^m(r) = \int_{\mathbb{S}^2} u(r\hat{x})\overline{Y_n^m(\hat{x})}\, ds(\hat{x})$$

and satisfy Parseval's equality

$$\int_{|x|=r} |u(x)|^2 ds = r^2 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left|a_n^m(r)\right|^2.$$

Our assumption (2.47) implies that

$$\lim_{r\to\infty} r^2 \left|a_n^m(r)\right|^2 = 0 \tag{2.48}$$

for all n and m.

Since $u \in C^2(\mathbb{R}^3 \setminus \bar{D})$, we can differentiate under the integral and integrate by parts using $\Delta u + k^2 u = 0$ and the differential equation (2.17) to conclude that the a_n^m are solutions to the spherical Bessel equation

$$\frac{d^2 a_n^m}{dr^2} + \frac{2}{r}\,\frac{da_n^m}{dr} + \left(k^2 - \frac{n(n+1)}{r^2}\right) a_n^m = 0,$$

that is,

$$a_n^m(r) = \alpha_n^m h_n^{(1)}(kr) + \beta_n^m h_n^{(2)}(kr)$$

where α_n^m and β_n^m are constants. Substituting this into (2.48) and using the asymptotic behavior (2.42) of the spherical Hankel functions yields $\alpha_n^m = \beta_n^m = 0$ for all n and m. Therefore, $u = 0$ outside a sufficiently large sphere and hence $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ by analyticity (Theorem 2.2). □

Rellich's lemma ensures uniqueness for solutions to exterior boundary value problems through the following theorem.

Theorem 2.13 *Let D be as in Lemma 2.12, let ∂D be of class C^2 with unit normal ν directed into the exterior of D, and assume $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ is a radiating solution to the Helmholtz equation with wave number $k > 0$ which has a normal derivative in the sense of uniform convergence and for which*

$$\operatorname{Im} \int_{\partial D} u \,\frac{\partial \bar{u}}{\partial \nu}\, ds \geq 0.$$

Then $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$.

Proof From the identity (2.11) and the assumption of the theorem, we conclude that (2.47) is satisfied. Hence, the theorem follows from Rellich's Lemma 2.12. □

Rellich's lemma also establishes the one-to-one correspondence between radiating waves and their far field patterns.

Theorem 2.14 *Let D be as in Lemma 2.12 and let $u \in C^2(\mathbb{R}^3 \setminus \bar{D})$ be a radiating solution to the Helmholtz equation for which the far field pattern vanishes identically. Then $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$.*

Proof Since from (2.13) we deduce

$$\int_{|x|=r} |u(x)|^2 ds = \int_{\mathbb{S}^2} |u_\infty(\hat{x})|^2 ds + O\left(\frac{1}{r}\right), \quad r \to \infty,$$

the assumption $u_\infty = 0$ on $\mathbb{S}^2$ implies that (2.47) is satisfied. Hence, the theorem follows from Rellich's Lemma 2.12. □

Theorem 2.15 *Let u be a radiating solution to the Helmholtz equation in the exterior $|x| > R > 0$ of a sphere. Then u has an expansion with respect to spherical wave functions of the form*

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m \, h_n^{(1)}(k|x|) \, Y_n^m\left(\frac{x}{|x|}\right) \tag{2.49}$$

that converges absolutely and uniformly on compact subsets of $|x| > R$. Conversely, if the series (2.49) converges in the mean square sense on the sphere $|x| = R$ then it also converges absolutely and uniformly on compact subsets of $|x| > R$ and u represents a radiating solution to the Helmholtz equation for $|x| > R$.

Proof For a radiating solution u to the Helmholtz equation, we insert the addition theorem (2.43) into Green's formula (2.9), applied to the boundary surface $|y| = \tilde{R}$ with $R < \tilde{R} < |x|$, and integrate term by term to obtain the expansion (2.49).

Conversely, L^2 convergence of the series (2.49) on the sphere $|x| = R$, implies by Parseval's equality that

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left| h_n^{(1)}(kR) \right|^2 \, |a_n^m|^2 < \infty.$$

Using the Cauchy–Schwarz inequality, the asymptotic behavior (2.39) and the addition theorem (2.30) for $R < R_1 \leq |x| \leq R_2$ and for $N \in \mathbb{N}$ we can estimate

$$\left[\sum_{n=0}^{N} \sum_{m=-n}^{n} \left| h_n^{(1)}(k|x|) \, a_n^m Y_n^m\left(\frac{x}{|x|}\right) \right| \right]^2$$

$$\leq \sum_{n=0}^{N} \left| \frac{h_n^{(1)}(k|x|)}{h_n^{(1)}(kR)} \right|^2 \sum_{m=-n}^{n} \left| Y_n^m\left(\frac{x}{|x|}\right) \right|^2 \sum_{n=0}^{N} \sum_{m=-n}^{n} \left| h_n^{(1)}(kR) \right|^2 \, |a_n^m|^2$$

$$\leq C \sum_{n=0}^{N} (2n+1) \left(\frac{R}{|x|}\right)^{2n}$$

for some constant C depending on R, R_1, and R_2. From this we conclude absolute and uniform convergence of the series (2.49) on compact subsets of $|x| > R$. Similarly, it can be seen that the term by term first derivatives with respect to $|x|$ are absolutely and uniformly convergent on compact subsets of $|x| > R$. To establish that u solves the Helmholtz equation and satisfies the Sommerfeld radiation condition, we show that Green's formula is valid for u. Using the addition Theorem 2.11, the orthonormality of the Y_n^m and the Wronskian (2.37), we indeed find that

$$\int_{|y|=\tilde{R}} \left\{ u(y) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y)\, \Phi(x,y) \right\} ds(y)$$

$$= ik\tilde{R}^2 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m\, h_n^{(1)}(k|x|)\, Y_n^m\left(\frac{x}{|x|}\right) k\, W(h_n^{(1)}(k\tilde{R}), j_n(k\tilde{R}))$$

$$= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m\, h_n^{(1)}(k|x|)\, Y_n^m\left(\frac{x}{|x|}\right) = u(x)$$

for $|x| > \tilde{R} > R$. From this it is now obvious that u represents a radiating solution to the Helmholtz equation. □

Let R be the radius of the smallest closed ball with center at the origin containing the bounded domain D. Then, by the preceding theorem, each radiating solution $u \in C^2(\mathbb{R}^3 \setminus \bar{D})$ to the Helmholtz equation has an expansion with respect to spherical wave functions of the form (2.49) that converges absolutely and uniformly on compact subsets of $|x| > R$. Conversely, the expansion (2.49) is valid in all of $\mathbb{R}^3 \setminus \bar{D}$ if the origin is contained in D and if u can be extended as a solution to the Helmholtz equation in the exterior of the largest closed ball with center at the origin contained in $\bar{D}$.

Theorem 2.16 *The far field pattern of the radiating solution to the Helmholtz equation with the expansion (2.49) is given by the uniformly convergent series*

$$u_\infty = \frac{1}{k} \sum_{n=0}^{\infty} \frac{1}{i^{n+1}} \sum_{m=-n}^{n} a_n^m\, Y_n^m. \tag{2.50}$$

The coefficients in this expansion satisfy the growth condition

$$\sum_{n=0}^{\infty} \left(\frac{2n}{ker}\right)^{2n} \sum_{m=-n}^{n} |a_n^m|^2 < \infty \tag{2.51}$$

for all $r > R$.

Proof We cannot pass to the limit $|x| \to \infty$ in (2.49) by using the asymptotic behavior (2.42) because the latter does not hold uniformly in n. Since by Theorem 2.6 the far field pattern u_∞ is analytic, we have an expansion

$$u_\infty = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_n^m\, Y_n^m$$

with coefficients

$$b_n^m = \int_{\mathbb{S}^2} u_\infty(\hat{x})\, \overline{Y_n^m(\hat{x})}\, ds(\hat{x}).$$

On the other hand, the coefficients a_n^m in the expansion (2.49) clearly are given by

$$a_n^m\, h_n^{(1)}(kr) = \int_{\mathbb{S}^2} u(r\hat{x})\, \overline{Y_n^m(\hat{x})}\, ds(\hat{x}).$$

Therefore, with the aid of (2.42) we find that

$$\begin{aligned} b_n^m &= \int_{\mathbb{S}^2} \lim_{r\to\infty} r\, e^{-ikr} u(r\hat{x})\, \overline{Y_n^m(\hat{x})}\, ds(\hat{x}) \\ &= \lim_{r\to\infty} r\, e^{-ikr} \int_{\mathbb{S}^2} u(r\hat{x})\, \overline{Y_n^m(\hat{x})}\, ds(\hat{x}) \;=\; \frac{a_n^m}{k\, i^{n+1}}\,, \end{aligned}$$

and the expansion (2.50) is valid in the L^2 sense.

Parseval's equation for the expansion (2.49) reads

$$r^2 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left|a_n^m\right|^2 \left|h_n^{(1)}(kr)\right|^2 = \int_{|x|=r} |u(x)|^2 ds(x).$$

From this, using the asymptotic behavior (2.40) of the Hankel functions for large order n, the condition (2.51) follows. In particular, by the Cauchy–Schwarz inequality, we can now conclude that (2.50) is uniformly valid on $\mathbb{S}^2$. □

Theorem 2.17 *Let the Fourier coefficients b_n^m of $u_\infty \in L^2(\mathbb{S}^2)$ with respect to the spherical harmonics satisfy the growth condition*

$$\sum_{n=0}^{\infty} \left(\frac{2n}{keR}\right)^{2n} \sum_{m=-n}^{n} \left|b_n^m\right|^2 < \infty \tag{2.52}$$

with some $R > 0$. Then

$$u(x) = k \sum_{n=0}^{\infty} i^{n+1} \sum_{m=-n}^{n} b_n^m h_n^{(1)}(k|x|)\, Y_n^m\left(\frac{x}{|x|}\right), \quad |x| > R, \tag{2.53}$$

is a radiating solution of the Helmholtz equation with far field pattern u_∞.

Proof By the asymptotic behavior (2.40), the assumption (2.52) implies that the series (2.53) converges in the mean square sense on the sphere $|x| = R$. Hence, by Theorem 2.15, u is a radiating solution to the Helmholtz equation. The fact that the far field pattern coincides with the given function u_∞ follows from Theorem 2.16.

□

The last two theorems indicate that the equation

$$Au = u_\infty \tag{2.54}$$

with the linear operator A mapping a radiating solution u to the Helmholtz equation onto its far field u_∞ is ill-posed. Following Hadamard [165], a problem is called *properly posed* or *well-posed* if a solution exists, if the solution is unique and if the solution continuously depends on the data. Otherwise, the problem is called *improperly posed* or *ill-posed*. Here, for Eq. (2.54), by Theorem 2.14 we have uniqueness of the solution. However, since by Theorem 2.16 the existence of a solution requires the growth condition (2.51) to be satisfied, for a given function u_∞ in $L^2(\mathbb{S}^2)$ a solution of Eq. (2.54) will, in general, not exist. Furthermore, if a solution u does exist it will not continuously depend on u_∞ in any reasonable norm. This is illustrated by the fact that for the radiating solutions

$$u_n(x) = \frac{1}{n}\, h_n^{(1)}(k|x|)\, Y_n\left(\frac{x}{|x|}\right),$$

where Y_n is a normalized spherical harmonic of degree n the far field patterns are given by

$$u_{n,\infty} = \frac{1}{ki^{n+1}n}\, Y_n.$$

Hence, we have convergence $u_{n,\infty} \to 0$, $n \to \infty$, in the L^2 norm on $\mathbb{S}^2$ whereas, as a consequence of the asymptotic behavior (2.40) of the Hankel functions for large order n, the u_n will not converge in any suitable norm. Later in this book we will study the ill-posedness of the reconstruction of a radiating solution of the Helmholtz equation from its far field pattern more closely. In particular, we will describe stable methods for approximately solving improperly posed problems such as this one.