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Paola Magnaghi-Delfino
Giampiero Mele
Tullia Norando *Editors*

# Faces of Geometry. From Agnesi to Mirzakhani

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*Indeed, I am fully convinced that in this age…*
*every Woman ought to exert herself,*
*and endeavor to promote the glory of her sex,*
*and to contribute her utmost to increase that luster…*

(Maria Gaetana Agnesi, *Analytical Institutions*, translated into English by the late Rev. John Colson M.A.F.R.S., London, 1801)

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# Preface

Some introductory remarks about the reasons that motivated the choice of the topics of the Conference *Faces of Geometry. From Agnesi to Mirzakhani.*

We have two purposes, equally important.

First, we have the intent of promoting interdisciplinary discussions and connections between theoretical researches and practical studies on geometric structures and its applications in architecture, arts, design, education, engineering, mathematics.

Indeed, we believe that these related fields of study might enrich each other and renew common interests on these topics through networks of common inspirations.

We invite researchers, teachers and students to share their ideas, to discuss their scientific opinions in teaching these disciplines, in order to enhance the quality of geometry and graphics education.

Second, but not less important. We are sure that the scientific community and mathematics, in particular, needs the contribution of women.

Women have made significant contributions to science from the earliest times. Historians with an interest in gender and science have illuminated the scientific endeavours and accomplishments of women, the barriers they have faced, and the strategies they have implemented to have their work peer-reviewed and accepted in major scientific journals and other publications. The historical, critical and sociological study of these issues has become an academic discipline in its own right.

In 2018, we celebrated, in Politecnico di Milano, the anniversary of Maria Gaetana Agnesi, Milanese mathematician, the first woman to write the first vernacular handbook of mathematics for learners.

Nowadays, we celebrate the first Women in Mathematics Day, dedicated to Maryam Mirzakhani, the first woman that wins the Fields Medal.

The Turkish mathematician Betul Tanbay, in her tribute to Mirzakhani, recalled the classic geometric problem, called illumination problem, and compared Maryam Mirzakhani to the candle lighting the path for others to follow. Quoting, she said "Maryam showed forever that excellence is not a matter of gender or geography. Maths is a universal truth that is available to us all".

During the conference, we commemorate Giuseppina Biggiogero, the first woman that taught Descriptive Geometry in the Faculty of Architecture at Politecnico di Milano.

The Organizing Board of the Conference announced the birth of The International Association in Mathematics and Art—Italy (IAMAI), promoted by Italian scholars from various academic, disciplinary and cultural backgrounds.

The Mission of the Association is the promotion of researches and the dissemination of results in the various application fields, in reference to national and international contexts, enhancing the plots and convergences between areas that link Mathematics to Art, opened to forms of collaboration and involvement of other subjects, institutions and organizations.

Mathematics is the fruit of the thought both creative and logical, inspired and deeply linked to the Beauty, recognizable in various expressions of Art, from Architecture to Design and Fashion, from Painting to Sculpture, from Music to Dance and Theatre, including their digital and virtual expressions. For centuries, Italy has been a land of promotion and encounter between Art and Science and our country is full of signs of the Italian Cultural Heritage. The aim of the association is to give the maximum sharing to these witness through the appropriate communication and publishing channels.

Milan, Italy — Paola Magnaghi-Delfino
Novedrate, Italy — Giampiero Mele
Milan, Italy — Tullia Norando

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# About the Editors

**Paola Magnaghi-Delfino and Tullia Norando** are Professor of Mathematical Analysis at Politecnico di Milano. They are members of the laboratory FDS (Formation, Science Communication, Didactics and Experimental Teaching) of the Department of Mathematics. Its mission includes the improvement of teaching techniques, research and services consistent with the broader missions of the Department of Mathematics for the mathematical achievement of high school students. It also supports e-learning courses to enable students to study specific areas of mathematics and projects for students to want to learn how to apply advanced mathematics. In this context, FDS promotes large-scale initiatives for the dissemination and the "demystification" of the difficulties of mathematics as awards, games, television programs, conferences. Science communication activities are devoted to public awareness of science with the promotion and/or organization of seminars, conferences and other initiatives, in relation to other activities that take place both in Italy and abroad.

The fields of research of Paola and Tullia are the relationships between mathematics and arts and history of mathematics. They are engaged in projects based on the idea of reconciling the various talents of creative processes. Projects arise as the result of a synergy between different scientific experiences or from the dialogue with the artistic world, like graphic art, architecture, music and theatre. Paola and Tullia are scientific advisors of the plays of TeatroInMatematica (PACTA.dei Teatri company). They were curators of the exhibits "Lezione di Galileo Galilei sulla struttura dell'Inferno", in Politecnico di Milano (Mai 2013), in Ravenna Dante's September and in BergamoScienza Festival in October 2013, following a collaboration project with Accademia di Brera.

In 2016–2017–2018, they announced the Math&Art contest for students of the High School. At the end of the competition, Tullia and Paola organized the exhibits of the select artworks in Politecnico di Milano. In 2018, they were promoters of the initiative in honour of the 300th anniversary of the birth of Maria Gaetana Agnesi and collaborated on events included in the calendar of the City of Milan. They participated in the creation and execution of the performance *Conversations* and were organizers and speakers at the conference *Maria Gaetana Agnesi: woman,* *mathematics, and benefactress*. In August, they were members of the Organizing Board of the 18th Conference on Geometry and Graphics.

In 2019, they were promoters, in Milan, of the congress Faces of Geometry. From Agnesi to Mirzakhani, which is inserted in the world celebrations of Maryam Mirzakhani birthday.

They are co-founder of IAMAI (International Association of Mathematics and Art—Italy).

Most recent articles in this research lines:

P. Magnaghi-Delfino, T. Norando *Maria Gaetana Agnesi: suggestions from the past to new way to teach calculus* to Aplimat 2019, STU Editor.
P. Magnaghi-Delfino, T. Norando *Teaching calculus with Maria Gaetana Agnesi.* G. Slovensky casopis pre geometriu a grafiku—2019 vol. 15 (30).
P. Magnaghi-Delfino, T. Norando *Luca Pacioli: Letters from Venice* ImageMath6. Springer 2018.
P. Magnaghi-Delfino, T. Norando *Geometrical analysis of a design artwork coffee table designed by the architect Augusto Magnaghi-Delfino* Aplimat 2018 STU Editor.
P. Magnaghi-Delfino, T. Norando *Tug of war: maths & sports project* IV International Conference on Higher Education Advances 2018.
P. Magnaghi-Delfino, T. Norando *Maria Gaetana Agnesi. New way to teach maths is in the past?* Atti del convegno *INTE* 2018, Parigi.
P. Magnaghi-Delfino, T. Norando *Maria Gaetana Agnesi. There is no innovation whithout memory* Atti del convegno *INTE* 2018, Parigi.
P. Magnaghi-Delfino, T. Norando *Alessandro Mazzucotelli: an artistic and educational project* Aplimat Journal of Applied Mathematics 2016. STU Editor.
P. Magnaghi-Delfino, T. Norando *Luca Pacioli's Alphabeto Dignissimo Antiquo: a geometrical reconstruction* Aplimat Journal of Applied Mathematics 2015 pp. 555–577.
P. Magnaghi-Delfino, T. Norando *The Size and Shape of Dante's Mount Purgatory* Journal of Astronomical History and Heritage 2015 vol. 18 (2).
A. Angelini, P. Magnaghi-Delfino, T.Norando *Galileo Galilei's Location, Shape and Size of Dante's Inferno: an Artistic and Educational Project* Aplimat Journal of Applied Mathematics 2014 STU Editor.

**Giampiero Mele** is an architect, received his Ph.D. in "Survey and Representation of architecture and the environment" at the University of Florence in 2000 and a Ph.D. in "Architectural and Urban Design" from the Université di Paris 8 in 2004. Since 2014, he has been an Associate Professor at the Università degli studi eCampus, and professor of descriptive geometry at the University of Florence, at the Politecnico di Milano and University of Ferrara. His fields of research are the relationships between geometry and arithmetic in historic architecture and drawing in architecture and design. He has given talks at various conferences in these fields and is the author of numerous papers.

In 2019, they were promoters, in Milan, of the congress Faces of Geometry. From Agnesi to Mirzakhani, which is inserted in the world celebrations of Maryam Mirzakhani birthday.

Founder and President of IAMAI (International Association of Mathematics and Art—Italy).

Most recent articles in this research lines:

Giampiero Mele, Sylvie Duvernoy, *Early Trompe-l'oeil Effects in the Last Supper Depictions by Domenico Ghirlandaio*, Italy, Nexus Network Journal, n. 1–17, Birkauser, Springer 2017.
Giampiero Mele, *Perspective and proportion in the Montefeltro altarpiece of Piero della Francesca*, in AAVV, 16h conference in applied mathematics proceedings (Aplimat 2017) STU Editor, Bratislava, SLOVACCHIA, 2017.
Giampiero Mele, *Geometrical analysis of Lecce's roman theatre and anphitheatre drawings*, in L'udovit Balko, Dagmar Szarková, Daniela Richtáriková, 15h conference in applied mathematics proceedings (Aplimat2016), STU Editor, Bratislava, SLOVACCHIA, 2016.
Giampiero Mele, Giorgia Maniglio, *The nonagon as a tool for the drawing of the Roman Theatre of Lecce*, DISEGNARECON, vol. 8, n. 14, L'Aquila, Italy, 2015
Giampiero Mele, *A Geometrical Analysis of the Layout of Acaya*, Italy. Nexus Network Journal, n. 1–14, Birkauser, Springer 2012.
Giampiero Mele, Interfaces for the e-learning of Descriptive Geometry in virtual space, in Michela Rossi, *Descriptive Geometry and Digital Representation: Memory and Innovation*, THE MCGRAW-HILL COMPANIES, Milano, New York, 2012.

# Unexpected Geometries Exploring the Design of the Gothic City

**Maria Teresa Bartoli**

Geometry and architecture have always collaborated in the project, experimenting in the second the achievements of the first. In every age great mathematicians have worked alongside the great architects, although we do not always have direct information about them. History has recognized this relationship in many cases where the literary tradition provided news for its evidence: Pyramids, Greek temples, the Pantheon, the architecture of Humanism etc. Can historical research discover less evident forms of this relationship by working exclusively with geometry, in examples where the literary tradition has not directly given the news? The research was focused on the design of the gothic city, which historiography relates by finding in the past the characters considered necessary for the development of the subsequent history rather than the objective requirements linked to the knowledge and intentions of the historical moment.

The study of Florence urban design in the Gothic era was addressed to two themes: the layout of the walls and the distribution of Great Convents. The conclusions reached can give a lot of information about the relationship between the geometric culture diffused at the time of the works we are dealing with, the scientific treatises and the culture transmitted by the social, historical and poetic literature, giving a more complete and truthful image of the mindset of the time.

## 1 Introduction

The pyramid of Giza, at the root of the history of both geometry and architecture, is emblematic of the relationship between the two fields of thought. Its defining paradigm is not the slope of its faces (h/l = 14/11) nor of its edges (h/d = 9/10),

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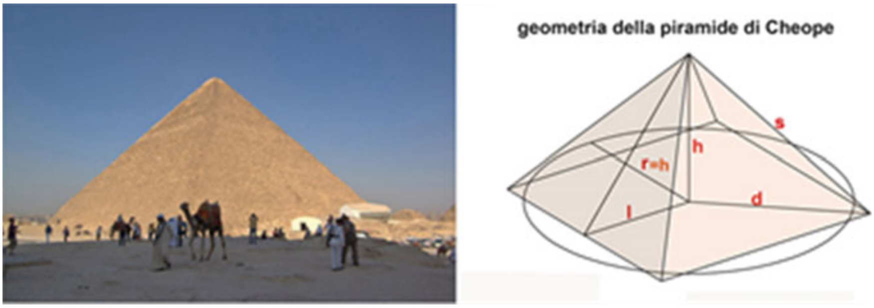



**Fig. 1** The pyramid of Cheops at Giza

where the secret of its fascination has been sought over time, but a hidden theorem written in an ancient hieroglyph which conveys the order given by priests to the architect: the square of the base has a perimeter which equals the circumference of a circle whose radius equals its height (r = h, Fig. 1, [1, pp. 59, 60]). Only one measurement need be remembered, the height, and the rest can be deduced. The height is 280 cubits (its circle of radius will therefore be 1760 cubits and the side of the base 440 cubits), each cubit is made up of 28 digits. Only a few geometric data are required to recall and convey the devised design.

One architectural theme that historiography has struggled to clarify, due to the inadequacy of the written tradition (which describes the events but not the rationale) is that of Florentine Gothic town planning (Fig. 2). The out-of-the-ordinary design of a city that at the time become one of the largest and most admired in Europe is very difficult to describe. It eludes many of the geometric paradigms that generally act as a point of reference for urban planning. The layout of its walls—we even know the name of their designer, Arnolfo—is difficult to describe and is generally explained with decisions that were made on the spot, safeguarding existing roads and directions [2, p. 35]; the internal road network, far removed from the rectangular mesh grid, follows the most disparate directions, forming irregular-shaped squares with their different orientations, set before large conventual churches. Yet the straight lines that define the roads and squares, like the steadfast direction of long stretches of wall, suggest that all this was strongly desired and corresponds to deliberate and shared intentions.

## 2 The Walls

The walls of Florence were built between 1284 (the year the main gates to the north of the river were founded) and 1324, the year in which the circuit was completed. Taking a ruler and compass to the last nineteenth-century plan, measured with scientific criteria before the walls were demolished (1865–70), reveals a circumstance that casts a surprising light on the relationship between geometry and urban design (Fig. 3):


**Fig. 2** The map of Florence in the mid-1800s

the gates to the north of the river, positioned at the apexes of the polygon of the wall layout, have special relationships with a particular point, the vertex of the pyramid of the Baptistery, with which they construct familiar geometries, creating an unexpected figure [3].

To the north of the river the vertexes of the polygon belong to a circle, at the centre of which is the baptistery (A), together with which they describe an equilateral triangle to the N.E. and a set square to the N.W. To the south the gates describe a right-angled triangle with 60° and 30° angles, whose hypotenuse belongs to an equilateral triangle inscribed within the circle of the northern gates. The stretches of walls between those described, on the north side, follow the course of the circle.

Observing the layout, it is not difficult to recognize the geometric strategies implemented for its creation: the north-south straight line that passes through the Baptistery and finishes at Porta San Giorgio crosses Orsanmichele (the first loggia was built together with the gates); the east-west straight line which ends at Porta la Croce to the east and Porta San Frediano to the west; the straight line that joins the gates of

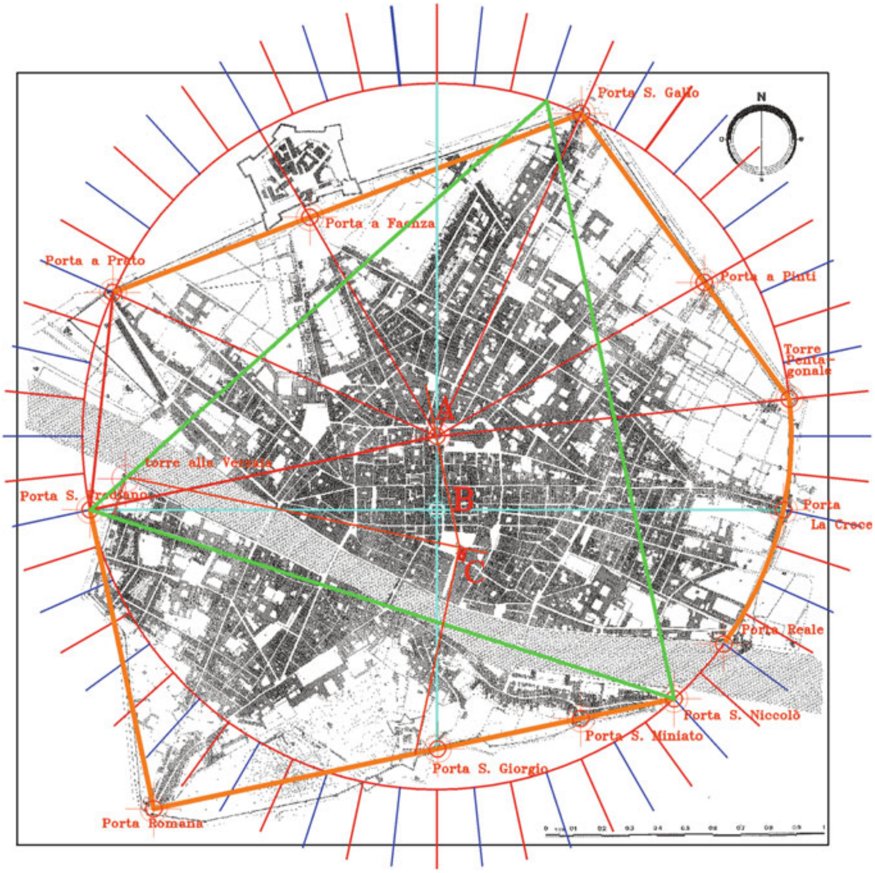



**Fig. 3** City gate distribution geometry; **a** Baptistery, **b** Orsanmichele

San Frediano and San Niccolò (the hypotenuse of the right-angled triangle, whose intermediate point passes through the southern end of the Ponte Vecchio).

Which figure is obscured in this layout? To my eyes it seemed quite clear when, by chance, I traced its lines with a pen: the head of a lion, facing the plain, from whose open mouth flows the river that brings life to the countryside in the valley.

The lion was already considered the totem of Florence at least from 1260, the year, according to Villani, of the miraculous event of the lion which, having escaped the menagerie in which it was kept, wandered around the urban centre sowing terror, later curling up next to a terrified widow with her young son in her arms. That same year, after the battle of Montaperti, Guittone d'Arezzo wrote the famous ode in which he compared the defeated Florence to a dejected lion. The city's identification with the lion is clear in its very strong propensity to invade all the most representative urban spaces of the city with this figure (I refer to Donatello's Marzocco and the golden weather-vane erected on the tower of Palazzo Vecchio). The lion of the walls has its nape facing east, its jaws, from which the river flows, facing west, and its chin to the south. Here, the complex symbology of the lion brings together all the faceted meanings of royal, ordering and providential power, interpreter of justice and dispenser of life. A family of lions was kept in the menagerie on Via dei Leoni and omens of the city's fortune were based on its fate.

On the inner face of the present-day north gate (by Andrea Pisano) of the Florentine baptistery there are 28 bronze panels with circular frames containing the faces of lions whose profile is very close to the design of the walls (Fig. 4). The question is: how could a similar design be created on the ground, within a circle whose diameter can be estimated as approximately 2560 m, equal to 4350 *braccia da panno*, indicated by Villani himself in his description of the walls?

An important medieval astronomical instrument, perhaps invented by the Arabs, was Jacob's staff or the baculo, which wassed to measure the angular distances between stars. It also became a land surveyor's instrument, useful for measuring linear distances (Fig. 5).

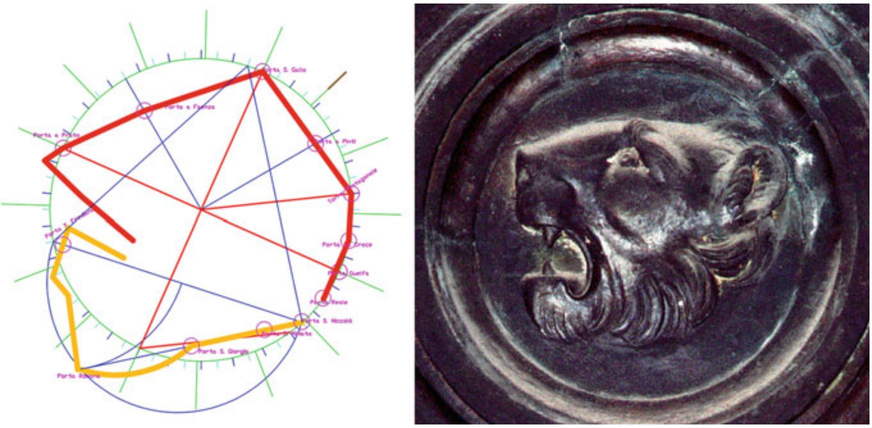



**Fig. 4** One of the 18 lion's heads inside the North door of the Baptistery, near the walls outline

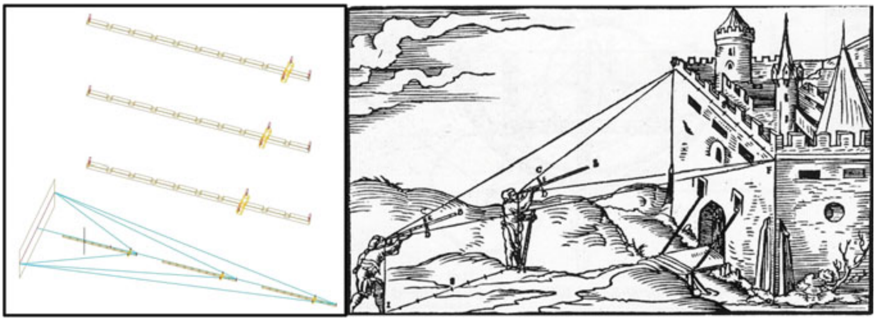


**Fig. 5** The Jacob's staff to measure distances

This instrument is a rod marked with graduations (for example a multiple of the *braccio* divided into 12 *once* or 20 *soldi*) and along its notches a strip as long as the distance between two notches can be placed to form a movable cross. Four sights are placed on the four vertices. The instrument is placed in front of a known width that acts as a base, which it is then moved away from while maintaining its central position. Moving the transom by 1, 2, 3 notches, the observation distance is 1, 2, 3 times the known width. This means it can measure distances from it that are multiples of that length. The Florentine braccio measures 0.5836 m. The diameter of the circle on which the gates to the north of the river stand measures approximately 4350 braccia (a number with no intrinsic merits, which is strange in a project where it represents the characteristic length). Careful measurement of the Baptistery reveals that if we consider the octagon as the result of two squares rotated by 45°, the side of each square is 43 braccia long; therefore the gates are 50 × 43 *braccia* away from the sides of the squares within the octagon of the Baptistery and the diameter of the circle is (50 × 2) × 43 + 43 = 4343 (Fig. 6).

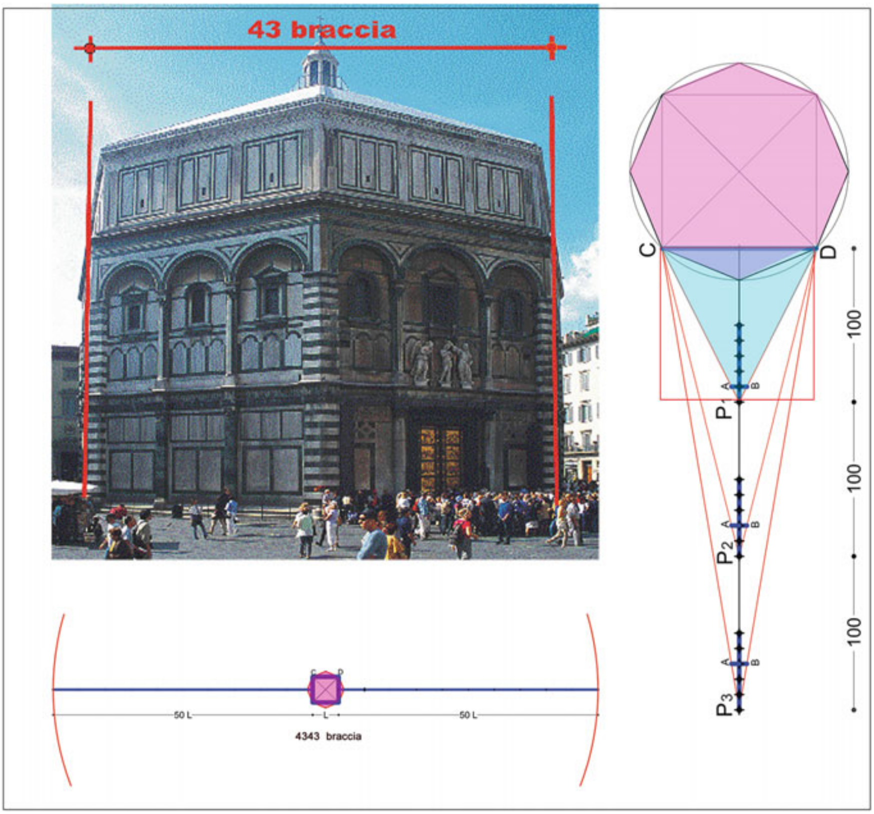
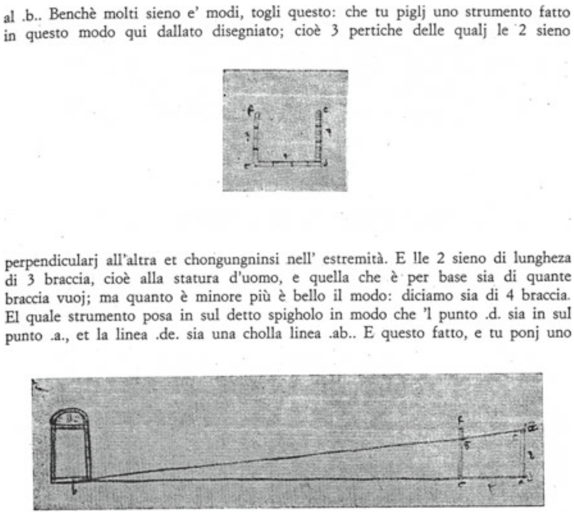



**Fig. 6** The Baptistery as the goal of the Jacob's staff

Tradition attributes the creation of the ribs to Arnolfo. The system of geometric relationships revealed by tracing the radii of the circle that incorporate the gates appears to be achievable by associating distance measurements and forward intersection operations according to preset angles (60° and 45°), from stations made up of a platform at the top of the Baptistery's pyramid and at the top level of towers and gates, with the help of mobile wooden towers. The form was not produced by exploration on the ground in order to find the most suitable direction, but by the firm intention to protect the city with the totemic image of a lion's head: magical thinking, shape design, geometry and technology came together to create the wall circuit.

## 3 Distribution of the Gothic Convents

Starting from the last decades of the 13th century, Florentine building activity also included the distribution of the convents within the urban fabric [4]. Their positions are spread out, while maintaining adequate distances between them. The logic is suggested by a papal bull issued in 1265 by Pope Clement IV in favour of the convents of the Franciscans, which established that no building of the other mendicant orders could be built at a distance of less than 300 Roman canne from their church; that is, the prescribed measurement could not be measured on the ground, and had to be measured per aerem. The technical meaning of this expression cannot be translated precisely, but it certainly refers to an optical method of indirect measurement. In Florence it is evidenced by Master Grazia de' Castellani, who between the 14th and 15th centuries produced some tacheometry exercises [5]. The instrument is a large C made up of 3 rods, the longest of which measures 4 braccia, while the other two, 3 braccia long, are positioned at the ends of the former (Fig. 7).


al .b.. Benchè molti sieno e' modi, togli questo: che tu piglj uno strumento fatto in questo modo qui dallato disegniato; cioè 3 pertiche delle qualj le 2 sieno

perpendicularj all'altra et chongungninsi nell' estremità. E lle 2 sieno di lungheza di 3 braccia, cioè alla statura d'uomo, e quella che è per base sia di quante braccia vuoj; ma quanto è minore più è bello il modo: diciamo sia di 4 braccia. El quale strumento posa in sul detto spigholo in modo che 'l punto .d. sia in sul punto .a., et la linea .de. sia una cholla linea .ab.. E questo fatto, e tu ponj uno

**Fig. 7** The instruments of M.o Grazia de'Castellani and its use

The instrument is used to measure urban or territorial distances. A sight is placed at the top of one of the cross roads, while the other rod has graduated markings like those of the braccio. A wire placed at the sight cuts across the graduated rod generating similar triangles.

The measurement of 300 Roman canne corresponds to 596 m (1 canna = 1.98 m); in Florence the same metric incisiveness for a similar length would have been expressed by 1000 braccia, equal to 583.6 m. To check compliance with this criterion in Florence, the reciprocal distances of Gothic-founded Florentine churches on the nineteenth-century map were analysed, starting with the two largest ones, Santa Croce and Santa Maria Novella. Figure 8 shows the result of this research.

The line joining the façades of the two churches is oriented with an exact angle of 60° from north and this direction runs along the long side of Santa Croce (**A**). 1000 braccia from the façade of Santa Croce, the line crosses a particular point of the ancient Via del Corso, where 3 towers still to this day form a sort of small square and two Gothic plaques name the corner of one of them as Canto alla Croce Rossa

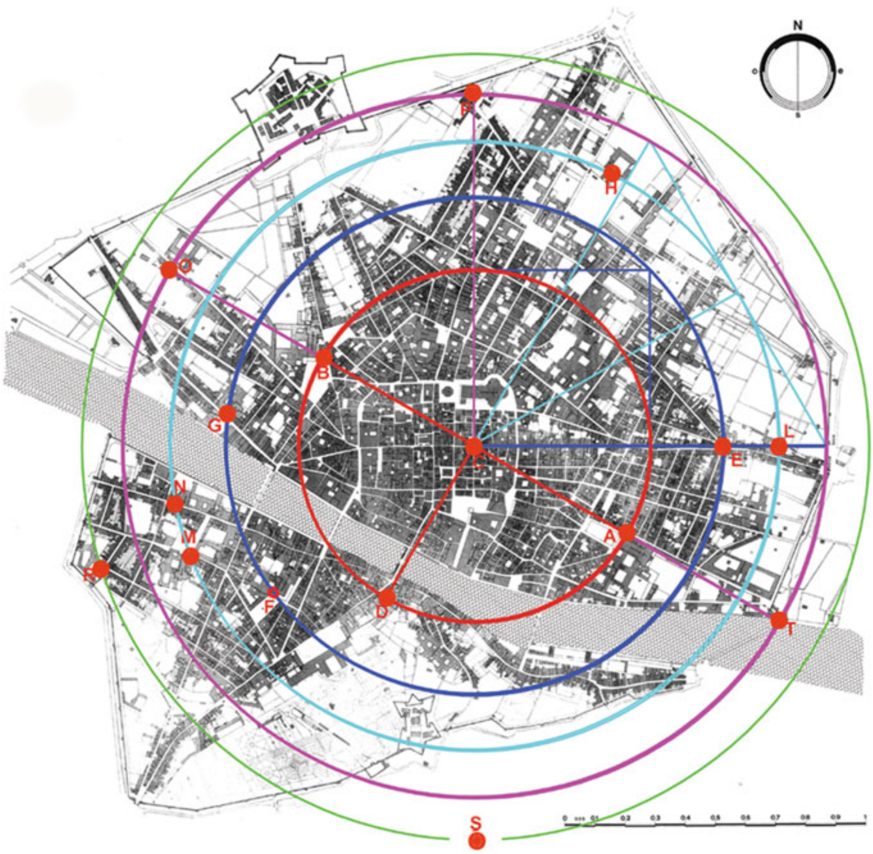
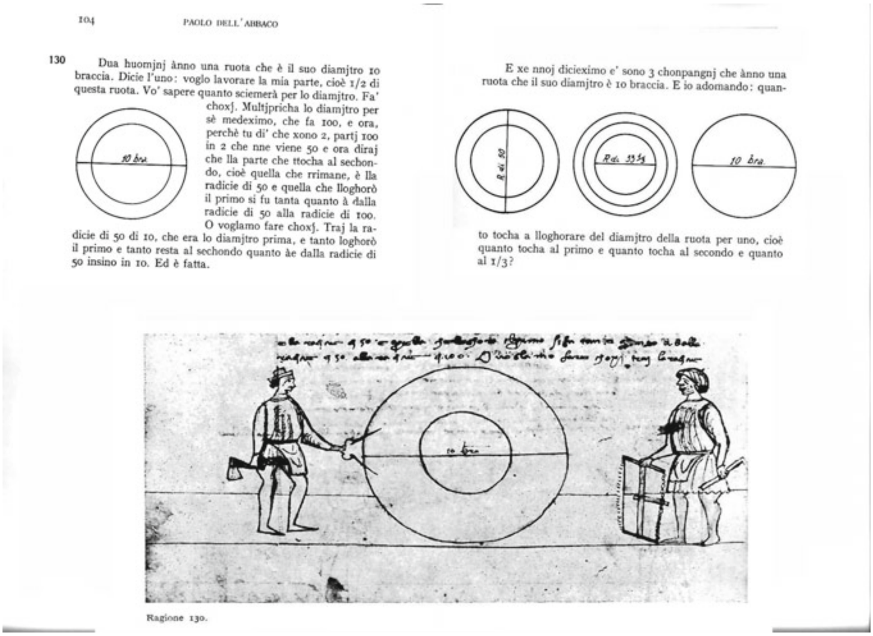



**Fig. 8** The distribution of the Gothic convents on the map of Florence

(**C**), a symbol of the popolano army of Florence. In the same direction, 1000 braccia from Canto alla Croce Rossa, the line continues up to the wall of the tombs of Santa Maria Novella (**B**), very close to the right-hand corner of the façade. The midpoint of the straight line orthogonal to this direction (the small square) crosses the river passing over the Ponte Vecchio and, after 1000 braccia, comes to rest on the façade of the church of Santa Felicita (**D**). For these three points we can describe a circle (r = 1000 br) and the square circumscribed to it, with a vertex oriented to the North. The east vertex touches the façade of the church of Sant'Ambrogio (**E**), which in the 1390s was shifted to the west, reaching that point at a distance of $r\sqrt{2}$ from Canto alla Croce Rossa. At the same distance we find the façades of Santo Spirito (**F**), on the other side of the river, and Ognissanti (**G**), on this side of the river. The circle of radius $r\sqrt{3}$ crosses the façade of the church of the Santa Teresa convent and the façade of the church of the Carmine convent. The equilateral triangle circumscribed to the thousand-braccia circle of radius is inscribed in the circle of radius $r\sqrt{4}$ = 2000 braccia, with a vertex to the north: the façade of the Gothic church of Santa Caterina d'Alessandria is situated on that vertex (**P**), at a distance of 2000 br from Canto alla Croce Rossa; on the eastern vertex of the triangle we find the ancient tower later named Zecca (**T**), a watchtower behind the wall circuit, which may have also been the start of all the construction. On the circle of this radius we find the Dominican convent of Via della Scala (**O**). Finally, on the circle of radius $r\sqrt{5}$, we find the church of the Camaldolese convent (**P**) and, at the same distance from the centre but now outside the walls, due south with respect to the centre of the design, the Gothic church of San Leonardo (**S**). The operations necessary for the construction are found in Grazia dei Castellani's exercises.

This graphic route has a precise geometric value described in Ragioni 130 and 131 of the arithmetic manuscript *Trattato d'aritmetica* by Paolo dell'Abbaco [6], who lived between the 13th and 14th centuries. The first one deals with the division of work between two stonemasons who must polish a stone wheel with a diameter of 10 braccia (Fig. 9).

The effort is proportional to the surface area, therefore it is a question of finding the radius of the half-area circle; the solution is $r1 = r/\sqrt{2}$. If the effort were to be divided up among three stonecutters, the radii of the circles inside the first would be equal to $r1 = r/\sqrt{3}$, $r2 = r1\sqrt{2}$. The second ragione deals with the inverse problem, which asks us to double and then triple and quadruple the area of a circle with circular rings; the solution is to multiply the radius r in sequence by $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$ (Fig. 10). The aim of the ragione is to divide up the effort and payment fairly. The precedent of these theorems can be found in Plato's Meno, which describes the doubling of the area of the square; in the Middle Ages it had become the norm for the proportions of conventual cloisters. It is not easy to understand the purpose of this operation, which is visually not apparent. A clue to its probable meaning is however offered by the Dantesque Empyrean, described in Canto XXX of the Divine Comedy, where Dante sees *light that took a river's form/light flashing, reddish-gold, between two banks/painted with wonderful spring flowerings*. The city of the blessed, gathered between two banks of green the centre of which is furrowed by a river of light, which then becomes a lake (like Florence during the then frequent floods), closed like a


104 PAOLO DELL'ABBACO

130 Dua huomjnj ànno una ruota che è il suo diamjtro 10 braccia. Dicie l'uno: voglo lavorare la mia parte, cioè 1/2 di questa ruota. Vo' sapere quanto sciemerà per lo diamjtro. Fa' choxj. Multjpricha lo diamjtro per sè medeximo, che fa 100, e ora, perchè tu di' che xono 2, partj 100 in 2 che nne viene 50 e ora diraj che lla parte che ttocha al sechondo, cioè quella che rrimane, è lla radicie di 50 e quella che lloghorò il primo si fu tanta quanto à dalla radicie di 50 alla radicie di 100. O voglamo fare choxj. Traj la radicie di 50 di 10, che era lo diamjtro prima, e tanto loghorò il primo e tanto resta al sechondo quanto àe dalla radicie di 50 insino in 10. Ed è fatta.

E xe nnoj dicieximo e' sono 3 chonpangnj che ànno una ruota che il suo diamjtro è 10 braccia. E io adomando: quanto tocha a lloghorare del diamjtro della ruota per uno, cioè quanto tocha al primo e quanto tocha al secondo e quanto al 1/3?



Ragione 130.

**Fig. 9** The reason 130 of the treatise of Paolo dell'Abbaco

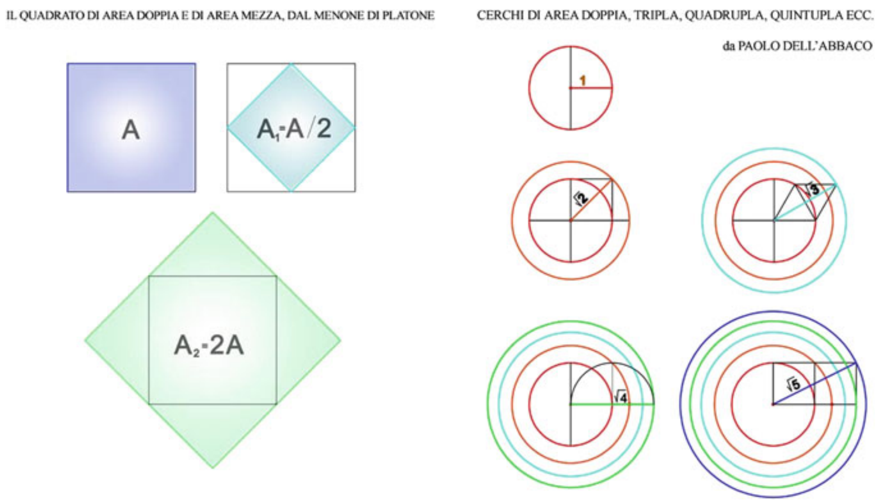



**Fig. 10** Plato's theorem and the regola of Paolo dell'Abbaco

flower, can only evoke Florence. The blessed are arranged on the grassy slope adorned with flowers, similar to a circular amphitheatre of one thousand steps. Beatrice shows them to Dante: *See how great is this council of white robes!/See how much space our city's circuit spans!* Perhaps Dante was partner to the aspiration to make Florence the symbolic city of a religious ideal (the *heavenly Jerusalem*) which had found its ideal form through the implementation of a theorem, not necessarily thought of as a real form.

## 4 Final Considerations

The historical roots of the geometric theorems highlighted in Florentine urban planning are integrated with science, geometry, immaginary and political vision. Herodotus related the birth of geometry to a physical fact (the floods of the Nile) and a political instance (the right proportion between income and taxes). In this matter the Mediterranean basin shared the same cultural heritage, including the Middle East. European historiography has not always taken contributions from the Middle East into account. The idea of a circular city had been achieved by the Abbasid Caliph al Mansur in 762 AD in the founding of the capital Baghdad, the magnificent City of Peace or the Round City. When the city was destroyed by the Mongols in 1259, the army of the besiegers, 150,000 soldiers strong, also had a large contingent of Franks. Perhaps news of his extraordinary figure had reached even Florence. Between 1020 and 1025 the Persian al Biruni (Al Khalili J. conceived of a brilliant experiment to calculate the radius of the earth. It started with the calculation of the height of a mountain, using an instrument (a square board with graduated edges) placed on a flat area (Fig. 11), from which it was possible to sight the top of the mountain on which a target was placed.

The instrument, which according to Al Biruni himself had been invented by the astronomer Sanad two centuries earlier, had the same use as that described by Master

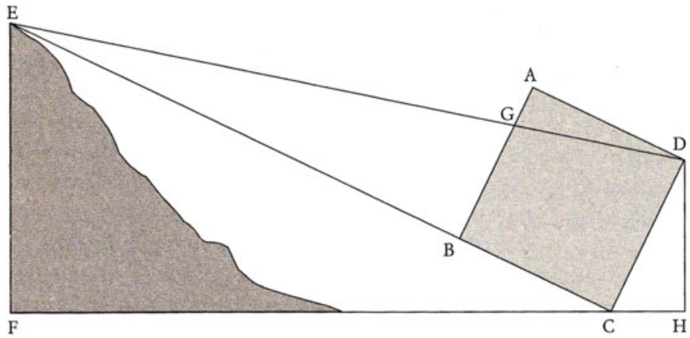



**Fig. 11** Measurement of the height of a mountain, by al Biruni [7, p. 226]

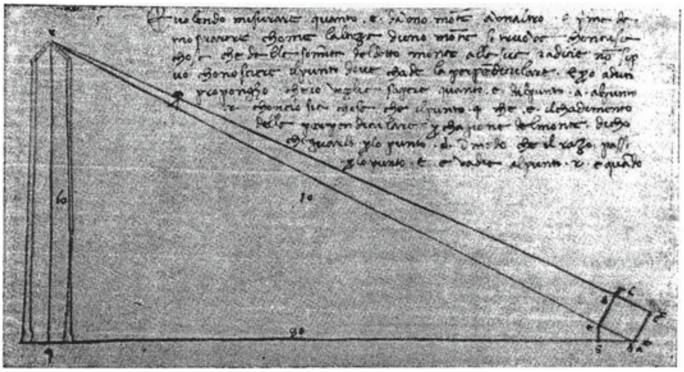


**Fig. 12** Measurement of the height of a mountain, by Grazia de'Castellani [5, p. 58]

Grazia dei Castellani to detect the height of a tower or a mountain (Fig. 12): this therefore derived from it.

The similarity between triangles used twice made it possible to determine the height sought. The radius of the earth was then determined by measuring the angle between the horizontal at eye level and the tangent from the eye to the surface of the sea.

All that we have recounted resulted from the convergence of an uninterrupted global scientific development occurring at a time and place destined to have an important impact on history.

## References


1. Goyon G (1980) Il segreto delle grandi piramidi. Newton Compton, Roma
2. Fanelli G (1988) Le città nella storia d'Italia, Firenze, Laterza 1980
3. Bartoli MT (2003) Un laboratorio dell'architettura gotica: Firenze, la città, le mura il Palazzo in Bartoli M.T., Bertocci S., Città e architettura, le matrici di Arnolfo, Firenze, Edifir, 8879701894, pp 17–40
4. Bartoli MT (2011) Vedi nostra città quant'ella gira. In: De Marchi A, Piraz G (eds) Santa Croce oltre le apparenze, ed. Gli ori, Pistoia 2011, pp 142–158
5. Arrighi G (1967) Un estratto del De Visu di M.o Grazia de'Castellani, in Atti della fondazione Giorgio Ronchi e contributi dell'Istituto nazionale di Ottica, 30/1 Arcetri, Firenze, pp 44–58
6. Arrighi G (ed) 1964, Paolo Dagomari, Trattato d'aritmetica secondo la lezione del codice magliabechiano XI, 86 della Bibliotaca nazionale di Firenze, Domus Galileiana, Pisa
7. Al-Khalili J (2013) La casa della saggezza – l'epoca d'oro della scienza araba, Bollati Boringhieri, Torino, pp 225–229
8. Porta G (ed) (1991) Nuova Cronica di Giovanni Villani. ed. Guanda, Parma
9. Pastoureau M (2012) Bestiari del Medioevo, Torino Einaudi
10. Pastoureau M (2010) Animali celebri, Firenze Giunti
11. Russo L (2013) La rivoluzione dimenticata. Il pensiero scientifico greco e la scienza moderna. Feltrinelli

# Observation, Drawing, Modeling. Elements of a Cognitive Process Between Analogic and Digital for Design Learning

**Federico Alberto Brunetti**

## 1 Introduction

Geometry accompanies the knowledge of space and places both intuitively and through cultural education. Some innate elements are at the base the instantaneous perception and processes of visual recognition of forms. Moreover, the theorical teaching and practical activities allow to structure visual and tactile and knowledge of primary geometric shapes, as well as the capacity for the mental modelling of space. In our Design Laboratory following a similar procedure, both analogue and digital modeling methods are coherently explored through the assigned project. The results of some case studies, recently concluded, are presented here, oriented to the composition of elementary geometric elements for the construction of reticular architectural structures, and for the radio centric enquiry of vegetal elements. The aim of these experiences concerns the possibility of experimenting visual understanding, on the bases of drawing, modeled in an analogue way by hand, and then verified through digital procedures by means of modeling and rendering software. A further example concerns a modeling exercise, based on the Made in Italy Design collection of Triennale di Milano. These specific training courses took place within the framework of the new training methods defined in Italy in the recent *Alternanza Scuola Lavoro* guidelines (in collaboration with Parco Nord Milano and the Accademia—Fondazione Fiera Milano), where the training focus on soft skills is integrated with the learning in specific and ordinary didactic disciplines.

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*… - eppure io ho costruito nella mia mente un modello di città da cui dedurre tutte le altre città possibili, - disse Kublai- Esso racchiude tutto quello che risponde alla norma. Siccome le città che esistono s'allontanano in vario modo dalla norma, mi basta prevedere le eccezioni alla norma e calcolarne le combinazioni più probabili. …*

Italo Calvino, *Le città invisibili*, p. 69 (1993)

*… - yet I have built in my mind a model of city from which to deduce all the other possible cities, - said Kublai- It contains everything that meets the norm. Since the cities that exist go away in various ways from the norm, it is enough for me to foresee the exceptions to the norm and calculate the most probable combinations. …*

## 2 Model, Etymological Elements

Modeling is a common practice in the modality of representation the project. Whether it is a first heuristic morphological verification phase, or an intermediate study finalized to evaluate the complexity of the volumetric intersections and constructive feasibility, or even in the final presentation of a project to the client's evaluation and to the general public, *the model it is a kind of physical concretization of specific design thinking*. This method of representation/presentation actually has profound methodological motivations and interesting etymological derivations. Deriving from the Latin *modus/modulus*, where it was originally proposed as an indication to an original form—and proportional element—to be acquired as an original and emblematic specimen to refer as a model to imitate, to be followed for the execution of an artifact (sculptural, architectural, iconographic, etc.). However, from this original concept derives the model term itself as a copy of the original, made according to the criteria of the sample artifact, derived from the reference unicum, so that it can serve itself as a replicable specimen [1]. In the history of sculpture, from the beginnings of classical times, the canonical model and its replicas alternated according to this meaning. In particular, and since nowadays, the practice of the didactic copy of the plaster models has accompanied the sculptural training in the teachings of the Fine Arts Academies. Starting from this historical background, which persists in influencing the current sense of the term model as a standard element of typological reference, the concept—closer to those of theory and vision—of the model is usually intended as a three-dimensional representation of simplified complexity and reduced scale of the project—or existing object—taken as a case study. An equivalent term in this sense is "maquette", deriving etymologically from "macula" intended as a sketch, or the first shareable and interlocutory sketch of the final design artifact.

The model can therefore can be briefly defined as a simplified representation of a complex element, which is thus reduced (in terms of dimensional scale, in the materials used—mimetic but not necessarily identical in the operation limited to morphology or ergonomics, and not necessarily working) for a preliminary assessment of the final realization. A more advanced term is that of a "*prototype*" which—normally on a scale of 1:1—anticipates the main ergonomic functions, operations and technologies prepared to verify a complete first simulation before the final realization or in series.

## 3 The Model Between Project and Epistemology

The model as a concept and as an actual practice (analogical, and today also virtual but, as we will see, above fundamentally as mental) that has been continuously effective in the design practice since the Renaissance, when the figure of the architect became autonomous, configured differently by the other construction workers, as an intellectual figure able in art of drawing and direct referent of the customer [2]; nowadays is definitely considered present in modern culture and since in contemporary design practice. A further meaning, to be considered particularly complex and articulated, requiring a specific treatment not possible here, derives from the meaning assumed in terms of *scientific epistemology*: model is intended as formalization (visual, descriptive, spatial but in any case heuristic) of an interpretative theory of complex phenomena that we are investigating. The scientific model is avowedly declared as non-mimetic, but as a logical-formal device that hypothesizes in a diagrammatic and potentially analogous and comparable—therefore falsifiable—way the characteristics of the phenomenon investigated [3]. It should also be emphasized that the debate concerning the descriptiveness of the mathematical and geometric disciplines has developed in history in a recursive trend between the desire to represent concrete objects and phenomena, deriving interpretative formulas, and equally cultivating the abstract speculation of logic-formal arithmetic and geometry—alphanumeric and graphic/visual—as a potentially self-referential disciplinary field, as long as it is consistent with its own axioms. Mathematical modeling would deserve an epistemological, historical and methodological treatment that is not possible to refer here. In today's digital computational era the forms implemented to represent mathematical functions[1] are a case study that feeds the extremely fertile relationship between mathematics [4] art and science. Nevertheless, we carefully inherit the heritage of analogical study models preserved in museums[2] where in the last century some of the main three-dimensional mathematical functions and mathematical equations were formalized in polished plaster and wood forms [5] (Fig. 1).

[1] http://mathematics-in-europe.eu/?p=746.

[2] https://www.museoscienza.org/dipartimenti/catalogo_collezioni/lista.asp?arg=Modelli%20matematici&c=10.


**Fig. 1** **a** Left: model in a white chalk shaped form, of a third degree polynomial equation (inv. No. 3998). **b** Center: surface place of points of space whose coordinates satisfy an equation of third degree (i.e. "cubic" or "third degree"), with a single singular point double uniplanar (inventory number 3978). **c** Right: third degree equation model (i.e. "cubic" or "third degree"), with two singular points (or surface irregularities), one conical and the other biplanar (inventory number 3974). Leonardo da Vinci National Museum of Science and Technology in Milan. Campedelli Luigi Collection 1951–1956; Contents developed with the contribution of the Lombardy Region and included in the regional catalog Lombardy Cultural Heritage. License Creative Commons Attribution—(CC BY-SA) 4.0

## 4 The Model Between Analog and Digital: A Synergistic Approach

Therefore, aspects of theoretical figurations, of effective materials, of effective processualism are predisposed to produce or present each model.

Similarly to the diagram that I presented in *Through the Design* (FB 2012), where around the term "Drawing" the trio "theory–hermeneutics–algorithm" [6] was tripartite, the same pattern can be hypothesized to arrange around the term "Model" by the trio "concept–materials–processes": infact to the modeling act converge both the quality of formal design ideals that inspire it, and concrete (or virtual) elements to realize the actual form and its perception, and finally the assembly procedure and/or modification and reconstruction that must remain potential in the design model as an evolvable hypothesis (Fig. 2).

"*Concept*": in fact the model must be able to represent an idea in its visible, tactile and interactive form, such that it allows the user/observer to understand the project—or object—through the model's artifact; "*Materials*": this factual sperimentation can be developed through the "construction" of the model that intrinsically allows a feasibility check; "*Processes*": since processing the material itself implies and allows the evaluation of the chronological and programmatic sequences of the realization, indicating the space-time characteristics intrinsic to the realization of the project. Idea, material, process: in this triad it is articulated both the realization of a model, but nevertheless the final artifact of which the model is a preliminary simulation; the material technical conditions—actual or simulated—are factors of evaluation and correction through the model towards the executable and final project.

**Fig. 2** Comparative diagram: trio drawing/model

Algorithm ↘ ↙ Theory

Drawing

↑

Hermeneutics

[FB 2012]

***

Concept ↘ ↙ Processes

Model

↑

Materials

[FB 2019]

It is interesting to recall here the exhibition "Sempering", contextual to the XX1 Triennale of 2016 in Milan curated by Luisa Collina and Cino Zucchi, in which concerning the contemporary architectural design, some key words are given as indicative of a technical action into morphological material design: *stacking, weaving, blowing, molding, connecting, folding, engraving, tiling, covering, arranging*.

This current finding derives by explicit analogy, in the generative project of architectural forms, from those definitions relationship between art and technique developed by Gottfried Semper in: *Der Stil* [7]. There the articulation of the four arts (textiles, ceramics, tectonics-carpentry and stereotomy) were schematically indicated the different original matrices of the architecture itself as a dialectic craft between forms and structure. It should be noted that this happens in the same age as the industrial arts [8] went to configure the post-artisan production of mass-produced artifacts, a prodrome of what will then be called "Design". This derivation of the forms of the project from the logic and procedures of the technique, present both in the final project and in simplified mode also in the executability of the model, underlines how in the construction of a maquette, are settled by some cognitive procedures similar to those necessary for the realization prepared of the final artefact to which it refers.

Here I would like to remember a motto attributed to Franco Albini where he recalled that: "*To lay the wood on the stone it requires metal*": i.e. even in the analogical construction of a model it is possible to experiment and understand the material and compositional syntaxes that will be part of the final artifact.

In the concise excursus, here only indicative, it is worth highlighting a long itinerary, which reaches up at least to the second half of the twentieth century, concerning the planning custom of analogical models not only in terms of verisimilitude, but also of performative checks, that have accompanied the design until a few decades ago and sometimes up to now, in order, to test the static and dynamic characteristics previously calculated in the laboratory: dams, skyscrapers, and domes in scale were real reliable simulators of the safety of the final artifacts.

Particularly emblematic are the images—only these remain—of the upside down turned cover model for the analogical calculation of the catenary vaults of the

Sagrada Familia of Barcellona by Antoni Gaudi, or the models by Pierluigi Nervi for static/dynamic testing and processualism in the constructive seriality of the prefabricated modules of roofs turned into reinforced and prestressed concrete [9].

Definitely interesting for the intuitive and methodological training of structural complexity is the "*Spaghetti & Structures*" competition held in 2005, within the course of Construction Technique of prof. Lorenzo Jurina, at the Milan Polytechnic.[3] Hereafter the structural calculation that organizes the three-dimensional grid, the students were invited to carry out experiments on scale models of bridges, designed by themselves, and realized with simple uncooked spaghetti pasta, in order to verify the limits of the load and/or dynamic resistances in fractures and failure in a completely harmless but experimentally effective way.

About the history of Design I must remember that Milan has been the main protagonist city of the "Made in Italy" design. In this town, in the years of economic recovery following the post 2nd war period, has been developed an industrial district for the improvements of technical and creative professionalism, even able to develop in analogue models and prototypes—antecedent to the digital age—for the best presentation of the projects entrusted to them. Some figures of experts leaved a masterful example: I remember for all of them Giovanni Sacchi[4] and Paolo Padova.[5]

Up to now in wind tunnels, and actually by appropriate digital sensors, the fluid dynamics of the wind resistance of bridges, buildings, cars, etc. are tested on scale models.[6] In the contemporary scenario, dominated by the computational potential of the digital algorithms, we can use devices that generate new morphologies in the design of the projected shapes [10], so the value of the model appears furtherly updated: in fact the projects—or objects—reaches complexity not always easily schematized in plans or sections; therefore the current functional and technological complexity benefits from integrated modeling of the various components (also according to the BIM protocols). This occurs thanks to interactive three-dimensional virtual models—which can be examinated on the computer screen or even in interactive stereoscopy using VR/AR viewers; but still these are—even if virtual—models.

But not only: in the most important Architecture and Design studios (we can simply mention Foster + partners and Renzo Piano's RPBW) there are permanently present analogical (solid) modeling departments of laboratories that, both with 3D printers and with activities manuals assisted by material processing equipment, accompany the various phases of concept, development and presentation of the projects. Finally I want to highlight that in the new Campus of the Milan Polytechnic conceived by Renzo Piano, an interdepartmental modeling structure is planned.[7]

So, the feature of our age show the evidence about not a *return to* the model, but a *return of* the model: passed by the era of modernity and post-modernity with varied

[3] http://civile.ing.unipi.it/it/news/107-spaghetti-structures.

[4] http://www.archiviosacchi.it/ access 2019.07.12.

[5] http://www.dipartimentodesign.polimi.it/lab-allestimenti-paolo-padova access 2019.07.12.

[6] http://www.windtunnel.polimi.it/ access 2019.07.12.

[7] https://www.polimi.it/en/the-politecnico/university-projects/construction-sites/new-architecture-campus/.

continuity, the cognitive validity of this means of representation and presentation appears decidedly renewed and vital.

## 5 Mental Model in the Fulcrum

What I would like below to point out briefly, both from a conceptual, experiential and training approach, is the centrality of the mental model of space: this "analogical" definition due to research on "mental images" by Stephen Kosslyn [11], developed by the same author recently with digital accuracy also through tomography and brain magnetic resonance techniques. Kosslyn "*investigated the images we mentally form to represent objects, environments, people not present, and on the neural substrate of that activity. In his theory …., starting from this propositional structure, two-dimensional images would form in our mind that it would then be possible to translate, rotate and inspect in the individual parts*".[8]

In this sense I would like to recall the idea that any model, concretely constructed in analogue mode or virtually in digital mode, is necessarily derived, founded or made perceptible by a mental model [12] by way of a three-dimensional image elaborated in our non-ocular visual abilities, but based on imaginative mental figuration which allows to conceive and recognize the three-dimensional shape understanding.[9]

Even for a child during the construction of a simple game, an idea is needed to pursue, or even to "discover in progress": any modeling apparently pertains to the outside world, but it can only be articulated and developed thanks to an inner capacity to organize forms and images. The well-known works of Bruno Munari,[10] an italian author famous as a designer of toys or rather as a player of design, exemplify this dynamic between matter, imagination, memory and compositional process, better than any abstract description.

In the fulcrum of this process the hand is the main means of gestural articulation of the body towards the space that surrounds the person: endowed with the finest tactile perception, it is the organ of operative abilities that, integrated with the three-dimensional view, and the hearing, allow the mind to construct and intervene in the shapes of the surrounding world. The anthropological value of the hand in the evolution, and learning of knowledge, is a crucial theme of the ability to construct models, both factual, mental and cognitive.

A large literature explores this nodal matter, I remember the fundamental one of Leroi Gourhan [13].

In Fig. 3 I propose a diagram to map the dynamics that I think we can detect, to summarize the evidence, of this very brief excursus—also in reference to the cited bibliographical sources—concerning of the actuality of the model.

[8] http://www.treccani.it/enciclopedia/stephen-michael-kosslyn/.

[9] http://www.mat.uniroma2.it/LMM/BCD/SSIS/Neurosc/Linguaggio/Linguaggio.htm by Catastini L. (Istituto d'Arte di Pisa "F. Russoli", Università di Roma "Tor Vergata").

[10] http://www.munart.org/ about the life, works and bibliographic source about Bruno Munari.

**Analogical model** ←→ **mental model** ←→ **virtual model**

Modifiable / not irreversible    perceptible - constructive mind - hand skills    interactive / reversible

Compositional process - Sequential and dynamic

Mimetic material textures ←→ Catalog to database rendering

sound of the material ←→ simulative contextual environment

Static simulations of analog dynamics ←→ Static simulations of digital dynamics

← 3D print ← → 3D scan →

Spherical camera – spherical scanning - spherical vision

Organism / simulacrum

**Fig. 3** Comparative diagram: Analog/mental/virtual model

At the center is placed the mental model of space: as an interactive/perceptive three-dimensional mnemonic design processing from which, and to which, every model is definitively originated and destined.

From this first indicative diagram we can highlight how the invisible presence of the mental model—essentially unconscious even to the same "user" or observer as correlated to his proprioception—is here proposed as the central condition for the experience or implementation of the analogical or virtual model.

Both in the practice of manual and technical execution of the analogical model—in which the drawings or constructive sections act as a guide for the assembly of elements in the spatial reconstruction of the final form—and in the digital production of a virtual model—computationally generated by Cartesian coordinates that define it through its interactive rendered rendering, possibly stereoscopic—the forms of design can be perceived three-dimensionally, and recognized as verisimilar, since they are captured by the same figurative processes that make three-dimensionality itself conceivable.

As in the vision and tactile perception of an analogical model—made by the user or manipulated when already realized—it is evident that the morphological understanding takes place through the senses but thanks to the mind that reconstructs it in an internal memory; also in the virtual experience—even if it can be optimized, by photorealistic and clear view—and it is carried out thanks to the same process of internalization that makes the pupillary vision flow to the mental memory of space.

It should be noted that actually the cinema visual effect softwares are at the forefront of figurative ability, thanks also to a generation of creatives endowed with profound technical skills able to create images, models and stories that are so congruent with human imagination, so that we can entrust images often more interesting and engaging in everyday reality itself. The experience by interactive and stereoscopic [14] vision, today already available, allow immersive and augmented experiences [15] of a considerable cognitive and illusionistic potential. At the end of the diagram (Fig. 3) I recalled the 3D scanning and 3D printing procedures which, despite the technically feasible limits, allow reversing engineering from an analogical object into a digital model [16].

In this case the fulcrum of the process obviously lies in the potential of the computational model, but which also must be directed by a technical operator that must be really able of interpreting the complexity of the real world, thanks to the personal skill of "see by modeling" the morphological articulations of the scene to be interpreted.

Finally three increasingly contiguous technologies can be mentioned here in this scenario: "Spherical camera—spherical scanning—spherical vision": these space optical processes, through their integration, make detectable, mappable and represented, in the form of morphological data integrated with three-dimensional images, not only of convex objects, but also all the surrounding 360° concave spatiality of environments in their entire globality [17]. The artifacts that can be created using digital 3D print modeling procedures (the concrete side on the presumable "analog" side) allows to realize objects at different scales, with morphological precision and in and environmental conditions that are sometimes unattainable for the capabilities of the human hand gestures [18]. We can hypothesize that the visual virtualization, linked to the robotics productive scenarios, will have only the limits, defined by the choices of profitability, ruled by with a new techno-ethics will that be able to direct them [19].

## 6 Morphology in Nature, Technique and Design, Through Modelling

This even concise and interdisciplinary dissertation is to motivate the cases of study of Didactics for Design that are presented here below. In this direction the following case studies show some experimentation in exercising a capacity of learning for a formalized attention method towards reality—naturalistic or technological—oriented by interpretative criteria derived from geometric-mathematical definitions. This training is based on the direct observation experiences, the drawing from true, the geometrical redesigned interpretation and the analogue—and also virtual—three-dimensional modeling. These skills have been developed in educational projects that I coordinated for the *Alternance school-work* protocols [20] in the Design sections at the Liceo Artistico di Brera in Milan, in particular with the students of the classes 3Dha, 4Dha and 5Dha in the years 2017–2018 and 2018–2019 in collaboration with ParcoNord Milano,[11] Fiera di Milano-Fondazione Fiera Milano-Accademia,[12] and Triennale di Milano.[13] Since these following case studies has already been recently presented, and published in the proceedings, of the previous APLIMAT2019 conference, I refer to that publication for a broader review of their images [21]. Following the visual language particularly suited to this artistic formative order, these experiences were conducted primarily through drawing from life and freehand concept, but soon oriented by a fundamental geometric competence for the realization of models to understand case studies examined. In fact the modeling presupposes an in-depth knowledge of the morphology that we intend to give back, and not least the inevitable simplifications—linked for example by the reduction of scale, or by the adoption of materials that at different sizes of the original emulate the structural or material simulation characteristics.

## 7 Radio Centric Morphologies in Natural Dynamics

The first experience has been carried out at the Parconord Milan, a public body that carried out by the renaturalization on the outskirts of a disused area, and now

[11] http://www.parconord.milano.it/ access 2019.07.12.

[12] https://www.accademiafieramilano.it/fondazione-fiera-milano.html. https://www.accademiafieramilano.it/ access 2019.07.12.

[13] https://www.triennale.org/eventi/museo-del-design-italiano/ access 2019.07.12.

recovered in green, where also the environmental culture is part of the mission of such a territorial structure: in fact the environmental recovery is accompanied by an incessant training and communication activity on sustainability issues. In this environmental context we have explored the dynamic and organizational qualities of the botanic radiocentric natural structures: with this reference we have learned to take attention to nature and the plant finds that we could observe; for example the leaves, or gems, flowers, fruits and seeds interpreted as temporal phases of the same vegetable organism that evolve in the short course of the season branching in the space, optimizing according to the species the osmotic interaction and capturing of the solar luminous radiation (Figs. 4, 5 and 6).

So, we have proposed, following accurate observations and naturalistic drawings from life, to invent and design fantastic plants based on radiocentric structure. As we would draw up a herbarium of finds, similarly we have built a "*wunderkammer*" of vegetables models imagined as an evolution of the geometries that the students were able to internalize and return as morphological patterns: designed, sectioned and constructed as models. In this way we have systematically explored the observation from

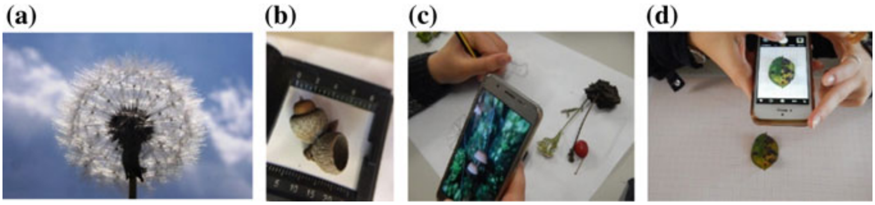


**Fig. 4** **a**, **b**, **c**, **d** Students workshop about naturalistic radiocentric structures ar ParcoNord Milano (picture of the Author 2018, with courtesy of Liceo Artistico Statale di Brera)

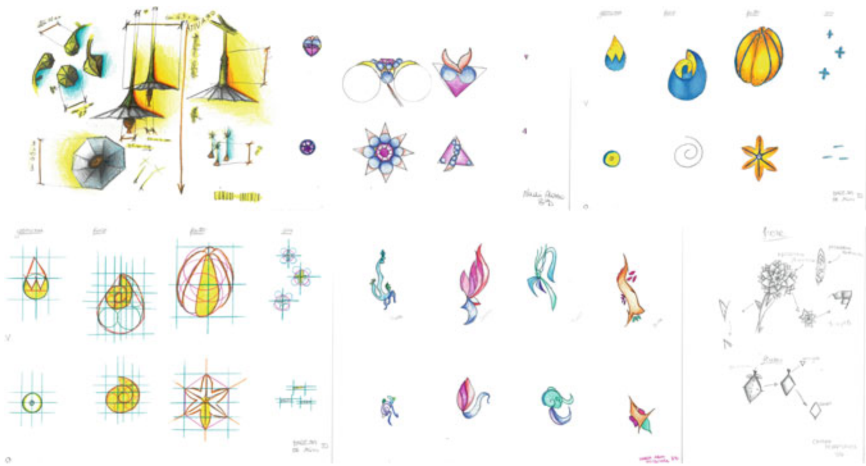

**Fig. 5** Students' didactic drawing of radial morphologies during the morphological workshop (with courtesy of Liceo Artistico Statale di Brera)

**Fig. 6** Students' didactic models of radial morphologies of fantastic vegetal

life, the drawing from life according to interpretative criteria guided by geometry, the possibility of deciphering the mathematical process underlying the diversification of specific forms (Fig. 6). A particular thanks and reference debt goes to the essay presented at last APLIMAT2018 by professors Salvadori and Brandi [22][14]: through the mathematical algorithm of the "superformula" conceived by Johan Gielis, which allows to simulate the configurations obtainable in a 3D virtual environment by modifying the value of the setting of some variables that govern the results of the algorithm that prefigures them. Just as mathematics and digital algorithms are able to figure out scenarios of possible worlds, so students have hypothesized fantastic forms of plants that, although essentially endowed with their own internal design logic, do not correspond to what is present in nature, or at least to what so far discovered and taxonomized (Fig. 5).

As far as mathematics can approach nature, this recent algorithm allows us to experiment and learn to understand the forms of equilibrium and the "design" of the partitions that radiate them. Moreover, even some different apps, which can also be managed simply by students' smartphones, it is possible to represent these morphological evolutions both in representation of rotations solids, delineating the wireframe generators, or as volumetric surfaces with chromatic and chiaroscuro variations. Analogously, the rotational configurations obtainable through the radial sequences of a single-axis on one centered pole were verified by sketch up software, where typical commands of the digital design can be used to obtain circular crown configurations.

## *7.1 Reticular Structures in the Triangular Tessellation Roof of the Fieramilano Vault Gallery*

Another configuration relates the study of the reticular structure that has been recently built for the long covered gallery path, in the Milan fair in Rho, designed by architect Massimiliano Fuksas, morphologically evoking the fluidity. This is an emblematic case of a project originally generated with analogically modeled forms in a plastic way, which later was interpreted in a virtual environment to discretize its components in order to verify its stability and gait, and finally engineered to make each element

[14] http://www.matematicaerealta.cloud/mediateca/ access 2019.07.12.

part of a modular structure, available to be realized with a serialized process, identifying by branding each individual element for the final assembly procedure into the building site.

The theme of the reticular structure shows—albeit in an essential and variable way: not only rigidly planar, but articulated in possible progressive concave/convex undulations—to the well-known geometric theme of tessellations; this is a fundamental argument of geometric speculations oriented to perception and logical-formal visual of the modular space.

With small groups of students—whose collaborative attitudes has been also improved as soft-skills—we organized a workshop of groups in which unitary modular elements (straws) have been mutually constrained in fulcrums (stapling points). In this way different configurations of planar space were executed (Fig. 7).

Thus the regular flat surface is created (indeed not so obvious to realize both by individuals and groups, given the tubular structure of a single element) and other case studies then easily derivable, were not only from the mere point of view of the mathematical scheme, but taking advantage of the manual interaction of the material and the operation of the process and the shapes to be emulated. Tubes, vortices, globes have thus progressively furnished the suspended wired with the pillars of the classroom where we hung the light models.

The final result was focused to understand and to reproduce a model of the structure in the undulated gallery and vortex dome gallery of Fieramilano: a morphology conceived as a unicum plastic, discretized and engineered by digital computation of a nubs of triangular elements defined by slight difference in the size of the sides, introducing the three-dimensional planar variation, obtained by the realization of the relative robotic line of production (Fig. 8).

The assembly of these elements, according to standardized and serialized procedures, simply obtained by the variation of the structural segments, obtain the formal result of the three-dimensional modulations in the course of the vault gallery and of the vortices of the entrance portals and inverse convolutions of the logo of the Service Center (Fig. 9).

So has been realized the reticular model of the vortex portal at the East door of the Fiera di Milano, as the middle section with asymmetric flaps and the logo dome of the service center. These models are currently exhibited at the entrance hall of the Academia of the Fondazione Fiera Milano. A later version in a digital

**Fig. 7** Basic reticular pattern morphologies. Collaboration for a serial construction of reticular beams with tubular structures (straws), construction of planar layers with a regular triangular/rhombus pattern; plane reticular structure, curved towards cylindrical shaped form (picture of the Author 2018, with courtesy of Liceo Artistico Statale di Brera)

**Fig. 8** Drawing survey at the reticular structures in the triangular tessellation roof of the Fieramilano vault gallery (picture of the Author, with courtesy of: Accademia Fondazione Fiera Milano)


**Fig. 9** **a**, **c** Left/right: Final reticular models of the vortex at East entrance, an dome in Service Center of Fieramilano. At center: Design workshop room at Liceo Artistico Statale di Brera (picture of the Author © 2018, with courtesy of Liceo Artistico Statale di Brera)

model, based on SketchUp software into a virtual model, was obtained by emulating the general shapes of the longitudinal and transversal sections, and has been made available as an interactive experience by Kubity app software (Fig. 10): so both on the smartphone display, in augmented reality or VR stereoscopy (available on the device of the teacher and available through QRC from the test version of the sw.). I like to note that methodologically we have even reached at the stereoscopic digital

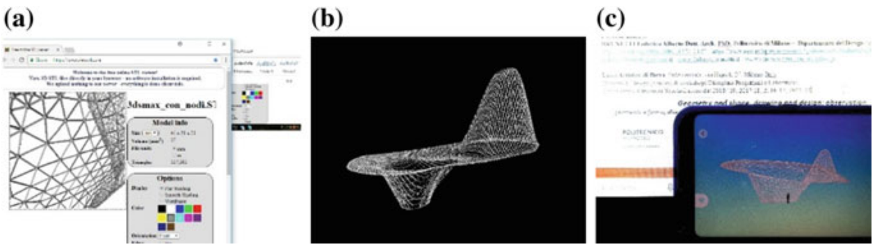


**Fig. 10** **a**, **b**, **c** Reticular study of the virtual model, interactively displayed on pc and smartphones, developed in collaboration with arch. Marco Valentino by the digital drawing educational project (2017–2018)

experience, through specific optical devices that allows the brain to process the paired visual images with the result of perceiving the proportionate three-dimensional depth of a model. This kind of experience can evidently be totally considered as a mental object, even if almost "tangible" in the view of its interactive verisimilitude.

So we can observe algorithms between nature and technique: i.e. to learn dealing with the form of nature through mathematical functions: the shape therefore reveal itself as a process. In the first case according to the simple evidence deriving from the radial and centrifugal directions derived from the dynamism that the biological growth introduces into space from the original fulcrum to the surrounding environment. In the second one with structures that from linear and discrete elements branch off planarly in the reticular space stabilizing the structure and allowing its modulations and possible topological involutions.

## *7.2 Centomodelli in Centogiorni (100 Models in 100 Days)*

The last case study here presented concern with an exhibition held at the Milan Triennale in 2018 [23] in the last set-up of the collection curated before the new inauguration of the new Museum of Design. Here has been proposed a chronological itinerary of the "Made in Italy" projects represented through a century of life and culture through an hundred of design objects symbolizing the development industry, of tastes and lifestyles: an intertwining between and the arts of our country. For these reasons, the exhibition proposed particularly emblematic objects from the point of view of the culture of the project and of the social history to which these case studies had been projected and destined. The origin of Design, despite the various critical chronologies that have fixed several milestones since the industrial revolution up to the '90s, now belongs to the ecosystem of objects in which contemporary generations of students were born and grew up in such artificial and anthropically functional landscape.

The relationship with things, instrumentally functional or even functionally symbolic, has accompanied the "sapiens" man in the tactile and visual interaction with the world, building a microcosm of tools, furniture, artifacts, decorations (and today devices) that constitutes a sort of concrete external memory by which the individual and social identity is accompanied and supported.

The study of these specimens was carried out through a drawing from life, the study of the designers and their professional excursus, the geometric reconstruction of the objects in the design configuration visible from the outside, and also in the hypothetical constructive sections and geometric drawings finalized to the reconstruction of a model in scale of the original case studies. This process of design learning, and of the morphological genesis from the concept to the modeled forms, has represented an interesting and direct formative path. Each student, from two different classes, had the commitment and the goal of analyzing and reproducing a maquette of the case study; therefore the groups of students have differently participated in a personal and collective experience, methodologically coordinated and

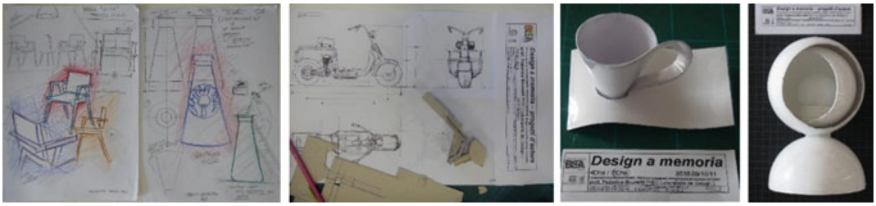


**Fig. 11** From drawing from life at the museum, to geometric scale structures, to study models


**Fig. 12** *Centomodelli in centogiorni*. Design Laboratory. Brera Art School, Design Address. prof. Federico Brunetti, classes 4Dha, 5Dha A. S. 2018/49 FuoriSalone 5vie—Exhibition at AIM, via S. Vincenzo 13 from 9 to 13 April 2019.04. *Design by heart—author projects—case studies* referred to the Triennale exhibitions in Milan: "stories of Design" and "A Castiglioni"

shared in the representation modeled on the history of 100 years of Italian design[15] (Fig. 11).

The result, briefly defined as "*centomodelli in centogiorni*" ("100 model in 100 days"), represented an intense and gratifying training experience through which the design culture originally presented in an institutional exhibition, definitely temporary, has been assimilated from young students. This experience has been so appreciated that a part of the collection of didact models has been invited by an important entity, the Associazione Interessi Metropolitani (A.I.M.) for a public exhibition in the context of the Fuorisalone 5Vie during the Salone del Mobile, where the entire city was sprinkled by with design-related events. Furthermore, it remained as a didactic kit for the class that was able to present this collective and personal result as a methodological topic in the final degree exams (Fig. 12).

## 8 Case Studies Conclusions

The opportunity demonstrated through these formative experiences has been the effective appropriating of meaningful concepts and drawings of samples of design by mean of the sketching from the truth, the geometric reconstruction, and in scale models realized by the exercise of accurate manual ability, that have allowed to put in common and interiorized what was seen, in a way that could hardly have been learned otherwise.

[15] http://www.liceoartisticodibrera.edu.it/100-modelli-in-100-giorni-aim-ospita-per-il-fuorisalone-gli-studenti-del-liceo-artistico-di-brera/ access 2019.07.12.

One challenge of our actual age has been recently defined as "datafication" of experiences [24, 25]: i.e. the progressive practice of embedding our everyday life, into algorithms, by the devices and digital clouds, that are surrounding us.

The contextual use of virtual modeling systems that pertains to these generations of students, is part of their level of digital literacy: an unavoidable skill for the professions they will undertake, but this artificialization of skills need to be enhanced by the conscious maturation of the inner substrate—both native than conceptual—of their mental modeling capacity of shaping forms in space, that is a substantial element of the conscious and operational learning of their designers training path.

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All the student of the class: 3Dha, 4Dha 5Dha A.S. 2017/'18–2018/19.

## References

1. Settis S, Anguissola A, Gasparotto D (2015) Serial/portable classic—The Greek Canon and its mutations. Progetto Prada Arte, Milano
2. Maquette numero monografico, Rassegna n. 32 dicembre 1987
3. Bonicalzi F. Il concetto di modello: aspetti storici e questioni teoriche
4. Bertacchini PA, Bilotta E, Francaviglia M, Pantano P (2005) Matematica, Arte e Industria Culturale—Mathematics, art and cultural industry. In: Atti del Convegno Nazionale "Matematica, Arte e Industria Culturale", Cetraro, 19–21 maggio 2005. (CD-Rom a cura di M.G. Lorenzi), EGS—Università della Calabria (Arcavacata di Rende, 2005)
5. Fischer G (1986) Mathematical models from the collections of universities and museums. Vieweg & Sohn, Braunschweig
6. Brunetti FA, Attraverso il Design, presentazione di un itinerario didattico. Esperienze per la formazione al Disegno, pp 1–6; in Brunetti FA (editor), Attraverso il Disegno. Relazioni tematiche, tracce metodologiche e atti didattici per l'architettura ed il design, CUSL, Milano, 2012, pp 1–6. [6 bis] Brunetti FA, Percezione, linguaggio, invenzione. Riferimenti teorici e metodologie operative. Leggere-scrivere/osservare-disegnare. Descrizione, interpretazione, prefigurazione. Identificazione delle componenti progettuali. "3 mondi", pp 7–14; in Brunetti FA (editor), Attraverso il Disegno. Relazioni tematiche, tracce metodologiche e atti didattici per l'architettura ed il design, CUSL, Milano, 2012, pp 7–14
7. Oechslin W, Hildebrand S (2003) Gottfried Semper 1803–1879. Achitektur und Wissenschaft. Prestel, München
8. de Quincy Q, Antoine-Chrysostome M, Farinati V, Teyssot G (1985) Modello. In: Dizionario storico di architettura: le voci teoriche. Marsilio, Venezia, pp 273–276
9. Barazzetta G (ed) (2007) Pier Luigi Nervi. Il modello come strumento di progetto e costruzione. Quodlibet Studio Recanati. ISBN 978-88-229-0074-6

10. Brunetti FA (2019) Tall buildings in Milano. Proportions and morphology between analogical and digital design procedure. Some considerations about Two case histories of contemporary construction sites: from the XX legacy to the XXI new century urban landscape. In: APLIMAT 2018. 17th Conference on Applied Mathematics, pp 137–153, 6–8 Feb 2018. Institute of Mathematics and Physics Faculty of Mechanical Engineering, Slovak University of Technology in Bratislava, Bratislava
11. Kosslyn MS (1980) Image and mind. Harvard University Press, Cambridge (Ma)
12. Brunetti F (2011) Morphology and mental imagery. Geometry and its application to architectural and design drawing learning. In: APLIMAT 2011, 10th international conference on applied mathematics February 1–4, 2011, Bratislava Faculty of Mechanical Engineering of Technology in Bratislava; APLIMAT—Journal of Applied Mathematics, 2012, pp 799–813
13. Leroi-Gourhan A, Le geste et la parola. Technique et langage. Albin Michel, Paris, 1964; ed. it. Il gesto e la parola, Einaudi, Torino, 1977
14. Brunetti FA (2014) Experience of three-dimensional vision in the era of digital interaction. New devices opportunity and challenges for applied mathematics and visual design. In: APLIMAT 2014. 13th conference on applied mathematics, pp 44–68. Proceedings Publishing House STU. Slovak University of Technology Bratislava, 2014, ISBN 978-80227-4140-8. Video lecture Conference participation: 5.02.2014
15. Brunetti FA (2016) Into the frame. The legacy of perspective in the technological design of visual digital devices. In: Proceedings of: ICGG2016 17th international conference on geometry and graphics 2016, Beijing ISGG 4–8, Aug 2016, article No. 149
16. Luhmann T, Robson S, Kyle S, Boehm J (2014) Close-range photogrammetry and 3D imaging. De Gruyter
17. Brunetti FA (2019) (Politecnico di Milano, Design Dept., Italy), Immersive visions. Some paradigm a of an exceeding iconography. Capture and monitoring by spherical cameras and interactive smartglasses. In: Cocchiarella L (ed) ICGG 2018—proceedings of the 18th international conference on geometry and graphics, pp 579–594. 40th Anniversary—Milan, Italy, August 3–7, 2018. Springer
18. Brunetti FA (2017) Augmented realities. In: Hammerling M, Cocchiarella L (ed) Informed architecture. Computational strategies in architectural design, pp 89–104. Springer. https://doi.org/10.1007/978-3-319-53135-9_9
19. Benanti P (2018) Oracoli. Tra algoretica e algocrazia. Luca Sossela, Roma
20. Brunetti FA (2018) Drawing as dialogue (Politecnico di Milano—Scuola del Design, Liceo Artistico di Brera, Milano, Italy). In: Proceedings of 4th international conference on higher education advances (HEAd '18). Universitat Politecnica de Valencia. http://dx.doi.org/10.4995/HEAd18.2018.8239
21. Brunetti FA (2018) The modulation of the shape. Radiocentric and reticular and morphology. Some didactic experiences in learning by doing through analogic nature and technical algorithm, pp. 97–108. In: APLIMAT 2019. 18th conference on applied mathematics, 5–7 Feb 2019, Bratislava. Institute of Mathematics and Physics. Faculty of Mechanical Engineering, University of Technology in Bratislava
22. Brandi P, Salvadori A (2017) Math-MAPS: a new road that opens up a world. Part I, Part II—Elementary dynamic models. In: Proceedings APLIMAT 16th conference on applied mathematics, pp 177–196, Bratislava, 31 Jan–2 Feb 2017
23. Alessi C, Dalla Mura M, de Giorgi M, Pasca V, Riccini R (2018) Storie. Il design italiano, catalogue of the 11th edition of the Triennale Design (RIPORTARE IN ITALIANO), Electa, Milano
24. Repčíková S (2013) Datafication of education enhancing teaching and learning through data mining and learning analyses. In: IEEE 11th international conference on emerging eLearning technologies and applications (ICETA), Slovakia Stará Lesná, 24–25 Oct 2013. IEEE Review, 2013. https://ieeexplore.ieee.org/document/6674454. Access 12 Sept 2019
25. Thoutenhoofd DE (2018) The datafication of learning: data technologies as reflection issue in the system of education. Stud Philos Educ 37(5):433–449

# Computational Process and Code-Form Definition in Design

**Giorgio Buratti**

In design process, drawing has always preceded the construction phase. The act of drawing, based on basic geometric elements such as lines, curves, surfaces and solid, allows to organize one's ideas, manage resources and predict results. In recent years, the increased level of digital literacy has led to a new kind of draw generated through the creation of algorithms. Form is not a priori defined, but it is consequence of a discrete rules set resulting from a refinement process of conceptual, communicative, structural and geometric instances, leading to the outcome that best meets the project hypotheses. This approach requires the adoption of theoretical analysis and understanding tools capable of managing a high level of complexity. In an age where the digital model can directly inform a machine able to manufacture it, the role of geometry is fundamental not only in understanding reality, but also in controlling the act of shaping matter. The paper analyses some experiences in design field where form is described and constructed by computational process.

## 1 Introduction

Computational design is a multidisciplinary area of study which, in general terms, can be defined as the application of computational strategies to design process and whose relevant aspect relates to the logical-creative nature of "calculation". In computation the real world's complexity is translated into elementary steps subsequently elaborated as algorithm, a systematic procedure based on a series of unambiguous instructions that explain how to achieve a specific objective. Combining computa-

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tional principles with design practice configures a new multidisciplinary area where the conscious use of IT tools is translated into procedures and rules for the project. In design context, computational project strategies are instantiated in the ability to manage complexity, understood both as a set of structures and relationships and as amount of data and information.

## 2 Computational Design and Complex Systems

The term "complex" does not simply mean "complicated"; rather, it is a precise definition that refers to the science of complexity, a field of research that has not yet been completely formalised but which is equipped with theoretical tools suitable for the new context. The term 'complicated' in fact, comes from the Latin *cum plicum*, which means "*paper crease*". A complicated problem can therefore be solved by explaining, or rather "smoothing the creases". 'Complex', meanwhile, derives from *cum plexum* which means "knot" or "weave". The solution to a "complex" problem lies in the intricate weave created by the knots, i.e. the relations among the elements. The study of complex systems implies the analysis of phenomena composed by a large number of elements, also diverse, that interact to create a dynamic that is not predictable when observing the behaviour of the individual elements.

This systems, apparently chaotic, can be described by non-linear and non-additive dynamics. In a linear system the effect of a group of elements is the sum of the effects considered separately. In the group there are no new properties that are not already present in the individual elements.

Meanwhile, in a non-linear system the whole may be greater than the sum of its parts as it is the connections between the various elements that determine the structure and organisation of the system. Collective properties that are not foreseeable a priori emerge as a result of the multiple interactions between the various agents that make up the system. These dynamics disappear as soon as the system is separated, materially and theoretically, into isolated elements.

In the systemic vision the units are relationship patterns, inserted within a broader network of connections. In design, for example, form may be considered the result of the interaction of precise formalisable and quantifiable conditions (formal aspect, materials, physical and temporal constraints, pre-established goal, interaction with the user, economic and production factors) and a creative instance that must be implemented. These determining factors interact reciprocally to achieve a common goal and so the design process has all of the typical characteristics of a complex system.

The revolution inherent computational design is the possibility to represent relations and processes. In this new dimension the various design instances can be organised in emerging relational structures that transfer typical characteristics of living systems to the design process, such as the ability to adapt and transform, and self-organisation. This behaviour cannot be controlled according to the classic linear method (topdown), which seeks to predict all possible situations and subsequently prescribes the solution for dealing with them. Only by defining the behaviour of entities on the basis of the design (bottom-up) and leaving the task of simulating the collective effect of the interactions to the calculating power of the computer is it possible to check the validity of the design hypothesis [1].

If we view design as a complex system based on the interconnection of various factors, computational design is the device capable of integrating the interaction of the various components, fostering the interaction of the physical context, cultural characteristics, social aspects and construction system [2].

## 3 Managing the Complexity: Triply Periodic Minimal Surfaces

This paragraph presents an experimentation that shows the potential of algorithmic generative modelling and check the veracity of the theoretical deductions.

These tests were carried out using Rhinoceros, a McNeel CAD software, which combines a powerful NURBS engine, ideal for creating and managing complex forms, with a complete programming environment based on Visual Basic language.

The use of the programming code was simplified thanks to the use of the Grasshopper plug-in. The application is based on already-compiled functions which, without requiring specific knowledge of the programming language, can be assembled directly in the graphic interface, inspired conceptually by flow diagrams.

The potential of the tool was applied to the study of minimal surfaces, geometric objects with very interesting characteristics, not only in design terms. In recent years, many scientific disciplines have been turning with great interest to the study of minimal surfaces. This focus is justified both by the problems of a mathematical nature that have been revealed by the research and by the discovery of a number of properties (mechanical, structural and associated with electrical conductivity) that are distinctive of them. Configurations of minimal surfaces have been found in a wide variety of different systems: from the arrangement of calcite crystals that form the exoskeleton of certain organisms and the composition of human tissue to the basic structure of synthetic foams and the theories that explain the nature of astronomical phenomena.

A minimal surface is a surface whose mean curvature is always zero. This definition is closely related to the Plateau Problem, also known as the first law: if a closed polygon or oblique plane (similar to a closed frame of any shape) is assigned, then there is always a system of surfaces, including all possible surfaces that touch the frame, which are able to minimise the area. In other words, the problem is to identify the shape which covers the largest surface with the same perimeter [3].

To make this principle clear it is necessary to gain a deeper understanding of the concept of the mean curvature of a surface: consider point **P** of a surface and the perpendicular to the surface at point **N**, which therefore intersects the surface with the plane $\boldsymbol{\pi}$ on which **N** lies (Fig. 1) [4].

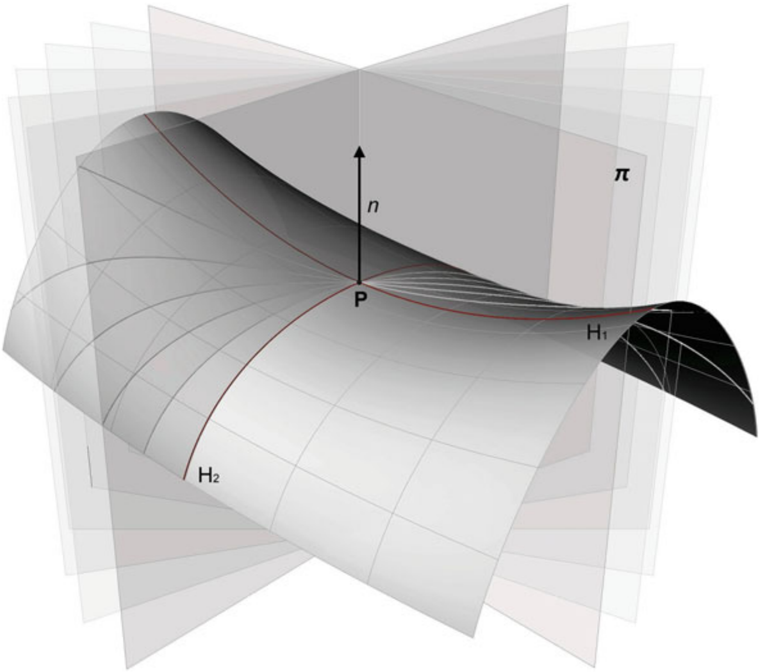



**Fig. 1** Perpendicular plane, tangent to the point P and the principal curvatures of a saddle minimal surface

In the intersection curve obtained consider the curvature at point **P**. Even at an intuitive level the curvature provides information on the behaviour of the curve: if a straight line is taken as the example of the curve, there is no inflection and the curvature in this case will be zero, whereas in the particular case where the curve is a circumference its inflection will be constant at every point. For a generic curve, the curvature, which varies from point to point, is defined by the construction of the osculating circle, i.e. the circle tangent to the curve that best approximates it, and will therefore be defined by the relation **Cp = 1/r** where **r** is the radius of the osculating circle.

If the plane **π** is rotated around the perpendicular **N**, then for each of the positions of the plane section curves are obtained that are characterised by a different value for the curvature at point **P** (obviously if the surface considered is not a sphere, in which case they will all be of equal value). In the case of a generic surface between the different curvatures the one whose value is the largest and the one whose value is the smallest are preferred, which are designated as the principal curvatures of the surface and indicated with **H1** and **H2**.

The mean curvature H is the algebraic sum of the two principal curvatures defined by the equation:

$$\mathbf{H} = \mathbf{H1} + \mathbf{H2/2}$$

It follows that the equation that characterises minimal surfaces (also called Lagrange's theorem), expressed in terms of the principal curvatures, becomes:

$$\mathbf{H1} + \mathbf{H2} = \mathbf{0}$$

This condition can be obtained either because both values are zero, as in the case of the plane which is therefore a minimal surface, or because:

$$\mathbf{H1} = -\mathbf{H2}$$

That is, at any point one of the principal curvatures is concave and one is convex, as in the case of a saddle surface.

## 4 Description and Genesis of Minimal Surfaces: Implicit Method

Minimal surfaces can be described in different way, in this work we will only talk about the formulation we used: the implicit method. The implicit form is appropriate to the digital description because it allows the handling of the large number of elements that characterize TPMS, without overload the calculation process and also does not allow self-intersections. Typically, an implicit surface is defined by an equation of the form:

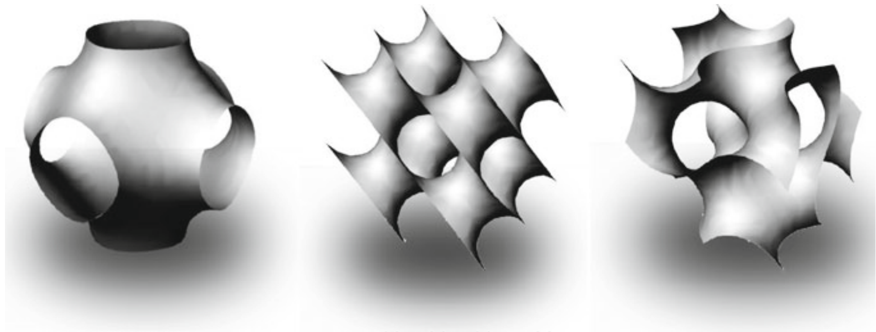
$$f(x, y, z) = 0$$

The implicit surfaces divide the space into three regions, where:

$f(x, y, z) < 0$ for points outside the surface
$f(x, y, z) > 0$ for points inside the surface
$f(x, y, z) = 0$ for points on the edge

Some minimal surfaces (Fig. 2 P, D and G surfaces) can be described implicitly to a good degree of approximation by the following equations:

$$\begin{aligned}
&\text{P: } \cos(x) + \cos(y) + \cos(z) = 0;\\
&\text{D: } \sin(x)\sin(y)\sin(z) + \sin(x)\cos(y)\cos(z)\\
&\quad + \cos(x)\sin(y)\cos(z) + \cos(x)\cos(y)\sin(z) = 0;\\
&\text{G: } \cos(x)\sin(y) + \cos(y)\sin(z) + \cos(z)\sin(x) = 0;
\end{aligned}$$



$P: \cos(x) + \cos(y) + \cos(z) = 0;$

$D: \sin(x)\sin(y)\sin(z) + \sin(x)\cos(y)\cos(z) + \cos(x)\sin(y)\cos(z) + \cos(x)\cos(y)\sin(z) = 0$

$G: \cos(x)\sin(y) + \cos(y)\sin(z) + \cos(z)\sin(x) = 0;$

**Fig. 2** Digital description of P-surface, D-surface and Gyroid

Ideal as it is for the rapid display of certain surfaces in digital representation, the implicit method provides no information on the topology.

## 5 Triply Periodic Minimal Surfaces

Triply Periodic Minimal Surfaces (Tpms) have interesting characteristics for project purposes. They are called periodic because they consist of a base unit that can be replicated in Cartesian space in three dimensions (triply), thus creating a new surface seamlessly and without intersections [5].

A uniform minimal surface is, usually, characterised by different curvatures; in other words, some surfaces are flatter than others. It follows that not all points of the surface support any concentrated loads equally well.

If the same surface is, however, associated with a periodic distribution, i.e. the individual units are repeated next to each other, the physical iteration between the modules causes a compensatory effect that greatly increases their structural efficiency.

This is achieved, by the definition of minimal surface, through the use of as little material as possible.

The advantages mentioned above are real when the surface obtained is a system under voltage or the material with which it is constructed is able to withstand tensile stresses and compression.

In summary:

1. Tpms have natural geometric rigidity
2. Allow optimum use of materials
3. Configure stable and resistant structures.

There is a large number of known embedded triply periodic minimal surfaces. Moreover, it seems that the examples come in 5-dimensional families, most of which are only partially understood [6]. Our lack of knowledge of these surfaces makes it very hard to put them into categories. For the moment, we use the genus of the quotient by the largest lattice of orientation preserving translations as a guide. In this thesis we study three–periodic minimal surfaces that have three lattice vectors, i.e., they are invariant under translation along three independent directions. Numerous examples are known with cubic, tetragonal, rhombohedral, and orthorhombic symmetries. The symmetries of a TPMS allow the surface to be constructed from a single asymmetric surface patch, which extends to the entire surface under the action of the symmetry group (Fig. 3).

The most important local symmetries of minimal surfaces are Euclidean reflections (in mirror planes) and two–fold rotations. Many triply periodic minimal surfaces can best be understood and constructed in terms of fundamental regions bounded by mirror symmetry planes. According to H. S. M. Coxeter[1] there are exactly seven types of such regions of finite size. Many triply periodic minimal surfaces have embedded straight lines, which of necessity must be C2 symmetry axes (180° rota-

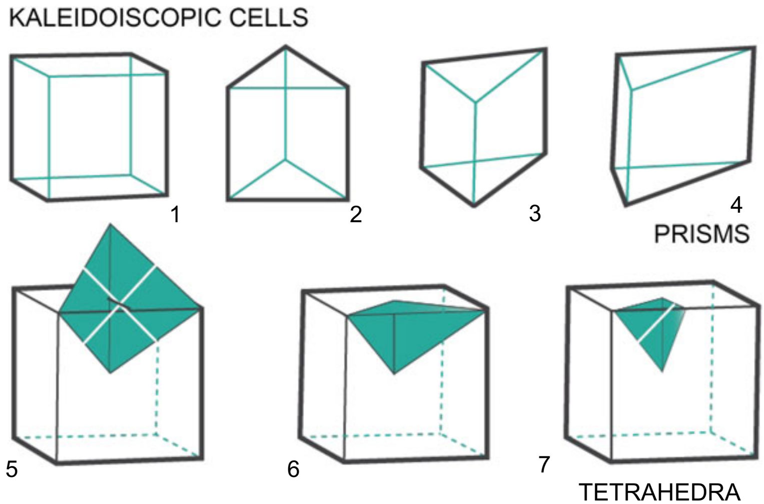



**Fig. 3** Kaleidoscopes cells: (1) Rectangular Parallelepiped; (2) Equilateral Prism; (3) Isosceles Prism; (4) Rectangular Prism. (5) Tetragonal Disphenoid, this tetrahedron can be viewed as two trirectangular tetrahedral stacked up. There are three possible C2 axes, shown in white and black; (6) Trirectangular Tetrahedron, this tetrahedron is shown as 1/24of a cube, (7) Quadrirectangular Tetrahedron, this tetrahedron is shown as 1/48 of a cube, it is the fundamental region for the full symmetry group of the cube. There is one possible C2 axis, shown in green

[1]Harold Scott MacDonald Coxeter, (1907–2003) was a British-born Canadian geometer. He was most noted for his work on regular polytopes and higher-dimensional geometries. He was a champion of the classical approach to geometry, in a period when the tendency was to approach geometry more and more via algebra.

tional symmetry). Possible C2 axes are shown in color below. There are two classes of kaleidoscopic cells: the prisms and the tetrahedra. A prism in the general sense is a plane polygon extended at right angles in the third dimension. A tetrahedron is a polyhedron with four flat faces. The relations of symmetry previously described structure the different Tpms (Fig. 4).

Now we have to find a way to generate and control the TPMS in digital environment. The computation played an essential role in the simulation and modelling process of such complex phenomena [7]. It was used Grasshopper, a graphical algorithm editor tightly integrated with Rhino's 3-D modelling tools in order to create an algorithm able to describe and to control various types of TPMS.

Using Grasshopper it's possible to define algorithms that are able to describe with good approximation any minimal surfaces directly from its implicit formulation. The algorithm translates the algebraic equation into a finished form that can be studied, manipulated and replicated. The process can be conceptually simplified imagining that, in the domain of Cartesian space, the equation "selects" points, belonging to the surface you decide to represent. The next algorithm's instruction connects them by triangulation creating the surface. It is now possible to exploit the symmetry characteristics of the single unit by replicating it in a symmetrical cell, which is suitable to further replication in a modular lattice and to study the processes of adaptation to any required morphology (Fig. 5).

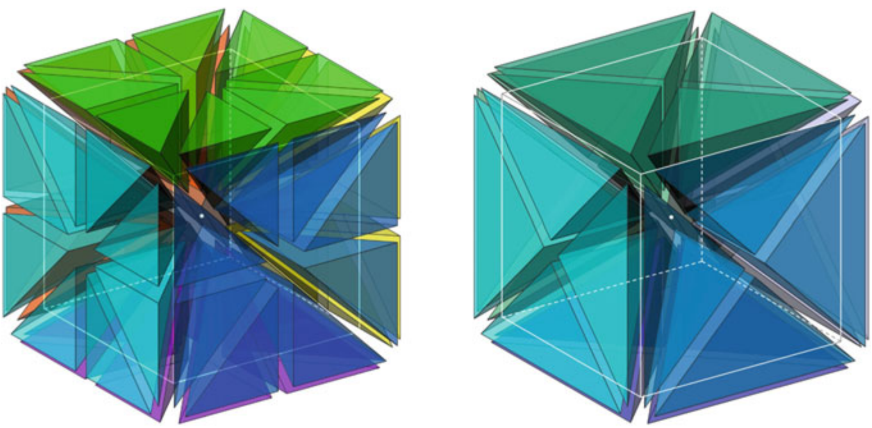


**Fig. 4** The cube has 13 axes of symmetry: 6*C*2 (axes joining midpoints of opposite edges), 4*C*3 (space diagonals), and 3*C*4 (axes joining opposite face centroids). It can be divided into 24 *Trirectangular Tetrahedron* or 48 *Quadrirectangular Tetrahedron*

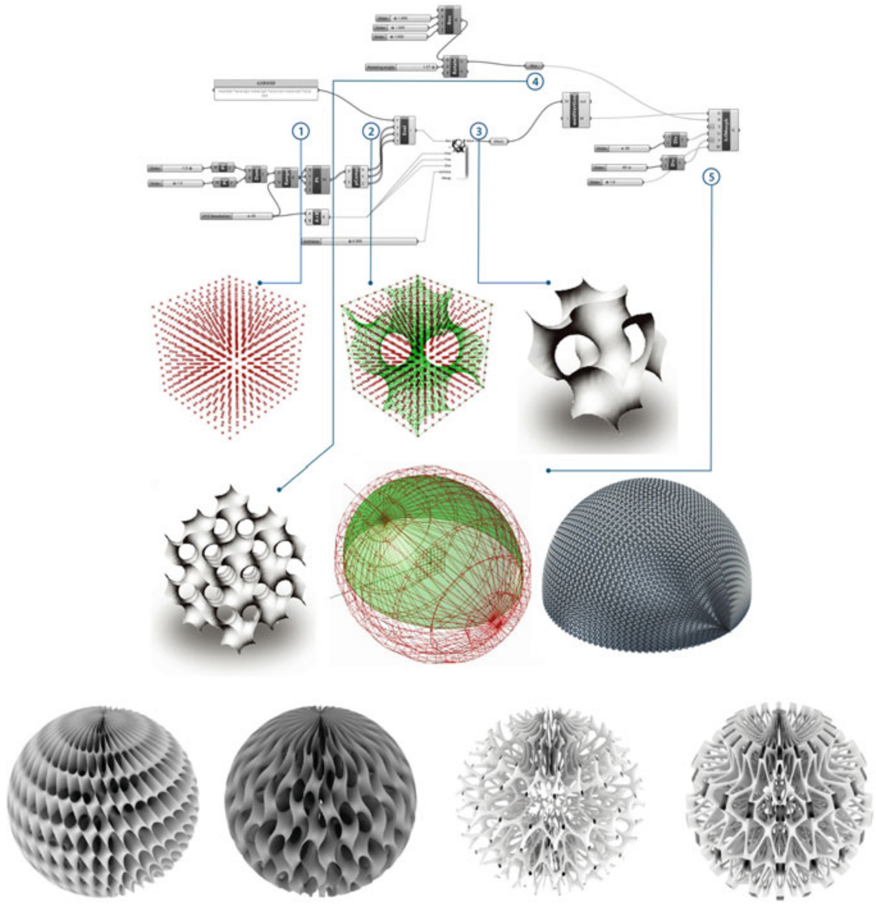



**Fig. 5** Step of the algorithm: (1) Definition of points in the fundamental cell; (2) Triangulation creates the surface; (3) Gyroid surface; (4) Invariant translation to create a TPMS based on Gyroid; (5) Discretization of the Hemispherical dome to obtain a surface composed by Gyroid. Below Experimentation on a sphere with a Diamond and a Gyroid surface and their complemenary

## References

1. Buratti G (2018) Algorithmic modelling of triply periodic minimal surface. In: Computational morphologies design rules between organic models and responsive architecture, pp 55–63. ISBN: 978-33-19609-18-8, 978-331-96091-95
2. Buratti G, Rossi M (2017) Form is matter. Triply periodic minimal surfaces structures by digital design tools. In: ShoCK!—sharing computational knowledge!—proceedings of the 35th eCAADe conference, pp 259–268. ISBN:978-94-91207-13-6
3. Buratti G (2012) Generative algorithms and associative modeling to design articulate surfaces In: Nexus Ph.D. Day relationships between architecture and mathematics, pp 94–95. ISBN: 978-8-83867-33-37

4. Chodosh O, Maximo D (2016) On the topology and index of minimal surfaces. J Differ Geom 104(3):399–418. https://doi.org/10.4310/jdg/1478138547; https://projecteuclid.org/euclid.jdg/1478138547
5. Dierkes U, Hildebrandt S, Sauvigny F (2010) Minimal surfaces, Second edn. Springer, 708 p
6. Emmer M (2009) Bolle di sapone. Tra arte e matematica, Bollati Boringhieri, collana Nuova Cultura, Torino
7. Lord E, Mackay A (2003) Periodic minimal surfaces of cubic symmetry. Curr Sci 85(3):346–362. Retrieved from http://www.jstor.org/stable/24108665

# To Observe, to Deduce, to Reconstruct, to Know

**Franca Caliò and Elena Marchetti**

This paper deals with a didactic experience gained in courses that are part of the first year of studies for the Schools of Architecture and Design at Politecnico di Milano. One of the themes of these courses concerns geometric problems, in order to introduce students to 3D space. The peculiarity of the didactic method used is, first of all, to induce the student to observe the real object by identifying its geometric characteristics (symmetries, proportions, contours, and surfaces enveloping it). Subsequently, the goal is to teach how to translate the observed form into mathematical language and to draw it on the computer, using a suitable tool of Computer Graphics. The method, consequently, allows to deepen and appreciate, with greater awareness, the characteristics of the studied form. To illustrate the process, we will present applications related to significant and fascinating objects of interest for the public we are addressing.

## 1 Introduction

Among the many personal educational experiences, the authors of this paper have addressed the teaching of mathematics in courses of the early years at the Schools of Architecture and Design. The experience acquired and the fruitful and continuous exchanges of opinions with colleagues in this environment have led to giving particular importance to geometry, taking a different aspect from the usual.

In general, in fact, mathematics is considered an abstract discipline very close to philosophy and consequently for those who live in the mathematical environment it

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is a high expression of thought and for those who are interested in it a valid exercise of the mind. However, math is not just this.

If we refer to the environment of applied sciences, such as physics, biology, chemistry, engineering, mathematics is recognized as a fundamental tool for the simulation and re-elaboration of natural phenomena. However, math is not just this.

Even as regards geometry, it is generally seen as means of classifying curves and surfaces and describing their characteristics. Once more, mathematics is not just this.

Rarely mathematics is considered as an aesthetic and creative intellectual activity. In other words, the idea that mathematics is linked to the artistic culture, if not even to the art itself, is still rarely accepted even if it currently finds increasing credit. This is the aspect of mathematics that is highlighted in this paper, seizing it from an educational point of view.

Why are we interested and want to pass on this interest to those who are culturally forming? The new technologies, heavily computer based, like 3D printers, renew the designer figure, by imposing an effort on common design between designers, architects, mathematicians and engineers. This fusion of artistic, scientific and technological cultures brings with it a creative project based on a revised geometric sensitivity (Pottmann et al. [12], for a very interesting formalization of this idea).

In this direction the didactic experience, here presented, is developed.

The first goal of this proposal is to learn, with the eye of the artist, to grasp, among the aesthetic aspects, in nature, in art, in buildings and in artifacts, the geometric component, which brings with it harmony, symmetry, dynamism and consequent beauty [11].

A second goal is to learn how to design, at any scale, by giving priority to harmony that is determined through geometric characteristics.

Finally, the goal is to find the right tool. An expressive and creative mathematical tool, as well as rigorous, is 3D analytical geometry [2]. Here lines, surfaces, classical and more generic shapes can be represented not only with pencil, ruler and compass, but also with a few equations. This vision requires a sensibility that is certainly different from that of a painter or writer, but still can be defined as artistic. With the advent of computer science, analytic geometry has perfected its language using vector calculus, becoming vectorial analytical geometry. Another example of mathematical tool is the matrix calculus [2]. Thanks to this instrument we are able not only to virtually represent, deform and move objects in space, but also to create, define and build them [3].

This experience, according to our opinion, can be extended to teaching in a good secondary school that has an interest in making people aware of the three-dimensional space, of the objects that can be represented in it, of the movements and deformations on these objects. Consequently, the student can get used, in an almost amusing way, to use modern and adequate language and tools to describe them and, why not, even create them.

## 2 Steps of the Didactic Method

The sequence of steps of our didactic method is illustrated hereinafter.

- The student observes the shape of a real or ideal "object" (some images of architectural buildings, design products, natural forms as well items of the everyday life will be presented) from an aesthetic point of view and comes to this consideration: each object, understood as a structure that defines a space, is delimited by surfaces. For example, let us observe the well-known work by Borromini Sant'Ivo alla Sapienza in Roma (Fig. 1).
- The student understands that, from the figurative point of view, besides other factors—static equilibrium, building materials, functional use, social impact and so on—the object is characterized by the configuration of its surfaces.
- The student points out the peculiarities and the geometric nature of the surfaces; in this case, he looks at the ribbon pattern that "rolls" on a conical surface (Fig. 2).
- Finally, the student describes the surface thanks to the mathematical tools, i.e. the 3D analytic parametric Geometry.

In this case the shape-describing expression is as follows:

$$\begin{cases} x = au\cos u & u \in [u_1, u_2] \\ y = bu\sin u & v \in [v_1, v_2] \\ z = cu = v & a, b, c \in R \end{cases}$$

The student, using a suitable software provided with precise instructions, virtually represents the surface (Fig. 3), obtaining in our case the following result:

**Fig. 1** Sant'Ivo alla Sapienza in Roma (*Photo* taken by the authors)

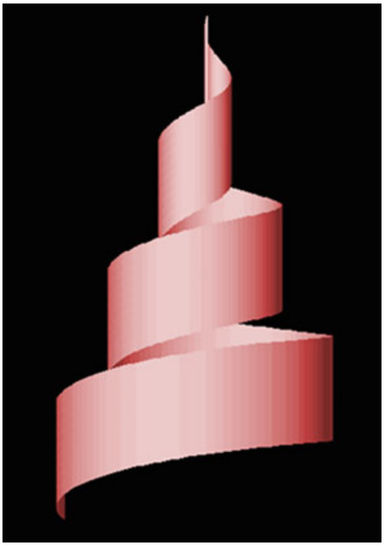


**Fig. 2** The geometrical surface

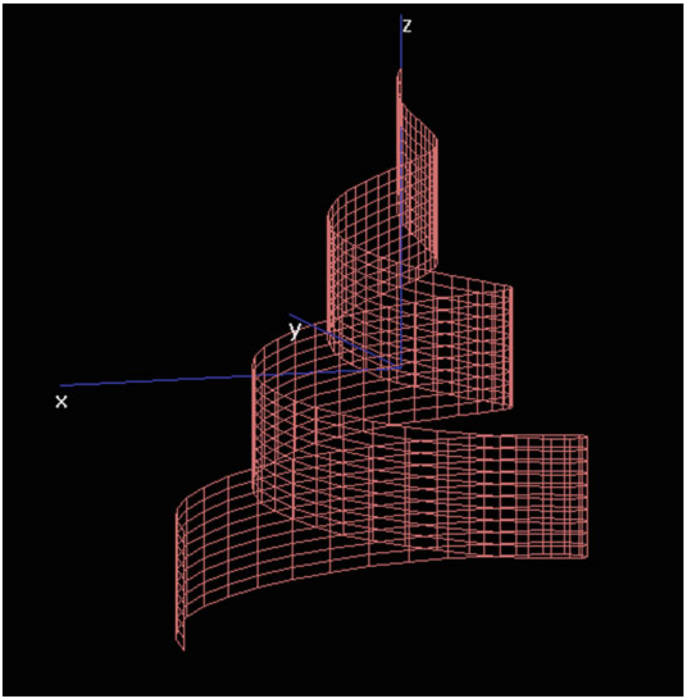



**Fig. 3** The virtual reconstruction of the surface

The sequence of steps in the didactic process can be summarized as follows: I observe, I deduce, I reconstruct and, consequently, I learn to know.

With the same tools with which we have learned to know and, therefore, to appreciate an object of whatever nature it is, through its geometric soul, we now learn to understand that we can create an object through its geometric characteristics. In this sense, we are not only able to deduce from the observation the harmony that geometry attributes, but also to exploit this possibility to pursue a project idea.

In this sense the didactic experience is focused on particularly interesting surfaces, called dynamic surfaces: their epistemological nature is identified and the mathematical language is able to interpret their dynamism.

## 3 Dynamic Surfaces

The dynamic surfaces are special surfaces obtained by continuous transformation of a planar or skew curve or by deformation of a given surface [4, 6, 10].

A curve that is subjected to a transformation is called a generatrix, a curve leading the transformation is directrix. A continuous transformation can be a rotation, a translation or their combination.

A surface acquires its physiognomy through the definition of the generatrix curve and through the law of movement.

The rotation surfaces (or surfaces of revolution) are generated by a generatrix curve rotating around an axis.

More contemporary rotations can lead to very famous and strange surfaces: Moebius strip or Klein bottle [3].

The translational surfaces are generated by the movement of a generatrix along a directrix and assume qualitatively different aspects, because they are strongly influenced by both curves (generatrix and directrix).

The combination of the two movements, rotation and translation, gives rise to shapes appreciated and adopted with enthusiasm in different sectors: helicoids of every type (dependent on generatrix, from the rotation axis and the directrix of translation).

Deformation of the simplest surfaces (plane and sphere) generates surfaces that acquire a remarkable significance from aesthetic point of view.

Obviously, every type of surface is accompanied by the observation of objects of various kinds that interpret them.

This transform-oriented vision of the reality is the principal subject here presented on which the didactic method is illustrated. This idea has proved to be successful thanks to the fact that not only the searched target can be easily and pleasantly reached but also it can be somehow overridden. In fact, it is intrinsic to this approach the capability of generating new unpredictable shapes whose aesthetic and validity can be a posteriori verified. It is a fantasy stimulating approach.

In the next Sections of the paper the generative process of the shape is described and an example of this procedure is illustrated.

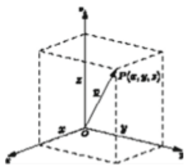
Final remarks will concern the skills of students acquired through this methodology.

## 4 Refresh on Parametric Geometry Language

What it is meant by 3D parametric geometry? The 3D parametric analytical geometry is understood as the evolution of 3D analytical geometry, which in turn constitutes the evolution of the synthetic geometry language, superimposing the algebraic language to the geometric language [5]. Precisely:

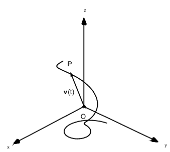
– given a Cartesian orthogonal reference system, a point P can be represented in 3D space through Cartesian coordinates;
– there is correspondence of point P as well as with an algebraic 3-component vector, as well as with a geometric vector, starting from the origin of the Cartesian system and terminating into the point:

$$P(x, y, z)$$
$$\mathbf{v} = [x \ \ y \ \ z]^T$$



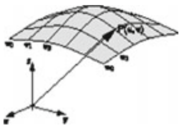
– a curve is expressed by means of an algebraic vector depending on a single parameter and it is geometrically described by the set of the terminating points of the geometric vector corresponding to the different values of the parameter:

$$P(x(t), y(t), z(t))$$
$$\mathbf{c}(t) = [x(t) \ \ y(t) \ \ z(t)]^T t \in [t_1 t_2]$$



– a surface is expressed by means of an algebraic vector depending on two parameters and is geometrically described by a set of generating curves related to each other through some given law:

$$P(x(t,u), y(t,u), z(t,u))$$
$$\mathbf{s}(t,u) = \begin{bmatrix} x(t,u) & y(t,u) & z(t,u) \end{bmatrix}^T \quad \begin{matrix} u \in [u_1, u_2] \\ t \in [t_1, t_2] \end{matrix}$$



An affine geometric transformation (i.e. translation, rotation, reflection, scaling, ...) is obtained from the application of an appropriate matrix to a vector and by imposing a translation to the result. It is possible to compose some transformations. Each linear transformation is described by a generic matrix equation:

$$A\mathbf{v} + \mathbf{b} = \mathbf{w}$$

where $A$ is the transformation matrix, **b** the translation vector, **v** the current vector that must be transformed and **w** the transformed vector.

## 5 Generative Process

What is it meant by the generative process of a surface? It is meant firstly the interpretation and secondly the representation of the genesis of the surface shape.

Using the dynamic surface definition and the mathematical tools introduced, we can describe the generative process. Precisely:

– a generation law of the shape is determined;
– a basis curve is selected (generatrix curve) and parametrically expressed;
– an action is applied to the curve by means of an algebraic parametric transformation (one parameter is introduced in the matrix of transformation); elementary transformations are composed if needed;
– the parametric equation (two parameters) of the surface is determined;
– finally, the surface is graphically obtained.

## 6 Laboratory Experience

The didactic experience here presented is integrated by laboratory activities. The main purpose of the laboratory is to graphically implement and visualize the theoretical results obtained. The open source software SCILAB® (http://www.scilab.org/) is used. The SCILAB® software allows very easy manipulations of one- and two-parameter equations and an immediate application of matrix calculus. Moreover, it gives dynamic rendering of the image during the generation of forms. It can be used as an introductory tool to more professional and complex products.

The following example illustrates the laboratory activities. The proposed example has a dual purpose: by comparing two different architectures for place, materials and purpose, it highlights two classic geometries that generate, through their form, equally different sensations.

Two architectural objects are to be observed in this sense: the Dome designed by Foster, covering of the Reichstag building in Berlin (Fig. 4) and the Treetop Experience (Fig. 5), born from an idea of the Danish Design and Architecture Studio Effekt, structure of a path immersed in the Glisselfeld forest (South of Copenhagen) [7].

The geometric shapes that are found in the described architectures can be interpreted as ellipsoid, hyperboloid of one sheet, dynamic surfaces.

The ellipsoid of revolution can be generated by rotating a half-ellipse of $yz$-plane around the $z$-axis. The parametric equations for the generatrix (ellipse) are as follows:

**Fig. 4** The dome of the Reichstag building in Berlin (*Photo* taken by the authors)

**Fig. 5** The Treetop Experience in the Glisselfeld forest (South of Copenhagen). Courtesy of Architecture Studio Effekt

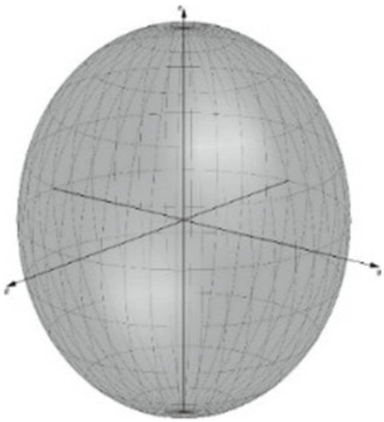
The matrix of revolution (around the z-axis) is as follows:

$$\begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix},\ 0 \le v < 2\pi$$

The resulting position vector for the ellipsoid is a modification of the position vector of the generatrix, which is given by the product of the rotation matrix by the position vector of the generatrix as follows (Fig. 6):

$$\begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a \cos u \\ b \sin u \end{bmatrix} = \begin{bmatrix} -a \cos u \sin v \\ a \cos u \cos v \\ b \sin u \end{bmatrix} \begin{matrix} v \in [0, 2\pi] \\ u \in [0, \pi] \end{matrix}$$

**Fig. 6** Ellipsoid of revolution (with b > a)



The hyperboloid of one sheet can be generated through a rotation of an arc of hyperbola in the *yz*-plane around the *z*-axis. The parametric equations for the generatrix (hyperbola) are as follows:

$$\begin{cases} x = 0 \\ y = c\,Chu,\ u \in R, \quad c,\ d \in \mathbb{R}^{+} \\ z = d\ Shu \end{cases}$$

where the asymptotes for the hyperbola are given (in the *yz*-plane) by:

$$z = \pm\frac{d}{c}y$$

and the functions *Chu* and *Shu* are the hyperbolic cosine and sine, respectively.

The resulting position vector for the hyperboloid (Fig. 7) is given by the product of the rotation matrix and the position vector of the generatrix as shown in the next mathematical expression.

$$\begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ c\,Chu \\ d\ Shu \end{bmatrix} = \begin{bmatrix} -c\ Chu\ \sin\ v \\ c\ Chu\ \cos\ v \\ d\ Shu \end{bmatrix} \begin{matrix} v \in 0, 2\pi \\ u \in \mathbb{R} \end{matrix}$$

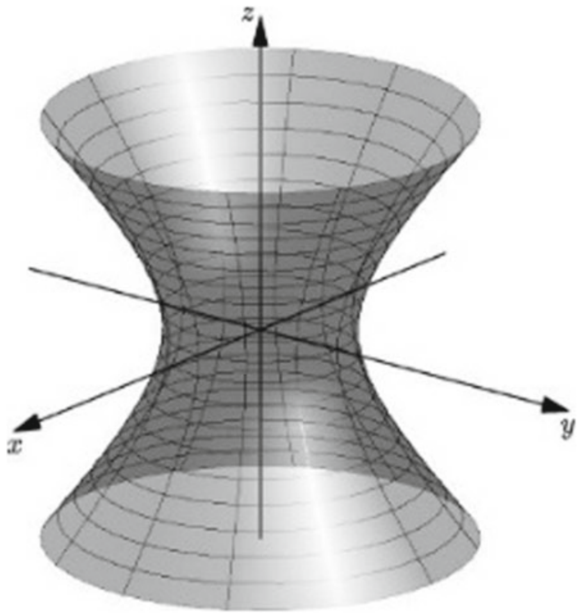



**Fig. 7** Hyperboloid of revolution


**Fig. 8** Reconstruction of the dome

It is geometry that gives the two objects shown in the next two figures a meaning, giving them a symbolic character.

The first (Fig. 8), which is geometrically a portion of an ellipsoid (convex surface), conveys the sensation of attracting attention to the underlying environment (the seat of the Bundestag) and protecting its history, without foreclosing, through the choice of materials, contact with the outside, or rather obtaining light and energy.

The second (Fig. 9), which is a hyperboloid of one sheet (concave surface), thanks to its shape reduces to a minimum the environmental impact, simulating the structure of a tree and, thus opening upwards like a crown and spreading downwards to simulate the root complex. Also in this case the choice of materials has a decisive role: materials in an ecological respect, but also with a disposition such as to strongly emphasize the geometric nature of the form. Moreover, we can observe that the hyperboloid is a ruled surface and this characteristic is highlighted by the wooden weaving that structures the architecture.

We conclude by observing that the geometry of a shape characterizes its beauty, regularity, functionality, but also very often it has an unexpected role: it determines a sensation, a state of mind, helps to reflect and to interpret; establishes an emotional relationship with the space in which it is located and with the world it serves.

The conclusion, as well as the heart of the laboratory's problem, invites the student to generate a form, in order to highlight the aspect that is not only aesthetic but also creative in geometry, as we presented it.

Now let's take a curve whose equation is known by you, and give it a movement or a deformation through the tools you learned to know. What do we get? If we like it we accept it otherwise we have learned how and where to intervene mathematically [8].

Let us now try to repeatedly apply (obviously with a control of the situation) some very precise transformations on a flat form we can get to build, through the

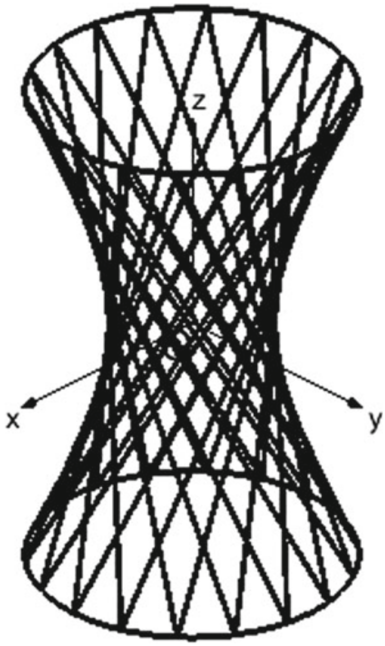



**Fig. 9** Reconstruction of the TreeTop

**Fig. 10** Creation of a frieze

mathematical tools, friezes or rosettes [9] and [1], or even a tessellation of a plane, freeing the imagination in creating decorations, textiles, patterns and so on. In the following some examples of them (Figs. 10, 11, 12 and 13).

## 7 Final Remarks

The didactic method in teaching the geometric component of mathematics in Architecture and Design Schools exemplified in this work leads to interesting results, suitable to the audience to which it is addressed.

Firstly, it educates the student to thoroughly observe the object, to be described or to be designed, with an aesthetic eye, focusing on the geometric aspects of the surfaces that characterize the object.

Secondly, the student incorporates the possibility of creating complex forms with the modern and agile mathematical language of the parametric 3D geometry. As a

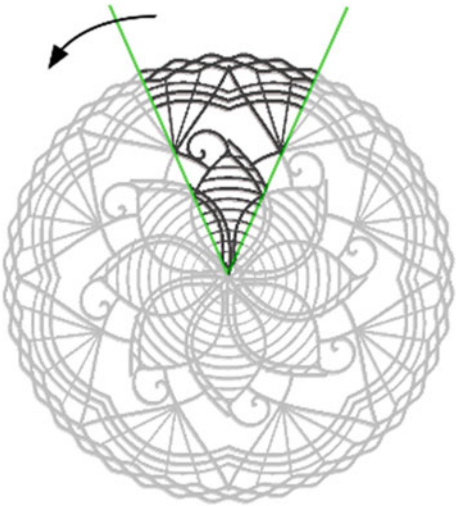


**Fig. 11** Creation of cyclic rosette

**Fig. 12** Creation of a dihedral rosette

result, students are sensitized to the potential of mathematical language itself, which will be an indispensable cultural basis for those who are oriented to virtual design.

Finally, by the virtual reconstruction of the object, the student understands that he has in his hand a tool that allows to manipulate the object, for example to correct its dimensions and shape.

This latter consideration allows the student to grasp the aspect of less obvious and unusual mathematics: the creativity. In other words, following the particular path proposed in this paper, the knowledge and study of the genesis of the surface stimulates the designer to develop creative abilities, which allow him to go further the possibility to observe, understand and then communicate the already realized projects.

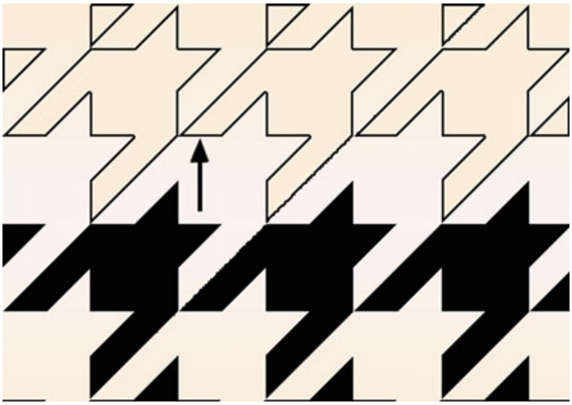


**Fig. 13** Creation of a tessellation of plane

This method seeks to consolidate the deep and ancient bond between art and science, contributing to the collapse of the separation between scientific culture that it observes and it studies and the humanistic culture that it thinks and creates.

## References


1. Betti R, Marchetti E, Rossi CL (eds) (2007) Simmetria una scoperta matematica, 2nd edn. Polipress, Milano. ISBN/ISNN: 97888-7398-031-5, ISBN/ISNN: 97888-7398-074-2
2. Caliò F, Lazzari A (2016) Elements of mathematics. Società Editrice Esculapio, Bologna. ISBN 978-88-9385-052-0
3. Caliò F, Scarazzini E (1997) Metodi matematici per la generazione di curve e superfici, 2nd edn. CittàStudiEdizione, Milano. ISBN 978-88-251-7302-4
4. Caliò F, Marchetti E (2000) Generation of architectural forms through linear algebra. Nexus III, Architecture and mathematics. Pacini Editori, Pisa, pp 9–22. Published. In: Williams K, Ostwald MJ (eds) Architecture and mathematics from antiquity to the future, vol II: The 1500 s to the future). Springer International Publishing Switzerland, pp 483–496. ISBN: 9783319001432
5. Caliò F, Argeri M, Lazzari A, Sesana D (2011) Geometria vettoriale per la grafica. DeAgostini Scuola, Torino. ISBN 978-88-251-7366-6
6. Caliò F, Miglio E (2012) City life: The dinamic of growth. In: Rossi M, Duvernoy S, Mele G (eds) Maths in the city: a mathematical tour of Milanese Architecture. Maggioli editore, pp 191–195. ISBN: 9788838761287
7. Caliò F, Marchetti E (2018) Two architectures: two compared geometries. Nexus Netw J 1:1–19. ISSN: 1522-4600
8. Caliò F, Lazzari A, Sesana D (2010) Curve e superfici: realizzazione di forme attraverso SCILAB. Pitagora Editrice, Bologna. ISBN 883-71-1818-X
9. Caliò F, Scotti A (2014) Mathematics: method and creativity in fabric creation. In: Proceedings of international congress CIMODE, pp 1–8 on CD ISBN 978-88-6493-027-5
10. Marchetti E, Costa LR (2002) The fire tower. Nexus Netw J 4(2):38–53.ISSN: 1590-5896/158

11. Marchetti E, Costa LR (2012) The Milan Cathedral (il Duomo). In: Rossi M, Duvernoy S, Mele G (eds) Maths in the city: a mathematical tour of Milanese architecture. Maggioli editore, pp 26–35. ISBN: 9788838761287
12. Pottmann H, Asperl A, Hofer M, Kilian A, Bentley D (2007) Architectural geometry, vol 724. Bentley Institute Press. ISBN 978-1-934493-04-5

# The Role of Geometry in the Architecture of Louis Kahn and Anne Tyng

**Cristina Càndito**

Among the salient features of the architecture of Louis I. Kahn, one recognizes the ability to reconcile the aesthetics of Modernism with monumentality and one of the elements used to implement this programmatic attitude is Geometry. In this regard, we cite the projects for the Trenton Bath House (1955–1956) and the unrealized Philadelphia City Tower (1952–1957), which are today attributed in the geometric conception to Kahn's collaborator, Anne Griswold Tyng. Her production of concepts of unrealized projects is less known, among which the one for the General Motors Exhibit 1964 (1960–1961), which adopted particular geometric figures, showing not only her knowledge, but also her ability to manipulate regular and semi-regular polyhedra. Tyng was also interested in women role in the culture and she expressed in her most appropriate form—the geometrical schemes in her texts written in 1989 and 1997—her own theory about the evolution of the woman's role towards an autonomous creative expression.

## 1 Introduction

A figure not sufficiently known in the twentieth century architecture panorama is that of Anne Griswold Tyng (1920–2011) who in the last twenty years has been rediscovered above all for having contributed to some projects by the well-known architect Louis I. Kahn (1901–1974).

The particular formation of Tyng, her passion for geometric subjects and the mutual esteem with Kahn lead to a fruitful collaboration starting from 1945. The social context is the start of the process of inclusion of women in the world of architecture, while the personal context is that of a romantic relationship with Kahn.

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After the end of the relationship and the collaboration with Kahn, Tyng had the opportunity to reflect on women role in the culture telling some stories of women who, although endowed with great skills, were able to play the function of inspiring muses for men who would become famous in different fields [7]. Tyng also gave a lucid analysis supported by documents that attest to her creative role in architecture [8].

Her original critical thinking can be identified in the path that pass through the project theory texts [6] up to the personal exhibition of 2011 [9], which Tyng herself helped to set up. It is through these testimonies and other recent studies that we try to understand not only Tyng's role in the affirmation of the geometric logic in some architectures conceived with Kahn, but also her ability to anticipate some instances of contemporary architecture.

## 2 Tyng's Geometric Imprint

The exhibition about Anne Griswold Tyng, *Inhabiting Geometry*, held at the University of Pennsylvania's Institute of Contemporary Art (13th January to 20th March 2011) and at the Graham Foundation of Chicago (15th April to 18th June 2011) [9] showed Platonic polyhedra as meaningful references to Tyng's original and pioneering ideas. The models exposed had their faces dematerialized because they are composed only of the wooden edges painted by white on the internal surfaces. They recall the representation adopted by Leonardo da Vinci for the famous illustrations of the Platonic polyhedra "vacui" (empty) in the text by Luca Pacioli (*De divina proportione*, Venice 1509). However, it does not seem to be just a quotation, because the objects are interpreted as architectures thanks to their relationship between external volume and internal space, but also because of their dimension and articulation. In fact, they are combined to create spatial suggestion, as the dodecahedron with nested cube and similar geometrical arrangement.

The freedom of thought with which Tyng expressed her architectural ideas finds perhaps its origin in her training and early professional experiences. Tyng spent her first eighteen years of life mainly in China, where she was born in 1920, in a family of two Episcopal missionaries from Boston. Back in the United States, she attended, in Cambridge (Massachusetts), the first School of Architecture, which offers training on architectural design for women only [8, p. 18].

The training continued at the Harvard Graduate School of Design, which at that time opens registration for women. It is always mentioned the presence of Walter Gropius and Marcel Breuer: two of the most important protagonists of the Bauhaus, the well-known German school of architecture with progressive conceptions that cost the closure by the Nazi regime.

In 1944 Tyng was in New York where she carried out drawing works but the following year she returned to her family in Philadelphia because, as she herself recounts [8, p. 27], architectural firms considered a woman's candidacy as "improper and outrageous". In 1945, she started the collaboration with the firm by Oscar G.

Storonov (1905–1970) and Louis I. Kahn. In 1947, Kahn left Storonov followed by Tyng who developed both collaborative and autonomy projects, encouraged by Kahn himself.

The work of Louis I. Kahn is appreciated for the characteristic union of rigor and creativity in his architectures, as well as the ability to reconcile the aesthetics of Modernism with monumentality and one of the means used to implement this programmatic requirement is geometry. Among the significant architectural examples in this regard there are, in addition to the concluding work of his career, the House of Parliament in Dhaka (from 1962), also the projects for the Salk Institute (1959–1965), the Yale University Art Gallery (1951–1953), the Philadelphia City Tower (1952–1957) and the Trenton Bath House (1955–1956) [1, 5]. Parts of these works, in different ways, are attributed in the global conception to Anne Griswold Tyng.

## 3 From the Triangle to the Tetrahedron in Tyng's Drawings

Tyng had a relationship with Kahn since the 1945 and collaborated professionally with him even beyond the end of that (about the 1960). Tyng then took a Ph.D. in Architecture at the University of Pennsylvania (1975) where she also taught for almost thirty years and she offered, in different writings, new keys of interpretation of some design she made with Louis Kahn, without obscuring his extraordinary role in twentieth century architecture.

Tyng reported that, since the first moments in Kahn's firm, she was involved in every phase of the work that allow her to experiment her abilities in a variety of subjects [8, p. 32].

Kahn himself supplies the source for their collaboration for the Philadelphia City Tower (1952–1957) (Fig. 1) project designed for Tomorrow's City Hall in Philadelphia of which she created the model [4]. There was an ambiguous occurrence before an exhibition at MoMA (*Visionary Architecture*, Museum of Modern Art, New York, September 29–December 4, 1960) organized few months after their sentimental separation, when Tyng realized she was not included in the attributions. Once Kahn was asked to insert her name, he agreed and Tyng was full recognized by him as the "geometric conceiver" of the tower, as he also wrote around a portrait of Tyng Kahn drawn in 1972 [8, p. 55, 202–204, 213].

The triangular reticular surfaces can recall the geodesic structure by Richard Buckminster Fuller (1895–1983), but in the Philadelphia Tower, we can see a spatial configuration. Moreover, Fuller researches into geodesic spheres (since the 1940s) were probably known by Kahn and they could occasionally meet [2, p. 70], but Fuller was professor at Yale University subsequently (between 1969 and 1970), when Fuller was in touch with Anne Tyng.

We can see a more evident reference: the reticular structures of Konrad Wachsmann (1901–1980), the German architect refugee in the United States due to Nazi political persecutions. Tyng had drawn for him (New York, 1944) perspectives of

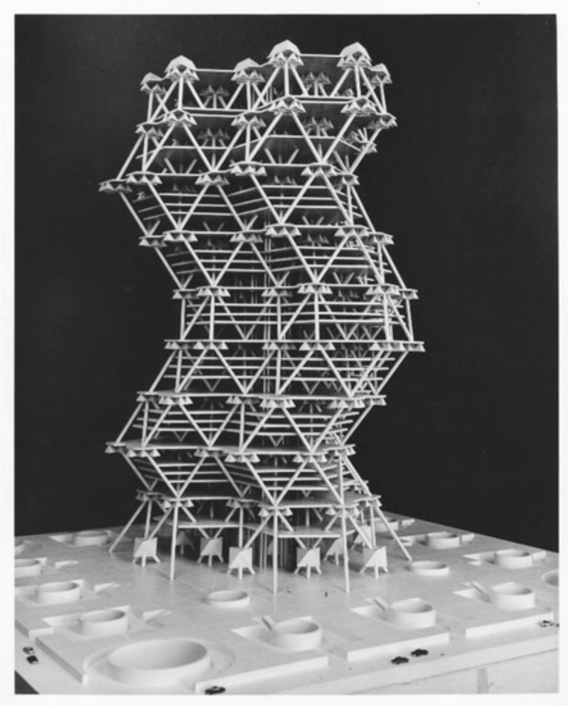

**Fig. 1** City tower model Philadelphia, PA Unbuilt, 1952–1957, Kahn and Tyng, associated architect. Louis I. Kahn collection, University of Pennsylvania and Pennsylvania historical and museum commission

compositions of elements for which Wachsmann is considered a true pioneer of prefabrication.

We observe that the Air Force Aircraft Hangars (1951) (www. atlasofplaces.com) present an evident similarity with some independent works by Tyng, such as those for the unrealized project for the Elementary School (Bucks County, Pennsylvania, 1949–1951) (Fig. 2), characterized by a triangular spatial grid, therefore constituted by tetrahedra, which extends to form three supports on the ground.

Tetrahedra cut in two equal parts are the basis of Tyng's design, of the Parents' house on Chesapeake Bay, Eastern Shore of Maryland (1953) (http://www.mmmlib.com/anne%20tyng%20maryland.html).

The triangular and tetrahedral elements are also present in the project for the Yale University Art Gallery (1951–1953) attributed to Kahn. Tyng made the model and participated in the design of the triangular stair in the cylindrical stairwell and in the most significant features in the exhibition hall: the ceiling composed of tetrahedra (Fig. 3). This significant detail is very close to Tyng's previous projects, as the elementary school and the parent's house cited before [8, p. 47–48] that could be interpreted as pioneering architectural experiments.

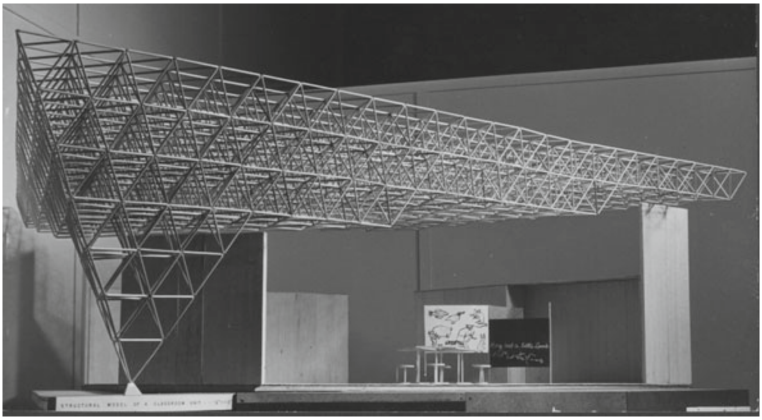

**Fig. 2** Model for proposed elementary school bucks county, PA Unbuilt, 1952, Anne Tyng, architect. The Architectural Archives, University of Pennsylvania, by the gift of Anne Griswold Tyng

**Fig. 3** Yale University Art Gallery. Left: Triangular staircase. Photo by Sage Ross (CC BY-SA 2.0; https://www.flickr.com/photos/ragesoss/369389643). Right: exhibition hall ceiling, Photo by Timothy Brown (CC 2.0; https://www.flickr.com/photos/atelier_flir/15108822322)

## 4 Spatial Developments of Plane Geometric Relationships

Tyng expressed her idea about the role of geometry in architecture in different contributes. According to her [6], geometry provides a repertoire of archetypes that derive from the same structure of the molecules and we can recognize different stages of geometric thought development, from the simpler forms of symmetries to the more complex ones, characterized by the combination of different kind of movement.

She evoked theories by Carl Gustav Jung and Henry Focillon, but the most interesting part is the original illustration of the three-dimensional interpretation of the

Golden spiral, where she drawn, in a one-point perspective, cubes translated along the z axis and crossed by a spiral (Fig. 4).

Geometry of the platonic solids was the basis of her spatial conceptions: tetrahedron and octahedron were both interpreted as translations of the cube and Tyng assign them a design potentials independent by the architectural style.

The variant of the pyramid with four regular triangular faces, the tetrahedron, is the square-based pyramid, which, besides, may constitute half of another of the five regular solids: the octahedron [8, p. 196].

Four square-based pyramids are used in the roofs of the Trenton Bath House (1955–56) (Fig. 5). The building is designed as part of a larger but unrealized project for the Jewish Community Center in Delaware Valley. The plan consists of a Greek cross with the four rooms covered by a pavilion roof, which surround an open atrium. At the corner of each space, a large open rectangular column supports the roof. Tyng wrote that Kahn was working on a different spatial configuration and that she suggested the four spaces for the Bath House, inspired by the structures she remembered from her childhood in China [8, p. 190]. William Whitaker, curator of the Architectural Archives of the University of Pennsylvania School of Design testified that it was a Tyng's project [3].

Tyng was also interested by the semi-regular polyhedra, as it is shown by the project (1960–1961) for the General Motors World's fair Exhibit 1964 [8, pp. 198, 199]. Her drawn, never realized, can be interpreted as a polar array of six half trun-

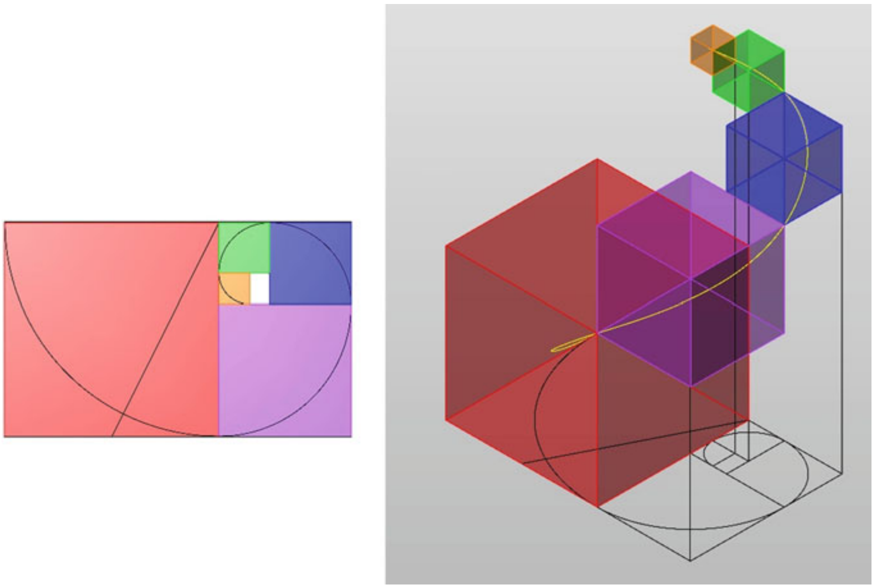

**Fig. 4** Golden spiral and its spatial development. Plan and isometric interpretation of the Tyng's drawing (C. Càndito)

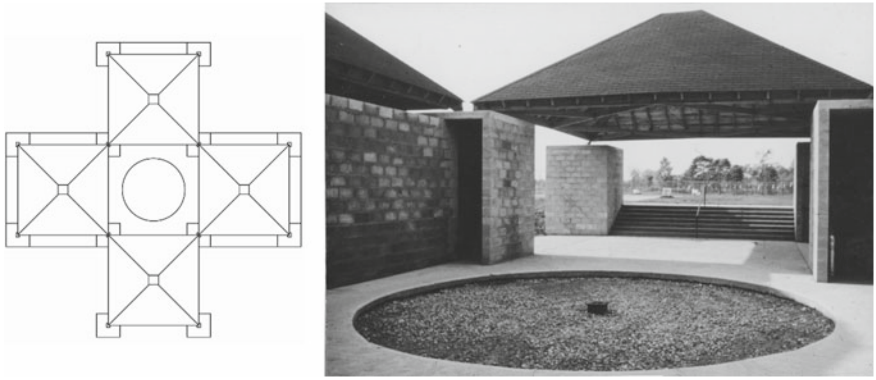

**Fig. 5** Jewish Community, Bathhouse Trenton, NJ 1959, Louis Kahn & Associates. Left: scheme of the plan; Right: Louis I. Kahn Collection. University of Pennsylvania and Pennsylvania Historical and Museum Commission

cated octahedra, Archimedean polyhedra with fourteen faces of which eight regular hexagons and six squares (Fig. 6).

That particular kind of geometrical form it is also called Kelvin polyhedra, by the name of Lord Kelvin (William Thompson 1848) who thought it was the form that could solve a problem related to the minimum surface with the maximum volume. Kelvin problem is beyond the present discussion, but it was well known by Tyng [8, p. 198]. The six polyhedra are joined by five low volumes with roofs following the lines of the inclined hexagonal faces and the model was made in steel rods and parachute silk.

## 5 Conclusions. Ambiguous Attributions, Geometric Clarity

The architectural role of Anne Griswold Tyng was obscured by the figure of Louis Kahn. She met him very early in her career: graduated from Harvard in 1944, Tyng start to work with him from the 1945 and their relationship until around fifteen years, during which they had a daughter (1954). They continued their collaboration until 1964 and occasionally also in the following years, so Tyng exercised a clear influence on Kahn work.

Tyng predisposition for a wise application of geometric principles and her specific contribution to Kahn's works were brought into light in several articles and, above all, thanks to the publication of their correspondence [8].

Although the romantic idea of love and work expressed by Kahn ("*Our wonderful way of love and work that is nothing but another form of love*" [8, p. 9], Kahn refused to assign her a clear role, but he wrote on different projects "*Anne Tyng reminded me of my own premises*" [8, p. 194].

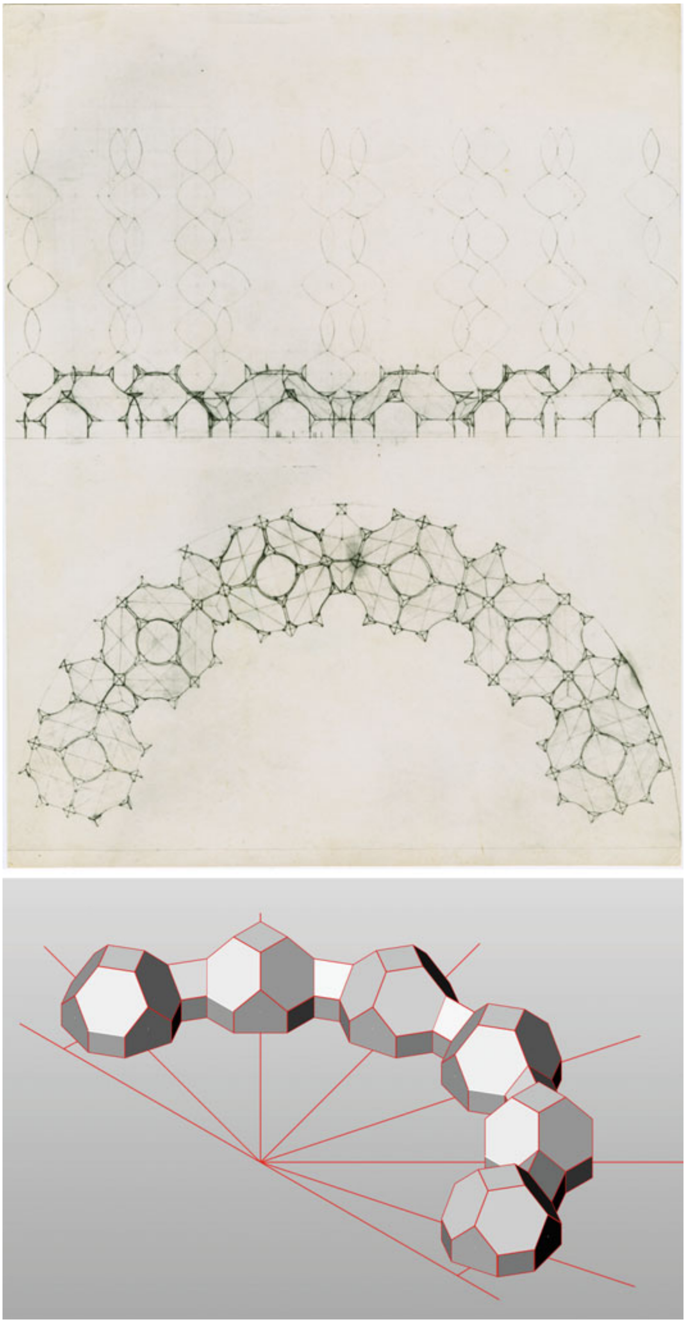

**Fig. 6** General Motors Pavilion, 1964 New York World's Fair Unbuilt, 1960–1961. Louis I Kahn & Associates. The Architectural Archives, University of Pennsylvania, gift of Anne Griswold Tyng. Below: Isometric image of the hypothetical model (C. Càndito)

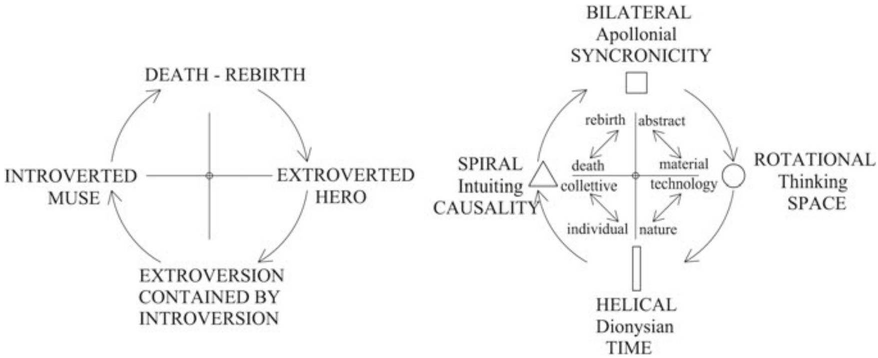


**Fig. 7** Schemes of geometrical drawings [7, p. 181 and 8, p. 212]

On several occasions, it is insisted on name only to Kahn—an indisputable genius of twentieth century architecture—as the creator of projects done in collaboration with Tyng and we all remember the evocative photo (https://www.philly.com/philly/columnists/inga_saffron/louis-kahn-retrospective-philadelphia-fabric-workshop-architecture-20170810.html) of the Philadelphia Tower model portrayed together with its creator, Louis Kahn, at the 1960 MoMA exhibition.

In 1989, Tyng wrote about women role in culture: she did not tell her own story but, at the same way, she explained her personal difficulties: "*Understanding the role of muse is a step in the psychic development of women and men*." [7, p. 171]. Tyng illustrated her theories with a circular scheme where—inspired by Carl Gustav Jung theories—she shown how introverted and extroverted individualization could reciprocally change passing through "death and rebirth", as a crisis that put the muse in the condition to become heroine herself.

A similar scheme was adopted by Kahn to illustrate the cycle of architectural creations (passing through History, Nature, Order and Design) and Tyng herself used a circle to represent the theory of creative process. The four phases (Bilateral, Rotational, Helical and Spiral) that schematize the components of the relationship between man and space, capable of regenerating themselves through design creations that reveal their geometric essence [8, p. 212] (Fig. 7).

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## References

1. Brownlee DB, De Long DG (1991) Louis I. Kahn in the realm of architecture. New York: Rizzoli, p. 448. ISBN 0847813304
2. Cummings LP (1991) Louis I. Kahn: i musei, Milano: Electa, p. 292. ISBN 9788843535729

3. Jacobs K (2018) Anne Tyng and her remarkable house. Architect 107(2): 81. (https://www.architectmagazine.com/design/anne-tyng-and-her-remarkable-house_o). Retrieved 15th February 2019
4. Kahn Louis I (1953) Toward a plan for midtown Philadelphia. Perspecta 2:10–27
5. Kries M, Eisenbrand J, Von Moos S (2013) Louis Kahn. The Power of Architecture. Weil am Rhein, Vitra Design, p. 370. ISBN 9783931936921
6. Tyng A (1969) Geometric extension of consciousness. *Zodiac*, 19
7. Tyng A (1989) From Muse to heroine, toward a visible creative identity. In Berkeley EP, McQuaid M, Architecture: a place for women. Washington: Smithsonian Institution Press, 1989, pp. 171–186. ISBN 9780874742312
8. Tyng A (1997) Louis Kahn to Anne Tyng: the Rome letters 1953–1954, New York: Rizzoli, p. 216. ISBN 9780847820092
9. Tyng A, Imperiale A, Sabin JE, Schaffner I (2011) Inhabiting geometry. Institute of Contemporary Art, p. 112. ISBN 9780884541219

# Thinking Architecture in Four Dimensions

**Alessandra Capanna**

## 1 Introduction

In mathematics, it is quite easy to define four-dimensional geometry. With their equations, in fact, mathematicians work without any difficulty with any "*n*" dimension. From this point of view, it is also quite easy to describe what shape in our 3d word a hypercube, for example, can assume, taking advantage of projections of the geometric figure in the lower dimension. We have to say that architects are accustomed to draw the space they imagine through orthogonal projections and therefore to see the 3d space through its 2d projection in plan and section.

Moreover, the perception of the physical space in architecture is experiential. It means that although the sense of sight is able to capture the geometric, figurative and aesthetic characters, ultimately the harmony of the shapes, the quality of the built environment is perceived with all the senses, in a dynamic approach, as a continuous sequence of space-events in so demonstrating the consistency of architecture space-time 4d reality.

As part of a research on the design theory and related analysis of the compositional process, as well as related changings of paradigms in Architecture, the paper presents the geometric consistency of multidimensional characteristics of a number of architectures in which the concept and the image of multi-dimensional geometry interpret the architectural thought of the XXI century, are investigated.

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## 2 This Thing Called Theory

Theory is intrinsic to architecture, and it is an indispensable part of the project.

In the framework of the relationships between architecture and mathematics, as well as with other arts, it is crucial to refer to the basic philosophy about the creative work of architects.

It concerns the complex relationships among ideas, memories, references and expression within the project, which develops as a succession of distinct actions. As a chaotic aggregation and overlapping of abstract needs and specific requirements, design is supported by the dialogue between form and matter. In this research, which consists in investigating a new spatiality that may be consistent with the thought of the third millennium, there are questions about the multi-dimensional geometry standing as a sort of axiom that it could be one of the possible expressions of contemporaneity.

Changings of paradigms and new technologies are only a part of the question. The starting point of the design thinking issues is rather to be found in the research interests of each architect, enclosed into the contemporary debate.

In other worlds, thinking and making Architecture are strictly linked, in a way that establishes Architecture as able to generate thinking as well as absorbing thinking from the outside, from other disciplines for instance, and in this case from the mathematical thought. In so doing, inquiring how Architecture Theory can be part of a common debate together with the development of the knowledge in the spatiality of geometry can be assumed as the environment in which the architectures here presented are born. Furthermore, the Architecture in four dimensions is not a consequence of researches that tend to show that the fourth dimension is the geometry of contemporary architecture or represents its most relevant aspect, but that this is an opportunity to develop an issue related to the relationships between mathematics and architecture.

Thinking Architecture in four dimensions is an act of will of the designer, that wants to test exactly those forms.

## 3 The Idea Comes First

If we define thinking process as the genesis of any compositional activity, architects have to admit that the idea comes first. The idea comes first and the architect makes use of the project as a tool to verify peculiar theories, that is to say "to practice our personal obsessions".

Explaining how ideas are born is therefore as difficult as discussing what creativity is, a topic that is a fundamental part of architectural theory.

If I had to explain to a student how to develop a project, I would give priority to the method: to analyze the site where the building should be placed, to study the theme from a functional and distributive point of view, to see the examples. But how did the idea arise?

For a designer the idea of a plan layout often comes before the idea of form, but it is the latter that belongs to a sort of structuring heritage for the architect's imagination. The story of the genesis of an architecture is often crossed by memories and figurative references. In 2013 Massimiliano Fuksas at a conference in Genoa entitled "One day one project" said: "I will tell a story that is not true, but I like to make it believe that it is: I was in Greece when, looking up, I saw a cloud in the sky, with the same shape as the one I later built. Hence my inspiration … Although this anecdote is not true, I sell it as such, and I often tell it because I believe that behind every project there must be a valid story."

The meaning of this statement is that everything can be transformed into creative inspiration to which the readings on the selected research topic contribute considerably, and especially those works of visual art that have interpreted the same subject of the study support the imagination.

Starting from the analysis of these works, which in the research on the four-dimensional architectural space were selected as examples, it is therefore of primary importance to "understand the form", recognize the theoretical framework, and then to draw the project as the development of the abstract idea. Falling the idea into reality, which is made of rules and matter, will require an adaptation that is the result of a close dialogue between theory and practice.

## 4 Figure Out the Form

So let's start from the act of will, which is the basis of design experimentation that has as its objective the realization of an architectural space in 4 dimensions. To design a space in fourth dimension, it is first needed to understand what the distinctive features of this geometric shape are. And to understand the form it is necessary to start from the mathematical definition of higher dimension geometries. Reading the treaties was fundamental. What we know is that mathematicians initially described the higher dimension geometries through their projection in the lower one. In the introduction of the book *The Fourth Dimension Simply Explained* we read: "The geometry studied in the schools is divided into two parts, Plane Geometry, or Geometry of Two Dimensions, and Solid Geometry, or Geometry of Three Dimensions, and the study of these geometries suggests an extension to geometry of four or more dimensions." [1]

Other texts have reported possible experiences for a "non-Vitruvian man", in a no longer Euclidean world perceiving a space in four dimensions in its projection in the third of the physical world in which we live. Mathematical tales that can be translated into architecture provide simple solutions that are at the same time daring, which would come to the mind of a child who, lacking scientific instruments, is forced to draw on imagination. In this regard, Edwin Abbott is quoted on several occasions. In his *Flatland* he made the Square unveil worlds of many more dimensions than the 2d that characterize his flat existence, in which in order to perceive the shapes, one is obliged to an effort of absolute abstraction and to settle a number of conventional references to understand the site and to identify people. The narrative contribution has therefore contributed to making the multi-dimensional space visible with a language understandable even to non-specialists.

## *4.1 Tales of Space in Four Dimensions*

Norman McLaren, Scottish Canadian filmaker, expert in visual-music and graphic-sound, produced a series of drawings on the theme of the fourth dimension, complemented by long handwritten considerations, in support to the drawings. In particular, *Four-Dimensional House* (Fig. 1) is completed by a long annotation that tells how to walk this house: *I go in the west door; you go in the east one. I walk down the corridor & go into the second room on the right on the ground floor; you go up one flight & take first room on the left. We find ourselves in the same room, the one with the window pane. (you are one floor higher than me but we are also in the same room; are we therefore also on the same floor??) For you, the bottom right hand pane is broken; for me top most right one. (for you are standing at right angles to me).If you had not been able to see my side of entry to the house, nor I yours, you would maintain it was a 3-storey house, while I would argue it was a 2-storey building; we*

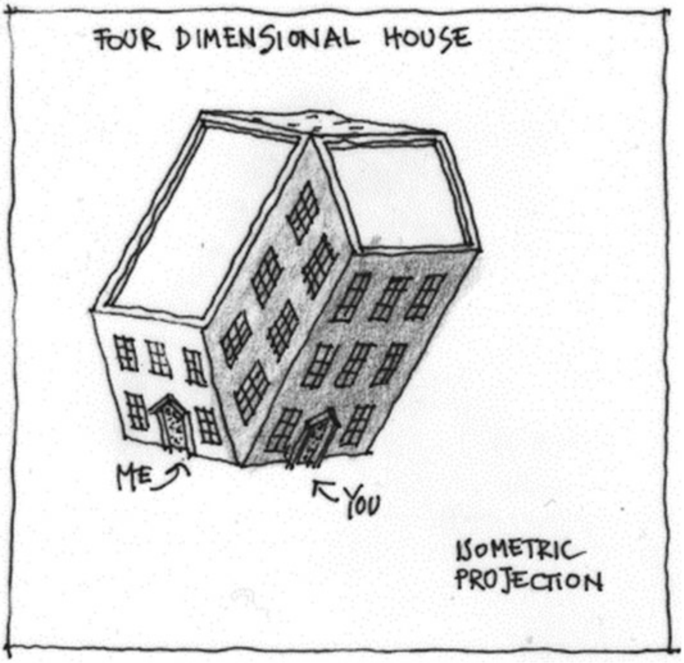


**Fig. 1** Norman McLaren's four-dimensional house in a sketch by A. Capanna

*are of course both correct. Only a knowledge of the additional dimension on both our parts would solve an apparently insolvable contradiction!*

At the foot of the drawing there is another annotation that reads: Isometric Projection Four Dimensional House.

Another mathematical novel.

From Heinlein's "And he built a Crooked house" [2]:

*A tesseract is a hypercube, a square figure with four dimensions, like a cube has three, and a square has two. Here, I'll show you. Quintus Tael (…) returned with a box of toothpicks which he spilled on the table stuck toothpicks and some plasticine. Rolled it into pea-sized balls into four of these and hooked them together into a square. Another one like it, four more toothpicks, and we make a cube. "The toothpicks were now arranged in the framework of a square box, a cube, with the pellets of clay holding the corners together. Now we make another cube just like the first one, and the two of them will be two sides of the tesseract. Now pay attention. You open up one corner of the first cube, interlock the second cube at the corner, and then close the corner. Then take eight more toothpicks and join the bottom of the first cube to the bottom of the second, on a slant, and the top of the first to the top of the second, the same way. That's a tesseract, eight cubes forming the sides of a hypercube in four dimensions.*

Here raises the question whether the experience of a space in 4 dimensions is only a question of visualization.

If in the first case the reference figure was the "compacted" shape of the hypercube, with each edge of the two cubes connected by rectilinear segments to form a cage in which the six cubes adjacent to the two regular ones are deformed, and two opposite faces are rhombus shaped, in the second story the four-dimensional house takes on the shape of the famous painting *Crucifixion (Corpus Hypercubus)* by Salvator Dalì. It is the most commonly known image of the tesseract in which four cubes are placed one above the other and the other four are each placed on the side face of the one on the second level of the column of four cubes. In Heinlein's story we witness the transformation of this figure represented in three dimensions into a tesseract because of its collapse due to the intervention of a slight earthquake that allows the protagonists of the story, once they enter the four-dimensional house, to experience a plural space in which, like Packman of the famous video game, one is inside and outside at the same time and crossing surfaces is possible to pass in other spaces not adjacent to the coming from (Fig. 2).

## 5 Imitating the Shape

It is not always easily identifiable if the inspiration for a particular project that is evidently inspired to the image of the hypercube is limited to the exterior or formal aspect, and because of this, it is necessary to look deeper at the compositional structure, and to see if, and to what degree, these elements are the same in both the architectural and the geometric-mathematic works. Some architectures are clearly


**Fig. 2** Question of visualization: Brandon & Davis's cardboard hypercube showing some images of the geometry of the hypercube. Picture 3: McLarens's four dimensional house shape reference. Picture 4: Heinlein's crooked house shape reference

inspired by the image of the hypercube, and do not present an interpretation of the spatiality typical of 4d geometry, in a way that seemingly translate only the external appearance.

This similarity is very evident in the Monument to the unknown political prisoner which Max Bill presented in the competition announced in 1952 by the Contemporary Art Institute of London. An impressive architectural sculptures conceived as the ordered composition of three identical open cubes shaped like steps leading to a small triangular courtyard. The walls had to be coated with reflective material in which visitors would have seen themselves. The mirroring walls that multiply the views by repeating the reflection in each other, together with the central void in the cubes and with the steps that indifferently join all the interior surfaces as if one were to ascend or descend from all four sides by rotating the position with respect to the common "regular" zenith, not only mimic the shape of the hypercube, but also propose a spatial interpretation (Fig. 3).

The Grand Arche de la Defense, by Johann Otto von Spreckelsen, built in Paris between 1982 and 1989, belongs to the same figurative universe. Like a door that opens out into the world, on the one hand it is the twentieth century version of the Arc de Triomphe with which it visually dialogues almost aligning along the axis of the Champs-Élysées, on the other hand, in its configuration of two squares, one inside the other, and 4 trapezoids to connect the inner and outer faces is a model of a Schlegel diagram of a hypercube (projection into the three-dimensional space of a four-dimensional cube).

An image of the hypercube is recognizable on the main façade of the Tesseract House built in 2017 in Atlanta, Georgia, by West Architecture Studio, which also hints at this spatiality through the use of windows placed on different planes (Fig. 4).

**Fig. 3** Johann Otto von Spreckelsen, Grande Arche de la Défense, 1982–1989. Photo A.C. 1992, left. Perceiving the space od the void cube. Photo A.C. 2012

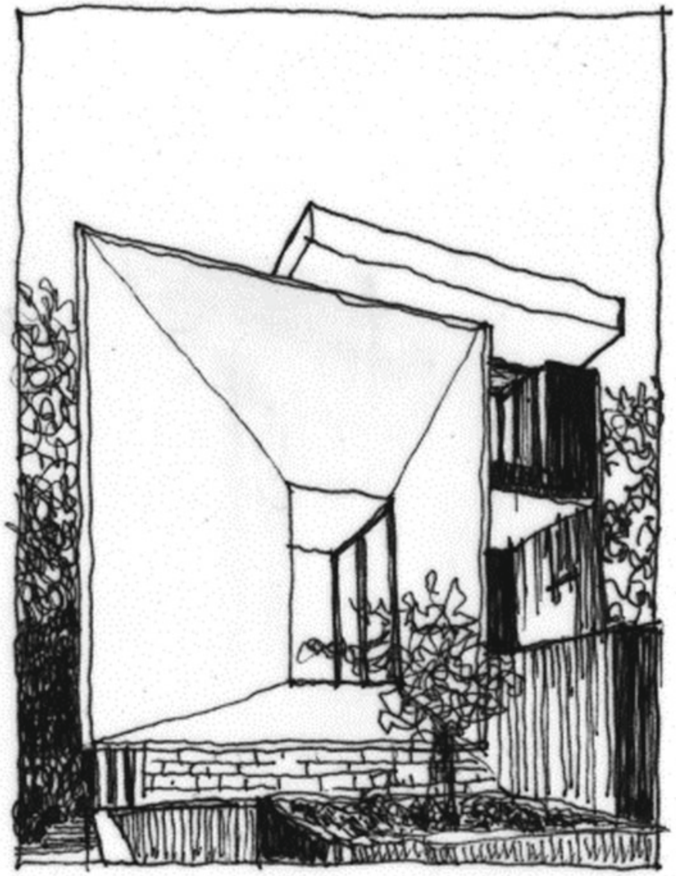

**Fig. 4** West Architecture Studio 2017—Tesseract house, Atlanta, Georgia in a sketch by A. Capanna

In Architecture it is almost impossible to imitate a form without understanding its geometric-mathematical features and evoking correctly and precisely a concept is not the same than to make a copy of something coming from different branch of learning.

## 6 How to Get Through the Looking Glass

The novel *Alice through the looking glass* tells a story about a reversed world where things have opposite references and frequent changes in time and spatial directions [3].

It is an imaginary world which has little to do with architecture, apart from imagining future spaces and environment. The romance is full of nonsense that are just what architects have to avoid. What connects it with the research for new spatiality in architecture is the question that clarifies the title of the second part of Alice's adventures. The complete title is in fact *Alice through the looking glass and What Alice Found There*. The other dimension that Alice finds beyond the mirror is the one we are looking for.

After the primary questions that motivate and support the research on the fourth dimension in architecture: why should Architecture have only three dimensions? why should Architecture have more than three dimensions? and how to perceive more than three dimensions? we have to outline the sizes of the four dimensional architecture space.

Of course there is the SPACE-TIME TYPE, expressed in measurements of length, width, height, plus one that measures the length of time (duration) to acquire complete information. Another type, more complex to find in our real world (on this side of the mirror), is the purely GEOMETRIC. According to Cartesian coordinates $x\ y\ z\ t$ [4, pp. 3–32], taking into consideration that in the fourth dimension a person may go in and out of a locked room at his pleasure crossing corners and borders, just like Packman entering and exiting through the pc's screen, and may also walk freely along the horizontal pavement as well as along the ceiling and along the walls, just like Fred Astaire in the movie "Royal Wedding".

### *6.1 Matters of Position*

Rem Koolhaas is the author of Prada Transformer a temporary pavilion located in the center of Seoul close to the 16th Century Gyeonghui Palace. The pavilion consisted of four basic geometric shapes—a circle, a cross, a hexagon, a rectangle—leaning together and wrapped in a translucent membrane. The pavilion has four differently shaped faces. Each shape is a potential floor plan. Each side plan is precisely designed to organize a different event installation creating a building with four identities. Walls became floors and floors walls as the pavilion was flipped over by three cranes after

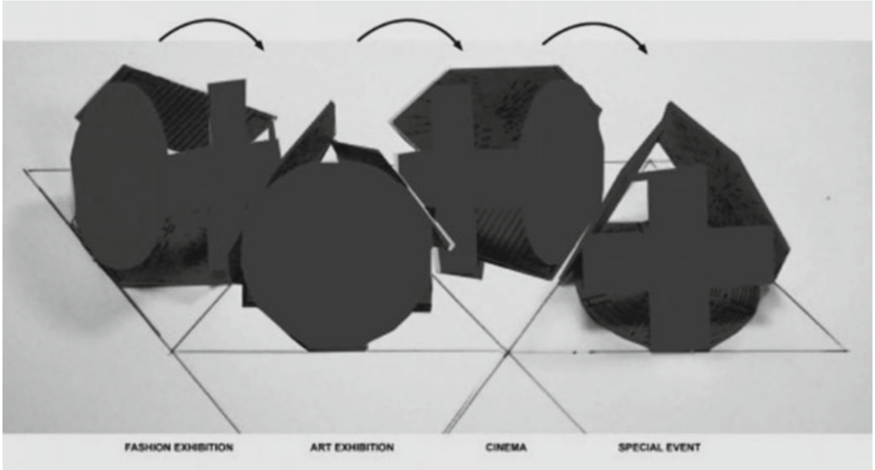


**Fig. 5** Rem Koolhaas 2009—Prada transformer temporary pavilion geometries

each event to set up the next. The Transformer, built and assembled in a "Fashion Exhibition" configuration for the opening April 25th, 2009, was hexagonal base. In a second phase, the space was transformed into a rectangular-based cinema. The third configuration was that with the cruciform base, set up for the "Beyond Control" contemporary art exhibition, organized by the Prada Foundation and curated by Germano Celant. Finally, the "Special Event" dedicated to Prada fashion, with the Transformer put on the circular base.

In this temporary architecture the change of horizontal/vertical references can be assimilated to a four-dimensional sequence (Fig. 5).

One of the characteristics in fact of living in a four-dimensional environment is related to the question of position inside-out, up-down.

For the project "Outlinet" three students in one of my courses at the University of Rome, developed an idea for a temporary installation. A sequence of six cubes, simply combined or inserted one inside another, have steel edges. The cubes have neither walls nor pavement or ceilings, not even transparent; people can freely cross all the surfaces because the geometry of the cubes is constructed only by means of the edges. Moreover, one can assume any position, upside down included, because some of the lacking surfaces are replaced with an elastic semi-transparent net on which it is possible to stand in any position (Fig. 6).

## *6.2 Matters of Interpretation*

A distorted cube that especially in the interiors and in the position of the windows, experiences the fourth dimension. But it is with Ex of In House that Holl with Dimitra

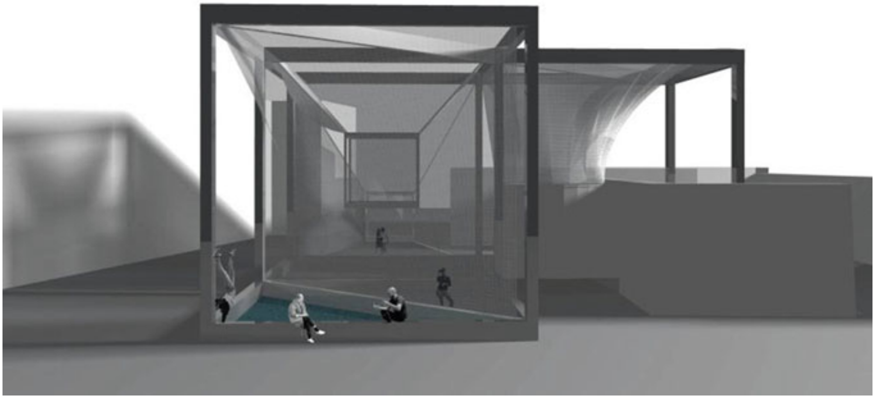

**Fig. 6** Cecilia Lalatta-Costerbosa, Alessandra Mantrici and Valentina Savarese, "outlinet", temporary installation at Teatro India in Rome, Atelier di Exhibit Design A. A. 2012–2013

Tsachrelia get the result of merging inside-out perception and space-time evolution. "The house's geometry is formed from spherical spaces intersecting with tesseract trapezoids intended as a catalyst of volumetric inner space. The geometry of the spherical intersections begins to be felt at the entry porch; an orb of wood carved out of the house volume welcomes the entrant". (from the description @ http://www.stevenholl.com/projects/ex-of-in-house).

We can say that it is a "more organic" interpretation of the topic of porous architecture that Holl practiced in a number of projects, for instance in the "U pavilion" in Amsterdam inspired by the Menger sponge, which merges internally the geometry presented on the façades.

The Ex of In House soak up light through a series of large circular openings that cut into the building and intersectinc each other so that light would filter through in section (Fig. 7).

Holl glossed the watercolor he drew at the beginning of the project for this house with the concise definition "spherical intersect". The same four-dimensional spatiality we can perceive within the site-specific projects known as "building cuts" which Gordon Matta Clark called Conical Intersect sculptural transformations of architecture produced through direct cuts into buildings scheduled for demolition (Fig. 8).

This intersection of voids, which apparently is a paradox, allows to lose some elements of orientation in space and to project the inside to the outside and vice versa, which is one of the geometrical characteristics of the fourth dimension.

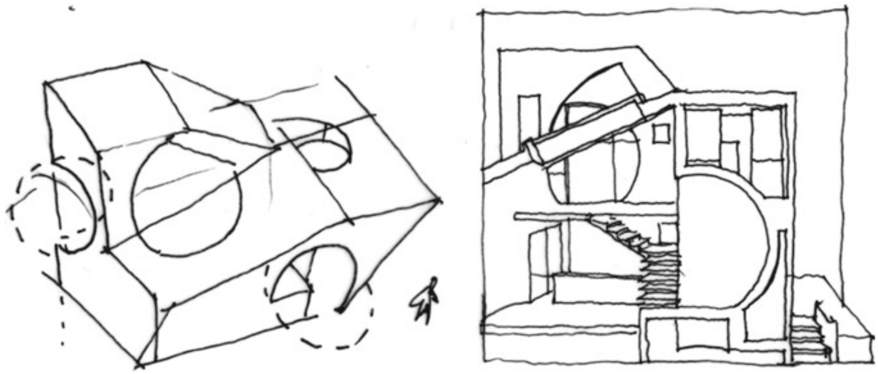

**Fig. 7** A. Capanna, sketches representing the four dimensional spatiality of Steven Holl's with Dimitra Tsachrelia's *Ex of In House*

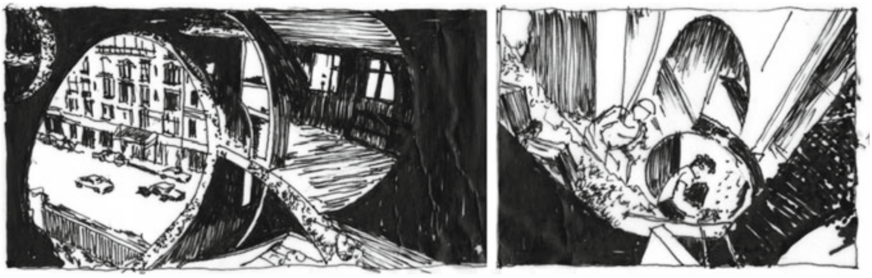

**Fig. 8** A. Capanna, sketches representing the four dimensional spatiality of Gordon Matta Clark's *Conical Intersect*—Paris 1975

## 7 Conclusions

Many other architectural projects are inspired by multi-dimensional geometry and the synthesis here presented is intended as an introduction to a particular area of research into the relationship between mathematics and architecture.

Understanding geometries and studying the changings of paradigms in Art and Science is crucial for a new approach to Architecture. We have to look beyond our physical and intellectual limits to get and reproduce the core of mathematical concepts. Architecture is a particular application of virtual reality—we can say it is virtual reality in concrete—whose aim and skill are to make visible the invisible and possible the impossible.

## References

1. Manning HP (1910) The fourth dimension simply explained. Munn & Company, New York
2. Heinlein Robert A (1941) … and he built a crooked house. Astounding Science Fiction Feb. 1941. http://www.westarchitecture.com/modern-home-atlanta-irwin. Accessed 14 July 2019; http://www.stevenholl.com/projects/ex-of-in-house. Accessed 14 July 2019
3. Carroll L (1865–1871) Alice's adventures in wonderland and through the looking-glass, and what Alice found there. MacMillan & Co., London
4. Hinton CH (1884) What is the fourth dimension? Swan Sonnenschein & Co., London. http://libguides.stir.ac.uk/archives/mclaren. Accessed 14 July 2019

# Reuleaux Triangle in Architecture and Applications

**Giuseppe Conti and Raffaella Paoletti**

The Reuleaux triangle is a figure with the remarkable property of having constant width, a typical property of the circle. It takes its name from Franz Reuleaux, a 19th century German engineer, who studied its properties, in particular the ones related to applications to mechanics. However, this figure was previously known: actually, we find it in the shape of the windows and in the ornaments of some Gothic architecture. Furthermore, Leonardo da Vinci, to represent the terrestrial globe, used eight Reuleaux triangles, each one corresponding to an octant of the spherical surface. Even the mathematician Euler encountered this figure in his study of geometric forms with constant width.

The Reuleaux triangle has numerous applications also in modern architecture, in jewelry design, in simple objects of everyday life, in the brands of many companies, in the shape of coins, in the mechanics of rotary engines and some cinema projectors, in the form of some water valve covers and of road signs. Furthermore, it is able to generate curious mechanisms.

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## 1 Introduction

The Reuleaux triangle is a figure of constant width[1]; it is obtained by intersecting three (equal) circles with the centers in the vertices of an equilateral triangle and the radius equal to the side of the triangle. The border of this figure is also called Reuleaux curve.

Reuleaux initially studied this triangle to demonstrate with a counterexample that a plane object having the same width in every direction does not necessarily have a circular boundary.[2] This question could be of crucial importance, as shown in the following episode: in 1986, the space shuttle Challenger exploded and the causes were identified in design errors of the O-rings (circular rubber seals) used to seal the rockets with the launch propellant. The material they were composed of was not suitable and the famous physicist Feynman, who was a member of the inquiry committee, also doubted that they were not perfectly circular. So he asked how the roundness of this seals had been verified and the answer was by measuring the length of three diameters. Feynman objected that this was not enough, since even the Reuleaux curves have constant diameters.

The Reuleaux triangle was already known to Leonardo da Vinci; moreover, it is found in some ornaments and windows of Gothic architecture; however, it was Reuleaux who characterized it as a curve with constant width and applied this property to many mechanical constructions.[3] It should, however, be kept in mind that

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[1]Given a plane, convex and closed figure, the distance between two parallel straight lines, each of them having at least one intersection point with the border of the figure but none with the interior of the figure, is called width relative to the direction of the straight lines. If the width is the same for every direction, the figure is said to be of constant width.

[2]See Bragastini [2].

[3]Franz Reuleaux was born on 30 September 1829 in Eschweiler (Germany). To complete his training, he worked from 1844 to 1846 in a foundry and then in a machinery assembly office. Later he enrolled at the Karlsruhe Polytechnic, completing his studies in two years; finally, he attended the Faculty of Philosophy in Berlin. After graduating he taught courses about machine constructions in Bonn. From 1856 to 1864 he was professor of machine design at the Zurich Federal Polytechnic, where he developed many of his ideas on kinematics. From 1864 he was professor at the Gewerbe Akademie in Berlin, later becoming its president. He attended numerous international fairs as a head of delegation. He died on May 20, 1905 in Charlottenburg. More than an inventor, Reuleaux can be defined as a “scientific engineer” and a machine theorist; he is considered the father of modern kinematics (the latter word, coined by Ampère). He criticized German militarism; in fact, after seeing a cannon built by Krupp, he said: “here is a murderer”.

Reuleaux had a certain reputation for his studies; proof of this is the fact that Wittgenstein wanted to enroll himself at the school where Reuleaux had taught in 1906.

His work is vast. His written works include: Konstruktionslehre für den Maschinenbau (1854–62); Theoretische Kinematik (1875); Kurzgefasste Gechichte der Dampfma-schine (1891); Die praktischen Beziehungen der Kinematik zur Geometrie und Mechanik (1900); he also directed the Buch der Erfindungen, Gewerbe und Industrien (vol. 8, Leipzig 1883–1889), translated into Italian with the title *Le grandi scoperte e le loro applicazioni* (vol. 13, Turin 1886–96). Reuleaux had created in Berlin a collection of over 800 models of mechanisms, many of which by means of his triangle, which were widely used in Europe before the Second World War. Most of them were lost

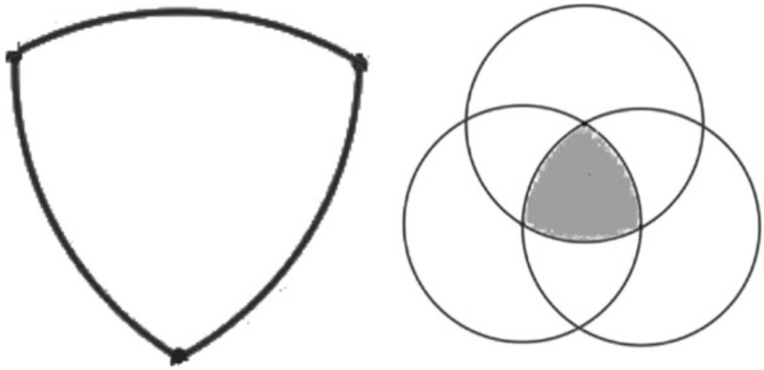

**Fig. 1** The Reuleaux triangle and its construction

some steam engines had a Reuleaux triangle-shaped cam since 1830 ([7], p. 240) (Fig. 1).

The first mathematician who studied constant width curves was Leonhard Euler. In a document, presented in 1771 and published in 1781 with the title *De curvis triangularibus*, Euler studied curvilinear triangles and constant width curves, which he called orbiform.

It is shown (Barbier theorem) that, if a closed and convex plane set $C$ has constant width $h$, then the measure of the length of its boundary is $\pi h$ [3] (Fig. 2).[4]

There are other curvilinear regular polygons with constant width; they are built like the Reuleaux triangle, but starting from a regular polygon with an odd number of sides (Reuleaux polygons) (Fig. 3).

Figures of constant width can also be obtained by starting from irregular polygons with an odd number of sides, provided the diagonals joining a vertex with the two opposite vertices have a constant measure. The construction of these polygons is very simple: it is sufficient to trace arcs of circumference having the center in each

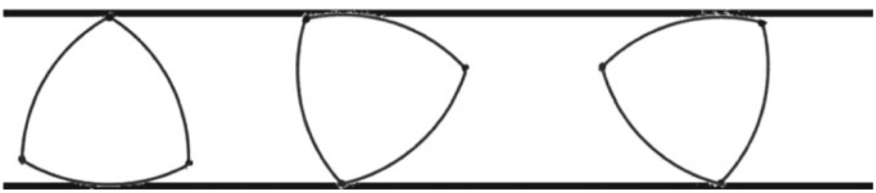

**Fig. 2** Property of Reuleaux triangle

in the destructions of 1941–45 war. The Reuleaux Collection of Kinematic Mechanisms, located at Cornell University, contains a series of 219 models, which are probably the last remaining.

[4]To prove this, just use Cauchy's formula $L = 1/2\int_0^{2\pi} B(\theta)d\theta$ where $B(\theta)$ is the length of the projection of $\boldsymbol{C}$ along a straight line with direction corresponding to the angle $\theta$.

**Fig. 3** Reuleaux (regular) polygon with 5 sides

vertex and the side opposite to the considered vertex as a chord; the radius of these circumferences is precisely the width of the figure (Fig. 4).

Evidently, when the number of sides tends to infinity, these polygons tend to become a circle (the only plane figure of constant width having a center of symmetry).

The Blaschke-Lebesgue Theorem ([5], p. 67) states that, among all the convex figures of constant width $h$, the circle is the one of maximum area while the Reuleaux triangle is the one of minimum area. Observe that the area of the Reuleaux triangle of width $h$ is: $\frac{h^2(\pi-\sqrt{3})}{2}$, so it's about 10% smaller than the area of the circle having the same width.

It is interesting to note that the solutions of the following algebraic inequality give the coordinates of the points of the plane of a figure with constant width (Fig. 5):

$$\begin{aligned}&(x^2+y^2)^4-45(x^2+y^2)^3-41{,}283(x^2+y^2)^2+7{,}950{,}960(x^2+y^2)\\&\quad+16(x^2-3y^2)^3+48(x^2+y^2)(x^2-3y^2)^2\\&\quad+x(x^2-3y^2)\left(16(x^2+y^2)^2-5544(x^2+y^2)+266{,}382\right)\leq 720^3\end{aligned}$$

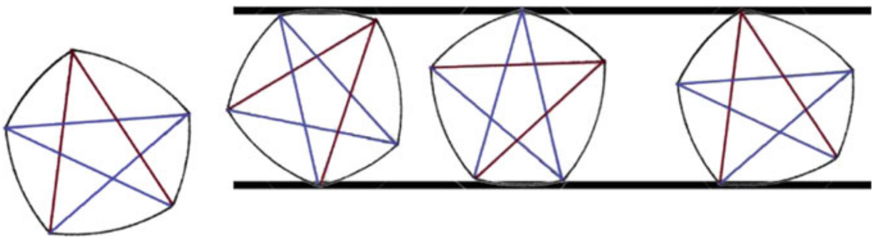

**Fig. 4** Reuleaux (irregular) polygon with 5 sides

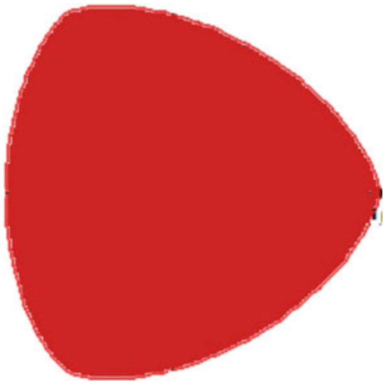

**Fig. 5** Plane figure with constant width

## 2 Applications

The Reuleaux triangle has numerous applications.

Leonardo da Vinci used this figure before 1514 to make a cartographic projection of the terrestrial globe,[5] now placed in the *Royal Collection* at Windsor. He divided the surface of the globe into eight equilateral spherical triangles, each of them delimited by the equator and two meridians separated by 90°. After that, he projected these spherical triangles on a plane, obtaining eight triangles very similar, in shape, to those of Reuleaux. In this representation Leonardo reproduced all the information reported by Amerigo Vespucci after his travels (1497–1504). We note that the name *America* appears in this map. It is interesting to note that Leonardo had already drawn the Reuleaux triangle before.[6]

In 1616 Nicolaas Geelkercken used a projection with Reuleaux triangles, similar to that of Leonardo, to represent the earth globe.

The coverage of the Kresge Auditorium (MIT Campus), designed by Eero Saarinen, also corresponds to an eighth of a sphere; therefore, its projection on the horizontal plane has a shape similar to that of Leonardo's triangles.

The Reuleaux triangle can be found in the shape of windows and ornaments of some Gothic and Neo-Gothic buildings. Here are some examples: the large window in the central apse of Milan cathedral; some of the windows of the Gothic church of Nôtre Dame (12th century) and of the cathedral in Bruges; the Sacred Heart cathedral in Bendigo, Australia (1896–2001).

In modern architecture, the Reuleaux triangle was used by Norman Foster (1990–1992) in the Collserola Communication Tower in Barcelona and by Dörte Gatermann (2006) in the Kölntriangle in Cologne. In addition, the base of the Donauturn observation tower in Vienna (by Hannes Lintl) and the section of the internal

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[5] Actually, not all scholars agree on the paternity of this map and even on the type of projection used.

[6] See Paris Manuscript A, 15v.

**Fig. 6** Ashtray in the shape of Releaux triangle

structure of the Mercedes-Benz Museum in Stuttgart (by Ben van Berkel and Caroline Bos's UNStudio) have the same shape.

There are everyday objects that can be in the shape of Reuleaux triangles, such as tables, chandeliers, baskets and ashtrays (Fig. 6).

The profile of many guitar picks is a Reuleaux triangle; in this case, the shape does not concern only the aesthetics but also the utility, since it combines a sharp point (to provide a strong articulation) with a wide tip (to produce a warm timbre). Moreover, due to this particular shape, the picks can be used more easily, being indifferent the angle that is used.

The Reuleaux triangle shape is used in jewels and in the external shape, as well as in the mechanism, of some watches.

In 1933, in Japan, a stamp (**Yubari-stamp**) appeared in the shape of a Reuleaux triangle (Fig. 7).

The Reuleaux triangle is represented in the brands of many (Italian and foreign) companies.

In the United States the Reuleaux triangle occurs in various contexts.

There are road signs with this shape, for example those along the historical and panoramic paths of the National Trail System.

Since the 1950s, Philadelphia firefighters have adopted this form for the taps of the hydrants found in the streets: in this way the hydrants are safe from tampering. In fact, in summer it often happens that someone opens them to take a shower; since the Reuleaux triangle is a curve of constant width, a common wrench does not manage to grip the sides, but turns in a void. Therefore, without the right tool it is difficult, if not impossible, to open the tap.

On Third Street in San Francisco and in the city of Newport Beach there are some Releaux triangle-shaped manhole covers. Apart from the circle, this is the right shape to make a manhole so that it cannot fall through the hole while turning it around.

Traditionally, coins have a circular profile. However, to save material, some of them have the shape of a Reuleaux triangle or of a regular Reuleaux polygon; we

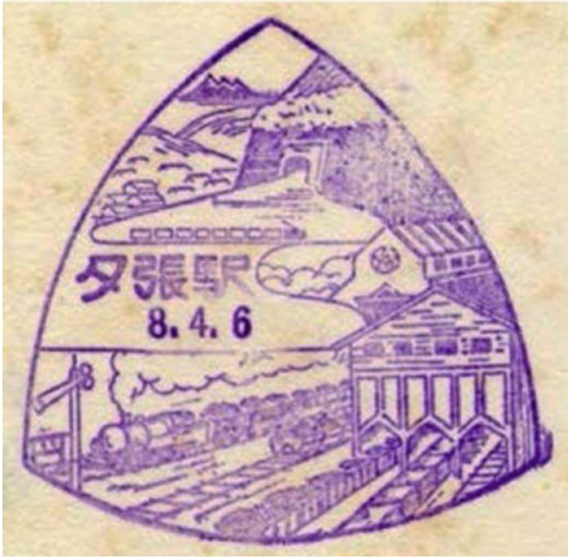


**Fig. 7** Yubari-stamp

mention, among many others, the 1996 60 dollar Bermuda coin of triangular shape, the 2004 10,000 crowns Slovak coin of pentagonal shape, the 1998 20 pence British coin of heptagonal shape. In every case, the Blaschke-Lebesgue theorem asserts that they are made with a smaller amount of metal than what is needed for a circular coin. Moreover, being figures of constant width, they behave like circular coins in automatic dispensers, since the currency detector will always measure the same width of the coin.

The Reuleaux triangle is used in numerous mechanisms. A very interesting application is found in some cinema projectors: the aim is to advance the motion picture film with a jerky movement, in which each frame of the film stops for a fraction of a second in front of the lens of the projector. This advancement can be done using a mechanism in which the rotation of a Reuleaux triangle, inside a square, is used to create a pattern of movement that, alternatively, lets the film slide quickly on each new frame and pauses the movement of the film while the frame itself is projected. A feed mechanism of this type is found in the Luch-28 mm projector of the Leningrad Optics and Mechanics Amalgamation of 1963.

Another very famous application of the Reuleaux triangle can be found in the Wankel rotating piston engine, designed and developed by the German engineer Felix Wankel starting from 1951. The shape of its rotating piston is very close to a Reuleaux triangle, with the arches slightly lowered to increase the volume of the combustion chamber.

The first production car to mount this type of engine was the NSU spider in 1963; with only 498 cc. of displacement, it developed a power of 50 HP and pushed the car to a speed of over 150 km/h. Later the NSU Ro 80 model was produced.

The Mazda Cosmo Sport coupé (1967) was the first Mazda to be powered by a Wankel engine. Subsequently, until 2012, the company produced the Rx-8 model, equipped with the Wankel engine. Another model of this brand, the 787B, equipped with this type of engine, won the 24 h of Le Mans in 1991; this fact caused a stir because it was the first Japanese car to win this important race (only in 2018 Toyota managed to win the 24 h of Le Mans). Currently, Mazda is developing a new model (RX-9) with a rotating piston engine.

Other car manufacturers have tested models with Wankel engines, both for cars and motorcycles and also for aircraft. This engine has many advantages: compared to the usual internal combustion engines, it is quiet and easy to build, both for the lack of valves and for the fact that it produces the rotating motion directly, without the need of a crankshaft. Its weak point is due to the three angular points, which wear out quickly, causing a lowering of compression and a high consumption of lubricant. Some car manufacturers are experimenting with appropriate mechanical measures to eliminate this problem (Fig. 8).

The profile of the inner part of the housing, in which the revolving piston is located, is a fine geometric figure: it is a plane curve called two-lobed epitrochoid (Fig. 9).

The Reuleaux triangle is used in mechanics to make square holes. For this purpose, we use a drill bit that has a Reuleaux-triangle shaped section perpendicular to the axis of the curve itself, which rolls without crawling inside a square, leaning on all sides (Fig. 10).

By rotating the center of gravity of this figure along a particular almost circular curve (in reality it is formed by four ellipse arcs), it is possible to make a square hole, even if the vertices of this square remain slightly rounded [8].

Starting from the equilateral triangle, we can construct a figure of constant width without angular points. Consider the equilateral triangle ABC with side a. Let b be an arbitrary quantity. With center in vertex A and radius a + b we trace the arc of circumference LM; with the same radius and centers in vertex B and then C we trace the arcs of circumference PQ and RS. Then we link these arcs together with three arcs of circumference having the same radius b and the centers respectively at points A, B, C. The constructed figure has constant width a + 2b and don't have angular points. In fact, the tangent lines in L both to the arc LM and to the arc LS

**Fig. 8** The phases of the Wankel engine; in order: suction, compression, burst, discharge

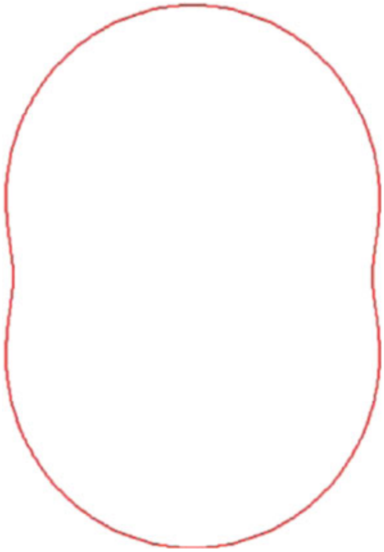

**Fig. 9** Two-lobed epitrochoid

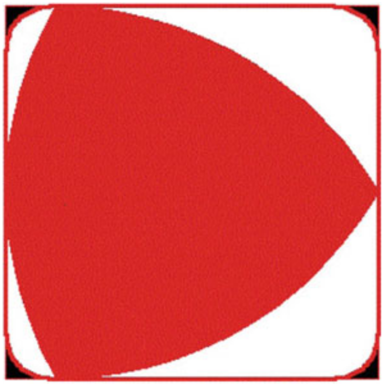

**Fig. 10** "Square" hole with Reuleaux triangle

are perpendicular to the line AB, so they coincide. The same consideration can be made in the other connecting points of the arches (Fig. 11).

In 2016 Panasonic launched the MC-RS1AW cleaning robot, renamed RULO by the company; its shape is similar to the "smoothed" Reuleaux triangle and the company asserts that this form makes it possible to exploit the brilliance of the kinematics to reach the narrowest corners and clean them more effectively.

In the space, the sphere is a figure with constant width; however, it is not the only one with this property ([6], pp. 281–282). Similarly to what was done in the plane with the Reuleaux triangle, in the space we can define the Reuleaux tetrahedron by intersecting four spheres with the same radius and whose centers are on a regular tetrahedron. This solid, however, does not have constant width. To obtain a figure

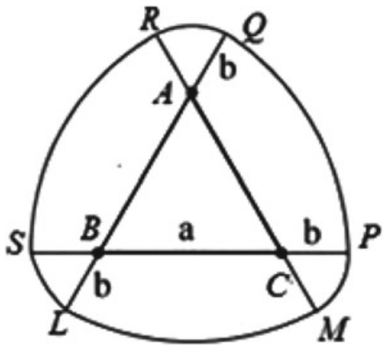


**Fig. 11** Reuleaux triangle with "smoothed" angles

of constant width, one can modify the Reuleaux tetrahedron by replacing three of its edge arcs (those converging in the same vertex or those belonging to the same face) with curved surfaces, that is surfaces obtained by the rotation of a circular arc. The solid that is obtained in this way has constant width and is called Meissner tetrahedron [1], pp. 150–151).

Hamlet (1949), one of the works by the American artist Man Ray (1890–1976), was based on a photograph in which a Meissner tetrahedron appears: in the artist's opinion, this solid resembles the skull of Yorick.

A solid of constant width can also be obtained by rotating a Reuleaux triangle around one of its three axes of symmetry. It is shown that, among all the rotation solids having constant width, this is the only one having minimum volume [4]. It is conjectured that among all the solids of constant width, Meissner tetrahedron is the one of minimum volume (Fig. 12).

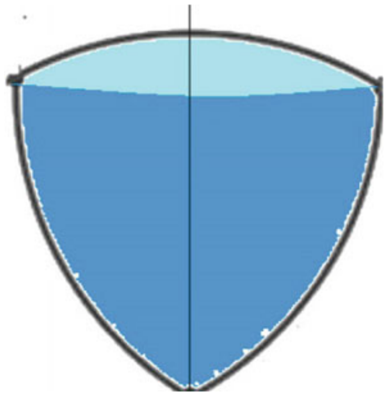

**Fig. 12** Solid obtained by rotating a Reuleaux triangle

## References

1. Balzarotti G, Lava PP (2010) 103 curiosità matematiche. Ulrico Hoepli Editore, Milano
2. Bragastini R (2003) Contributo per una interpretazione filosofica dell'opera di Franz Reuleaux. Università degli Studi di Milano, Milano
3. Bayen T, Hiriart-Urruty J-B (2012) Objets convexes de largeur constante (en 2D) ou d'epaisseur constante (en 3D): du neuf avec du vieux. Annales Mathématiques, Quebec 36(2):333–358
4. Campi S, Colesanti A, Gronchi P (1996) Minimum problems for volumes of convex bodies. Partial differential equations ad applications: collected papers in honor of Carlo Pucci. Lecture Notes in Pure and Applied Mathematics, n. 177, Marcel Dekker, New York, Base, Hong Kong, pp 43–55
5. Gruber PM (1983) Convexity and its Applications. Birkhäuser, Basilea
6. Hilbert D, Cohn-Vossen S (1960) Geometria intuitiva. Boringhieri, Torino
7. Moon FC (2007) The Machines of Leonardo Da Vinci and Franz Reuleaux, Kinematics of Machines from the Renaissance to the 20th Century. Springer, Berlin
8. Smith GS (1993) Drilling square holes. Math Teach 86(7):579–583

# Interplays of Geometry and Music: How to Use Geometry to Analyze an Artwork in Order to Compose a Musical Piece

**Chiara de Fabritiis**

This paper, which summarizes a collaboration with D. Amodio (Conservatorio Benedetto Marcello, Venezia), describes how geometric techniques can be used to analyze an artwork and to obtain parameters employed for the composition of a musical score; in particular these approach was applied to a painting by Jackson Pollock and a poem by Giacomo Leopardi. In the first case, the initial task is the study of the graphic structure of the canvas, looking for forms and their spatial organization; this work is followed by the choice of the mathematical techniques used to examine the different classes of objects previously singled out; the last step is the computation of the parameters which will be used by the composer to orchestrate the score. In the second case, the starting point is the analysis of the phonetic structure of the poem, looking for consonances; then a combinatoric approach is used in order to create a representation of the idyll as a plane graph to highlight the permutations of some group of letters; a second image of the poem as a cylindrical helix is used to measure time distances in the occurrence of syllables; again, the parameters obtained by these computations are used for the draft of the music.

## 1 Introduction

In the last years, the idea of extracting music from different kind of objects (e.g., stars, DNA of cells, numbers) has spread out in several fields of research, from physics to biology, including mathematics. Indeed, on March 2019 NASA released a video with a "sonification" of an image showing a cluster of galaxies which was taken on August 13th 2018 by the telescope Hubble (see [13]): time flows from left to right and the frequency of sound changes from bottom to top, ranging from 30 to 1000

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hertz; the objects near the bottom of the image produce lower notes, while those near the top produce higher ones.

The composer Peter Gena collaborated during several years with the physician Charles Strom in writing scores inspired by some DNA molecules (see [8–10]); also Antonella Prisco, a researcher in genetics at CNR, Naples, produced pieces of music derived from the DNA of a cell (see [14, 15]).

Michael Blake, a songwriter and producer based in Portland, translated the initial part of sequence of digits of $\pi$ into notes (see [3]), though Lars Erickson already had the idea to compose a "symphony" based on the number $\pi$ (see [6]); there is also a "dodecaphonic $\pi$" due to Jim Zamerski (see [16]).

To be more precise, the majority of these tracks are not true "compositions" but they are mere translations of a sequence (of stars, nucleobases, digits, respectively) into a sequence of notes.

Our research (see [1, 2, 4, 5] for more details) is marked by a different approach: the piece we create is not a mere "mathematical reproduction" of the object, as there is not an a priori-given sequence to translate into music. Our paradigm of work, which can be applied to various sorts of artistic products (e.g. paintings, poems) is the overlap of three different levels of discretional interpretation:

(1) We start by an analysis of the structure in search of the "forms" that appear in the artwork and of their organization. In the case of a painting we look for the patterns of spatial organization of the images; in the case of a poem we consider the phonetic and syllabic arrangements of the sound and their recurrence.
(2) After the analysis of the (spatial or phonetic) configuration has been carried out, we choice of the mathematical techniques to be used for the study of each of the structures detected in (1) and we perform the computation of the parameters which are given by the different pieces of the artwork.
(3) At last, since based on the parameters computed in (2), we write the score and orchestrate it, and we finally perform the work obtained by this process.

Of course, the various classes of artworks we use as source of inspiration require distinct analytical techniques: when dealing with a painting, the structures to be investigated are the graphical and morphological forms; if the artwork is a poem our attention will be mainly addressed to phonetical and acoustical patterns.

We underline that each of the above steps entrains an arbitrariness of choice by the authors, also in the mathematical part of the investigation because of the election of the mathematical techniques we carry out. On the other hand, once choices have been made, the computations are performed following rigorous mathematical techniques.

## 2 Analysis of Jackson Pollock's "Summertime n. 9"

The first artwork we considered is a creation of the American painter Jackson Pollock, owned by Tate Gallery, London. The reason why we choose this masterpiece is the fact that the author used a technique, called "dripping", to realize many of his abstract designs. Indeed, Pollock laid the canvas on the floor of his atelier and he walked here and there, with a brush in his hands, letting the paint drip on the ground. In the meanwhile, according to several witnesses, he used to listen to music; so we wonder if the music he was listening to remains, in a certain sense, encompassed into the pattern of drops and stains appearing on the painting.

The unusual procedure that Pollock used to realize Summertime n. 9 yields as a relevant consequence a particular texture of the painting: there are several layers of different paints which are stratified on the cloth (a fact that is more evident when looking to pictures taken in grazing light), showing different forms arranged in varied spatial patterns.

We started our analysis by classifying the drops or stains of various kind of paints in separate groups, according to their colours and geometric forms, with the purpose to deal with them by means of different mathematical approaches.

The list of these different groups is given by:

1. blue regions;
2. yellow regions;
3. red regions;
4. black and grey “patches”;
5. black and grey “thick” structure;
6. black and grey “thin” structure;
7. coloured dots;
8. short curves (or “long” points).

Indeed, the forms we consider are obtained by distinct pictorial material and techniques which produce a big difference in rendering. Black and grey patches are wall paints (water distemper) dripped on the canvas; blue, yellow and red regions were obtained by brushing oil colours on the areas delimited by the dripping of grey and black paint, coloured dots and short curves are strokes of the brush with oil colours, while black and grey structures arise from the dripping.

To measure up the quantities needed for the composition, we put on the painting a Cartesian reference frame in which the $x$-axis is the horizontal lower side and the $y$-axis is with the vertical left side. The form of Summertime n. 9, which is much longer than higher, recalls a stave, so we identified the $x$-variable as time and settled the unit of measurement so that the painting is 24 min long. The choice of the $y$-variable requires deeper considerations: the parallel with the stave would lead us to measure pitch on the vertical, but this choice would give no freedom in the modulation of sounds, hence we decided to put loudness on the vertical. Human perception of volume is quite wide, but usually the variation within the same piece does not exceed the ratio 1:3, so we decided that points on the upper horizontal side of the picture would sound 3 times louder than points on the lower horizontal side. In this way we associated to each point in the painting coordinates (a; b), where a is the time elapsed from the beginning and b measures the loudness of the sound to which the point is associated in the score.

Now that we have a reference frame on the painting, we should interpret its parts by means of different mathematical techniques. For instance, the occurrence of patches of the same colour in different positions suggests the idea of a theme that is played repeatedly at certain times, being modified according to suitable rules which keep track of the modifications of the patch itself.

Black and grey patches are the first objects appearing on the canvas in Pollock's creation of the painting; they have the form of large circular or elliptic drops (Fig. 1).

First of all, we number these drops increasingly from left to right, then we draw the ellipse which better suits each drop (in order to get a better visual approximation we only trace its contour, using a transparent filling); we also show the rectangle which bounds the selected ellipse. To establish the position of each of the ellipses and to deduce from it the parameters for the composition, we measure the horizontal and vertical coordinates of the lower left vertex of the rectangle which bound the ellipse, the width and the height of this rectangle; we also determine the rotation of the axis of the ellipse; these parameters allow us to determine the starting time of each patch, its duration and loudness.

Now we turn to the coloured patches: first of all we choose to consider each colour (yellow, blue, red) separately; then we associate to the first region of each colour a theme (see [1] for a detailed report of the draft of the score); all remaining patches are assigned a theme which is obtained from the original one according to the modification of the coloured region.

The mathematical problem is to choose a suitable theory which describes the parameters which allow us to follow the variations in the shape of the patches. We decided to look to the regions in the painting as a subset of the complex plane and to use holomorphic maps to compute the quantities needed to describe the modifications in the patches. The parameters required for the composition should keep track of the variations of each region with respect to the first one: the easiest way to do so is

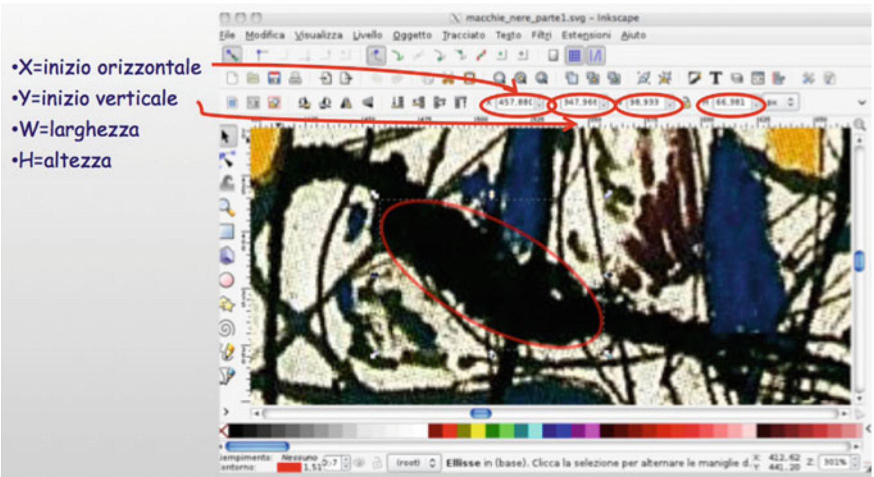


**Fig. 1** Parameters given by an ellipse

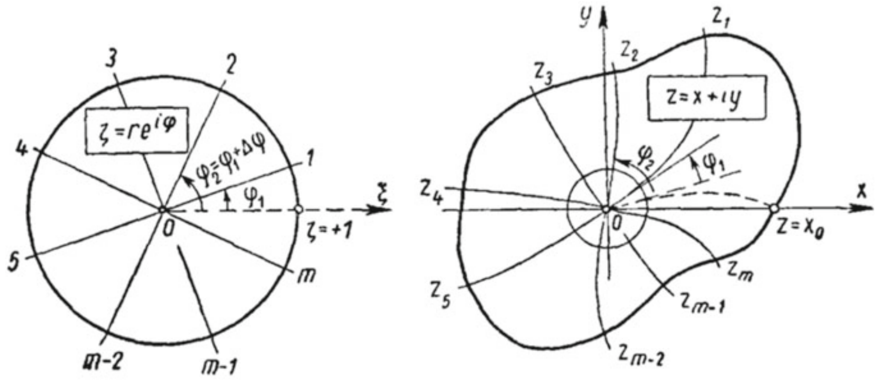


**Fig. 2** Modifications of a disk according to Filchakov (see [7])

to take, in each of the chosen directions, the ratio between the length of the given segment and the length of the first one (Fig. 2).

This procedure, introduced in full generality by Filchakov in a chapter of his book [7] (many thanks to Prof. E. Pervova, University of Pisa, for the translation), gives an approximation of the coefficients of the biholomorphism, provided the map fixes the origin.

The remaining parts of the painting are used to create the background accompaniment which is realized by percussions. Since these instruments cannot play different notes, a simpler analysis, based on a frequency principle, will be enough for the compositional purposes. We recall that in the picture there are coloured dots and short curves, which are obtained by small touches of the brush dampened in oil colours, a "thick" structure and a "thin" structure, which are both obtained from the dripping of the grey and black paint; they arise from the fact that when the liquid starts dripping on the canvas, first it produces the large drops, then a strip of colour which becomes a thread as the dripping goes on and on. These different pieces are all treated with the same technique, with minor adaptations to the different situations: the idea is to consider a measure given by some kind of density; indeed it appears natural that a richer covering of the canvas in the picture entails a denser musical tissue. In the composition each of these forms will be associated to a percussion instrument and for each of them one by one we "follow the instructions" given by the painting.

First, we divide the painting into 12 vertical strips, lasting 2 min each.

As for the coloured dots and the short curves, we simply count their number in each strip and compute the ratio between the number of the objects in the given strip and the number of objects in the first one. In this way we obtain a parameter which evaluates how much the associated instrument should increase or decrease its presence. In the case of short curves, we also provide additional information, by specifying how many of them are placed at the bottom, at the center and at the top of the strip.

The thick lines present in the structure are "weighed" by counting their number and considering their area, which is computed by approximating them with suitable rectangles or parallelograms. In order to obtain a parameter which tells us how much we have to increase or decrease the presence of sounds produced by the instruments associated to the thick structure, we compute the ratio between the measure obtained in each strip and the measure of the first one.

Finally, the lines of the thin structure are counted by taking a close grid of parallel lines at constant pace and counting the number of intersections they create with the parallels. Since the directions of the thin lines of the painting are substantially random, the number of intersections with the lines of the grid does not on depend on the orientation of the grid itself and it estimates both the number of the thread of the structure and their length.

## 3 Analysis of "L'infinito" by G. Leopardi

After this first research, our investigation moved its attention towards the famous poem "L'infinito", composed by Giacomo Leopardi while staying in Recanati between 1818 and 1819: our aim was to unveil the wonderful rhythmical and phonetic patterns which are deeply hidden inside it; the structures of the lyric are then interpreted through mathematical techniques and the parameters obtained with these tools are used by Davide Amodio to create a musical piece inspired by the idyll.

The syntactic structure of the lyric is very simple: there are four sentences which are mainly based on parataxis (that is, on coordinate clauses), the only subordinate clauses are either relative ones or subordinates governed by a gerund which work as an attribute. A powerful sensation of indefiniteness, which Leopardi looked for intensely, as he claimed in his work "Zibaldone", is obtained by the so-called *enjambement*, a poetical technique in which a sentence does not stop at the end of a line but flows continuously to the beginning of the next one. E.g., at lines 4–5 the pause of the new line comes between the adjective *interminati* and the noun *spazi*, the same happens at lines 5–6 with *sovrumani* and *silenzi*, at lines 9–10 (*quello*/*infinit*o) and finally at lines 13–14 (*questa/immensità*).

The sensation of vagueness and indeterminacy is strengthened by the fact that the words belonging to the semantic field of *infinity* (*interminati*, *sovrumani*, *profondissima quiete*, *eterno*, *immensità*) appear mostly at the beginning or at the end of the lines, just before or just after the pause of the end-line. We also have to take into account the alliterations which appear mainly in the central part of the poem (the couple *sedendo-mirando* above all, but also *voce-vo*, *tan-ta par-te*). For example, Fig. 3, taken from [5], illustrates the position of the tonic vowels in the last six lines of the poem, highlighting the links between the locations the same vowels hold in different lines.

More generally, we look for the correspondences of the same sounds in the idyll: we consider the different vowels (or groups of vowels) in their different occurrences and draw the graphs which connect the places where they appear (in some cases

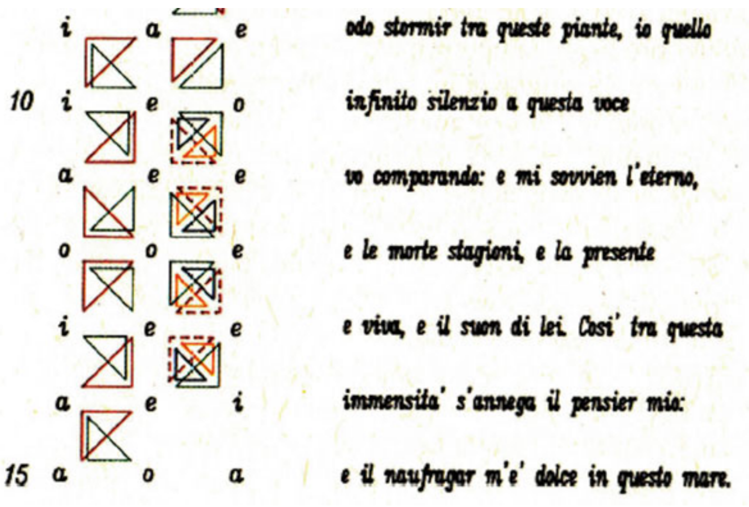


**Fig. 3** The position of the tonic vowels in the last six lines of the poem

associated with the same consonant), as we were "chasing" them in a diagram (see e.g. [11]). This strategy is carried out by creating a model of the lyric via an image, underlining the representation of the points corresponding to the same group of letters (if necessary permuted) and connecting them with segments, thus creating a graph. The outlined procedure allows us to translate the combinatorics of sounds into images that are used by the composer to draft a part of the score which will work as a musical base for the performance of the poem as a song.

Figure 4 depicts the image of first 8 lines of "L'infinito": here we consider the presence of the vowel "u" (only when it does not follow the letter "q" because in Italian "q" is always followed by "u" and their union constitutes a special sonority which is different from the other syllables where "u" appears): the symmetry of this image evokes the balance of the cross-references in the lyric.

Nonetheless, for our purposes the study of the patterns of the association of sounds within the idyll is not enough. Indeed, Musti in his paper [12], points out that the analysis of a lyric must consider also a different point of view because when considering a poem, we can either listen to it or read it silently.

The first approach, which is related to the phonetic field, is the one we used so far, while the second method we are now going to investigate must deal with the "spatial" structure of the verses as lines on the written page. When reading aloud, time flows linearly onwards; on the contrary when looking to the printed page, at the end of each line the eye in a certain sense "comes back", even if not exactly at the same place, because the beginning of a new line lies below the previous verse. We notice that this common suggestion of circularity, for this particular poem is strengthened by the consonance of the first and the last line: the words "questo colle" occurring in the first verse are strictly related, both under the sonorous and the conceptual viewpoint, to the words "questo mare" appearing in the last one.

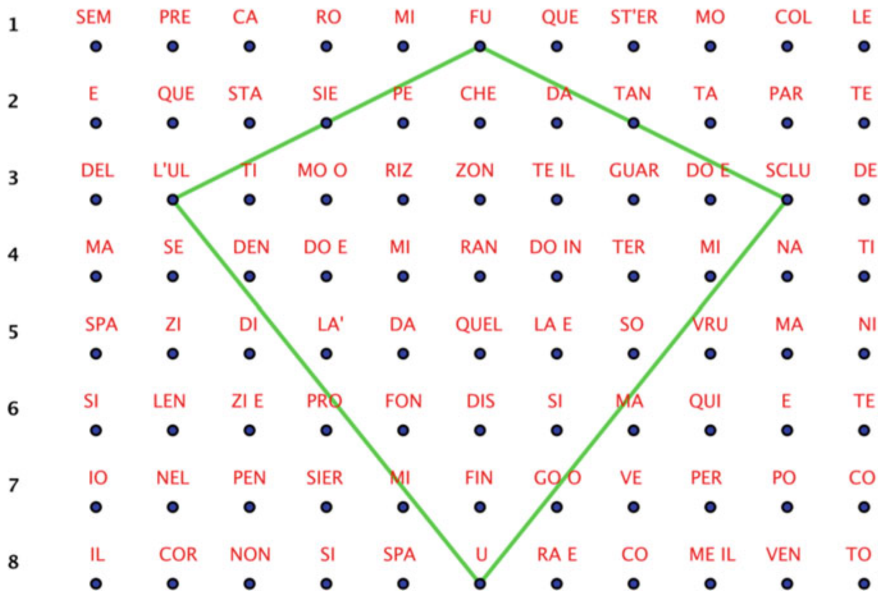


**Fig. 4** The kite with the occurrences of the vowel "u"

This fact emphasizes the idea of "return with modification" which is the same that is incorporated in a circular helix: the choice of this particular curve is due to a twofold motivation: first of all, each line has the same length (because any verse contains eleven syllables), thus suggesting to pick a curve lying on a right circular cylinder, moreover the vertical distance between two consecutive lines is the same, and this points towards a curve of constant pitch. We now need to establish suitable values for the radius r and the pitch p; we decided to set $r = 10$ and $p = 8$: in this way the rectangle we are going to consider has width $20\pi \approx 63$ and height $8 * 15 = 120$; this corresponds to a proportion 1:2 which approximately the same of the written text. The length of the theme will be equal to 10 notes, so that it is clearly recognizable and gives to the composition of reasonable duration (Fig. 5).

To control the modifications of the theme, we need a measure of the distance between the points on this helix which is again associated to the two different ways of enjoyment of the poem: the first is when we listen to the idyll and thus time flows increasingly, a situation that corresponds to measuring the intrinsic metric on the curve parametrized by unitary arclength, the second interpretation is when we silently read the written text: the carriage return at the end of each line implies that the distance between two syllables can eventually decrease, an approach which is equivalent to taking into account the ambient metrics in the three dimensional space for the points lying on the curve. In particular, in order to obtain the parameters needed for the composition, we examine two different metrics on the ambient space, the first being the standard Euclidean distance, the second is the so-called Manhattan


**Fig. 5** The beginning of the score for the “helix”

(or taxicab) one and we apply them to some groups of letters which have a peculiar relevance in the text, such as “qu”.

## References


1. Amodio D (2013) From Pollock’s Summertime to Jacksontime. In: Imagine math 2. Springer, Milano, pp 151–159
2. Amodio D (2015) Sonorous Geometries of Infinity. In: Imagine maths 4. UMI, Bologna, pp 113–132
3. Blake M (2019) https://www.youtube.com/watch?v=wK7tq7L0N8E, Visited May 5th 2019
4. de Fabritiis C (2013) From Canvas to Music: Mathematics as a Tool for the Composition of Jackson Time. In: Imagine math 2. Springer, Milano, pp 161–169
5. de Fabritiis C (2015) Simple geometries of infinity. In: Imagine maths 4. UMI, Bologna, pp 123–133
6. Erikson L (2019) http://pisymphony.com/gpage.html, Visited May 28th 2019
7. Filchakov PF (1964) Approximated methods for conformal mapping. USSR, Kyiv (Russian)
8. Gena P (2011) Apropos sonification: a broad view of data as music and sound. AI & Soc 27:197–205
9. Gena P (2019) http://www.petergena.com/DNAmus.html. Visited May 5th, 2019
10. Gena P, Strom C (2019) http://www.petergena.com/docs/gena-strom-DNA.pdf. Visited May 5th, 2019
11. Gentile MT (1991) Leopardi e la forma della vita (genesi, formazione, tradizione). Bulzoni, Roma
12. Musti M (1980) La sintassi sonora: analisi delle fonie leopardiane in «A Silvia» (vv. 1–6), Misure critiche 9–10, pp 57–74
13. NASA (2019) Hubble Treasury Trove Sonification, http://svs.gsfc.nasa.gov/13061. Visited May 5th, 2019
14. Prisco A (2012) DNA in sound—from Emergency Index 2011. Ugly Duckling Presse, Brooklyn

15. Prisco A (2019) DNA in musica, http://www.youtube.com/watch?v=FmmitflrxNw. Visited May 5th, 2019
16. Zamerski J (2019) https://www.youtube.com/watch?v=AOaR4NS7ObI. Visited May 5th, 2019

# Harmony in Space

**Biagio Di Carlo**

In their simplest and most elementary form, the structures used in design science are the three-dimensional version of the planar interlacing (biaxial and triaxial) that has always been used for the construction of gratings, baskets weavings and textures. The reciprocal frames can be considered as a premise to the tensegrity structures, which in turn can be considered as a premise to the geodesic structures. Geodesic structures arise from the correct subdivision of polyhedral shapes. The nascent reciprocal joint as a simple, natural and economic form, can be reworked towards the starred joint where the rods contribute towards a single junction point. The structural stability of natural structures is guaranteed by the presence of the triangle. A triangulated structure, optimized for use, does not require additional materials to ensure its resistance.

## 1 Introduction

> Space is not a passive vacuum, but has properties that impose powerful constrain on any structure that inhabits it. (A. Loeb)

The knowledge of polyhedra is essential to properly design an architectural work. Children can make a tower putting on various cubes, but a tower can be also made with octahedrons or other space modules; in this case we observe new potentiality of form.

We must educate children from an early age to the concept of space considering not only the plane geometry but also the solid geometry that is the synergetic geometry of polyhedra as space forms. The polyhedra in the space are like the notes for the music, so the polyhedra can be used to create architectural music. Plato wrote that

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to know the nature of the universe we have to concentrate on the unity of all things and dive ourselves in the study of music, astronomy, geometry and the number (the so-called ‘quadrivium’).

The Platonic and Archimedean solids are not only objects in the three-dimensional space but mainly the modules that create the space texture. Enclosing the space using only four equal regular polygons we will only have four equilateral triangles of the tetrahedron form and not another one.

Music is the exact miniature of the laws operating in the entire universe. There is music both inside and outside the human body. The space delineated by polyhedra and the musical harmony represent the language of beauty. Music and the harmonic space of the polyhedra can open up and strengthen our intuitive faculties.

The five Platonic solids symbolize the four elements (earth, fire, air, water) and the universe. Plato had already suggested that the space had its existence and its very specific rules. The modern scientific research has validated Plato’s ancient thesis.

The origin of polyhedral shapes is very ancient. Almost certainly, the Egyptians knew the tetrahedron, the cube and the octahedron. Some dodecahedron shaped objects of Etruscan origin were found near Padova, in Italy.

Considering the five Platonic solids at the microscale, the tetrahedron can be identified in silicates, in methane, in quartz, in diamond, in the water molecule; the octahedron in gold, in aluminium crystals; the cube in the sodium chloride, in pyrite crystals; the icosahedron in the hydrides of boron, the fullerenes (truncated icosahedron) in space; the dodecahedron in pyrite, in dodecahedron, in quasicrystals and in radiolarians.

## 2 Geodesic Structures Construction

In 2004 I published my book in Italian named “Strutture Geodetiche” including for the first time the tetrahedral ideogram D.S.T. (Design Science Tetrahedron). Then in 2013, I presented its English version at Delft Symmetry Festival within the article ‘Design Science Structures’. The tetragram synthetically shows four structural groups having the golden section as a common denominator: (1) platonic and Archimedean solids (2) R. B. Fuller’s geodesic structures (3) Kenneth Snelson’s tensegrity structures (4) reciprocal frame structures by Leonardo Da Vinci. The polyhedral space modules relating to the design science are the starting point for a correct design tuned with nature. Following the existing micro, medium and macro scale rules, we face the natural method that characterizes the design science.

In their most simple and basic form, the structures used in the design science are the three-dimensional version of the planar weaving (biaxial and triaxial), always used for the manufacture of baskets and weavings. Reciprocal frames can be considered as an introduction to tensegrity structures, which in turn can be considered as an introduction to geodesic structures. Geodesic structures derive from the proper division of polyhedral shapes. In this way, we obtain a reciprocal frame joint in a natural and economic form, it can be modified into the 'stellated' joint where all the rods join to a single point.

Sacred geometry is the universal language contemplated in the macrocosm reflected in the microcosm that is the language of harmony, of beauty, of faith, of proportion, of the universal order and rhythm. The person living within a spherical space thinks to be the centre of things and feels the presence of the geometric rigor linked to the beauty and the perfection of the golden ratio structures. Observing a geodesic structure and its shadow from the inside, you get a spatial perception comparable to the magical suggestions of mandalas and sacred geometry.

The 'takraw ball' is used in Malaysia and Thailand for a game called Sepak Takraw.

In Southeast Asia, there is still an intense debate about its origin. It probably originated in China and is related to the decorative spheres used in the Temari's art in ancient China and Japan. It is possible to recognize the basic elements of design science in the Takraw Ball: polyhedra (icosidodecahedron and truncated icosahedron), the great and small circles of the geodesic sphere, the triaxial weaving of reciprocal frames, the non-resonant quality of tensegrity. It is not easy to find the data for its construction, because this art is handed on orally by the basket maker masters. The original version is made of bamboo or rattan while the current version is made of plastic. The most common versions are two: the first one is referred to geodesic geometry of an icosidodecahedron (Icosa alt 2v) with 6 great circles. The second one is much more complex because there are 12 small circles added to the 6 big ones, to get a geodesic sphere Icosa Alt. 4V, so we have 18 circles in all. All circles are interwoven in semi-reciprocal way.

In the various workshops I held in different Italian and foreign universities and schools, I always felt the participant's ludic and creative involvement. Above all, children aged 8 years and older were excited. They worked with big belief, responsibility and a lot of fun, anxious to see the result. Kids always want to be involved and not excluded. They often invent new forms, showing intuition, creativity and a great desire to learn while having fun.

The workshops were done on polyhedra, geodesic domes, tensegrity structures, the bridge, the arch and Leonardo's ring. We often used cardboard, cardboard tubes, rolled paper rods, bamboos, jute canes and 'arundo donax' reeds. The rods were connected by Leonardo's reciprocal frame joint. The workshop targets were:

- to be aware of the shapes that make up the space geometry;
- to learn the design science concept in harmony with the natural world;
- to consider math, geometry, science and architecture both scientifically rigorous tools and instruments for a ludic activity;
- to get the shape optimization concept, and the dimensional stability of the structures;
- to stimulate creativity and intuition;
- to promote socialization.

## 3 Workshops

In my workshops I regularly use the poster 'CARTA DEI POLIEDRI' I first designed in 1983. Later it was published in 'Dome Magazine', 1999 and 'Bioarchitettura', 2000. It is a poster containing: the five Platonic solids, the 13 Archimedean solids and the 13 duals of Archimedean, all divided in families. For example, looking at the family of the icosahedron we immediately see that the pentakis dodecahedron is a triangulated dodecahedron, corresponding to a geodesic 2v. The 'Carta dei Poliedri' is a very useful tool to quickly locate a polyhedron and its immediate family.

Bamboo is a sustainable material par excellence, being rugged, inexpensive, easily available and highly resilient. Bamboo plants are the living example of resilience: during a hurricane, they can bend but do not break. For centuries, Japanese used the bamboo, which is very similar to a tensegrity structure. In fact, its internal cells are able to withstand both the traction and the compression forces. In nature, the tensile and the compressive forces always interact each other. Our body, made of rigid bones and flexible ligaments, is an example of tensegrity structure. Recently Stephen Levin, Donald Ingber, Graham Scarr, Tom Flemons and others innovated the biology science with biotensegrity, proposing a new paradigm that revises and expands the understanding of the kinematics, biomechanics and functional anatomy.

## 4 Conclusions

In my workshops, I always propose structures belonging to the design science. Design science is considered as a bridge between art and science: geometry becomes an intermediary between harmony and unity of the natural world. The geometry is a creation of nature not a human invention. Men can learn from nature itself. According to Fuller, all the natural forms tend toward the curved shape. Nature refers to the value of the golden section, not to the phi- Greek or the Cartesian axes.

In the transition from the micro to the macro scale, the form starts with a point and ends with a sphere (new point), passing through endless space modules. Kandinsky's methods in his art have a scientific verification in aggregation theories about compact balls (close packing) and in the structural chemistry.

All natural forms are the result of interactions between the physical forces of the external environment and the fundamental laws that govern them. The natural structures reach stability by means of triangulation. A structure built with triangles, does not need additional materials to be reinforced. It is very important to build and manipulate scale models because they contain all the necessary information to build the object in its actual size. By manipulating a model, everything becomes clearer, we can guess the results and the touch of the model parts is stored forever.

# Caterina Marcenaro + Franco Albini for the Love of Art

**Kay Bea Jones**

I was immediately smitten when I visited *Il Tesoro di San Lorenzo* in Genoa many years ago (Fig. 1). I happened to discover the buried treasure that holds the sacraments and remnants of the crusades beneath the Duomo, and I was baffled by the fact that I was completely unfamiliar with it. I knew immediately that *this* modern architecture suited me, and I began a quest to learn more. When I discovered that it was designed by the Milanese Rationalist architect, Franco Albini, and he had three other museums in Genoa, I chose to study his work in depth, and only then did I learn about his client, Caterina Marcenaro. As director of the *Belle Arti* for 21 years, Marcenaro commissioned Albini to design all four Genoese museums on 'recycled' sites.[1] In addition, the same architect had designed Marcenaro's apartment in the garret of one of these 4 museums where she lived for 20 years.

My objective today is to describe the professional relationship between Albini and his formidable client, and offer reflections on the importance women have played, and in particular this Genoese woman, in modern museum architecture and curation. I will conclude with a brief observation of the Treasury of San Lorenzo. Marcenaro was the primary catalyst and source for the Italian postwar museographic movement that reverberated around the world. She had to fight battles to defend her vision, and the results—highly praised by her most erudite critics—have proved very durable. I will consider three cases as evidence of her fortitude, her vision, her intelligence and her scruples. I will aim to present Marcenaro in her own words.

[1] The Palazzo Bianco, Palazzo Rosso, Tesoro di San Lorenzo e Museo di Sant'Agostino are modern municipal museums of the City of Genoa that required radical interventions into existing structures and sites after World War II. Three of the four museums were installed into pre-existing monumental buildings that had suffered from allied bombing in the war. The Treasury Museum resulted from postwar reorganization of ecclesiastical artifacts.

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**Fig. 1** The Treasury of San Lorenzo by Franco Albini, Genoa, 1952

While shining light on Marcenaro, her principle architect Franco Albini, deserves credit for having listened to a woman (Fig. 2). Not that he would dwell in her shadow; his formal ideas preceded his work with her and provided inspiration for her radical innovations, but as Marcenaro's collaborator, he realized her vision for a series of

**Fig. 2** Caterina Marcenaro (left) with Franco Albini (third from left) at the opening of the Palazzo Bianco Gallery

museums as it coincided with his own. And here, the fact that Albini had already received critical acclaim for his painting installation designs from the most renowned critics of the day, it is even more significant that his own ideas did not dominate the discourse. Marcenaro's writings and authority role lead me to believe that she governed the relationship and took responsibility for challenges to their radical interventions. Some of her projects with other renowned architects, including Ignazio Gardella, did not come to fruition. In addition, there are other women who had formative, professional relations with Albini that I will briefly mention in this context. It is useful to reflect on this unusual attribute of a mid-century modern architect who worked well with women. Yet each relationship has a story of its own.

I will conclude by honoring the theme of geometry by illustrating the influence of and Marcenaro's completed museums and installation designs by referring to the example of one highly revered project beyond Genoa.

## 1 Marcenaro's Career

Caterina Marcenaro grew up fatherless in a popular neighborhood of Genoa. After high school, she attended Rome's La Sapienza where she studied history of art. She graduated in 1937 with a thesis on the Italian travels of Antonio Van Dyck. Little is known about her full involvement with partisans during the war, but we do know that she hosted meetings of the CNL-*Comitato Nationale di Liberazione*. She taught history of art in a Genoese high school until 1948 and had already begun publishing articles during her first teaching job. In 1938 Marcenaro started working with Orlando Grosso, then the director of the *Ufficio Belle arti di Genova* with whom she curated exhibitions of 17th and 18th century painting.[2] During Fascism, Grosso was a member of the PNF (*Partito Nazionale Fascista*), and was removed from his leadership after the war, but he is credited for having protected Genoa's magnificent artistic patrimony.

In 1945 Marcenaro became the first female faculty member of Genoa's *Facolta di Magistero* teaching history of art, with courses she introduced into the curriculum. She left that position in 1951; in 1950 she succeeded Grosso as director of Genoa's *Belle Arti.* She had by that time already begun working on the major renovation of the Palazzo Bianco Museum. During Marcenaro's long and conflictual reign as head of the *Belle Arti*, earning her the less than flattering name, *la Zarina*, she oversaw four major archives, published prolifically and rebuilt the museums of Genoa, many having been damaged during allied bombing in World War II. She paid a stiff price, and in doing so she changed Italian museum culture and installation practices, brought international renown to Genoa's collections, and modernized the viewing of historic artifacts.[3]

[2] For more on the life of Caterina Marcenaro, see [1].

[3] See also [2] and "Palazzo Rosso dai Brignole-Sale a Caterino Marcenaro: luci ed ombre di un caposaldo della museologia italiana," by Piero Boccardo. Genova e il Collezionismo nel Novecento

A few words about Albini are due here, since his story proceeds his work with Marcenaro. He left his imprint on modern Italy before and after the war in several sectors, including his late project for the Eremitani museum of Padua, which would be inconceivable without his prior Genoese experiences and his intensive work with Marcenaro. Direct testimony from his professional partner of 25 years, Franca Helg, provides insight into the affinity between Albini and Marcenaro. Helg was involved in the designs of three of the four Genoa museums. She later wrote.

"Working with Caterina Marcenaro, a woman of exceptional sensitivity, tenacity, and rigor, was often difficult on account of the severity of the demands she imposed, but Albini's working methodology was characterized by a desire to understand to the greatest degree possible the problems at stake, delving into them thoroughly. He responded to her insightful criticisms, strengthening his work with new images and new suggestions."[4]

A side comment about Albini's female collaborators—Franca Helg entered the studio in 1951 and eventually became an equal partner of Studio Albini carrying on its direction after Franco died in 1977 (Fig. 3). In 1959, Albini wrote a letter to Ernesto Nathan Rogers that appeared in *Domus* defending the importance of her work in the studio. A female partner was unheard of in Italian architecture studio's

**Fig. 3** Franca Helg and Franco Albini with Studio Albini

(Torino: Umberto Allemandi) and *Medioevo Demolito, Genova* 1860–1940 (Genova: Pirella editore, 1990).

[4]Helg [3], p. 551. Cited in *Franco Albini Architecture and Design* 1934–1977 by Stephen Leet (New York: Princeton Architectural Press, 1990) p. 16.

**Fig. 4** The Scipione Black & White installation at the Pinacoteca Brera in Milan by Franco Albini, 1941

in the 1950s. Stories surround that relationship, too, but Helg's own 1979 testimony about her long career with Albini published in *l'Architettura* is a reverent and honest portrayal of their shared work. Erroneously, architecture historians sometimes refer to them as husband and wife, which they were not.

When I began the research for my book on Studio Albini, I was fortunate enough to meet Matilde Baffa, whose generous insights were essential to my work. She had studied with Albini at IUAV in Venice in 1958 and eventually published the conversations of the MSA—*Il Movemento di Studi per l'Architettura* 1945–61—where Albini's voice can best be heard.[5] Albini selected Baffa as his research assistant. She later became *professore ordinario* at the Politecnico di Milano. She told me that Albini was a very supportive, if severe, mentor, and her assistance to me was invested in seeing that his legacy was better known beyond Italy.

Another female architect who shared Albini's sensibilities was Italian/Brazilian architect Lina Bo Bardi, who has finally received due acclaim as a world-renowned modernist. Some scholars have argued that Albini borrowed from her 1956 installation of art in her MASP—Sao Paulo Museum of Art. The scale of this place is certainly impressive, and Bo Bardi introduced modern innovations in viewing art into Brazil. Her project follows decades of Albini's installations to which it refers. Lina Bo knew Albini's 1941 Scipione show at the Pinacoteca Brera, as she lived in

[5] Baffa et al. [4].

Milan working for Gio Ponti at that time (Fig. 4). She was well aware of Albini's 1950 Palazzo Bianco exhibit with its floating paintings, perhaps from well-circulated publications. In fact, she invited Albini to speak in Brazil in the early 50s [when his plane crashed into the ocean and he was among the few survivors.[6] The pair shared mutual admiration, no doubt, and I think it's clear that the younger Bo Bardi was directly inspired by Albini.

During fertile years in Genoa, Marcenaro and Albini completed the Palazzo Bianco in 1950 and the Treasury of San Lorenzo Museum in 1956, the same year Marcenaro moved into the apartment he designed for her in the Palazzo Rosso. The latter gallery itself opened in 1961. The Sant'Agostino Museum, their largest project, was begun in the 1960s but not completed until after both had died. Nonetheless, the work completed by Studio Albini at Sant'Agostino is a masterpiece and shows the maturing sensibilities of a seasoned designer. It is the only one of his Genoese projects with Marcenaro that includes dominant facades on a public square, the Piazza Sarzano. The overt, carefully-crafted expression of the structural steel exoskeleton, reminiscent of Studio Albini's *La Rinascente* Department Store in Rome, offers an example of his mature tectonics and window design. By moving from his interior spatial expressions of lightweight suspension to public scale massing of balance and harmony, he also demonstrated his great sensitivity to the unique urban fabric and networks of *creuze* (alleys) characteristic of the historic center of Genoa (Fig. 5).

## 2 Palazzo Bianco

In 1956 shortly after the Museum reopened to the public, Marcenaro published a lucid explanation of her intentions for Palazzo Bianco in the UNESCO journal, *Museums*, in French and English.[7] She wrote to set the record straight on several issues. One can readily read between the lines that she had detractors, among them, those asserting that she had taken too much liberty in making a modern, abstract gallery from a historic baroque palace previously functioning as a domestic museum. They claimed she was guilty of erasing history. While the resulting modern Palazzo Bianco museum would later bring national acclaim to Albini and was immediately lauded by luminaries including Giulio Carlo Argan and George E. Kidder Smith, the *Belle Arti* director was under scrutiny. She was being criticized for having achieved her very intentions—that is, for extracting modern spaces from traditional sites while choosing how to locate both in evidence and in relationship, thereby reviving the historic collections of art they accommodated. Her objective to participate in the ongoing evolution of culture and her search for truth in the confrontation between old and new would condition all her subsequent work ( Fig. 6).

[6] From conversation between author and Marco Albini, the son of Franco, in Columbus, Ohio, March 2006.

[7] Marcenaro [5].

**Fig. 5** Sant' Agostino Museum by Studio Albini in Genoa, 1979

As Marcenaro explains, the Palazzo Bianco dates from the 17th century, originally built as the home of one of Genoa's noble families, the Brignole Sale. The baroque structure had been given to the city Genoa in 1889 by the Duchess of Galliera after it had succumbed to a series of alterations for a succession of residents. As Marcenaro documents, the palace had never been an aristocratic residence with its own 18th century collection of paintings. Instead, its furnishings and holdings had already been redistributed by the time the municipality accepted the gift, and a wide-ranging series of eclectic interventions followed with exhibits of unrelated artifacts that changed frequently over the decades. Three years after allied bombing left the Palazzo roof and upper floors badly damaged, the baroque palazzo was reconstructed in 1945 returning it to its monumental form with enfilade rooms and cortile proportions that reflected its best and most original state. Marcenaro described the condition to which the building had been reduced and the chaotic collage of past curatorial strategies, followed by full disclosure of her objectives, along with the actions of her architect. Her description is thorough, precise, generous, and intended as the last word on any controversy.

In the text she credits her collaborator:

"The architect, Franco Albini, gave considerable and indeed brilliant help in solving the problem of presentation. In the *interest of education*, the palace concept was

**Fig. 6** Suspended paintings in the Palazzo Bianco Gallery, Genoa

abandoned and the museum criterion strictly adhered to. In other words, the works of art were treated not as the decorative part of a setting, but as a world in themselves, sufficient to absorb the visitors' full attention."[8]

Franco Albini was a taciturn designer, and his few public addresses including one at the Turin Polytechnic in 1954, offer insights into his design intentions. He emphasized relationships of works of art to their distinct surroundings with equanimity of value in any exhibited subject that "only needs to be exhibited properly."[9] Albini defended the modern installation of a work of art that has been detached from its original context as it "acquires its essential autonomy" and "becomes a source of spiritual pleasure by way of contemplation."[10]

We affirm the *educational function of the museum* and the necessity to insert it into modern life. With attention to both, architecture tries to mediate between the two. Architecture must acclimatize the public as well as the artifact. Regarding the architectural problem, whether new construction or adapting an existing historic structure, while respecting the curatorial criteria, the building must be alive and autonomous.

[8]Marcenaro, p. 263. Italics included for this essay.

[9]Lecture at the Turin Ploytechnic at the beginning of the 1954–55 academic year, "The functions an architecture of the museum: some experiences," reprinted in *Zero Gravity*: *Franco Albini, Costruire le Modernitá,* Irace and Bucci (Milan: Mondadori, 2006) p. 72.

[10]Ibid.

Let's examine the coincidence of each partner's reference to the educational function of the museum. In a 1949 issue of Adriano Olivetti's journal, *Comunitá*, Giulio Carlo Argan published his essay "*Il museo come scuola*."[11] He argues in the text that the experience of art is to educate, and museums must become places of social utility. Olivetti published the California open air school by Richard Neutra in the same *Comunitá* issue, while Argan jointly put forth the American ideal of the "living museum." Also, in *Comunitá*, Licisco Magagnato wrote "a modern museum organizes exhibits, films, scholastic visits, published books, documentaries, slides, and photographs." This living museum gets compared in contrast to a "cemetery of artworks" where artifacts are locked away and protected for scholars and posterity.[12]

As Marcenaro and Albini embraced this challenge facing a Genoese public reluctant to change, they saw in the revived palazzo an opportunity to depart from the typical domestic gallery model in which great rooms host great things staged exclusively for select people who remain like ghosts in darkened salons. All such a museum serves to do is remember the past. In this scenario those ancient artifacts were meant to match or be situated in like surroundings, the implication being to memorialize the pastness of the past.[13] The Palazzo Bianco designer/curator pair successfully sought to interrupt and replace expectations for history's temporal distancing by employing four formal trends:

1. **Mobility**: Artifacts were intended to be moved in the context of the gallery so collections cycle and maintain a fresh exhibit experience for local patrons and citizens. With the wealth of each museum's collection, a flexible storage area would make accessing these works an essential part of the overall project. Albini's storage system in the attic of Palazzo Bianco remains in use. He also invented novel installation armatures so viewers could situate paintings in the best light. He designed lightweight seating easily repositioned by gallery visitors. Marcenaro wrote: "For obvious reasons, mobility is an even more important consideration in the room set aside for educational exhibitions, where the greatest possible freedom has been ensured thanks to a system by which the vertical supports carrying the pictures can be extended at either end so as to press firmly against any one of a series of points on the floor and the ceiling."[14]
2. **Visibility**: Seeing with modern eyes demanded upsetting the status quo. This invited interrupting the expected relations of historic artworks paired to match époque historic settings. Marcenaro wrote: "Another important consideration was that of visibility. The arrangement of the pictures is regulated by their horizon instead of their lower edge, and this median line has been set at the average height of the human eye—that is, at 1.50 m."[15]

[11] Argan [6], pp. 64–66.

[12] Magagnato, Licisco *Comunitá*, no. 6 gennaio-febbraio 1950, pp. 31–34.

[13] Affirmation that Marcenaro successfully redefined the image of the Palazzo Bianco Gallery: "E' stato programmaticamente abbandonato il concetto di palazzo ed è stato rigorosamente perseguito il museo". Cfr Emiliani [7], pp. 153–154.

[14] Marcenaro [5], p. 267.

[15] See Footnote 14.

Paintings were removed from non-original frames to lighten their apparent load and focus only on content and painterly qualities. They were hung with cables left visible.

"Perhaps you cannot say that the frame is necessary or that it is useless: but you can say almost always that it is an opportunity for space to act as the intermediary between the image and the environment as a frame or a wall, on the surface or background, or volume of air assigned to the painting, almost a zone of influence in its pictorial space."[16]

3. **Lightness**: Of transparent volumes, day light, and air were intangible qualities of Albini's modern museum designs. The monumental spaces of the original palace architecture provided cubic rooms that were maintained uniformly white. Daylight was controlled with venetian blinds and Albini's sleek suspended artificial lighting bars continued the theme of invisible infrastructure. Refined details of glass enclosures isolated the cortile with elegant brass hardware on glass doors between galleries, which maximized visual connections and extended filtered daylight throughout the galleries. Carlo Scarpa praised the Palazzo Bianco electric lighting design in a letter to Albini asking about its fabricator.
4. **Abstract space with minimal color**: The essence of the black and white galleries allowed the collection of paintings to introduce all color and differentiate the museum experience from an ordinary dwelling. Some walls were covered with *ardesia* (slate) to provide a neutral tone background that would set off white sculpture. The white walls were accented by the typical Genoese geometric pattern of black floors. Albini's leather-covered umbrella chairs introduced a warm hue.

Among all of the pair's dislocated and floating artifacts, the harshest critics pointed to the precious medieval fragment resting on a piston available for audience's maneuvers. Although lauded in international contexts by Kidder Smith (1954) and Michael Brawne (1965), the animosity expressed in local editorials for this museum installation was unprecedented.

Marcenaro addressed in *Museum* this most controversial component at Palazzo Bianco (Fig. 7):

"Special methods were adopted for the display of certain particularly distinctive and important items. For instance. The fragments from the tomb of *Margarita di Brabante*, by Pisano, has been mounted on a cylindrical steel support which can be raised and swiveled as desired. This solution has been much discussed, not always with approval. Apart from the fact that there were no original designs or later documentary evidence to show how the work had been set up in the first place, it consists simply of a fragment, and considerable permission is thus permissible in its display. Moreover, it is, though a fragment, of such quality that it was essential for it to be easily viewable; it had, therefore, to be mobile and to be set in a place apart. The fact that mobility was obtained by the use of a revolving, electrically-operated steel

[16] Franco Albini, comments titled "Le funzioni e l'architettura del museo: alcune esperienze" were given as at the Turin Polytechnic for the opening of the 1954–55 academic year, printed in *Zero Gravity*, pp. 71–73.

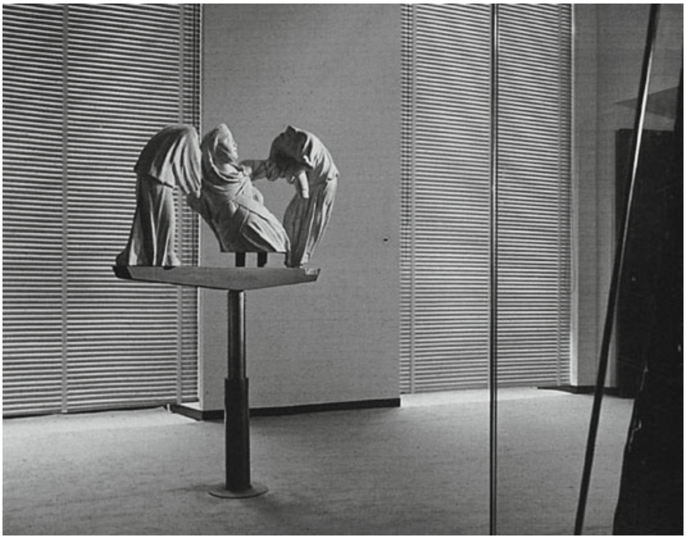

**Fig. 7** Manual piston installation of Pisano's Margarita di Brabante as originally installed at Palazzo Bianco

cylinder is due, not to lack respect for Giovanni, but to simplicity and humility of approach in respect to a great work of art. To have placed the fragment on a pedestal or in the shadow of a marble niche would have been, not only to resort to arbitrary treatment and revive the thorny question of the genuine versus the spurious, but to bring undue influence to bear on the work, especially regards proportion, thus confusing the general public and disturbing the atmosphere or purity and tranquility which I consider essential when a visitor—particularly an uninformed visitor—approaches a real masterpiece."[17]

While the piston installation for Margherita di Brabante has been long since dismantled, it inspired a similar placement for Eleanora di Toledo by Carlo Scarpa at the Palazzo Abatellis in Palermo, where a female bust floats as the main protagonist of the gallery sequence and stands in evidence of Marcenaro and Albini's influence (Fig. 8).

Another of Marcenaro's highly criticized display methods at Palazzo Bianco involved placing paintings on moveable armatures with bases salvaged from architectural ruins. Genoa had lost several medieval churches to bombing, disuse, and urban renewal, and she decided that storerooms of these fragments were unnecessary. The origins of this suspension motif date to Albini's own apartment in Milan;

"Certain paintings stand mounted on iron supports fixed into the capitals and bases of Roman or Gothic pillars. This solution has given rise to some criticism on the grounds of what is held to be too close an association of the new with the old. In my opinion, however, this view is unfounded; every cultural problem—it cannot be

[17] Caterina Marcenaro, Museum V.5 (1952), p. 266.

**Fig. 8** Studies of Carlo Scarpa's installation of Eleonora di Toledo at the Palazzo Abatellis, Palermo

denied that museums are part of culture—must be solved in terms, not of what is old and what is new, but of what is true and what is false. It is the business of culture to search for truth, no matter whether the truth will in fact be discovered.[18]

… for ancient art, the Italian museological equation is necessarily nearly always one in which the binominal "monument-museum" has its place. I don't know whether in the specific case of the Palazzo Bianco the unknown x factor has been found. All I can say is it has been sought after—by careful planning with a full sense of responsibility, without any desire to create controversy and in complete good faith. What matters… is not so much to find a truth as to seek it, and to do so unremittingly, even if there are no real prospects of ever finding it.[19]

In his 1952 article for *Metron*, G.C. Argan unequivocally praised the entirety of the Palazzo Bianco Museum renovation, from the quality of the experience of viewing art to the new storeroom in the attic, calling the intervention "unquestionably the most modern Italian museum" of the day.[20] Most notably, he recognized the importance of Albini's collaboration with Marcenaro, whose vision complemented the architect's own perseverance, courage and rigor. Tafuri called Palazzo Bianco a "masterpiece of museological function and neutrality and a patient reconstruction of textual fragments."[21] When Kidder Smith published Italy Builds in 1954, his inclusion of the Palazzo Bianco project alerted an international audience to what was to come. Luigi Moretti commented that Albini's renovation was music to "somewhat

[18] See Footnote 14.

[19] See Footnote 14.

[20] Argan, *Metron,* p. 39

[21] Tafuri underscored the success of the model and its role in modern history: "the design of Palazzo Bianco by Albini immediately became a necessary point of reference for a culture intent on safeguarding, in all situations, reassuring equilibrium. Albini's design is a masterpiece of its kind: the extreme and rigorously developed museological function accompanied by a refined neutrality of the décor displaying works; at the same time, it allows other signs to shine through like filigree, reducing them to respectful interlinear glosses of patiently reconstructed textual fragments." p. 49.

deafened ears" as a clarion call for more continuity after the war with Rationalist practices. Nothing comparable was being done to revitalize Renaissance antiquities in Moretti's Rome or anywhere else in Italy—yet.

## 3 Marcenaro's Apartment in Palazzo Rosso

All the themes expressed in the public galleries of Palazzo Bianco are in evidence in Caterina's garret apartment under the concrete beams of the modern roof with small framed views over Genoa's slate rooftops (Fig. 9). She again came under severe public scrutiny and criticism when Gio Ponti published the architectural interior as "The house of an art lover, on the last floor of an historic palazzo" in Domus in 1955, intending for its owner to remain anonymous.[22] Readers quickly figured it was the home of Marcenaro, and unfairly and inaccurately assumed that the artworks hanging in her apartment belonged to the museum. Instead they were her private collection, which she donated at the time of her death and are now held in two Milanese collections: Cassa di Risparmio Cariplo and the Museo Diocesano. Her collection also included furniture and shelving designed by Albini. The entire apartment was an exemplary *gesamtkunswerk* of the pair's installation vision to lighten and free historic works from the weight of history.

When I began to investigate the relationship between the powerful arts administrator and her architect, I was disappointed to discover that her private space had

**Fig. 9** Caterina Marcenaro's apartment in the attic of Palazzo Rosso by Franco Albini, 1954

[22]Ponti, Gio, "La casa di un amatore di arte, all'ultimo piano di un antico palazzo," *Domus*, no. 307, June 1955, pp. 11–18.

become an ill-kept warehouse. Fortunately, the primary elements of Albini's interior including the floating hearth and ladder stair were still in place, although her furniture and collections were gone. Following recent renovations, the suggestive interior designed by Albini for Marcenaro, and only for her, are now part of the collection and *percorso* of the Palazzo Rosso Gallery.

## 4 Marcenaro's Formal Inspiration for the Treasury of San Lorenzo

While Albini was designing the Palazzo Rosso residence, he was also working on the Treasury of San Lorenzo crypt museum. This project began immediately upon completion of the acclaimed Bianco Gallery, and we know that Marcenaro gave Albini his source of inspiration for the subsequent museum of buried treasures. She suggested to him that he take his students to visit the Treasury of Atreus in Mycenae. Both a burial crypt and container for artifacts representing untold wealth, the 1250 B.C. tholos provided a model for submerged construction that symbolically would retain Genoa's treasure. I have not seen the apocryphal postcard he sent her from Mycenae simply declaring "you were right!" but an eye witness, Bruno Gabrielli, recounted the story to me. Subsequently Albini's design for four geometrically linked subterranean cylinders became the quintessential diagram for the precious holdings of the Genoese Duomo collection.

This story of inspiration and collaboration for San Lorenzo resulted in a truly remarkably intervention, ideal for the collection of gem-studded silver, gold, glass and textiles as sacred artifacts. Albini's choice of the matte-finished local gray stone called *promontorio*, the lighting and electrical plan with subdued intensity, and the accessible storage system, resulted in a uniquely sublime "living museum." Apparently, American architect Philip Johnson agreed. He borrowed Albini's plan, along with mobility characteristics of paintings on swivel hinges when he submerged his painting collection on his own 40 acres in New Canaan, Connecticut. In describing his inspiration for his design, Johnson referenced not Albini's sublime spaces, but the same Treasury of Atreus, and he convinced both Vincent Scully and Francesco dal Co of his selected historic inspiration for his very personal gallery.[23] It is, however, most obvious that he and we have Caterina Marcenaro to thank (Fig. 10).

My final observation—I can think of no better example than the works produced by the collaboration between Caterina Marcenaro and Franco Albini to represent the existential 'truth' of the Italian postwar achievement in architecture. By marrying modernity to tradition, each exerting their relative expertise to the question, each playing a fertile and essential role in the culture of modern exhibition design, our pair of lovers left a lasting legacy in modern museology.

[23] Jones [8], pp. 172–175.

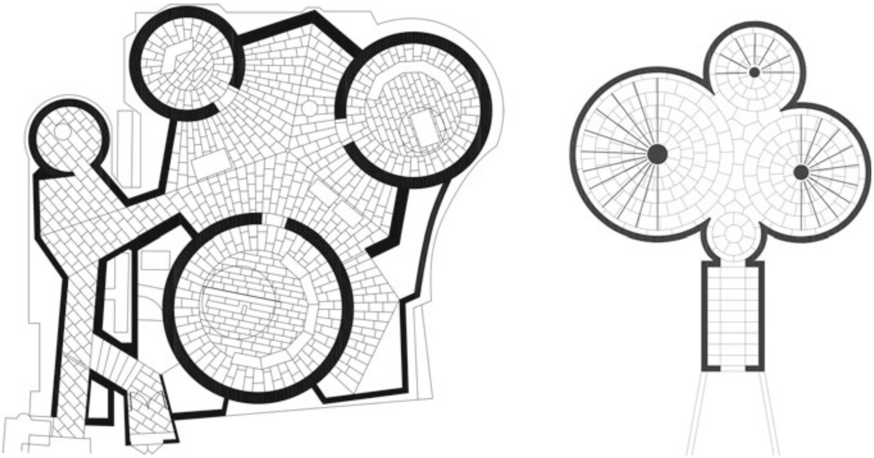

**Fig. 10** Plans of the treasury of San Lorenzo by Albini, 1954, and the submerged Painting Gallery by Philip Johnson, 1965

## Bibliography


1. Fontanarossa R (2015) La Capostipite di sé: Una donna alla guida dei musei. Caterina Marcenaro a Genova (1948–71). Etgraphiae ed., Roma
2. Di Fabio C (1990) Una protagonista della scena culturale genovese fra 1950–1970: Caterina Marcenaro fra casa e musei. In: Genova e il Collezionismo nel Novecento. Umberto Allemandi, Torino
3. Helg F (1997) Testamonianza su Franco Albini. L'Architettura, cronica e storia, October 1979. Huber, Antonella, Il museo italiano. La trasformazione di spazi storici in spazi espositivi: attualità dell'esperienza museografica degli anni '50, Milano, Lybra immagine
4. Baffa M, Morandi C, Protasoni S,Rossari A (1995) Il Movimento per gli Studi di Architettura 1945–1961[Movement for the study of architecture 1945–1961]. Laterza & Figli, Rome
5. Marcenaro C (1952) The museum concept and the rearrangement of the Palazzo Bianco, Genoa. Museum, UNESCO V:250–267
6. Argan GC (1949) Il Museo come scuola. Comunitá 3:64–66. maggio-giugno
7. Emiliani MD (1982) I musei della ricostruzione in Italia, tra disfatta e rivincita della storia, in Licisco Magagnato (a cura di). Carlo Scarpa a Castelvecchio, Milano, Edizioni di Comunità
8. Jones KB (2014) Suspending modernity: the architecture of Franco. Studies in Architecture Series, Surrey, Albini, Ashgate
9. Argan GC (1952) La Galleria di Palazzo Bianco a Genova. Metron 45:25–39. giugno
10. Argan GC (1962) Franco Albini. Milan
11. Meyer de EC (2002) After the Flood, Lina Bo Bardi's Glass House. Back Issue Hard/Soft, Cool/Warm, n. 16, Winter/Spring
12. De Seta C (1979) Franco Albini, architect: between rationalism and technology, Franco Albini 1930–1970. Centro Di, Firenze
13. Ellard G, Johnstone S (2019, Febuary 11) An aerial sensibility: notes on Franco Albini, Caterina Marcenaro and Museum of Floating Objects, *Afterall/On-Line.* https://www.afterall.org/online/an-aerial-sensibility-notes-on-franco-albini-caterina-marcenaro-and-a-museum-of-floating-objects#footnote14041
14. Magagnato L (1962) La storia dalla museografia fa corpo con la storia stessa dell'architettura. Comunitá 99:54–63

15. Magagnato L (1982) a cura di. Edizioni di Comunità, Carlo Scarpa a Castelvecchio, Milano
16. Tafuri M (1989) The history of Italian architecture, 1944–1985, translated by Jessica Levine, MIT Press, Cambridge, MA
17. Tovaglieri P Franco Albini e Palazzo Bianco: Nuove Idee for I Musei Italiani. Dopoguerra—Postwar Architecture in Italy and Germany, Academia. https://www.academia.edu/38246338/Franco_Albini_and_Palazzo_Bianco

# How to Solve Second Degree Algebraic Equations Using Geometry

**Paola Magnaghi-Delfino and Tullia Norando**

## 1 Introduction

One of the most complicated problems faced by mathematicians was to calculate the solutions of the algebraic equations of each degree. First examples of first-degree equation solutions are reported in an Egyptian papyrus dating back to 1650 BC. In some Babylonian tablets, we find methods of resolution of some second-degree equations by geometric construction. Euclid, around 300 BC, described a geometric method for solving equations.

The concept of equation, as we know it, was born and developed in the Arab world, above all thanks to Al-Khuwarizmi, which distinguishes six types of first and second-degree equations and resolves them using squared completion.

Omar Khayyam, in his book *Treatise on the proof of algebra problems*, published in 1070, deals with the transformation of geometric problems into algebraic problems and vice versa, and set in a general way how to bring them back to equations at the maximum of third degree for which geometric solutions are proposed [3].

In 1748, Maria Gaetana Agnesi, a Milanese mathematician, published her main book *Analytical Institutions for the use of Italian Youth* [1].

The first volume deals with the analysis of finite quantities and the second of the infinitesimal analysis. Maria Gaetana dedicates her work to the Empress Maria Theresa of Austria, an enlightened woman. In chapter II of the first book, Maria Gaetana deals with the study of first and second-degree equations by providing a method of solving second-degree equations by geometric means. Furthermore, she proposes some problems and exercises that can be solved with equations.

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## 2 Second-Degree Equations on a Geometric Way

The second chapter of the first volume of the *Analytical Institutions* is entitled *of Equations, and of Plane Determinate Problems*.

Maria Gaetana gives the following definition of the equation: an equation is a relation of equality, which two or more quantities, whether numerical, geometrical, or physical, have with one another when compared together; or which they have with nothing when compared to that [2].

The aggregate of all those terms which are wrote before the mark of equality, is called the First Member of the Equation; and the aggregate of all those which are wrote after it, is called the Second Member, or the Homogeneous Comparations.

Then Maria Gaetana enumerates the axioms useful for solving the equations:

– If two equal things we shall add equals, or if we shall subtract equals from the, the sums or the reminders will also be equal
– If equal things are multiplied or divided by equals, the products or quotients will be also equal
– If from equals a root be extracted with an equal index, the roots or quantities resulting will be equal
– If equals are raised to a power with an equal index, those powers or resulting quantities will be equal.

Then, Maria Gaetana, being an excellent teacher, describes step by step how to solve an equation: first of all if in the denominator there is the unknown, she reduces to the common denominator, secondly she makes positive the end of the maximum power of the unknown, and, written by a part of the sign of equal all the terms that contain the unknown in their order, write the known ones on the other side. Thirdly, if the first term, that is the maximum power of the unknown, has a denominator, she gets rid of the fraction and if it had a coefficient, she divides it so as not to have any.

All the infinite number of affected quadratic equations may be comprehended and expressed by this formula

$$x^2 \pm ax \pm b^2 = 0 \quad \text{with a, b} > 0$$

that is the following four different combinations of their signs.

(1) $x^2 + ax - b^2 = 0$
(2) $x^2 - ax - b^2 = 0$
(3) $x^2 + ax + b^2 = 0$
(4) $x^2 - ax + b^2 = 0$

First, Maria Gaetana considers the first two equations.

She takes AC = *a*/2, AB = *b* at the right angles to AC. With radius AC let a circle AED be described and from the point B let the right line BD be drawn, terminating in the periphery at D, and passing through the centre C. Then BE will be the positive value of the unknown quantity, in the first equation and BD its negative value; as on the contrary in the second equation, BD will be the positive value and BE the negative value (Fig. 1).

In effect, by resolving the two equations, we have $x = \pm \frac{a}{2} \pm \sqrt{\frac{a^2}{4} + b^2}$. In addition, by the construction, being AC = CE = CD = *a*/2, AB = *b*, it will be CB $= \sqrt{\frac{a^2}{4} + b^2}$ and therefore BE $= -\frac{a}{2} + \sqrt{\frac{a^2}{4} + b^2}$, which is the positive value of the first equation and BD $= -\frac{a}{2} - \sqrt{\frac{a^2}{4} + b^2}$, the negative value and conversely for the second equation.

Then Maria Gaetana considers the third and the fourth equation (Fig. 2).

She takes AC = *a*/2, AB = *b* at the right angles to AC and with a radius AC she describes a semicircle AHD and draw BD parallel to AC. The two right line BE and BD will be the two values, or the two negative roots of the third equation or the two positive values in the fourth equation.

Now resolving the equations, the third will give us $x = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b^2}$ and the fourth $x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b^2}$. Therefore, drawing the right lines CD, CE and CI

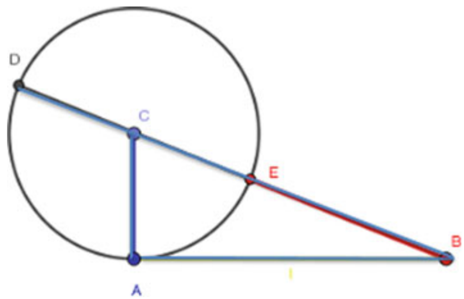


**Fig. 1** Equation (1)

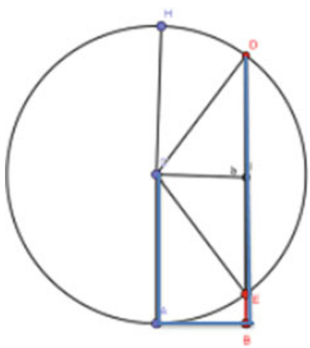


**Fig. 2** Equation (2)

perpendicular to BD, it will be ID $= $ IE $= \sqrt{\frac{a^2}{4} - b^2}$, BE $= -\frac{a}{2} + \sqrt{\frac{a^2}{4} - b^2}$, negative value in the third equation because BI > IE; and BD $= -\frac{a}{2} - \sqrt{\frac{a^2}{4} - b^2}$ the other negative value. On the contrary, BD will be positive and BE positive, both being the positive values of the unknown quantity in the fourth equation.

Therefore, to construct any affected quadratic equation, it suffices to assume the radius CA equal to half the coefficient of the second term, and the tangent AB equal to the square-root of the last term and the rest as in one or the other of the two figures.

Thus, for example, to construct the equation

$$x^2 + ax - bx + c^2 = 0,$$

we have AC $= \frac{a-b}{2}$, AB $= \sqrt{a^2 - c^2}$, if $a >$ c, AB $= \sqrt{c^2 - a^2}$, if $a < c$.

It may happen that in the construction of the figure, the right line BD shall not cut nor touch the circle. It will touch it when it is AC = AB that is $a/2 = b$ and the two values of the unknown quantity of the equation BD and BE will be equal one positive and the other negative.

It will neither touch it nor cut when BA > AC that is $b > a/2$. The unknown quantities will not have any value at all but will be impossible or imaginary.

Now we consider the cases in which the last term is equal to a rectangle. The equations are the following:

(1) $x^2 + ax - bc = 0$
(2) $x^2 - ax - bc = 0$
(3) $x^2 + ax + bc = 0$
(4) $x^2 - ax + bc = 0$

We consider the first two equations.

Let the circle BAD be described with any diameter, provided it will be not less either $a$ or $b - c$ (supposing $b > c$, where $b$ is the greater side of the rectangle and $c$ the lesser side).

Now, from every point A in the periphery let the two chords AB $= a$ and AD $= b - c$ be inscribed in the circle, and let this last be produced to F so that DF $= c$. With centre C of the first circle, and with radius CF, let a second circle FGH be described, which may cut the chords AD, AB produced, in the points F, G, E, and H. Then AG will be the positive value or root, and AH the negative, in the first equation and on the contrary for the second equation (Fig. 3).

To apprehend the reason of this, it is necessary to have recourse to two properties of the circle, which are demonstrated by geometricians; which are, that the right lines EA and DF are equal and also GA and BH and that the rectangles EA × AF and GA × AH are also equal. Now we bisect the line BA by the point M.

By the second theorem of Euclid, MG$^2$ = MA$^2$ + BG × GA, that is HA × AG, and FA × AE.

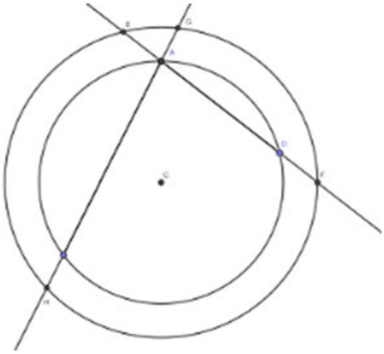

**Fig. 3** Equation (1)

But

$$\mathrm{MA}^2 = \sqrt{\frac{a^2}{4} + bc} \Rightarrow \mathrm{AG} = -\frac{a}{2} + \sqrt{\frac{a^2}{4} + bc} > 0$$
$$\Rightarrow \mathrm{AH} = -\frac{a}{2} - \sqrt{\frac{a^2}{4} + bc} < 0$$

We consider the third and the fourth equation.

Let any circle RAD be described with a diameter not less than $a$ or $b + c$.

From any point A of the periphery, let two chords be inscribed in it, that is AR $= a$ and AD $= b + c$. Let DF $= c$ and with centre C and radius CF let another circle GHF be described which shall cut the two chords AR, AD in the points F, G, E, H. AG and AH will be the two negative values in the third equation and the two positives in the fourth (Fig. 4).

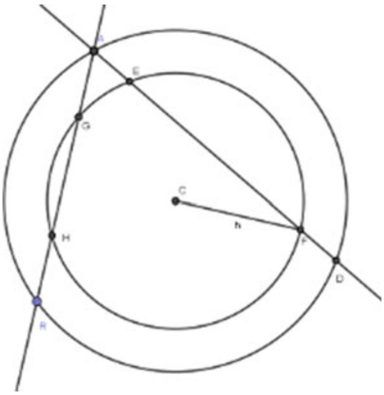

**Fig. 4** Equation (2)

For the demonstration, we bisect RA with the point M.

For the second theorem of Euclid, we have $MA^2 = HA \times AG = RG \times GA = DE \times EA + MG^2$

Then $\frac{a^2}{4} = bc + MG^2 \Rightarrow GA = MA + MG - \frac{a}{2} + \sqrt{\frac{a^2}{4} - bc}\, e$

$MG - MR = GR = -\frac{a}{2} - \sqrt{\frac{a^2}{4} - bc} \Rightarrow MG + MR = \frac{a}{2} + \sqrt{\frac{a^2}{4} - bc} = RG > 0$ e

$MA - MG = +\frac{a}{2} - \sqrt{\frac{a^2}{4} - bc} = AG > 0.$

When $bc = \frac{a^2}{4}$, the circle HGEF will be tangent to the straight-line HA and the two values will be equal and, when $bc > \frac{a^2}{4}$ they will be imaginary.

## 3 Problem: To Inscribe a Cube in a Given Sphere

Let KQEP be a great circle of a sphere, A its centre and AT = $a$ its radius, AR half of the height or of the side of the cube to be inscribed and therefore make AR = $x$. Through the point R let there be conceived to pass a plane perpendicular to AT, the common section of which, with the sphere shall be the circle QNSKFQ and the square inscribed in this circle shall be one face or one plane of the parallelepiped inscribed in the sphere (Fig. 5).

But, because this parallelepiped ought to be a cube, it will therefore follow that RG = SN = NO or AR = RI = IO; and besides, that the planes which enclose it should be at right angles.

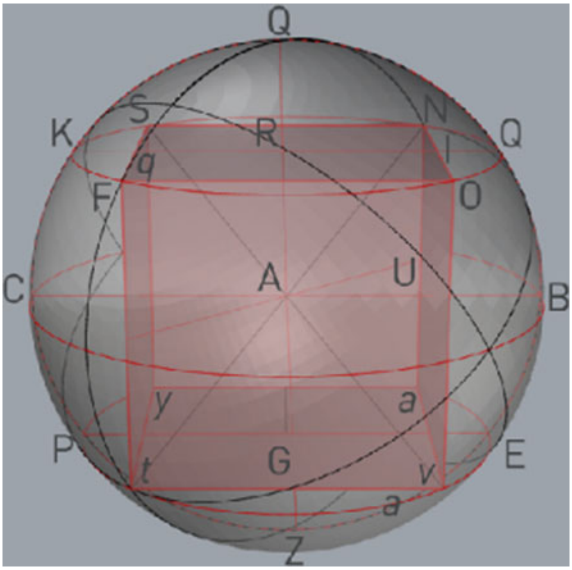


**Fig. 5** Figure made by Giampiero Mele

In the circle KPEQ, the ordinate will KR $= $ RQ $= \sqrt{a^2 - x^2}$ and taking RI $=$ RA $= x$, it will be KI $= \sqrt{a^2 - x^2} + x$ and IQ $= \sqrt{a^2 - x^2} - x$.

In the circle NKOQ, the ordinate

IO $= \sqrt{KIxIQ} = \sqrt{a^2 - 2x^2}$ Therefore, the equation will be $\sqrt{a^2 - 2x^2} = x$, and hence $a^2 = 3x^2$ or $x = \pm\sqrt{\frac{a^2}{2}}$.

Now, taking AU equal to the third part of the radius AB, upon the diameter CU describe the semicircle CRU; the point R in which it cuts the radius AT shall be the point required. In addition, it will be AR $= \sqrt{\frac{a^2}{3}}$, half of the side of the cube, taking its positive value on the side of T and the negative towards Z.

Whence taking AG = AR and through the points R and G, the sphere being cut by two planes perpendicular to RG; and taking RH = RI = RA and through the points I and H, the sphere being cut by two others planes perpendicular to HI and by two others through SN and FO, perpendicular NO, the cube will be inscribed.

For, because, by the construction, as it plainly appears, the planes are perpendicular to one another, and it being AR $=$ RI $= \sqrt{\frac{a^2}{3}}$, it will be by the property of the circle KQEP, the ordinate RQ $= \sqrt{\frac{2a^3}{3}}$, and therefore IQ $= \sqrt{\frac{2a^3}{3}} - \sqrt{\frac{a^2}{3}}$ and IO $= \sqrt{\frac{a^2}{3}}$ and consequently all the sides are equal, as was to be demonstrated.

From the construction of this problem arises a pretty simple synthetically demonstration.

Since AU $= \frac{1}{3}$ AC, the rectangle CAU that is the square of AR, will be a third part of the square of the radius, and therefore AR = RI.

If, from the centre A of the sphere be drawn a right line AI to the point I, the square of AI will be double the square of AR, that is, two third parts of the square of the radius.

And if from the centre, a radius AO be supposed to be drawn, the square of IO will be equal to the square of AO, lessened by the square of AI; that is, equal to the square of the radius, lessened by two third parts of the same square, and therefore equal to one third part of the square of the radius, and consequently IO is equal to AR.

## 4 Problem II

The velocities of two bodies being given their distance and the difference of time on which they begin to move in a right line $s$ the point in that line and the time is required in which the bodies will meet.

Let the first body be at A, the velocity of which is such that it would be described the space $c$ in the time $f$ and B the second body with such a velocity that it would describe the space $d$ Let the difference of time in which they being to move be $h$ and let their distance AB be $e$.


*First case*: A and B move in the same way and let them come together at the point D. Make AD = x and BD = x − a.

The body A describe the space $c$ in a time $f$, in what time will be describe the space $x$?

That is.

$$c : f = x : t(A)\ t(A) = (xf)/c$$

The body B will employ a time t(B) to arrive in D

$$d : g = (x - e) : t(B)\ t(B) = (xg - eg)/d$$

If the body A begin to move after the body B by a time $h$, we find that

$$\frac{fx}{c} + h = \frac{xg - eg}{d}$$

$$x = \frac{chd + ceg}{cg - fd}$$

If the body A begin to move before B by a time $h$, we find that

$$\frac{fx}{c} = h + \frac{xg - eg}{d}$$

$$x = \frac{ceg - chd}{cg - fd}$$

*Second case*: A and B move contrary ways, or towards each other and let them meet in M.

We put $AM = x\ BM = e - x$

The body M will employ a time t(B) to reach M

$$t(B) = \frac{ge - gx}{d}$$

If A begin its motion after B, we will find

$$\frac{fx}{c} + h = \frac{ge - gx}{d}$$

If A begin its motion before B, we will find

$$\frac{fx}{c} = h + \frac{ge - gx}{d}$$

Then Maria Gaetana Agnesi applies the formulas found to numerical examples.

## 4.1 First Example

Let the body A have such a velocity as to move 9 miles in 1 h, and the body B to move 15 miles in 2 h; and let them distant from each other 18 miles, and let B begin to move 1 h before A.

Then we have

b = 1, f = 1, c = 9, g = 2, d = 15, e = 18

$$x = \frac{324 + 135}{18 - 15} = 153, \ t = 18$$

Therefore, the two moving bodies will be together at the distance from the point A of 153 miles after 18 h from the beginning of the motion.

## 4.2 Second Example

Let the body A have such a velocity as to move 4 miles in 1 h, and the body B to move 5 miles in 1 h, and let them be distant 6 miles and A begin to move 2 h before B. Therefore

b = 2, f = 1, c = 4, g = 1, d = 5, e = 6

$$x = \frac{24 - 40}{6 - 5} = 16, \ t = 4.$$

Therefore, the two bodies A and B will be together at the distance of 16 miles from the point A, after 4 h from the beginning of the motion.

## 4.3 Third Example

Let the body A have such a velocity as to describe 7 miles in two hours, and the body B 8 miles in 3 h and let them be distant 59 miles, and A begin to move 1 h before B towards. Substituting the values in the formula, we find that x = 35. Therefore, the two bodies will meet each other at the distance of 35 miles from the point a, after 10 h from the beginning of motion.

## 5 Conclusions

We think that pre-university students can acquire the fundamental mathematical ideas using also everyday problems. From this point of view, we can use many suggestions and examples, contained in Agnesi's Books. If we propose these arguments or problems by means of laboratory instruments, flipped classroom techniques, or by didactical methods that you prefer, we think that the media are different, but the meaning is the same.

## References

1. Agnesi MG (1748) Instituzioni Analitiche. Tomo I, Tomo II. Milano. https://archive.org/details/bub_gb_xDF_ksE24HUC; https://archive.org/details/BUSA298_184. Accessed 20 June 2019
2. Agnesi MG (2011) Analytical institutions in four book (trans: Colson J). Scholar's Choice. http://books.google.com/books?id=o54AAAAMAAJ&oe=UTF-8. Accessed 20 June 2019
3. Silvio Maracchia (2005) Storia dell'algebra. Liguori Editore, Napoli

# Teatro Comunale, Ferrara: The Question of the Curve. From the Debate to the Geometric Analysis

**Giampiero Mele and Susanna Clemente**

The Teatro Comunale was built in Ferrara at the end of the 18th century, at a time when modern theatre was gradually leaving the space of the Duke's Court and Academy to become part of the urban fabric, shifting from representing the elite to turning towards wider communities. The models of court theatre and public theatre with several levels of boxes coexisted for a long time, until the complete codification of the "teatro all'italiana", of which the Comunale represents one of the clearest examples. Over time there have been several renovations. However, the plan has never been strongly altered and has come almost intact to this day. This makes the comparison between measured survey and available historical sources particularly significant and interesting.

The construction of the Teatro Comunale, which lasted over a decade, started under the papal domination and ended at the time of the Cispadan Republic. Around 1786, at night, the dwellings on the so-called Isola del Cervo were demolished, to start the construction of a first project by Giuseppe Campana. However, the built theatre follows the design by Cosimo Morelli, which includes several oval curves for the shape of other spaces such as the courtyard for carriages and the hall. His design also recalls the neighboring oval church of San Carlo designed by Aleotti.

From the written sources we can see that the question over the shape that the curve of the theatre cavea should have followed has been intensely debated.

The measured survey of the Ferrara Theatre and the analysis of the actual geometrical layout has been carried out in parallel with the studies of these papers. The Biblioteca Ariostea archives contain documents in which the relationship between

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geometry and functionality is discussed by designers and experts nominated by the City who financed the construction.

The Ferrara Theatre was built in the same period as the Milano Teatro alla Scala, so we may assume that this debate extended beyond the borders of the city.

## 1 Introduction

The Teatro Comunale was built in Ferrara at the end of the 18th century, at a time when modern theatre was gradually abandoning the space of the court and the academy to become part of the urban fabric, passing from representing an elite to turning towards wider communities. The models of court theatre and public theatre with overlapping boxes coexisted for a long time, until the complete codification of the "teatro all'italiana", of which the Comunale represents one of the clearest examples. Over time there have been several renovations that have affected the decorative apparatus by Migliari [1], technological systems and structures. However, the plan has never been strongly altered, and has come almost intact to this day. This characteristic makes the activities of survey, verification and comparison of theatre spaces with descriptive and figurative historical sources particularly significant (Fig. 1).

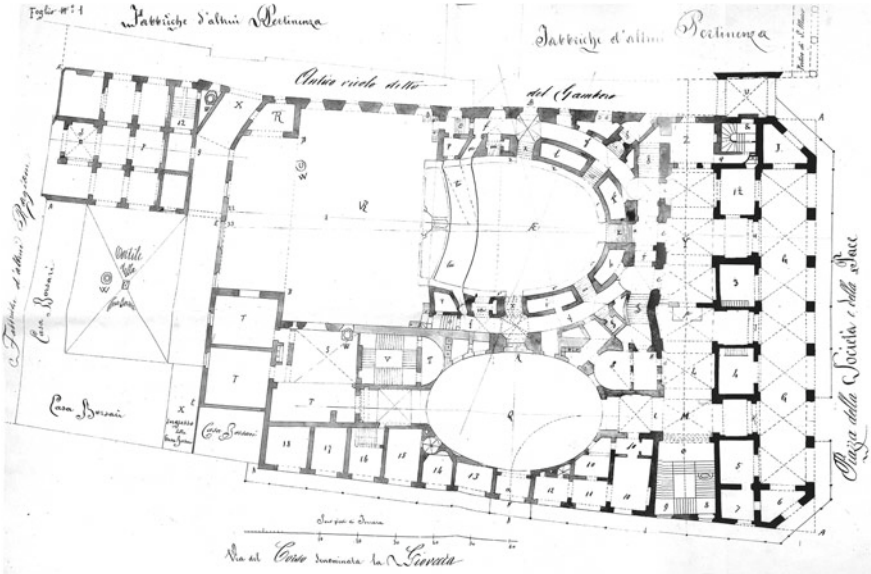

**Fig. 1** Plan of the ground floor of the Municipal Theater of Ferrara with the metric scale in historical unit of measure (foot of Ferrara). One can see the two ovals, one for the theater hall and the other for the courtyard of the carriages [2]

The construction site of the Teatro Comunale, which lasted over a decade, was started under the papal domination and ended at the time of the Cispadan Republic. At night, at the behest of the Pope, the dwellings on the so-called Isola del Cervo were demolished, to create a first project by Giuseppe Campana [3]. The building respects the creation by Cosimo Morelli, the author of an interesting correspondence between the two parallel axes of the oval courtyard for carriages and the hall, as well as the neighboring oval church of San Carlo designed by Aleotti. The building, inserted harmoniously into the consolidated urban fabric, determines a strong interpenetration between the inside and the outside, being the oval courtyard facing along Via Giovecca, on which there is the largest, in terms of size, of the two longitudinal elevations. The interior is characterized by the absence of the royal box, as well as the absence of the proscenium boxes, although all the four orders of 23 boxes each, and the gallery, are directly joined to the semielliptic arch of the proscenium. A lowered vault covers the room. These characteristics were modified several times during the construction, which began parallel to that of the Teatro alla Scala, with the same protagonists.

## 2 On the Generating Curve of the Teatro Comunale

A long debate has remained unresolved, related to the shape that the curve of the theatre should have taken and to the definitive authorship of the building [4]. To complete the work on the Campana construction site, soon interrupted, Antonio Foschini, whose role in the design of the theatre was claimed over time, and Cosimo Morelli anonymously submitted their own solution to the opinions of Piermarini and Stratico [5]. Of the tables, lost, marked with the letters AA and BB, the paternity was never identified.

Retracing Piermarini's opinion, in favor of the BB solution, we obtain important observations of geometry, of the study of visuals and acoustics. Piermarini emphasizes that the design of the platfond of the BB project is certainly preferable as it describes a regular whole figure, elliptical, and that the view is better, since it is not impeded, as in the solution AA, by the first three boxes, too advanced compared to the proscenium. In terms of acoustics, Piermarini underlines the importance of the proscenium, and of the impediments represented by the scenic backdrops in the propagation of sound. Finally, Piermarini states that the BB design curve is "la medesima che si è posta in uso in uno dei grandi Teatri d'Italia" [5], at Teatro alla Scala, and since it worked "a meraviglia" [5], he sees no reason to attempt new configurations.

The opinion by Simone Stratico, Professor of the University of Padua, is dated May 25th, 1791. From the text, the main measures of what was hitherto constructed and the geometric, spatial, functional and acoustic characteristics of the two design solutions presented can be deduced. First of all, compared to the Campana's work with 19 boxes for order, in addition to the two at the proscenium, the solution is affirmed, then realized, with 23 boxes per order, the absence of the proscenium, and the greater breadth of the Prince's stage, today absent [6]. Stratico suggests that the aforementioned conditions have been respected in both designs, and that they differ exclusively “nel modo di descrivere la curva che deve servire al contorno della Sala Teatrale, o Uditorio e nella disparità di alcune dimensioni” [7]. You can also read in the opinion: […] “trovo che nel disegno AA gli archi circolari CI, EH, i quali uniscono gli altri archi BC, LI e BE, HK, non sono descritti da centri posti nelle rette, che passino per i punti C, A: I, F: E, A: H, F, nelle quali sono i centri A, F de’ due cerchi BEDC, SIGH. Quindi invece di continuarsi la curva in una flessione regolare BCIL, BEHK risultano necessariamente quattro angoli d’intersezione degli archi ai punti I, C, E, H, i quali comunque nel lavoro possono essere con industria occultati, formeranno non pertanto una centina irregolare, e d’effetto spiacevole alla vista, che nella figura in piccolo non si può per avventura discernere, ma nella figura reale ed in grande verrà sentito, ancorché a colpo d’occhio da tutti non sia per apprendersene la ragione” [7]. Instead in the BB drawing the arcs constituting the curve present the same tangent in the point of their union, determining a continuous theatrical curve, not disturbed by the angles of intersection of the arcs. The AA curve is comparable to a “poligono di lati curvi” [7]. Stratico, with regard to dimensional disparities, describes two types of problems. In fact, there are measures that should be the same in both designs, as they relate to the pre-existing buildings, and others that differ precisely because of the different thinking of the designers. “Rispetto alle prime: parmi di rilevare che le muraglie principali che chiudono l’arco del Teatro siano già costruite. Ciò posto: nel disegno AA trovo la larghezza totale di quest’ area misurata nella linea MN, comprendendo la grossezza delle muraglie, di piedi 64: e la lunghezza totale misurata nella linea GB compresa la grossezza delle muraglie di piedi 63. Nel disegno BB trovo la prima di queste misure di piedi 63: la seconda di piedi 62 e ½. Non m’ arresterei a questa osservazione, se non mi guidasse a dell’altro. Un palchetto corrispondente nel disegno AA di diametro UT ha piedi 3 e ½ di sfondato, e così anche il palchetto del Principe. Nel disegno BB il palchetto corrispondente al diametro KK ha piedi 4 e ½ di sfondato e quello del Principe ha piedi 5 di sfondato” [7]. These differences make it difficult to understand the real dimensions of the future boxes, especially in the depth, and should therefore be eliminated. As regards to the differences dictated by the designers’ intuitions, there are in Ferrara theater:

Distance of the maximum width of the Room from the parapet of the Prince’s box:
AA: feet 47 + ½;
BB: feet 47 + ½;
Maximum width of the theatre room:
AA: feet 38;
BB: feet 39;
Opening of the scene:
AA: feet 38;
BB: feet 33;
Distance of the maximum width of the Hall from the Curtain:
AA: feet 20;
BB: feet 25;
Distance of the maximum width of the Room from the parapet of the Prince’s box:

AA: feet 27 + ½;
BB: feet 22 + ½.

In summary, the curve of the AA solution is excessively elongated, as well as significantly narrower, with more oblique views than those guaranteed by the BB curve instead. "Se poi si riguardi il tratto dal diametro di massima larghezza all'apertura di scena sarà facile dall'addotte misure di computare, quanto più rapidamente convergano i lati della Curva nel disegno AA di quello che nel BB. Una retta condotta per l'estremità del diametro di massima larghezza, e per l'estremità dell'apertura di Scena dalla stessa parte, va ad incontrare il diametro di lunghezza della Sala teatrale alla distanza dal diametro di larghezza, di piedi 88 nel disegno AA, e di piedi 138 nel disegno BB" [7].

In the field of acoustic Stratico represents that not enough theories have been developed to favor one curve over the other. They are therefore considered similar, given that the halls have small dimensions compared to the limits identified for the propagation of the human voice, and that the construction materials and ornaments are chosen to favor the diffusion of sounds.

Finally, Cardinal Spinelli opted for a realization that would put together the best elements of the two alternatives, that is to say, without the proscenium boxes and at the same time not too long [8].

## 3 Survey and Geometrical Analysis of the Theatre

The integrated analysis, done with different methods and survey systems, was the tool and the means to achieve this dual objective. The manual survey, the one with 3D Laser Scanner (Fig. 2) and that with a 3D laser EDM were integrated to produce a model, a two-dimensional database, which had the metric quality to fulfill the scientific requirements. A database that can process more themes and analysis, useful not only to the project, including the restoration, but also to the management of the monument. An informative document of this kind, accurate and detailed, produces knowledge and can give answers and set many questions as well to all scholars who wish to investigate it.

The metric analysis of the survey, based on the historical unit of measure, the "piede ferrarese" (0.403854 mt), made it possible to relate number and measure, to identify the geometric scheme that generated the so much discussed oval by historians. The result, obtained through the indirect survey carried out by laser distance meter 3D, was very helpful. By this instrument, the profile that faces in the audience, at the first order of boxes, where the oval had to have the intact measures—net of the moldings—has been detected (Fig. 3). The resulting dxf file has been analyzed in a CAD environment to find the oval centers and their metrics relations. We have a very precise scheme that shows a figure whose major axis is 59 feet, it is orthogonal to the proscenium and this cuts at its midpoint; the minor axis that bisects the main one, is parallel to the proscenium and is 39 + 1/10 feet. On the major axis, symmet-

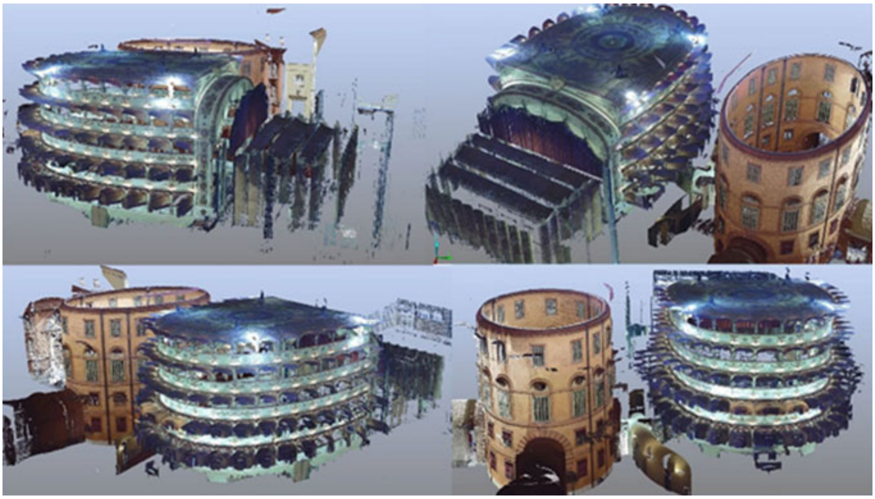

**Fig. 2** Tridimensional point cloud of the inside of the theater and of Foschini's rotunda, obtained through a laserscan 3D Faro Cam 2 focus 3D survey. 9 scans were combined to obtain the complete cloud

rically with respect to the transverse axis, are placed at a distance of 23 feet from one another, the centers A and B of the minor circumferences, which have a radius equal to 18 feet. The distance between the centers C, E, D, situated on the minor axis, is 80 + 9/10 feet (32.67 mt). The two isosceles triangles that are generated by connecting the four centers have angles in C and D whose measure is 32°. The sides intersects the two minor circumferences at the points 1, 2, 3, 4, identifying the measurement of 60 feet of the radius of the major circumferences (Fig. 3). This oval is intercepted by the proscenium in the points 3 and 4. The width of the boxes and of the corridor are identified by two concentric ovals that are obtained by increasing the radii of the circumferences of 5 feet to get the first, the one of the boxes, and other 5 feet to obtain the second, that of the corridor.

The measurement of the perimeter of the oval is essential to divide it, and identify the rhythm of the boxes. The problem is solved as the sum of the perimeters of the respective arches of circumference. The formula that puts in relation the measure of the arch l with that of the circumference C and the one of the central angle of the radius, that delimit the arch φ with that of the round angle, is $l/C = \varphi/(360°)$ from which it results that $l = \varphi/360 \times 2\pi r$.

The perimeter of the portion of the oval to be divided is 113 + 1/2 (33.5 × 2 = 67; 67 + 46.5 = 113.5). The central stage box is 5 + 1/2 foot and the other 22 minor boxes (11 + 11) are 4 + 9/10 foot (113.5 − 5.5 = 108; 108: 22 = 4.9).

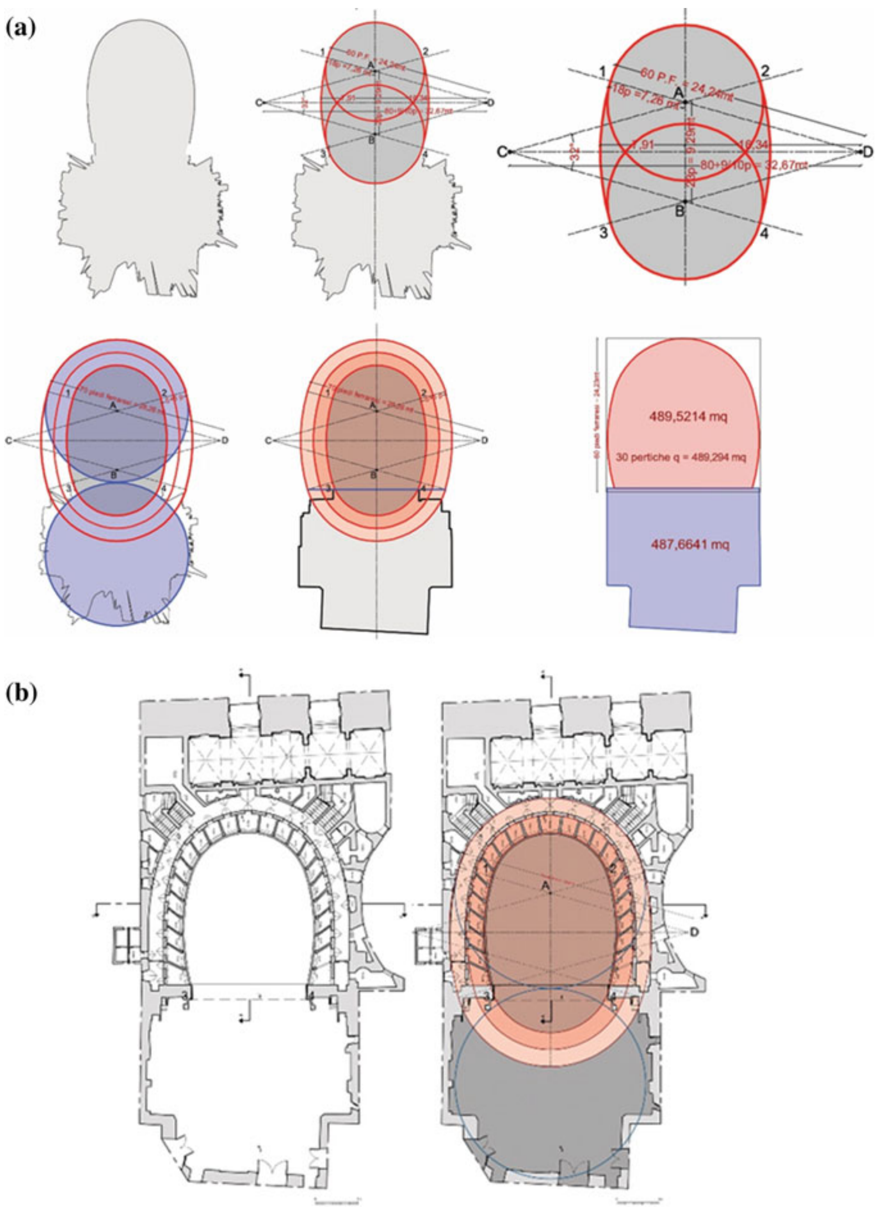


**Fig. 3** **a** Geometrical schemes of the oval describing the hall of the Teatro Comunale di Ferrara. Scheme A, survey obtained using the 3D laser distance meter. Scheme B, E, C, superposition of the oval to the survey and its metrics relations. Scheme D, E parallel ovals that define the depth of the boxes and of the corridor. Scheme F distinction between the space for the spectators and that for the stage. The sum of the areas of the two is 60 pertiche quadre ferraresi. **b** Drawing of the survey of the first row of the stage boxes. Superimposition of the geometric scheme of the oval to the survey of the first row of boxes

# 4 Conclusions

The comparison between the documents found at the Biblioteca Ariostea archive in Ferrara and the data obtained from the metric analysis allow us to state with absolute certainty that the solution of the oval adopted in the construction of the Ferrara Theatre is neither the one called AA nor the one called BB. The third solution proposed by Cardinal spinelli that we will call CC, adopted in the construction, is remarkably different in size from those commented and analyzed by Piermarini and Stratico. The dimensions of the solutions AA, BB and that found thanks to the CC metric analysis are reported below.

– Distance of the maximum width of the Room from the parapet of the Prince's box is in AA: feet 47 + ½; in BB: feet 47 + ½; while in CC is 44 + 1/2 foot, that is almost 17.99 m.
– Maximum width of the theatre room is in AA: feet 38; BB: feet 39; in CC solution is 38 feet (15.34 mt.).
– Opening of the scene is in AA: feet 38; BB: feet 33; CC: feet 33 + 9/10.
– Distance of the maximum width of the Hall from the Curtain is in AA: feet 20; BB: feet 25; CC: feet 18 (7.26 mt.)
– Distance of the maximum width of the Room from the parapet of the Prince's box is in AA: feet 27 + ½; BB: feet 27 + ½; CC: feet 30 + 2/3.

From the few notes above it is clear that the final solution adopted as the shape of the oval does not correspond to those described by Piermarini and Stratico. It is clear that a third solution was studied and implemented. At this stage and on the basis of these two documents it is not possible to establish the authorship of the CC solution, however it is possible to see that the architect's goal is to build a well proportioned portion of an oval to have good acoustics, and that it had the same surface of the stage (30 + 30 = 60 square poles), following the requirements for the type of *teatro all'italiana* [9].

# References

1. Teatro Comunale di Ferrara ristaurato l'anno 1851 dal celebre artista Prof. Francesco Migliari e successiva morte di questi—sec. XIX [Biblioteca Comunale Ariostea, Fondo Antolini, busta n. 120]
2. Piante due del Teatro Comunale di Ferrara—sec. XIX [Biblioteca Comunale Ariostea, Fondo Antolini, busta n. 70]
3. Teatro Comunale—Brano di relazione intorno all'operato dell'architetto Campana. Attivo e passivo del Teatro Comunale. Nota dei proprietari dei palchi nel dic. 1802. Calcoli intorno alla curva teatrale e note relative alla fabbrica suddetta - sec. XIX [Biblioteca Comunale Ariostea, Fondo Antolini, busta n. 69]
4. Paolo Fabbri e Maria Chiaria Bertiteri, 2004. I teatri di Ferrara. Il Comunale, LIM Editrice, Lucca

5. Pareri degli ing. e prof. Piermarini e Stratico sulla curva del Teatro Comunale di Ferrara e la distribuzione dei palchi—sec. XVIII [Biblioteca Comunale Ariostea, Fondo Antolini, busta n. 76]
6. Schizzetti tre della curva del Teatro Com. di Ferrara. I. Distribuzione de' palchi del 1° ordine che si trovava in questo Teatro vecchio. II. Distribuzione dei palchi del 1° ordine secondo la nuova riforma; conservata la solita distanza dalla scena a quelli che ne hanno la proprietà giusta al primo sistema. III. Distribuzione de' palchi del 1° ordine in questo Teatro secondo la nuova riforma conservata la solita distanza dalla scena—sec. XVIII [Biblioteca Comunale Ariostea, Fondo Antolini, busta n. 77]
7. Pareri gegli ing. e prof. Piermarini e Stratico sulla curva del Teato Comunale di Ferrara e la distribuzione dei palchi – sec. VIII [Biblioteca Comunale Ariostea, Fondo Antolini, busta n. 76]
8. Foschini Antonio—Biglietto con cui dichiara al March. Camillo Bevilacqua di eseguire la curva del Teatro come allo schizzetto inviatogli, 15 agosto 1791. Lettera al March. Bevilacqua intorno al consegnare la pianta del Teatro ai costruttori imolesi, 11 luglio 1791. Medaglioni pubb. nel 1816, incisore F. Padovani—sec. XVIII [Biblioteca Comunale Ariostea, Fondo Antolini, busta n. 73]
9. Saggio sopra l'architettura teatrale, ossia dell'ordinanza la più vantaggiosa ad una sala di teatro relativamente ai principii dell'ottica e dell'acustica—sec. XIX [Biblioteca Comunale Ariostea, Fondo Antolini, busta n. 84]

# Women and Descriptive Geometry in Italian University

**Barbara Messina**

This paper aims to analyse the women's contribution to the teaching of Descriptive Geometry in the Italian Faculty of Architecture and Engineering.

Starting from the analysis of current data collected by ministerial archives and by retrieving, back in time, further information, such as the sources of the Italian Association of Drawing Professors (Italian Drawing Union-UID), the paper proposes a diachronic reading that can illustrate, in general, the role of the teaching by women in the specific scientific-disciplinary field ICAR 17/Disegno. An area of interest in which many different cultures coexist. In particular, we draw attention to Descriptive Geometry, firstly highlighting—through appropriate graphs that re-elaborate the acquired data—the contribution, the position and the incidence of the female figure in the field.

Then, focusing on some key figures for the university teaching evolution of this discipline, we intend to honour those who have distinguished themselves, by leaving a mark both in the didactic and in the scientific field.

## 1 Introduction

In recent years, several studies have investigated, from different points of view, the link that exists between the problems typical of female identity and the difficulties of women in asserting their role in the professional sphere.

In this line arises the present essay, which is part of a wider interdisciplinary

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investigation project promoted by the OGEPO[1] Interdepartmental Research Centre of Salerno University. The project, titled *Gender and Professions. Contexts, Languages, Representations from The 14th Century to The Present*, intends to analyse the relationship between gender and professions, starting from those that still show strong stereotypes in roles, or significant differences in terms of career progression. Particular attention is paid, here, to the presence of women in the field of university career, with specific regard to the teaching of Descriptive Geometry in the Italian Faculties or Departments of Engineering and Architecture.

## 2 The Career Progression of Italian Academic Professors

The issue must certainly be placed within a broader context, since, despite the progress recorded in recent years, the female academic professors remains a minority. This situation generally affects the whole of Europe—excluding Finland, where there is perfect gender equality, or countries as Norway, United Kingdom, Portugal, Sweden, which almost achieve gender equality. In particular, analysing the Italian situation, the statistics obtaining from the MIUR database [5], from December 31st, 2011 to today, show a significant growth in the presence of women. In this time frame they increase by about a third: nevertheless the percentage of female teachers remains at around 37%, far from a hoped gender equality.

Yet, monitoring the progression of students, starting from school education and following them at the university and post-graduate level, the data would seem to lead to different results. In fact, considering all the academic courses without distinction, there is an average female presence of 56%. Moreover, are women 59% of the graduates, 51% of students enrolled in Ph.D. courses and 52% of those who achieve the Ph.D. title. By shifting attention from training to academic careers, there are encouraging data at the first step of fellowship researchers. Instead, the teaching staff highlights a gradual decrease in female presences, depending on the growth of the academic hierarchy (46% researchers, 36% associate professors, 21% full professors) [6, 7].

In this regard, it should be noted that the gender percentage in the student distribution, for each training level, varies in a very significant way depending on the disciplinary area (75% of female presence for 'Human Sciences', 31% 'Engineering and Technology'). Also, in academic careers, the presence of women is very prevalent in 'Humanities', showing an inversion of tendency in technical-scientific areas [6, 7].

[1]The *Osservatorio per la diffusione degli studi di Genere e la cultura delle Pari Opportunità* (OGEPO), was established at the University of Salerno in 2011 and recognized as an Interdepartmental Research Centre since 2014. It deals with equal opportunities and gender studies, promoting interdisciplinary research and comparison on investigations and statistics related to gender, to equality and equal opportunities, to the presence of women in the history and society, as well as legal questions, historical, social, economic, political and cultural aspects, inherent to these issues. Director of the Centre and coordinator of the researches is Professor Marisa Pelizzari.

Considering the ‘Graphic Representation’ area we firstly premise that it includes a great variety of knowledge branches. Moreover, the Drawing disciplines are present in various courses of Architecture, Engineering, Design, but also of Literature or Psychology (Fig. 1).

With specific reference to the Drawing’s academic teaching staff, in its totality, about 39% of professors, independently of the role,[2] are women: a value fully consistent with that national. However, it should be noted that about 66% of our female professors are affiliated in architectural departments; while about 30% are in engineering structures. This could influence the women percentage distribution at the various academic level since the female presence in Architecture courses is generally more substantial.[3]

Furthermore, with reference to the gender distribution trend, over the years, the considered time frame shows encouraging data, represented in the graphs. Considering all the professors, since 2001[4] there has been a gradual and constant convergence of the gender curves, with a minimum difference in the last 5 years (to date, women

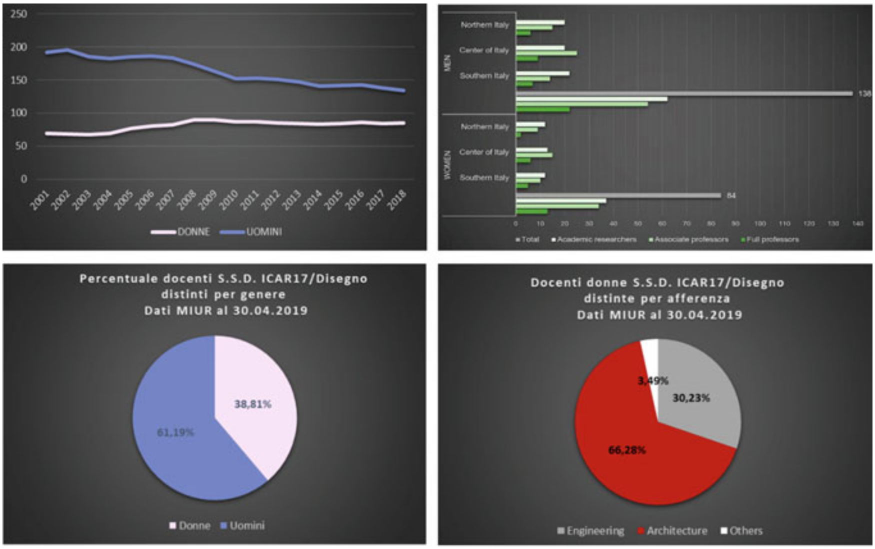


**Fig. 1** Statistical data relating to the presence of female professors in the scientific disciplinary area of ‘Drawing’. Above: Trend over the years (not distinguished by role) (left) and comparison between men and women professors (distinguished by role) (right). Below: Percentage of men and women professors (left) and Faculty/Department distribution (right)

[2]The percentage value shows a slight fluctuation, according to the academic role: the female full professors correspond to 39.47%, the associate professors correspond to 37.37%, while the academic researchers are 40.24% ( MIUR data, as of 4 May 2019).

[3]The Architectural courses degrees placing themselves between the humanistic and technical areas, thus balancing the data referred to the purely engineering courses and bringing back on national average the overall area values.

[4]In that year the female presence is about 26% with 69 women out of 261.

professors correspond to 85 out of a total of 219). Disaggregating data in according to the academic role, the obtained graphs show a substantially similar trend over time.

Finally, an interesting observation concerns the presence of women in top management and institutional roles. Indeed, considering the *Glass Ceiling Index*, an international index that measures the gender equality at the top level of academic career, the Drawing area reveals a very strong parity in reaching the so-called grade A (that is the role of full professor).[5]

This positive result is confirmed if we analyze the female presence in institutional roles, with particular reference to the positions of Dean, which show how the women of the Drawing area have much relevance. In fact, out of 6 Faculty Deans, so far elected in the area of Drawing, 4 of which in Architecture and 2 in Engineering, 2 were women. That is a third of the total number.

The balance is even more interesting if we look at Architecture alone, in which there is an absolute gender equality. If we equate, to the role of Dean of Faculty, the Direction of the Departments that, since 2010, have replaced the Faculties, we register about 43% of female presences (3 women out of a total of 7), which even reaches 60% considering only Architecture (here the balance is reversed, becoming in fact 3 women out of a total of 5).

Still with reference to the scientific-disciplinary area of Drawing, a more detailed investigation was then conducted regarding Descriptive Geometry which, as already stated, is one of the possible fields of interest in research and teaching of the disciplines of representation in general.

To this end, in the definition of the statistical data useful for quantifying the incidence of female presence with respect to the teaching staff considered in its entirety, firstly all the teachers were identified who, in a specific or generic way, deal with Descriptive Geometry. The survey, which at this stage took into account the contents provided in all Italian courses of the Drawing's area, highlighted how Descriptive Geometry finds significant consideration within its scientific-disciplinary sphere, being taught—sometimes together with other contents[6]—by more than of 57% of Italian professors (with a distribution of male and female teachers equal to 35 and 22% respectively) (Figs. 2 and 3).

[5]The Glass Index analyses the following ratio:

$$\text{GCI} = \frac{(\text{Women Grade ABC})/(\text{Women \& Men Grade ABC})}{(\text{Women Grade A})/(\text{Women \& Men Grade A})}$$

Particularly, according to the obtained value, the index evaluates the gender equality at the top level of careers, as specified below:

GCI = 1 No gender difference in reaching grade A

GCI < 1 Over representation of women at grade A

GCI > 1 Under representation of women at grade A

Referring to the scientific area of Drawing, this is the result:

GCI of Drawing Area = (85/219)/(15/38) = 0.388/0.395 = 0.98.

[6]Generally, Architecture courses include specific classes in Descriptive Geometry, together with other generic Drawing classes, while the Engineering courses mainly involve wider courses, in which however specific issues of Descriptive Geometry are addressed.


| | Totale Docenti | Impegno specifico su GD (didattica specifica e ricerca) % | Ordinari | | Associati | | Ricercatori | |
|---|---|---|---|---|---|---|---|---|
| Donne | 19 | 30,16% | 5 | 45,45% | 8 | 22,22% | 6 | 37,50% |
| Uomini | 44 | 69,84% | 6 | 54,55% | 28 | 77,78% | 10 | 62,50% |
| | 63 | 100,00% | 11 | 100,00% | 36 | 100,00% | 16 | 100,00% |
| | **Totale Docenti** | **Impegno complessivo su GD (generico e specifico) %** | **Ordinari** | | **Associati** | | **Ricercatori** | |
| Donne | 49 | 39,20% | 11 | 57,89% | 24 | 35,29% | 14 | 36,84% |
| Uomini | 76 | 60,80% | 8 | 42,11% | 44 | 64,71% | 24 | 63,16% |
| | 125 | 100,00% | 19 | 100,00% | 68 | 100,00% | 38 | 100,00% |

**Fig. 2** Statistical data relating to the presence of the teaching of Descriptive Geometry in Italy

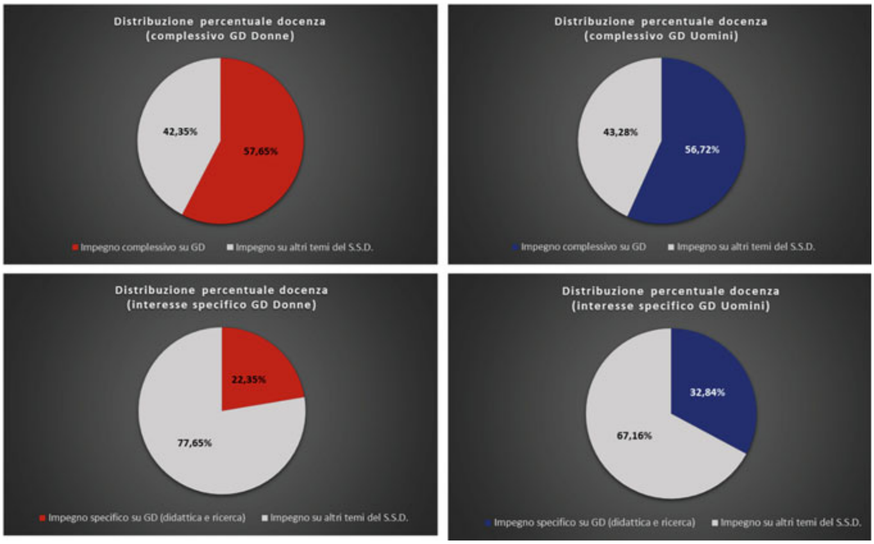


**Fig. 3** Percentage distribution, by gender, of academic professors involved in Descriptive Geometry. Above: values related to "generic" female (left) and male (right) professors. Below: values relating to "specific" professors only

If, instead, we analyse only the "specific" professors, that is, those who particularly teach Descriptive Geometry or who, although "generic" professors, do research in this particular field of the scientific-disciplinary area, the percentages are reduced quite clearly: of all teaching staff, only 28% deal specifically with Descriptive Geometry, with a certain gender difference that shows a 20% presence of men compared to 8% of women.

And again, if we evaluate the percentage of professors of the same gender—that is, by disaggregating the data for the male or female category only—it emerges that while on the overall data ("generic" and "specific" teaching) there is a substantial congruence of values with respect to the average calculated on the entire Drawing's professors staff (for both subgroups the professors of Descriptive Geometry is attributable to approximately 43% of the total), with reference to only specific teaching and to research interests, the professors involved in Descriptive Geometry stand at 33% for men and 22% for women (Fig. 3).

## 3 Searching Our Roots: The Ladies of Descriptive Geometry

If the quantitative data does not appear to be entirely satisfactory, shifting the attention to qualitative aspects and contents it is certain that the presence of the female gender in the teaching of Descriptive Geometry it's very significant.

In particular, turning to the recent past, many female figures have left a profound mark: some of them have held important institutional roles and prominent positions in the management of the university system, receiving for this also acknowledgments and lifetime achievement awards. But, regardless of this, it is clear that all of them have succeeded in making a cultural contribution of great value, often pursuing didactic and research paths already traced by their great masters, in other cases creating real "schools" (Fig. 4).

Considering, in this context, only the women who have achieved the role of full professor, and retracing a timeline that, from the pioneers of Descriptive Geometry leads to our days, first of all, must be included Anna Sgrosso Neapolitan by adoption and a pivotal figure in the teaching of Descriptive Geometry, still today a cultural and scientific reference point for many researchers. She graduated in Architecture in the immediate post-war period (1950), and then she worked at the University of Naples "Federico II" with Mario Giovanardi, one of the fathers of the Neapolitan School of Descriptive Geometry of the Faculty of Architecture. She was a volunteer assistant (until 1960), an in-charge assistant (until 1966) and then an ordinary assistant (until 1980), reaching the maximum level in the academic hierarchy in 1980, when she became an ordinary professor. From 1991 to 2002 he coordinated the Ph.D. of Drawing area at the University of Naples "Federico II" [4].

In 2005 she was awarded the *UID Certificate of Magister*, maximum recognition for the career, "Because of her tireless work of discovery and reinvention of

**Fig. 4** Above: Anna Sgrosso (left), Rosa Penta (in the middle), Mariella Dell'Aquila (right). Below: Maria Teresa Bartoli (left), Maura Boffito (in the middle), Laura De Carlo (right)

descriptive geometry, for the generous dedication to teaching, for humanity and the confidentiality of her presence in the school".[7] And again, in 2017 it was awarded the *UID Gold Plaque* "[…] for the significant results achieved in research and teaching in the Representation area".

It's evident, in her scientific approach, the ability to synthesize the analytical rigor with an extraordinary graphic sensibility: by focusing precisely on the deep study of projective principles, the founding basis of this discipline, Anna Sgrosso gives new strength to Descriptive Geometry, revitalizing it and reorganizing "the traditional methods of representation in an unconventional way"[8] (Fig. 5).

She has divulged this discipline, embracing and reinterpreting it with great sensibility and a recognized originality. But above all giving a very personal imprint to the research methodology with which she operates, which becomes a distinctive and recognizable sign of the school of which she is the initiator [2, 3]. In particular, by the so-called "geometrical-structural" representation—one of her hallmarks—she proposes "an innovative interpretation of architecture, whether it is realized or drawn,

[7] Mention of *UID Certificate of Magister 2005* [1].

[8] Mention of *UID Gold Plaque 2017.*

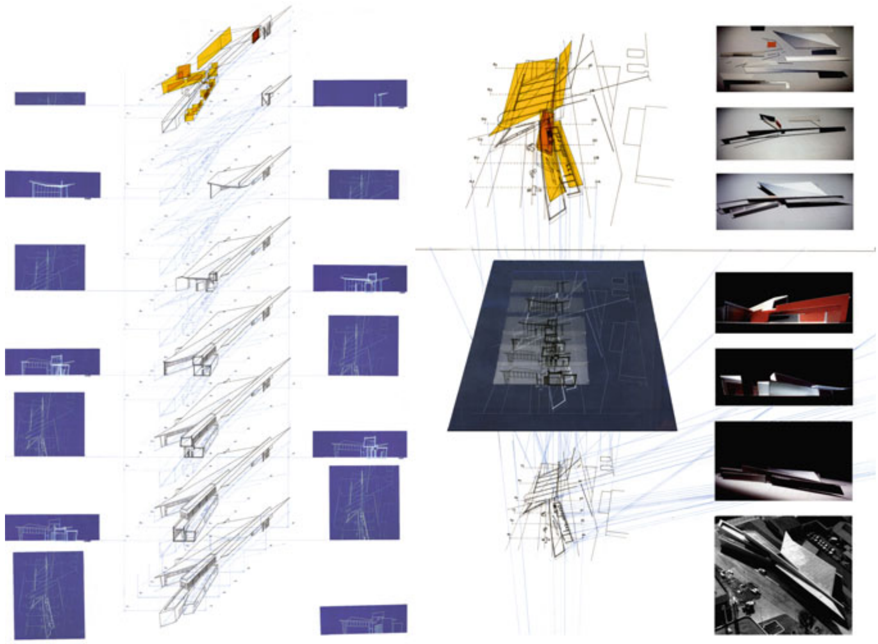

**Fig. 5** The fire station of Zaha Hadid for the Vitra Campus, in Weil am Rhein. A so-called "geometrical-structural" representation by Barbara Messina (degree thesis. Title: Real space and virtual space: the architecture of Guarino Guarini and Zaha Hadid. Supervisor: Anna Sgrosso; Co-supervisor: Agostino De Rosa, July 1998)

of which it manages to provide the geometric structure as well as the configurative genesis of the spaces".[9]

Rosa Penta (who died in 2014) is still of Neapolitan education and belongs to the same generation. She graduates in Architecture in 1958 in Naples and, as Anna Sgrosso, immediately is part of the entourage of Mario Giovanardi, engaging in research and teaching of Descriptive Geometry. Initially she worked as a volunteer assistant (until 1963), later as an ordinary assistant and finally as academic professor. In 1986 she obtained the title of full professor. Her career continues, since 1991, in Aversa, at the Faculty of Architecture of the Second University of Naples,[10] of which she is co-founder and where she will be, firstly, Department Director and, from 1991 to 2004, coordinator of a Ph.D. specific of the Drawing area [9].

Her research, marked by the strong scientific rigor, proposes a graphic layout very close to that of Anna Sgrosso, addressing however more on the survey of architecture and the environment. The geometrical-descriptive approach, which precisely bases on the configurative and morphological interpretation of the artefacts, in fact, becomes

[9] See Footnote 8.

[10] The Second University of Naples (SUN) changed its name in November 2016: today it is known as University of Campania "Luigi Vanvitelli".

the indispensable premise for representing the architectural or urban built space, to which she dedicates most of her activity. Very interesting, for example, are the research projects she coordinated on the Neapolitan portals and staircases, or on some Neapolitan squares: in these examples, starting from the survey of the artefacts analysed, she succeeds in restoring the space compositional logic through a rigorous geometric representation (Fig. 6).

Pupil of Anna Sgrosso is, instead, Mariella Dell'Aquila: graduated in Architecture in 1971 at the University "Federico II" of Naples, she starts and continues her academic career here, first following her "master", as a collaborator and assistant, and then as a professor (associate since 1994 and full professor since 2000). In 2003, and until 2010, she took over from Anna Sgrosso in the coordination of the specific Ph.D. of the Drawing area.

Her didactic activity, always related to research, sees her engaged with great dedication on themes as the geometric representation—aimed at the correct interpretation of the reality investigated—or as the reading of drawn architectures of which, thanks to inverse perspective procedures, traces to backward the genesis of space, starting from the image. "Her studies, focusing on the survey and descriptive representation of architecture, never ignore the logical and deductive rigor of mathematics, incor-

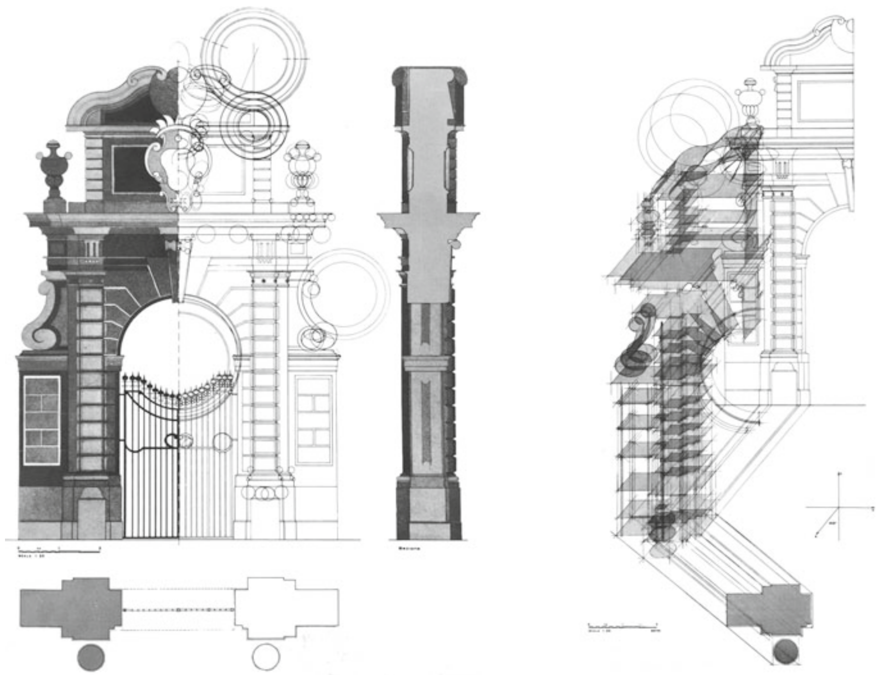

**Fig. 6** Survey and representation, in orthographic and isometric projections, of the palazzo Cellamare portal, in Naples, with identification of geometric matrices (elaborated by the students of the course of "Disegno e Rilievo", a.y. 1987–1988, prof. Rosa Penta)

porated within the Science of Representation. In her teaching there is […] authentic respect for the mathematical character of the discipline, in a continuous fading of Science in Art and Art in Science”[11] (Fig. 7).

Instead, from the Florentine school is Maria Teresa Bartoli: she graduated in Architecture in Florence in 1971 working in the academic field first as an assistant and, since 1983, as a researcher; her career continues, always at the University of Florence, as associate professor (since 2000) and full professor (since 2002). From 2014 to 2016, she coordinates the Florentine Ph.D. in Architecture, which is linked to the National School of Ph.D. in *Scienza della Rappresentazione e dell’Ambiente*.

In the didactic field, she is involved as much on the Architecture Survey as on the Descriptive Geometry: this explains her propensity to integrate measure and form, proposing a “metrological” approach as the basis on which to set up more properly geometric surveys. She deals in particular with the Renaissance perspective. This allows her to analyse the space built on the basis of a rigorous theoretical apparatus, with the aim of identifying the symbolic forms underlying it. The numerous surveys, conducted with this approach on Florentine Gothic and Renaissance architecture, have allowed her to highlight new aspects of architectural and urban contexts investigated, rediscovering often hidden design intentions (Fig. 8).

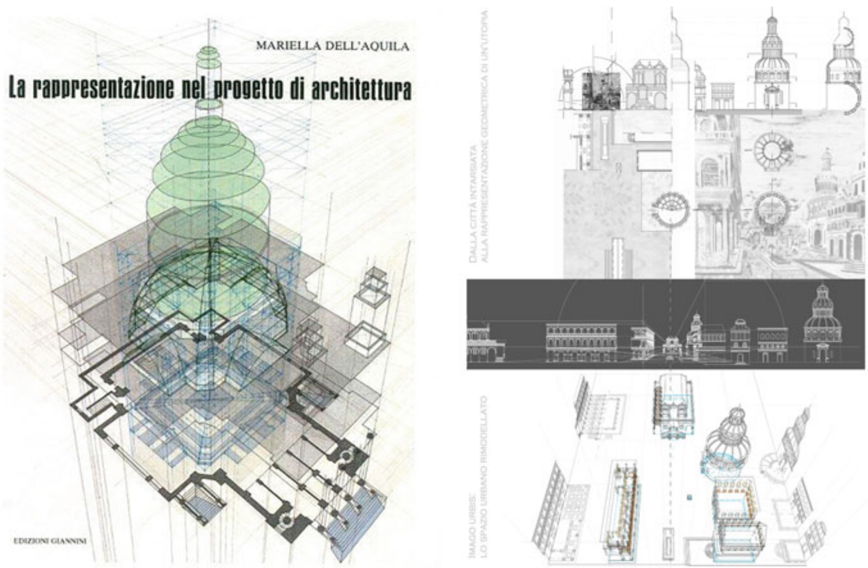


**Fig. 7** Geometric approach to the representation of built and imaginary space. Left, the front cover of La rappresentazione del progetto in architettura. Right: reconstruction of a drawn urban space (by Stefano Chiarenza, Ph.D. thesis in “Survey and Representation of architecture and the environment”. Title: Le città immaginarie: le tarsie lignee nella Certosa di S. Martino a Napoli, XV cycle, March 2003. Tutor: Mariella Dell’Aquila)

[11] See Pascariello [8, p. 7].

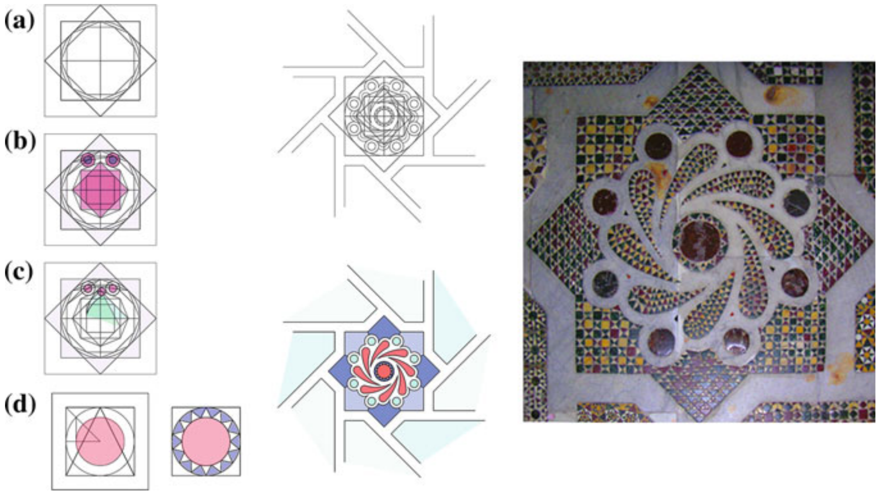


**Fig. 8** Between research and didactics: the study of the graphic path that draws with continuity one of the flooring inlays of the Cathedral of Monreale (by Maria Teresa Bartoli)

An eclectic figure, for the multiplicity of her interdisciplinary interests, is that of Maura Boffito. Graduated in Architecture in Turin (1971) and in Philosophy in Genoa (1988), in 1975 she began her career as a contractor in the field of Drawing disciplines at the University of Genoa. In this university she went through the various academic stages, becoming a researcher in 1980, associate professor in 1992 and full professor in 2000. Since 1990 she has been specifically engaged in teaching Descriptive Geometry, carrying out her didactic activity mainly at the Faculty of Architecture of the Genoa University.[12]

Her fields of interest range from the survey of architecture to archival research, from the cataloguing of the Genoese artistic and iconographic heritage to the interpretation of painted architecture, from descriptive geometry to the history of representation, intertwining—with interdisciplinary approaches—geometrical-mathematical investigations with historical-critical-anthropological ones.

In 1997 she was awarded the UID Silver Plaque with the following motivation: "A new way of tackling the problems of the basics and applications of Descriptive Geometry, an original and fun way of presenting the didactics, a humanistic and scientific culture together, which traverses research and teaching, a set of results and answers, by the students, of exceptional interest […] Maura Boffito enriches […] her inner world and her didactics not only with a profound humanistic spirit, but also of a particular knowledge of the philosophy and rituals of the American Indian

[12] She also taught at the Faculties of Architecture in Milan and Mantua, at the Faculty of Engineering of Brescia and, in her University of origin, at the School of Specialization of Restoration of Monuments. Finally, again at the University of Genoa, she held important institutional roles, working within the Faculty of Architecture, as well as the Council and Board for the Degree in Architecture.

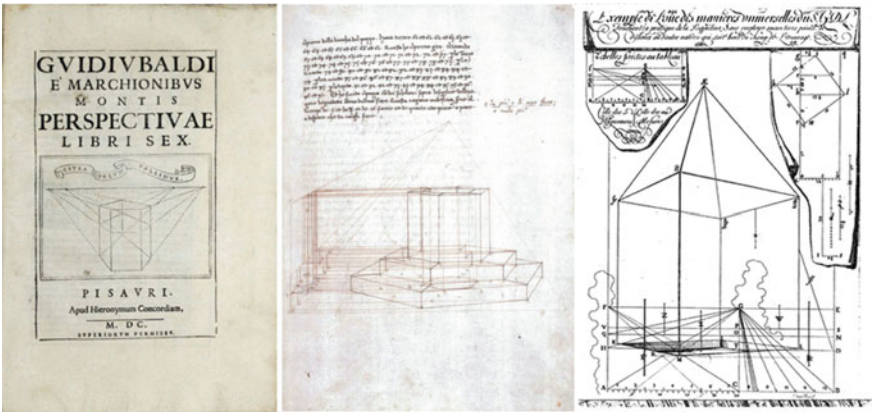


**Fig. 9** Among the main interests of Maura Boffito is the treatises on issues related to descriptive geometry. Left: frontispiece of the Perspectivae libri sex by Guidobaldo del Monte (1600). In the middle: an inner page of the treatise De prospectiva pingendi by Piero della Francesca (1482 ca.). Right: inside page of the treatise Exemple de l'une des manieres universelles du S.G.D.L. touchant la pratique de la perspective by Girad Desargues (1636)

populations, thus widening, even more, the already broad field of her teaching"[13] (Fig. 9).

Laura De Carlo is from the Roman school: in 1970 she graduated in Architecture at the University of Rome "La Sapienza", where only a year later her academic career began. Here, over the time, she holds the various roles, until she became, in 2002, full professor for the scientific disciplinary are ICAR/17. From 2004 to 2010 she coordinated a Ph.D. specific to the area of Drawing of "Sapienza" University, which joined since 2005 of the National Ph.D. School in *Scienza della Rappresentazione e dell'Ambiente*.

In 2008 she promotes and implements, together with Riccardo Migliari, the '*LA-BO-RA-TO-RI-O nazionale per il rinnovamento della geometria descrittiva*', whose purpose is to develop researches aimed at the use of computer technologies as a tool with which to visualize, in new forms, the classic themes of descriptive and projective geometry [2].

Her research activity is focused on themes related to Descriptive Geometry, with the aim of combining current digital representation techniques with the scientific foundations of representation. In fact, she deals with applications specifically relevant to this field of research—for example aimed at *quadraturismo*, stereotomy, the morphogenesis of complex forms in architecture—as well as with analysis and reading of architecture conducted through the three-dimensional digital model, not leaving out the possibilities of multimedia communication offered by the new technological systems (Fig. 10).

[13] Mention of *UID Silver Plaque 1997*. See Cundari [1].

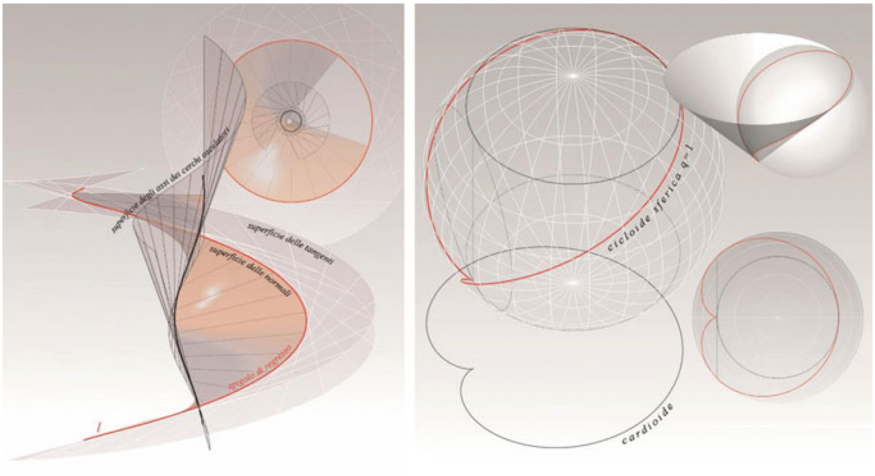


**Fig. 10** Digital representation of complex surfaces, through forms and languages specific to infographic representation (by Laura De Carlo)

## 4 Conclusions

The short excursus, though not exhaustive, intends therefore to rediscover the cultural roots common to the many academic professors who are today dealing with Descriptive Geometry in Italy, and to highlight the female contribution given to this discipline.

A personal and authentic contribution, that of women professors, who, with great strength—even in historical moments not particularly easy for their career progression—have imposed themselves in the academic sphere. Their presence has enriched the didactics and the research by the sensitivity and passion that is typical of the female inner world, succeeding in merging, in all cases, scientific rigor and humanity in the relationship, first, with her own students.

**Acknowledgements** Several colleagues of the Drawing scientific-disciplinary are have contributed to providing useful data and information for this contribution. Cristina Càndito, University of Genoa, Alessandra Pagliano e Ines Pascariello, University of Naples “Federico II”, Marta Salvatore, “Sapienza” University of Rome, Ornella Zerlenga, University of Campania “Luigi Vanvitelli”. My heartfelt thanks go to them.
I am also grateful, for the availability shown, to Maria Teresa Bartoli and Maura Boffito, with whom I was able to speak directly, and Laura De Carlo, who provided me with the images and information relating to their profiles. Thanks again to Andrea Giordano for the valuable advice offered.
Finally, my thoughts go to Vito Cardone, who was able to direct my curiosity on this research topic.

## References

1. Cundari C (ed) (2008) Una riflessione per continuare. Contributi per il XXX Convegno dell'UID vol. III – I riconoscimenti dell'UID. MG Sistemi editoriali S.r.l., Roma, 70 p
2. De Carlo L, Migliari R, Carlevaris L (eds) (2012) Attualità della geometria descrittiva. Gangemi editore, Roma, 512 p. ISBN: 9788849223057
3. De Rosa A, Giordano A (2018) La geometria, lo spazio, la configurazione: un incontro con Anna Sgrosso. Diségno 2:9–15. https://doi.org/10.26375/disegno.2.2018.2
4. Gravagnuolo B (ed) (2008) La Facoltà di architettura dell'Ateneo fridericiano di Napoli, 1928–2008. Clean Edizioni, Napoli, 432 p. ISBN: 9788884971302
5. https://cercauniversita.cineca.it/php5/docenti/cerca.php. Accessed 31 May 2019
6. Italia in cifre 2016 (2016). ISTAT, Roma, 60 p. https://www.istat.it/it/files/2016/12/ItaliaCifre2016.pdf. Accessed 31 May 2019
7. Morana MT, Sagramora S (2016) Focus 'Le carriere femminili nel settore universitario'. MIUR, Roma, 16 p. http://ustat.miur.it/media/1091/notiziario_1_2016.pdf. Accessed 13 March 2018
8. Pascariello MI (2014). L' insegnamento della geometria descrittiva nella Facoltà di architettura di Napoli. Le lezioni di Mariella dell'Aquila. Nane Edizioni, Napoli, 204 p. ISBN: 9788896790021
9. Zerlenga O (2015) Profesora Rosa Penta. EGA. Revista de Expresión Gráfica Arquitectónica 25:19

# Witch of Agnesi: The True Story

**Tullia Norando and Paola Magnaghi-Delfino**

The recent celebration of the three hundredth anniversary of the birth of Maria Gaetana Agnesi, offers an opportunity to reflect on how we have understood and misunderstood her legacy to the history of mathematics. Maria Gaetana was the author of an important vernacular textbook, *Instituzioni analitiche ad uso della gioventú italiana* (Milan, 1748), the first book dedicated to learners of mathematics and one of the best-known women natural philosophers and mathematicians of her generation.

Most popularly and erroneously, Agnesi is known as the woman who discovered a cubic curve that the English mathematician John Colson, while occupying the Lucasian professorship of Mathematics, called "the witch", leading to its modern description as "the witch of Agnesi". Colson inflicted dual infamy to Agnesi, crediting her with a result that belonged to the preceding generations of mathematicians, while damning her for the ages by presenting her no discovery as a product of diabolic female power. This article is dedicated to restoring the truth and giving Agnesi the right place it deserves in the history of mathematics and its teaching.

## 1 Introduction

Maria Gaetana Agnesi, born on 1718, was the eldest daughter of a wealthy Milanese family. Her father's Pietro vaster ambition to vault the family to the centre Milanese society included Maria Gaetana's education and the perpetual display of her learning in the salons (*conversazioni*) held in Palazzo Agnesi. From a tender age, she was surrounded by multiple tutors, most of them in religious orders, who nurtured her talents.

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Her learning was to be a supremely public demonstration of the enlightened experiment with women's learning. It is to Mazzotti's credit [11] that he has fully realized an account of Agnesi as a peculiar product of Milan's version of the Catholic Enlightenment. She was also demonstrating early signs of her pious proclivities which, when combined with a certain psychological complexity and persistent health problems, frustrated her family, many admirers, and subsequent biographers though they would eventually take great pride in her charitable activities. Don Pietro encouraged Agnesi's scientific tutors (Count Carlo Belloni, the Pavian physics professor Father Francesco Manara, and the philosopher Father Michele Casati) to transform Agnesi into an even more proficient example of the woman natural philosopher. They began to instruct her in this subject in 1733, including allied instruction in geometry and algebra, though it is worth noting that her contemporary biographer Giovanni Maria Mazzucchelli dates the inauguration of "a glorious theatre in her own home" in which Agnesi defended various philosophical arguments. Foreigners were warmly invited to dust off their rusty Latin to debate with Agnesi the many fine points of modern philosophy, and local gazettes gleefully reported her triumphs and the succession of her distinguished foreign admirers as a measure of the city's growing prestige. During the period in which she gave up public displays of scientific learning, she completed her mathematics knowledge, developing relationships with most of Italy's important natural philosophers and mathematicians. She expanded her network of scholarly correspondents. Agnesi distinguish herself from those women who were simply casual consumers of knowledge, since she aspired to a deeper level of understanding that ostensibly distinguished the serious philosopher from the philosophical dilettante. She unequivocally indicated her desire to be placed in the former category [5].

In 1740, Agnesi retreated from society but he had not yet left the mathematical correspondences and relationships, but she engaged in the very ambitious enterprise: the composition of the first vernacular textbook of mathematics. In 1748, Maria Gaetana Agnesi published *Instituzioni analitiche ad uso della gioventù italiana* (Analytical Institutions), the early textbook on Calculus. Agnesi described her own textbook as a book for "Italian youth," but she also personalized this description by invoking her duty to educate her younger brothers. Agnesi described her "idea of facilitating for young people, as much as possible, a study which is unto itself so difficult and laborious, reducing to it to the order and clarity of which it is capable, which to my knowledge no one had yet tried to do". Agnesi had written a beautifully organized book, using preferably the Leibniz notation without forget Newtonian fluxions and legacy of Cartesian analysis, filled with well-chosen examples.

## 2 The Witch of Agnesi: Myth and Reality

In Analytical Institutions, after first considering two other curves, Maria Gaetana includes a study of a particular curve. She defines the curve geometrically as the locus of points satisfying a certain proportion, determines its algebraic equation, and finds its vertex, asymptotic line, and inflection points. She named the curve according to Guido Grandi, *versiera* (1718), in Latin *versoria*. Coincidentally, at that time in Italy it was common to speak of the Devil, the adversary of God, through other words like *avversiero* or av*versiera*, derived from Latin *adversarius*.

*Versiera*, in particular, was used to indicate the wife of the devil, or "witch". Cambridge professor John Colson mistranslated the name of the curve as "witch", because he had learned Italian language only for the translation of the Agnesi's book. Different modern works about Agnesi and about the curve suggest slightly different guesses how exactly this mistranslation happened. Stephen Stigler suggests that Grandi himself "may have been indulging in a play on words", a double pun connecting the devil to the versine and the sine function to the shape of the female breast (both of which can be written as "seno" in Italian) [15]. The last hypothesis is funny rather than credible.

Before Grandi and Agnesi, the curve was studied from a different point of view by Pierre de Fermat, *Treatise of Quadratures* (1659). S. Stigler also reports that Newton worked on this curve some time before 1718, but this work was not published until 1779. Stigler does not identify this work, but it could have been *Geometria Analytica*. We can find the Newton's classification of the cubic curves in the chapter *Curves by Sir Isaac Newton* in *Lexicon Technicum* by John Harris (London, 1710). Other investigator was C. Huygens, who rediscovered the same Fermat's result on the quadrature [8]. A version of this curve was used by Gottfried Wilhelm Leibniz to derive the Leibniz formula for $\pi$.

## 3 The Witch in Pierre de Fermat

The *Treatise of Quadratures* (1659) of Fermat [3, 4], contains the first known proof of the computation of the area under a higher parabola or under a higher hyperbola with the appropriate limits of integration in each case. The second part of the *Treatise* is obscure and difficult to read and mostly unnoticed by Fermat's contemporaries. Fermat reduced the quadrature of a great number of algebraic curves in implicit form to the quadrature of known curves: the higher parabolas and hyperbolas of the first part of the paper. Others, he reduced to the quadrature of the circle. Fermat made the clever use of two procedures, quite novel at the time: the change of variables and a particular case of the formula of integration by parts. With these tools, Fermat was able to square some well-known curves as the folium of Descartes, the cissoid of Diocles or the witch of Agnesi. Fermat writes that the last curve was suggested to him "*ab erudito geometra*" [by a learned geometer]. Paradís et al. [12] speculate that the geometer who suggested this curve to Fermat might have been Antoine de Laloubère, the first mathematician to study the properties of the helix.

Fermat introduced the witch of Agnesi in his Treatise by means the cubic equation

$$b^3 = xy^2 + xb^2 \tag{1}$$

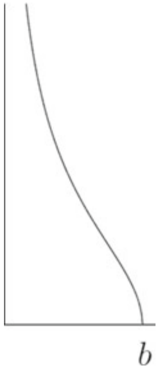


**Fig. 1** Modern graphic corresponding to Eq. (1)

where $b$ is a positive parameter.

We can see the modern graphic corresponding to Eq. (1) in Fig. 1.

Fermat determines the area of the plane region included from the curve and asymptote. In the *Treatise* p. 234, Fig. 148, we see a graphic as in Fig. 2.

In this picture, we observe that the $x$, $y$ axis are inverted, as it was usual in the mathematical works of this period. The figure explains the first change of variables

$$V.C.1 \quad x = \frac{z^2}{b} \tag{2}$$

that transforms the Eq. (1) in

$$b^4 = z^2y^2 + z^2b^2 \tag{3}$$

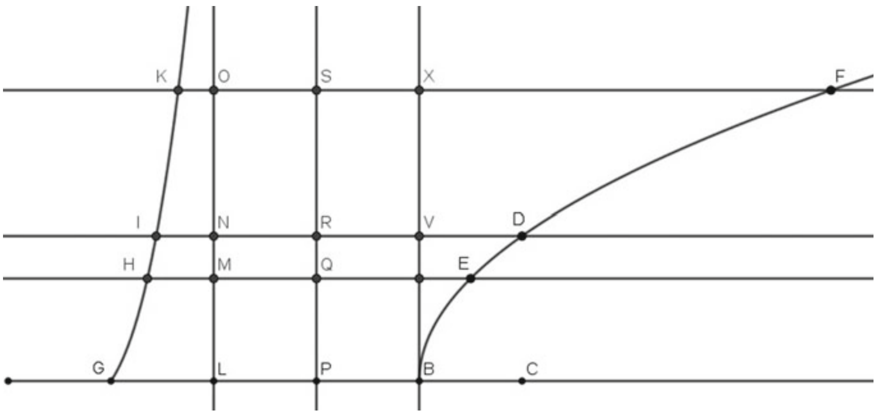


**Fig. 2** Graphic in the Fermat's Treatise

After, Fermat uses the second change of variables

$$V.C.2 \quad y = \frac{ub}{z} \tag{4}$$

that transforms the Eq. (3) in

$$b^2 = u^2 = z^2 \tag{5}$$

Now Fermat claims that he has reduced the problem of the quadrature of the cubic curve to the quadrature of the circle.

From modern point of view, Fermat obtains the result by means two integration by parts

$$\int_0^\infty xdy = V.C.1 = \frac{1}{b}\int_0^\infty z^2 dy = G.R. = \frac{2}{b}\int_0^b yzdz = V.C.2$$
$$= 2\int_0^b udz = 2\int_0^b \sqrt{b^2 - z^2}dz = \frac{\pi}{2}b^2 \tag{6}$$

In formula (6), *G.R.* (General Rule) is a rule not proved by Fermat. This property is due to Blaise Pascal in Traité des trilignes rectangles et de leurs onglets [13].

## 4 The Witch in Guido Grandi

The name *versiera* appears for the first time in the *Notes to the Galileo Treatise on the naturally accelerated motion* of Grandi [7], where we read that *versiera*, in Latin versoria, derives from the words *sinus versus*, and that the curve itself was obtained for the first time by Grandi in the work entitled *Quadratura Circuli et Hyperbolae* [6].

The attribution to Grandi is confirmed by the passage of the *Exercitatio geometrica in qua agitur de dimensione omnium conicarum sectionum, curvae parabolicae* by Lorenzo Lorenzini, a disciple of Viviani, in which he says

> … sit pro exemplo curve illa, quam Doctissimus magnusque Surveyor Guido Grandus versoriam nominat, quamque describit in suis quadraturis prop. IV et coroll. VI prop III in notis ad Galilei librum de motu naturaliter accelerated … [9]

Giovanni Vacca and Gino Loria also confirmed the attribution [10, pp 93–99, 16].

In his book *Quadratura Circuli et Hyperbolae*, Grandi gave the name *Scala*, the scale curve, to the locus that is associated with Maria Gaetana Agnesi. He justified the name *Scala* because this curve can serve as a measure of light intensity, having in

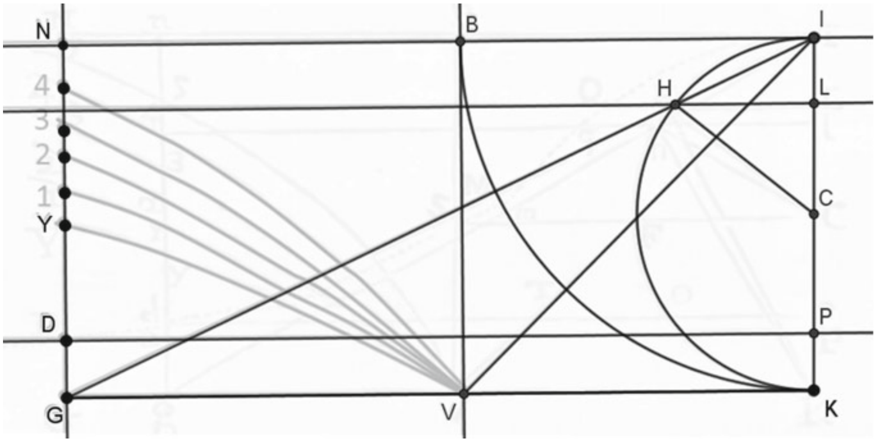


**Fig. 3** Grandi's Proposition III

mind the properties of the light's rays, as exposed in the *Opticae*, Liber III by Claude Francois Milliet Dechales.

Grandi gave the first definition of the locus in Proposition III (Fig. 3).

> Given a semicircle of diameter IK, the tangent KG and IG intersecting the periphery at H. this determines the sine HL of the angle HCL. Let $(GK)^2$ be to $(KI)^2$ as the diameter is to YN and this to 1N. In this way is had the infinity of terms 2N, 3N, 4N, etc. I affirm that the sum of all the differences of these terms taken alternately Y1, 23, 45, etc. equals the versed sine IL of the intercepted arc IH.

Then in Proposition IV, Grandi derives the Cartesian equation of the *Scala*. Taking angle ICH = φ, KI = a, KG = x, with GD = LI = y, he proves that

$$IL = \frac{a}{2}(1 - \cos\varphi) = \frac{a}{2} vers\varphi = \frac{a^3}{x^2 + a^2}$$

The use of the versine or versed sine is important because the versine was considered one of the most important trigonometric functions, above all because in the calculations, its logarithm could always be calculated and, in absence of modern tools of calculation, logarithm simplified the operations by means of the logarithmic tables.

The versine appears as an intermediate step in the application of the half-angle formula, derived by Ptolemy, that was used to construct such tables. Tables of the values of the versine were available and, in particular, the use of the versine was very important for the calculation of the distances between points on the surface of the Earth, calculation based on the knowledge of latitude and longitude by the formula called formula of the haversine.

In 1718, Grandi returns to this curve, now as

> scale of velocities … that curve which I describe in my book of quadratures, proposition 4, derived from the versed sine, which I am wont to call the Versiera but in Latin (is) Versoria.

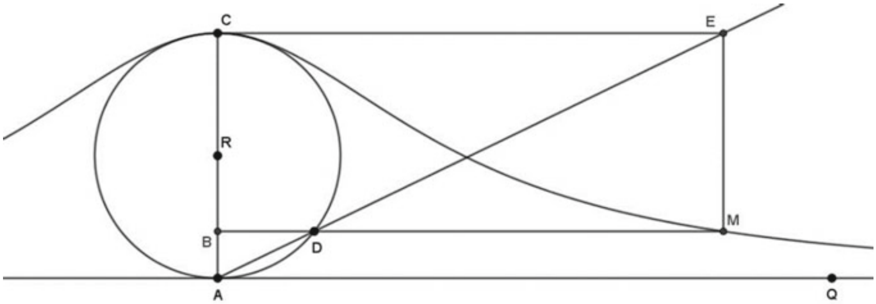


**Fig. 4** Agnesi's Versiera

In this *Note*, to show the usefulness that can be derived from its curve in mechanics, Grandi gives a second definition of the curve which is the same that will give, about thirty years later, Maria Gaetana Agnesi.

## 5 The Witch in Instituzioni Analitiche

Versiera appears in *Analytical Institutions* twice the Tomo I (Chapter V, Problem III, n. 238 and Example III) and twice in Tomo II (Book II, Chapter IV, Example II, n. 100 and Book III, Chapter I, n. 26) [1] (Fig. 4).

Description of the curve: *The secant AE trough a selected point A of the fixed circle cuts the circle in D. DM is drawn perpendicular to the diameter AC, EM parallel to it. The path of the point M is the versiera.*

In Tomo I, Maria Gaetana finds the Cartesian equation of the curve and the drawing method. In Tomo II, she finds the vertex, asymptotic line, flex points via differential method and antiderivative or primitive integral, mixing geometrical and differential methods. Maria Gaetana cannot calculate he area between the versiera and the asymptote, via the differential method that we know as second fundamental theorem of calculus, that states that the integral of a function $f$ over some interval can be computed by using any one of its infinitely many antiderivatives. Only in 1825, Augustin-Louis Cauchy started the project of introducing the concept of function and limit into the infinitesimal analysis.

## 6 The Fate of the Witch

It is curious that Maria Gaetana is remembered for her versiera, one of the many curves present in her book, rather than for the importance of her entire work.

Further evidence that Italian mathematicians do not think of the versiera of Agnesi as a "witch" is furnished by a curious slip in G. Peano's *Applicazioni geometriche* *del calcolo infinitesimale* [14], where he notes that the locus of a particular equation is a curve "called the visiera of Agnesi." He give the equation of this curve in polar coordinates. But this would seem to be a double slip of memory on Peano's part, since this curve is not the versiera of Agnesi! Gino Loria concludes [10]: "It is thus a curve quite distinct from the versiera and so may keep the name visiera given it by Peano."

A scaled version of the curve is the probability density function of the Cauchy distribution. The Cauchy distribution has a peaked distribution visually resembling the normal distribution, but its heavy tails prevent it from having an expected value by the usual definitions, despite its symmetry. In terms of the witch itself, this means that the coordinate of the centroid of the region between the curve and its asymptotic line is not well defined, despite this region's symmetry and finite area.

The versiera finds many applications in physics, especially in resonance phenomena. An example is the direct monochromatic light that strikes an atom: the intensity of the radiation emitted by the atom has the shape of a versiera as a function of the difference in frequency (between the external one and the resonance one).

The witch of Agnesi approximates the spectral energy distribution of spectral lines, particularly X-ray lines. We can find other applications in electrical circuits and in fluid dynamics. The cross-section of a smooth hill has a similar shape to the witch. Curves with this shape have been used as the generic topographic obstacle in a flow in mathematical modelling. Solitary waves in deep water can also take this shape.

## 7 Conclusions

In the annals of mathematics, the Milanese mathematician Maria Gaetana Agnesi occupies a peculiar niche. Most popularly and erroneously, Agnesi is known as the woman who discovered a cubic curve that the English mathematician John Colson, called the "witch," leading to its modern description as the witch of Agnesi.

The mathematical community only gradually acknowledged her withdrawal from the scientific community. Political economists, reformers, and philosophes at the vanguard of the Milanese Enlightenment, perceived Agnesi's humble existence to be a defect of his own society rather than a personal and voluntary act of charity.

Gradually her name was forgotten, although the great mathematician Vincenzo Riccati saw that his *Analytical Institutions* was continuing the tradition of Italian mathematics textbooks, in which Maria Gaetana played a signal role. In the introduction to the first volume of his textbook, Riccati let his readers know that he was advancing a project begun by Agnesi by explicitly comparing the content of the two works. He reminded readers of the significance of Agnesi's accomplishment in writing a clear book on a difficult subject for the "necessity and utility for Italian youth. Nowadays, the women's contribution to advance of Mathematics is universally recognized and many scholars brought back to Maria Gaetana Agnesi her right place in the History of Mathematics [2].

## References


1. Agnesi MG (1748) Instituzioni Analitiche. Tomo I, Tomo II, Milano. https://archive.org/details/bub_gb_xDF_ksE24HUC; https://archive.org/details/BUSA298_184
2. Bagni GT (1996) La didattica dell'Analisi Matematica nel Settecento: le Institutiones Analyticae di V. Riccati e G. Saladini. http://www.syllogismos.it/history/Riccati-Saladini.pdf
3. Fermat P (1891) Oeuvres de Fermat. Paris, Tome I, pp 279–280. https://gallica.bnf.fr/ark:/12148/bpt6k6213144d.texteImage
4. Fermat P (1896) Oeuvres de Fermat. Paris, Tome III, pp 233–234. https://gallica.bnf.fr/ark:/12148/bpt6k62145354/f11.image
5. Findlen P (2011) Calculation of faith: mathematics, philosophy and sanctity in 18-th century in Italy (new work on Maria Gaetana Agnesi. Historia Mathematica 38:248–291
6. Grandi G (1703) Quadratura Circuli et Hyperbolae, Propositio III, IV. Pisa. https://www.e-rara.ch/zut/content/titleinfo/1338091
7. Grandi G (1718) Note al Trattato del galileo del moto naturalmente accelerato. In: Opere di Galileo Galilei. Firenze, Tomo III, p 393. https://www.e-rara.ch/zut/content/structure/2501380
8. Huygens C (1905) Oeuvres Complètes de Christiaan Huigens. La Haye, Tome X, pp 364–373. https://gallica.bnf.fr/ark:/12148/bpt6k77858m/f1.image
9. Lorenzini L (1721) Exercitatio geometrica in qua agitur de dimensione omnium conicarum sectionum, curvae parabolicae. Firenze. https://archive.org/details/bub_gb_G27cugxjkyEC/page/n4
10. Loria G (1930) Curve speciali algebriche e trascendenti, vol 1. Ulrico Hoepli, Milano (cap. XII)
11. Mazzotti M (2007) The world of Maria Gaetana Agnesi, mathematician of god. The Johns Hopkins University Press, Baltimore
12. Paradís J, Pla J, Viader P (2008) Fermat's method of quadrature. Revue d'Histoire des Mathématiques 14(1):5–51, MR 2493381
13. Pascal B (1819) Traité des trilignes rectangles et de leurs onglets In: Oeuvres completes de Blaise Pascal. Physique et Mathématiques. Digital edition. https://gallica.bnf.fr/ark:/12148/bpt6k6459589t/f400.item.r=Trait%C3%A9%20de%20trilignes%20rectangles%20et%20leur%20onglets
14. Peano G (1887) Applicazioni Geometriche del Calcolo Infinitesimale. Bocca, Torino
15. Stigler SM (1974) Studies in the history of probability and statistics. XXXIII. Cauchy and the witch of Agnesi: an historical note on the Cauchy distribution. Biometrika 61(2):375–380. MR 0370838
16. Vacca G (1901) Sulla versiera. Bollettino di bibliografia e storia delle scienze matematiche, Torino 4:33–34

# The Dividing of the Sphere in Domes of Medieval Anatolia

**Sibel Yasemin Özgan and Mine Özkar**

The stylistic language of art and architecture in medieval Anatolia largely consists of geometric features with various levels of mathematical complexity. Whereas the two-dimensional graphic designs employ certain geometric relations and rules, their making, in three-dimensional space, relies on the spatial material qualities and the overall architectural form more than just visual transformations. For understanding how their architectonic harmony was implemented, it is crucial to consider not only the geometric design but also other parameters such as the surface geometry, the physical properties of the material, and the crafting technique. Under the patronage of Seljuks in Anatolia, the rigorous application of the decoration program on historical buildings manifests a collaboration coordinated by a master builder between mathematicians, designers, and craftsmen. Geometric patterns were applied to all kinds of building surfaces. Dome decorations particularly addressed challenges of building with spherical geometry. We investigate the historical ways to construct continuous patterns on dome surfaces and how each simultaneously handles aspects of geometrical calculation, the design, and construction processes.

## 1 Introduction

Medieval Anatolia was ruled by the Seljuks up to the fourteenth, and in some parts to the early fifteenth century. The Seljuks, named after their dynastic ancestor- a chief named Seljuk b. Duqaq, were initially a Turkish nomadic group living in Eurasian

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steppes [27, p. 1]. They founded an empire that at its height expanded its realm from the borders of modern western China to the eastern Mediterranean between the eleventh and fourteenth centuries. The Seljuks eventually reached Anatolian territories and inhabited these new lands [11, pp. 1–4].

Artistic and architectural traditions of Anatolian Seljuks were influenced partially by their predecessors. The Seljuks in Anatolia created their own design conventions and built a unique and multifaceted artistic culture [21, 22, 28, 32, 33]. The remnants of Seljuk architecture show exceptional building characteristic and spatial proportions and continue to be sources of inspiration. Yet, these monuments are also accredited for another distinctive feature, i.e. the decorations that adorn the buildings and artworks. In addition to the calligraphic art and arabesques (floral patterns), geometric patterns that consist of polygons, stars and lines seen in medieval Islamic Art were inseparable features of Seljuk art and architecture. The lace-like manifestation and geometrical rules beneath patterns, especially those that usually adorn planar architectural surfaces acquired much attention [8–10, 12, 15–18, 20, 37, 38]. Differently, this text concentrates on geometric configurations that are implemented on the interior surfaces of a few domes of the said period and geography.

## 2 Handling the Hemispherical Surface

The tradition of decorating the inner surfaces of dome structures existed in monuments of the predecessors of the Anatolian Seljuks. Some of the earlier examples constructed during the Great Seljuk dynasty in Iran reveal that master builders already handled the spherical surface as early as the 11th century. The practice of patterned domes was performed by many artisans of different dynasties and geographies. Many craftsmen pioneered in the application of geometric designs on interior surfaces of various domes. The Zangids and Ayyubids in Syria, the Nasrid and Christian Mude´jar artists in Spain, the Mamluks in Egypt, the Muzaffarids and Timurids in Persia and Central Asia, and the Mughals in India built complex examples of dome surfaces [7, p. 98].

The greater part of the ornamented dome examples in Anatolian Seljuks followed the Persian tradition of brick ornamentation. In these examples, the pattern designs are created through different arrangements of interlocking bricks, some of which are also glazed (in turquoise or black color). The Anatolian Seljuk brick domes expose either a rosette that whirls to match a spherical surface or an arrangement of rotating bricks. Peker [28–31, 33] confirms that cosmological meanings were attributed to Anatolian Seljuk architecture and the dome presented the celestial sphere—“a gate of earth and sky” [33, p. 80]. Baer [2, pp. 99–103] suggests that the ornaments in the dome convey two ideas: the first one alludes to the stellar firmament that creates a presence of luminary bodies, while the other one presents the revolving movement of the world and the constant transformation between day and night (Fig. 1).

Spherical geometry studies “figures on the surface of a sphere” [39, 40]. Basic concepts of plane geometry such as points, angles or lines still exist on the three-

**Fig. 1** Patterned brick domes *Left*: Sahib Ata Hankah, Konya (1283–93) *Right*: A vaulted room from Karatay Madrasa, Konya (1251–1253)

dimensional geometry of the sphere. Yet, the analogues of the straight lines on a plane are the great circles that are the intersections of the spherical surface with any plane that goes through its center. Mathematical geography and astronomy make the most use of this geometry in the medieval Islamic world [6, p. 157]. Learning from these fields, it is possible to approach the making process of a medieval ornamented dome more systematically. The design process can be divided into two main steps: first, the evaluation and accurate partition of the surface and second, applying a fitting geometric design based on the surface partition.

The handling of inner or outer facades of a dome starts initially with a useful division of the hemispherical surface. Literature [8, 37, pp. 52–53] suggests that Medieval artisans developed two alternative methods for the spherical surface division: The first method is based on tessellating *vertical segments on a spherical surface.* The second method is based on the construction of a *spherical polyhedron in order to tile the surface with polygonal panels.* We observe the possibility of another method that divides the hemispherical surface for decorating it. This type engages *latitudes and longitudes*, and is based on the partitioning of the sphere into circles.

While it is possible to geometrically divide dome surfaces with these methods, construction techniques require additional understanding of the geometry dependent on the material at hand. In our study, we are addressing various matters that are subject to the building process.

## 3 Designs Based on the System of Latitudes and Longitudes

The brick dome of the Ulu Mosque of Malatya, built in 1247, has a design that is based on the division of the hemispherical surface to circles (Fig. 2). The lower structure of the building is constructed out of cut stone while the upper structures are in brick. The craft that draws attention to the use of bricks in the dome is known as "the naked brick style" or "exposed brick". In this technique, bricks are laid both for load bearing construction and as ornaments on these. Decorations are created using

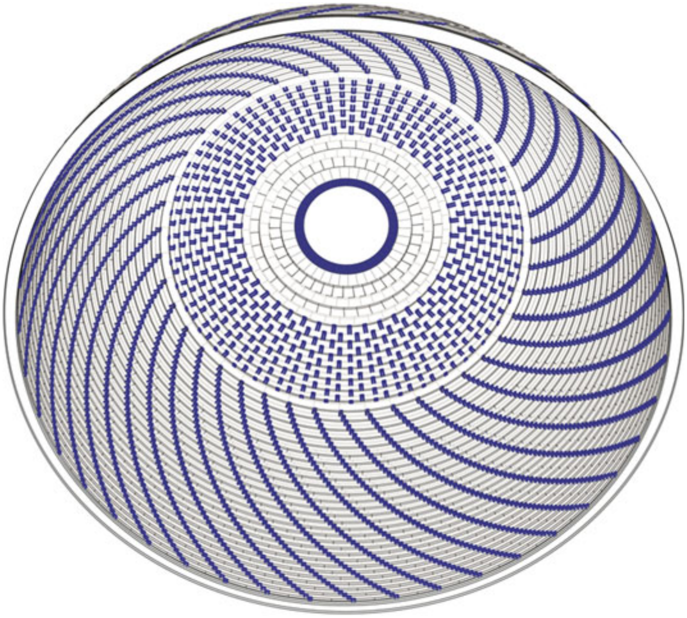

**Fig. 2** Approximated model of the dome in Ulu Mosque in Malatya (1247) as seen from below

plain and glazed bricks alternatingly and the mortar that controls the background [3–5].

The rightful implementation of the pattern in the Ulu Mosque in Malatya is handled by the division of the hemisphere into 76 latitudes (horizontal layers). Starting from the bottom, the first two latitudes are constructed only as rows of plainly lain bricks. Next are the fifty-five rows that comprise glazed as well as plain bricks to create a swirling effect that brings to light some imagined spirals on the hemispherical surface. This visual effect is realized by the geometric brickwork which places the smaller turquoise glazed plug bricks in between the plain bricks. The turquoise pieces are on the intersections of imaginary longitudes with the latitudes and for the spherical spirals. A spherical spiral, more commonly known as a rhumb, or a loxodrome, is "*a line on a sphere of constant bearing that cuts across all meridians at any constant angle except a right one*" [35, pp. 91–92]. There are 41 of these rhumb lines that the turquoise bricks form. One latitude of plain bricks separates the layers with the whirling effect from the upper structure, the thirteen rows of which are decorated with alternating plain bricks and turquoise plug bricks. An accurate material implementation of the whirling effect alone requires a precise calculation. Sizes and the number of plain bricks between two glazed bricks vary as the pattern moves up the dome. It would have been necessary to precalculate these before the construction to achieve an uninterrupted pattern (Fig. 3).

A comparison with similar brickwork from the Seyid Mahmud Hayrani Turba in Akşehir (1268–1269) illustrates how the pattern is craftily altered when applied on a

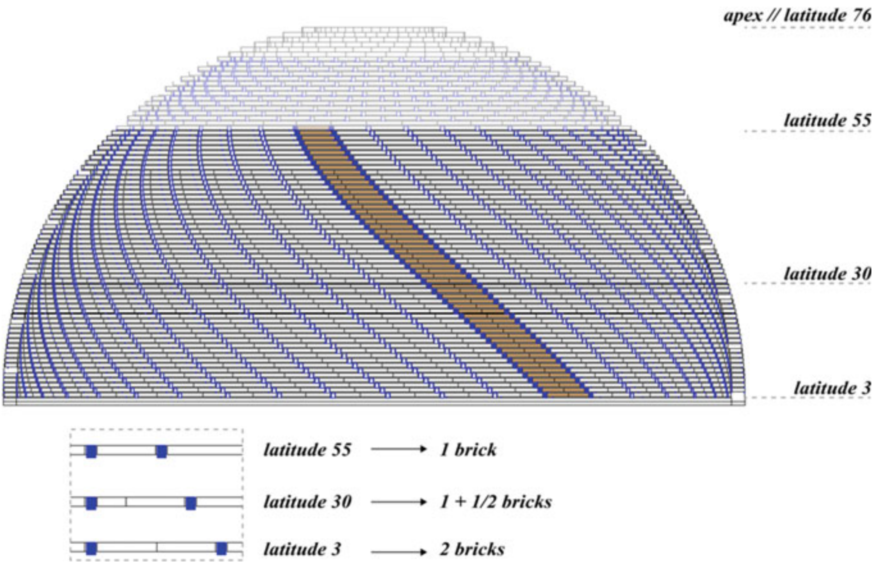


**Fig. 3** Both the quantity and the dimensions of the bricks between plug bricks change to achieve precision in the whirling pattern in Ulu Mosque in Malatya (1247)

dome and on a cylindrical surface (Fig. 4). The number of the bricks in between the turquoise plug bricks never changes on different levels of the single-curved surface. Differently in the dome, both the sizes and the numbers of the brick sizes change gradually in accordance with the tapering surface. According to Bakırer's [4] detailed survey, while the number of the bricks between turquoise plug bricks are 2 at the very bottom, this number changes to 1 + ½ bricks in the middle area and to 1 in the upper parts (Fig. 3). The resulting effect is acquired through this calculation of the brick sizes and numbers.

Other designs exist where parts of rhumb lines contribute to the pattern on the dome. The design in the Sahib Ata Hankah in Konya built in 1279–1280 has a

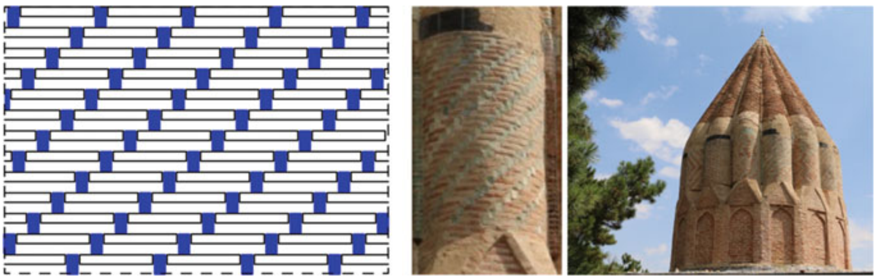

**Fig. 4** Detail from the brickwork in Seyid Mahmud Hayrani Turba in Aksehir, Turkey (1268–1269) illustrated based on Bakırer's documentation [4, f. 27]

geometrical structure as such. In this example, five left-handed and five right-handed spirals are surrounding the entire surface and the intersections of the spirals form ten pointed stars. The materialization of the dome structure is handled by a combination of horizontal and vertical brick bonds. In order to achieve the concentric execution of the ten-pointed stars, builders rotated the glazed bricks by 90° and combined these with the plain bricks that follow the latitudes (Fig. 5).

More layouts that employ partial loxodromes exist but other features come to the foreground in most of these examples. Both in Ince Minareli Madrasa and Esrefoglu Mosque, robust motifs are created out of segments of loxodromes (Fig. 6). The continuity of the spiral is not there anymore but all the motifs are in line and in scale as they followed the geometry of the loxodromes. In the pattern from the dome of the Taş (Alaaddin) Mosque from Çay, Afyon (b. 1258) diamond shape arrangements of bricks are the main motifs but the sixteen-pointed concentric stars that they embellish (Fig. 7) are similar to the layout in Malatya. Individual bricks that shape the diamond brickworks are assembled separately in wooden boxes and later applied to the surface.

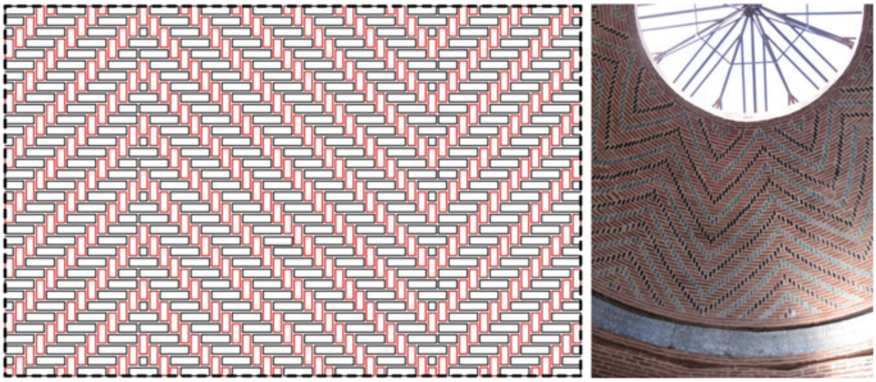

**Fig. 5** The brickwork on the dome in Sahib Ata Hankah (1279–1280) illustrated based on Bakırer's documentation [4, f. 60]

**Fig. 6** Patterned brick domes *Left*: Ince Minareli Madrasa, Konya (1264–1265) *Right*: Eşrefoglu Mosque in Beyşehir, Konya (1297–1301)

**Fig. 7** Patterned brick dome and details from the brickwork from Taş (Aladdin) Mosque in Çay, Afyon (1258)

## 4 Designs Based on Vertical Segments of the Sphere

Bonner [7, 8] identifies the vertical repetitive parts as radial gore segments and suggests that this methodology became the historically preferred system for applying both geometric and floral patterns onto domes. The vertical divisions can be processed in different numbers. Examples from the Islamic world frequently favored 8-, 12-, and occasionally 16-fold segmentation of the dome. Some rare cases show 6-fold and even 24-fold segmentation. As a rule, the divisions follow the symmetry of the supporting chamber from which the dome raises. Since most structures are based on a square floor plan, the segmentation numbers are almost always multiples of four [8, p. 531].

Vertical segmentation of the sphere is a befitting methodology for both inner and outer surfaces of diverse dome typologies. The Persian brick dome in Masjid-i Jami' of Ardistan, famous masonry domes in Cairo from the Mamluk dynasty [8, p. 533, 37, pp. 52–53] and subsequent polychromatic cut-tile mosaic examples from Persia and Central Asia such as the interior dome of the mausoleum of Turabek-Khanym in Konye-Urgench, Turkmenistan (Fig. 8) and the Safavid dome at the Aramgah-i Ni'mat Allah Vali in Mahan, Iran [8, pp. 536–538] are among the many examples based on vertical segmentation.

The vertical division of the spherical surface is also a historically documented methodology. Hankin [15] illustrated dome patterns on vertical surface segments during his visits to India. Similarly, Pope and Ackerman's [34] historical photos from the reconstruction of the Madar-ı Shah Madrasa Dome built in Isfahan Province of Iran in 1930 reveal the use of this method in similar Persian examples. To the best of our knowledge, there is not a clear application of the method in a medieval Anatolian Seljuk example. Based on the assumed knowledge of the builders of the dome geometry consisting of latitudes and longitudes as discussed in the examples of the previous section, this method is still a point of interest for research in Anatolia especially with regards to the application of patterns with materials other than brick.

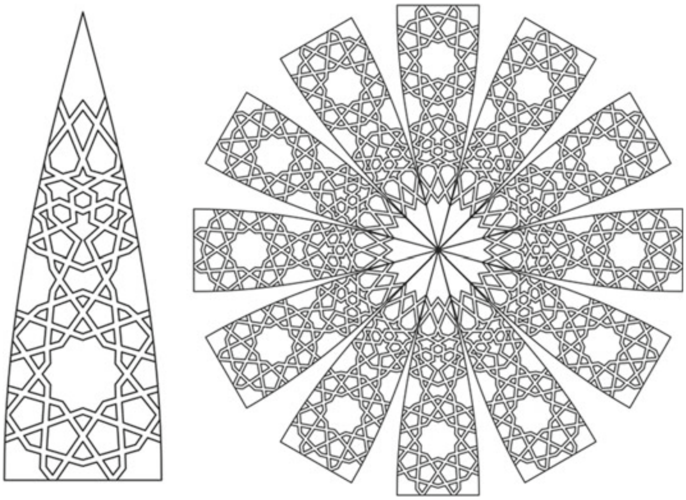

**Fig. 8** *Left* The illustration of the geometric design on a single segment of the dome surface of Turabek-Khanym in Konye-Urgench, Turkmenistan based on Bonner's analysis [8, p. 537]. *Right* The representation of the 12-fold vertical division of the dome surface of Turabek-Khanym and the matching pattern

## 5 Designs Based on Spherical Polyhedra

Medieval builders developed another approach for handling the spherical surfaces. These, described as "the most geometrically interesting and visually arresting" by Bonner [8, p. 537] employ a spherical polyhedron geometry. A spherical polyhedron is a set of arcs on the surface of a sphere that correspond to the projections of the edges of a polyhedron [39, 40].

The first apparent historical use of this method is observed in the north dome chamber of Terkan Khatun in the Friday Mosque of Isfahan, Iran (1088–1089 A.D.). The geometric pattern arrangement on the dome uses a polyhedron as the base of its repetitive schema. The brickwork is arranged in a five-fold rotational symmetry, so that the surface division clearly stands as a spherical dodecahedron. The brickwork on this prominent dome is applied as a perfect spherical dodecahedron. This has led the French archeologist and art historian Grabar [14, pp. 64–65] to assume that the renowned mathematician-astronomer Omar Khayyam, who lived in Isfahan at that time, was involved in the design. Additionally, Abu'l-Wafa' Al-Buzjani's (ca. 940–998) treatise *On the Geometric Constructions Necessary for the Artisan* (*Kitab Fī mā yaḥtaj ilayhi al-ṣāni' min al-a'māl al-handasiyya*) deals with the problem of deconstructing a spherical surface into regular spherical polygons [36, p. 175]. Buzjani's instructions for constructing convex regular and quasiregular spherical polyhedra attest to the exchange of knowledge between mathematicians and builders.

Subsequent monuments followed this tradition of pattern design founded on an underlying polyhedron [7]. Examples produced by different cultures from the

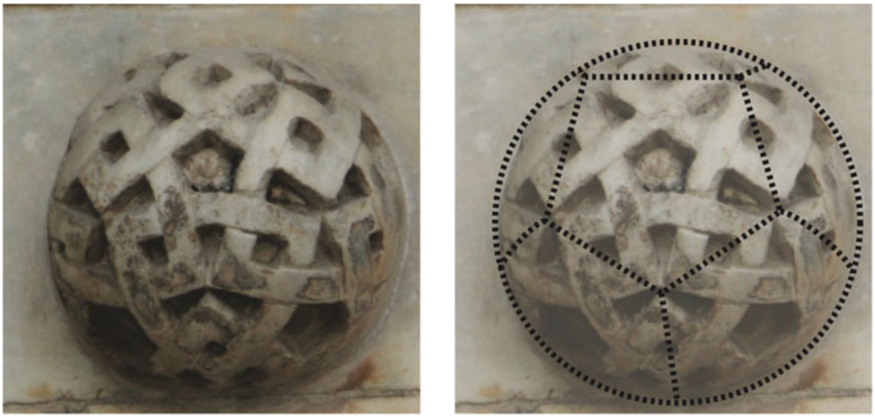

**Fig. 9** The geometric design on the hemispherical stone relief from Sahib Ata Mosque (1258) is based on a spherical dodecahedron

medieval Islamic world vary depending on the materials used. The hemisphere-shaped stone reliefs common in Anatolian Seljuk buildings are based on spherical polyhedron geometry and are little models that evidence the understanding of a sphere through a polyhedron. Figure 9 demonstrates such a geometric design that uses a spherical dodecahedron to encapsulate a repetitive motif. The hemispherical stone relief is on the entry portal of the Sahib Ata Mosque (1258) in Konya [8, p. 539].

The scale of the hemispherical stone from the Sahib Ata Mosque is much smaller compared to a dome surface, yet underlying geometrical principle for surface division is the same. Nevertheless, the making is based on stone carving, hence the craftsman responsible for the design can improvise on the pattern as he carves. On the other hand, a surface decoration based on a cladding system as with the bricks requires more accurate calculation for surface divisions. The detailing level in the making alters correspondingly as the base material change. Each material corresponds to another type of polyhedral geometry. While most of the carved or brickwork examples depend either on Platonic or Archimedean polyhedra, domes that are covered with a cladding, an additional layer of severe materials (ceramic, tile mosaic, wood parquet etc.) require more sophisticated geometries, consequently more subdivisions of the spherical surface. Existing research [13, 19] on cut-tile mosaics that adorn the interior surface of Konya Karatay Madrasa (1251–1253) suggests that intricate spherical polyhedral geometry was possibly the key feature in the application technique. Cut-tile mosaics must have been prepared as panels and applied onto the surface to complete the whole. Some of these panels are polygonal and although missing details, suggestive of the pieces of a polyhedral division of the dome. These demand further inquiry into how the polyhedral geometry played a role in the development and implementation of ornamental patterns with local materials on Anatolian domes.

## 6 Conclusion

The making of geometric patterns out of physical materials on architectural surfaces is a complicated issue and deemed as sophisticated craftsmanship. In this study, we expose how the materialization of decorations on hemispherical surfaces require further thinking on behalf of the makers in terms of relating the geometrical calculations and the craft techniques used. With reverse engineering, we investigate three different methods that were used by the medieval artisans for governing the interior surface of domes for applying adornments. Our analyses show that geometry and artistic techniques are both of great consequence for realization.

There is an acknowledged relation between the arts of building and mathematics [23–26]. Buzjani's aforementioned work *On the Geometric Constructions Necessary for the Artisan* [1] a document on geometry written especially for architect-artisans, is a key reference and evidence for the mathematician's involvement in solving problems with the geometric designs. Similarly, based on Omar Khayyam's writings, Özdural [23] reports that mathematicians and artisans gathered in meetings and discussed several design problems. Geometricians developed visual instructions to offer knowledge and strategies to artisans to simplify geometrical challenges. Even if there are not many texts that survived from the said period and geography, it is apparent that the geometrical knowledge then was not limited to plane geometry. Theorems concerning spherical geometry and spherical trigonometry were put to practice. From the various dome surface designs this study looks at, one may infer that mathematicians' involvement was constant during not only the design processes but also the construction planning and execution was constant. Practical geometry aided medieval builders to construct domes, even if there is little evidence for on-site tools and techniques they may have used to do so, and at the same time adorning their interior surfaces with meaningful and meticulously laid-out motifs that they were able to adapt to different sites.

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## References

1. Abu'l-Wafa Al-Būzjānī (ca. 940–998) Kitab Fī mā yaḥtaj ilayhi al-ṣāni' min al- a'māl al-handasiyya (On the Geometric Constructions Necessary for the Artisan). Süleymaniye Kütüphanesi (Süleymaniye Library), Ms. Ayasofya 2753, Istanbul
2. Baer E (1998) Islamic ornament. Edinburgh University Press, Edinburgh
3. Bakırer Ö (1980) A study on the use of brickbonds in Anatolian Seljuk architecture. M.E.T.U J Fac Archit 6–2:143–181
4. Bakırer Ö (1981) Selçuklu öncesi ve selçuklu dönemi anadolu mimarisinde tuğla kullanımı: Metin, vol 1. Orta Doğu Teknik Üniversitesi, Ankara
5. Bakırer Ö (2017) Reflections of the "Naked Brick Style" in Seljuk Anatolia. Selçuklu Medeniyeti Araştırmaları Dergisi [J Seljuk Civilizational Stud (Jscs)] 2:193–225

6. Berggren JL (2003) Episodes in the mathematics of medieval Islam. Springer, New York
7. Bonner J (2016) The historical significance of the geometric designs in the Northeast Dome chamber of the friday Mosque at Isfahan. Nexus Netw J 18(1):55–103. https://doi.org/10.1007/s00004-015-0275-3
8. Bonner J (2017) Islamic geometric patterns: their historical development and traditional methods of construction. Springer, New York
9. Bourgoin J (1879) Les elements de l'art arabe: le trait des entrelacs. Librairie de Firmin-Didot et Cie, Paris
10. Broug E (2008) Islamic geometric patterns. Thames & Hudson, London
11. Canby SR, Beyazit D, Rugiadi M, Peacock ACS (2016) Court and cosmos: the great age of the Seljuqs. Yale University Press
12. Critchlow K (1976) Islamic patterns: an analytical and cosmological approach. Schocken Books, New York
13. Erdemir Y (2001) Karatay medresesi çini eserleri müzesi. Konya Valiliği İl Kültür Müdürlüğü Yayınları, Konya
14. Grabar O (1990) The great Mosque of Isfahan. New York University Press, New York
15. Hankin EH (1905) On some discoveries of the methods of design employed in Mohammedan art. J Soc Arts 53:461–477
16. Hankin EH (1925) Examples of methods of drawing geometrical arabesque patterns. Math Gaz 12(176):370–373
17. Lee AJ (1987) Islamic star patterns. Muqarnas 4:182–197
18. Makovicky E (2016) Symmetry: through the eyes of old masters. De Gruyter, Berlin, Boston
19. Mülayim S (1981) Konya Karatay Medresesi'nin Ana Kubbe Geometrik Bezemesi. Sanat Tarihi Yıllığı, XI, pp 111–132
20. Mülayim S (1982) Anadolu Türk mimarisinde geometrik süslemeler: Selçuklu Çağı. Hacettepe Sosyal ve İdari Bilimler Fakültesi döner sermaye işletmesi, Ankara
21. Ögel S (1987) Anadolu Selçukluları'nın taş tezyinatı. Türk Tarih Kurumu Basımevi, Ankara
22. Ögel S (1994) Anadolu'nun Selçuklu Çehresi. Akbank Yayınları, Istanbul
23. Özdural A (1995) Omar Khayyam, mathematicians, and "conversazioni" with artisans. J Soc Archit Hist 54(1):54–71
24. Özdural A (1998) A mathematical sonata for architecture: Omar Khayyam and the Friday Mosque of Isfahan. Technol Cult 39(4):699–715
25. Özdural A (2000) Mathematics and arts: connections between theory and practice in the medieval Islamic world. Historia Mathematica 27(2):171–201. https://doi.org/10.1006/hmat.1999.2274
26. Özdural A (2010) On interlocking similar or corresponding figures and ornamental patterns of cubic equations. Muqarnas 13(1996):191–211
27. Peacock ACS (2010) Early Seljūq history: a new interpretation. Routledge
28. Peker AU (1996a) Anadolu'nun Onüçüncü Yüzyılını Anlamak. In: Semra Ögel'e Armağan. Mimarlık ve Sanat Tarihi Yazıları, pp 107–117
29. Peker AU (1996b) Anadolu Selçukluları'nın anıtsal mimarisi üzerine kozmoloji temelli bir anlam araştırması. Unpublished Ph.D. Thesis, Istanbul Technical University, Graduate School of Social Sciences, Istanbul
30. Peker AU (1998) Ortaçağ Anadolu Mimarisinde Anlam. Arkeoloji ve Sanat 20(85):29–38
31. Peker AU (2006) Evrenin Binası: Mimarîde Yazı ve Kozmolojik Anlam. In: Peker AU, Bilici K (eds) Selçuklu Uygarlığı: Sanat ve Mimarlık. Kültür ve Turizm Bakanlığı Yayınları, Ankara, pp 31–41
32. Peker AU (2008) Anadolu Selçuklu Mimarisini Etkileyen Evren ve Estetik Anlayışı. In: Esen A, Karpuz H, Eravşar O (eds) Anadolu Selçuklu Şehirleri ve Uygarlığı Sempozyumu. Selçuklu Belediyesi, Konya, pp 85–107
33. Peker AU (2009) Anadolu Selçuklu Mimari Tarihinde Anlam Araştırmaları. Türkiye Araştırmaları Literatür Dergisi 7(13):67–80
34. Pope AU, Ackerman P (1981) A survey of Persian art: from prehistoric times to the present, vol 3. Architecture (Text). SoPA, Ashiya

35. Popko ES (2012) Divided spheres: geodesics and the orderly subdivision of the sphere. CRC Press, Boca Raton
36. Scriba CJ, Schreiber P (2010) 5000 Jahre Geometrie: Geschichte, Kulturen, Menschen. Fünftausend Jahre Geometrie, 3rd edn. Springer, Berlin
37. Sutton D (2011) Faszination Orientalischer Ornamente. (trans: Goldmann C). Artemis & Winkler, Mannheim
38. Wade D (1976) Pattern in Islamic art. Overlook Press, Woodstock, New York
39. Weisstein EW (2019) Spherical geometry. From *MathWorld*, A Wolfram Web Resource. http://mathworld.wolfram.com/SphericalGeometry.html. Accessed 30 May 2019
40. Weisstein EW. (2019) Spherical polyhedron. From *MathWorld*, A Wolfram Web Resource. http://mathworld.wolfram.com/SphericalPolyhedron.html. Accessed 30 May 2019

# Organic Reference in Design. The Shape Between Invention and Imitation

**Michela Rossi**

Historical treatises pursued the search for a rule in geometric laws that envisage the relationship between numbers and forms, fixing the articulation and the measure of architecture. Despite an inevitable inertia, architectural research always showed in formal and structural canons the concepts expressed by geometry, which like any science evolves in an attempt to explain increasingly complex facts, as the man's ability to observe the nature's world progresses. New geometries coincide with new space-structural conceptions that refer to inspirational models, which are based on the commitment of nature: on one hand it asks questions to explain, on the other hand it offers solutions to design problems.

The reference to the nature is a constant explaining the strong relationship between geometry and design. It finds a scientific reason in the perfect efficiency of biological equilibria, which men imitated first in the external forms of ornament, then in the balance of structures, and finally in the orderly control of chaos.

## 1 Introduction

Many historical treatises refer to the myth that the architecture was born from the imitation of the nature.

The reference to natural models evolved together with the knowledge of physical and biological phenomena and the mathematical laws created to describe and measure them. The different interpretations of the organic model reveal the implicit relationships between the evolution of mathematics and geometry, the knowledge of the natural world and the work of man, emphasizing the relationship between the observation of nature—from which all sciences were born—the arts and the

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mathematics. With geometry, since the classical world, that was the first theoretical construction to explain natural phenomena, running together with the Western philosophical thought [1].

Euclidean geometry is the essence of Western knowledge, the primitive origin from which first developed the science of numbers and then the computer science. In facts the mathematics has evolved by looking for ways to explain new problems that accompanied the increase in observation skills, constructing numerical or geometric models capable of quantifying and reproducing natural phenomena, as well as the refinement of the ability to observe them [2]. We can trace this evolution in artificial realizations, which always follow an intentional project. This is particularly true in architecture, which more than any other art expresses in the articulation of its design the synthesis of science and technology. In human activities we recognize many models that are differently inspired by nature, mediated by the mathematics in the game of drawing with numbers and shapes [3].

The organic model in architecture evolves along with the mathematics, developing new geometries alongside the Euclidean one. The last dominated until the Baroque, which exalted the achievements of Projective geometry in the complex structures that applied the study of the conics. A non-Euclidean geometry lies at the basis of Escher's and Fuller's research. As well it is the reference to topology in the suggestions of Deconstructivist Architecture [4], up to the celebration of computer science in the dynamic reactions of last responsive structures. While new geometries explain increasingly complex phenomena, new forms shape architecture and design, which joining science and technology [5]. The concept develops the physical reference to a natural structure or phenomenon, which is copied into a non-mimetic imitation. Imitation is in fact an instinctive learning strategy and perhaps the first teaching method. Nature was the first model available, which has remained current because it is looked at with new eyes.

The artificial imitation reinterprets natural patterns and readjusts them to a different context. With the refinement of the knowledge it concerned more complex aspects, which we can articulate in five successive phases:

1. at the beginning, the focus was the *shape* with its properties resumed in the archaic articulations of the ornament [6];
2. with the birth of architecture, which reworked the formal concepts of everyday objects [7], pointed on *composition*, pursuing balance and harmony among the parts;
3. later the scientific method focused on *structures*, their mechanical behaviour and structural properties of forms [8];
4. after the explanation of the chemical-physical laws of natural forms [9] the attention felt on the *relation* in transformation (growth) and the functional properties of organs;
5. eventually the reference is the biological adaptation *processes*, associated with the topological relativity of apparently disordered forms [10].

Each approach corresponds to a different awareness level of natural phenomena, life and, as we will see, find a motivation in the explanation and in the geometric-mathematical description of the same.

## 2 The Closed Form (Unit and Decomposition)

Plato was the first who built a geometric model of knowledge. Through regular solids he referred the reality o a few physical elements, composed by only two triangles. He sensed the concept of the chemical structure of matter from the observation of regular forms in nature, such as crystals and elementary symmetries of many animal and plant organisms. Regular solids exist in the skeletons of the radiolarians and, but the dodecahedron, in the crystalline aggregates as well. In fact, the chemical structure of the molecules builds spatial lattices, which determine the regular shapes of crystals, such as snowflakes [11] as a direct consequence of the geometry of the molecular structure.

The first form of imitation was the copying, sometimes with symbolic abstraction, which implies a conceptual reference that is mediated by reinvention. Geometric motifs of natural inspiration with realistic or stylized shapes are common in ornamental decorations since pre-historic cultures. The ornament is a secondary aspect, but it is significant because the decoration is integrated into elementary shapes, which respond to the function and workmanship of the simplest artefacts, enhancing their purity through geometric references (axes, orientations, isometries…).

The two archetypes are the closed form and of the divisions of unity in regular polygons, hence cyclic or dihedral symmetrical scans. They are still the base of the design theory (basic design) and of ornament, which bases on the regularity of Euclidean symmetries [6]. In its primitive stage the imitation therefore concerned the most evident: the form and its organization, according to the fundamental references to spatial orientation through the fundamental entities of geometry.

## 3 The Harmony of the Parts (Repetition and Multiplication)

The architecture articulates in composite forms enclosing complex spaces. According to the classical theory, the drawing gives harmony to the whole by controlling the relationship in between the parts. The Alberti's principle of *concinnitas* lies in the correct proportioning of the whole, in which nothing can be added or removed unless for worse. The concept expresses the Vitruvian triad, *firmitas*, *utilitas*, *venustas* which defines the conditions for a good project. The Classicism explicitly called the nature as a model of harmony. The proportioning, regulated by shapes and numbers, refers to the divine perfection through the golden ratio. It is recurrent in the architecture as well in the living organisms. The *concinnitas* sums up the logic of form: "*Concinnitas is the fundamental and most exact law of nature. Artificial beauty must imitate the model of nature, creator of the best forms*" [12].

The imitation of nature appears in the structural conception of architecture, which is rich in aesthetic and formal values: the architectural order is the main case of theory, but it is only the most evident example since all human creations base on the observation and imitation of natural ones. As well in the ornament, the search for beauty is related to the presence of signs in harmony with the shape. The decoration of the architectural order recalls the original wooden constructive matrix: the imitation is pretty direct. It re-elaborates the form that is expressed by the Vitruvian *utilitas* in the adaptation to the stone construction.

In fact, nature works with materials of a predetermined size, like masons. The concepts of harmony, order and rule are related to the relationship between *form* and *number*, which are the recurring terms in the search for beauty through the application of universal laws, mainly the golden ratio. In the articulation of organisms, however, other concepts of geometry also appear, such as symmetry, addition, multiplication, division. The counting unity, strictly linked to number, expresses the central importance of the concept of *module*.

The formal archetypes refers to the repetitive juxtaposition of equal elements, according to the elementary symmetries in a modular lattice with rational proportions, which may grow to the infinite in the three directions of the Cartesian space.

In general terms, the form is defined by absolute or relative magnitudes in the different directions, and it depends from the chemical and physical laws of matter.

## 4 The Balance of Forces (Order and Symmetry)

Modern science is born from the systematic observation of nature after Galileo stated that the book of nature is written with the characters of Geometry: "*Philosophy is written in this grand book, the universe… It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures*" [13]. Descartes reconciled numbers and forms with the Analytical Geometry. The understanding that the scale of quantities is a discriminating factor and in nature it is not possible to increase a shape beyond any limit, keeping proportions and materials unchanged.

Construction science and mathematical calculation take the place of the geometric sizing of the structures. The existence of thresholds in a form's growth because of mechanical, technological or constructive problems, is frequent the application to architecture.

The formal archetype is the intrinsic equilibrium, with the determination of its static centre of gravity, underlined by the articulation of the architecture itself.

D'Arcy Thompson showed that the form articulation is the effect of the physical forces of the system. It is due to molecular pressure in the cell, to mechanical stress in bigger structures. The force of gravity controls the shape of the bigger organisms and the activities of superior beings, while in the field of the infinitely small it loses importance compared to the surface tension, which induces the semi-fluid bodies to assume spherical shape. The natural shape of the cell keeps the least external surface with respect to the internal volume, but the external forces induce transformations to balance the system according to the maximum efficiency, which is typical of nature. In higher organisms, the skeleton's mechanical structure performs static functions.

Galileo explained the mechanical principle of resistance by form and his observations allowed the scientific method and the science to born. Even in the eggshell, which is the model of thin-structure domes, resistance is in relation to size. The hollow shell is suitable for small animals, in the larger ones it is stiffened with ribs as in tortoises. This fact is well known to structural scientists, whose solutions often take the shape of bone skeletons, where the substance thickens in the most stressed points.

## 5 The Organism (Growth and Transformation)

The crucial fact is the transition from the inanimate world to life. The border seems to be due to the form geometry: the breaking of the symmetry as a factor of balanced stability, which implies imbalance, therefore movement or changes.

In the organic model the most important change is growth that is implicit in life.

The archetype is the spiral, an open and continuous form that associates the linear aggregation and the radial lattice, geometrically expressed by the gnomon in the golden rectangle, on which rivers of ink have been spent, above all in relation with architecture. In the spiral growth, the cells alignment takes place by moving the upper layer with respect to the lower one with a rotation with respect to the vertical axis and sometimes even with a shift with respect to the reference plane. The geometry of this form is a logarithmic function that grows without changing of shape. Each increment is similar and similarly located with respect to the previous one. The same result by adding what Aristotle call *gnomon*, which is the part you shall add for self-similarity, analogous to that of fractals.

The principle of growth is connected to the evolutionary concept of transformation, which characterizes the interpretation of the organic architecture of the Modern Movement, focused on formal and functional analogies between the architecture and living organisms. The growth process demonstrates the importance of the system balances and it was the basis of the brilliant intuitions of R. Buckminster Fuller and of P. Frei Otto. The first with geodesic domes, the other with tensile structures, they pursued the maximum efficiency through the imitation of organic model.

Fuller's domes are light structures with a regular organization. This is inspired by the shapes of radiolarians, supported by an exoskeleton that distributes the forces over the cell surface. Their shape is the goal of a research, which started from the study of the balance of forces and from the properties of regular polyhedra [14]. That allowed him to transpose the principle of cell balance into a greater dimension. The solution derives from regular geodetic divisions of the sphere surface, inspired by geographical projections and the microcosm of single-celled individuals. Fuller himself points out that his experiments investigate the balance of nature, trying to learn its secrets.

## 6 The Dynamic System (Transformation and Responsiveness)

The identification of the mathematical relationships linking form and growth explains the formal evolution. D'Arcy W. Thompson explains that biological processes manifest asymmetries in the balance of system forces, with lines of less resistance along which the development is faster.

The forces that alters the growth in the different parts of the same organism, determine diversifications in the forms. In the small the balance responds to the surface tension on the cell's membrane, in the great to the force of gravity. Asymmetry is the main difference between vital and non-vital phenomena. Pasteur stated that the production of exclusively asymmetric compounds is a life's prerogative. It manifests itself in ordered organisms, in which symmetry reappears as a factor linked to cellular multiplication. Cells' proliferation means controlled growth, which is geometrically organized but not uniform and has some stopping points. The shape therefore adapts to external conditions.

Organic growth means a change of dimensions, sometimes even of proportions and shape. In organic tissues the increase occurs through repeated cellular multiplication, with the tendency to be placed in linear series. Histological aggregates present repetitive formations of similar cells. The arrangement is linked to the optimization filling the space and induces adaptations of the tendential sphericity of the cell into a squared or a hexagonal lattice (cubic or tetrahedral in 3D).

Variants and invariants between species have been explained with deformed Cartesian diagrams that highlight formal homologies also in species that are not close in their evolution. In morphologically similar individuals, the deformation of a two-directions grid that connect corresponding points, measures in its deformation the adaptation of the same organization to different growing conditions. Lines assume different curvatures maintaining the same topology in their functional layout.

Contemporary evolution of Deconstructivist Architecture gave new vitality to organic design. Today visual computation allows the imitation of life-responsive processes for the computerized search for optimized and therefore environmentally sustainable solutions. New organic forms derive from the simulation of complex and apparently disordered phenomena, regulated by recursive algorithms, which underline the maturity of computation in design: '*Computers outgrown their servile function in the digital drawing room, where the real design was still done far away from the machines, sketched by hand, guided by genius…*' [10].

## 7 Conclusions

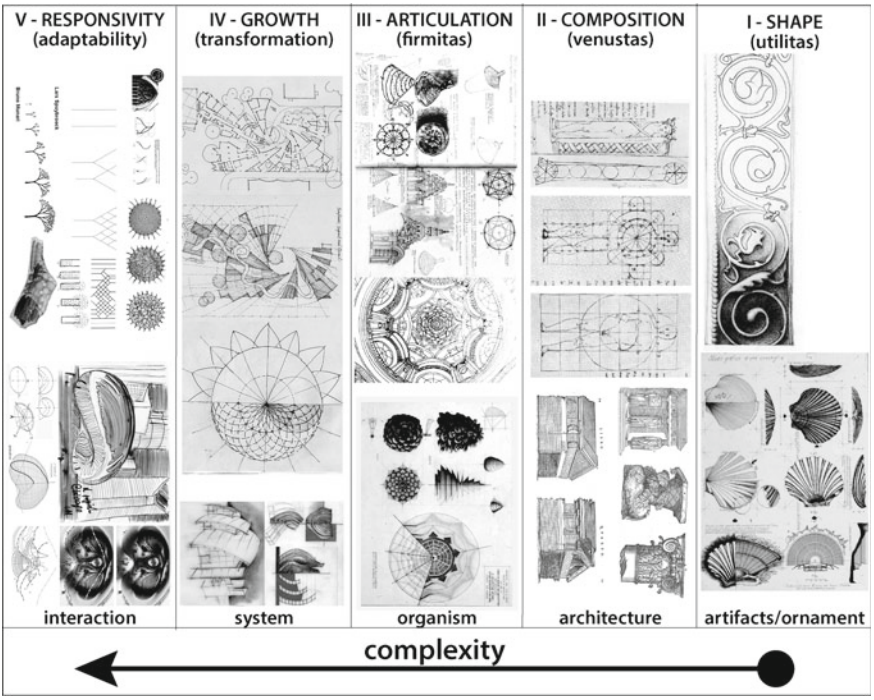
The principle of nature as the main inspiration of the project accompanies the theory of architecture from its origin. The digital computation granted the success of Organic Design as a sustainable approach inspired by natural organisms.

Today the imitation of the nature pursues the optimization in the search for the best solution in the project. Nature, which the ancients considered perfect in its things, pursues the maximum efficiency of its systems with a continuous evolutionary process and therefore remains the best model.

In the evolution of the imitative concept of nature we recognize 5 steps. Each new approach keeps the previous principles by adding new design references in a more sophisticated interpretation, from the exterior appearance to the adaptation processes:

1. the essential geometry and shape references;
2. the composition and the parts sizing;
3. the structural functionality in the relationships pattern;
4. the growth of the organism and the law of transformation;
5. the responsive system in the relationship between cause and effect.

Each of these phases exemplifies a step in the development of a computational model that optimizes the variables of a problem, solving it through recursive algorithms, scripting the process code. Taken together, they show the design intent that governs the concept layout of artefacts.




Organic references in contemporary architecture, form and geometry. Students' drawings

## References


1. Rossi M (2006) Natural architecture and constructed forms: structure and surfaces from idea to drawing. Nexus Netw J 8(1):112–122
2. Penrose R (2004) The road to reality. A complete guide to the laws of the universe. Jonathan Cape, London
3. Ferrero G, Cotti C, Rossi M, Tedeschi C (2009) Geometries of the imaginary space: architectural developments of ideas of M.C. Escher and Buckminster Fuller. Nexus Netw J 11(2):305–316
4. Wigley M (1993) The architecture of deconstruction: Derrida's Haunt. The MIT Press, Cambridge
5. Rossi M (2018) On rules and roots: the organic reference in design. In: Rossi M, Buratti G (eds) Computational morphologies. Design rules between organic models and responsive architecture. Springer, Cham, pp 23–38
6. Gombrich E (1979) The sense of order. A study in the psychology of decorative art. Phaidon, Oxford
7. Semper G (1860–1863) Der Stil in den technischen und tektonischen Künsten. München
8. Galileo G (1632) Dialogo sopra i due massimi sistemi del mondo. Giovan Batista Landini, Firenze
9. D'Arcy Thompson W (1917) On growth and form. Cambridge University Press, Cambridge
10. Spuybroek L (2004) Nox. Machining architecture. Thames & Hudson, London
11. Kepler J (1619) Harmonices Mundi. Linz

12. Alberti LB (1485) De re aedificatoria. Nicolai Laurentii Alamani, Florentia
13. Drake S (1953) Dialogue concerning the two chief world system (trans: Drake S). University of California Press, Berkeley
14. Fuller RB (1975) Synergetics. Explorations in the geometry of thinking. Macmillan Publishing, London

# Inverse Formulas: From Elementary Geometry to Differential Calculus

**Anna Salvadori and Primo Brandi**

## 1 Introduction

The aim of our contribution is to give a didactical perspective to the subject of the meeting. In particular I would like to illustrate a didactic path that starting from the inverse formulas of elementary geometry, in continuity through school of different degree, reaches the differential equations.

Before starting I would give my personal tribute to Leonardo Da Vinci (1452–1519), in his five hundredth centenary. In "Trattato della pittura" Leonardo writes: "*nessuna investigazione si può dimandare vera scienza, s'essa non passa per le matematiche dimostrazioni…nessuna certezza è dove non si può applicare una delle scienze matematiche*."[1]

About one century after Galileo Galilei (1564–1642) in "Saggiatore" confirms the opinion, with more strength and perhaps more awareness: "*La filosofia naturale è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi agli occhi, io dico l'universo, ma non si può intendere se prima non s'impara a intender*

[1] No investigation can be asked for true science, if it does not pass through mathematical proofs … no certainty is where one of the mathematical sciences cannot be applied.

[2] Natural philosophy is written in this very great book which is continually open to us before the eyes, I say the universe, but it cannot be understood if one does not first learn to understand the language and know the characters in which it is written. It is written in mathematical language, and the characters are triangles, circles and other geometric figures, without which means it is impossible to understand humanly words, without them it is a wandering around for a dark labyrinth.

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*la lingua e conoscer i caratteri nei quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi ed altre figure geometriche, senza i quali mezzi è impossibile a intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro labirinto*."[2]

The well known Galileo's sentence marks the origin of scientific method that places mathematics at the center of research as the language of science. Naturally, the technology that will be developed in the following, since it is based on science, equally adopts mathematics as the language.

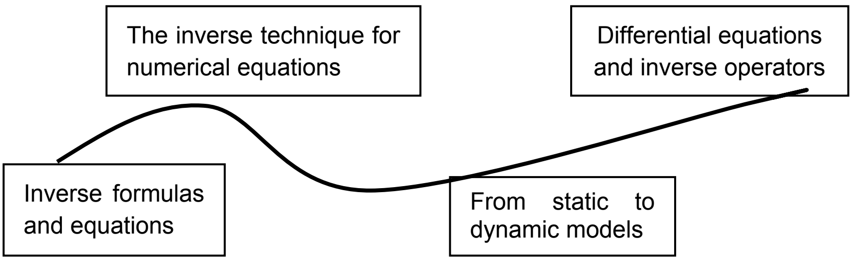
We are wondering how much of the spirit of the two great Scientists remain in the knowledge of the modern students, at any school level. Even the students of Mathematics, who will be the future teachers, often ignore this aspect of the discipline. This is one of the reason why about twenty five years ago, on the behalf of the Department of Mathematics and Informatics of the University of Perugia, we started the didactical project Mathematics&Real-life (M&R) which promotes the dynamic interaction between the real world and mathematics as startup to educational innovation (www.matematicaerealta.eu). Every year M&R organizes many activities devoted to Schools of all levels, from primary one to high school.

We do believe that the connection between mathematics and daily life could be the key to awaken the students' interest for the discipline.

Of course this new perspective requires a drastic change in the didactical organization. In particular imposes a different order of priority of the topics and, what is fundamental, highlights some guidelines that should connect the various School orders.

M&R produced Math-Maps [1–3, 6], a kind of Google map for Mathematics, where the guideline are presented and illustrated with the scientific and didactical references. One of the guideline is devoted to the theme of the equation; a fundamental topic which takes up the students for a long time.

M&R path on equations proposes four main stations, referred to different School levels. We devote a section to each station.



## 2 Inverse Formulas and Equations

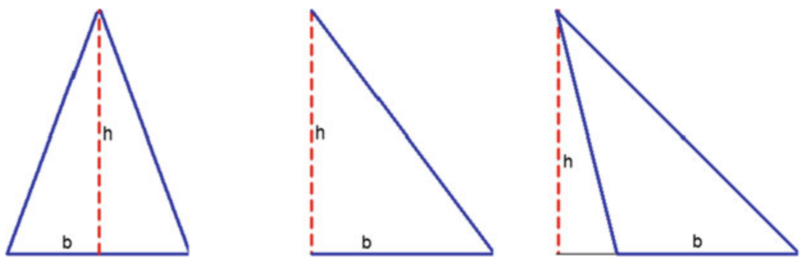
The inverse formulas of elementary geometry often constitute an hard educational "rock". The temptation for students, often endorsed by the teachers, is to memorize them as a long sequence of disconnected formulas. Thus they forget them very soon and make a big mess.

Actually they are the first case in which students face the problem of the inverse, which is not elementary at all. In this case they must reverse a formula, successively they will have to reverse functions, metrics, operators, … but often they will do it in an unconscious way. The textbooks and often also the teachers provide only operative techniques, one different from the other (depending on the contingent problem) and do not highlighting the underlying problematic which unifies all them. Thus students get the opinion that mathematics is just a sequence of ad hoc techniques, that they have to memorize.

When they face a problem the first question is: how to solve it? Instead they should ask themselves: what is the current situation? What is the goal I want to achieve? Which is my plane to get the solution?

In our experience within M&R, the inverse formulas reveals the suitable starting point for the path of equatons.

**Area of a triangle** Let's start with a very easy example.




$$A = \frac{b \cdot h}{2}$$

The formula of the area of a triangle is an *equality* that involves three elements: area, base, height. Every equality is a *balance relationship*; the two hand sides of the equality are *in equilibrium* whatever the triangle.

Assume now that we want to invert the formula with respect to the base. This means that we have to rewrite the equality in a different form. Precisely we look for an equality of the type

$$b = ???$$

where at the right hand side must compare the other variables (A and h). Of course we have to maintain the balance. How to do it?

At the Secondary School an algebraic manipulation is proposed: by virtue of the equivalence principles the equality can be re-written in different form preserving the balance. In this case by multiplying or dividing both the sides for the same factors, we get the following relations that solve the problem.

$$A = \frac{b \cdot h}{2} \quad \Leftrightarrow \quad 2A = \cancel{2}\,\frac{b \cdot h}{\cancel{2}} \quad \Leftrightarrow \quad \frac{2A}{h} = \frac{\cancel{h} \cdot b}{\cancel{h}}$$

$$b = \frac{2A}{h}$$

Actually, even if the students often don't realize it, the technique is the same adopted to solve linear equations!

In other word, to invert the formula is equivalent to look at the equality as an equation, where b is the unknown and the other variables are parameters.

This is true for every inverse formulas: we have to look at the equality as an equation and just to solve it!

M&R proposes to unify the approach of the inverse formulas with those of the equations and to insert them in a common problematic: that of the *inverse function*.

We will see it in detail for linear equations, but the process is common to all the elementary equations (see [5, Chaps. 9, 10] for the details).

## 3 The Inverse Technique for Numerical Equations

A numerical equation is a relation of the type

$$f(x) = c$$

where the function $f : Dom_f \to R$ and the number $c \in R$ are given. The resolution of the equation consists on determining the value (values) of the variable $x$, if they exist, such that the equilibrium is assured.

We can look at the equation in the following way: the unknown is "closed into the box" constituted by the function $f$. To solve the equation means to free the contents of the box without breaking it, i.e. maintaining the balance.

The solution is just to open the box! Roughly speaking, this means to apply the inverse function $f^{-1}$ to both the side of the equation:

$$x = f^{-1}[f(x)] = f^{-1}[c]$$

As we proved in [3, Chap. 9], most of the devices adopted to solve numerical equations hide this process, which plays the role of unifying approach to resolution. In force of the *inverse technique* the relationship between reverse processes and equations get deeper. Moreover the same technique can be adapted also to functional equations, in particular to the differential ones, as we will show in Sect. 5.

## *3.1 Linear Models and Equations*

Of course the inverse technique need a suitable functional approach to equations. We will present it into details for the linear equations. As it is customary for M&R we begin with real life situations.

**Fast … Like the Wind**
On 22 June 2018 (Meeting de Atletismo Madrid) Filippo Tortu just twenty years hold became the Italian record holder of the 100 m with the time of 9′9″, breaking the record of Pietro Mennea who had resisted since 1979 and becoming the first Italian in history to fall below 10′.

**As it is well known, the word record 9′58″ belongs to Usain Bolt (Berlin, 2009). How long can Tortu run in Bolt's time?**

**Bassano Bridge Needs a Helping Hand**
… and not just paint. The last structural restoration of the bridge dates back to 1966, after the ruinous flood of November 4th of that year. And it comes the day before yesterday, when the monitoring tools revealed a situation no longer sustainable: in the last three months the bridge was lowering at a speed of 3 cm a month.

Source: Famiglia Cristiana, 11 marzo 2016

The delivery report of the restoration works was signed yesterday.

Source: Il Gazzettino Vicenza-Bassano, 3 marzo 2017

**How much has the bridge been lowered while waiting for the restoration work?**

**Read Speed**
Charles Osgood, is a retired American radio and television commentator and writer.

Osgood is best known for being the host of *CBS News Sunday Morning*, a role he held for over 22 years (1994–2016). Osgood also hosted *The Osgood File*, a series of daily radio commentaries, from 1971 until December 29, 2017. Osgood claimed to take about 1 min to read 15 lines (double-spaced bars).

**How long does he take to read an A4 page?**

Three different situations a single basic concept: the speed. As it is well known, in the case of constant speed, it is the ratio between space and time. Thus we get that space and time are proportional (and speed is the constant of proportionality)

$$v = \frac{s}{t} \quad \Leftrightarrow \quad s = v \cdot t$$

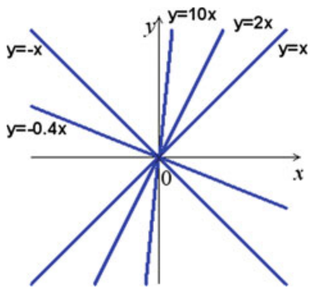
Note that the same relationship holds for the area and the base of a triangle

$$A = \frac{h}{2} \cdot b$$

where the constant of proportionality is $h/2$.

The previous four different problematic situations (one from elementary geometry, three from real life) admit one single model: the linear one

$$y = k\,x \quad k \neq 0$$




The linear model is adopted to describe all the linear phenomena that students meet in various disciplines, where the proportionality coefficient $k$ takes different names: i.e. speed, conversion factor, specific gravity, reaction speed, interest rate, discount rate, exchange rate, zoom-in or zoom-out factor.

To face the different question of the previous situations we have just to solve a linear equation of the type

$$k\,x = a.$$

In order to introduce the inverse technique for linear equation, let us introduce first the *space of linear functions*.

## 3.2 *The Space of Linear Functions*

In the numerical approach we use strongly the algebraic structure of numbers N, Z, Q (sum and product) or the analogous algebra for polynomial. We can introduce an analogous structure in the set of linear functions

$$L = \{f(x) = k\,x, \quad k \in R - \{0\}\}$$

equipped with the composition of function. In order to introduce the operation of composition between two linear function, an example form real life can be usefull.

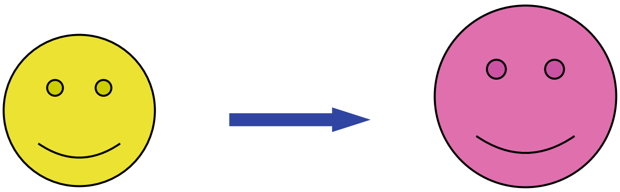
**Christmas Shopping**
A shoe seller before Christmas increases prices by 20%, after the holidays sells all the goods with a 20% discount.

**A pair of shoes with an initial cost of € 80 in January are sold for € 80**
**True ☐ False ☐**

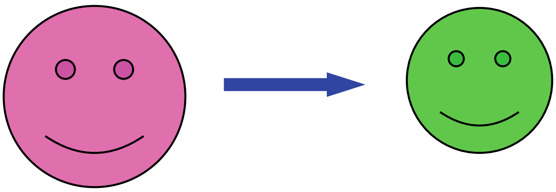
Source: Terni city competition, 2004
Let us suggest to face the subject by means of the graphic approach.

*Graphic Approach*

– from the list price to the Christmas increase (zoom-out)



– January sales on the Christmas price (zoom-in)



– comparison: list price and January price: as we can see, the starting image looks bigger that the final one.

In order to confirm or not what appears we must move from a qualitative approach to a quantitative one. Thus we can adopt a functional approach.

*Functional Approach*

The "list price" and "Christmas price" are two classes of proportional quantities, with proportionality factor 120/100 = 1.2. In other words the proportionality relation is represented by the function (see Fig. 1).

$$p_{Christmas}(x) = 1.2\ x$$

where $x$ denotes the list price.

Similarly "Christmas price" and "January price" constitute two classes of proportional quantities, with proportionality factor 80/100 = 0.8

More precisely, we have the function (see Fig. 2).

$$p_{Jauary}(y) = 0.8\ y$$

where $y$ denotes Christmas price.

If we compose the two transformations

$$x \quad \to \quad 1.2 \cdot x \quad \to \quad 0.8\,(1.2 \cdot x) = 0.8 \cdot 1.2 \cdot x = 0.96 \cdot x$$

we get the function (see Fig. 3).

$$P_{January}(x) = 0.96\ x$$

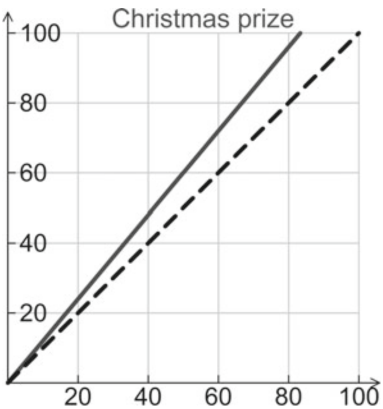



**Fig. 1** Comparison between Christmas price and list price

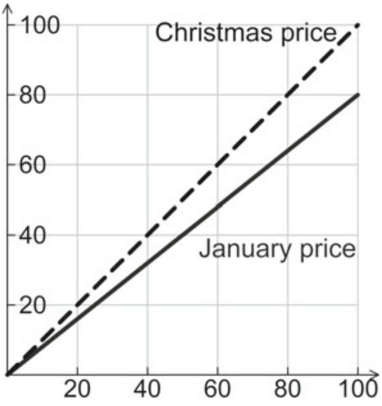



**Fig. 2** Comparison between January price and Christmas price

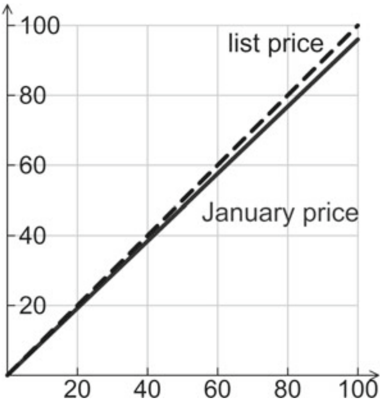



**Fig. 3** Comparison between January price and list price

where $x$ is the list price.

**The composition of two linear functions**. In the light of what we have seen in the example, we can introduce the operation of composition between two linear function. Given two linear functions $f(x) = kx$ and $g(x) = mx$, the composite is the linear function whose coefficient is given by the product of the two coefficients

$$f \circ g(x) = kmx$$

It is easy to prove that the operation is commutative and associative, moreover the neutral element is the identity function $y = x$.

Let us discuss now the existence of inverse element, again starting from an example of the real life.

**Water and Ice**
The water, freezing, increases its volume by 1/11.

**How much does the volume of ice decrease when water melts back?**

Source: Terni city competition, 2003

We can face the problem adopting the three approaches: numeric, graphic and functional.

*Numeric Approach*

Water and ice are directly proportional quantities

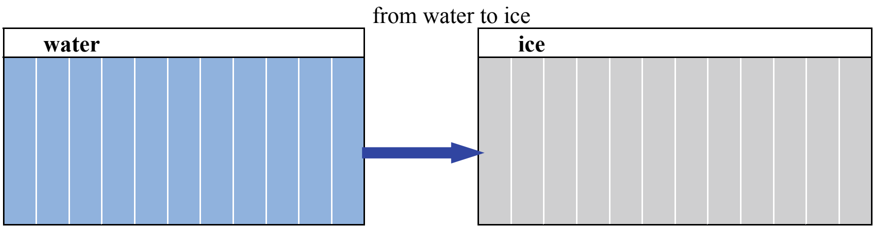
| water volume | ice volume |
|---|---|
| 1 | $1+\frac{1}{11}=\frac{12}{11}$ |
| *unknown volume* | 1 |

thus we can apply a proportion

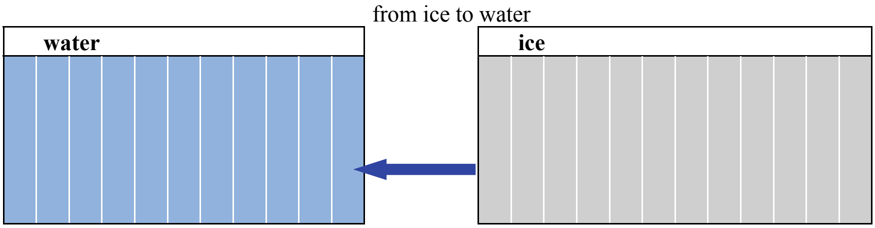
$$1 : 1 + \frac{1}{11} = unknown\ volume : 1 \quad \Rightarrow \quad unknown\ volume = \frac{11}{12} = 1 - \frac{1}{12}$$

from which the answer follows: when ice melt into water it loss 1/12 of its volume.

This approach is not easy at all for students who could prefer the *graphic approach* which translates into images what we have proved by means of the proportion.




$$1=\frac{11}{11} \quad \Rightarrow \quad 1+\frac{1}{11}=\frac{12}{11}$$




$$1-\frac{1}{12}=\frac{11}{12} \quad \Leftarrow \quad 1=\frac{12}{12}$$

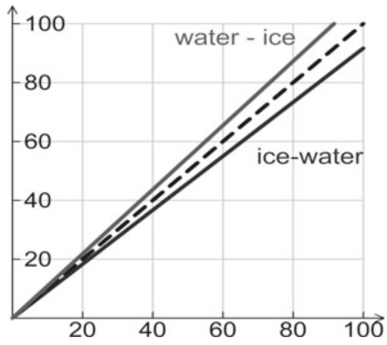



**Fig. 4** Functions water-ice and ice-water compared

We can finally confirm our intuition by means of the *functional approach* (see also Fig. 4):

$$\begin{array}{ll} function\ water-ice & function\ ice-water \\ i = i(w) = \left(1 + \frac{1}{11}\right)w = \frac{12}{11}w & w = w(i) = \frac{11}{12}i \\ i(w) = \frac{12}{11}w & w(i) = \frac{11}{12}i = \left(1 - \frac{1}{12}\right)i \end{array}$$

From this example we get the following definition.

**The inverse element**. Given a linear function $f(x) = kx \quad k \neq 0$ the inverse element (if it exists) is a function $f^*(x) = mx$ such that

$$f^* \circ f(x) = x$$

since $f^* \circ f(x) = k\, m\, x$, the coefficient $m$ must satisfy the condition

$$km = 1 \quad \Rightarrow \quad m = \frac{1}{k}.$$

Thus every linear function admits an inverse element which is the linear function whose coefficient is the reciprocal of the given coefficient

$$f^{-1}(x) = \frac{x}{k}$$

It is easy to see that the graphics of the function $f$ and its inverse $f^{-1}$ are symmetric with respect to the identity function $y = x$ (Fig. 5).

## 3.3 *The Inverse Technique for Linear Equations*

Let us consider a linear equation

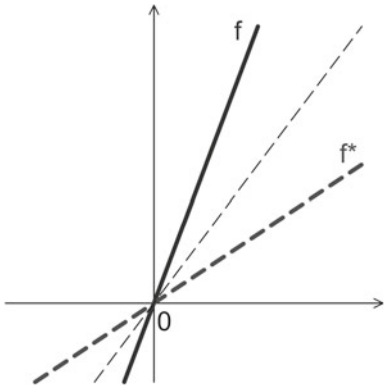



**Fig. 5** A linear function and its inverse

$$kx = a$$

Denoted by $f(x) = kx$, from Fig. 6 we can easily seen that the solution exists and is unique.

In order to get this solution we can apply to both the side of the equation the inverse function

$$x = f^{-1}[f(x)] = f^{-1}[a] = \frac{a}{k}$$

*Functional Approach*

The previous equation is the particular case of the general equation

$$f(x) = a$$

where the function $f$ is linear. The unknown *x is enclosed in a box*, thus to solve the equation we have to open the box! How can do it? Just open the box, using the *inverse*

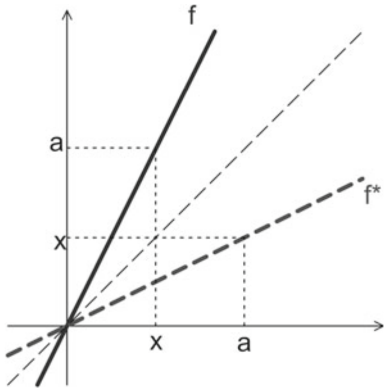



**Fig. 6** A graphic representation of inverse technique

*function*. Of course for regions of equilibrium we must apply the inverse function to both members of the equation. Formally we have

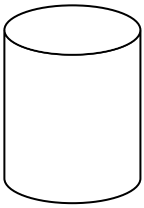
$$f(x) = a \quad \Rightarrow \quad f^{-1}[f(x)] = f^{-1}[a] \quad \Rightarrow \quad x = f^{-1}[a]$$

If we want to solve the linear equation we have to find the inverse function of the linear function involved.

## *3.4 Other Examples of Inverse Technique for Equations*

We wish to present some other examples of inverse technique applied to elementary equations and inverse formulas.

**Area of a Cylinder**



Assume we wish to invert the formula of the surface of a cylinder

$$S = h \cdot 2\pi r + \pi r^2$$

with respect to the height. The function which enclose the unknown is of the type

$$f(h) = m\,h + q$$

where $m$ and $q$ are parameters.

The function $f$ is a translated linear function, represented by a straight line (not passing from the origin). A new model is introduced to face the situation (see Fig. 7).

It is easy to see that the composition operation in the set of translated linear functions

$$L_{tr} = \{f(x) = m\,x + q, \quad m \in R - \{0\} \quad q \in R\}$$

is associative, non-commutative. The neural element is still the identity function $y = x$. Every function admits an inverse element (see Fig. 8) of the same type

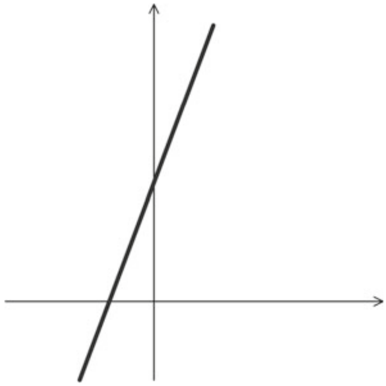


**Fig. 7** The graphic of a linear function

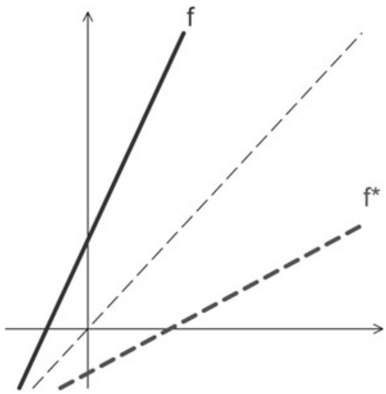



**Fig. 8** A translated linear function and its inverse

$$y = f(x) = m\,x + q \quad \Leftrightarrow \quad x = f^{-1}(y) = \frac{1}{m}y - \frac{q}{m}$$

Again the graphics of the function $f$ and its inverse $f^{-1}$ are symmetric with respect to the identity function $y = x$.

Thus also in this case we can solve the equation $f(x) = a$ by applying to both the side the inverse function (see Fig. 9).

$$f(x) = m\,x + q = a \quad \Leftrightarrow \quad x = f^{-1}(a) = \frac{a - q}{m}$$

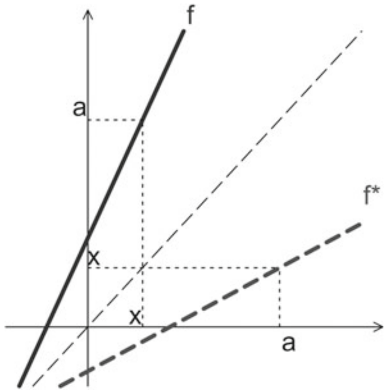
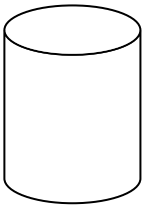



**Fig. 9** A graphic representation of inverse technique

## Volume of a Cylinder



Assume we wish to invert the formula of the volume of a cylinder

$$V = \pi r^2 h$$

with respect to the radius.

Now the unknown is enclosed into the quadratic box

$$f(r) = k r^2$$

where $k > 0$ is a parameter.

Again a new model to take into account, the quadratic one. Note that the function $f : R \to R$ defined by

$$f(x) = k x^2 \quad k \neq 0$$

is not invertible (see Fig. 10).

How can we adopt the inverse technique to solve quadratic equation?

Of course we can. We have just to put into play the partial inverse functions!

More precisely, we consider the two partial functions (Figs. 11 and 12).

**Fig. 10** Function $f$

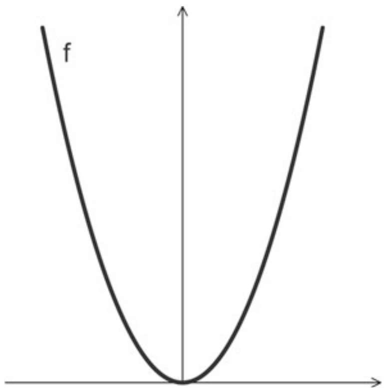



**Fig. 11** Function $f_1$

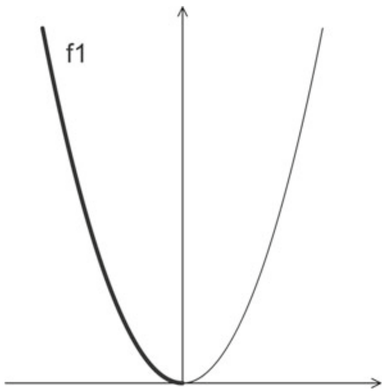
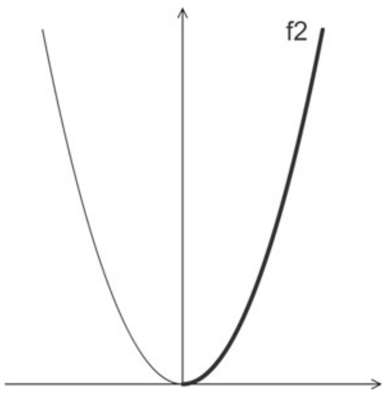



**Fig. 12** Function $f_2$



$$f_1 = f_{/R^-} \quad \text{and} \quad f_2 = f_{/R_0^+}$$

The two function $f_1$ and $f_2$ are invertible and the inverse functions represents two partial inverse of the quadratic function $f$; they are respectively (Figs. 13 and 14).

$$x = f_1^{-1}(y) = -\sqrt{\frac{y}{k}} \qquad x = f_2^{-1}(y) = \sqrt{\frac{y}{k}}.$$

Thus to solve the equation

$$f(r) = k\,r^2 = a \quad a > 0$$

by applying he inverse technique, we have to consider two "partial" equations

$$\begin{cases} k\,r^2 = a \\ r < 0 \end{cases} \quad \vee \quad \begin{cases} k\,r^2 = a \\ r \geq 0 \end{cases}$$

**Fig. 13** Function $f_1$ and its inverse

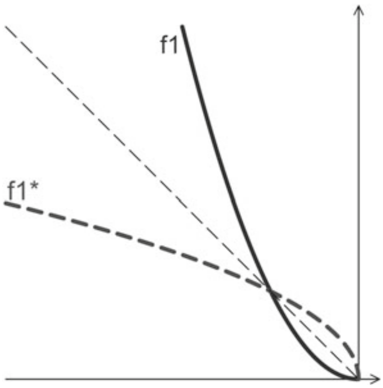
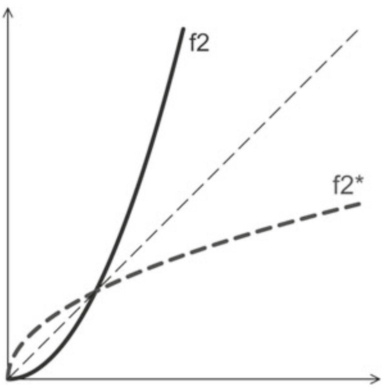



**Fig. 14** Function $f_2$ and its inverse



and apply to each equation the respective inverse function.

The result, as it is well known, gives two solutions

$$r = -\sqrt{\frac{a}{k}} \quad \vee \quad r = \sqrt{\frac{a}{k}}$$

In the case of the cylinder, we can just consider the partial equation

$$V = \pi r^2 h \quad r \geq 0 \quad \Rightarrow \quad r = \sqrt{V/\pi h}$$

The linear and quadratic models are useful to face the following real situation.

## 3.5 BMI Index

The Body Mass Index (BMI) is a parameter widely used to obtain a general assessment of one's body weight. It relates the height to the weight of the subject with a simple mathematical formula. It is obtained by dividing the weight in kg of the subject with the square of the height expressed in meters

$$BMI = \frac{weight}{(height)^2}$$

In mathematical terms the formula becomes very similar to those of the elementary geometry:

$$BMI = \frac{w}{h^2}$$

We can look at the formula from different point of views.

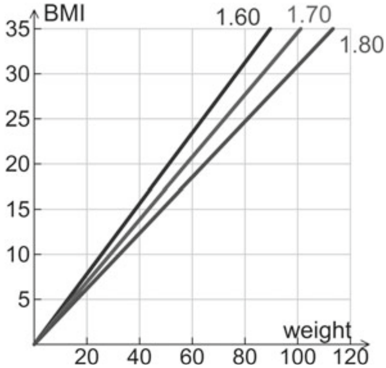



**Fig. 15** Function weight-BMI

**With respect to the weight**: BMI index is directly proportional to the weight (see Fig. 15).
**With respect to the height**: BMI index is inversely proportional to the square of the height (see Fig. 16).
**With respect to the BMI**. It could be interesting to note that the nutritionists use the following representation (see Fig. 17) which is different from both the previous ones. Is it correct or not?

Actually, as we can see Fig. 17, the two axes-variable are height and weight respectively; thus BMI play the role of parameter. As a consequence the curves are parabola arches which represents the lines of equal BMI.

## 4 From Static to Dynamic Models

The models we have adopted in the previous sections to describe various situations of the real can be considered as *photos* of the situation. When we want to describe an evolving phenomenon as long as it accurs, we need to pass from the photos to a *movie*. In other word we have to adopt a dynamic model.

The transition from static situations (the system is immutable in time) to dynamic ones (the system evolution changes in time) turns numerical equations (the unknown is a number) into functional equations (the unknown is a function).

Let us start with two basic examples.

**Malthus Model in Population Dynamics**
The English economist Thomas Malthus proposed in 1798 a first model to study the evolution of an isolated population, based on the assumption:

**(M) the growth rate is directly proportional to the number of individuals**.

If we adopt a real function $P : R_0^+ \to R$ to describe the population with respect to time, Malthusian principle is translated into the relation

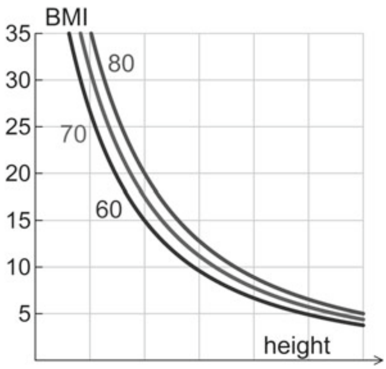



**Fig. 16** Function height-BMI

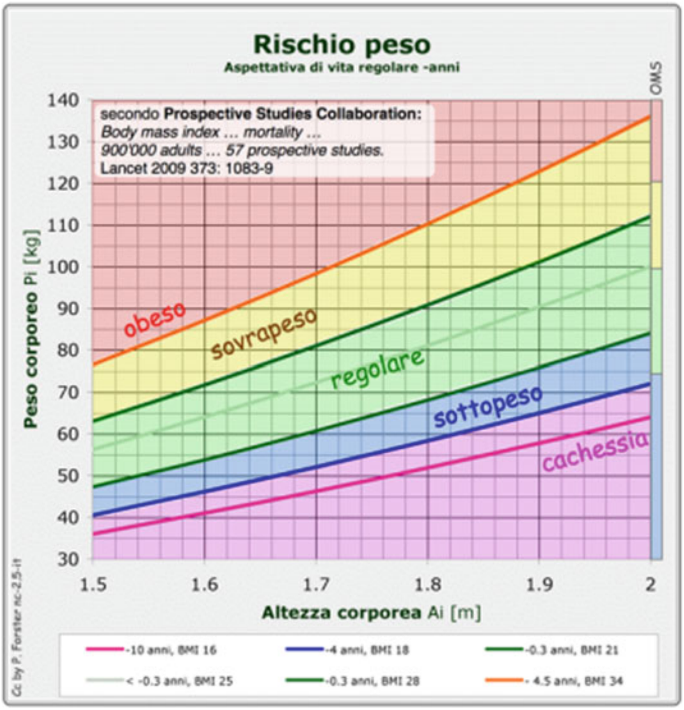



**Fig. 17** Lines of equal BMI

$$P'(t) = k\,P(t)$$

It is still an equilibrium equation … but a *functional* one, more precisely a *differential equation*, since the unknown function $P$ appears in the relation together with its derivative $P'$. If we take into account the population at the starting time, the model becomes

$$\begin{cases} P'(t) = k\,P(t) \\ P(0) = P_0 \end{cases}$$

**Newton's law on heating and cooling**

According to Newton's law

**(N) the rate of change of temperature is proportional to the thermal gradient (the difference between the fluid temperature and the environment temperature)**

Denoted by $T : R_0^+ \to R$ the function that describe the temperature with respect to the time, the statement is translated into the differential problem

$$\begin{cases} T'(t) = k\,[T(t) - T_a] \quad k \neq 0 \\ T(0) = T_0 \end{cases}$$

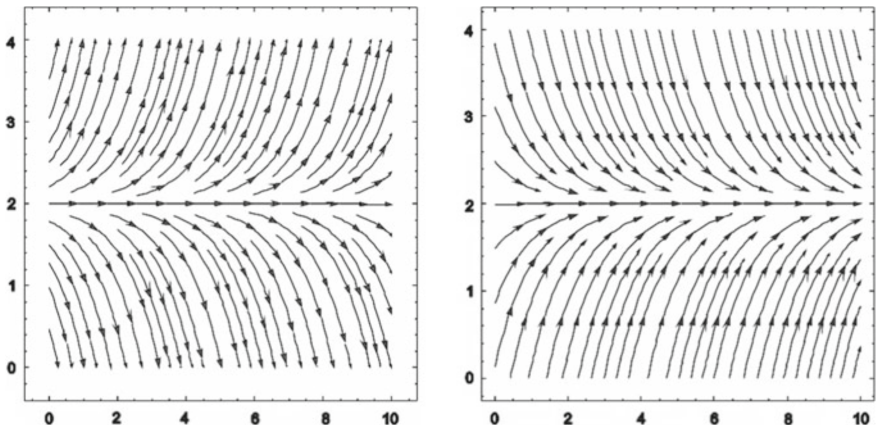



**Fig. 18** The direction fields of Malthus-Newton equations

where $T_a$ denotes the environment temperature.

Two different situation, one single model: a linear differential equations!

An elementary differential equation is a relation of the type

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases} \quad t \in [t_0, t_0 + a]$$

where the function $f(t, x)$ is continuous and the solution $x = x(t)$ is a $C^1$ function (continuous with continuous derivative). The function $f$ is called *direction field* since it works as a GPS: at any "time" $t$, when the solution has reached the position $(t, x(t))$, function $f$ indicates the direction to advance.

In the following images we can see the direction fields of the differential equations of Malthus-Newton equations (Fig. 18).

Now a big question arises: how to approach the study of differential equations?

We propose here an approach to differential equations that mimics the resolution of a numerical equation, based on the *inverse technique.*

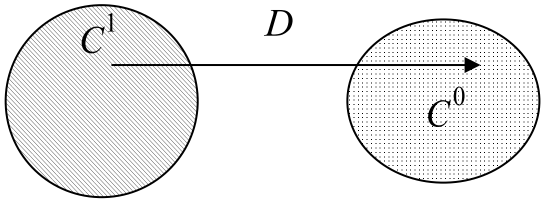
## 5 Differential Equations and Inverse operators

In order to adopt the inverse technique for differential equations, we must first discuss the invertibility of derivative operator.

## *5.1 The Inverse of Derivative Operator*

The derivative operator $D$ associates to any derivable function $h : I \to R$, its derivative $Dh$.

We can restrict our interest to the space $C^1$ (derivable functions with continuous derivative).

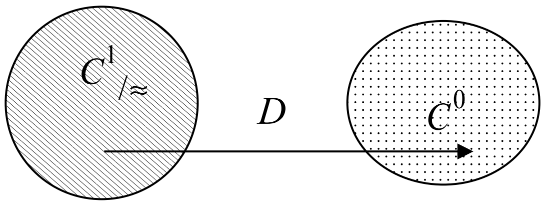



Operator $D$ is linear, but unfortunately it is not injective. In fact its kernel is constituted by the constant functions

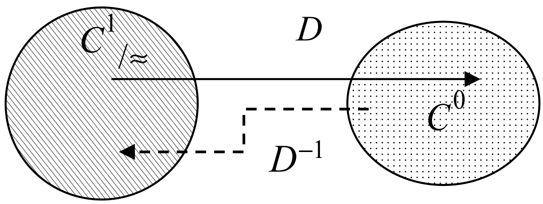
$$ker D = \left\{h \in C^1 \quad Dh = 0\right\}$$

Thus, to get injective the linear operator $D$ it is sufficient to equipped $C^1$ by means of the equivalence relation

$$f \approx h \quad \Leftrightarrow \quad f - h = const$$




The inverse linear operator $D^{-1}$ acts backward: given a function $g : I \to R$, $D^{-1}g$ is a derivable function $h$ such that $Dh = g$. Now the problem arise: how to select function $h$.



As is known, the elementary theory of integration gives a satisfactory solution, i.e. given a function $g \in C^0$, its integral function

$$h(x) = \int_{x_0}^{x} g(t)\mathrm{d}t$$

satisfies the properties:

(i) $h \in C^1$
(ii) $Dh = g$

As a consequence the inverse operator $D^{-1}$ (also called anti-derivative operator) admit the following representation

$$D^{-1}g = \int_{x_0}^{x} g(t)dt \quad g \in C^0$$

We are now able to transport the inverse technique to differential equations.

## 5.2 *The Inverse Technique for Differential Equations*

**First Case** Let us consider first the differential equations

$$x'(t) = f(t)$$

where $f : Dom\, f \to R$ is a given continuos function. To solve the equation we have to reconstruct the function $x(t)$ from its derivative.

Adopting the *inverse technique*, we apply the *integral function* (i.e. the inverse operator of derivative) to both the sides

$$\int_0^t x'(s)ds = x(t) - x_0 = \int_0^t f(s)ds$$

In this way we obtain the *integral form* of the differential equation

$$x(t) = x_0 + \int_0^t f(s)ds$$

**Second Case** Let us consider now an equation of the type

$$x'(t) = f(t)g(x(t))$$

where $f$ and $g$ are given continuous function.

Note that both Malthus and Newton equations belongs to this family.

Assumed $g(x) \neq 0$ for every $x$, the equation can be written

$$\frac{x'(t)}{g(x(t))} = f(t).$$

Applying the *integral function* to both the sides

$$\int_0^t \frac{x'(s)}{g(x(s))} ds = \int_0^t f(s) ds.$$

by virtue of the integration by substitution, we get

$$\int_0^t \frac{x'(s)}{g(x(s))} ds = \left[ \int_{x_0}^x \frac{dy}{g(y)} \right]_{x=x(t)}.$$

Finally, if we put

$$G(x) = \int_{x_0}^x \frac{1}{g(y)} dy$$

the solution of the equation (in implicit form) is

$$G(x(t)) = \int_{t_0}^t f(s)\, ds + c \quad c \in R.$$

We can now discuss the solution of Malthus and Newton differential equations.

**Malthus model**
In the particular case of Malthus equation

$$\begin{cases} P'\{t\} = kP(t) \\ P(0) = P_0 \end{cases}$$

the inverse technique gives an exponential function as a solution (see Figs. 19 and 20).

$$\int_0^t \frac{P'(s)}{P(s)} ds = \int_0^t k\, ds \quad \Rightarrow \quad \log P(t) = k\, t + \log P_0 \quad \Rightarrow \quad P(t) = P_0\, e^{k\, t}$$

**Fig. 19** Case $k < 0$

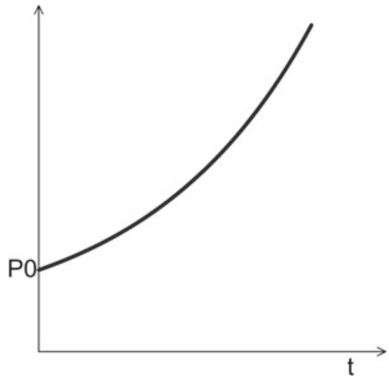
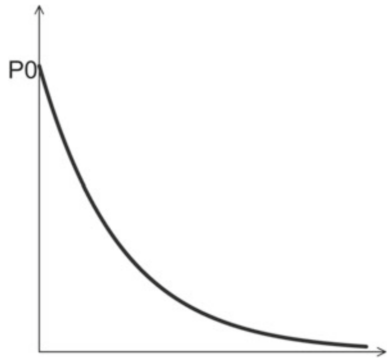



**Fig. 20** Case $k < 0$




**Newton model**

Let us solve now the general growing/decay equation inspired by Newton's law

$$\begin{cases} T'(t) = k\,[T(t) - T_a] \\ T(0) = T_0 \end{cases} \quad t \geq 0 \quad k < 0$$

First note that if $T_0 = T_a$ the solution is a constant function $T(t) = T_a$ (called *equilibrium solution*). If $T(t) \neq T_a$ for every $t$, the equation can be written

$$\frac{T'(t)}{T(t) - T_a} = k$$

and the *inverse technique* leads to the family of solutions

$$T(t) = (T_0 - T_a)\,e^{k(t-t_0)} + T_a$$

It is easy to see that two possible evolutions occur (see Figs. 21 and 22).

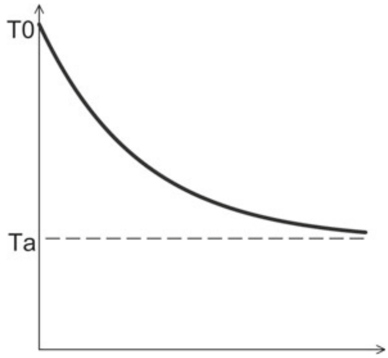



**Fig. 21** Case—cooling $T_a < T_0$

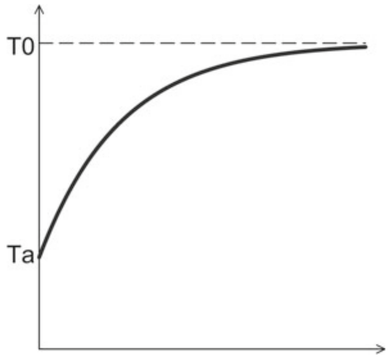



**Fig. 22** Case—heating $T_a > T_0$

## *5.3 Some examples*

Let us discuss the dynamic models for three different situations from real life (see [4] for the details).

**Coffee Cooling**

Let us describe, according to Newton law, the temperature evolution of a coffee poured into a cap.

If $T : R_0^+ \to R$ is the function which describe the coffee temperature, with respect to time, applying the formula in Sect. 5.2, we obtain (see also Fig. 23).

$$T(t) = (T_0 - T_a)\, e^{k\,t} + T_a \quad t \geq 0 \quad k < 0$$

where $T_a$ and $T_0$ denote the room temperature and the temperature of the coffee just poured, respectively.

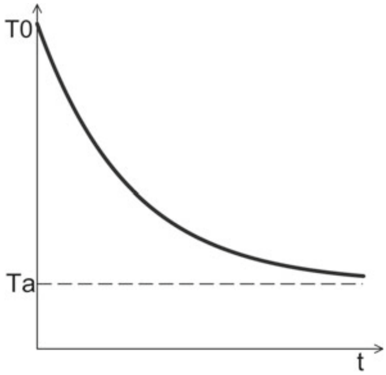



**Fig. 23** Coffee cooling

**Fish growth**
According to the biologist Ludwig von Bertalanffy the growth rate of the length of some fish species is proportional to the difference between the maximum length of the species and the present length. In particular, for the North Sea cod, we can take into account the following estimates:

The maximum length $L^*$ of the species 53 $cm$, the length of newborn fish $L_0 = 10$ $cm$, the growth factor $k = -0.2$.

**The constraction of the model**. Denoted by $L : R_0^+ \rightarrow R$ the function that describe the length of the fish in time, the assumption of von Bertalanffy is translated into the Cauchy problem

$$\begin{cases} L'(t) = k(L^* - L(t)) & t \geq 0 \\ L(0) = L_0 \end{cases} \quad k > 0.$$

Note that the model is very similar to Newton model. By applying the inverse technique, we obtain the solution

$$L(t) = L^* - \left(L^* - L_0\right)e^{-kt}$$

Thus the growth of North Sea cod is described by the function (see also Fig. 24)

$$L(t) = 53 - 43e^{-0,2t}$$

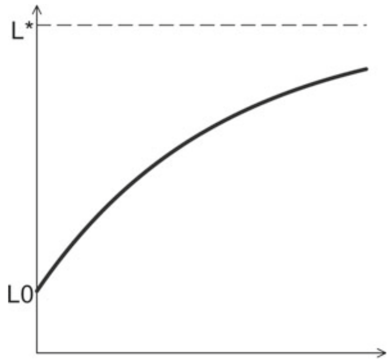



**Fig. 24** Fish growth

**The Liquid Mixing**

In a tank containing salt water at concentration $\alpha$ a salt water at a different concentration $\beta$ is introduced, at a constant rate $r$. In order to keep the liquid volume constant, water exits from the tank with the same speed. The solution is mixed continuously. Discuss the evolution of salt concentration.

**The construction of the model**. Denoted by $C : R_0^+ \to R$ the function that describes the salt concentration in time, the phenomenon can be described by the model

$$C(t + \Delta t) = C(t)(1 - \lambda_1 \Delta t) + \beta\lambda_1 \Delta t \quad \Rightarrow \quad C'(t) = -C(t)\lambda_1 + \beta\lambda_1$$

where $V$ denote the liquid volume and $\lambda_1 = \frac{r}{V}$.

Thus we get the Cauchy problem

$$\begin{cases} C'(t) = -C(t)\lambda_1 + \beta\lambda_1 \\ C(0) = \alpha \end{cases}$$

By applying the inverse technique, we obtain the solution

$$C(t) = \beta + (\alpha - \beta)\, e^{-\lambda_1 t}$$

**The discussion of the solution**. It is easy to see that two possible evolutions occur, according to $\alpha < \beta$ (Fig. 25) or $\alpha > \beta$ (Fig. 26).

In both the cases the concentration is monotone and evolves asymptotically up (or down) to the concentration $\beta$ of the liquid incoming.

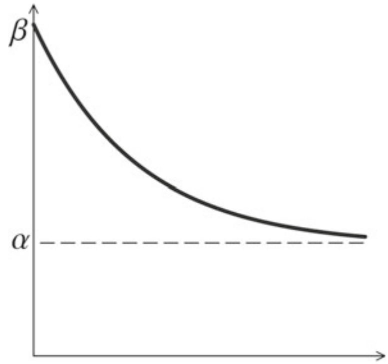



**Fig. 25** First case

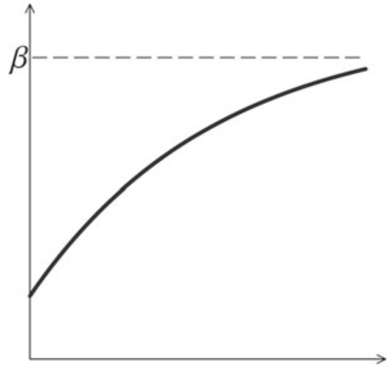



**Fig. 26** Second case

## References

1. Brandi P, Salvadori A (2017) Math-MAPS a new road that opens up a world. Part I Dynamic models without derivative. In Proceedings APLIMAT, pp 187–196
2. Brandi P, Salvadori A (2016) MATH Maps. Itinerari per le competenze, II volume, Quaderni Alice&Bob - PRISTEM Bocconi, 45, Egea, p 124
3. Brandi P, Salvadori A (2015) MATH Maps. Itinerari per le competenze, I volume, Quaderni Alice&Bob - PRISTEM Bocconi, 40, Egea, p 124
4. Brandi P, Salvadori A (2015) Nuovi Percorsi di Matematica, Introduzione al Calcolo di Newton secondo Matematica&Realtà, 2 volums, Aguaplano-Officina del libro, Passignano s.T. (PG), p 357, 295
5. Brandi P, Salvadori A (2011) Prima di iniziare. Conoscenze e competenze di base per l'Università. Aguaplano-Officina del libro, Passignano s.T. (PG), p 316
6. https://www.matematicaerealta.eu

# A Concrete Approach to Geometry

**Emanuela Ughi**

A concrete approach to geometry is barely accepted in primary school but is not usually used and proposed in higher school levels. I present some examples to show how the activity of touching, building, manipulating mathematical objects can stimulate thoughts, observations and questions that are not at all elementary, at every level.

Geometry—at its very beginning—is concrete: even its name—how to measure the ground—clearly refers to a material application.

As an example, think about the idea of a straight line. Even people without any mathematical background know several properties of a straight line: we all know that it is the shortest path between two points, that it is the path of a falling stone, that a straight line can glide over itself. We do not know exactly when we learnt those properties, nor in which order. But in a child letting a pacifier fall repeatedly we can clearly see the seed of a researcher making experiments on geometry and physics.

The concrete experiences we had in our childhood were necessary to build this knowledge. I had the chance to work with a very smart blind girl, aged 19, about the geometry of the first book of the Elements of Euclid, and discovered that she had no idea about the path of a falling thing—she thought that things fell along a curve, similar to an arch of an hyperbola, since people usually asked "where did the pen end up?". She completely missed the visual experience of a falling thing and consequently she missed also a correct and complete concept of a straight line.

After the concrete approach, anyway, it is possible to visualize a geometric shape in our mind, to rotate it or to cut it. A mathematician is usually skilful in performing this task: I like to recall a joke, saying that it is hard to see the difference between a working mathematician and a sleeping one. Indeed, the mathematician (a geometer, especially) often thinks with their eyes closed. Like Paul Gauguin, even the mathematician "closes his eyes to see better".

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But this skill is not common, and even between undergraduate students in mathematics, I met people who were not able to follow a lecture, when required to imagine a shape or to manipulate it in their mind.

It is from this experience that I started, more than 20 years ago, to make mathematical objects in a concrete way, to let my student “see”; my objects always embody a mathematical idea, definition or theorem, and can be touched, reversed, moved in all possible ways.

From the very beginning, the interest raised by my models has been much greater than I expected and led me to develop my work towards different directions; in particular, I set up a mathematical hands-on museum, the Galleria di Matematica at the Polo Museale Universitario dell’Ateneo di Perugia.

I have often asked myself the reason behind this success; my answer is that people need to understand and even more, they feel that they need to believe themselves able to understand, so that they can enjoy every mathematical experience while growing this belief.

A concrete approach in teaching can be a way to communicate a vision in an interesting and durable way. The memories of Maria Montessori’s teaching materials and activities, that I used nearly 60 years ago, are still fascinating for me, and the way in which I think about the number line directly stems from them.

And a concrete object, well planned and realized, can offer an alternative way to understand, making clear some ideas that had remained obscure, perhaps after a notional and repetitive schooling: it is a joy for me to hear the surprised exclamation “Ohhh!”, when someone has finally grasped a mathematical concept through one of my objects.

Especially in recent years, a concrete approach to teaching has been suggested and urged by many. In particular, the importance of the Mathematics Laboratory is emphasized, as a kind of Renaissance workshop in which it is possible to learn by doing and seeing the others who do.

But in practice, in the school, except for wonderful exceptions, knowledge too often continues to be transmitted as a transfer of notions, repeating the path received by teachers many years before: just static work on books and paper.

The use of concrete material support in the school is tolerated at the primary school (and not always); interesting in this regard is the observation on the use of fingers in the representation of numbers: on the one hand the neurosciences underline the importance of using them and encourage such an approach; on the other hand, in classes—not only in Italy!—children are scolded if they use their fingers to help with the calculation. Jo Boaler clearly notes that teachers think that the use of fingers is babyish and wrong, and hence it should be avoided [1].

This lack of concrete experience is more crucial nowadays: indeed, until a few years ago, the child still had the possibility—and the need—to act on the concrete world, while this aspect—at least in Italy—is today less and less present in the growth process: Tullio De Mauro speaks of “child without hands”: a child who does not cut the salami (and does not discover the conical sections that change shape when the knife is tilted), who does not tie up the shoes (and does not fight with the intertwining of the strings, the top-down, the lateralization), who does not build a model of analog clock in cardboard as he learns to read it, and so on.

The public using my concrete objects often say that they are beautiful. But, what does it mean?

Beauty in mathematics is usually referred to the elegance and deepness of a proof—a theoretical aesthetic appeal of arguments explaining "why", or to the pleasure of the enlightening moment, when something becomes suddenly clear [2, 3].

I suggest adding the pleasure of "the last tile of a puzzle", the kinesthetic pleasure of fulfilling a necessary shape, according to a mathematical inevitability. This of course helps in having positive emotions, in this way strengthening the learning of the related mathematical concepts (Fig. 1).

I would like to remark that a concrete approach doesn't allow to cheat; teaching this way is necessarily "honest", in the sense of Bruner's proposal: "We begin with the hypothesis that any subject can be taught effectively in some intellectually honest form to any child at any stage of development".

A way to use a concrete approach in teaching is to ask children to make mathematical objects themselves.

And yet, when something is really built, it turns out that geometry is always present, with specific problems inherent in every project, and construction often passes between phases of theoretical design thinking, and moments of searching for concrete solutions to implement the project itself.

This happens at every level: I like to recall that, during the last restoration of the Last Supper, the hole of a nail was discovered in the forehead of Christ; Leonardo himself used it to stretch strings to define the vanishing point of a series of parallel lines in the construction of the structure of the painting.

At this regard, Maria Montessori wrote "Quando la mano si perfeziona in un lavoro scelto spontaneamente, e nasce la volontà di riuscire, di superare un ostacolo, la coscienza si arricchisce di qualcosa di ben diverso da una semplice cognizione: è la coscienza del proprio valore." (When the hand becomes more skilled in a spontaneously chosen work, and the will to succeed, to overcome an obstacle, is born, the

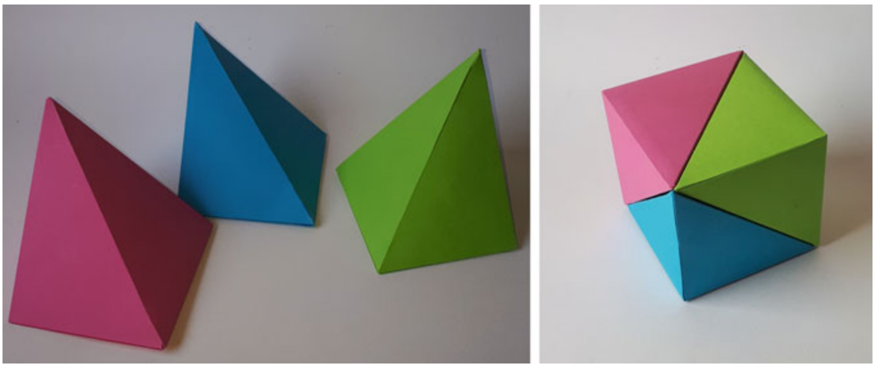

**Fig. 1** A mathematical puzzle: 3 equal pyramids whose union is a cube

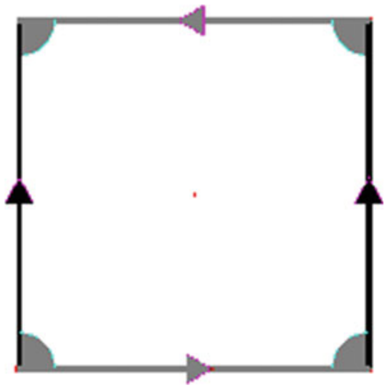

**Fig. 2** A Klein bottle as a quotient of a square

consciousness is enriched with something quite different from a simple cognition: it is the awareness of one's own value.)

Incidentally, a concrete approach is sometimes the only possible access path for some disabled student to fully access mathematical contents. For instance, only concrete models can honestly and carefully explain parallelism, or the perspective theory, to a blind child.

I am going to show a couple of examples to support my belief that the activity of constructing, touching, manipulating objects can provide motivation, and even stimulate thoughts, observations and generate newer and deeper mathematical questions.

**Example 1: The Klein Bottle**

Undergraduate students know the definition of the Klein bottle as a quotient of the square, obtained by glueing opposite edges as shown in Fig. 2.

Very often they see also the image in Fig. 3, to represent the same Klein bottle.

My student Lucia Minchielli and I interviewed many of them (about 60 students, after 5 years of mathematical studies). We wanted to explore their consciousness about why the previous definition corresponds to the image, and what happens in the intersection part, but unfortunately most of them simply ignored the need to unify their information. In her thesis Lucia Minchielli then designed a model to make a topological Klein bottle in cardboard and translucent sheet. The activity of mounting it is challenging, but forces to think about the structure of abstract topological variety, and the need of thinking the two parts in the intersection detail as belonging to distinct coordinate charts. I like to see how to glue together the gray corners in Fig. 2 in order to obtain a chart containing the unique point corresponding the four vertices of the original square.

**Example 2: The Helicoid**

In architecture there are examples of "double stairs", like the Pozzo di San Patrizio in Orvieto and the Grand Escalier in Chambord, in which there are two distinct paths, in such a way that people going up, and people going down, do not meet. The surprise for this apparently confusing fact can be used to fire attention on the geometric shape of the helicoid.

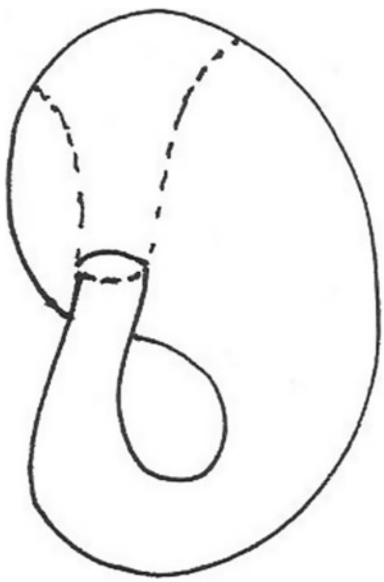

**Fig. 3** Sketch of a Klein bottle

The steps start from the observation of a single spiral staircase, obtained by the movement of a segment (or of a wooden stick), orthogonal to the z-axis, and having a vertex on it. The segment rotates and at the same time goes up along the z-axis, so generating the spiral staircase. The curve described by the free vertex of the stick is a circular helix.

So, we can now pass to explore what happens when the segment (or the stick) is moving the same way, but rotating respect its medium point. The result is now a helicoid (Fig. 4).

Moreover, the shape of the helicoid can be found even in some fusilli pasta: some brand indeed makes fusilli having two distinct paths, that children can experience by fulfilling them by play dough of two distinct colours. Other brands make fusilli with three paths, and to distinguish them is a first challenge requiring observation (Fig. 5).

To make the difference between two such "ideal" fusilli is topologically simple: a 2-way ideal fusillo's convex closure is a cylinder, and the 2 distinct paths correspond to the 2 connected components of the complement of the fusillo in the cylinder; analogously, the complement of the 3-way ideal fusillo has 3 connected components.

I propose here an (open) question about what happens with true fusilli? They are 3-dimensional objects, always homeomorphic to a sphere, often skew and irregular. Nevertheless, everybody accepts to recognize an "axis", and 2 or 3 paths along it. So, my question is: what do they see? How do they detect—for example—2 paths? What is—geometrically—a path in this case? I suppose that similar questions can be related to some problems in artificial intelligence, about how to learn to recognize shapes.

Fig. 4 From a pile of sticks to a helicoid

Fig. 5 Fusilli

The beauty of the object can be better appreciated by the physical act of realizing the model of a helicoid by rotating the sticks. We can start from a pile of sticks, initially arranged as a plane (of course dividing the space in two not-connected half spaces). The manual motion helps in imagining the homotopy of the space that modify the original plane in the helicoid, and the two semi-spaces in the two non-connected paths.

Moreover, putting the axis of the object as horizontal, in high school it can be exciting to recognize the graph of sinus and cosinus functions (and, perhaps, teachers can ask to provide motivations for this fact) and to see waves while rotating the object.

## References

1. Boaler J, Chen L (2018) Why kids should use their fingers in math class. In: Pitici M (ed) The best writings on mathematics 2017. Princeton University Press, Princeton and Oxford, pp 76–81. ISBN 978-0-69117-863-9
2. Cellucci C (2015) Mathematical beauty, understanding, and discovery. Found Sci 20:339–355
3. Rota G-C (1997) The phenomenology of mathematical beauty. Synthese 111(2):171–182

# Tessellation

**Mária Ždímalová**

In this contribution we discuss tessellation. We analyze basic tessellation, types of tessellation, geometric approach and applications of tessellation in geometry as well as in architecture and art. We study as well as groups of tessellation used in Spanish Alhambra [1–3]. Finally we open possibilities how to use tessellation for aggregations, aggregations functions and aggregate tessellation. We discuss how we can use weighted Voronoi diagram for tessellation and we consider as well weighted Voronoi tessellation [4, 5].

## 1 Introduction

A tessellation [6, 7] (or tiling) is a pattern of geometrical objects that covers the plane. The geometrical objects must leave no holes in the pattern and they must not overlap. It should be able to extend the pattern to infinity. It makes a tessellation by starting with one or several figures and then rotate it, translate or reflect them; or do a combination of transformations, in order to get a repeating pattern. If there is interest only want to use one regular polygon to make a tessellation, there are only three possible polygons to use: triangle, square and hexagon.

The tessellation of the plane by these objects has a lot of nice properties that have been widely studied, see e.g. [8]. Starting with a tiling of regular polygons, we can distort it. It is possible to distort it in many different ways. In the example above:

- The triangle tiling is distorted to a tiling of two different tiles.
- The square tiling is distorted to a tiling of one tile. All tiles are translations of the tile in the centre.

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- The hexagon tiling is also distorted to a tiling of one tile. Every tile is rotated relative to its neighbors.

A tessellation of a flat surface is the tiling of a plane using one or more geometric shapes, called tiles, with no overlaps and no gaps. In mathematics, tessellation can be generalized to higher dimensions and a variety of geometries. A periodic tiling has a repeating pattern. Some special kinds include regular tilings with regular polygonal tiles all of the same shape, and semi-regular tilings with regular tiles of more than one shape and with every corner identically arranged. The patterns formed by periodic tilings can be categorized into 17 wallpaper groups. A tiling that lacks a repeating pattern is called "non-periodic". An aperiodic tiling uses a small set of tile shapes that cannot form a repeating pattern. In the geometry of higher dimensions, a space-filling or honeycomb is also called a *tessellation of space*. A real physical tessellation is a tiling made of materials such as cemented ceramic squares or hexagons. Such tilings may be decorative patterns, or may have functions such as providing durable and water-resistant pavement, floor or wall coverings. Historically, tessellations were used in Ancient Rome and in Islamic art such as in the decorative geometric tiling of the Alhambra palace. In the twentieth century, the work of M. C. Escher often made use of tessellation, both ordinary Euclidean geometry and in hyperbolic geometry, for artistic effect. Tessellation are sometimes employed for decorative effect in quilting. Tessellations form a class of patterns in nature, for example in the arrays of hexagonal cells found in honeycombs [7, 9]. A temple mosaic from the ancient Sumerian city of Uruk IV (3400–3100 BC), showed a tessellation pattern in coloured tiles. Tessellation were used by the Sumerians (about 4000 BC) in building wall decorations formed by patterns of clay tiles. Decorative mosaic tilings [9] made of small squared blocks called tesserae were widely employed in classical antiquity, sometimes displaying geometric patterns. In 1619 Johannes Kepler made an early documented study of tessellations. He wrote about regular and semi-regular tessellation in his *Harmonices Mundi*; he was possibly the first to explore and to explain the hexagonal structures of honeycomb and snowflakes.

Roman geometric mosaic

Two hundred years later in 1891 [7, 9], the Russian crystallographer Yevgraf Fyodorov proved that every periodic tiling of the plane features one of seventeen different groups of isometries. Fyodorov's work marked the unofficial beginning of the mathematical study of Tessellations. Other contributors include Aleksei Shubnikov and Nikolai Belov (1964), and Heinrich Heesch and Otto Kienzle (1963).

## 2 Etymology

In Latin, *tessella* is a small cubical piece of clay, stone or glass used to make mosaics. The word "tessella" means "small square" (from *tessera*, square, which in turn is from the Greek word τέσσερα for *four*). It corresponds to the everyday term *tiling*, which refers to applications of tessellations, often made of glazed clay. For example, there are eight types of semi-regular tessellation, made with more than one kind of regular polygon but still having the same arrangement of polygons at every corner. Irregular tessellation can also be made from other shapes such as pentagons, polynoms and in fact almost any kind of geometric shape. The artist M. C. Escher is famous for making tessellation with irregular interlocking tiles, shaped like animals and other natural objects. If suitable contrasting colours are chosen for the tiles of differing shape, striking patterns are formed, and these can be used to decorate physical surfaces such as church floors [7, 9]. A tessellation or tiling is a cover of the Euclidean plane by a countable number of closed sets, called *tiles*, such that the tiles intersect only on their boundaries. These tiles may be polygons or any other shapes. Tessellation are formed from a finite number of prototiles in which all tiles in the tessellation are congruent to the given prototiles. If a geometric shape can be used as a prototile to create a tessellation, the shape is said to *tessellate* or to *tile the plane*. The Conway criterion is a sufficient but not necessary set of rules for deciding if a given shape tiles the plane periodically without reflections. No general rule has been found for determining if a given shape can tile the plane or not, which means there are many unsolved problems concerning tessellation.

Mathematically, [7, 9] tessellation can be extended to spaces other than the Euclidean plane. The Swiss geometer Ludwig Schläfli pioneered this by defining *polyschemes*, which mathematicians nowadays call polytopes. These are the analogues to polygons and polyhedra in spaces with more dimensions. He further defined the Schläfli symbol notation to make it easy to describe polytopes. For example, the Schläfli symbol for an equilateral triangle is {3}, while that for a square is {4}. The Schläfli notation makes it possible to describe tilings compactly. For example, a tiling of regular hexagons has three six-sided polygons at each vertex, so its Schläfli symbol is {3, 6}. Other methods also exist for describing polygonal tilings. When the tessellation is made of regular polygons, the most common notation is the vertex configuration, which is simply a list of the number of sides of the polygons around a vertex.

## 3 What Are the Types of Tessellations?

**Regular Tessellation**: Regular tessellation are tile patterns made up of only single shape placed in some kind of pattern. There are three types of regular tessellation: triangles, squares and hexagons. Regular tessellations have interior angles that are divisors of 360°. For example, a triangle's three angle total 180°; which is divisor of 360. A hexagon contains six angles whose measurement total 720°. This is also a divisor of 180, because 180 fits even 720.

**Semi-Regular tessellations**: When two or three types of polygons share a common vertex, a semi-regular tessellation is forms [7]. There are nine different types of semi-regular tessellations including combining a hexagon and a square that both contain a 1-inch side. Another example of a semi-regular tessellation is formed by combining two hexagons with two equilateral triangles.

**Demi-Regular Tessellation**: There are 20 different types of demi-regular tessellation. These are Tessellations that combine two or three polygon arrangements [7]. A demi-regular tessellation can be formed by placing a row of squares, then a row of equilateral triangles that are alternated up and down forming a line of squares when combined. Demi-regular tessellation always contains two vertices.

**Non-Regular Tessellation**: A non-regular tessellation is a group of shapes that have the sum of all interior angles equaling 360°, see [7]. There are again, no over loops or gaps, and non-regular tessellations are formed many times using polygons that are not regular.

**Other Types**: There are two other types of tessellation which are three-dimensional Tessellation and non-periodic Tessellation. A three-dimensional tessellation uses three-dimensional forms of shapes, such as octahedrons. A non-periodic tessellation is a tilling that does not have a repetitious pattern. That tilling evolves as it is created, yet still contains no overlapping or gaps.

## 4 Deeper About Tessellation and Tiling

Mathematicians use some technical terms when discussing tilings. An *edge* is the intersection between two bordering tiles; it is often a straight line. A *vertex* is the point of intersection of three or more bordering tiles. Using these terms, an *isogonal* or vertex-transitive tiling is a tiling where every vertex point is identical; that is, the arrangement of polygons about each vertex is the same. The fundamental region is a shape such as a rectangle that is repeated to form the tessellation. For example, a regular tessellation of the plane with squares has a meeting of four squares at every vertex. The sides of the polygons are not necessarily identical to the edges of the tiles. An **edge-to-edge tiling** is any polygonal tessellation where adjacent tiles only share one full side, i.e. no tile shares a partial side or more than one side with any other tile. In an edge-to-edge tiling, the sides of the polygons and the edges of the tiles are the same. The familiar "brick wall" tiling is not edge-to-edge because the long side of each rectangular brick is shared with two bordering bricks.

A **normal tiling** is a tessellation for which every tile is topologically equivalent to a disk, the intersection of any two tiles is a single connected set or the empty set, and all tiles are uniformly bounded. This means that a single circumscribing radius and a single inscribing radius can be used for all the tiles in the whole tiling; the condition disallows tiles that are pathologically long or thin [7, 9].

A **monohedral tiling** is a tessellation in which all tiles are congruent; it has only one prototile. A particularly interesting type of monohedral tessellation is the spiral monohedral tiling. The first spiral monohedral tiling was discovered by Heinz Voderberg in 1936; the Voderberg tiling has a unit tile that is a nonconvex enneagon. The **Hirschhorn tiling**, published by Michael D. Hirschhorn and D. C. Hunt in 1985, is a pentagon tilingusing irregular pentagons: regular pentagons cannot tile the

Euclidean plane as the internal angle of a regular pentagon, $3\pi/\boxed{5}$, is not a divisor of $2\pi$. An isohedral tiling is a special variation of a monohedral tiling in which all tiles belong to the same transitivity class, that is, all tiles are transforms of the same prototile under the symmetry group of the tiling.

A regular tessellation is a highly symmetric, edge-to-edge tiling made up of regular polygons, all of the same shape. There are only three regular tessellations: those made up of equilateral triangles, squares, or regular hexagons. All three of these tilings are isogonal and monohedral.

A semi-regular (or Archimedean) tessellation uses more than one type of regular polygon in an isogonal arrangement. There are eight semi-regular tilings (or nine if the mirror-image pair of tilings counts as two). These can be described by their vertex configuration; for example, a semi-regular tiling using squares and regular octagons has the vertex configuration $4.8^2$ (each vertex has one square and two octagons). Many non-edge-to-edge tilings of the Euclidean plane are possible, including the family of Pythagorean tilings, tessellations that use two (parameterised) sizes of square, each square touching four squares of the other size. An edge tessellation is one in which each tile can be reflected over an edge to take up the position of a neighbouring tile, such as in an array of equilateral or isosceles triangles. Penrose tilings, which use two different quadrilateral prototiles, are the best known example of tiles that forcibly create non-periodic patterns. They belong to a general class of aperiodic tilings, which use tiles that cannot tessellate periodically. The recursive process of substitution tiling is a method of generating aperiodic tilings. One class that can be generated in this way is the rep-tiles; these tilings have surprising self-replicating properties. Pinwheel tilingsare non-periodic, using a rep-tile construction; the tiles appear in infinitely many orientations. It might be thought that a non-periodic pattern would be entirely without symmetry, but this is not so. Aperiodic tilings, while lacking in translational symmetry, do have symmetries of other types, by infinite repetition of any bounded patch of the tiling and in certain finite groups of rotations or reflections of those patches. A substitution rule, such as can be used to generate some Penrose patterns using assemblies of tiles called rhombs, illustrates scaling symmetry. A Fibonacci word can be used to build an aperiodic tiling, and to study quasicrystals, which are structures with aperiodic order.

## 5 Wallpaper Groups and Symmetries

The wallpaper groups are the 17 possible plane symmetry groups. They are commonly represented using Hermann-Mauguin-like symbols or in orbifold notation (Zwillinger 1995, p. 260) [7, 9, 10].

Translational symmetry is just one type of symmetry [7, 10]. There is also rotational and reflection symmetry. An image has a **rotational symmetry** if you can rotate the image around some point and get the same image. An image has a **reflection symmetry** if you can reflect the image in some line and get the same image.

## 6 Tessellation in Different Areas

**Tessellation and colour**: If the colours of this tiling are to form a pattern by repeating this rectangle as the fundamental domain, see [7, 9], at least seven colours are required; more generally, at least four colours are needed. Sometimes the colour of a tile is understood as part of the tiling; at other times arbitrary colours may be applied later. When discussing a tiling that is displayed in colours, to avoid ambiguity one needs to specify whether the colours are part of the tiling or just part of its illustration. This affects whether tiles with the same shape but different colours are considered identical, which in turn affects questions of symmetry. The four colour theorem states that for every tessellation of a normal Euclidean plane, with a set of four available colour [7, 9].

**Tessellation in higher dimensions**: Tessellating in the three-dimensional space: the rhombic dodecahedron is one of the solids that can be stacked to fill space exactly, for more details see [7, 9]. Tessellation can be extended to three dimensions. Certain polyhedra can be stacked in a regular crystal pattern to fill (or tile) three-dimensional space, including the cube (the only Platonic polyhedron to do so), the rhombic dodecahedron, the truncated octahedron, and triangular, quadrilateral, and hexagonal prisms, among others. Any polyhedron that fits this criterion is known as a plesiohedron, and may possess between 4 and 38 faces. Similarly, in three dimensions there is just one quasiregular honeycomb, which has eight tetrahedra and six octahedra at each polyhedron vertex. However, there are many possible semi-regular honeycombs in three dimensions. Schmitt-Conway biprism is a convex polyhedron with the property of tiling space only aperiodically [7, 9]. Using the geometrical representation of the objects and their localisation into a square (or other) grid helps to describe their position in the plane, figure out their symmetry and last but not least solve some practical problems—see e.g. [11].

**In art and manufacturing**: Tessellation are also a main genre in origami (paper folding). Tessellation is used in manufacturing industry to reduce the wastage of material (yield losses) such as sheet metal when cutting out shapes for objects like car doors or drinks cans.

**In nature**: In botany, the term "tessellate" describes a checkered pattern, for example on a flower petal, tree bark, or fruit. Flowers including the fritillary and some species of *Colchicum* are characteristically tessellate. Many patterns in nature are formed by cracks in sheets of materials. These patterns can be described by Gilbert Tessellation, known as random crack networks. Other natural patterns occur in foams; these are packed according to Plateau's laws. In 1887, Lord Kelvin proposed a packing using only one solid, the bitruncated cubic honeycomb with very slightly curved faces. In 1993, Denis Weaire and Robert Phelan proposed the Weaire–Phelan structure, like in [7, 9].

**In puzzle and recreational mathematics**: Tessellation have given rise to many types of tiling puzzle, from traditional jigsaw puzzles (with irregular pieces of wood or cardboard) and the tangram to more modern puzzles which often have a mathematical basis. For example, polyiamonds and polyominoes are figures of regular triangles and squares, often used in tiling puzzles. Authors such as Henry Dudeney and Martin Gardner have made many uses of tessellation in recreational mathematics, see e.g. [3, 9].

## 7 Tessellation in Computer Games

Tessellation is used even now in computer games. It is popular and spread around the world of computer gamer. With the recent buzz around DirectX 11, see [12], the gamer probably heard a lot about one of its biggest new features: tessellation. As a concept, tessellation is fairly straight forward—we can take a polygon and divide it into smaller pieces. How does it benefit games? We will take a look at why tessellation is bringing profound changes to 3D graphics on the PC, and how the NVIDIA® GeForce® GTX 400 series GPUs provide breakthrough tessellation performance. In its most basic form, tessellation is a method of breaking down polygons into finer pieces. For example, if you take a square and cut it across its diagonal, you've "tessellated" this square into two triangles. By itself, tessellation does little to improve realism. For example, in a game, it does not really matter if a square is rendered as two triangles or two thousand triangles-tessellation only improves realism if the new triangles are put to use in depicting new information [12]. The simplest and most popular way of putting the new triangles to use is a technique called displacement mapping. A displacement map is a texture that stores height information. When applied to a surface, it allows vertices on the surface to be shifted up or down based on the height information. For example, the graphics artist can take a slab of marble and shift the vertices to form a carving. Another popular technique is to apply displacement maps over terrain to carve out craters, canyons, and peaks. Like tessellation, displacement mapping [12], has been around for a long time, but until recently, it has never really caught on. The reason is that for displacement mapping to be effective, the surface must be made up of a large number of vertices. In essence—displacement mapping needs tessellation, and vice versa.

## 8 Islamic Patterns, Alhambra and Escher

Islamic decoration, which tends to avoid using figurative images, makes frequent use of geometric patterns which have developed over the centuries. For more see [1–3, 6 and 10]. The geometric designs in Islamic art are often built on combinations of repeated squares and circles, which may be overlapped and interlaced, as can arabesques (with which they are often combined), to form intricate and complex patterns, including a wide variety of tessellations. These may constitute the entire decoration [2], may form a framework for floral or calligraphic embellishments, or may retreat into the background around other motifs. The complexity and variety of patterns used evolved from simple stars and lozenges in the ninth century, through a variety of 6—to 13—point patterns by the 13th century, and finally to include also 14—and 16—point stars in the sixteenth century [4–6]. Geometric patterns appears in different forms in Islamic art and architecture including kilim carpets, Persian girih and Moroccan zelligetile work, muqarnas decorative vaulting, jali pierced stone screens, ceramics, leather, stained glass, woodwork, and metal work.

West, both among craftsmen and artists including M. C. Escher in and among mathematicians and physicists including Peter J. Lu and Paul Steinhardt who controversially claimed in 2007 that tilings at the Darb-e Imam shrine in Isfahan could generate quasi-periodic patterns like Penrose tilings.

The tillings in the Alhanbra [4, 5] in Sain were laid out by Moor and by Vhristian artisans inspired by the Moor's style in 14th century. They are made of coloured tiles forming patterns, many truly symmetrical and beautiful. Some were not tessellations because they didn't cover a surface with repetitive design without gaps or overlaps. However, many of the Alhambra's patterns were true tessellation. They inspired the young M. C. Echer, who copied these geometric tessellations into his notebook and later tweaked some into tessellations that resembled animals or people. As example we can mention "China Boy" 1936 and "Strong Men" 1936. Echer defined "tessellation" as "the regular division of a plane". The shape in a tessellation can be geometric like squares an triangles, or shaped like animals and people. The Alhambra artists made many beautiful tessellation art much more popular. Esher noted that the Alhambra tilings never included animal or plants. One of Escher's biggest contributions to tessellation art was to make designs with people and animals instead of stiff geometric shapes like squares and triangles. Escher used to ask the audience if they knew of any tessellations done by others artist in the pasts. He was sent details of a tapestry design by Koloman Moser entitled "Forellenreigen" ("trout farm"), depicting a fish tessellation completed around 1899–1902. We note that Esher was born in 1898. We can not see nothing like this here before Escher. There were Egyptian, Hindu, Chinese and English Tessellations, all "abstract" style rather than looking like animals and plants and people [4–6].

## 9 Aggregate Tessellations

Now we want to show direction to aggregate Tessellation and as well as we will consider weighted Voronoi diagram in the connection of tessellations and aggregation [1, 2]. A tessellation of $R^d$ is a countable collection of closed bounded sets called cells such that

(a) Union of all cells is the whole space;
(b) Intersection of any two different cells has d-Lebesgue measure zero;
(c) Each bounded set intersect a finite number of cells.

Tessellations are used top model different cellular systems. We assume that each cell $C_i$ is associate with a unique nucleus $x(C_i)$ according to a certain rule satisfying an obvious compatibility condition: $\Phi x C_i = \Phi x(C_i)$. For any shift transformation $\Phi$ in $R^d$. For example, the Voronoi tessellation [1, 2] has cells defined as

$$C(x_i) = \left\{x \in R^d \|x - x_i\| \Phi \leq \left\|x - x_{ij}\right\|, \quad j \neq i\right\},$$

where $\|\,\|$ is the Euclidean norm. Thus, the cell with nucleus $x_i$ consists of the points that are closed to $x_i$ than to any other nucleus. A random tessellation with nuclei can be viewed as marked point process $M = \{x_i, \ C(x_i)\}$. In this paper we deal with stationary Tessellations, i.e. $M$ is stationary with respect to shifts Φ. Recently, random Voronoi Tessellations were used as models of services zones of telecommunications stations. This has many advantages: the main advantage of this model is that reduce the number of structuring parameters of the model to just a few parameters of underlying stochastic process and often for an analytical treatment of complex networks characteristic. We can tell that model using Voronoi diagrams, weighted Voronoi diagrams as well as Voronoi Tessellations [1, 2] over—simplify the complex geometry of service zone. For instance, in the case of wireless communications the base station that will handle a call from a mobile terminal is determined by the signal strength rather than Euclidian distance to the stations. Affected by the wave phenomena, the zones boundaries have extremely irregular, distorted shapes. The motivation of more complex tessellation models that are still described in terms of a small number of parameters and patterns. Then they can be simple for analytical solutions [1, 2]. For this the authors [1, 2] introduces an operation of aggregation on independent stationary Tessellations equipped with nuclei. Let $\Theta^0 = \left\{C^0\left(x_i^0\right)\right\}$ and $= \left\{C^1\left(x_i^1\right)\right\}$ be two such Tessellations. Define the aggregate cells of $\Theta_0^1 = \Theta^0\ \Theta^1$ as

$$C_0^1\left(x_i^0\right) = U_{j:xj1 \in C0(xi0)}\left\{C^1\left(x_j^1\right)\right\}.$$

In words $\left\{C_0^1\left(x_i^0\right)\right\}$ is the union of the cells of $\Theta^1$ whose nuclei lie in $C^0\left(x_i^0\right)$. Due to the independence and stationary assumptions, with probability 1 every $x_j^1$ lies in unique cell of $\Theta^0$. Therefore, $\Theta_0^1$ is again a tessellation, even if some of the cells can be empty. Let $\{\Theta^n\}_{n \in N}$ be a sequence of independent stationary Tessellations with the nuclei sets $\Pi_n = \{X_i^n\}, \quad n \in N$. The aggregation of the first in terms of the sequence yields the aggregate tessellation of order n: $\Theta^n = \Theta^0 \Theta^2 \cdots \Theta^n$ with the nuclei set $\Pi_0 = \{X_i^0\}$. The cells of this tessellation will be called aggregate n-cells an denote by $\left\{C_0^n\left(x_i^0\right)\right\}$. In this condition we follow with Voronoi tessellations and aggregations and weighted Voronoi diagrams.

## 10 Tessellation with Polygons

Any triangle or quadrilateral (even non-convex) can be used as a prototile to form a monohedral tessellation, often in more than one way. If only one shape of tile is allowed, tilings exists with convex $N$-gons for $N$ equal to 3, 4, 5 and 6. For $N = 5$, see Pentagonal tiling, for $N = 6$, see Hexagonal tiling, for $N = 7$, see Heptagonal tiling and for $N = 8$, see octagonal tiling [3, 7, 9].

## 11 Voronoi Tilings

Voronoi or Dirichlet tilings are Tessellations where each tile is defined as the set of points closest to one of the points in a discrete set of defining points. (Think of geographical regions where each region is defined as all the points closest to a given city or post office.) The *Voronoi cell* [3, 9] for each defining point is a convex polygon. The Delaunay triangulation is a tessellation that is the dual graph of a Voronoi tessellation. Voronoi tilings with randomly placed points can be used to construct random tilings of the plane.

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## References

1. https://www.annenewman.com.au/the-alhambra-and-escher/
2. http://rixosmagazine.com/the-alhambra-palace/
3. http://www.Tessellations.org/tess-beginnings.shtml
4. Tchoumathenko K, Zuyev S (2001) Aggregate and fractal tessellations. Probab Theory Relat Field 121:198–218
5. Zabarina K (2018) Quantitative methods in economics, Tessellation as an alternative aggregation method. 78–91
6. Grunbau B (2006) What symmetry groups are present in the Alhambra? Notice of the AMS, vol 53, Num 6, pp 670–673
7. https://en.wikipedia.org/wiki/Tessellation
8. Erika Fecková Škrabuľáková, Elena Grešová (45/2019) Costs Saving via Graph Colouring Approach. Accepted in Scientific Papers of the University of Pardubice, Series D
9. Montesinis JM (1987) Classical tessellations and threefolds. Springer, New York
10. http://mathworld.wolfram.com/WallpaperGroups.html
11. Przybylo J, Schreyer J, Škrabuľáková E (2016) On the facial thue choice number of plane graphs via entropy compression method. Gr Combinat 32(3):1137–1153
12. https://www.nvidia.com/object/tessellation.html