On the Antecedent Normal Form of Conditional Knowledge Bases

Christoph Beierle[(✉)] and Steven Kutsch

FernUniversität in Hagen, 58084 Hagen, Germany
{christoph.beierle,steven.kutsch}@fernuni-hagen.de

Abstract. Desirable properties of a normal form for conditional knowledge are, for instance, simplicity, minimality, uniqueness, and the respecting of adequate equivalences. In this paper, we propose the notion of antecedentwise equivalence of knowledge bases. It identifies more knowledge bases as being equivalent and allows for a simpler and more compact normal form than previous proposals. We develop a set of transformation rules mapping every knowledge base into an equivalent knowledge base that is in antecedent normal form (ANF). Furthermore, we present an algorithm for systematically generating conditional knowledge bases in ANF over a given signature. The approach is complete in the sense that, taking renamings and equivalences into account, every consistent knowledge base is generated. Moreover, it is also minimal in the sense that no two knowledge bases are generated that are antecedentwise equivalent or that are isomorphic to antecedentwise equivalent knowledge bases.

Keywords: Conditional · Knowledge base · Equivalence · Antecedentwise equivalence · Antecedent normal form · ANF · Renaming · Knowledge base generation

1 Introduction

A core question in knowledge representation and reasoning is what a knowledge base consisting of a set of conditionals like "If A then usually B", formally denoted by $(B|A)$, entails [20]. For investigating this question and corresponding properties of a knowledge base, for comparing the inference relations induced by different knowledge bases, for implementing systems realizing reasoning with conditional knowledge bases, and for many related tasks a notion of normal form for knowledge bases is advantageous. Desirable properties of a normal form for conditional knowledge bases are, for instance, simplicity, minimality, uniqueness, and the respecting of adequate equivalences of knowledge bases. Normal forms of conditional knowledge bases have been investigated in e.g. [3,4]. In this paper, we propose the new notion of antecedentwise equivalence of conditional knowledge bases and the concept of antecedent normal form (ANF) of a knowledge base. Antecedentwise equivalence identifies more knowledge bases as being equivalent and allows for a simpler and more compact normal form

G. Kern-Isberner and Z. Ognjanović (Eds.): ECSQARU 2019, LNAI 11726, pp. 175–186, 2019.
https://doi.org/10.1007/978-3-030-29765-7_15

than previous proposals. As an effective way of transforming every knowledge base $\mathcal{R}$ into an equivalent knowledge base being in ANF, we develop a set of transformation rules Θ achieving this goal. Furthermore, we present an algorithm KB^{ae}_{gen} enumerating conditional knowledge bases over a given signature. The algorithm is complete in the sense that every consistent knowledge base is generated when taking renamings and antecedentwise equivalences into account. Moreover, KB^{ae}_{gen} is also minimal: It will not generate any two different knowledge bases $\mathcal{R}$, $\mathcal{R}'$ such that $\mathcal{R}$ and $\mathcal{R}'$ or any isomorphic images of $\mathcal{R}$ and $\mathcal{R}'$ are antecedentwise equivalent. This algorithm is a major improvement over the approach given in [9] because it generates significantly fewer knowledge bases, while still being complete and minimal. Systematic generation of knowledge bases as achieved by KB^{ae}_{gen} is fruitful for various purposes, for instance for the empirical comparison and evaluation of different nonmonotonic inference relations induced by a knowledge base (e.g. [5,17,20,22]) with the help of implemented reasoning systems like InfOCF [6].

For illustrating purposes, we will use ranking functions, also called ordinal conditional functions (OCF) [23,24], as semantics for conditionals. However, it should be noted that all notions and concepts developed in this paper are independent of the semantics of ranking functions we use in this paper. They also apply to every semantics satisfying system P [1,17], e.g., Lewis' system of spheres [21], conditional objects evaluated using Boolean intervals [12], possibility distributions [10], or special classes of ranking functions like c-representations [15]. A common feature of these semantics is that a conditional $(B|A)$ is accepted if its verification $A \wedge B$ is considered more plausible, more possible, less surprising, etc. than its falsification $A \wedge \neg B$.

After recalling required basics in Sect. 2, antecedentwise equivalence and ANF is introduced in Sect. 3. The system Θ transforming a knowledge base into ANF is presented in Sect. 4. Orderings and renamings developed in Sect. 5 are exploited in knowledge base generation by KB^{ae}_{gen} in Sect. 6, before concluding in Sect. 7.

2 Background: Conditional Logic

Let $\mathcal{L}$ be a propositional language over a finite signature Σ of atoms $a, b, c, \ldots$. The formulas of $\mathcal{L}$ will be denoted by letters $A, B, C, \ldots$. We write AB for $A \wedge B$ and $\overline{A}$ for $\neg A$. We identify the set of all complete conjunctions over Σ with the set Ω of possible worlds over $\mathcal{L}$. For $\omega \in \Omega$, $\omega \models A$ means that $A \in \mathcal{L}$ holds in ω, and the set of worlds satisfying A is $\Omega_A = \{\omega \mid \omega \models A\}$. By introducing a new binary operator $|$, we obtain the set $(\mathcal{L} \mid \mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}$ of *conditionals* over $\mathcal{L}$. For a conditional $r = (B|A)$, $ant(r) = A$ is the *antecedent* of r, and $cons(r) = B$ is its *consequent*. The *counter conditional* of $r = (B|A)$ is $\overline{r} = (\overline{B}|A)$. As semantics for conditionals, we use *ordinal conditional functions (OCF)* [24]. An OCF is a function $\kappa : \Omega \rightarrow \mathbb{N}$ expressing degrees of plausibility of possible worlds where a lower degree denotes "less surprising". At least one world must be regarded as being normal; therefore, $\kappa(\omega) = 0$ for at least one $\omega \in \Omega$. Each κ uniquely extends to a function mapping sentences to $\mathbb{N} \cup \{\infty\}$ given by

$\kappa(A) = \min\{\kappa(\omega) \mid \omega \models A\}$ where $\min \emptyset = \infty$. An OCF κ *accepts* a conditional $(B|A)$, written $\kappa \models (B|A)$, if the verification of the conditional is less surprising than its falsification, i.e., if $\kappa(AB) < \kappa(A\overline{B})$; equivalently, $\kappa \models (B|A)$ iff for every $\omega' \in \Omega_{A\overline{B}}$ there is $\omega \in \Omega_{AB}$ with $\kappa(\omega) < \kappa(\omega')$. A conditional $(B|A)$ is trivial if it is *self-fulfilling* ($A \models B$) or *contradictory* ($A \models \overline{B}$); a set of conditionals is self-fulfilling if every conditional in it is self-fulfilling. A finite set $\mathcal{R} \subseteq (\mathcal{L}|\mathcal{L})$ of conditionals is called a *knowledge base*. An OCF κ accepts $\mathcal{R}$ if κ accepts all conditionals in $\mathcal{R}$, and $\mathcal{R}$ is *consistent* if an OCF accepting $\mathcal{R}$ exists [14]. We use $\diamond$ to denote an inconsistent knowledge base. $\mathit{Mod}(\mathcal{R})$ denotes the set of all OCFs κ accepting $\mathcal{R}$. Two knowledge bases $\mathcal{R}, \mathcal{R}'$ are *model equivalent*, denoted by $\mathcal{R} \equiv_{mod} \mathcal{R}'$, if $\mathit{Mod}(\mathcal{R}) = \mathit{Mod}(\mathcal{R}')$. We say $(B|A) \equiv (B'|A')$ if $A \equiv A'$ and $AB \equiv A'B'$. Example 1 presents a knowledge base we will use for illustration.

Example 1 ($\mathcal{R}_{car}$ [4]). Let $\Sigma_{car} = \{c, e, f\}$ where c indicates whether something is a car, e indicates whether something is an e-car, and f indicates whether something needs fossil fuel. The knowledge base $\mathcal{R}_{car}$ contains seven conditionals:
q_1: $(f|c)$ "*Usually cars need fossil fuel.*"
q_2: $(\overline{f}|e)$ "*Usually e-cars do not need fossil fuel.*"
q_3: $(c|e)$ "*E-cars usually are cars.*"
q_4: $(e|e\overline{f})$ "*E-cars that do not need fossil fuel usually are e-cars.*"
q_5: $(e\overline{f}|e)$ "*E-cars usually are e-cars that do not need fossil fuel.*"
q_6: $(\overline{e}|\top)$ "*Usually things are no e-cars.*"
q_7: $(cf \vee \overline{c}f \,|\, ce \vee c\overline{e})$ "*Things that are cars and e-cars or cars but not e-cars are cars that need fossil fuel or are no cars but need fossil fuel.*"

3 Antecedentwise Equivalence of Knowledge Bases

For comparing or generating knowledge bases, it is useful to abstract from merely syntactic variants. In particular, it is desirable to have minimal versions and normal forms of knowledge bases at hand. The following notion of equivalence presented in [4] employs the idea that each piece of knowledge in one knowledge base directly corresponds to a piece of knowledge in the other knowledge base.

Definition 1 (equivalence $\equiv_{ee}$ [4]). *Let $\mathcal{R}$, $\mathcal{R}'$ be knowledge bases.*

- *$\mathcal{R}$ is an elementwise equivalent sub-knowledge base of $\mathcal{R}'$, denoted by $\mathcal{R} \ll_{ee} \mathcal{R}'$, if for every conditional $(B|A) \in \mathcal{R}$ that is not self-fulfilling there is a conditional $(B'|A') \in \mathcal{R}'$ such that $(B|A) \equiv (B'|A')$.*
- *$\mathcal{R}$ and $\mathcal{R}'$ are* strictly elementwise equivalent *if $\mathcal{R} \ll_{ee} \mathcal{R}'$ and $\mathcal{R}' \ll_{ee} \mathcal{R}$.*
- *$\mathcal{R}$ and $\mathcal{R}'$ are* elementwise equivalent*, denoted by $\mathcal{R} \equiv_{ee} \mathcal{R}'$, if either both are inconsistent, or both are consistent and strictly elementwise equivalent.*

Elementwise equivalence is a stricter notion than model equivalence. In [3], as a simple example the knowledge bases $\mathcal{R}_1 = \{(a|\top), (b|\top), (ab|\top)\}$ and $\mathcal{R}_2 = \{(a|\top), (b|\top)\}$ are given which are model equivalent, but not elementwise equivalent since for $(ab|\top) \in \mathcal{R}_1$ there is no corresponding conditional in $\mathcal{R}_2$.

The idea of the notion of antecedentwise equivalence we will introduce here is to take into account the set of conditionals having the same (or propositionally equivalent) antecedent when comparing to knowledge bases.

Definition 2 (**$Ant(\mathcal{R})$, $\mathcal{R}_{|A}$, ANF**). *Let $\mathcal{R}$ be a knowledge base.*

- $Ant(\mathcal{R}) = \{A \mid (B|A) \in \mathcal{R}\}$ *is the set of* antecedents of $\mathcal{R}$.
- *For $A \in Ant(\mathcal{R})$, the set $\mathcal{R}_{|A} = \{(B'|A') \mid (B'|A') \in \mathcal{R}$ and $A \equiv A'\}$ is the set of A-*conditionals in $\mathcal{R}$.
- $\mathcal{R}$ *is in* antecedent normal form (ANF) *if either $\mathcal{R}$ is inconsistent and $\mathcal{R} = \diamond$, or $\mathcal{R}$ is consistent, does not contain any self-fulfilling conditional, contains only conditionals of the form $(AB|A)$, and $\left|\mathcal{R}_{|A}\right| = 1$ for all $A \in Ant(\mathcal{R})$.*

Definition 3 (**$\ll_{ae}$, equivalence $\equiv_{ae}$**). *Let $\mathcal{R}$, $\mathcal{R}'$ be knowledge bases.*

- *$\mathcal{R}$ is an antecedentwise equivalent sub-knowledge base of $\mathcal{R}'$, denoted by $\mathcal{R} \ll_{ae} \mathcal{R}'$, if for every $A \in Ant(\mathcal{R})$ such that $\mathcal{R}_{|A}$ is not self-fulfilling there is an $A' \in Ant(\mathcal{R}')$ with $\mathcal{R}_{|A} \equiv_{mod} \mathcal{R}'_{|A'}$.*
- *$\mathcal{R}$ and $\mathcal{R}'$ are* strictly antecedentwise equivalent *if $\mathcal{R} \ll_{ae} \mathcal{R}'$ and $\mathcal{R}' \ll_{ae} \mathcal{R}$.*
- *$\mathcal{R}$ and $\mathcal{R}'$ are* antecedentwise equivalent, *denoted by $\mathcal{R} \equiv_{ae} \mathcal{R}'$, if either both are inconsistent, or both are consistent and strictly antecedentwise equivalent.*

Note that any two inconsistent knowledge bases are also antecedentwise equivalent according to Definition 3, e.g., $\{(b|a), (\overline{b}|b)\} \equiv_{ae} \{(b|b), (a\overline{a}|\top)\}$, enabling us to avoid cumbersome case distinctions when dealing with consistent and inconsistent knowledge bases. In general, we have:

Proposition 1 (**$\equiv_{ae}$**). *Let $\mathcal{R}, \mathcal{R}'$ be consistent knowledge bases.*

1. *If $\mathcal{R} \ll_{ae} \mathcal{R}'$ then $Mod(\mathcal{R}') \subseteq Mod(\mathcal{R})$.*
2. *If $\mathcal{R} \equiv_{ae} \mathcal{R}'$ then $\mathcal{R} \equiv_{mod} \mathcal{R}'$.*
3. *If $\mathcal{R} \ll_{ee} \mathcal{R}'$ then $\mathcal{R} \ll_{ae} \mathcal{R}'$.*
4. *If $\mathcal{R} \equiv_{ee} \mathcal{R}'$ then $\mathcal{R} \equiv_{ae} \mathcal{R}'$.*
5. *None of the implications (1.)–(4.) holds in general in the reverse direction.*

Proof. (1.) If $\mathcal{R} \ll_{ae} \mathcal{R}'$, Definition 3 implies that there is a function $f : Ant(\mathcal{R}) \to Ant(\mathcal{R}')$ with $\mathcal{R}_{|A} \equiv_{mod} \mathcal{R}'_{|f(A)}$ for each $A \in Ant(\mathcal{R})$. Thus, $\mathcal{R} = \bigcup_{A \in Ant(\mathcal{R})} \mathcal{R}_{|A} \equiv_{mod} \bigcup_{A \in Ant(\mathcal{R})} \mathcal{R}'_{|f(A)} \subseteq \mathcal{R}'$ implies $Mod(\mathcal{R}') \subseteq Mod(R)$. Employing (1.) in both directions, we get (2.).

(3.) If $\mathcal{R} \ll_{ee} \mathcal{R}'$, Definition 1 ensures a function $f : \mathcal{R} \to \mathcal{R}'$ with $\{(B|A)\} \equiv_{mod} \{f((B|A))\}$ for each $(B|A) \in \mathcal{R}$. Hence, $A \equiv A'$ must hold if $(B'|A') = f((B|A))$. Thus, $\{(B|A) \mid (B|A) \in \mathcal{R}_{|A}\} \equiv_{mod} \{f((B|A)) \mid (B|A) \in \mathcal{R}_{|A}\}$ for each $A \in Ant(\mathcal{R})$. Together with $\mathcal{R} = \bigcup_{A \in Ant(\mathcal{R})} \mathcal{R}_{|A}$ and $\{f((B|A)) \mid (B|A) \in \mathcal{R}_{|A}\} \subseteq \mathcal{R}'$ this implies $\mathcal{R} \ll_{ae} \mathcal{R}'$. Employing (3.) in both directions yields (4.).

For proving (5.) w.r.t. both (1.) and (2.), consider $\mathcal{R}_3 = \{(c|a), (c|b)\}$ and $\mathcal{R}_4 = \{(c|a), (c|b), (c|a \vee b)\}$. Then $\mathcal{R}_3 \equiv_{mod} \mathcal{R}_4$ and $\mathcal{R}_3 \ll_{ae} \mathcal{R}_4$, but $\mathcal{R}_4 \not\ll_{ae} \mathcal{R}_3$ and therefore $\mathcal{R}_3 \not\equiv_{ae} \mathcal{R}_4$. For (5.) w.r.t. both (3.) and (4.), consider again $\mathcal{R}_1 = \{(a|\top), (b|\top), (ab|\top)\}$ and $\mathcal{R}_2 = \{(a|\top), (b|\top)\}$. We have $\mathcal{R}_1 \equiv_{ae} \mathcal{R}_2$ because $\mathcal{R}_{1|\top} \equiv_{mod} \mathcal{R}_{2|\top}$, but $\mathcal{R}_1 \not\ll_{ee} \mathcal{R}_2$ and therefore $\mathcal{R}_1 \not\equiv_{ee} \mathcal{R}_2$. $\square$

In the proof of Proposition 1 $\mathcal{R}_1 \not\equiv_{ee} \mathcal{R}_2$ and $\mathcal{R}_1 \equiv_{ae} \mathcal{R}_2$ holds, but also $\mathcal{R}_2 \ll_{ee} \mathcal{R}_1$. The following example shows that two knowledge bases may be antecedentwise equivalent even if they are not comparable with respect to $\ll_{ee}$.

Example 2 ($\equiv_{ae}$). Let $\mathcal{R}_5 = \{(bc|a), (cd|a)\}$ and $\mathcal{R}_6 = \{(bd|a), (bcd|a)\}$. Then $\mathcal{R}_5 \equiv_{ae} \mathcal{R}_6$, but $\mathcal{R}_5 \not\equiv_{ee} \mathcal{R}_6$, $\mathcal{R}_5 \not\ll_{ee} \mathcal{R}_6$, and $\mathcal{R}_6 \not\ll_{ee} \mathcal{R}_5$.

4 Transforming Knowledge Bases into ANF

In order to be able to deal with normal forms of formulas in $\mathcal{L}$ without having to select a specific representation, we assume a function ν mapping a propositional formula A to a unique normal form $\nu(A)$ such that $A \equiv A'$ iff $\nu(A) = \nu(A')$. We also use a function Π with $\Pi(\mathcal{R}) = \diamond$ iff $\mathcal{R}$ is inconsistent; Π can easily be implemented by the tolerance test for conditional knowledge bases [14]. Using Π and the propositional normalization function ν, the system Θ given in Fig. 1 contains four transformation rules:

(SF) removes a self-fulling conditional $(B|A)$ with $A \not\equiv \bot$.
(AE) merges two conditionals $(B|A)$ and $(B'|A')$ with propositionally equivalent antecedents to a conditional having this antecedent and the conjunction of the consequents.
(NO) transforms a conditional $(B|A)$ by sharpening its consequent to the conjunction with its antecedent and propositionally normalizes the antecedent and the resulting consequent.
(IC) transforms an inconsistent knowledge base into $\diamond$.

Example 3 ($\mathcal{N}(\mathcal{R}_{car})$). Consider the knowledge base $\mathcal{R}_{car}$ from Example 1.

(SF) As $ef \models e$, q_4 is self-fulfilling, and the application of (SF) removes q_4.
(AE) Applying this rule to q_3 and q_5 yields $q_8 : (ce\overline{f}|e)$.

$$(SF)\ \textit{self-fulfilling}: \quad \frac{\mathcal{R} \cup \{(B|A)\}}{\mathcal{R}} \quad A \models B,\ A \not\equiv \bot$$

$$(AE)\ \textit{antecedence equivalence}: \quad \frac{\mathcal{R} \cup \{(B|A), (B'|A')\}}{\mathcal{R} \cup \{(BB'|A)\}} \quad A \equiv A'$$

$$(NO)\ \textit{normalization}: \quad \frac{\mathcal{R} \cup \{(B|A)\}}{\mathcal{R} \cup \{(\nu(AB)|\nu(A))\}} \quad A \neq \nu(A) \text{ or } B \neq \nu(AB)$$

$$(IC)\ \textit{inconsistency}: \quad \frac{\mathcal{R}}{\diamond} \quad \mathcal{R} \neq \diamond, \Pi(\mathcal{R}) = \diamond$$

Fig. 1. Transformation rules Θ and their applicability conditions for the normalization of knowledge bases respecting antecedence equivalence; Π is a consistency test, e.g. the tolerance criterion [14], and ν a normalization function for propositional formulas.

(NO) Applying this rule to q_1 or to q_7 yields $\widetilde{q_1} : (\nu(cf)|\nu(c))$ in both cases, applying it to q_2 or to q_5 yields $\widetilde{q_2} : (\nu(e\overline{f})|\nu(e))$, applying it to q_3 yields $\widetilde{q_3} : (\nu(ce)|\nu(e))$, and applying it to q_6 yields $\widetilde{q_6} : (\nu(\overline{e})|\nu(\top))$. Applying (NO) to $q_8 : (ce\overline{f}|e)$ yields $\widetilde{q_8} : (\nu(ce\overline{f})|\nu(e))$; note that first applying (AE) to $\widetilde{q_2}$ and $\widetilde{q_3}$ and then (NO) to the result also yields exactly $\widetilde{q_8}$.
(IC) As $\mathcal{R}_{car}$ is consistent, (IC) can not be applied to $\mathcal{R}_{car}$.

Thus, applying Θ exhaustively and in arbitrary sequence to $\mathcal{R}_{car}$ gives us the knowledge base $\Theta(\mathcal{R}_{car}) = \{\widetilde{q_1}, \widetilde{q_6}, \widetilde{q_8}\}$. In contrast, the transformation system $\mathcal{T}$ given in [4] would yield $\mathcal{T}(\mathcal{R}_{car}) = \{\widetilde{q_1}, \widetilde{q_2}, \widetilde{q_3}, \widetilde{q_6}\}$ containing more conditionals.

Proposition 2 (properties of Θ). *Let $\mathcal{R}$ be a knowledge base.*

1. (termination) *Θ is terminating.*
2. (confluence) *Θ is confluent.*
3. ($\equiv_{mod}$ correctness) *$\mathcal{R} \equiv_{mod} \Theta(\mathcal{R})$.*
4. ($\equiv_{ae}$ correctness) *$\mathcal{R} \equiv_{ae} \Theta(\mathcal{R})$.*
5. ($\equiv_{ae}$ minimizing) *If $\mathcal{R}$ is inconsistent then $\Theta(\mathcal{R}) = \diamond$. If $\mathcal{R}$ is consistent, then for all knowledge bases $\mathcal{R}'$ it holds that $\mathcal{R}' \subsetneqq \Theta(\mathcal{R})$ implies $\mathcal{R}' \not\equiv_{ae} \mathcal{R}$.*
6. (ANF) *$\Theta(\mathcal{R})$ is in antecedent normal form.*

Proof. (1.) (SF), (AE), and (IC) remove at least one conditional, and (NO) can be applied at most once to any conditional. Hence, Θ is terminating.

(2.) Since Θ is terminating, local confluence of Θ implies confluence of Θ; local confluence of Θ in turn can be shown by ensuring that for every critical pair obtained form superpositioning two left hand sides of rules in Θ reduces to the same knowledge base [2,16]: Any critical pair obtained from (IC) and another rule in Θ reduces to $\diamond$ since all rules preserve the consistency status of a knowledge base. Any critical pair obtained from (SF) with (NO) reduces to the same knowledge base since applying (NO) to a self-fulfilling conditional yields again a self-fulfilling conditional. Regarding critical pairs with respect to (NO), we observe that if $\mathcal{R}$ contains two distinct conditionals $(B|A)$ and $(B'|A')$ with $(\nu(AB)|\nu(A)) = (\nu(A'B')|\nu(A'))$, then applying (NO) first to either of the conditionals and second to the other one yields the same result. Critical pairs between (AE) and (NO) reduce to the same result because propositional normalization commutes with (AE). For a critical pair of (SF) and (AE) consider $\mathcal{R}_0 = \mathcal{R} \cup \{(B|A), (B'|A')\}$ with $A \equiv A'$ and $A' \models B'$. Applying (SF) yields $\mathcal{R}_1 = \mathcal{R} \cup \{(B|A)\}$, and applying (AE) yields $\mathcal{R}_2 = \mathcal{R} \cup \{(BB'|A)\}$. Applying (NO) to both $\mathcal{R}_1$ and $\mathcal{R}_2$ yields the same result because $A \equiv A'$, $A' \models B'$ and therefore $AB \equiv ABB'$. Thus, we are left with critical pairs obtained from (AE) which arise from $\mathcal{R} \cup \{(B|A), (B'|A'), (B''|A'')\}$ with $A \equiv A' \equiv A''$ so that (AE) could be applied to $\{(B|A), (B'|A')\}$ and to $\{(B'|A'), (B''|A'')\}$. Applying (AE) to the result followed by (NO) yields $\mathcal{R} \cup \{(\nu(BB'B'')|\nu(A))\}$ in both cases.

(3.) By Proposition 1, (3.) will follow from the proof of (4.).

(4.) We will show that $\equiv_{ae}$-equivalence is preserved by every rule in Θ.

(IC) Since Π is a consistency test, $\mathcal{R} \equiv_{ae} \diamond$ because all inconsistent knowledge bases are $\equiv_{ae}$-equivalent. Because all other rules preserve the consistency status of $\mathcal{R}$, we assume that $\mathcal{R}$ is consistent when dealing with the other rules in Θ. (SF) By Definition 3 we get $\mathcal{R} \cup \{(B|A)\} \equiv_{ae} \mathcal{R}$. (AE) This rule preserves $\equiv_{ae}$-equivalence because $A \equiv A'$ implies $\{(B|A), (B'|A')\} \subseteq (\mathcal{R} \cup \{(B|A), (B'|A')\})_{|A}$, $(BB'|A) \in (\mathcal{R} \cup \{(BB'|A)\})_{|A}$, and $Mod(\{(B|A), (B'|A')\}) = Mod(\{(BB'|A)\})$. (NO) This rule preserves $\equiv_{ae}$-equivalence because $(B|A) \in (\mathcal{R} \cup \{(B|A)\})_{|A}$, $(\nu(AB)|\nu(A)) \in (\mathcal{R} \cup \{(\nu(AB)|\nu(A))\})_{|A}$, and $Mod(\{(B|A)\}) = Mod(\{(\nu(AB)|\nu(A))\})$.

(5.) The $\equiv_{ae}$-minimizing property will follow from the proof of (6.).

(6.) From (1.) and (2.) we conclude that $\Theta(\mathcal{R})$ is well defined. If $\Theta(\mathcal{R})$ was not in ANF then at least one of the rules in Θ would be applicable to $\Theta(\mathcal{R})$, contradicting that Θ has been applied exhaustively. □

Proposition 2 ensures that applying Θ to a knowledge base $\mathcal{R}$ always yields the unique normal form $\Theta(\mathcal{R})$ that is in ANF. This provides a convenient decision procedure for antecedentwise equivalence and thus also for model equivalence.

Proposition 3 (antecedentwise equivalence). *Let $\mathcal{R}$, $\mathcal{R}'$ be knowledge bases. Then $\mathcal{R} \equiv_{ae} \mathcal{R}'$ iff $\Theta(\mathcal{R}) = \Theta(\mathcal{R}')$.*

5 Orderings and Renamings for Conditionals

For developing a method for the systematic generation of knowledge bases in ANF, we will represent each formula $A \in \mathcal{L}$ uniquely by its set Ω_A of satisfying worlds. The two conditions $B \subsetneq A$ and $B \neq \emptyset$ then ensure the falsifiability and the verifiability of a conditional $(B|A)$, thereby excluding any trivial conditional [8]. This yields a propositional normalization function ν, giving us:

Proposition 4 ($NFC(\Sigma)$ [9]). *For $NFC(\Sigma) = \{(B|A) \mid A \subseteq \Omega_\Sigma,\ B \subsetneq A,\ B \neq \emptyset\}$, the set of* normal form conditionals *over a signature Σ, the following holds:*

- **(nontrivial)** *$NFC(\Sigma)$ does not contain any trivial conditional.*
- **(complete)** *For every nontrivial conditional over Σ there is an equivalent conditional in $NFC(\Sigma)$.*
- **(minimal)** *All conditionals in $NFC(\Sigma)$ are pairwise non-equivalent.*

For instance, for $\Sigma_{ab} = \{a, b\}$ we have $(\{ab, a\overline{b}\}|\{ab, \overline{a}\overline{b}\}) \equiv (\{ab\}|\{ab, \overline{a}\overline{b}\})$ where the latter is in $NFC(\Sigma_{ab})$. Out of the different 256 conditionals over Σ_{ab} obtained when using sets of worlds as formulas, only 50 are in $NFC(\Sigma_{ab})$ [9].

For defining a linear order on $NFC(\Sigma)$, we use the following notation. For an ordering relation $\leqslant$ on a set M, its lexicographic extension to strings over M is denoted by $\leqslant_{lex}$. For ordered sets $S, S' \subseteq M$ with $S = \{e_1, \ldots, e_n\}$ and $S' = \{e'_1, \ldots, e'_{n'}\}$ where $e_i \leqslant e_{i+1}$ and $e'_j \leqslant e'_{j+1}$ its extension $\leqslant_{set}$ to sets is:

$$S \leqslant_{set} S' \text{ iff } n < n', \text{ or } n = n' \text{ and } e_1 \ldots e_n \leqslant_{lex} e'_1 \ldots e'_{n'} \tag{1}$$

For Σ with ordering $\lessdot$, $[\![\omega]\!]_\lessdot$ is the usual interpretation of a world ω as a binary number; e.g., for Σ_{ab} with $a \lessdot b$, $[\![ab]\!]_\lessdot = 3$, $[\![a\overline{b}]\!]_\lessdot = 2$, $[\![\overline{a}b]\!]_\lessdot = 1$, and $[\![\overline{a}\overline{b}]\!]_\lessdot = 0$.

Definition 4 (induced ordering on formulas and conditionals). *Let Σ be a signature with linear ordering $\lessdot$. The orderings induced by $\lessdot$ on worlds ω, ω' and conditionals $(B|A), (B'|A')$ over Σ are given by:*

$$\omega \overset{w}{\leqslant} \omega' \text{ iff } [\![\omega]\!]_{\lessdot} \geqslant [\![\omega']\!]_{\lessdot} \tag{2}$$

$$(B|A) \overset{c}{\leqslant} (B'|A') \text{ iff } \Omega_A \overset{w}{\lessdot}_{set} \Omega_{A'}, \text{ or } \Omega_A = \Omega_{A'} \text{ and } \Omega_B \overset{w}{\leqslant}_{set} \Omega_{B'} \tag{3}$$

In order to ease our notation, we will omit the upper symbol in $\overset{w}{\lessdot}$ and $\overset{c}{\lessdot}$, and write just $\lessdot$ instead, and analogously $\leqslant$ for the non-strict variants. For instance, for Σ_{ab} with $a \lessdot b$ we have $ab \lessdot a\overline{b} \lessdot \overline{a}b \lessdot \overline{a}\overline{b}$ for worlds, and $(ab|ab \vee a\overline{b}) \lessdot (ab|ab \vee \overline{a}\overline{b})$ and $(ab \vee \overline{a}\overline{b}|ab \vee a\overline{b} \vee \overline{a}\overline{b}) \lessdot (\overline{a}\overline{b}|ab \vee a\overline{b} \vee \overline{a}b \vee \overline{a}\overline{b})$ for conditionals.

Proposition 5 ($\mathit{NFC}(\Sigma)$, $\lessdot$ [9]**).** *For a linear ordering $\lessdot$ on a signature Σ, the induced ordering $\lessdot$ according to Definition 4 is a linear ordering on $\mathit{NFC}(\Sigma)$.*

Given the ordering $\lessdot$ on $\mathit{NFC}(\Sigma)$ from Proposition 5, we will now define a new ordering $\prec$ on these conditionals that takes isomorphisms (or *renamings*) $\rho : \Sigma \to \Sigma$ into account and prioritizes the $\lessdot$-minimal elements in each isomorphism induced equivalence class. As usual, ρ is extended canonically to worlds, formulas, conditionals, knowledge bases, and to sets thereof. We say that X and X' are *isomorphic*, denoted by $X \simeq X'$, if there exists a renaming ρ such that $\rho(X) = X'$. For a set M, $m \in M$, and an equivalence relation $\equiv$ on M, the set of equivalence classes induced by $\equiv$ is denoted by $[M]_{/\equiv}$, and the unique equivalence class containing m is denoted by $[m]_{\equiv}$. For instance, for Σ_{ab} the only non-identity renaming is the function ρ_{ab} with $\rho_{ab}(a) = b$ and $\rho_{ab}(b) = a$, $[\Omega_{\Sigma_{ab}}]_{/\simeq} = \{[ab], [a\overline{b}, \overline{a}b], [\overline{a}\overline{b}]\}$ are the three equivalence classes of worlds over Σ_{ab}, and we have $[(ab|ab \vee a\overline{b})]_{\simeq} = [(ab|ab \vee \overline{a}b)]_{\simeq}$.

Definition 5 ($\mathit{cNFC}(\Sigma)$, $\prec$ [9]**).** *Given a signature Σ with linear ordering $\lessdot$, let $[\mathit{NFC}(\Sigma)]_{/\simeq} = \{[r_1]_{\simeq}, \ldots, [r_m]_{\simeq}\}$ be the equivalence classes of $\mathit{NFC}(\Sigma)$ induced by isomorphisms such that for each $i \in \{1, \ldots, m\}$, the conditional r_i is the minimal element in $[r_i]_{\simeq}$ with respect to $\lessdot$, and $r_1 \lessdot \ldots \lessdot r_m$. The* canonical normal form conditionals *over Σ are $\mathit{cNFC}(\Sigma) = \{r_1, \ldots, r_m\}$. The* canonical ordering *on $\mathit{NFC}(\Sigma)$, denoted by $\prec$, is given by the schema*

$$r_1 \prec \ldots \prec r_m \prec [r_1]_{\simeq} \setminus \{r_1\} \prec \ldots \prec [r_m]_{\simeq} \setminus \{r_m\}$$

where $r \prec r'$ iff $r \lessdot r'$ for all $i \in \{1, \ldots, m\}$ and all $r, r' \in [r_i]_{\simeq} \setminus \{r_i\}$.

Proposition 6 ($\mathit{NFC}(\Sigma)$, $\prec$ [9]**).** *For a linear ordering $\lessdot$ on a signature Σ, the induced ordering $\prec$ according to Definition 5 is a linear ordering on $\mathit{NFC}(\Sigma)$.*

While $\mathit{NFC}(\Sigma_{ab})$ contains 50 conditionals, there are 31 equivalence classes in $[\mathit{NFC}(\Sigma_{ab})]_{/\simeq}$; hence $\mathit{cNFC}(\Sigma_{ab})$ has 31 elements [9]. The three smallest elements in $\mathit{NFC}(\Sigma_{ab})$ w.r.t. $\prec$ are $(\{ab\}|\{ab, a\overline{b}\})$, $(\{a\overline{b}\}|\{ab, a\overline{b}\})$, $(\{ab\}|\{ab, \overline{a}\overline{b}\})$, and their corresponding equivalence classes are $[(\{ab\}|\{ab, a\overline{b}\}), (\{ab\}|\{ab, \overline{a}b\})]$, $[(\{a\overline{b}\}|\{ab, a\overline{b}\}), (\{\overline{a}b\}|\{ab, \overline{a}b\})]$, and $[(\{ab\}|\{ab, \overline{a}\overline{b}\})]$.

6 Generating Knowledge Bases in ANF

The algorithm KB^{ae}_{gen} (Algorithm 1) generates all consistent knowledge bases up to antecedentwise equivalence and up to isomorphisms. It uses pairs $\langle \mathcal{R}, C \rangle$ where $\mathcal{R}$ is a knowledge base and C is a set of conditionals that are candidates for extending $\mathcal{R}$ to obtain a new knowledge base. For extending $\mathcal{R}$, conditionals are considered sequentially according to their $\prec$ ordering. Note that in Line 3, only the *canonical* conditionals (which are minimal with respect $\prec$) are used for initializing the set of one-element knowledge bases. In Line 3 (and in Line 11, respectively), a conditional r is selected for initializing (or extending, respectively) a knowledge base. In Lines 4–6 (and in lines 13–15, respectively), in the set D conditionals are collected that do not have to be considered as candidates for further extending the current knowledge base: D_1 contains all conditionals that are smaller than r w.r.t. $\prec$, D_2 contains all conditionals having the same antecedent as r (since R should be ANF), and $\overline{r}$ would make $\mathcal{R}$ inconsistent. The consistency test used in Line 12 can easily be implemented by the well-known tolerance test for conditional knowledge bases [14].

Proposition 7 **(KB^{ae}_{gen}).** *Let Σ be a signature with linear ordering $\lessdot$. Then applying KB^{ae}_{gen} to it terminates and returns $\mathcal{KB}$ for which the following holds:*

1. (correctness) *If $\mathcal{R} \in \mathcal{KB}$ then $\mathcal{R}$ is a knowledge base over Σ.*
2. (ANF) *If $\mathcal{R} \in \mathcal{KB}$ then $\mathcal{R}$ is in ANF.*

Algorithm 1. KB^{ae}_{gen} – Generate knowledge bases over Σ up to $\equiv_{ae}$

Input: signature Σ with linear ordering $\lessdot$
Output: set $\mathcal{KB}$ of knowledge bases in ANF of over Σ that are consistent, pairwise antecedentwise non-equivalent and pairwise non-isomorphic

1: $L_1 \leftarrow \emptyset$
2: $k \leftarrow 1$
3: **for** $r \in cNFC(\Sigma)$ **do** ▷ only canonical conditionals for initialization
4: $\quad D_1 \leftarrow \{d \mid d \in NFC(\Sigma), d \preccurlyeq r\}$ ▷ conditional d can not extend $\{r\}$
5: $\quad D_2 \leftarrow \{(B|A) \mid (B|A) \in NFC(\Sigma), A = ant(r)\}$ ▷ $(B|A)$ can not extend $\{r\}$
6: $\quad D \leftarrow D_1 \cup D_2 \cup \{\overline{r}\}$ ▷ $\overline{r}$ can not extend $\{r\}$
7: $\quad L_1 \leftarrow L_1 \cup \{\langle \{r\}, NFC(\Sigma) \setminus D \rangle\}$
8: **while** $L_k \neq \emptyset$ **do**
9: $\quad L_{k+1} \leftarrow \emptyset$
10: $\quad$ **for** $\langle \mathcal{R}, C \rangle \in L_k$ **do** ▷ $\mathcal{R}$ knowledge base, C candidates for extending $\mathcal{R}$
11: $\quad\quad$ **for** $r \in C$ **do**
12: $\quad\quad\quad$ **if** $\mathcal{R} \cup \{r\}$ is consistent **then** ▷ extend $\mathcal{R}$ with conditional r
13: $\quad\quad\quad\quad D_1 \leftarrow \{d \mid d \in C, d \preccurlyeq r\}$ ▷ conditional d can not extend $\mathcal{R} \cup \{r\}$
14: $\quad\quad\quad\quad D_2 \leftarrow \{(B|A) \mid (B|A) \in C, A = ant(r)\}$ ▷ $(B|A)$ can not extend $\mathcal{R} \cup \{r\}$
15: $\quad\quad\quad\quad D \leftarrow D_1 \cup D_2 \cup \{\overline{r}\}$ ▷ $\overline{r}$ can not extend $\mathcal{R} \cup \{r\}$
16: $\quad\quad\quad\quad L_{k+1} \leftarrow L_{k+1} \cup \{\langle \mathcal{R} \cup \{r\}, C \setminus D \rangle\}$
17: $\quad k \leftarrow k + 1$
18: **return** $\mathcal{KB} = \{\mathcal{R} \mid \langle \mathcal{R}, C \rangle \in L_i, i \in \{1, \ldots, k\}\}$

3. ($\equiv_{ae}$ minimality) *If $\mathcal{R}, \mathcal{R}' \in \mathcal{KB}$ and $\mathcal{R} \neq \mathcal{R}'$ then $\mathcal{R} \not\equiv_{ae} \mathcal{R}'$.*
4. ($\simeq$ minimality) *If $\mathcal{R}, \mathcal{R}' \in \mathcal{KB}$ and $\mathcal{R} \neq \mathcal{R}'$ then $\mathcal{R} \not\simeq \mathcal{R}'$.*
5. (consistency) *If $\mathcal{R} \in \mathcal{KB}$ then $\mathcal{R}$ is consistent.*
6. (completeness) *If $\mathcal{R}$is a consistent knowledge base over Σ then there is $\mathcal{R}' \in \mathcal{KB}$ and an isomorphism ρ such that $\mathcal{R} \equiv_{ae} \rho(\mathcal{R}')$.*

Proof. The proof is obtained by formalizing the description of KB^{ae}_{gen} given above and the following observations. Note that KB^{ae}_{gen} exploits the fact that every subset of a consistent knowledge base is again a consistent knowledge base. Thus building up knowledge bases by systematically adding remaining conditionals according to their linear ordering $\prec$ ensures completeness; the removal of candidates in Lines 5 and 14 does not jeopardize completeness since Proposition 2 ensures that for each knowledge base an antecedentwise equivalent knowledge base exists that for any propositional formula A contains at most one conditional with antecedent A. Checking consistency when adding a new conditional ensures consistency of the resulting knowledge base. ANF is ensured because all conditionals in $NFC(\Sigma)$ are of the form $(AB|A)$. Because for all A, each generated $\mathcal{R}$ contains at most one conditional with antecedent A, $\equiv_{ae}$-minimality is guaranteed, and $\simeq$-minimality can be shown by induction on the number of conditionals in a knowledge base. □

Note that KB^{ae}_{gen} generates significantly fewer knowledge bases than the algorithm *GenKB* given in [9]. For each formula A, each $\mathcal{R} \in GenKB(\Sigma)$ may contain up to half of all conditionals in $NFC(\Sigma)$ with antecedent A,[1] while $\mathcal{R} \in KB^{ae}_{gen}(\Sigma)$ may contain at most one conditional with antecedent A.

For instance, $KB^{ae}_{gen}(\Sigma_{ab})$ will generate the knowledge base $\mathcal{R}_7 = \{(\{\overline{a}\overline{b}\}|\{a\overline{b}, \overline{a}\overline{b}\}), (\{a\overline{b}\}|\{ab, a\overline{b}, \overline{a}b\})\}$, but it will not generate the knowledge base $\mathcal{R}_8 = \{(\{\overline{a}\overline{b}\}|\{a\overline{b}, \overline{a}\overline{b}\}), (\{ab, a\overline{b}\}|\{ab, a\overline{b}, \overline{a}b\}), (\{a\overline{b}, \overline{a}b\}|\{ab, a\overline{b}, \overline{a}b\})\}$ which is antecedentwise equivalent to $\mathcal{R}_7$, i.e., $\mathcal{R}_8 \equiv_{ae} \mathcal{R}_7$. Furthermore, $KB^{ae}_{gen}(\Sigma_{ab})$ will also not generate, e.g., the knowledge bases $\mathcal{R}_9 = \{(\{a\overline{b}, \overline{a}b\}|\{ab, a\overline{b}, \overline{a}b\}), (\{\overline{a}\overline{b}\}|\{\overline{a}b, \overline{a}\overline{b}\}), (\{ab, \overline{a}b\}|\{ab, a\overline{b}, \overline{a}b\})\}$ or $\mathcal{R}_{10} = \{(\{\overline{a}\overline{b}\}|\{\overline{a}b, \overline{a}\overline{b}\}), (\{\overline{a}b\}|\{ab, a\overline{b}, \overline{a}b\})\}$ which are both antecedentwise equivalent to $\mathcal{R}_7$ when taking isomorphisms into account; specifically, we have $\rho_{ab}(\mathcal{R}_{10}) = \mathcal{R}_7$, and $\rho_{ab}(\mathcal{R}_9) = \mathcal{R}_8$ and hence also $\rho_{ab}(\mathcal{R}_9) \equiv_{ae} \mathcal{R}_7$.

7 Conclusions and Further Work

Aiming at a compact and unique normal form of conditional knowledge bases, we introduced the new notion of antecedentwise equivalence. We developed a system Θ transforming every knowledge base into its unique antecedent normal form. The algorithm KB^{ae}_{gen} is complete in the sense that it generates, for any signature Σ, knowledge bases in ANF such that all knowledge bases over Σ are

[1] Note that it can not be more than half of these conditionals with the same antecedent because otherwise there would be a conditional together with its counter conditional, leading to inconsistency of the knowledge base.

covered up to isomorphisms and antecedentwise equivalence. Furthermore, the set of knowledge bases returned by KB^{ae}_{gen} is minimal because no two different knowledge bases are generated such that they or any isomorphic images of them are antecedentwise equivalent. Currently, we are working with KB^{ae}_{gen} and the reasoning system `InfOCF` [6] for empirically evaluating different nonmonotonic inference relations induced by a conditional knowledge base and for computing the full closures of such inference relations [18]. Another part of our future work is the investigation of inferential equivalence of ANF (for another normal form see [3,7]) with respect to semantics that are not syntax independent like rational closure (cf. [11,13]), but that are syntax dependent like lexicographic closure [19].

References

1. Adams, E.W.: The Logic of Conditionals: An Application of Probability to Deductive Logic. Synthese Library. Springer, Dordrecht (1975). https://doi.org/10.1007/978-94-015-7622-2
2. Baader, F., Nipkow, T.: Term Rewriting and All That. Cambridge University Press, Cambridge (1998)
3. Beierle, C.: Inferential equivalence, normal forms, and isomorphisms of knowledge bases in institutions of conditional logics. In: Hung, C., Papadopoulos, G.A. (eds.) The 34th ACM/SIGAPP Symposium on Applied Computing (SAC 2019), Limassol, Cyprus, 8–12 April 2019, pp. 1131–1138. ACM, New York (2019)
4. Beierle, C., Eichhorn, C., Kern-Isberner, G.: A transformation system for unique minimal normal forms of conditional knowledge bases. In: Antonucci, A., Cholvy, L., Papini, O. (eds.) ECSQARU 2017. LNCS (LNAI), vol. 10369, pp. 236–245. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-61581-3_22
5. Beierle, C., Eichhorn, C., Kern-Isberner, G., Kutsch, S.: Skeptical, weakly skeptical, and credulous inference based on preferred ranking functions. In: Kaminka, G.A., et al. (eds.) Proceedings 22nd European Conference on Artificial Intelligence, ECAI-2016. Frontiers in Artificial Intelligence and Applications, vol. 285, pp. 1149–1157. IOS Press, Amsterdam (2016)
6. Beierle, C., Eichhorn, C., Kutsch, S.: A practical comparison of qualitative inferences with preferred ranking models. KI Künstliche Intelligenz **31**(1), 41–52 (2017)
7. Beierle, C., Kern-Isberner, G.: Semantical investigations into nonmonotonic and probabilistic logics. Ann. Math. Artif. Intell. **65**(2–3), 123–158 (2012)
8. Beierle, C., Kutsch, S.: Computation and comparison of nonmonotonic skeptical inference relations induced by sets of ranking models for the realization of intelligent agents. Appl. Intell. **49**(1), 28–43 (2019)
9. Beierle, C., Kutsch, S.: Systematic generation of conditional knowledge bases up to renaming and equivalence. In: Calimeri, F., Leone, N., Manna, M. (eds.) JELIA 2019. LNCS (LNAI), vol. 11468, pp. 279–286. Springer, Cham (2019). https://doi.org/10.1007/978-3-030-19570-0_18
10. Benferhat, S., Dubois, D., Prade, H.: Possibilistic and standard probabilistic semantics of conditional knowledge bases. J. Log. Comput. **9**(6), 873–895 (1999)
11. Booth, R., Paris, J.B.: A note on the rational closure of knowledge bases with both positive and negative knowledge. J. Log. Lang. Comput. **7**(2), 165–190 (1998)
12. Dubois, D., Prade, H.: Conditional objects as nonmonotonic consequence relationships. Spec. Issue Conditional Event Algebra IEEE Trans. Syst. Man Cybern. **24**(12), 1724–1740 (1994)

13. Giordano, L., Gliozzi, V., Olivetti, N., Pozzato, G.L.: Semantic characterization of rational closure: from propositional logic to description logics. Artif. Intell. **226**, 1–33 (2015)
14. Goldszmidt, M., Pearl, J.: Qualitative probabilities for default reasoning, belief revision, and causal modeling. Artif. Intell. **84**, 57–112 (1996)
15. Kern-Isberner, G. (ed.): Conditionals in Nonmonotonic Reasoning and Belief Revision. LNCS (LNAI), vol. 2087. Springer, Heidelberg (2001). https://doi.org/10.1007/3-540-44600-1
16. Knuth, D.E., Bendix, P.B.: Simple word problems in universal algebra. In: Leech, J. (ed.) Computational Problems in Abstract Algebra, pp. 263–297. Pergamon Press, Oxford (1970)
17. Kraus, S., Lehmann, D., Magidor, M.: Nonmonotonic reasoning, preferential models and cumulative logics. Artif. Intell. **44**, 167–207 (1990)
18. Kutsch, S., Beierle, C.: Computation of closures of nonmonotonic inference relations induced by conditional knowledge bases. In: Kern-Isberner, G., Ognjanović, Z. (eds.) ECSQARU 2019. LNAI, vol. 11726, pp. 226–237. Springer, Cham (2019)
19. Lehmann, D.: Another perspective on default reasoning. Ann. Math. Artif. Intell. **15**(1), 61–82 (1995)
20. Lehmann, D.J., Magidor, M.: What does a conditional knowledge base entail? Artif. Intell. **55**(1), 1–60 (1992)
21. Lewis, D.: Counterfactuals. Harvard University Press, Cambridge (1973)
22. Paris, J.: The Uncertain Reasoner's Companion - A Mathematical Perspective. Cambridge University Press, Cambridge (1994)
23. Spohn, W.: Ordinal conditional functions: a dynamic theory of epistemic states. In: Harper, W., Skyrms, B. (eds.) Causation in Decision, Belief Change, and Statistics, II, pp. 105–134. Kluwer Academic Publishers, Dordrecht (1988)
24. Spohn, W.: The Laws of Belief: Ranking Theory and Its Philosophical Applications. Oxford University Press, Oxford (2012)