Chapter 6
Generalized Langevin Equation

The Langevin equation is connected to the Brownian motion formulated by Einstein and Smoluchowski. The Langevin equation for a free particle with mass m is given by Langevin [35] (for details, see Ref. [11])

$$m\dot{v}(t) + \gamma v(t) = \xi(t), \tag{6.1}$$
$$\dot{x}(t) = v(t),$$

where $x(t)$ is the particle displacement, $v(t)$ is its velocity, γ is the friction coefficient, and $\xi(t)$ is a Gaussian random noise with zero mean $\langle\xi(t)\rangle = 0$ (so-called white noise). Its correlation has the form

$$\langle\xi(t)\xi(t')\rangle = 2\gamma k_B T\delta(t' - t), \tag{6.2}$$

where k_B is the Boltzmann constant, T is the absolute temperature of the environment in which the particle is immersed, and $2\gamma k_B T$ is the so-called spectral density. The notation $\langle\cdot\rangle$ means ensemble averaging, i.e., statistical averaging over an ensemble of particles at a given moment of time t. Relation (6.2) represents the second fluctuation-dissipation theorem, which is valid only in case of internal noise $\xi(t)$. The Langevin equation (6.1) actually is obtained from the second Newton law of motion of a particle in presence of viscous dynamic friction force $-\gamma\dot{x}(t)$ and an internal random noise $\xi(t)$, which is a residual force due to the interaction of the surrounding molecules on the particle. For a free particle, the MSD at long times reads

$$\langle x^2(t)\rangle = \frac{2k_B T}{\gamma}t,$$

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which is Einstein relation for the Brownian motion. From the MSD, one concludes that the Langevin equation (6.1) describes normal diffusion process, with diffusion coefficient given by

$$D = \lim_{t\to\infty} \frac{\langle x^2(t)\rangle}{2t} = \frac{k_B T}{\gamma}.$$

For the same process, the VACF has exponential decay in respect of time (for details, see next section)

$$\langle v(t)v(0)\rangle = \frac{k_B T}{m} e^{-\frac{\gamma}{m}t}.$$

For a particle in a given potential $V(x)$, the corresponding Langevin equation becomes

$$\begin{aligned} m\dot{v}(t) + \gamma v(t) + \frac{\mathrm{d}V(x(t))}{\mathrm{d}x} &= \xi(t), \\ \dot{x}(t) &= v(t), \end{aligned} \tag{6.3}$$

where

$$F(x) = -\frac{\mathrm{d}V(x(t))}{\mathrm{d}x}$$

is an additional force which acts on the particle due to the potential $V(x)$. For harmonic potential

$$V(x) = \frac{m\omega^2 x^2}{2},$$

the Langevin equation (6.3) turns to

$$\begin{aligned} m\ddot{x}(t) + \gamma\dot{x}(t) + m\omega^2 x(t) &= \xi(t), \\ \dot{x}(t) &= v(t), \end{aligned} \tag{6.4}$$

where ω is the frequency of the oscillator, and m is its mass.

For an internal noise whose correlation is not of the form (6.2), then the Langevin equation (6.3) becomes a GLE [34],

$$\begin{aligned} \ddot{x}(t) + \int_0^t \gamma(t-t')\dot{x}(t')\mathrm{d}t' + \frac{\mathrm{d}V(x(t))}{\mathrm{d}x} &= \xi(t), \\ \dot{x}(t) &= v(t), \end{aligned} \tag{6.5}$$

where we set $m = 1$, and $\gamma(t)$ is the generalized friction memory kernel. The internal noise $\xi(t)$ is of a zero mean ($\langle\xi(t)\rangle = 0$), whose correlation is given by

$$\langle\xi(t)\xi(t')\rangle = C(t' - t). \tag{6.6}$$

When the system reaches an equilibrium state, i.e., the noise is internal, the correlation is related to the friction memory kernel via the second fluctuation-dissipation theorem [34, 42, 72] in the following way:

$$C(t) = k_B T\gamma(t), \tag{6.7}$$

This means that fluctuation and dissipation come from the same source. The friction memory kernel satisfies [12]

$$\lim_{t\to\infty}\gamma(t) = \lim_{s\to 0} s\hat{\gamma}(s) = 0,$$

where $\hat{\gamma}(s) = \mathscr{L}[\gamma(t)](s)$ is the Laplace transform of $\gamma(t)$. If the fluctuation and dissipation do not come from the same source (in case of external noise), then the second fluctuation-dissipation theorem (6.7) is not satisfied, and the system does not reach a unique equilibrium state. The GLE (6.5) for a free particle ($V(x) = 0$) in case of a stationary Gaussian random force $\xi(t)$, in case when the second fluctuation-dissipation theorem holds, describes a stationary, Gaussian, non-Markovian process [19, 20].

GLE has been used to describe anomalous diffusion processes. In the pioneer work of Mainardi and Pironi [42], the authors introduced fractional Langevin equation and showed that it is a special case of a GLE. The M-L function appears in the analysis of the MSD and VACF for a given GLE. Thus, Mainardi and Pironi [42] for the first time in the literature represented the velocity and displacement correlation functions in terms of the M-L functions, and generalized the results for the standard Brownian motion (see also Ref. [40]).

6.1 Free Particle: Generalized M-L Friction

In this section we consider anomalous diffusion of a free particle with mass $m = 1$, driven by stationary random force $\xi(t)$ [34, 42, 72]:

$$\dot{v}(t) + \int_0^t \gamma(t - t')v(t')\mathrm{d}t' = \xi(t), \tag{6.8}$$
$$\dot{x}(t) = v(t),$$

where the noise $\xi(t)$ is internal noise. Therefore, the second fluctuation-dissipation theorem (6.7) holds.

The anomalous diffusion process can be modeled by GLE with internal noise, which correlation is of power-law form [5, 6, 12, 39, 68]

$$C(t) = C_\lambda \frac{t^{-\lambda}}{\Gamma(1-\lambda)},$$

where C_λ is a proportionality coefficient independent on time and which dependents on the exponent λ ($0 < \lambda < 1$ or $1 < \lambda < 2$). In some investigations [40, 42] the friction memory kernel is represented as a superposition of Dirac delta and power-law function.

Generalization of the power law correlation function is the one parameter M-L correlation function [7, 66, 67]

$$C(t) = \frac{C_\lambda}{\tau^\lambda} E_\lambda(-(t/\tau)^\lambda),$$

where τ is the characteristic memory time, $0 < \lambda < 2$, and $E_\lambda(z)$ is the one parameter M-L function (1.1). Furthermore, more generalized friction memory kernel of the form

$$C(t) = \frac{C_\lambda}{\tau^\lambda} t^{\nu-1} E_{\lambda,\nu}(-(t/\tau)^\lambda),$$

was introduced [8, 16], where $E_{\lambda,\nu}(z)$ is the two parameter M-L function (1.4).

We have introduced the three parameter M-L friction memory kernel [59]

$$C(t) = \frac{C_{\alpha,\beta,\delta}}{\tau^{\alpha\delta}} t^{\beta-1} E_{\alpha,\beta}^{\delta}\left(-\frac{t^\alpha}{\tau^\alpha}\right), \tag{6.9}$$

where τ is the characteristic memory time, $C_{\alpha,\beta,\delta}$ may depend on α, β, and δ ($\alpha > 0$, $\beta > 0$, $\delta > 0$), and $E_{\alpha,\beta}^{\delta}(z)$ is the three parameter M-L function (1.14). This noise (6.9) contains several parameters and a number of limiting cases, which means that the obtained results can be used for better description and fits of experimental data. Note that, from relation (1.29), for the generalized M-L noise (6.9) one has

$$\gamma(t) \simeq t^{-\alpha\delta+\beta-1}, \quad t \to \infty. \tag{6.10}$$

For fulfillment of the condition the friction memory kernel $\gamma(t)$ goes to zero for $t \to \infty$ [12],

$$\lim_{t\to\infty} \gamma(t) = \lim_{s\to 0} s\hat{\gamma}(s) = 0, \tag{6.11}$$

where $\hat{\gamma}(s) = \mathscr{L}[\gamma(t)](s)$, one should consider such values of parameters for which $\beta < 1 + \alpha\delta$ is satisfied.

The three parameter M-L noise (6.9) is a generalization of the two parameter M-L noise, which is obtained for $\delta = 1$. For $\beta = \delta = 1$ it yields the one parameter M-L noise. From the asymptotic behavior of three parameter M-L noise (6.9), for $\tau \to 0$, $\beta = \delta = 1$ and $\alpha \neq 1$, one recovers the power-law correlation function. Setting $\alpha = \delta = 1$, the correlation function corresponds to the one for the standard Ornstein-Uhlenbeck process

$$C(t) = \frac{C_{1,1,1}}{\tau} e^{-t/\tau},$$

which for $\tau \to 0$ turns to the correlation function for the standard Brownian motion.

6.1.1 Relaxation Functions

In order to find the MSD and VACF we use the Laplace transform method [40, 42], and the so-called relaxation functions. Thus, from Eq. (6.8) it follows

$$\mathscr{L}[v(t)] = v_0 \frac{1}{s + \mathscr{L}[\gamma(t)]} + \frac{1}{s + \mathscr{L}[\gamma(t)]} \mathscr{L}[\xi(t)]. \tag{6.12}$$

From relation (6.12) for the displacement $x(t)$ and velocity $v(t) = \dot{x}(t)$ one obtains

$$x(t) = \langle x(t) \rangle + \int_0^t G(t - t')\xi(t')\,\mathrm{d}t', \tag{6.13}$$

$$v(t) = \langle v(t) \rangle + \int_0^t g(t - t')\xi(t')\,\mathrm{d}t', \tag{6.14}$$

where

$$\langle x(t) \rangle = x_0 + v_0 G(t), \tag{6.15}$$

$$\langle v(t) \rangle = v_0 g(t) \tag{6.16}$$

and

$$G(t) = \int_0^t g(t')\,\mathrm{d}t'. \tag{6.17}$$

The function $g(t)$ represents inverse Laplace transform of $\hat{g}(s)$,

$$\hat{g}(s) = \frac{1}{s + \hat{\gamma}(s)}, \tag{6.18}$$

where

$$\hat{\gamma}(s) = \mathscr{L}[\gamma(t)](s) = \frac{C_{\alpha,\beta,\delta}}{k_B T \tau^{\alpha\delta}} \frac{s^{\alpha\delta-\beta}}{\left(s^{\alpha} + \tau^{-\alpha}\right)^{\delta}}$$

is obtained from Laplace transform formula (1.17) for the three parameter M-L function. The function

$$I(t) = \int_0^t G(t')\,\mathrm{d}t' \tag{6.19}$$

is also of interest in the analysis of the velocity and displacement correlation functions as we will see later. Therefore,

$$\hat{G}(s) = s^{-1}\hat{g}(s) = \frac{1}{s^2 + s\hat{\gamma}(s)}, \tag{6.20}$$

and

$$\hat{I}(s) = s^{-1}\hat{G}(s) = \frac{s^{-1}}{s^2 + s\hat{\gamma}(s)}. \tag{6.21}$$

These functions $I(t)$, $G(t)$, and $g(t)$ are known as relaxation function, and by analysis of their behavior one can show the existence of anomalous diffusion.

From relation (6.18) it follows

$$\hat{g}(s) = \frac{1}{s + \gamma_{\alpha,\beta,\delta}\frac{s^{\alpha\delta-\beta}}{\left(s^{\alpha}+\tau^{-\alpha}\right)^{\delta}}} = \frac{s^{\frac{1+\beta}{2}-1}}{s^{\frac{1+\beta}{2}} + \gamma_{\alpha,\beta,\delta}\frac{s^{\alpha\delta-\frac{1+\beta}{2}}}{\left(s^{\alpha}+\tau^{-\alpha}\right)^{\delta}}}, \tag{6.22}$$

where $\gamma_{\alpha,\beta,\delta} = \frac{C_{\alpha,\beta,\delta}}{k_B T \tau^{\alpha\delta}}$. Relaxation function $g(t)$ can be obtained by applying relation (1.18) with $\alpha \to \frac{1+\beta}{2}$, $\rho \to \alpha$, $\gamma \to \delta$, $\lambda \to -\gamma_{\alpha,\beta,\delta}$, $\nu \to -\tau^{-\alpha}$, $\mu \to 1$ to (6.22). Thus, we obtain [59]

$$g(t) = \sum_{k=0}^{\infty}(-1)^k \gamma_{\alpha,\beta,\delta}^k t^{(1+\beta)k} E_{\alpha,(1+\beta)k+1}^{\delta k}\left(-(t/\tau)^{\alpha}\right). \tag{6.23}$$

By using relation (1.19) in (6.19) and (6.17), one finds

$$G(t) = \sum_{k=0}^{\infty}(-1)^k \gamma_{\alpha,\beta,\delta}^k t^{(1+\beta)k+1} E_{\alpha,(1+\beta)k+2}^{\delta k}\left(-(t/\tau)^{\alpha}\right), \tag{6.24}$$

$$I(t) = \sum_{k=0}^{\infty}(-1)^k \gamma_{\alpha,\beta,\delta}^k t^{(1+\beta)k+2} E_{\alpha,(1+\beta)k+3}^{\delta k}\left(-(t/\tau)^{\alpha}\right). \tag{6.25}$$

The mean velocity (6.16) and mean particle displacement (6.15) then become

$$\langle v(t)\rangle = v_0 \sum_{k=0}^{\infty}(-1)^k \gamma_{\alpha,\beta,\delta}^k t^{(1+\beta)k} E_{\alpha,(1+\beta)k+1}^{\delta k}\left(-(t/\tau)^{\alpha}\right), \tag{6.26}$$

$$\langle x(t)\rangle = x_0 + v_0 \sum_{k=0}^{\infty}(-1)^k \gamma_{\alpha,\beta,\delta}^k t^{(1+\beta)k+1} E_{\alpha,(1+\beta)k+2}^{\delta k}\left(-(t/\tau)^{\alpha}\right). \tag{6.27}$$

Note that for $\tau \to 0$, by using relation (1.28), for the relaxation functions we have

$$g(t) = E_{1+\beta-\alpha\delta}\left(-\frac{C_{\alpha,\beta,\delta}}{k_B T} t^{1+\beta-\alpha\delta}\right), \tag{6.28}$$

$$G(t) = t E_{1+\beta-\alpha\delta,2}\left(-\frac{C_{\alpha,\beta,\delta}}{k_B T} t^{1+\beta-\alpha\delta}\right), \tag{6.29}$$

$$I(t) = t^2 E_{1+\beta-\alpha\delta,3}\left(-\frac{C_{\alpha,\beta,\delta}}{k_B T} t^{1+\beta-\alpha\delta}\right), \tag{6.30}$$

from where for $\beta = \delta = 1$, which corresponds to the power-law correlation function, we obtain the well-known results (see, for example, [39])

$$I(t) = t^2 E_{2-\alpha,3}\left(-\frac{C_{\alpha,1,1}}{k_B T} t^{2-\alpha}\right) \simeq \frac{k_B T}{C_{\alpha,1,1}} \frac{t^{\alpha}}{\Gamma(1+\alpha)} \quad \text{for} \quad t \to \infty. \tag{6.31}$$

Remark 6.1 ([59]) The function $g(t)$ given by (6.23) is uniformly convergent series with argument t/τ for all $t \in \mathbb{R}$. This can be shown in the following way. The function $g(t)$ is a double series of form

$$g(t) = \sum_{k=0}^{\infty} b_k t^{(1+\beta)k} \sum_{m=0}^{\infty} f_{k,m}(t), \tag{6.32}$$

where $b_k = (-1)^k \gamma_{\alpha,\beta,\delta}^k$, and

$$f_{k,m}(t) = \frac{(\delta k)_m}{\Gamma(\alpha m + (1+\beta)k + 1)} \frac{(-1)^m}{m!} \left(\frac{t}{\tau}\right)^{\alpha m}.$$

To show that the series (6.32) converges uniformly, we have to demonstrate that both series with respect to columns (keeping k fixed and summing m) and the series with respect to the rows (summing k for fixed m) lead to uniformly convergent series. In that case the resulting function $g(t)$ is continuous within the radius of convergence and can be integrated within the interval of convergence. As the three parameter M-L function (1.14) defines an absolutely converging function, which is easily demonstrated by a ratio test, we only need to verify the summation over k with fixed m. Let us use

$$a_k = b_k \frac{(\delta k)_m}{\Gamma(\alpha m + (1+\beta)k + 1)}.$$

By using [31]

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1 + \frac{(a-b)(a+b-1)}{2z} + O\left(\frac{1}{z^2}\right)\right], \tag{6.33}$$

$$(|z| \to \infty, |\arg(z)| \le \pi - \varepsilon, |\arg(z+a)| \le \pi - \varepsilon, 0 < \varepsilon < \pi)$$

we find that

$$\begin{aligned}
\left|\frac{a_{k+1} t^{(1+\beta)(k+1)}}{a_k t^{(1+\beta)k}}\right| &= \left|\frac{\gamma_{\alpha,\beta,\delta}\Gamma(\delta(k+1)+m)\Gamma(\delta k)\Gamma(\alpha m + (1+\beta)k + 1) t^{1+\beta}}{\Gamma(\delta(k+1))\Gamma(\delta k + m)\Gamma(\alpha m + (1+\beta)(k+1)+1)}\right| \\
&= \left|\gamma_{\alpha,\beta,\delta} t^{1+\beta}\right| \times \left|\frac{\Gamma(\delta k + \delta + m)}{\Gamma(\delta k + \delta)}\right| \times \left|\frac{\Gamma(\delta k)}{\Gamma(\delta k + m)}\right| \\
&\quad \times \left|\frac{\Gamma((1+\beta)k + \alpha m + 1)}{\Gamma((1+\beta)k + \alpha m + 1 + (1+\beta))}\right| \\
&\simeq \left|\gamma_{\alpha,\beta,\delta} t^{1+\beta}\right| |\alpha m + (1+\beta)k + 1|^{-(1+\beta)},
\end{aligned} \tag{6.34}$$

which goes to zero if $k \to \infty$. Thus we prove that the series is uniformly convergent. The convergence of series in M-L functions has been extensively studied by Paneva-Konovska in a series of works [49–51].

6.1.2 Velocity and Displacement Correlation Functions

From the general expressions for the velocity and displacement correlation functions [12, 52]

$$\langle v(t)v(t')\rangle = k_B T g(|t-t'|) + \left(v_0^2 - k_B T\right) g(t)g(t'), \tag{6.35a}$$

$$\begin{aligned}
\langle x(t)x(t')\rangle &= x_0^2 + \left(v_0^2 - k_B T\right) G(t)G(t') + C_0 v_0 \left(G(t) + G(t')\right) \\
&\quad + k_B T \left(I(t) + I(t') - I(|t-t'|)\right),
\end{aligned} \tag{6.35b}$$

we obtain the following exact results for the generalized M-L memory kernel [59]

$$\begin{aligned}
\langle v(t)v(t')\rangle &= k_B T \sum_{k=0}^{\infty}(-1)^k \gamma_{\alpha,\beta,\delta}^k (|t-t'|)^{(1+\beta)k} E_{\alpha,(1+\beta)k+1}^{\delta k}\left(-(|t-t'|/\tau)^{\alpha}\right) \\
&+ \left(v_0^2 - k_B T\right) \sum_{k=0}^{\infty}(-1)^k \gamma_{\alpha,\beta,\delta}^k t^{(1+\beta)k} E_{\alpha,(1+\beta)k+1}^{\delta k}\left(-(t/\tau)^{\alpha}\right) \\
&\times \sum_{l=0}^{\infty}(-1)^l \gamma_{\alpha,\beta,\delta}^l t'^{(1+\beta)l} E_{\alpha,(1+\beta)l+1}^{\delta l}\left(-(t'/\tau)^{\alpha}\right),
\end{aligned} \tag{6.36}$$

$$\begin{aligned}
\langle x(t)x(t')\rangle &= x_0^2 + \left(v_0^2 - k_B T\right) \sum_{k=0}^{\infty}(-1)^k \gamma_{\alpha,\beta,\delta}^k t^{(1+\beta)k+1} E_{\alpha,(1+\beta)k+2}^{\delta k}\left(-(t/\tau)^{\alpha}\right) \\
&\times \sum_{l=0}^{\infty}(-1)^l \gamma_{\alpha,\beta,\delta}^l t'^{(1+\beta)l+1} E_{\alpha,(1+\beta)l+2}^{\delta l}\left(-(t'/\tau)^{\alpha}\right) \\
&+ x_0 v_0 \sum_{k=0}^{\infty}(-1)^k \gamma_{\alpha,\beta,\delta}^k \\
&\times \Big[t^{(1+\beta)k+1} E_{\alpha,(1+\beta)k+2}^{\delta k}\left(-(t/\tau)^{\alpha}\right) \\
&+ t'^{(1+\beta)k+1} E_{\alpha,(1+\beta)k+2}^{\delta k}\left(-(t'/\tau)^{\alpha}\right)\Big] \\
&+ k_B T \sum_{k=0}^{\infty}(-1)^k \gamma_{\alpha,\beta,\delta}^k \\
&\times \Big[t^{(1+\beta)k+2} E_{\alpha,(1+\beta)k+3}^{\delta k}\left(-(t/\tau)^{\alpha}\right) \\
&+ t'^{(1+\beta)k+2} E_{\alpha,(1+\beta)k+3}^{\delta k}\left(-(t'/\tau)^{\alpha}\right)\Big] \\
&- k_B T \sum_{k=0}^{\infty}(-1)^k \gamma_{\alpha,\beta,\delta}^k (|t-t'|)^{(1+\beta)k+2} \\
&\times E_{\alpha,(1+\beta)k+3}^{\delta k}\left(-(|t-t'|/\tau)^{\alpha}\right).
\end{aligned} \tag{6.37}$$

For $t = t'$ it eventually leads to

$$\langle v^2(t))\rangle = k_B T + \left(v_0^2 - k_B T\right)\left(\sum_{k=0}^{\infty}(-1)^k \gamma_{\alpha,\beta,\delta}^k t^{(1+\beta)k} E_{\alpha,(1+\beta)k+1}^{\delta k}\left(-(t/\tau)^{\alpha}\right)\right)^2, \tag{6.38}$$

$$\langle x^2(t)\rangle = x_0^2 + \left(v_0^2 - k_BT\right)\left(\sum_{k=0}^{\infty}(-1)^k\gamma_{\alpha,\beta,\delta}^k t^{(1+\beta)k+1}E_{\alpha,(1+\beta)k+2}^{\delta k}\left(-(t/\tau)^\alpha\right)\right)^2$$
$$+ 2x_0v_0\sum_{k=0}^{\infty}(-1)^k\gamma_{\alpha,\beta,\delta}^k t^{(1+\beta)k+1}E_{\alpha,(1+\beta)k+2}^{\delta k}\left(-(t/\tau)^\alpha\right)$$
$$+ 2k_BT\sum_{k=0}^{\infty}(-1)^k\gamma_{\alpha,\beta,\delta}^k t^{(1+\beta)k+2}E_{\alpha,(1+\beta)k+3}^{\delta k}\left(-(t/\tau)^\alpha\right). \tag{6.39}$$

Thus, for the time-dependent diffusion coefficient [42, 53]

$$D(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle x^2(t)\rangle, \tag{6.40}$$

by using relation (1.19) we obtain

$$D(t) = \left(v_0^2 - k_BT\right)\left[\sum_{k=0}^{\infty}(-1)^k\gamma_{\alpha,\beta,\delta}^k t^{(1+\beta)k+1}E_{\alpha,(1+\beta)k+2}^{\delta k}\left(-(t/\tau)^\alpha\right)\right]$$
$$\times\left[\sum_{l=0}^{\infty}(-1)^l\gamma_{\alpha,\beta,\delta}^l t^{(1+\beta)l}E_{\alpha,(1+\beta)l+1}^{\delta l}\left(-(t/\tau)^\alpha\right)\right]$$
$$+ x_0v_0\sum_{k=0}^{\infty}(-1)^k\gamma_{\alpha,\beta,\delta}^k t^{(1+\beta)k}E_{\alpha,(1+\beta)k+1}^{\delta k}\left(-(t/\tau)^\alpha\right)$$
$$+ k_BT\sum_{k=0}^{\infty}(-1)^k\gamma_{\alpha,\beta,\delta}^k t^{(1+\beta)k+1}E_{\alpha,(1+\beta)k+2}^{\delta k}\left(-(t/\tau)^\alpha\right). \tag{6.41}$$

Here we consider thermal initial conditions $x_0 = 0$ and $v_0 = k_BT$. From the general expressions of the velocity and displacement correlation functions (6.35a) and (6.35b), one finds that the relaxation functions, under the assumption (6.11), are connected to the MSD, time dependent diffusion coefficient and VACF in the following way [12, 42, 66], respectively,

$$\left\langle x^2(t)\right\rangle = 2k_BTI(t), \tag{6.42}$$

$$D(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle x^2(t)\right\rangle = k_BTG(t), \tag{6.43}$$

$$C_V(t) = \frac{\langle v(t)v(0)\rangle}{\left\langle v^2(0)\right\rangle} = g(t). \tag{6.44}$$

Furthermore, these relaxation functions can be used to find variances [12, 17, 66, 68]

$$\sigma_{xx} = \langle x^2(t)\rangle - \langle x(t)\rangle^2 = k_BT\left[2I(t) - G^2(t)\right], \tag{6.45a}$$

$$\begin{aligned}\sigma_{xv} &= \langle (v(t) - \langle v(t)\rangle)(x(t) - \langle x(t)\rangle)\rangle \\ &= \frac{1}{2}\frac{\mathrm{d}\sigma_{xx}}{\mathrm{d}t} = k_BTG(t)\left[1 - g(t)\right],\end{aligned} \tag{6.45b}$$

$$\sigma_{vv} = \langle v^2(t)\rangle - \langle v(t)\rangle^2 = k_BT\left[1 - g^2(t)\right]. \tag{6.45c}$$

Therefore, for thermal initial conditions, $x_0 = 0$ and $v_0^2 = k_BT$, for the MSD (6.39), $D(t)$ (6.41) and VACF (6.36), we obtain [59]

$$\langle x^2(t)\rangle = 2k_BT\sum_{k=0}^{\infty}(-1)^k\gamma_{\alpha,\beta,\delta}^k t^{(1+\beta)k+2}E_{\alpha,(1+\beta)k+3}^{\delta k}\left(-(t/\tau)^\alpha\right) = 2k_BTI(t), \tag{6.46}$$

$$D(t) = k_BT\sum_{k=0}^{\infty}(-1)^k\gamma_{\alpha,\beta,\delta}^k t^{(1+\beta)k+1}E_{\alpha,(1+\beta)k+2}^{\delta k}\left(-(t/\tau)^\alpha\right) = k_BTG(t), \tag{6.47}$$

$$C_V(t) = \frac{\langle v(t)v(0)\rangle}{k_BT} = \sum_{k=0}^{\infty}(-1)^k\gamma_{\alpha,\beta,\delta}^k t^{(1+\beta)k}E_{\alpha,(1+\beta)k+1}^{\delta k}\left(-(t/\tau)^\alpha\right) = g(t), \tag{6.48}$$

respectively. Graphical representation of the MSD (6.46) and VACF (6.48), in case of thermal initial conditions is given in Figs. 6.1, 6.2 and 6.3.

6.1.3 Anomalous Diffusive Behavior

The anomalous diffusive behavior of the particle can be obtained either from the exact results by using properties of the three parameter M-L function or by using the Tauberian theorems [18] (see Appendix B), as it was done by Gorenflo and Mainardi in Ref. [24]. From relation (1.28) it follows that

$$\gamma(t) \simeq \frac{\gamma_{\alpha,\beta,\delta}\tau^{\alpha\delta}}{\Gamma(\beta - \alpha\delta)} \times t^{-\alpha\delta+\beta-1}$$

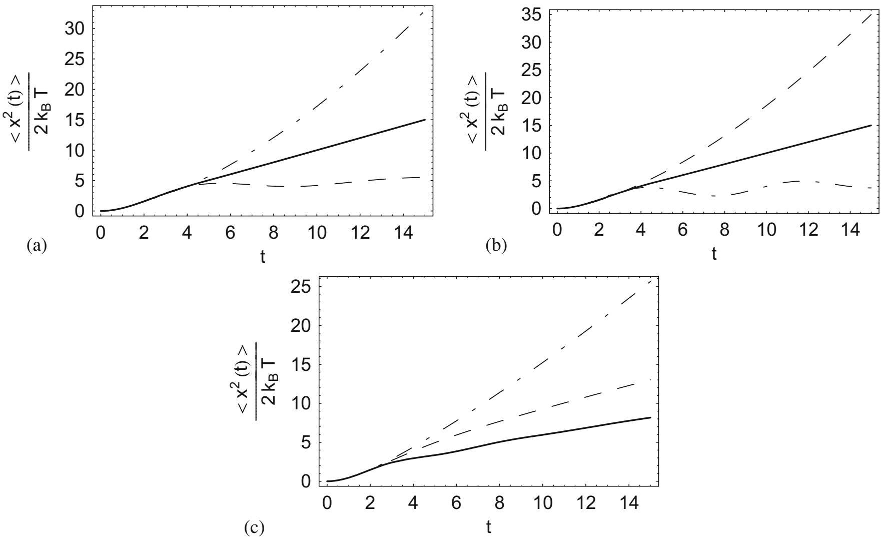

Fig. 6.1 Graphical representation of the MSD (6.46) for $\tau = 1$, $C_{\alpha,\beta,\delta} = 1$, $k_BT = 1$; (**a**) $\beta = \delta = 1$: $\alpha = 1$ (solid line), $\alpha = 1/2$ (dashed line), $\alpha = 3/2$ (dot-dashed line); (**b**) $\alpha = \delta = 1$: $\beta = 1$ (solid line), $\beta = 1/2$ (dashed line), $\beta = 3/2$ (dot-dashed line); (**c**) $\alpha = 3/2$, $\beta = 1$, $\delta = 1/2$ (solid line), $\alpha = \beta = 1/2$, $\delta = 3/4$ (dashed line), $\alpha = 3/4$, $\beta = 1/2$, $\delta = 1$ (dot-dashed line). Reprinted from Physica A, 390, T. Sandev, Z. Tomovski, and J.L.A. Dubbeldam, Generalized Langevin equation with a three parameter Mittag-Leffler noise, 3627–3636, Copyright (2011), with permission from Elsevier

for long times ($\alpha\delta \neq \beta$), so from the Tauberian theorems, one obtains [59]

$$\hat{\gamma}(s) \simeq \frac{C_{\alpha,\beta,\delta}}{k_BT} \cdot s^{\alpha\delta-\beta}, \quad s \to 0. \tag{6.49}$$

From (6.18), (6.17), (6.19), (6.112), and (1.17) it follows

$$\hat{g}(s) \simeq \frac{1}{s + \frac{C_{\alpha,\beta,\delta}}{k_BT} \cdot s^{\alpha\delta-\beta}} = \frac{s^{\beta-\alpha\delta}}{s^{1+\beta-\alpha\delta} + \frac{C_{\alpha,\beta,\delta}}{k_BT}}, \quad s \to 0, \tag{6.50}$$

$$g(t) \simeq E_{1+\beta-\alpha\delta}\left(-\frac{C_{\alpha,\beta,\delta}}{k_BT} \cdot t^{1+\beta-\alpha\delta}\right), \quad t \to \infty, \tag{6.51}$$

$$G(t) \simeq tE_{1+\beta-\alpha\delta,2}\left(-\frac{C_{\alpha,\beta,\delta}}{k_BT} \cdot t^{1+\beta-\alpha\delta}\right), \quad t \to \infty, \tag{6.52}$$

$$I(t) \simeq t^2E_{1+\beta-\alpha\delta,3}\left(-\frac{C_{\alpha,\beta,\delta}}{k_BT} \cdot t^{1+\beta-\alpha\delta}\right), \quad t \to \infty. \tag{6.53}$$

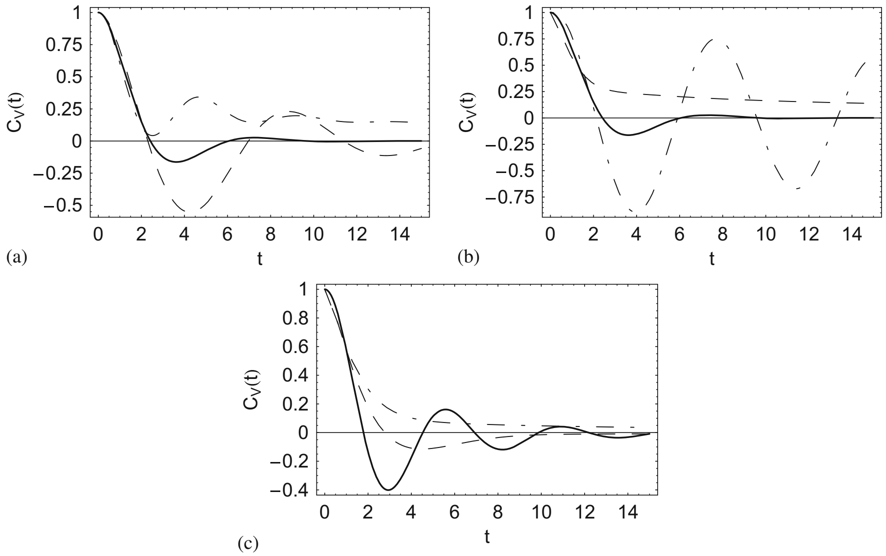

Fig. 6.2 Graphical representation of the VACF (6.48) for $\tau = 1$, $C_{\alpha,\beta,\delta} = 1$, $k_B T = 1$; (**a**) $\beta = \delta = 1$: $\alpha = 1$ (solid line), $\alpha = 1/2$ (dashed line), $\alpha = 3/2$ (dot-dashed line); (**b**) $\alpha = \delta = 1$: $\beta = 1$ (solid line), $\beta = 1/2$ (dashed line), $\beta = 3/2$ (dot-dashed line); (**c**) $\alpha = 3/2$, $\beta = 1$, $\delta = 1/2$ (solid line), $\alpha = \beta = 1/2$, $\delta = 3/4$ (dashed line), $\alpha = 3/4$, $\beta = 1/2$, $\delta = 1$ (dot-dashed line). Reprinted from Physica A, 390, T. Sandev, Z. Tomovski, and J.L.A. Dubbeldam, Generalized Langevin equation with a three parameter Mittag-Leffler noise, 3627–3636, Copyright (2011), with permission from Elsevier

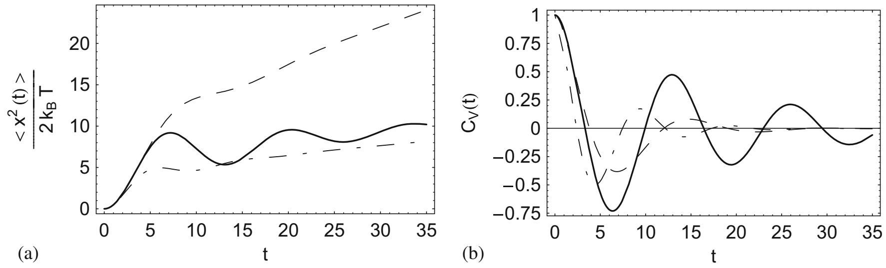

Fig. 6.3 Graphical representation of exact results (6.46) and (6.48), respectively, for $\tau = 10$, $C_{\alpha,\beta,\delta} = 1$, $k_B T = 1$, $\alpha = 1/2$, and $\beta = \delta = 1$ (solid line), $\beta = 3/4$, $\delta = 1$ (dashed line), $\beta = 3/4$, $\delta = 1/2$ (dot-dashed line); (**a**) MSD (6.46); (**b**) VACF (6.48). Reprinted from Physica A, 390, T. Sandev, Z. Tomovski, and J.L.A. Dubbeldam, Generalized Langevin equation with a three parameter Mittag-Leffler noise, 3627–3636, Copyright (2011), with permission from Elsevier

From the asymptotic expansion formula (1.28) of the three parameter M-L function, one finds

$$g(t) \simeq \frac{k_B T}{C_{\alpha,\beta,\delta}\Gamma(\alpha\delta - \beta)} \cdot t^{\alpha\delta-\beta-1}, \tag{6.54}$$

$$G(t) \simeq \frac{k_B T}{C_{\alpha,\beta,\delta}\Gamma(\alpha\delta - \beta + 1)} \cdot t^{\alpha\delta-\beta}, \tag{6.55}$$

$$I(t) \simeq \frac{k_B T}{C_{\alpha,\beta,\delta}\Gamma(\alpha\delta - \beta + 2)} \cdot t^{\alpha\delta-\beta+1}. \tag{6.56}$$

Thus, the time-dependent diffusion coefficient gets the form [59]

$$D(t) \simeq \frac{(k_B T)^2}{C_{\alpha,\beta,\delta}\Gamma(\alpha\delta - \beta + 1)} \cdot t^{\alpha\delta-\beta}. \tag{6.57}$$

From (6.57) we conclude that for $\beta - 1 < \alpha\delta < \beta$ in the long time limit the particle motion is subdiffusive, and for $\beta < \alpha\delta$—superdiffusive [59]. Note that for $\beta = 1$ the obtained results are same as those in Ref. [57] (where $\beta = 1$, $\omega = 0$). For $\beta = \delta = 1$, the results obtained in Refs. [39, 53, 59] are recovered. The case with $\alpha = \beta = \delta = 1$ corresponds to the one considered in Refs. [42, 53, 59]. For $\delta = 1$ one derives the relaxation functions obtained in Ref. [8] ($\omega = 0$, $\alpha = 2$, $\beta = 1$). Comparison of the asymptotic and exact results for the MSD and VACF for thermal initial conditions is given in Fig. 6.4. In Fig. 6.5 comparison with the results for the Brownian motion is given.

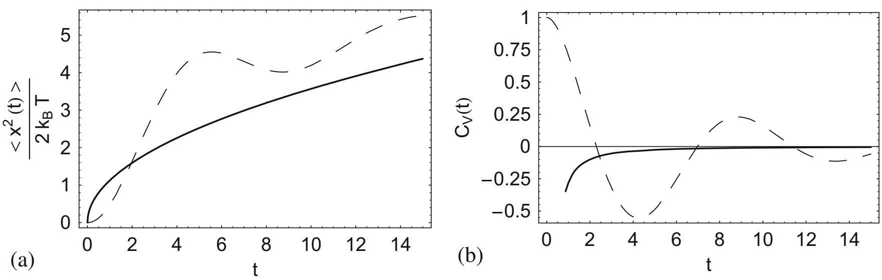

Fig. 6.4 Graphical representation of asymptotic and exact results for $\tau = 1$, $C_{\alpha,\beta,\delta} = 1$, $k_B T = 1$, $\alpha = 1/2$, $\beta = \delta = 1$; (**a**) MSD; asymptotic solution (6.56) (solid line), exact solution (6.25) (dashed line); (**b**) VACF, asymptotic solution (6.54) (solid line), exact solution (6.23) (dashed line). Reprinted from Physica A, 390, T. Sandev, Z. Tomovski, and J.L.A. Dubbeldam, Generalized Langevin equation with a three parameter Mittag-Leffler noise, 3627–3636, Copyright (2011), with permission from Elsevier

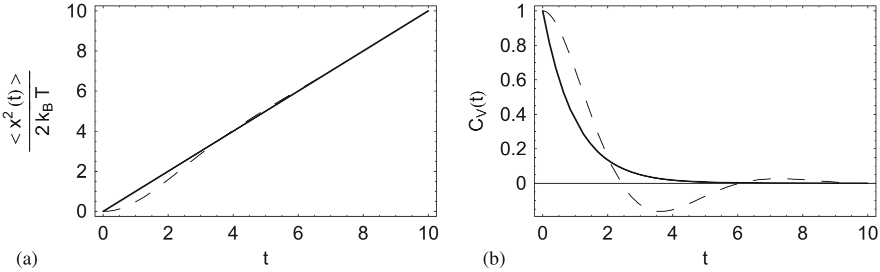

Fig. 6.5 Graphical representation of MSD and VACF, respectively, for $k_B T = 1$; (**a**) standard Brownian motion $\frac{\langle x^2(t)\rangle}{2k_B T} = t$ (solid line) and exact result (6.25) for $\alpha = \beta = \delta = 1$, $\tau = 1$, $C_{\alpha,\beta,\delta} = 1$ (dashed line); (**b**) standard Brownian motion $C_V(t) = e^{-t}$ (solid line) and exact result (6.23) for $\alpha = \beta = \delta = 1$, $\tau = 1$, $C_{\alpha,\beta,\delta} = 1$ (dashed line). Reprinted from Physica A, 390, T. Sandev, Z. Tomovski, and J.L.A. Dubbeldam, Generalized Langevin equation with a three parameter Mittag-Leffler noise, 3627–3636, Copyright (2011), with permission from Elsevier

In the short time limit, the relaxation functions behave as [59]

$$I(t) \simeq \frac{t^2}{2} - \frac{C_{\alpha,\beta,\delta}}{k_B T \tau^{\alpha\delta}} \frac{t^{\beta+3}}{\Gamma(\beta+4)}, \tag{6.58}$$

$$G(t) \simeq t - \frac{C_{\alpha,\beta,\delta}}{k_B T \tau^{\alpha\delta}} \frac{t^{\beta+2}}{\Gamma(\beta+3)}, \tag{6.59}$$

$$g(t) \simeq 1 - \frac{C_{\alpha,\beta,\delta}}{k_B T \tau^{\alpha\delta}} \frac{t^{\beta+1}}{\Gamma(\beta+2)}. \tag{6.60}$$

These results can be obtained either by using Tauberian theorems or from the exact results by using the first two terms in the corresponding series. For $\beta = 1$ the results from Ref. [57] are obtained ($\beta = 1$, $\omega = 0$), and for $\delta = 1$ those given in Ref. [8] ($\omega = 0$, $\alpha = 2$, $\beta = 1$).

6.2 Mixture of Internal Noises

6.2.1 Second Fluctuation-Dissipation Theorem

Let us now consider a stationary Gaussian internal noise $\xi(t)$ with a zero mean $\langle \xi(t)\rangle = 0$, represented as a mixture of N independent noises [58]

$$\xi(t) = \sum_{i=1}^{N} \alpha_i \xi_i(t),$$

for which $\left\langle \xi_i(t)\xi_j(t')\right\rangle = 0$ $(i \neq j)$, each of zero mean $\langle \xi_i(t)\rangle = 0$, with correlation functions of the form

$$\langle \xi_i(t)\xi_i(t')\rangle = \zeta_i(t' - t). \tag{6.61}$$

Thus, for the correlation function $C(t)$ we have

$$\langle \xi(t)\xi(t')\rangle = \left\langle \sum_{i=1}^{N} \alpha_i \xi_i(t) \sum_{j=1}^{N} \alpha_j \xi_j(t') \right\rangle = \sum_{i=1}^{N} \alpha_i^2 \langle \xi_i(t)\xi_i(t')\rangle. \tag{6.62}$$

Therefore, the second fluctuation-dissipation theorem (6.7) gives

$$\sum_{i=1}^{N} \alpha_i^2 \zeta_i(t) = k_B T \gamma(t). \tag{6.63}$$

Two $(N = 2)$ distinct independent noises (white noise and an arbitrary noise) were analyzed in Ref. [65], and various diffusive regimes are observed. Such situations with two types of noises have been shown to govern the motion of the tracked particles in several experimental works by Weigel et al. [69], Tabei et al. [64], and Jeon et al. [27]. Therefore, our investigation of GLE (6.8) for a particle driven by mixture of noises is justified with such experimental observations.

6.2.2 Relaxation Functions

Here we use the known relations for the relaxation function (6.18), (6.20), and (6.21) in order to analyze the diffusive behavior of the particle. The Laplace transformation to relation (6.63) yields

$$\hat{\gamma}(s) = \frac{1}{k_B T} \sum_{i=1}^{N} \alpha_i^2 \hat{\zeta}_i(s). \tag{6.64}$$

In what follows we consider different forms of the noise that are of importance in the anomalous diffusion theory.

6.2.3 White Noises

First, let us consider the motion of a free particle driven by N internal white noises, i.e., $\zeta_i(t) = \delta(t)$ $(\hat{\zeta}_i(s) = 1)$. Relation (6.63) then becomes

$$\hat{\gamma}(s) = \frac{1}{k_B T} \sum_{i=1}^{N} \alpha_i^2.$$

The inverse Laplace transform for the relaxation function $G(t)$ gives

$$G(t) = \mathscr{L}^{-1}\left[\frac{1}{s^2 + s\frac{\sum_{i=1}^{N}\alpha_i^2}{k_BT}}\right](t) = \frac{1 - e^{-\frac{\sum_{i=1}^{N}\alpha_i^2}{k_BT}t}}{\frac{1}{k_BT}\sum_{i=1}^{N}\alpha_i^2}, \tag{6.65}$$

from where the MSD (6.42) and VACF (6.44) read

$$\left\langle x^2(t)\right\rangle = \frac{2\,(k_BT)^2\,t}{\sum_{i=1}^{N}\alpha_i^2} - 2k_BT\frac{1 - e^{-\frac{\sum_{i=1}^{N}\alpha_i^2}{k_BT}t}}{\left(\frac{1}{k_BT}\sum_{i=1}^{N}\alpha_i^2\right)^2}, \tag{6.66}$$

$$C_V(t) = e^{-\frac{\sum_{i=1}^{N}\alpha_i^2}{k_BT}t}. \tag{6.67}$$

From relation (6.66), one concludes that in the long time limit ($t \to \infty$), the MSD has a linear dependence on time

$$\left\langle x^2(t)\right\rangle \simeq \frac{2\,(k_BT)^2}{\sum_{i=1}^{N}\alpha_i^2}t,$$

i.e., normal diffusive behavior of the particle, as it was expected, with diffusion coefficient

$$D = \frac{(k_BT)^2}{\sum_{i=1}^{N}\alpha_i^2},$$

and exponential relaxation of the VACF. Graphical representation of the MSD and VACF for different values of N is given in Fig. 6.6.

6.2.4 *Power Law Noises*

Next we analyze the case of N independent noises with power-law correlation functions

$$\zeta_i(t) = \frac{1}{\Gamma(1-\lambda_i)}t^{-\lambda_i}, \quad \text{i.e.,} \quad \hat{\zeta}_i(s) = s^{\lambda_i - 1},$$

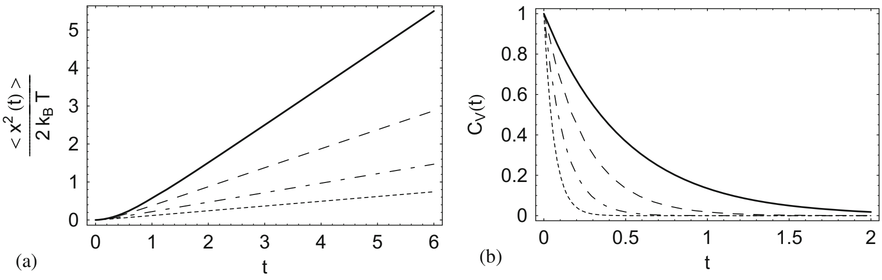

Fig. 6.6 Graphical representation of: (**a**) MSD (6.66), (**b**) VACF (6.67) for $\alpha_i^2 = 2$, in case of thermal initial conditions $v_0 = k_B T = 1$, $x_0 = 0$, and a mixture of Dirac delta noises $N = 1$ (solid line), $N = 2$ (dashed line); $N = 3$ (dot-dashed line); $N = 4$ (dotted line). Reprinted from Phys. Lett. A, 378, T. Sandev and Z. Tomovski, Langevin equation for a free particle driven by power law type of noises, 1–9, Copyright (2014), with permission from Elsevier

$i = 1, 2, \ldots, N$, $0 < \lambda_1 < \ldots < \lambda_N < 1$, $\lambda_i \neq 1$. From relation (6.63) one gets

$$\hat{\gamma}(s) = \frac{1}{k_B T} \sum_{i=1}^{N} \alpha_i^2 s^{\lambda_i - 1}.$$

Here we note that we can extend the analysis for the case of $\hat{\gamma}(s)$ with $0 < \lambda_1 < \ldots < \lambda_N < 2$, but in such a case the memory kernel $\gamma(t)$ is defined only in the sense of distributions [9, 13, 22, 42, 43, 70]. For the relaxation function $G(t)$, by using the approach given in Refs. [26, 37], we obtain

$$\begin{aligned} G(t) &= \mathscr{L}^{-1}\left[\frac{1}{s^2 + \sum_{i=1}^{N} A_i s^{\lambda_i}}\right](t) = \mathscr{L}^{-1}\left[\frac{s^{-2}}{1 - \sum_{i=1}^{N} \frac{(-A_i)}{s^{2-\lambda_i}}}\right](t) \\ &= t \sum_{j=1}^{\infty} \sum_{k_1 \geq 0, k_2 \geq 0, \ldots, k_N \geq 0}^{k_1 + k_2 + \ldots + k_N = j} \binom{j}{k_1 \quad k_2 \quad \ldots \quad k_N} \frac{\prod_{i=1}^{N} \left(-A_i t^{2-\lambda_i}\right)^{k_i}}{\Gamma\left(2 + \sum_{i=1}^{N} (2 - \lambda_i) k_i\right)} \\ &= t E_{(2-\lambda_1, 2-\lambda_2, \ldots, 2-\lambda_N), 2}\left(-A_1 t^{2-\lambda_1}, -A_2 t^{2-\lambda_2}, \ldots, -A_N t^{2-\lambda_N}\right), \end{aligned} \tag{6.68}$$

where $A_i = \frac{\alpha_i^2}{k_B T}$,

$$\binom{j}{k_1 \quad k_2 \quad \ldots \quad k_N} = \frac{j!}{k_1! k_2! \cdots k_N!}$$

are the so-called multinomial coefficients, and $E_{(a_1, a_2, \ldots, a_N), b}(z_1, z_2, \ldots, z_N)$ is the multinomial M-L function (1.35).

Let us analyze the case with $0 < \lambda_1 < \lambda_2 < 2$, $\lambda_1, \lambda_2 \neq 1$. From (6.68) we obtain

$$G(t) = tE_{(2-\lambda_1,2-\lambda_2),2}\left(-A_1 t^{2-\lambda_1}, -A_2 t^{2-\lambda_2}\right)$$
$$= \sum_{n=0}^{\infty}(-A_1)^n t^{(2-\lambda_1)n+1} E_{2-\lambda_2,(2-\lambda_1)n+2}^{n+1}\left(-A_2 t^{2-\lambda_2}\right), \quad (6.69)$$

and thus

$$I(t) = \sum_{n=0}^{\infty}(-A_1)^n t^{(2-\lambda_1)n+2} E_{2-\lambda_2,(2-\lambda_1)n+3}^{n+1}\left(-A_2 t^{2-\lambda_2}\right), \quad (6.70)$$

$$g(t) = \sum_{n=0}^{\infty}(-A_1)^n t^{(2-\lambda_1)n} E_{2-\lambda_2,(2-\lambda_1)n+1}^{n+1}\left(-A_2 t^{2-\lambda_2}\right), \quad (6.71)$$

where $E_{\alpha,\beta}^{\delta}(z)$ is the three parameter M-L function (1.14) [54].

For the long time limit behavior, from (1.28), we obtain

$$I(t) \simeq \frac{t^{\lambda_2}}{A_2} E_{\lambda_2-\lambda_1,\lambda_2+1}\left(-\frac{A_1}{A_2}t^{\lambda_2-\lambda_1}\right) \simeq \frac{1}{A_1}\frac{t^{\lambda_1}}{\Gamma(1+\lambda_1)}. \quad (6.72a)$$

$$G(t) \simeq \frac{t^{\lambda_2-1}}{A_2} E_{\lambda_2-\lambda_1,\lambda_2}\left(-\frac{A_1}{A_2}t^{\lambda_2-\lambda_1}\right) \simeq \frac{1}{A_1}\frac{t^{\lambda_1-1}}{\Gamma(\lambda_1)}. \quad (6.72b)$$

$$g(t) \simeq \frac{t^{\lambda_2-2}}{A_2} E_{\lambda_2-\lambda_1,\lambda_2-1}\left(-\frac{A_1}{A_2}t^{\lambda_2-\lambda_1}\right) \simeq \frac{1}{A_1}\frac{t^{\lambda_1-2}}{\Gamma(\lambda_1-1)}. \quad (6.72c)$$

From the MSD

$$\frac{\langle x^2(t)\rangle}{2k_B T} \simeq \frac{1}{A_1}\frac{t^{\lambda_1}}{\Gamma(1+\lambda_1)},$$

we conclude that the particle shows anomalous diffusive behavior with the lower diffusion exponent λ_1 ($0 < \lambda_1 < \lambda_2 < 2$). Therefore, subdiffusion appears for $0 < \lambda_1 < 1$ and superdiffusion for $1 < \lambda_1 < 2$. VACF becomes

$$C_V(t) \simeq \frac{1}{A_1}\frac{t^{\lambda_1-2}}{\Gamma(\lambda_1-1)}.$$

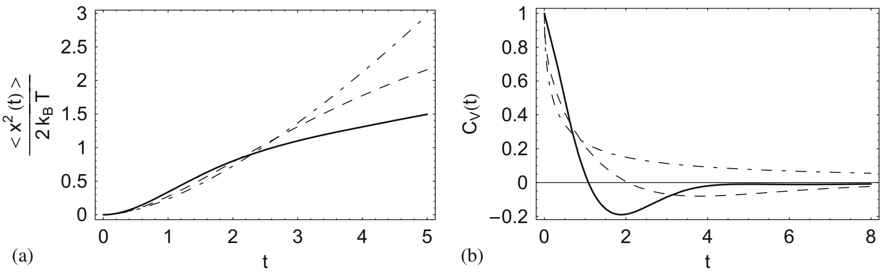

Fig. 6.7 Graphical representation of: (**a**) MSD (6.70), (**b**) VACF (6.71) for $A_1 = A_2 = 1$, in case of thermal initial conditions $v_0 = k_BT = 1$, $x_0 = 0$, and a mixture of two power law noises, for $\lambda_1 = 1/2$, $\lambda_2 = 3/4$ (solid line), $\lambda_1 = 1/2$, $\lambda_2 = 3/2$ (dashed line); $\lambda_1 = 5/4$, $\lambda_2 = 3/2$ (dot-dashed line). Reprinted from Phys. Lett. A, 378, T. Sandev and Z. Tomovski, Langevin equation for a free particle driven by power law type of noises, 1–9, Copyright (2014), with permission from Elsevier

For the short time one finds

$$I(t) \simeq \begin{cases} \frac{t^2}{2} - \frac{A_2 t^{4-\lambda_2}}{\Gamma(5-\lambda_2)} - \frac{A_1 t^{4-\lambda_1}}{\Gamma(5-\lambda_1)}, & \text{for } \lambda_2 \leq 1 + \frac{\lambda_1}{2}, \\ \frac{t^2}{2} - \frac{A_2 t^{4-\lambda_2}}{\Gamma(5-\lambda_2)} + \frac{A_2^2 t^{6-2\lambda_2}}{\Gamma(7-2\lambda_2)}, & \text{for } \lambda_2 > 1 + \frac{\lambda_1}{2}. \end{cases} \tag{6.73}$$

Thus, we conclude that the noise with the greater exponent λ_2 has dominant contribution to the particle behavior in the short time limit. For the variance in the short time limit we have

$$\frac{\sigma_{xx}}{2k_BT} \simeq (3 - \lambda_2)\,\frac{A_2 t^{4-\lambda_2}}{\Gamma(5 - \lambda_2)}.$$

Graphical representation of the MSD and VACF for different values of parameters λ_1 and λ_2 is given in Fig. 6.7. The anomalous diffusive behavior of the particle is evident.

6.2.5 Distributed Order Noise

Furthermore, let us instead of mixture of noises consider an internal noise of distributed order, i.e.,

$$k_BT\gamma(t) = \alpha^2 \int_0^1 \frac{t^{-\lambda}}{\Gamma(1 - \lambda)}\,\mathrm{d}\lambda.$$

Such memory kernel was used by Kochubei [33] in the theory of evolution equations with distributed order derivative, which is a useful tool for modeling ultraslow relaxation and diffusion processes. The Laplace transform of the memory kernel then becomes

$$\hat{\gamma}(s) = \frac{\alpha^2}{k_B T} \frac{s-1}{s \log s}.$$

We note that the assumption (4.32) is satisfied for this memory kernel since

$$\lim_{s \to 0} s\hat{\gamma}(s) = \frac{\alpha^2}{k_B T} \lim_{s \to 0} \frac{s-1}{\log s} = 0.$$

Thus, we have

$$\hat{G}(s) = \frac{1}{s^2 + \frac{\alpha^2}{k_B T}\frac{s-1}{\log s}} = \sum_{n=0}^{\infty} \left(-\frac{\alpha^2}{k_B T}\right)^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{s^{n+k+2} \log^n s}, \tag{6.74}$$

from where, by inverse Laplace transform, the relaxation function $G(t)$ becomes

$$G(t) = t + \sum_{n=1}^{\infty} \left(-\frac{\alpha^2}{k_B T}\right)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k \mu\left(t, n-1, n+k+1\right). \tag{6.75}$$

Here

$$\mu\left(t, \beta, \alpha\right) = \int_0^{\infty} \frac{t^{\alpha+\tau} \tau^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+\tau+1)} \mathrm{d}\tau, \tag{6.76}$$

whose Laplace transform reads

$$\mathscr{L}\left[\mu\left(t, \beta, \alpha\right)\right](s) = \frac{1}{s^{\alpha+1} \log^{\beta+1} s},$$

$\Re(\alpha) > -1$, $\Re(s) > -1$ [14]. For detailed properties and relations of these and related Volterra functions, we refer to the literature [2–4, 21, 46]. Thus, the relaxation functions are represented in terms of series in special functions $\mu\left(t, \beta, \alpha\right)$, and their representation in a closed form is an open problem. Here we use Tauberian theorems (see Appendix B and Refs. [24, 41] for details) to find the asymptotic behavior of the relaxation functions. In the long time limit ($t \to \infty$, i.e., $s \to 0$ according to the Tauberian theorems) we obtain

$$\begin{aligned} I(t) \simeq \mathscr{L}^{-1}\left[\frac{s^{-1}}{\frac{\alpha^2}{k_B T}\frac{s-1}{\log s}}\right](t) &= \frac{1}{A}\mathscr{L}^{-1}\left[\frac{\log s}{s-1} - \frac{\log s}{s}\right](t) \\ &= \frac{1}{A}\left[\gamma + \log t - e^t \mathrm{Ei}(-t)\right] \\ &= \frac{1}{A}\left[\gamma + \log t + e^t \mathrm{E}_1(t)\right], \end{aligned} \tag{6.77}$$

where $A = \frac{\alpha^2}{k_B T}$, $\gamma = 0.577216$ is the Euler-Mascheroni (or Euler's) constant,

$$\mathrm{Ei}(t) = -\int_{-t}^{\infty} \frac{e^{-x}}{x}\,\mathrm{d}x$$

is the exponential integral function [14], and

$$\mathrm{E}_1(t) = -\mathrm{Ei}(-t) = \int_{t}^{\infty} \frac{e^{-x}}{x}\,\mathrm{d}x.$$

From the asymptotic expansion formula

$$\mathrm{E}_1(t) \simeq \frac{e^{-t}}{t}\sum_{k=0}^{n-1}(-1)^k\frac{k!}{t^k},$$

for $t \to \infty$ [41], which has error of order $O\left(n!t^{-n}\right)$, for the relaxation function we obtain $I(t) \simeq \frac{\gamma}{A} + \frac{1}{A}\log t$. The MSD has logarithmic dependence on time

$$\frac{\left\langle x^2(t)\right\rangle}{2k_B T} \simeq \frac{\gamma}{A} + \frac{1}{A}\log t,$$

and therefore the particle shows ultraslow diffusive behavior. In the same way, for the VACF in the long time limit ($t \to \infty$) we find

$$C_V(t) \simeq -\frac{1}{At}\left[1 + e^t t\mathrm{Ei}(-t)\right] = -\frac{1}{At}\left[1 - e^t t\mathrm{E}_1(t)\right] \simeq -\frac{1}{At^2}.$$

Similar relaxation functions were obtained by Mainardi [41] in analysis of fractional relaxation equation of distributed order. The short time limit ($t \to 0$, i.e., $s \to \infty$) becomes

$$\begin{aligned} I(t) &= \mathscr{L}^{-1}\left[\frac{s^{-1}}{s^2 + A\frac{s-1}{\log s}}\right](t) = \mathscr{L}^{-1}\left[\frac{1}{s^3}\left(1 - \frac{As - A}{s^2\log s + As - A}\right)\right](t) \\ &\simeq \mathscr{L}^{-1}\left[\frac{1}{s^3}\left(1 - \frac{As - A}{s^2\log s}\right)\right](t) = \frac{t^2}{2} - A\mu\left(t, 0, 3\right) + A\mu\left(t, 0, 4\right). \end{aligned} \tag{6.78}$$

In the same way, for the VACF in the short time limit we obtained

$$C_V(t) \simeq 1 - A\mu\left(t, 0, 1\right) + A\mu\left(t, 0, 2\right).$$

Here we note that the same result can be obtained directly from the series expression (6.75).

A more general distributed order internal noise is of form

$$k_BT\gamma(t) = \int_0^1 p(\lambda)\frac{t^{-\lambda}}{\Gamma(1-\lambda)}\,\mathrm{d}\lambda, \quad \text{i.e.,} \quad k_BT\hat{\gamma}(s) = \int_0^1 p(\lambda)s^{\lambda-1}\,\mathrm{d}\lambda,$$

where $p(\lambda)$ is the weight function. The case with $p(\lambda) = \alpha^2$ yields the already considered uniformly distributed noise

$$k_BT\gamma(t) = \alpha^2\int_0^1 \frac{t^{-\lambda}}{\Gamma(1-\lambda)}\,\mathrm{d}\lambda.$$

For $p(\lambda) = \sum_{i=1}^{N}\alpha_i^2\delta(\lambda-\lambda_i)$, where $\delta(\lambda)$ is the Dirac delta, $0 < \lambda_i < 1$, $i = 1, 2, \ldots, N$, the mixture of N internal power-law noises

$$k_BT\gamma(t) = \sum_{i=1}^{N}\alpha_i^2\int_0^1 \delta(\lambda-\lambda_i)\frac{t^{-\lambda}}{\Gamma(1-\lambda)}\mathrm{d}\lambda = \sum_{i=1}^{N}\alpha_i^2\frac{t^{-\lambda_i}}{\Gamma(1-\lambda_i)},$$

is recovered.

6.2.6 Mixture of White and Power Law Noises

As an addition, we analyze the GLE with mixture of P white noises, and Q power-law noises, where $P + Q = N$,

$$\gamma(t) = \frac{1}{k_BT}\sum_{i=1}^{P}\alpha_i^2\delta(t) + \frac{1}{k_BT}\sum_{j=1}^{Q}\beta_j^2\frac{t^{-\lambda_j}}{\Gamma\left(1-\lambda_j\right)},$$

whose Laplace transform pair is given by

$$\hat{\gamma}(s) = \frac{1}{k_BT}\sum_{i=1}^{P}\alpha_i^2 + \frac{1}{k_BT}\sum_{j=1}^{Q}\beta_j^2 s^{\lambda_j-1}.$$

Here we also note that we can extend our analysis to exponents between 1 and 2, but in such a case the memory kernel is defined only in the sense of distributions [9, 13, 22, 42, 43, 70]. In the same way as previously described, we obtain

$$G(t) = \mathscr{L}^{-1}\left[\frac{1}{s^2 + \sum_{i=1}^{P}A_is + \sum_{j=1}^{Q}B_js^{\lambda_j}}\right](t), \tag{6.79}$$

where $A_i = \frac{\alpha_i^2}{k_B T}$ and $B_j = \frac{\beta_j^2}{k_B T}$. We rewrite relation (6.79) in the following way

$$G(t) = \mathscr{L}^{-1}\left[\frac{1}{s^2 + \sum_{i=1}^{Q+1} C_i s^{\tilde{\lambda}_i}}\right](t), \tag{6.80}$$

where $0 < \tilde{\lambda}_1 < \ldots < \tilde{\lambda}_{Q+1} < 2$. $\tilde{\lambda}_i$ actually have the values of λ_j and 1. For a given value $r = i \in \{1, 2, \ldots, Q+1\}$,

$$C_r s^{\tilde{\lambda}_r} = \sum_{i=1}^{P} A_i s, \quad \text{i.e.,} \quad C_r = \sum_{i=1}^{P} A_i, \quad \tilde{\lambda}_r = 1.$$

Note that if $0 < \lambda_j < 1$ then $r = Q+1$, and if $1 < \lambda_j < 2$ then $r = 1$. Therefore, the relaxation function $G(t)$ is represented through the multinomial M-L function (1.35),

$$\begin{aligned} G(t) &= \mathscr{L}^{-1}\left[\frac{1}{s^2 + \sum_{i=1}^{Q+1} C_i s^{\tilde{\lambda}_i}}\right](t) \\ &= t E_{\left(2-\tilde{\lambda}_1, 2-\tilde{\lambda}_2, \ldots, 2-\tilde{\lambda}_{Q+1}\right),2}\left(-C_1 t^{2-\tilde{\lambda}_1}, -C_2 t^{2-\tilde{\lambda}_2}, \ldots, -C_{Q+1} t^{2-\tilde{\lambda}_{Q+1}}\right). \end{aligned} \tag{6.81}$$

Mixture of white and power law noises of the form

$$\gamma(t) = \frac{1}{k_B T}\left[\alpha^2 \delta(t) + \beta^2 \frac{t^{-\lambda}}{\Gamma(1-\lambda)}\right],$$

was considered in Ref. [65], for $0 < \lambda < 1$. From (6.81) we obtain

$$G(t) = \sum_{n=0}^{\infty} (-B)^n t^{(2-\lambda)n+1} E_{1,(2-\lambda)n+2}^{n+1}(-At), \tag{6.82}$$

where $A = \frac{\alpha^2}{k_B T}$ and $B = \frac{\beta^2}{k_B T}$. This relation yields

$$I(t) = \sum_{n=0}^{\infty} (-B)^n t^{(2-\lambda)n+2} E_{1,(2-\lambda)n+3}^{n+1}(-At), \tag{6.83}$$

$$g(t) = \sum_{n=0}^{\infty} (-B)^n t^{(2-\lambda)n} E_{1,(2-\lambda)n+1}^{n+1}(-At). \tag{6.84}$$

By using the asymptotic expansion formula (1.28), in the long time limit we obtain

$$I(t) \simeq \frac{t}{A} E_{1-\lambda,2}\left(-\frac{B}{A}t^{1-\lambda}\right) \simeq \frac{1}{B}\frac{t^{\lambda}}{\Gamma(1+\lambda)}, \tag{6.85a}$$

$$G(t) \simeq \frac{1}{A} E_{1-\lambda,1}\left(-\frac{B}{A}t^{1-\lambda}\right) \simeq \frac{1}{B}\frac{t^{\lambda-1}}{\Gamma(\lambda)}, \tag{6.85b}$$

$$g(t) \simeq \frac{1}{B}\frac{t^{\lambda-2}}{\Gamma(\lambda-1)}. \tag{6.85c}$$

Thus, the MSD becomes

$$\frac{\langle x^2(t)\rangle}{2k_BT} \simeq \frac{1}{B}\frac{t^{\lambda}}{\Gamma(1+\lambda)}, \quad 0 < \lambda < 1,$$

which means that the particle shows subdiffusive behavior. From (6.85a) we note that the power law noise has dominant contribution to the particle behavior in the long time limit. These results are in agreement with those obtained in Ref. [65]. The short time limit yields

$$I(t) \simeq \frac{t^2}{2} - \frac{At^3}{6} - \frac{Bt^{4-\lambda}}{\Gamma(5-\lambda)}, \tag{6.86a}$$

$$G(t) \simeq t - \frac{At^2}{2} - \frac{Bt^{3-\lambda}}{\Gamma(4-\lambda)}, \tag{6.86b}$$

$$g(t) \simeq 1 - At - \frac{Bt^{2-\lambda}}{\Gamma(3-\lambda)}, \tag{6.86c}$$

from where we conclude that both noises contribute to the particle behavior. The contribution of the white noise to the particle behavior in the short time limit is dominant. For variance (6.45a) we recovered the result obtained in Ref. [65],

$$\frac{\sigma_{xx}}{2k_BT} \simeq \frac{At^3}{3} + (3-\lambda)\frac{Bt^{4-\lambda}}{\Gamma(5-\lambda)},$$

for $t \to 0$. Graphical representation of the MSD and VACF is given in Fig. 6.8.

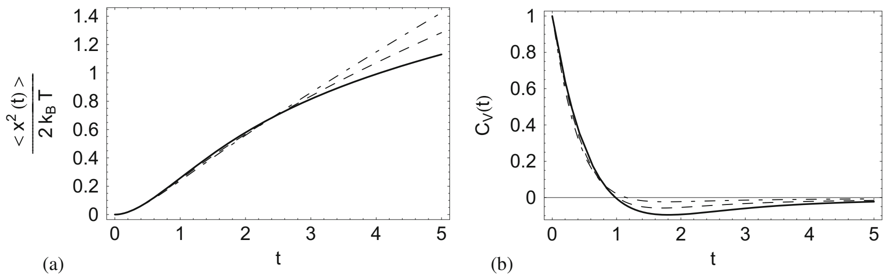

Fig. 6.8 Graphical representation of: (**a**) MSD (6.83), (**b**) VACF (6.84), in case of thermal initial conditions $v_0 = k_B T = 1$, $x_0 = 0$, and a mixture of Dirac delta ($A = 2$) and power law ($B = 1$) noises, for $\lambda = 1/4$ (solid line), $\lambda = 1/2$ (dashed line); $\lambda = 3/4$ (dot-dashed line). Reprinted from Phys. Lett. A, 378, T. Sandev and Z. Tomovski, Langevin equation for a free particle driven by power law type of noises, 1–9, Copyright (2014), with permission from Elsevier

In the same way, from (6.81), the case $1 < \lambda < 2$ yields

$$G(t) = \sum_{n=0}^{\infty} (-A)^n t^{n+1} E_{2-\lambda,n+2}^{n+1}\left(-Bt^{2-\lambda}\right), \tag{6.87}$$

and thus

$$I(t) = \sum_{n=0}^{\infty} (-A)^n t^{n+2} E_{2-\lambda,n+3}^{n+1}\left(-Bt^{2-\lambda}\right), \tag{6.88}$$

$$g(t) = \sum_{n=0}^{\infty} (-A)^n t^{n} E_{2-\lambda,n+1}^{n+1}\left(-Bt^{2-\lambda}\right). \tag{6.89}$$

From the asymptotic expansion formula we obtain the asymptotic behavior of relaxation functions

$$I(t) \simeq \frac{t^{\lambda}}{B} E_{\lambda-1,\lambda+1}\left(-\frac{A}{B}t^{\lambda-1}\right) \simeq \frac{1}{A}t, \tag{6.90}$$

so the MSD is

$$\frac{\langle x^2(t)\rangle}{2k_B T} \simeq \frac{1}{A}t.$$

It means that the particle shows normal diffusive behavior. Therefore, the white noise has dominant contribution to the particle behavior in the long time limit. This result is obtained by Mainardi et al. [42, 43] in case of friction memory kernel

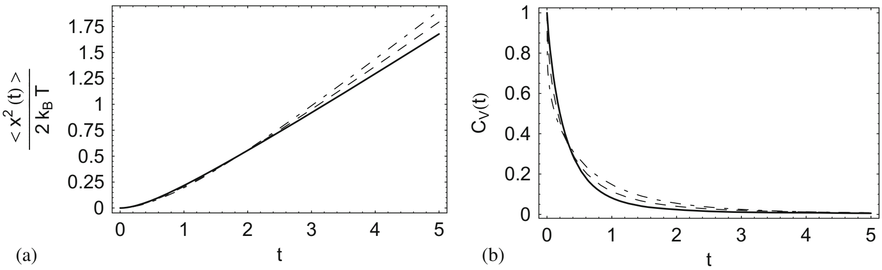

Fig. 6.9 Graphical representation of: (**a**) MSD (6.88), (**b**) VACF (6.89), in case of thermal initial conditions $v_0 = k_B T = 1$, $x_0 = 0$, and a mixture of Dirac delta ($A = 2$) and power law ($B = 1$) noises, for $\lambda = 5/4$ (solid line), $\lambda = 3/2$ (dashed line); $\lambda = 7/4$ (dot-dashed line). Reprinted from Phys. Lett. A, 378, T. Sandev and Z. Tomovski, Langevin equation for a free particle driven by power law type of noises, 1–9, Copyright (2014), with permission from Elsevier

represented as superposition of white and power law noises. For the short time limit it follows

$$I(t) \simeq \begin{cases} \frac{t^2}{2} - \frac{Bt^{4-\lambda}}{\Gamma(5-\lambda)} - \frac{At^3}{6} & \text{for } \lambda \leq 3/2, \\ \frac{t^2}{2} - \frac{Bt^{4-\lambda}}{\Gamma(5-\lambda)} + \frac{B^2 t^{6-2\lambda}}{\Gamma(7-2\lambda)} & \text{for } \lambda > 3/2, \end{cases} \tag{6.91}$$

so the power-law noise has dominant contribution to the particle behavior in the short time limit. Here we note that the friction memory kernel, which represents superposition of white and power law noises in sense of distributions, was considered by Mainardi et al. [42, 43] for $\lambda = 3/2$ and it was shown that the VACF behaves as $C_V \simeq t^{-3/2}$. This result can be obtained from asymptotic expansion of relation (6.90),

$$C_V \simeq \frac{t^{\lambda-2}}{B} E_{\lambda-1,\lambda-1}\left(-\frac{A}{B}t^{\lambda-1}\right) = -\frac{t^{-1}}{A} E_{\lambda-1,0}\left(-\frac{A}{B}t^{\lambda-1}\right) \simeq -\frac{B}{A^2}\frac{t^{-\lambda}}{\Gamma(1-\lambda)},$$

$\lambda = 3/2$, and represents a proof of the computer simulations of the VACF observed by Alder and Wainwright [1]. Graphical representation of the MSD and VACF is given in Fig. 6.9.

Let us now consider mixture of three noises, one of which is the white noise,

$$\hat{\gamma}(s) = \frac{1}{k_B T}\left[\alpha^2 + \beta_1^2 s^{\lambda_1 - 1} + \beta_2^2 s^{\lambda_2 - 1}\right],$$

where $0 < \lambda_1 < 1$ and $1 < \lambda_2 < 2$. From relation (6.81) we obtain

$$G(t) = tE_{(2-\lambda_1,1,2-\lambda_2),2}\left(-B_1 t^{2-\lambda_1}, -At, -B_2 t^{2-\lambda_2}\right), \tag{6.92}$$

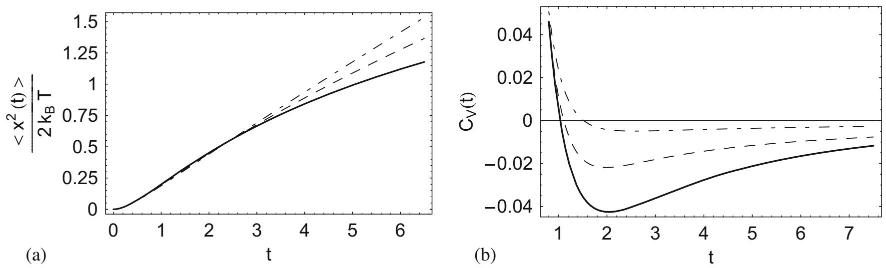

Fig. 6.10 Graphical representation of: (**a**) MSD (6.94), (**b**) VACF $g(t) = G'(t)$, in case of thermal initial conditions $v_0 = k_BT = 1$, $x_0 = 0$, and a mixture of Dirac delta ($A = 2$) and power law ($B_1 = B_2 = 1$) noises, for $\lambda_1 = 1/4$, $\lambda_2 = 5/4$ (solid line); $\lambda_1 = 1/2$, $\lambda_2 = 5/4$ (dashed line); $\lambda_1 = 3/4$, $\lambda_2 = 5/4$ (dot-dashed line). Reprinted from Phys. Lett. A, 378, T. Sandev and Z. Tomovski, Langevin equation for a free particle driven by power law type of noises, 1–9, Copyright (2014), with permission from Elsevier

i.e.,

$$G(t) = \sum_{n=0}^{\infty}(-A)^n t^{n+1} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{B_1}{A}\right)^k t^{(1-\lambda_1)k} E_{2-\lambda_2,n+2+(1-\lambda_1)k}^{n+1}\left(-B_2 t^{2-\lambda_2}\right), \tag{6.93}$$

and

$$I(t) = \sum_{n=0}^{\infty}(-A)^n t^{n+2} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{B_1}{A}\right)^k t^{(1-\lambda_1)k} E_{2-\lambda_2,n+3+(1-\lambda_1)k}^{n+1}\left(-B_2 t^{2-\lambda_2}\right), \tag{6.94}$$

where $A = \frac{\alpha^2}{k_BT}$, $B_1 = \frac{\beta_1^2}{k_BT}$ and $B_2 = \frac{\beta_2^2}{k_BT}$. The long time limit yields

$$I(t) \simeq \frac{t^{\lambda_2}}{B_2} E_{\lambda_2-\lambda_1,\lambda_2+1}\left(-\frac{B_1}{B_2} t^{\lambda_2-\lambda_1}\right) \simeq \frac{1}{B_1} \frac{t^{\lambda_1}}{\Gamma(1+\lambda_1)}, \tag{6.95}$$

which means that dominant contribution to the particle behavior in the long time limit has the noise with the exponent $0 < \lambda_1 < 1$. Thus, the particle shows a subdiffusive behavior. The short time limit, again, yields ballistic motion $I(t) \simeq \frac{t^2}{2}$. Graphical representation of the MSD and VACF is given in Fig. 6.10.

Here we note that combinations of white noise and anomalous diffusion were studied by Eule and Friedrich [15] and Jeon et al. [29].

6.2.7 More Generalized Noise

The mixture of white and two parameter M-L noise of form

$$\zeta_{ML}(t) = \frac{1}{\tau^{\mu}} t^{\nu-1} E_{\mu,\nu}\left(-\frac{t^{\mu}}{\tau^{\mu}}\right),$$

for which

$$\hat{\zeta}_{ML}(s) = \frac{1}{\tau^{\mu}} \frac{s^{\mu-\nu}}{s^{\mu} + \tau^{-\mu}}$$

is further generalization of the previous cases of white and power-law noises. For $\nu = 1$ we obtain the one parameter M-L noise, and for $\tau \to 0$—the power law noise. The case $\mu = \nu = 1$ gives the exponential noise, and the case $\mu = \nu = 1$ with $\tau \to 0$ recovers the Dirac delta noise. Similar M-L noises have been introduced in the literature to describe complex data related to anomalous diffusion [7, 12, 55, 57, 59, 60]. In case of the Dirac delta and the two parameter M-L noise,

$$\gamma(t) = \frac{1}{k_B T}\left[\alpha^2 \delta(t) + \beta^2 \frac{1}{\tau^{\mu}} t^{\nu-1} E_{\mu,\nu}\left(-\frac{t^{\mu}}{\tau^{\mu}}\right)\right],$$

the relaxation function $G(t)$ becomes

$$\begin{aligned} G(t) &= \mathscr{L}^{-1}\left[\frac{1}{s^2 + As + B\tau^{-\mu}\frac{s^{\mu-\nu+1}}{s^{\mu}+\tau^{-\mu}}}\right](t) \\ &= \mathscr{L}^{-1}\left[\frac{s^{-\mu-2}\left(s^{\mu} + \tau^{-\mu}\right)}{1 + \tau^{-\mu}s^{-\mu} + As^{-1} + A\tau^{-\mu}s^{-1-\mu} + B\tau^{-\mu}s^{-1-\nu}}\right](t) \\ &= tE_{(\lambda_1,\ldots,\lambda_4),2}\left(-C_1 t^{\lambda_1}, \ldots, -C_4 t^{\lambda_4}\right) \\ &\quad + \frac{t^{\mu+1}}{\tau^{\mu}} E_{(\lambda_1,\ldots,\lambda_4),\mu+2}\left(-C_1 t^{\lambda_1}, \ldots, -C_4 t^{\lambda_4}\right), \end{aligned} \tag{6.96}$$

where $A = \frac{\alpha^2}{k_B T}$, $B = \frac{\beta^2}{k_B T}$, $C_i \in \left\{\tau^{-\mu}, A, A\tau^{-\mu}, B\tau^{-\mu}\right\}$, and $\lambda_i \in \{\mu, 1, \mu+1, \nu+1\}$. Same approach can be performed in case of combination of the power-law and M-L noises,

$$\gamma(t) = \frac{1}{k_B T}\left[\alpha^2 \frac{t^{-r}}{\Gamma(1-r)} + \beta^2 \frac{1}{\tau^{\mu}} t^{\nu-1} E_{\mu,\nu}\left(-\frac{t^{\mu}}{\tau^{\mu}}\right)\right],$$

since in this case for the relaxation function one finds

$$G(t) = \mathscr{L}^{-1}\left[\frac{1}{s^2 + As^r + B\tau^{-\mu}\frac{s^{\mu-\nu+1}}{s^\mu+\tau^{-\mu}}}\right],$$

which can be represented in terms of the multinomial M-L functions (1.35).

6.3 Harmonic Oscillator

In this section we analyze the behavior of a harmonic oscillator driven by generalized M-L internal noise (6.9). The corresponding GLE for the harmonic oscillator with mass $m = 1$ and frequency ω driven by stationary random force $\xi(t)$ is given by:

$$\begin{aligned}
&\ddot{x}(t) + \int_0^t \gamma(t-t')\dot{x}(t')\,\mathrm{d}t' + \omega^2 x(t) = \xi(t),\\
&\dot{x}(t) = v(t),
\end{aligned} \tag{6.97}$$

The GLE describes the particle dynamics bounded in the harmonic potential well and immersed in complex or viscoelastic media. The internal noise $\xi(t)$ is of a zero mean ($\langle\xi(t)\rangle = 0$). Again we apply the second fluctuation-dissipation theorem since the considered noise is internal.

GLE (6.97) represents a suitable model for description of anomalous dynamics within proteins. Within given protein, the movements are bounded in small domains, thus the potential energy can be well approximated by the harmonic potential. Furthermore, the movements of the proteins are in a given complex liquid environment and its influence on the particle movement can be described by appropriate friction memory kernel. The high viscous damping, which is characteristic for the proteins in a liquid environment, will be described by neglecting the inertial term in Eq. (6.97). Information for the behavior of the oscillator will be obtained from the MSD, time dependent diffusion coefficient, and VACF. The normalized displacement correlation function, which is an experimental measured quantity, will be analyzed as well.

6.3.1 *Harmonic Oscillator Driven by an Arbitrary Noise*

Let us formally solve the GLE (6.8). From the initial condition $x(0) = x_0$ and $\dot{x}(0) = v(0) = v_0$, one obtains

$$\hat{X}(s) = x_0\frac{s+\hat{\gamma}(s)}{s^2+s\hat{\gamma}(s)+\omega^2} + v_0\frac{1}{s^2+s\hat{\gamma}(t)+\omega^2} + \frac{1}{s^2+s\hat{\gamma}(s)+\omega^2}\hat{F}(s), \tag{6.98}$$

where $\hat{F}(s) = \mathscr{L}[\xi(t)](s)$ and $\hat{\gamma}(s) = \mathscr{L}[\gamma(t)](s)$. From Eq. (6.98) for $x(t)$ and $v(t) = \dot{x}(t)$ one finds

$$x(t) = \langle x(t)\rangle + \int_0^t G(t-t')\xi(t')\,\mathrm{d}t', \tag{6.99}$$

$$v(t) = \langle v(t)\rangle + \int_0^t g(t-t')\xi(t')\,\mathrm{d}t', \tag{6.100}$$

where

$$\langle x(t)\rangle = v_0 G(t) + x_0[1-\omega^2 I(t)], \tag{6.101}$$

$$\langle v(t)\rangle = v_0 g(t) - x_0\omega^2 G(t), \tag{6.102}$$

are the average displacement and average velocity, respectively. The function $G(t)$ is the Laplace pair of

$$\hat{G}(s) = \frac{1}{s^2 + s\hat{\gamma}(s) + \omega^2}. \tag{6.103}$$

The same relations for the relaxation functions are valid, $I(t) = \int_0^t G(t')\,\mathrm{d}t'$ and $g(t) = \frac{\mathrm{d}G(t)}{\mathrm{d}t}$, as previously.

The MSD, time dependent diffusion coefficient, and VACF are related with the relaxation functions as previously, i.e., $\left\langle x^2(t)\right\rangle = 2k_B T I(t)$, $D(t) = k_B T G(t)$ and $C_V(t) = g(t)$, respectively [12]. These relations are valid for friction memory kernels which satisfy the assumption (6.11).

6.3.2 *Overdamped Motion*

From relation (6.103) we note that for the M-L noise (6.9) very complex expressions for the relaxation functions are obtained, and exact results are very difficult to be obtained. For simpler friction memory kernels of the Dirac delta type (standard Langevin equation), power-law type (fractional Langevin equation), one and two parameter M-L types the corresponding relaxation functions can be found exactly. In case of the three parameter M-L noise (6.9) the calculations become very complex, and thus one analyzes the asymptotic behavior of the oscillator in the short and long time limit. Therefore, instead of that, we analyze the overdamped motion, which means that there is high viscous damping, i.e., the inertial term $\ddot{x}(t)$ vanishes. This case of high friction leads to same asymptotic behavior in the long time limit as the one for the GLE, so the overdamped motion can be used to analyze the anomalous diffusive behavior of the oscillator in the long time limit. This case of high viscous damping appears in the analysis of conformational dynamics of proteins, due to the liquid environment in which the proteins are immersed [10]. Thus, the relaxation functions $\hat{g}(s)$, $\hat{G}(s)$ and $\hat{I}(s)$ become

$$\hat{g}(s) = \frac{s}{s\hat{\gamma}(s) + \omega^2}, \quad \hat{G}(s) = s^{-1}\hat{g}(s), \quad \hat{I}(s) = s^{-1}\hat{G}(s). \tag{6.104}$$

By substitution of the friction memory kernel (6.9) in (6.104), by applying the Laplace transform formula (1.18), for $\hat{I}(t)$ we obtain [55]

$$\begin{aligned} I(t) &= \mathscr{L}^{-1}\left[\frac{1}{\omega^2}\frac{s^{\frac{\beta-1}{2}-1}}{s^{\frac{\beta-1}{2}} + \frac{\gamma_{\alpha,\beta,\delta}}{\omega^2}\frac{s^{\alpha\delta-\frac{\beta-1}{2}}}{(s^\alpha+\tau^{-\alpha})^\delta}}\right] \\ &= \frac{1}{\omega^2}\sum_{k=0}^{\infty}\left(-\frac{\gamma_{\alpha,\beta,\delta}}{\omega^2}\right)^k t^{(\beta-1)k} E_{\alpha,(\beta-1)k+1}^{\delta k}\left(-(t/\tau)^\alpha\right). \end{aligned} \tag{6.105}$$

For the long time limit ($s \to 0$), one finds the asymptotic behavior

$$\begin{aligned} I(t) &= \frac{k_BT}{C_{\alpha,\beta,\delta}} t^{1+\alpha\delta-\beta} E_{1+\alpha\delta-\beta,2+\alpha\delta-\beta}\left(-\frac{k_BT\omega^2}{C_{\alpha,\beta,\delta}} t^{1+\alpha\delta-\beta}\right) \\ &= \frac{1}{\omega^2}\left[1 - E_{1+\alpha\delta-\beta}\left(-\frac{k_BT\omega^2}{C_{\alpha,\beta,\delta}} t^{1+\alpha\delta-\beta}\right)\right]. \end{aligned} \tag{6.106}$$

Therefore, the MSD reads

$$\begin{aligned} \langle x^2(t)\rangle = 2k_BT I(t) &\simeq \frac{2k_BT}{\omega^2}\left[1 - E_{1+\alpha\delta-\beta}\left(-\frac{k_BT\omega^2}{C_{\alpha,\beta,\delta}} t^{1+\alpha\delta-\beta}\right)\right] \\ &\simeq \frac{2k_BT}{\omega^2}\left[1 - \frac{C_{\alpha,\beta,\delta}}{k_BT\omega^2}\frac{t^{-(1+\alpha\delta-\beta)}}{\Gamma(\beta-\alpha\delta)}\right], \end{aligned} \tag{6.107}$$

and the VACF becomes

$$C_V(t) = g(t) \simeq -\frac{C_{\alpha,\beta,\delta}}{\omega^4}\frac{(1+\alpha\delta-\beta)(2+\alpha\delta-\beta)t^{-(1+\alpha\delta-\beta)-2}}{\Gamma(\beta-\alpha\delta)}. \tag{6.108}$$

At long times $t \to \infty$, the MSD reaches the equilibrium value

$$\langle x^2(t)\rangle_\infty = \frac{2k_BT}{\omega^2}.$$

For a free particle ($\omega = 0$) from (6.107) one obtains

$$
\begin{aligned}
I(t) &\approx \lim_{\omega\to 0} \frac{1}{\omega^2}\left[1 - E_{1+\alpha\delta-\beta}\left(-\frac{k_B T\omega^2}{C_{\alpha,\beta,\delta}} t^{1+\alpha\delta-\beta}\right)\right] \\
&= \lim_{\omega\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}\omega}\left[1 - E_{1+\alpha\delta-\beta}\left(-\frac{k_B T\omega^2}{C_{\alpha,\beta,\delta}} t^{1+\alpha\delta-\beta}\right)\right]}{\frac{\mathrm{d}}{\mathrm{d}\omega}\omega^2} = \frac{k_B T}{C_{\alpha,\beta,\delta}} \frac{t^{1+\alpha\delta-\beta}}{\Gamma(2+\alpha\delta-\beta)},
\end{aligned}
\tag{6.109}
$$

which is identical to (6.56) for the GLE for a free particle. In a similar way, for the relaxation functions $G(t)$ and $g(t)$, which are directly related to the time dependent diffusion coefficient and VACF, follow results (6.55) and (6.54), respectively.

Remark 6.2 Previous studies [57] showed that analytical treatment of the GLE with internal three parameter M-L noise with correlation function of the form

$$
C(t) = \frac{C_{\alpha,\beta,\delta}}{\tau^{\alpha\delta}} E_{\alpha,\beta}^{\delta}(-(t/\tau)^{\alpha}),
\tag{6.110}
$$

where $C_{\alpha,\beta,\delta}$ does not depend on time, and can depend on α, β and δ, where $\alpha > 0$, $\beta > 0$, $\delta > 0$, $0 < \alpha\delta < 2$, is very complex. The difficulty of analytical treatment of the GLE with an internal noise with correlation (6.110) is due to the Laplace transform of three parameter M-L function, see relation (1.61) for $\kappa = 1$. Therefore, we only analyze the asymptotic behavior of relaxation functions by using Tauberian theorems (see Appendix B). For the Laplace pair of $\gamma(t)$, from Eq. (1.61), we have

$$
\hat{\gamma}(s) = \frac{C_{\alpha,\beta,\delta}}{k_B T\tau^{\alpha\delta}} \frac{s^{-1}}{\Gamma(\delta)} \sum_{k=0}^{\infty} \frac{\Gamma(1+\alpha k)\Gamma(\delta+k)}{\Gamma(\beta+\alpha k)k!} \frac{(-1)^k}{(s\tau)^{\alpha k}}.
\tag{6.111}
$$

For the long time limit ($t \to \infty$) the frictional memory kernel has the following behavior

$$
\gamma(t) \simeq \frac{C_{\alpha,\beta,\delta}}{\Gamma(\beta-\alpha\delta)k_B T} \cdot t^{-\alpha\delta},
$$

so the Tauberian theorem yields

$$
\hat{\gamma}(s) \simeq \gamma_{\alpha,\beta,\delta} \cdot s^{\alpha\delta-1}, \quad s \to 0,
\tag{6.112}
$$

where

$$
\gamma_{\alpha,\beta,\delta} = \frac{C_{\alpha,\beta,\delta}}{k_B T} \frac{\Gamma(1-\alpha\delta)}{\Gamma(\beta-\alpha\delta)}.
$$

Here we use that $\beta \neq \alpha\delta$, $\beta \neq \alpha\delta - 1$ and $\alpha\delta \neq 1$. By substitution of (6.112) in the relaxation function

$$\hat{I}(s) = \mathscr{L}[I(t)] = \frac{s^{-1}}{s^2 + s\hat{\gamma}(s) + \omega^2}, \tag{6.113}$$

for $0 < \alpha\delta < 2$, one obtains

$$\hat{I}(s) \simeq \frac{s^{-1}}{\gamma_{\alpha,\beta,\delta} s^{\alpha\delta} + \omega^2} = \frac{1}{\omega^2}\left(\frac{1}{s} - \frac{s^{\alpha\delta-1}}{s^{\alpha\delta} + \omega^2/\gamma_{\alpha,\beta,\delta}}\right). \tag{6.114}$$

From the Laplace transform formula (1.3) for the one parameter M-L function, it follows [57]

$$I(t) \simeq \frac{1}{\omega^2}\left[1 - E_{\alpha\delta}\left(-\frac{\omega^2}{\gamma_{\alpha,\beta,\delta}} t^{\alpha\delta}\right)\right] \simeq \frac{1}{\omega^2}\left[1 - \frac{\gamma_{\alpha,\beta,\delta}}{\omega^2}\frac{1}{\Gamma(1-\alpha\delta)} t^{-\alpha\delta}\right]. \tag{6.115}$$

The MSD and VACF then read [57]

$$\left\langle x^2(t)\right\rangle \simeq \rho(\infty)\left[1 - \frac{\gamma_{\alpha,\beta,\delta}}{\omega^2}\frac{1}{\Gamma(1-\alpha\delta)} t^{-\alpha\delta}\right], \tag{6.116}$$

$$C_V(t) \simeq -\frac{\gamma_{\alpha,\beta,\delta}}{\omega^4}\frac{\alpha\delta(\alpha\delta+1)}{\Gamma(1-\alpha\delta)} t^{-\alpha\delta-2}. \tag{6.117}$$

respectively. The case with $\beta = \delta = 1$ corresponds to the results obtained in [12, 66, 67]. For a free particle ($\omega = 0$) we obtain [57]

$$I(t) = \mathscr{L}^{-1}\left[\frac{s^{-1-\alpha\delta}}{s^{2-\alpha\delta} + \gamma_{\alpha,\beta,\delta}}\right] = t^2 E_{2-\alpha\delta,3}\left(-\gamma_{\alpha,\beta,\delta} t^{2-\alpha\delta}\right), \tag{6.118}$$

where we apply the Laplace transform formula (1.6). The MSD then becomes

$$\left\langle x^2(t)\right\rangle = 2k_B T t^2 E_{2-\alpha\delta,3}\left(-\gamma_{\alpha,\beta,\delta} t^{2-\alpha\delta}\right) \simeq \frac{2k_B T}{\gamma_{\alpha,\beta,\delta}\Gamma(1+\alpha\delta)} t^{\alpha\delta}. \tag{6.119}$$

and the time dependent diffusion coefficient and VACF turn to

$$D(t) = k_B T t E_{2-\alpha\delta,2}\left(-\gamma_{\alpha,\beta,\delta} t^{2-\alpha\delta}\right) \simeq \frac{k_B T}{\gamma_{\alpha,\beta,\delta}\Gamma(\alpha\delta)} t^{\alpha\delta-1}, \tag{6.120}$$

$$C_V(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} t^2 E_{2-\alpha\delta,3}\left(-\gamma_{\alpha,\beta,\delta} t^{2-\alpha\delta}\right)$$
$$= E_{2-\alpha\delta}\left(-\gamma_{\alpha,\beta,\delta} t^{2-\alpha\delta}\right) \simeq \frac{1}{\gamma_{\alpha,\beta,\delta}\Gamma(\alpha\delta-1)} t^{\alpha\delta-2}, \tag{6.121}$$

respectively. Same result can be obtained from the L'Hôpital's rule, i.e., [57]

$$I(t) \simeq \lim_{\omega\to 0} \frac{1}{\omega^2}\left[1 - E_{\alpha\delta}\left(-\frac{\omega^2}{\gamma_{\alpha,\beta,\delta}} t^{\alpha\delta}\right)\right]$$
$$= \lim_{\omega\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}\omega}\left[1 - E_{\alpha\delta}\left(-\frac{\omega^2}{\gamma_{\alpha,\beta,\delta}} t^{\alpha\delta}\right)\right]}{\frac{\mathrm{d}}{\mathrm{d}\omega}\omega^2} = \frac{1}{\gamma_{\alpha,\beta,\delta}\Gamma(1+\alpha\delta)} t^{\alpha\delta}. \tag{6.122}$$

Thus, the particle shows anomalous diffusive behavior. The well-known result for $\beta = \delta = 1$ was obtained in Ref. [39, 52]. For $\alpha = \beta = \delta = 1$ one obtains $\rho(t) \simeq t$ and $C_V(t) \simeq E_1\left(-\gamma_{1,1,1} t\right) = e^{-\gamma_{1,1,1} t}$, which in fact is the result for Brownian motion [42, 52]. The case with $\alpha\delta = 1/2$ gives $C_V(t) \simeq t^{-\frac{3}{2}}$, which is theoretically obtained in Ref. [42] for superposition of the Dirac delta and power-law memory kernel, and previously confirmed by computer simulations for the VACF [1]. We can show, as well, that in case of a friction memory kernel which is a sum of the generalized M-L noise (6.110) and Dirac delta noise, the VACF has a form $C_V(t) \simeq t^{-\alpha\delta}$, so for $\alpha\delta = \frac{3}{2}$ and $\beta = 1$ again we obtain the same result $C_V(t) \simeq t^{-\frac{3}{2}}$ [57].

Remark 6.3 Let us now consider the following thermal initial conditions $\langle x_0^2\rangle = \frac{k_B T}{\omega^2}$, $\langle x_0 v_0\rangle = 0$, and $\langle \xi(t) x_0\rangle = 0$ for the GLE for a harmonic oscillator. For the normalized displacement correlation function, which is an experimental measured quantity, and which is defined by Burov and Barkai [5, 6]

$$C_X(t) = \frac{\langle x(t) x_0\rangle}{\langle x_0^2\rangle},$$

one obtains

$$C_X(t) = 1 - \omega^2 I(t). \tag{6.123}$$

For the friction memory kernel of form (6.9) in the limit $\tau \to 0$, for $C_X(t)$ we find

$$C_X(t) = 1 - \sum_{k=0}^{\infty} \left(-\omega^2\right)^{k+1} t^{2k+2} E_{2-(1+\alpha\delta-\beta),2k+3}^{k+1}\left(-\frac{C_{\alpha,\beta,\delta}}{k_B T} t^{2-(1+\alpha\delta-\beta)}\right). \tag{6.124}$$

The graphical representation of the normalized displacement correlation function (6.124) is given in Fig. 6.11. Note that for $\omega = 0.3$, $C_X(t)$ is a decreasing monotone function and $C_X(t) > 0$. For $\omega = 3$ and $\omega = 1$, $C_X(t)$ has an

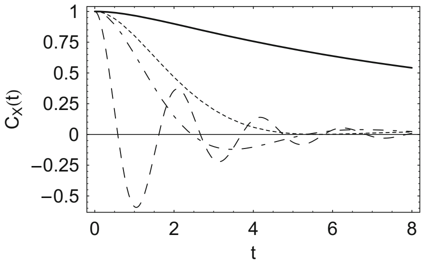

Fig. 6.11 Graphical representation of $C_X(t)$ (6.124) for $C_{\alpha,\beta,\delta} = 1$, $k_BT = 1$, $\alpha = 1/2$, $\beta = 7/16$, $\delta = 3/4$; $\omega = 0.3$ (solid line), $\omega = 3$ (dashed line); $\omega = 1$ (dot-dashed line); $\omega = 0.74$ (dotted line), see Ref. [55]

oscillation-like behavior passing the zero line, and goes asymptotically to zero. For $\omega = 0.74$, $C_X(t) > 0$, but it is a non-monotone function. It approaches the zero line asymptotically. These results are different than those obtained for the Langevin equation for a harmonic oscillator, for which the oscillator has only two different behaviors; either overdamped motion with $\langle x(t)\rangle > 0$ for all t under the condition $\langle x_0\rangle > 0$, for which $C_X(t)$ is monotone function, or underdamped motion when $\langle x(t)\rangle$ has oscillation-like behavior passing the zero line [5, 6]. The frequency at which the oscillator turns from overdamped to underdamped motion is the so-called critical frequency. For the GLE for a harmonic oscillator there is a need for definition to additional critical frequencies on which $C_X(t)$ changes its behavior, and their computation is a non-trivial problem [5, 6]. Such behaviors of $C_X(t)$ were observed in the molecular dynamic simulations of fluctuation of the donor-acceptor distance within proteins [38]. Moreover, such oscillation-like behavior and power law decay of the fluorescein-tyrosine distance within a protein are experimentally observed in Ref. [47].

6.4 GLE with Prabhakar-Like Friction

As we showed before, the regularized Prabhakar derivative (2.88) is a special case of the generalized derivative (2.89), therefore we conclude that the GLE with regularized Prabhakar friction memory kernel of the form

$$\gamma(t) = \gamma_{\mu,\rho,\delta}\, t^{-\mu} E_{\rho,1-\mu}^{-\delta}\left(-\left(\frac{t}{\tau}\right)^{\rho}\right). \tag{6.125}$$

has the form [56]

$$\ddot{x}(t) + \gamma_{\mu,\rho,\delta}\, {}_{C}\mathscr{D}_{\rho,-\nu,t}^{\delta,\mu}\, x(t) = \xi(t), \quad \dot{x}(t) = v(t). \tag{6.126}$$

Here ${}_{C}\mathscr{D}_{\rho,-\nu,t}^{\delta,\mu}$ is the regularized Prabhakar derivative (2.88), $0 < \mu, \delta < 1$, $0 < \mu/\delta < 1$, $0 < \mu/\delta - \rho < 1$, $\nu = \tau^{-\mu}$, τ is a time parameter, and $\gamma_{\mu,\rho,\delta}$ is the generalized friction coefficient. This equation is a generalization of the fractional Langevin equation considered by Lutz [39], which is recovered by setting $\delta = 0$.

The Laplace transform of the friction memory kernel (6.125) reads

$$\hat{\gamma}(s) = \gamma_{\mu,\rho,\delta}\, s^{-\rho\delta+\mu-1}\left(s^{\rho}+\tau^{-\rho}\right)^{\delta} \tag{6.127}$$

By asymptotic expansion of the three parameter M-L function (1.28) and the Laplace transform of the friction memory kernel (6.127), we show that the assumption (4.32) is satisfied for $\mu > \rho\delta$. We consider that the noise $\xi(t)$ is internal, i.e., the second fluctuation-dissipation theorem of the form [56]

$$\left\langle \xi(t)\xi(t')\right\rangle = k_b T \gamma_{\mu,\rho,\delta}\, |t-t'|^{-\mu} E_{\rho,1-\mu}^{-\delta}\left(-\left(\frac{|t-t'|}{\tau}\right)^{\rho}\right), \tag{6.128}$$

is satisfied.

6.4.1 Free Particle

From the general formulas for the relaxation functions, the MSD, $D(t)$, and VACF become [56]

$$\left\langle x^2(t)\right\rangle = 2k_B T \sum_{n=0}^{\infty}(-\gamma_{\mu,\rho,\delta})^n t^{(2-\mu)n+2} E_{\rho,(2-\mu)n+3}^{-\delta n}\left(-\left(\frac{t}{\tau}\right)^{\rho}\right), \tag{6.129}$$

$$D(t) = k_B T \sum_{n=0}^{\infty}(-\gamma_{\mu,\rho,\delta})^n t^{(2-\mu)n+1} E_{\rho,(2-\mu)n+2}^{-\delta n}\left(-\left(\frac{t}{\tau}\right)^{\rho}\right), \tag{6.130}$$

$$C_V(t) = \sum_{n=0}^{\infty}(-\gamma_{\mu,\rho,\delta})^n t^{(2-\mu)n} E_{\rho,(2-\mu)n+1}^{-\delta n}\left(-\left(\frac{t}{\tau}\right)^{\rho}\right), \tag{6.131}$$

respectively.

The asymptotic expansion of the three parameter M-L function (1.29) for the long time limit yields

$$\left\langle x^2(t)\right\rangle \simeq 2k_B T\, t^2 E_{2-\mu+\rho\delta,3}\left(-\bar{\gamma} t^{2-\mu+\rho\delta}\right) \simeq \frac{2k_B T}{\bar{\gamma}} \frac{t^{\mu-\rho\delta}}{\Gamma(1+\mu-\rho\delta)}, \tag{6.132}$$

$$D(t) \simeq k_B T\, t E_{2-\mu+\rho\delta,2}\left(-\bar{\gamma} t^{2-\mu+\rho\delta}\right) \tag{6.133}$$

$$C_V(t) \simeq E_{2-\mu+\rho\delta}\left(-\bar{\gamma} t^{2-\mu+\rho\delta}\right), \tag{6.134}$$

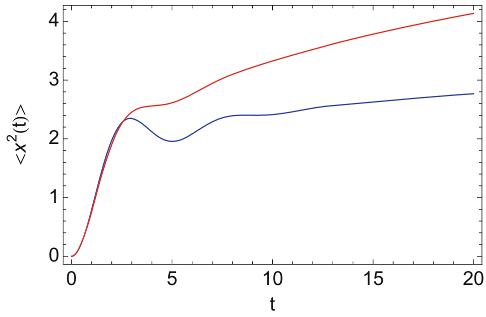

Fig. 6.12 Graphical representation of the MSD (6.129) for $k_B T = 1$, $\gamma_{\mu,\rho,\delta} = 1$, $\tau = 1$, $\rho = 1/2$, $\delta = 3/4$, and $\mu = 1/2$ (blue line), $\mu = 5/8$ (red line), see Ref. [56]

where $\bar{\gamma} = \gamma_{\mu,\rho,\delta}\tau^{-\rho\delta}$. Therefore, one concludes that in the system exists subdiffusion $\langle x^2(t)\rangle \simeq t^{\alpha}$ with anomalous diffusion exponent $\alpha = \mu - \rho\delta$, where $0 < \alpha < \delta < 1$.

Graphical representation of the MSD (6.129) is given in Fig. 6.12. From the figure we see that the MSD shows oscillation-like behavior for intermediate times which can be explained as a result of the cage effect of the environment represented by the M-L memory kernel [5].

6.4.2 High Friction

The high viscous damping, corresponding to vanishing of the inertial term $\ddot{x}(t) = 0$, yields [56]

$$\left\langle x^2(t)\right\rangle = \frac{2k_B T}{\gamma_{\mu,\rho,\delta}}\, t^{\mu} E^{\delta}_{\rho,\mu+1}\left(-\left(\frac{t}{\tau}\right)^{\rho}\right) \simeq \frac{2k_B T}{\gamma_{\mu,\rho,\delta}} \begin{cases} \frac{t^{\mu}}{\Gamma(\mu+1)}, & t \to 0 \\ \frac{t^{\mu-\rho\delta}}{\tau^{-\rho\delta}\Gamma(1+\mu-\rho\delta)}, & t \to \infty. \end{cases} \tag{6.135}$$

Therefore, we conclude that decelerating subdiffusion exists in the system, since the anomalous diffusion exponent from μ for the short time limit turns to $\mu - \rho\delta$ in the long time limit.

6.4.3 Tempered Friction

We further consider the GLE with a friction term represented through the tempered regularized Prabhakar derivative (2.92), i.e.,

$$\ddot{x}(t) + \gamma_{\mu,\rho,\delta}\ {}_{TC}\mathscr{D}^{\delta,\mu}_{\rho,-\nu,t}x(t) = \xi(t), \quad \dot{x}(t) = v(t), \tag{6.136}$$

where $b > 0$, and all the parameters are the same as in Eq. (6.126). From definition (2.92) one concludes that the friction memory kernel is given by [56]

$$\gamma(t) = \gamma_{\mu,\rho,\delta}\, e^{-bt} t^{-\mu} E_{\rho,1-\mu}^{-\delta}\left(-\left(\frac{t}{\tau}\right)^{\rho}\right). \tag{6.137}$$

The second fluctuation-dissipation theorem then reads

$$\left\langle \xi(t)\xi(t')\right\rangle = k_b T \gamma_{\mu,\rho,\delta}\, e^{-b|t-t'|}|t-t'|^{-\mu} E_{\rho,1-\mu}^{-\delta}\left(-\left(\frac{|t-t'|}{\tau}\right)^{\rho}\right). \tag{6.138}$$

For the MSD, we find [56]

$$\left\langle x^2(t)\right\rangle = \sum_{n=0}^{\infty}\left(-\gamma_{\mu,\rho,\delta}\right)^n\, I_{0+}^{n+3}\left(e^{-bt} t^{(1-\mu)n-1} E_{\rho,(1-\mu)n}^{-\delta n}\left(-\left(\frac{t}{\tau}\right)^{\rho}\right)\right), \tag{6.139}$$

where I_{0+}^{α} is the R-L integral (2.2). In absence of truncation ($b = 0$), from (6.139), by using that [30]

$$I_{0+}^{\zeta}\left(t^{\beta-1} E_{\alpha,\beta}^{\delta}\left(-\nu t^{\alpha}\right)\right) = t^{\zeta+\beta-1} E_{\alpha,\zeta+\beta}^{\delta}\left(-\nu t^{\alpha}\right),$$

we recover the result (6.129).

For high viscous damping, $\ddot{x}(t) = 0$, the following result for the MSD is obtained

$$\left\langle x^2(t)\right\rangle = \frac{2k_B T}{\gamma_{\mu,\rho,\delta}}\, I_{0+}^{2}\left(e^{-bt} t^{\mu-2} E_{\rho,\mu-1}^{\delta}\left(-\left(\frac{t}{\tau}\right)^{\rho}\right)\right). \tag{6.140}$$

Therefore, the short time limit yields subdiffusion

$$\left\langle x^2(t)\right\rangle \simeq \frac{2k_B T}{\gamma_{\mu,\rho,\delta}} \frac{t^{\mu}}{\Gamma(1+\mu)},$$

and the long time limit normal diffusion $\left\langle x^2(t)\right\rangle \simeq t$. This means that accelerating diffusion—from subdiffusion to normal diffusion—exists in the system. Such crossover from subdiffusion to normal diffusion has been observed, for example, in complex viscoelastic systems [28].

Graphical representation of the MSD (6.139) is given in Fig. 6.13. From the figure, one observes the influence of the truncation parameter b on the MSD behavior. The case with no truncation ($b = 0$) shows subdiffusive behavior (blue line), and the case with truncation (red and green lines) normal diffusion in the long time limit.

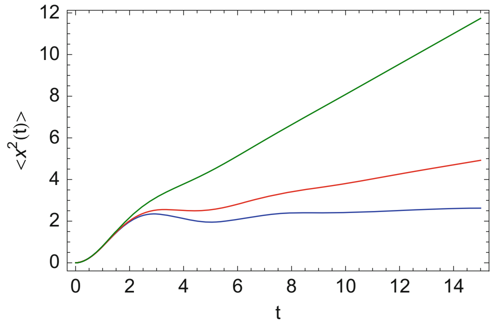

Fig. 6.13 Graphical representation of the MSD (6.139), for $k_B T = 1$, $\gamma_{\mu,\rho,\delta} = 1$, $\tau = 1$, $\rho = 1/2$, $\mu = 1/2$, $\delta = 3/4$, and $b = 0$ (blue line), $b = 0.1$ (red line) and $b = 0.5$ (green line), see Ref. [56].

6.4.4 *Harmonic Oscillator*

We further consider the GLE (6.141) for a harmonic oscillator with tempered regularized Prabhakar friction [56]

$$\ddot{x}(t) + \gamma_{\mu,\rho,\delta}\ {}_{TC}\mathscr{D}^{\delta,\mu}_{\rho,-\nu,t}\, x(t) + \omega^2 x(t) = \xi(t), \quad \dot{x}(t) = v(t), \tag{6.141}$$

where ω is the frequency of the oscillator. From the Laplace transform method we find exact result for the MSD

$$\begin{aligned} \frac{\langle x^2(t)\rangle}{2k_B T} &= \sum_{n=0}^{\infty} \left(-\gamma_{\mu,\rho,\delta}\right)^n \int_0^t (t-t')^{n+2} E^{n+1}_{2,n+3}\left(-\omega^2 (t-t')^2\right) \\ &\quad \times e^{-bt'} t'^{(1-\mu)n-1} E^{-\delta n}_{\rho,(1-\mu)n}\left(-\left(\frac{t'}{\tau}\right)^{\rho}\right) dt' \\ &= \sum_{n=0}^{\infty} \left(-\gamma_{\mu,\rho,\delta}\right)^n \mathbf{E}^{n+1}_{2,n+3,-\omega^2,0+}\left(e^{-bt} t^{(1-\mu)n-1} E^{-\delta n}_{\rho,(1-\mu)n}\left(-\left(\frac{t}{\tau}\right)^{\rho}\right)\right), \end{aligned} \tag{6.142}$$

where $\left(\mathbf{E}^{\delta}_{\alpha,\beta,-\omega^2,0+} f\right)(t)$ is the Prabhakar integral (2.46). For $\omega = 0$, the Prabhakar integral corresponds to the R-L integral (2.2), therefore, from (6.142) one finds the previously obtained result for a free particle (6.129).

We are particularly interested in the normalized displacement correlation function

$$C_X(t) = \frac{\langle x(t) x_0\rangle}{\langle x_0^2\rangle} = \frac{s + \hat{\gamma}(s)}{s^2 + s\hat{\gamma}(s) + \omega^2} = 1 - \omega^2 I(t), \tag{6.143}$$

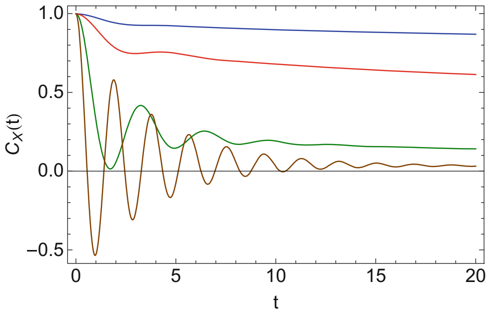

Fig. 6.14 Graphical representation of the normalized displacement correlation function, Eq. (6.145), for $\gamma_{\mu,\rho,\delta} = 1$, $\tau = 1$, $\rho = 1/5$ $\mu = 1/2$, $\delta = 3/4$, and $\omega = 0.25$ (blue line), $\omega = 0.5$ (red line), $\omega = 1.44$ (green line), $\omega = 3$ (brown line), see Ref. [56]

under the conditions $x_0^2 = \frac{k_B T}{\omega^2}$, $\langle x_0 v_0 \rangle = 0$, and $\langle \xi(t) x_0 \rangle = 0$ [5]. $C_X(t)$ then becomes [56]

$$C_X(t) = 1 - \omega^2 \sum_{n=0}^{\infty} \left(-\gamma_{\mu,\rho,\delta}\right)^n \mathbf{E}^{n+1}_{2,n+3,-\omega^2,0+} \times \left(e^{-bt} t^{(1-\mu)n-1} E^{-\delta n}_{\rho,(1-\mu)n}\left(-\left(\frac{t}{\tau}\right)^{\rho}\right)\right), \tag{6.144}$$

and the case with no truncation ($b =$) yields

$$C_X(t) = 1 - \omega^2 \sum_{n=0}^{\infty} \left(-\gamma_{\mu,\rho,\delta}\right)^n \mathbf{E}^{n+1}_{2,n+3,-\omega^2,0+} \left(t^{(1-\mu)n-1} E^{-\delta n}_{\rho,(1-\mu)n}\left(-\left(\frac{t}{\tau}\right)^{\rho}\right)\right). \tag{6.145}$$

Graphical representation of the $C_X(t)$ (6.145) and (6.144) is given in Figs. 6.14 and 6.15, respectively. In Fig. 6.14 different behaviors of $C_X(t)$ are observed, such as monotonic or non-monotonic decay without zero crossings (for $\omega < 1.44$), critical behavior between the situations with and without zero crossings (at critical frequency $\omega \approx 1.44$), and oscillation-like behavior with zero crossings (for $\omega > 1.44$), which appear due to the cage effect of the environment [5]. The friction, depending on the memory kernel parameters, forces either diffusion or oscillations. In Fig. 6.15 we note that with increasing of tempering, oscillation behavior with zero crossings appears. Thus, by tuning the values of friction parameters contained in the tempered Prabhakar derivative, we increase the versatility to fit complex experimental data.

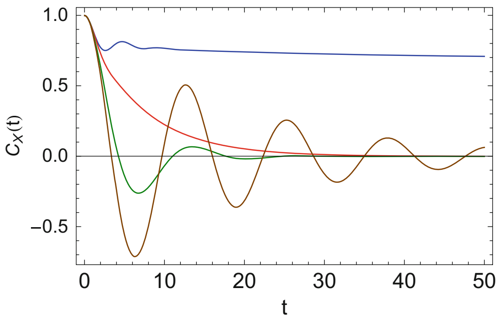

Fig. 6.15 Graphical representation of the normalized displacement correlation function, Eq. (6.144), for $\gamma_{\mu,\rho,\delta} = 1$, $\tau = 1$, $\rho = 1/2$ $\mu = 1/2$, $\delta = 3/4$, $\omega = 0.5$ and $b = 0$ (blue line), $b = 1$ (red line), $b = 10$ (green line), $b = 100$ (brown line), see Ref. [56]

6.5 Tempered GLE

Here we consider truncated three parameter M-L memory kernel of the form [36]

$$\gamma(t) = \frac{\gamma}{\tau^{\alpha\delta}} e^{-bt} t^{\beta-1} E_{\alpha,\beta}^{\delta}\left(-\frac{t^{\alpha}}{\tau^{\alpha}}\right), \tag{6.146}$$

where $\gamma > 0$ is a constant, $b \geq 0$, $\delta \geq 0$, $\tau > 0$ is a time parameter, and $E_{\alpha,\beta}^{\delta}(z)$ is the three parameter M-L function (1.14) [54]. Tempered diffusion with memory kernel of the form (6.146) with $\delta = 1$ was obtained within the CTRW theory in Ref. [61]. Similar kernels were considered in Refs. [48, 62, 63] in the context of tempered subdiffusion.

The Laplace transform of the kernel is given by

$$\hat{\gamma}(s) = \frac{\gamma}{\tau^{\alpha\delta}} \frac{(s+b)^{\alpha\delta-\beta}}{\left((s+b)^{\alpha} + \tau^{-\alpha}\right)^{\delta}}, \tag{6.147}$$

where we use the shift rule $\mathscr{L}\left[f(t)e^{-at}\right] = \hat{F}(s+a)$, $\mathscr{L}[f(t)] = \hat{F}(s)$, and the Laplace transform of the three parameter M-L function. It is obvious that the tempered memory kernel (6.146) satisfies the assumption (4.32). The tempered memory kernel is quite general and contains a number of limiting cases. For example, for $\tau \to 0$ ($\tau^{-1} \to \infty$) it becomes truncated power-law memory kernel

$$\gamma(t) = \gamma\, e^{-bt} \frac{t^{\beta-\alpha\delta-1}}{\Gamma(\beta-\alpha\delta)},$$

such that

$$\hat{\gamma}(s) = \gamma\, (s+b)^{-\beta+\alpha\delta}.$$

For $\delta = 1$ and $\delta = \beta = 1$, one finds the truncated two parameter and one parameter M-L kernel, respectively. In absence of truncation ($b = 0$), it yields the three parameter M-L memory kernel [59]

$$\gamma(t) = \frac{\gamma}{\tau^{\alpha\delta}}\, t^{\beta-1} E_{\alpha,\beta}^{\delta}\left(-\frac{t^{\alpha}}{\tau^{\alpha}}\right),$$

which for $\alpha = \beta = 1$ corresponds to the Kummer's confluent hypergeometric memory function

$$\gamma(t) = \frac{\gamma}{\tau^{\delta}}\, E_{1,1}^{\delta}\left(-\frac{t}{\tau}\right) = \frac{\gamma}{\tau^{\delta}}\, \phi\left(\delta, 1, -t/\tau\right),$$

considered in Ref. [32].

6.5.1 Free Particle: Relaxation Functions

The relaxation functions for the truncated three parameter M-L memory kernel (6.146) becomes

$$\begin{aligned} I(t) &= \mathscr{L}^{-1}\left[\frac{s^{-3}}{1 + \frac{\gamma}{\tau^{\alpha\delta}}\, s^{-1} \frac{(s+b)^{\alpha\delta-\beta}}{\left((s+b)^{\alpha}+\tau^{-\alpha}\right)^{\delta}}}\right] \\ &= \mathscr{L}^{-1}\left[\sum_{n=0}^{\infty}\left(-\frac{\gamma}{\tau^{\alpha\delta}}\right)^{n} s^{-(n+3)} \frac{(s+b)^{(\alpha\delta-\beta)n}}{\left((s+b)^{\alpha}+\tau^{-\alpha}\right)^{\delta n}}\right] \\ &= \sum_{n=0}^{\infty}\left(-\frac{\gamma}{\tau^{\alpha\delta}}\right)^{n} I_{0+}^{n+3}\left(e^{-bt} t^{\beta n-1} E_{\alpha,\beta n}^{\delta n}\left(-\frac{t^{\alpha}}{\tau^{\alpha}}\right)\right), \end{aligned} \tag{6.148}$$

where $I_{0+}^{\alpha} f(t)$ is the R-L integral (2.2) of order $\alpha > 0$. Respectively, the other relaxation functions read

$$G(t) = \sum_{n=0}^{\infty}\left(-\frac{\gamma}{\tau^{\alpha\delta}}\right)^{n} I_{0+}^{n+2}\left(e^{-bt} t^{\beta n-1} E_{\alpha,\beta n}^{\delta n}\left(-\frac{t^{\alpha}}{\tau^{\alpha}}\right)\right), \tag{6.149}$$

$$g(t) = \sum_{n=0}^{\infty}\left(-\frac{\gamma}{\tau^{\alpha\delta}}\right)^{n} I_{0+}^{n+1}\left(e^{-bt} t^{\beta n-1} E_{\alpha,\beta n}^{\delta n}\left(-\frac{t^{\alpha}}{\tau^{\alpha}}\right)\right). \tag{6.150}$$

In absence of truncation ($b = 0$), one finds the results obtained in Ref. [59]. We note that the relaxation functions (6.148)–(6.150) can also be written without the R-L integral in terms of the confluent hypergeometric function ${}_1F_1(a; b; z)$ [14], as

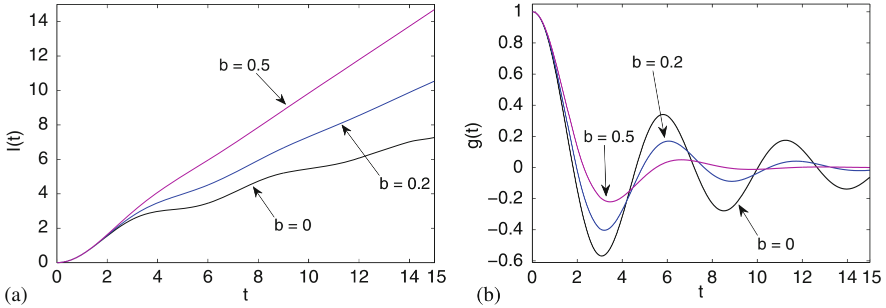

Fig. 6.16 Graphical representation of: (**a**) relaxation function $I(t)$ (6.148), (**b**) relaxation function $g(t)$ (6.150), for $\alpha = 1.5$, $\beta = 1.2$, $\delta = 0.6$, $\tau = 1$, $\gamma = 1$, $b = 0$ (black line), $b = 0.2$ (blue line), $b = 0.5$ (violet line). Reprinted from Physica A, 466, A. Liemert, T. Sandev and H. Kantz, Generalized Langevin equation with tempered memory kernel, 356–369, Copyright (2017), with permission from Elsevier

follows [36]

$$I(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+k}\tau^{-\alpha(\delta n+k)}\gamma^{n}\frac{(\delta n)_k}{k!}\frac{t^{\alpha k+\beta n+n+2}}{\Gamma(\alpha k+\beta n+n+3)} \times {}_1F_1(\alpha k+\beta n;\alpha k+\beta n+n+3;-bt), \tag{6.151}$$

$$G(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+k}\tau^{-\alpha(\delta n+k)}\gamma^{n}\frac{(\delta n)_k}{k!}\frac{t^{\alpha k+\beta n+n+1}}{\Gamma(\alpha k+\beta n+n+2)} \times {}_1F_1(\alpha k+\beta n;\alpha k+\beta n+n+2;-bt), \tag{6.152}$$

$$g(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{n+k}\tau^{-\alpha(\delta n+k)}\gamma^{n}\frac{(\delta n)_k}{k!}\frac{t^{\alpha k+\beta n+n}}{\Gamma(\alpha k+\beta n+n+1)} \times {}_1F_1(\alpha k+\beta n;\alpha k+\beta n+n+1;-bt). \tag{6.153}$$

Graphical representation of the relaxation function (6.148) for different values of parameters is given in Fig. 6.16. From the figure one concludes that in the case of truncation the relaxation function, which is proportional to the MSD, has a linear dependence on time in the long time limit. In absence of truncation for the chosen parameters the MSD shows subdiffusive behavior of the form $t^{0.7}$. Due to the complex form of the memory kernel in the intermediate times the MSD has an oscillation-like behavior. Such behavior can be explained due to the cage effects [5], which appear as a result of influence of the environment (represented by the friction memory kernel) on the particle motion.

From the exact result for the relaxation functions, we analyze the MSD and VACF. For the short time limit one finds

$$\frac{\langle x^2(t)\rangle}{2k_BT} \simeq \sum_{n=0}^{\infty}\left(-\frac{\gamma}{\tau^{\alpha\delta}}\right)^n I_{0+}^{n+3}\left(\frac{t^{\beta n-1}}{\Gamma(\beta n)}\right) = \sum_{n=0}^{\infty}\left(-\frac{\gamma}{\tau^{\alpha\delta}}\right)^n \frac{t^{(\beta+1)n+2}}{\Gamma((\beta+1)n+3)}$$
$$= t^2 E_{\beta+1,3}\left(-\frac{\gamma}{\tau^{\alpha\delta}}\, t^{\beta+1}\right) \simeq \frac{t^2}{\Gamma(3)} - \frac{\gamma}{\tau^{\alpha\delta}}\frac{t^{\beta+3}}{\Gamma(\beta+4)}, \tag{6.154}$$

while the VACF becomes

$$C_V(t) \simeq E_{\beta+1}\left(-\frac{\gamma}{\tau^{\alpha\delta}} t^{\beta+1}\right) \simeq 1 - \frac{\gamma}{\tau^{\alpha\delta}}\frac{t^{\beta+1}}{\Gamma(\beta+2)}. \tag{6.155}$$

The long time limit yields normal diffusion

$$\frac{\langle x^2(t)\rangle}{2k_BT} \simeq \sum_{n=0}^{\infty}\left(-\frac{\gamma}{\tau^{\alpha\delta}}\,\frac{b^{\alpha\delta-\beta}}{(b^\alpha+\tau^{-\alpha})^\delta}\right)^n \frac{t^{n+2}}{\Gamma(n+3)} = t^2 E_{1,3}\left(-\frac{\gamma}{\tau^{\alpha\delta}}\,\frac{b^{\alpha\delta-\beta}}{(b^\alpha+\tau^{-\alpha})^\delta} t\right)$$
$$= \frac{\exp\left(-\frac{\gamma}{\tau^{\alpha\delta}}\,\frac{b^{\alpha\delta-\beta}}{(b^\alpha+\tau^{-\alpha})^\delta} t\right) + \frac{\gamma}{\tau^{\alpha\delta}}\,\frac{b^{\alpha\delta-\beta}}{(b^\alpha+\tau^{-\alpha})^\delta} t - 1}{\left(-\frac{\gamma}{\tau^{\alpha\delta}}\,\frac{b^{\alpha\delta-\beta}}{(b^\alpha+\tau^{-\alpha})^\delta}\right)^2} \simeq \frac{b^\beta}{\gamma}(\tau^\alpha+b^{-\alpha})^\delta t, \tag{6.156}$$

$$C_V(t) \simeq E_{1,1}\left(-\frac{\gamma}{\tau^{\alpha\delta}}\,\frac{b^{\alpha\delta-\beta}}{(b^\alpha+\tau^{-\alpha})^\delta} t\right) = \exp\left(-\frac{\gamma}{\tau^{\alpha\delta}}\,\frac{b^{\alpha\delta-\beta}}{(b^\alpha+\tau^{-\alpha})^\delta} t\right) \to 0. \tag{6.157}$$

Therefore, characteristic crossover dynamics from ballistic motion to normal diffusion is observed.

6.5.2 High Viscous Damping Regime

Let us now consider high viscous damping, which means that $\dot{v}(t) = 0$. The GLE (6.8) then reads

$$\int_0^t \gamma(t-t')v(t')\,\mathrm{d}t' = \xi(t), \quad \dot{x}(t) = v(t). \tag{6.158}$$

The relaxation functions become

$$\hat{g}(s) = \frac{1}{\hat{\gamma}(s)}, \quad \hat{G}(s) = \frac{s^{-1}}{\hat{\gamma}(s)}, \quad \hat{I}(s) = \frac{s^{-2}}{\hat{\gamma}(s)}. \tag{6.159}$$

For the truncated memory kernel (6.146), we find exact result for the MSD

$$\begin{aligned}\frac{\langle x^2(t)\rangle}{2k_BT} &= \frac{\tau^{\alpha\delta}}{\gamma}\mathscr{L}^{-1}\left[s^{-2}\frac{(s+b)^{-\alpha\delta+\beta}}{\left((s+b)^{\alpha}+\tau^{-\alpha}\right)^{-\delta}}\right] \\ &= \frac{\tau^{\alpha\delta}}{\gamma} I_{0+}^{2}\left(e^{-bt}t^{-\beta-1}E_{\alpha,-\beta}^{-\delta}\left(-\frac{t^{\alpha}}{\tau^{\alpha}}\right)\right).\end{aligned} \tag{6.160}$$

Here we note that the MSD can also be written in terms of the regularized hypergeometric function [14], i.e.,

$$\frac{\langle x^2(t)\rangle}{2k_BT} = \frac{\tau^{\alpha\delta}}{\gamma}e^{-bt}t^{1-\beta}\sum_{k=0}^{\infty}(-1)^k(t/\tau)^k\frac{(-\delta)_k}{k!}{}_1\tilde{F}_1(2;2+\alpha k-\beta;bt), \tag{6.161}$$

where

$${}_1\tilde{F}_1(a;b;z) = \sum\nolimits_{k=0}^{\infty}(a)_k\, z^k/[k!\Gamma(k+b)].$$

For $\delta = 1$ we obtain the same result as the one obtained in Ref. [61] within the CTRW theory. Therefore, two different diffusion models which describe different stochastic processes may give same results for the MSD. For the short time limit subdiffusive behavior is observed,

$$\frac{\langle x^2(t)\rangle}{2k_BT} \simeq \frac{\tau^{\alpha\delta}}{\gamma}\frac{t^{1-\beta}}{\Gamma(2-\beta)},$$

while for the long time limit—normal diffusive behavior

$$\frac{\langle x^2(t)\rangle}{2k_BT} \simeq \frac{b^{\beta}}{\gamma}\left(\tau^{\alpha}+b^{-\alpha}\right)^{\delta}t,$$

which is same as (6.156).

The case with $b = 0$, for the short time limit gives subdiffusive behavior

$$\frac{\langle x^2(t)\rangle}{2k_BT} \simeq \frac{\tau^{\alpha\delta}}{\gamma}\frac{t^{1-\beta}}{\Gamma(2-\beta)},$$

while for the long time limit diffusive behavior of the form

$$\frac{\langle x^2(t)\rangle}{2k_BT} \simeq \frac{1}{\gamma}\frac{t^{1+\alpha\delta-\beta}}{\Gamma(2+\alpha\delta-\beta)}.$$

Therefore, the MSD has subdiffusive behavior for $\alpha\delta < \beta$, normal for $\alpha\delta = \beta$, and superdiffusive for $\alpha\delta > \beta$. This means that the particle shows accelerating diffusion, from subdiffusion it turns either to subdiffusion with greater anomalous diffusion exponent, normal diffusion or superdiffusion. Note that in the long time limit in both cases, with and without inertial term, same behavior for the MSD is obtained.

6.5.3 Harmonic Oscillator

For a particle bounded in a harmonic potential we use the previously presented general expressions for the relaxation functions, see (6.103). For the tempered memory kernel (6.146) one finds exact result for the relaxation function,

$$\begin{aligned} I(t) &= \mathscr{L}^{-1}\left[\frac{s^{-1}}{s^2+\omega^2}\frac{1}{1+\frac{\gamma}{\tau^{\alpha\delta}}\frac{s}{s^2+\omega^2}\frac{(s+b)^{\alpha\delta-\beta}}{\left((s+b)^\alpha+\tau^{-\alpha}\right)^\delta}}\right] \\ &= \mathscr{L}^{-1}\left[\sum_{n=0}^{\infty}\left(-\frac{\gamma}{\tau^{\alpha\delta}}\right)^n\frac{s^{n-1}}{\left(s^2+\omega^2\right)^{n+1}}\frac{(s+b)^{(\alpha\delta-\beta)n}}{\left((s+b)^\alpha+\tau^{-\alpha}\right)^{\delta n}}\right] \\ &= \sum_{n=0}^{\infty}\left(-\frac{\gamma}{\tau^{\alpha\delta}}\right)^n\int_0^t (t-t')^{n+2}E_{2,n+3}^{n+1}\left(-\omega^2(t-t')^2\right)e^{-bt'}t'^{\beta n-1} \\ &\quad \times E_{\alpha,\beta n}^{\delta n}\left(-\frac{t'^\alpha}{\tau^\alpha}\right)\mathrm{d}t' \\ &= \sum_{n=0}^{\infty}\left(-\frac{\gamma}{\tau^{\alpha\delta}}\right)^n \mathbf{E}_{2,n+3,-\omega^2,0+}^{n+1}\left(e^{-bt}t^{\beta n-1}E_{\alpha,\beta n}^{\delta n}\left(-\frac{t^\alpha}{\tau^\alpha}\right)\right). \end{aligned} \tag{6.162}$$

For the special case $\beta = \delta = 1$, and $\tau \to 0$, we obtain the result for tempered power-law memory kernel $\gamma(t) = e^{-bt}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, $0 < \alpha < 1$,

$$I(t) = \sum_{n=0}^{\infty}(-\gamma)^n\,\mathbf{E}_{2,n+3,-\omega^2.0+}^{n+1}\left(e^{-bt}\frac{t^{(1-\alpha)n-1}}{\Gamma((1-\alpha)n)}\right). \tag{6.163}$$

The normalized displacement correlation function is represented through $I(t)$, as $C_X(t) = 1 - \omega^2 I(t)$. Therefore, we have [55]

$$C_X(t) = 1 - \omega^2 \sum_{n=0}^{\infty} \left(-\frac{\gamma}{\tau^{\alpha\delta}}\right)^n \mathbf{E}^{n+1}_{2,n+3,-\omega^2,0+} \left(e^{-bt} t^{\beta n-1} E^{\delta n}_{\alpha,\beta n}\left(-\frac{t^\alpha}{\tau^\alpha}\right)\right). \tag{6.164}$$

For tempered power-law memory kernel $\gamma(t) = e^{-bt}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, $0 < \alpha < 1$, the normalized displacement correlation function $C_X(t)$ reduces to

$$C_X(t) = 1 - \omega^2 \sum_{n=0}^{\infty} (-\gamma)^n \mathbf{E}^{n+1}_{2,n+3,-\omega^2,0+} \left(e^{-bt} \frac{t^{(1-\alpha)n-1}}{\Gamma((1-\alpha)n)}\right). \tag{6.165}$$

Graphical representation of the normalized displacement correlation function (6.165) for different values of parameters is given in Fig. 6.17. From the figures one concludes that the normalized displacement correlation function shows different behaviors: monotonic decay, non-monotonic decay without zero crossings, critical behavior which distinguishes the cases with and without zero crossings, and oscillation-like behavior with zero crossings. These behaviors are based on the cage effects of the environment as shown by Burov and Barkai [5]. This means that, depending on the values of the friction memory kernel parameters, the friction caused by the complex environment may force either diffusion or oscillations. These effects are observed in the analysis of the relaxation functions as well (Fig. 6.16). From Fig. 6.17 one concludes that the critical frequencies in case of truncated power-law memory kernel are different than those in case of no truncation. Thus, for example, for $\alpha = 1/2$ the critical frequency in case of no truncation is 1.053 [5], while in case of truncation $b = 1/2$ it is equal to 0.903. The truncation decreases the critical frequency for $\alpha = 3/4$ from 0.965 [5] to 0.825, while for $\alpha = 1/5$ from 1.035 [5] to 0.889. Note that in case of classical harmonic oscillator two types of motion are observed, monotonic decay of $C_X(t)$ without zero crossings, and oscillation-like behavior with zero crossings. These two types of motions are separated at a critical frequency equal to $\gamma/2$.

6.5.4 Response to an External Periodic Force

It has been shown that the stochastic force either in classical oscillator [23], fractional oscillator [71], or in the GLE [5, 44, 45] yields some interesting behaviors in the system, such as stochastic resonance, and the double-peak phenomenon. Similar phenomena are observed if one considers the GLE with tempered memory kernel [36]. The external periodic force is of the form $A_0 \cos(\Omega t)$, where A_0 and Ω are the amplitude and frequency of the periodic driving force, respectively.

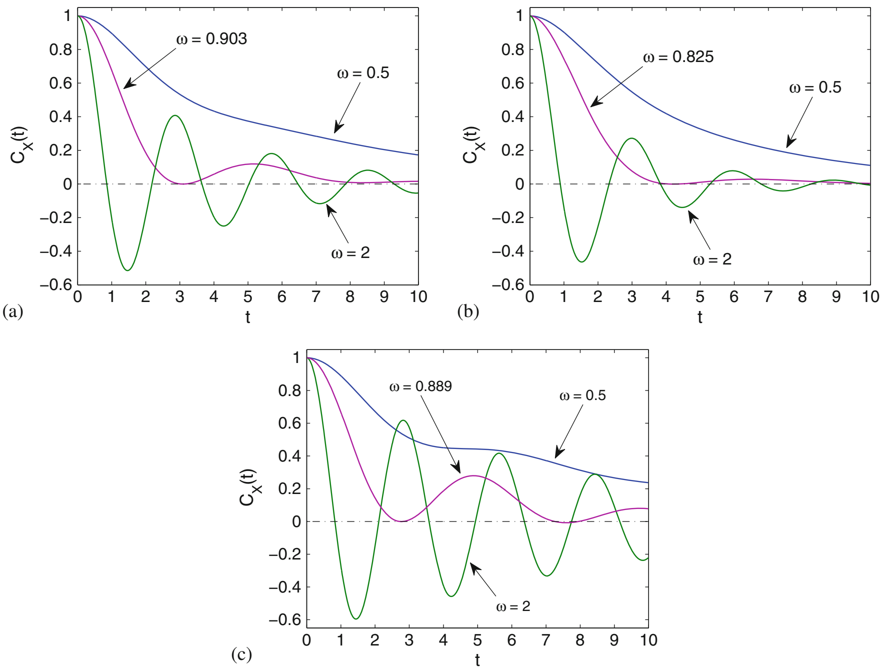

Fig. 6.17 Graphical representation of the normalized displacement correlation function (6.165) for truncated power-law memory kernel with $b = 1/2$ and different frequencies ω; (**a**) $\alpha = 1/2$, (**b**) $\alpha = 3/4$, (**c**) $\alpha = 1/5$. Reprinted from Physica A, 466, A. Liemert, T. Sandev and H. Kantz, Generalized Langevin equation with tempered memory kernel, 356–369, Copyright (2017), with permission from Elsevier

Therefore, we consider the following GLE

$$\ddot{x}(t) + \int_0^t \gamma(t-t')\dot{x}(t')\,\mathrm{d}t' + \omega^2 x(t) = A_0 \cos(\Omega t) + \xi(t), \tag{6.166}$$
$$\dot{x}(t) = v(t).$$

By using the Laplace transform method, for the mean displacement one finds

$$\langle x(t)\rangle = x_0\left[1 - \omega^2 \int_0^t h(t')\,\mathrm{d}t'\right] + v_0\, h(t) + A_0 \int_0^t \cos\big(\Omega(t-t')\big)\, h(t')\,\mathrm{d}t', \tag{6.167}$$

where

$$h(t) = \mathscr{L}^{-1}\left[\hat{h}(s)\right] = \mathscr{L}^{-1}\left[\frac{1}{s^2 + s\hat{\gamma}(s) + \omega^2}\right].$$

From here, for the long time limit ($s \to 0$, $t \to \infty$) it follows [5]

$$\langle x(t)\rangle \simeq A_0 \int_0^t \cos\left(\Omega(t-t')\right) h(t')\,\mathrm{d}t' \;\to\; \langle x(t)\rangle = R(\Omega)\cos\left(\Omega t + \theta(\Omega)\right), \tag{6.168}$$

where the response $R(\Omega)$ and the phase shift $\theta(\Omega)$ will be defined below. Here we consider the complex susceptibility

$$\chi(\Omega) = \chi'(\Omega) + \imath\chi''(\Omega) = \hat{h}(-\imath\Omega) = \frac{1}{\frac{\gamma}{\tau^{\alpha\delta}}(-\imath\Omega)\frac{(-\imath\Omega+b)^{\alpha\delta-\beta}}{\left((-\imath\Omega+b)^{\alpha}+\tau^{-1}\right)^{\delta}} + \omega^2 - \Omega^2}, \tag{6.169}$$

where

$$\hat{h}(-\imath\Omega) = \int_0^\infty e^{\imath\Omega t} h(t)\,\mathrm{d}t,$$

$$\chi'(\Omega) = \Re\left[\chi(\Omega)\right],$$

and

$$\chi''(\Omega) = \Im\left[\chi(\Omega)\right].$$

The real and imaginary parts of the complex susceptibility are experimental measured quantities. From the complex susceptibility, one finds the response

$$R(\Omega) = |\chi(\Omega)|, \tag{6.170}$$

and the space shift

$$\theta(\Omega) = \arctan\left(-\frac{\chi''(\Omega)}{\chi'(\Omega)}\right). \tag{6.171}$$

Particularly, we consider the special case of tempered power-law memory kernel $\gamma(t) = e^{-bt}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, which for $b = 0$ corresponds to the case considered in Ref. [5]. Therefore, for the complex susceptibility we find

$$\chi(\Omega) = \hat{h}(-\imath\Omega) = \frac{1}{\frac{\gamma}{\tau^{\alpha\delta}}(-\imath\Omega)(-\imath\Omega+b)^{\alpha-1} + \omega^2 - \Omega^2}, \tag{6.172}$$

which for $b = 0$ reduces to [5]

$$\chi(\Omega) = \hat{h}(-\imath\Omega) = \frac{1}{\gamma\,(-\imath\Omega)^{\alpha} + \omega^2 - \Omega^2}.$$

From Fig. 6.18 one concludes that resonance appears even for a free particle driven by truncated power-law noise, and that the resonant behavior depends on the truncation parameter b. We observe that the resonant peak which exists for $b = 0$ becomes smaller for $b = 0.5$, and disappear for $b = 1.0$ and $b = 1.5$. Here we note that the response function for the Brownian motion is a monotonic decaying function and resonance does not appear. In Fig. 6.19 same situation is observed for the harmonic oscillator driven by truncated power-law noise. The imaginary part of the complex susceptibility, or the so-called *loss*, shows double peak phenomenon. In

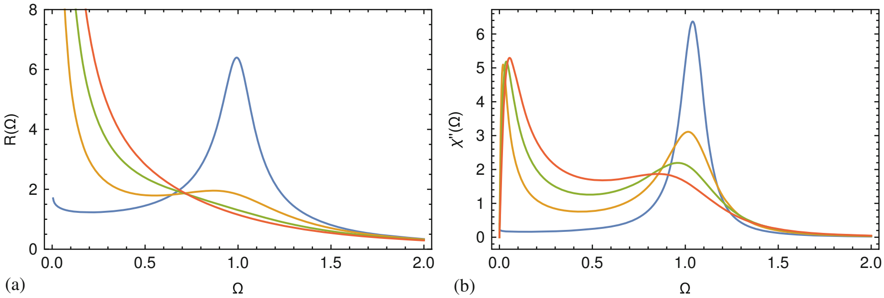

Fig. 6.18 Graphical representation of the (**a**) response $R(\Omega)$, (**b**) loss $\chi''(\Omega)$, for a free particle with tempered power-law memory kernel for $\alpha = 0.1$, $\gamma = 1$, and different values of b, $b = 0$ (blue line), $b = 0.5$ (brown line), $b = 1.0$ (green line), $b = 1.5$ (red line). Reprinted from Physica A, 466, A. Liemert, T. Sandev, and H. Kantz, Generalized Langevin equation with tempered memory kernel, 356–369, Copyright (2017), with permission from Elsevier

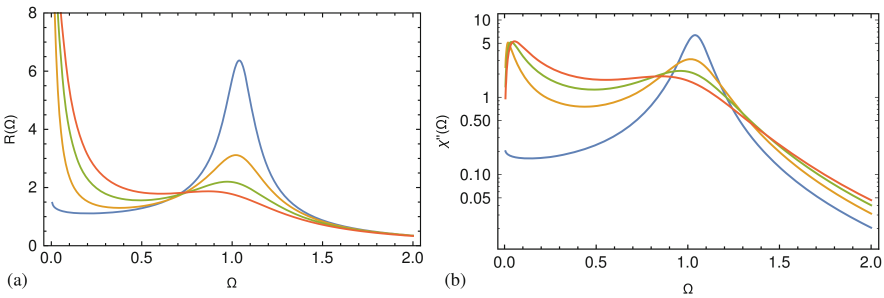

Fig. 6.19 Graphical representation of: (**a**) response $R(\Omega)$, (**b**) loss $\chi''(\Omega)$, for tempered power-law memory kernel with $\alpha = 0.1$, $\omega = 0.3$, $\gamma = 1$, and different values of b, $b = 0$ (blue line), $b = 0.2$ (brown line), $b = 0.4$ (green line), $b = 0.6$ (red line). Reprinted from Physica A, 466, A. Liemert, T. Sandev and H. Kantz, Generalized Langevin equation with tempered memory kernel, 356–369, Copyright (2017), with permission from Elsevier

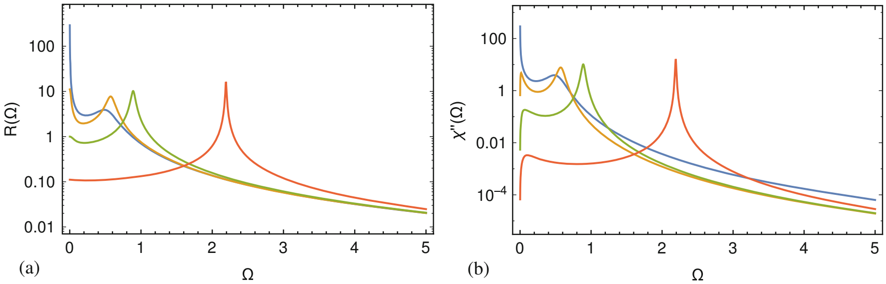

Fig. 6.20 Graphical representation of: (**a**) response $R(\Omega)$, (**b**) loss $\chi''(\Omega)$, for tempered Mittag-Leffler memory kernel with $\alpha = 0.1$, $\beta = 1$, $\delta = 3/4$, $\tau = 1$, $\gamma = 1$, $\omega = 0.1$, and different values of b, $b = 0$ (blue line), $b = 0.3$ (brown line), $b = 1$ (green line), $b = 3$ (red line). Reprinted from Physica A, 466, A. Liemert, T. Sandev and H. Kantz, Generalized Langevin equation with tempered memory kernel, 356–369, Copyright (2017), with permission from Elsevier

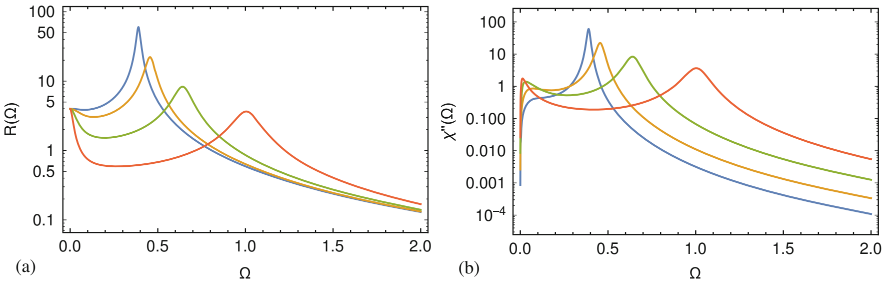

Fig. 6.21 Graphical representation of: (**a**) response $R(\Omega)$, (**b**) loss $\chi''(\Omega)$, for the tempered Mittag-Leffler memory kernel with $\alpha = 0.1$, $\beta = 1$, $\delta = 3/4$, $\tau = 1$, $b = 1/2$, $\omega = 0.1$, and different values of γ, $\gamma = 0.1$ (blue line), $\gamma = 0.3$ (brown line), $\gamma = 1$ (green line), $\gamma = 3$ (red line). Reprinted from Physica A, 466, A. Liemert, T. Sandev and H. Kantz, Generalized Langevin equation with tempered memory kernel, 356–369, Copyright (2017), with permission from Elsevier

Fig. 6.20 we observe similar behavior for the harmonic oscillator driven by truncated M-L noise. We find that by increasing the truncation parameter the resonance frequency is increasing. The dependence of the response and the loss on parameter γ for fixed values of α, β, δ, b is given in Fig. 6.21. By increasing parameter γ, the resonant frequency is increasing. One also concludes that by increasing parameter γ, from one peak the loss exhibits double-peak phenomena. Such double-peak phenomena have been observed in the investigation of relaxation processes in supercooled liquids [25].

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