Curvature in noncommutative geometry

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Dedicated to Alain Connes with admiration, affection, and much appreciation

Abstract Our understanding of the notion of curvature in a noncommutative setting has progressed substantially in the past 10 years. This new episode in noncommutative geometry started when a Gauss-Bonnet theorem was proved by Connes and Tretkoff for a curved noncommutative two torus. Ideas from spectral geometry and heat kernel asymptotic expansions suggest a general way of defining local curvature invariants for noncommutative Riemannian type spaces where the metric structure is encoded by a Dirac type operator. To carry explicit computations however one needs quite intriguing new ideas. We give an account of the most recent developments on the notion of curvature in noncommutative geometry in this paper.

1 Introduction

Broadly speaking, the progress of *noncommutative geometry* in the last four decades can be divided into three phases: *topological, spectral,* and *arithmetical.* One can also notice the pervasive influence of quantum physics in all aspects of the subject. Needless to say, each of these facets of the subject is still evolving, and there are many deep connections among them.

In its topological phase, noncommutative geometry was largely informed by index theory and a real need to extend index theorems beyond their classical realm of smooth manifolds, to what we collectively call *noncommutative spaces*. Thus

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K-theory, K-homology, and KK-theory in general were brought in and with the discovery of cyclic cohomology by Connes [10, 11], a suitable framework was created by him to formulate noncommutative index theorems. With the appearance of the groundbreaking and now classical paper of Connes [12], results of which were already announced in Oberwolfach in 1981 [10], this phase of the theory was essentially completed. In particular a noncommutative Chern-Weil theory of characteristic classes was created with Chern character maps for both K-theory and K-homology with values in cyclic (co)homology. To define all these a notion of Fredholm module (bounded or unbounded, finitely summable or theta summable) was introduced which essentially captures and brings in many aspects of smooth manifolds into the noncommutative world. These results were applied to noncommutative quotient spaces such as the space of leaves of a foliation, or the unitary dual of noncompact and nonabelian Lie groups. Ideas and tools from global analysis, differential topology, operator algebras, representation theory, and quantum statistical mechanics were crucial. One of the main applications of this resulting noncommutative index theory was to settle some long-standing conjectures such as the Novikov conjecture and the Baum-Connes conjecture for specific and large classes of groups.

Next came the study of the geometry of noncommutative spaces and the impact of *spectral geometry*. Geometry, as we understand it here, historically has dealt with the study of spaces of increasing complexity and metric measurements within such spaces. Thus in classical differential geometry one learns how to measure distances and volumes, as well as various types of curvature of Riemannian manifolds of arbitrary dimension. One can say the two notions of Riemannian metric and the Riemann curvature tensor are hallmarks of classical differential geometry in general. This should be contrasted with topology where one studies spaces only from a rather soft homotopy theoretic point of view. A similar division is at work in noncommutative geometry. Thus, as we mentioned briefly above, while in its earlier stage of development noncommutative geometry was mostly concerned with the development of topological invariants like cyclic cohomology, Connes-Chern character maps, and index theory, starting in about 10 years ago noncommutative geometry entered a new truly geometric phase where one tries to seriously understand what a *curved noncommutative space* is and how to define and compute curvature invariants for such a noncommutative space.

This episode in noncommutative geometry started when a Gauss-Bonnet theorem was proved by Connes and Cohen for a *curved noncommutative torus* in [22] (see also the MPI preprint [8] where many ideas are already laid out). This paper was immediately followed in [30] where the Gauss-Bonnet was proved for general conformal structures. The metric structure of a noncommutative space is encoded in a (twisted) spectral triple. Giving a state-of-the-art report on developments following these works, and on the notion of curvature in noncommutative geometry, is the purpose of our present review.

Classically, geometric invariants are usually defined explicitly and algebraically in a local coordinate system, in terms of a metric tensor or a connection on the given manifold. However, methods based on local coordinates, or algebraic methods

This is not a quantum curved torus

based on commutative algebra, have no chance of being useful in a noncommutative setting, in general. But other methods, more analytic and more subtle, based on ideas of spectral geometry are available. In fact, thanks to spectral geometry, we know that there are intricate relations between Riemannian invariants and spectra of naturally defined elliptic operators like Laplace or Dirac operators on the given manifold. A prototypical example is the celebrated *Weyl's law* on the asymptotic distribution of eigenvalues of the Laplacian of a closed Riemannian manifold M^n in terms of its volume:

$$N(\lambda) \sim \frac{\omega_n \mathrm{Vol}(M)}{(2\pi)^n} \lambda^{\frac{n}{2}} \qquad \lambda \to \infty. \tag{1}$$

Here $N(\lambda)$ is the number of eigenvalues of the Laplacian in the interval $[0, \lambda]$ and ω_n is the volume of the unit ball in $\mathbb{R}^n$. In the spirit of Marc Kac's article [39], one says one can hear the volume of a manifold. But one can ask what else about a Riemannian manifold can be heard? Or even we can ask: what can we learn by listening to a noncommutative manifold? Results so far indicate that one can effectively define and compute, not only the volume, but in fact the scalar and Ricci curvatures of noncommutative curved spaces, at least in many examples.

In his Gibbs lecture of 1948, *Ramifications, old and new, of the eigenvalue problem*, Hermann Weyl had this to say about possible extensions of his asymptotic law (1): *I feel that these informations about the proper oscillations of a membrane, valuable as they are, are still very incomplete. I have certain conjectures on what* *a complete analysis of their asymptotic behavior should aim at; but since for more than 35 years I have made no serious attempt to prove them, I think I had better keep them to myself.*

One of the most elaborate results in spectral geometry is Gilkey's theorem that gives the first four non-zero terms in the asymptotic expansion of the heat kernel of Laplace type operators in terms of covariant derivatives of the metric tensor and the Riemann curvature tensor [36]. More precisely, if P is a Laplace type operator, then the heat operator e^{-tP} is a smoothing operator with a smooth kernel $k(t, x, y)$, and there is an asymptotic expansion near $t = 0$ for the heat kernel restricted to the diagonal of $M \times M$:

$$k(t, x, x) \sim \frac{1}{(4\pi t)^{m/2}}(a_0(x, P) + a_2(x, P)t + a_4(x, P)t^2 + \cdots),$$

where $a_i(x, P)$ are known as the Gilkey-Seeley-DeWitt coefficients. The first term $a_0(x, P)$ is a constant. It was first calculated by Minakshisundaram and Pleijel [51] for $P = \Delta$ the Laplace operator. Using Karamata's Tauberian theorem, one immediately obtains Weyl's law for closed Riemannian manifolds. Note that Weyl's original proof was for bounded domains with a regular boundary in Euclidean space and does not extend to manifolds in general. The next term $a_2(x, P)$, for $P = \Delta$, was calculated by MacKean and Singer [50] and it was shown that it gives the scalar curvature:

$$a_2(x, \triangle) = \frac{1}{6}S(x).$$

This immediately shows that the scalar curvature has a spectral nature and in particular the total scalar curvature is a spectral invariant. This result, or rather its localized version to be recalled later, is at the heart of the noncommutative geometry approach to the definition of scalar curvature. The expressions for $a_{2k}(x, P)$ get rapidly complicated as k grows, although in principle they can be recursively computed in a normal coordinate chart. They are reproduced up to term a_6 in the next section.

It is this analytic point of view on geometric invariants that play an important role in understanding the geometry of curved noncommutative spaces. The algebraic approach almost completely breaks down in the noncommutative case. Our experience so far in the past few years has been that in the noncommutative case spectral and hard analytic methods based on pseudodifferential operators yield results that are in no way possible to guess or arrive at from their commutative counterparts by algebraic methods. One just needs to take a look at our formulas for scalar, and now Ricci curvature, in dimensions two, three, and four, in later sections to believe in this statement. The fact that in the first step we had to rely on heavy symbolic computer calculations to start the analysis shows the formidable nature of this material. Surely computations, both symbolic and analytic, are quite hard and are done on a case-by-case basis, but the surprising end results totally justify the effort.

The spectral geometry of a *curved noncommutative two torus* has been the subject of intensive studies in recent years. As we said earlier, this whole episode started when a Gauss-Bonnet theorem was proved by Connes and Tretkoff (formerly Cohen) in [22] (see also [8] for an earlier version), and for general conformal structures in [30]. A natural question then was to define and compute the scalar curvature of a curved noncommutative torus. This was done, independently, by Connes-Moscovici [21] and Fathizadeh-Khalkhali [31]. The next term in the expansion, namely the term a_4, which in the classical case contains explicit information about the analogue of the Riemann tensor, is calculated and studied in [17]. A version of the Riemann-Roch theorem is proven in [41] and the study of local spectral invariants is extended to all finite projective modules on noncommutative two tori in [47].

A key idea to define a curved noncommutative space in the above works is to conformally perturb a flat spectral triple by introducing a noncommutative Weyl factor. The complex geometry of the noncommutative two torus, on the other hand, provides a Dirac operator which, in analogy with the classical case, originates from the Dolbeault complex. By perturbing this spectral triple, one can construct a (twisted) spectral triple that can be used to study the geometry of the conformally perturbed flat metric on the noncommutative two torus. Then, using the pseudodifferrential operator theory for C^*-dynamical systems developed by Connes in [9], the computation is performed and explicit formulas are obtained. The spectral geometry and the study of scalar curvature of noncommutative tori have been pursued further in [23, 32, 28].

Finally, for the latest on interactions between noncommutative geometry, number theory, and arithmetic algebraic geometry, the reader can start with the article by Connes and Consani [16] in this volume and the references therein.

2 Curvature in noncommutative geometry

This section is of an introductory nature and is meant to set the stage for later sections and to motivate the evolution of the concept of curvature in noncommutative geometry from its beginnings to its present form. Clearly we have no intention of giving even a brief sketch of the history of the development of the curvature concept in differential geometry. That would require a separate long article, if not a book. We shall simply highlight some key concepts that have impacted the development of the idea of curvature in noncommutative geometry.

2.1 A brief history of curvature

Curvature, as understood in classical differential geometry, is one of the most important features of a geometric space. It is here that geometry and topology differ in the ways they probe a space. To talk about curvature we need more than just topology or smooth structure on a space. The extra piece of structure is usually encoded in a (pseudo-)Riemannian metric, or at least a connection on the tangent bundle, or on a principal G-bundle. It is remarkable that Greek geometers missed the curvature concept altogether, even for simple curves like a circle, which they studied so intensely. The earliest quantitative understanding of curvature, at least for circles, is due to Nicole Oresme in the fourteenth century. In his treatise, *De configurationibus*, he correctly gives the inverse of radius as the curvature of a circle. The concept had to wait for Descartes' analytic geometry and the Newton-Leibniz calculus before to be developed and fully understood. In fact the first definitions of the (signed) curvature κ of a plane curve $y = y(x)$ are due to Newton, Leibniz, and Huygens in the seventeenth century:

$$\kappa = \frac{y''}{(1 + y'^2)^{3/2}}.$$

It is important to note that this is not an intrinsic concept. Intrinsically any one-dimensional Riemannian manifold is locally isometric to $\mathbb{R}$ with its flat Euclidean metric and hence its intrinsic curvature is zero.

Thus the first major case to be understood was the curvature of a surface embedded in a three-dimensional Euclidean space with its induced metric. In his magnificent paper of 1828 entitled *disquisitiones generales circa superficies curvas,* Gauss first defines the curvature of a surface in an *extrinsic* way, using the Gauss map and then he proves his *theorema egregium*: the curvature so defined is in fact an *intrisic* concept and can solely be defined in terms of the first fundamental form. That is the Gaussian curvature is an isometry invariant, or in Gauss' own words:

> Thus the formula of the preceding article leads itself to the remarkable Theorem. If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged.

Now the first fundamental form is just the induced Riemannian metric in more modern language. As we shall see, in the hands of Riemann, Theorema Egregium opened the way for the idea of intrinsic geometry of spaces in general. Surfaces, and manifolds in general, have an intrinsic geometry defined solely by metric relations within the space itself, independent of any ambient space.

If $g = e^h(dx^2 + dy^2)$ is a locally conformally flat metric, then its Gaussian curvature is given by

$$K = -\frac{1}{2}e^{-h}\Delta h,$$

where Δ is the flat Laplacian. We shall see later in this paper that the analogous formula in the noncommutative case, first obtained in [21, 31], takes a much more complicated form, with remarkable similarities and differences.

Another major result of Gauss' surface theory was his *local uniformization theorem*, which amounts to existence of *isothermal coordinates*: any analytic

Riemannian metric in two dimensions is locally conformally flat. The result holds for all smooth metrics in two dimensions, but Gauss' proof only covers analytic metrics. Since conformal classes of metrics on a two torus are parametrized by the upper half plane modulo the action of the modular group, this justifies the initial choice of metrics for noncommutative tori by Connes and Cohen in their Gauss-Bonnet theorem in [22], and for general conformal structures in our paper [30]. By all chances, in the noncommutative case one needs to go beyond the class of locally conformally flat metrics. For recent results in this direction, see [35].

A third major achievement of Gauss in differential geometry is his local *Gauss-Bonnet theorem*: for any geodesic triangle drawn on a surface with interior angles α, β, γ, we have

$$\alpha + \beta + \gamma - \pi = \int K dA,$$

where K denotes the Gauss curvature and dA is the surface area element. By using a geodesic triangulation of the surface, one can then easily prove the global Gauss-Bonnet theorem for a closed Riemannian surface:

$$\frac{1}{2\pi} \int_M K dA = \chi(M),$$

where $\chi(M)$ is the Euler characteristic of the closed surface M. It is hard to overemphasize the importance of this result which connects geometry with topology. It is the first example of an index theorem and the theory of characteristic classes.

To find a true analogue of the Gauss-Bonnet theorem in a noncommutative setting was the motivation for Connes and Tretkoff in their groundbreaking work [22]. After conformally perturbing the flat metric of a noncommutative torus, they noticed that while the above classical formulation has no clear analogue in the noncommutative case, its spectral formulation

$$\zeta(0) + 1 = \frac{1}{12\pi} \int_M K dA = \frac{1}{6}\chi(M),$$

makes perfect sense. Here

$$\zeta(s) = \sum \lambda_j^{-s}, \qquad \mathrm{Re}(s) > 1, \tag{2}$$

is the *spectral zeta function* of the scalar Laplacian $\triangle_g = d^*d$ of (M, g). The spectral zeta function has a meromorphic continuation to $\mathbb{C}$ with a unique (simple) pole at $s = 1$. In particular $\zeta(0)$ is defined. Thus $\zeta(0)$ is a topological invariant, and, in particular, it remains invariant under the conformal perturbation $g \to e^h g$ of the metric. This result was then extended to all conformal classes in the upper half plane in our paper [30].

After the work of Gauss, a decisive giant step was taken by Riemann in his epoch-making paper *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen,* which is a text of his *Habilitationsvortrag* of June 1854. The notion of *space*, as an entity that exists on its own, without any reference to an ambient space or external world, was first conceived by Riemann. Riemannian geometry is intrinsic from the beginning. In Riemann's conception, a space, which he called a *mannigfaltigkeit*, manifold in English, can be discrete or continuous, finite or infinite dimensional. The idea of a geometric space as an abstract *set* endowed with some extra structure was born in this paper of Riemann. Local coordinates are just labels without any intrinsic meaning, and thus one must always make sure that the definitions are independent of the choice of coordinates. This is the *general principle of relativity*, which later came to be regarded as a cornerstone of modern theories of spacetime and Einstein's theory of gravitation. This idea quickly led to the development of tensor calculus, also known as the *absolute differential calculus*, by the Italian school of Ricci and his brilliant student Levi-Civita.

Riemann also introduced the idea of a Riemannian metric and understood that to define the bending or curvature of a space one just needs a Riemannian metric. This was of course directly inspired by Gauss' theorema egregium. In fact he gave two definitions for curvature. His *sectional* curvature is defined as the Gaussian curvature of two-dimensional submanifolds defined via the geodesic flow for each two-dimensional subspace of the tangent space at each point. For his second definition he introduced the geodesic coordinate systems and considered the Taylor expansion of the metric components $g_{ij}(x)$ in a geodesic coordinate. Let

$$c_{ij,kl} = \frac{1}{2} \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}.$$

He shows that sectional curvature is determined by the components $c_{ij,kl}$, and vice versa. Also, one knows that the components $c_{ij,kl}$ are closely related to Riemann curvature tensor.

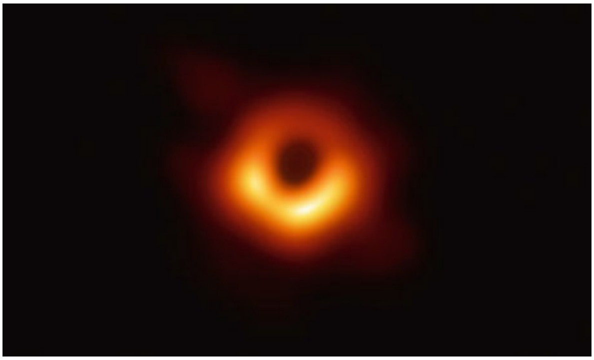
The Riemann curvature tensor, in modern notation, is defined as

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

where ∇ is the Levi-Civita connection of the metric, and X and Y are vector fields on the manifold. The analogue of this curvature tensor of rank four is still an illusive concept in the noncommutative case. However, the components of the Riemann tensor appear in the term a_4 in the small time heat kernel expansion of the Laplacian of the metric, the analogue of which was calculated and studied in [17] for noncommutative two tori and for noncommutative four tori with product geometries.

It is hard to exaggerate the importance of the *Ricci curvature* in geometry and physics. For example, it plays an indispensable role in Einstein's theory of gravity and Einstein field equations. In particular, it directly leads, thanks to Schwarzschild solution, to the prediction of black holes. It is also fundamental for the Ricci flow. Ricci curvature can be formulated in spectral terms and this opened up the possibility of defining it in noncommutative settings [34]. The reader should consult later sections in this survey for more on this.

The first black hole image by Event Horizon Telescope, April 2019



Although they won't be a subject for the present exposition, let us briefly mention some other aspects of curvature that have found their analogues in noncommutative settings. These are mostly *linear* aspects of curvature, and have much to do with representation theory of groups. They include Chern-Weil theory of characteristic classes and specially the Chern-Connes character maps for both K-theory and K-homology, Chern-Simons theory, and Yang-Mills theory. Riemannian curvature, whose noncommutative analogue we are concerned with here, is a *nonlinear* theory and from our point of view that is why it took so long to find its proper formulation and first calculations in a noncommutative setting.

2.2 Laplace type operators and Gilkey's theorem

At the heart of spectral geometry, Gilkey's theorem [36] gives the most precise information on asymptotic expansion of heat kernels for a large class of elliptic PDEs. Since this result and its noncommutative analogue play such an important role in defining and computing curvature invariants in noncommutative geometry, we shall explain it briefly in this section. Let M be a smooth closed manifold with a Riemannian metric g and a vector bundle V on M. An operator $P : \Gamma(M, V) \to \Gamma(M, V)$ on smooth sections of V is called a Laplace type operator if in local coordinates it looks like

$$P = -g^{ij}\partial_i\partial_j + \text{lower orders}.$$

Examples of Laplace type operators include Laplacian on forms

$$\Delta = (d + d^*)^2 : \Omega^p(M) \to \Omega^p(M),$$

and the Dirac Laplacians $\Delta = D^*D$, where $D : \Gamma(S) \to \Gamma(S)$ is a generalized Dirac operator.

Now if P is a Laplace type operator, then there exist a unique connection ∇ on the vector bundle V and an endomorphism $E \in \mathrm{End}(V)$ such that

$$P = \nabla^*\nabla - E.$$

Here $\nabla^*\nabla$ is the connection Laplacian which is locally given by $-g^{ij}\nabla_i\nabla_j$. For example, the Lichnerowicz formula for the Dirac operator, $D^2 = \nabla^*\nabla - \frac{1}{4}R$, gives

$$E = \frac{1}{4}R,$$

where R is the scalar curvature. Now e^{-tP} is a smoothing operator with a smooth kernel $k(t, x, y)$. There is an asymptotic expansion near $t = 0$

$$k(t, x, x) \sim \frac{1}{(4\pi t)^{m/2}}(a_0(x, P) + a_2(x, P)t + a_4(x, P)t^2 + \cdots),$$

where $a_{2k}(x, P)$ are known as the Gilkey-Seeley-De Witt coefficients. Gilkey's theorem asserts that $a_{2k}(x, P)$ can be expressed in terms of universal polynomials in the metric g and its covariant derivatives. Gilkey has computed the first four non-zero terms and they are as follows:

$$\begin{aligned} a_0(x, P) &= tr(1), \\ a_2(x, P) &= tr\left(E - \frac{1}{6}R\right), \\ a_4(x, P) &= \frac{1}{360}tr\Big(\big(-12R_{;kk} + 5R^2 - 2R_{jk}R_{jk} + 2R_{ijkl}R_{ijkl}\big) \\ &\qquad -60RE + 180E^2 + 60E_{;kk} + 30\Omega_{ij}\Omega_{ij}\Big). \end{aligned}$$

$$\begin{aligned} a_6(x, P) = tr\Big\{\frac{1}{7!}\Big(&-18R_{;kkll} + 17R_{;k}R_{;k} - 2R_{jk;l}R_{jk;l} - 4R_{jk;l}R_{jl;k} \\ &+ 9R_{ijku;l}R_{ijku;l} + 28RR_{;ll} - 8R_{jk}R_{jk;ll} + 24R_{jk}R_{jl;kl} \\ &+ 12R_{ijkl}R_{ijkl;uu}\Big) \\ + \frac{1}{9\cdot 7!}\Big(&-35R^3 + 42RR_{lp}R_{lp} - 42RR_{klpq}R_{klpq} + 208R_{jk}R_{jl}R_{kl} \\ &- 192R_{jk}R_{ul}R_{jukl} + 48R_{jk}R_{julp}R_{kulp} - 44R_{ijku}R_{ijlp}R_{kulp} \\ &- 80R_{ijku}R_{ilkp}R_{jlup}\Big) \end{aligned}$$

$$+\frac{1}{360}\Big(8\Omega_{ij;k}\Omega_{ij;k}+2\Omega_{ij;j}\Omega_{ik;k}+12\Omega_{ij}\Omega_{ij;kk}-12\Omega_{ij}\Omega_{jk}\Omega_{ki}$$
$$-6R_{ijkl}\Omega_{ij}\Omega_{kl}+4R_{jk}\Omega_{jl}\Omega_{kl}-5R\Omega_{kl}\Omega_{kl}\Big)$$
$$+\frac{1}{360}\Big(6E_{;iijj}+60EE_{;ii}+30E_{;i}E_{;i}+60E^3+30E\Omega_{ij}\Omega_{ij}$$
$$-10RE_{;kk}-4R_{jk}E_{;jk}-12R_{;k}E_{;k}-30RE^2-12R_{;kk}E$$
$$+5R^2E-2R_{jk}R_{jk}E+2R_{ijkl}R_{ijkl}E\Big)\Big\}.$$

Here R_{ijkl} is the Riemann curvature tensor, R is the scalar curvature, Ω is the curvature matrix of two forms, and ; denotes the covariant derivative operator.

As we shall later see in this survey, the first two terms in the above list allow us to define the scalar and Ricci curvatures in terms of heat kernel coefficients and extend them to noncommutative settings.

Alternatively, one can use spectral zeta functions to extract information from the spectrum. Heat trace and spectral zeta functions are related via Mellin transform. For a concrete example, let $\triangle$ denote the Laplacian on functions on an m-dimensional closed Riemannian manifold. Define

$$\zeta_\triangle(s)=\sum\lambda_i^{-s}\qquad \mathrm{Re}(s)>\frac{m}{2}.$$

The spectral invariants a_i in the heat trace asymptotic expansion

$$\mathrm{Trace}(e^{-t\triangle})\sim(4\pi t)^{\frac{-m}{2}}\sum_{j=0}^{\infty}a_jt^j\qquad\qquad (t\to 0^+)$$

are related to residues of spectral zeta function by

$$\mathrm{Res}_{s=\alpha}\zeta_\triangle(s)=(4\pi)^{-\frac{m}{2}}\frac{a_{\frac{m}{2}-\alpha}}{\Gamma(\alpha)},\qquad \alpha=\frac{m}{2}-j>0.$$

To get to the local invariants like scalar curvature we can consider localized zeta functions. Let $\zeta_f(s):=\mathrm{Tr}\,(f\triangle^{-s}),\ f\in C^\infty(M)$. Then we have

$$\mathrm{Res}\,\zeta_f(s)|_{s=\frac{m}{2}-1}=\frac{(4\pi)^{-m/2}}{\Gamma(m/2-1)}\int_M f(x)R(x)dvol_x,\quad m\geq 3,$$

$$\zeta_f(s)|_{s=0}=\frac{1}{4\pi}\int_M f(x)R(x)dvol_x-\mathrm{Tr}(fP),\qquad m=2,$$

where P is projection onto zero eigenmodes of $\triangle$. Thus the scalar curvature R appears as the density function for the localized spectral zeta function.

2.3 Noncommutative Chern-Weil theory

Although it is not our intention to review this subject in the present survey, we shall nevertheless explain some ideas of noncommutative Chern-Weil theory here. Many aspects of Chern-Weil theory of characteristic classes for vector bundles and principal bundles over smooth manifolds can be cast in an algebraic formalism and as such is even used in commutative algebra and algebraic geometry [5]. Thus one can formulate notions like de Rham cohomology, connection, curvature, Chern classes, and Chern character, over a commutative algebra and then for a scheme. This is a commutative theory which is more or less straightforward in the characteristic zero case. But there seemed to be no obvious extension of de Rham theory and the rest of Chern-Weil theory to the noncommutative case.

In [9] Connes realized that many aspects of Chern-Weil theory can be implemented in a noncommutative setting. The crucial ingredient was the discovery of cyclic cohomology that replaces de Rham homology of currents in a noncommutative setting [11, 12]. Let A be a not necessarily commutative algebra over the field of complex numbers. By a *noncommutative differential calculus* on A we mean a triple (Ω, d, ρ) such that (Ω, d) is a differential graded algebra and $\rho : A \to \Omega^0$ is an algebra homomorphism. Given a right A-module $\mathcal{E}$, a *connection* on $\mathcal{E}$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_A \Omega^1$ satisfying the Leibniz rule $\nabla(\xi a) = \nabla(\xi)a + \xi \otimes da$, for all $\xi \in \mathcal{E}$ and $a \in A$. Let $\hat{\nabla} : \mathcal{E} \otimes_{\mathcal{A}} \Omega^\bullet \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^{\bullet+1}$ be the (necessarily unique) extension of ∇ which satisfies the graded Leibniz rule $\hat{\nabla}(\xi\omega) = \hat{\nabla}(\xi)\omega + (-1)^{\deg \xi} \xi d\omega$ with respect to the right Ω-module structure on $\mathcal{E} \otimes_{\mathcal{A}} \Omega$. The *curvature* of ∇ is the operator of degree 2, $\hat{\nabla}^2 : \mathcal{E} \otimes_A \Omega^\bullet \to \mathcal{E} \otimes_A \Omega^\bullet$, which can be easily checked to be Ω-linear.

Now to obtain Connes' Chern character pairing between K-theory and cyclic cohomology, $K_0(A) \otimes HC^{2n}(A) \to \mathbb{C}$, one can proceed as follows. Given a finite projective A-module $\mathcal{E}$, one can always equip $\mathcal{E}$ with a connection over the universal differential calculus ΩA. An element of $HC^{2n}(A)$ can be represented by a closed graded trace τ on $\Omega^{2n} A$. The value of the pairing is then simply $\tau(\hat{\nabla}^{2n})$. Here we used the same symbol τ to denote the extension of τ to the ring $End_{\Omega^\bullet}(\mathcal{E} \otimes_A \Omega^\bullet)$. One checks that this definition is independent of all choices that we made [12]. Connes in fact initially developed the more sophisticated Chern-Connes pairing in K-homology with explicit formulas that do not have a commutative counterpart. For all this and more, the reader should check Connes' book and his above cited article [12, 14] as well as the book [40].

2.4 From spectral geometry to spectral triples

The very notion of Riemannian manifold itself is now subsumed and vastly generalized through Connes' notion of *spectral triples*, which is a centerpiece of noncommutative geometry and applications of noncommutative geometry to particle physics.

Let us first motivate the definition of a spectral triple. During the course of their heat equation proof of the index theorem, it was discovered by Atiyah-Bott-Patodi [2] that it is enough to prove the theorem for Dirac operators twisted by vector bundles. The reason is that these twisted Dirac operators in fact generate the whole K-homology group of a spin manifold and thus it suffices to prove the theorem only for these first order elliptic operators. This indicates the preeminence of Dirac operators in topology. As we shall see below, Dirac operators also encode metric information of a Riemannian manifold in a succinct way. Broadly speaking, spectral triples, suitably enhanced, are noncommutative spin manifolds and form a backbone of noncommutative geometry, specially its metric aspects. One precise formulation of this idea is *Connes' reconstruction theorem* [15] which states that a commutative spectral triple satisfying some natural conditions is in fact the standard spectral triple of a spin^c manifold described below.

Recall that the Dirac operator D on a compact Riemannian spin^c manifold acts as an unbounded selfadjoint operator on the Hilbert space $L^2(M, S)$ of L^2-spinors on M. If we let $C^\infty(M)$ act on $L^2(M, S)$ by multiplication operators, then one can check that for any smooth function f, the commutator $[D, f] = Df - fD$ extends to a bounded operator on $L^2(M, S)$. The metric d on M, that is the geodesic distance of M, can be recovered, thanks to the *distance formula* of Connes [14]:

$$d(p, q) = \mathrm{Sup}\{|f(p) - f(q)|; \parallel [D, f] \parallel \leq 1\}.$$

The triple $(C^\infty(M), L^2(M, S), \not{D})$ is a commutative example of a spectral triple.

The general definition of a spectral triple, in the odd case, is as follows.

Definition 2.1 *Let A be a unital algebra. An odd spectral triple on A is a triple $(A, \mathcal{H}, D)$ consisting of a Hilbert space $\mathcal{H}$, a selfadjoint unbounded operator D : $Dom(D) \subset \mathcal{H} \to \mathcal{H}$ with compact resolvent, i.e., $(D-\lambda)^{-1} \in \mathcal{K}(\mathcal{H})$, for all $\lambda \notin \mathbb{R}$, and a representation $\pi : A \to \mathcal{L}(\mathcal{H})$ of A such that for all $a \in A$, the commutator $[D, \pi(a)]$ is defined on $Dom(D)$ and extends to a bounded operator on $\mathcal{H}$.*

A spectral triple is called *finitely summable* if for some $n \geq 1$

$$|D|^{-n} \in \mathcal{L}^{1,\infty}(\mathcal{H}).$$

Here $\mathcal{L}^{1,\infty}(\mathcal{H})$ is the Dixmier ideal. It is an ideal of compact operators which is slightly bigger than the ideal of trace class operators and is the natural domain of the Dixmier trace. Spectral triples provide a refinement of Fredholm modules. Going from Fredholm modules to spectral triples is similar to going from the conformal class of a Riemannian metric to the metric itself. Spectral triples simultaneously provide a notion of *Dirac operator* in noncommutative geometry, as well as a Riemannian type *distance function* for noncommutative spaces. In later sections we shall define and work with concrete examples of spectral triples and their conformal perturbations.

3 Pseudodifferential calculus and heat expansion

In this section we discuss the classical pseudodifferential calculus on the Euclidean space and will then provide practical details of the pseudodifferential calculus of [9] that we use for heat kernel calculations on noncommutative tori.

3.1 Classical pseudodifferential calculus

In the Euclidean case we follow the notations and conventions of [36] as follows. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$ of non-negative integers and coordinates $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ we set:

$$|\alpha| = \alpha_1 + \cdots + \alpha_m, \qquad \alpha! = \alpha_1! \cdots \alpha_m!, \qquad x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m},$$

$$\partial_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m}, \qquad D_x^\alpha = (-i)^{|\alpha|} \partial_x^\alpha.$$

Also we normalize the Lebesgue measure on $\mathbb{R}^m$ by a multiplicative factor of $(2\pi)^{-m/2}$ and still denote it by dx. Therefore we have:

$$\int_{\mathbb{R}^m} \exp\left(-\frac{1}{2}|x|^2\right) dx = 1.$$

The main idea behind pseudodifferential calculus is that it uses the *Fourier transform* to turn a differential operator into multiplication by a function, namely the *symbol* of the differential operator. The Fourier transform $\hat{f}$ of a *Schwartz function* f on $\mathbb{R}^m$ is defined by the following integration:

$$\hat{f}(\xi) = \int_{\mathbb{R}^m} e^{-ix \cdot \xi} f(x)\, dx, \qquad \xi \in \mathbb{R}^m.$$

This integral is convergent because, by definition, the set of Schwartz functions $\mathcal{S}(\mathbb{R}^m)$ consists of all complex-valued smooth functions f on the Euclidean space such that for any multi-indices α and β of non-negative integers

$$\sup_{x \in \mathbb{R}^m} |x^\alpha D^\beta f(x)| < \infty.$$

It turns out that the Fourier transform preserves the L^2-norm, hence it extends to a unitary operator on $L^2(\mathbb{R}^m)$.

The differential operator D_x^α turns in the Fourier mode to multiplication by the monomial ξ^α, in the sense that:

$$\widehat{(D_x^\alpha f)}(\xi) = \xi^\alpha \hat{f}(\xi).$$

The monomial ξ^α is therefore called the symbol of the differential operator D_x^α. Then, the *Fourier inversion formula,*

$$f(x) = \int_{\mathbb{R}^m} e^{i\xi\cdot x} \hat{f}(\xi)\, d\xi, \qquad f \in \mathcal{S}(\mathbb{R}^m),$$

implies that

$$D_x^\alpha f(x) = \int_{\mathbb{R}^m} e^{ix\cdot\xi} \xi^\alpha \hat{f}(\xi)\, d\xi = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} e^{i(x-y)\cdot\xi} \xi^\alpha f(y)\, dy\, d\xi. \tag{3}$$

It is now clear from the above facts that the symbol of any differential operator, given by a finite sum of the form $\sum a_\alpha(x) D_x^\alpha$, is the polynomial in ξ of the form $\sum a_\alpha(x)\xi^\alpha$, whose coefficients are the functions $a_\alpha(x)$ (which we assume to be smooth). Using the notation $\sigma(\cdot)$ for the symbol it is an easy exercise to see that given two differential operators P_1 and P_2, the symbol of their composition $\sigma(P_1 \circ P_2)$ is given by the following expression:

$$\sum_{\alpha\in\mathbb{Z}^m_{\geq 0}} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma(P_1)\, D_x^\alpha \sigma(P_2), \tag{4}$$

which is a finite sum because only finitely many of the summands are non-zero.

By considering a wider family of symbols, one obtains a larger family of operators which are called *pseudodifferential operators*. A smooth function $p : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{C}$ is *a pseudodifferential symbol of order* $d \in \mathbb{R}$ if it satisfies the following conditions:

- $p(x, \xi)$ has compact support in x,
- for any multi-indices $\alpha, \beta \in \mathbb{Z}^m_{\geq 0}$, there exists a constant $C_{\alpha,\beta}$ such that

$$|\partial_\xi^\beta \partial_x^\alpha p(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{d-|\beta|}. \tag{5}$$

Clearly the space of pseudodifferential symbols possesses a filtration because, denoting the space of symbols of order d by S^d, we have:

$$d_1 \leq d_2 \implies S^{d_1} \subset S^{d_2}.$$

Existence of symbols of arbitrary orders can be assured by observing that for any $d \in \mathbb{R}$ and any compactly supported function f_0, the function $p(x,\xi) = f_0(x)(1 + |\xi|^2)^{d/2}$ belongs to S^d.

Given a symbol $p \in S^d$, inspired by formula (3), the corresponding pseudodifferential operator P is defined by

$$Pf(x) = \int_{\mathbb{R}^m} e^{ix\cdot\xi} p(x,\xi) \hat{f}(\xi)\, d\xi, \qquad f \in \mathcal{S}(\mathbb{R}^m). \tag{6}$$

The space of pseudodifferential operators associated with symbols of order d is denoted by $\Psi^d(\mathbb{R}^m)$. Searching for an analog of formula (4) for general pseudodifferential operators leads to a complicated analysis which, at the end, gives *an asymptotic expansion* for the symbol of the composition of such operators. The formula is written as

$$\sigma\left(P_1 P_2\right) \sim \sum_{\alpha \in \mathbb{Z}^m_{\geq 0}} \frac{1}{\alpha!} \partial^\alpha_\xi \sigma(P_1)\, D^\alpha_x \sigma(P_2). \tag{7}$$

It is important to put in order some explanations about this formula. If $\sigma(P_1) \in S^{d_1}$ and $\sigma(P_2) \in S^{d_2}$, then there is a symbol in $S^{d_1+d_2}$ that gives $P_1 \circ P_2$ via formula (6). However $\sigma(P_1 \circ P_2)$ has a complicated formula which involves integrals, which can be seen by writing the formulas directly. The trick is then to use Taylor series and to perform analytic manipulations on the closed formula for $\sigma(P_1 \circ P_2)$ to derive the expansion (7). The error terms in the Taylor series that one uses in the manipulations are responsible for having an asymptotic expansion rather than a strict identity. The precise meaning of this expansion is that given any $d \in \mathbb{R}$, there exists a positive integer N such that

$$\sigma\left(P_1 P_2\right) - \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \partial^\alpha_\xi \sigma(P_1)\, D^\alpha_x \sigma(P_2) \in S^d.$$

Therefore, as one subtracts the terms $\frac{1}{\alpha!} \partial^\alpha_\xi \sigma(P_1)\, D^\alpha_x \sigma(P_2)$ from $\sigma(P_1 \circ P_2)$, the orders of the resulting symbols tend to $-\infty$. Regarding this, it is convenient to introduce the space $S^{-\infty} = \cap_{d \in \mathbb{R}} S^d$ of the *infinitely smoothing pseudodifferential symbols*. For example, for any compactly supported function f_0, the symbol $p(x, \xi) = f_0(x) e^{-|\xi|^2}$ belongs to $S^{-\infty}$.

The composition rule (7) is a very useful tool. For instance, it can be used to find a *parametrix* for *elliptic* pseudodifferential operators. Important geometric operators such as Laplacians are elliptic, and by finding a parametrix, as we shall explain, one finds an approximation of the fundamental solution of the partial differential equation defined by such an important operator. Intuitively, a pseudodifferential symbol $p(x, \xi)$ of order $d \in \mathbb{R}$ is *elliptic* if it is non-zero when ξ is away from the origin (or invertible in the case of matrix-valued symbols), and $|p(x, \xi)^{-1}|$ is bounded by a constant times $(1 + |\xi|)^{-d}$ as $\xi \to \infty$. For our purposes, it suffices to know that a differential operator $D = \sum a_\alpha(x) D^\alpha_x$ of order $d = \max_\alpha |\alpha|$ is elliptic if its *leading symbol*,

$$\sigma_L(D) = \sum_{|\alpha| = d} a_\alpha(x) \xi^\alpha,$$

is non-zero (or invertible) for $\xi \neq 0$. Given such an elliptic differential operator one can use formula (7) to find an inverse for D, called a *parametrix*, in the quotient $\Psi / \Psi^{-\infty}$ of the algebra of pseudodifferential operators Ψ by infinitely smoothing operators $\Psi^{-\infty}$. This process can be described as follows. One makes the natural assumption that the symbol of the parametrix has an expansion starting with a leading term of order $-d$ and other terms whose orders descend to $-\infty$, namely terms of orders $-d-1, -d-2, \ldots$, and one continues as follows. The formula given by (7) can be used to find these terms recursively and thereby find a parametrix R such that

$$DR - I \sim RD - I \sim 0.$$

We will illustrate this carefully in Section 3.2 in a slightly more complicated situation, where a parameter λ and a parametric pseudodifferential calculus are involved in deriving heat kernel expansions. We just mention that invertibility of $\sigma_L(D)$ is the crucial point that allows one to start the recursive process, and to continue on to find the parametrix R.

3.2 Small-time heat kernel expansion

For simplicity and practical purposes we assume that P is a positive elliptic differential operator of order 2 with

$$\sigma(P) = p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi),$$

where each p_j is (homogeneous) of order j in ξ. We know that $p_2(x,\xi)$ is non-zero (or invertible) for non-zero ξ. The first step in deriving a small time asymptotic expansion for $\mathrm{Tr}(\exp(-tP))$ as $t \to 0^+$ is to use the Cauchy integral formula to write

$$e^{-tP} = \frac{1}{2\pi i}\int_\gamma e^{-t\lambda}(P-\lambda)^{-1}\,d\lambda, \tag{8}$$

where the contour γ goes clockwise around the non-negative real axis, where the eigenvalues of P are located. The term $(P-\lambda)^{-1}$ in the above integral can now be approximated by pseudodifferential operators as follows. We look for an approximation R_λ of $(P-\lambda)^{-1}$ such that

$$\sigma(R_\lambda) \sim r_0(x,\xi,\lambda) + r_1(x,\xi,\lambda) + r_2(x,\xi,\lambda) + \cdots,$$

where each r_j is a symbol of order $-2-j$ in the parametric sense which we will elaborate on later. For now one can use formula (7) to find the r_j recursively out of the equation

$$R_\lambda(P-\lambda) \sim I.$$

This means that the terms r_j in the expansion should satisfy

$$\sum_j r_j \circ ((p_2 - \lambda) + p_1 + p_0) \sim 1, \tag{9}$$

where the composition $\circ$ is given by (7). By writing the expansion one can see that there is only one leading term, which is of order 0, namely $r_0(p_2 - \lambda)$ and needs to be set equal to 1 so that it matches the corresponding (and the only term) on the right-hand side of the Equation (9). Therefore the leading term r_0 is found to be

$$r_0 = (p_2 - \lambda)^{-1}. \tag{10}$$

Here the ellipticity plays an important role, because we need to be ensured that the inverse of $p_2 - \lambda$ exists. Since, in our examples, P will be a Laplace type operator, the leading term p_2 is a positive number (or a positive invertible matrix in the vector bundle case) for any $\xi \neq 0$. Therefore for any λ on the contour γ, we know that $p_2 - \lambda$ is invertible. One can then proceed by considering the term that is homogeneous of order -1 in the expansion of the left-hand side of (9) and set it equal to 0 since there is no term of order -1 on the right-hand side. This will yield a formula for the next term r_1. By continuing this process one finds recursively that for $n = 1, 2, 3, \ldots,$ we have

$$r_n = -\left(\sum_{\substack{|\alpha|+j+2-k=n, \\ 0\le j<n,\, 0\le k\le 2}} \frac{1}{\alpha!} \partial_\xi^\alpha r_j \, D_x^\alpha p_k \right) r_0. \tag{11}$$

It turns out that the r_n calculated by this formula have the following homogeneity property:

$$r_n(x, t\xi, t^2\lambda) = t^{-2-n} r_n(t, \xi, \lambda).$$

Having an approximation of the resolvent $R_\lambda \sim (P - \lambda)^{-1}$ via the symbols r_n, one can use the formulas (8) and (6) to approximate the kernel K_t of the operator e^{-tP}, namely the unique smooth function such that

$$e^{-tP} f(x) = \int K_t(x, y) \, f(y) \, dy, \qquad f \in \mathcal{S}(\mathbb{R}^m).$$

Since $\mathrm{Tr}(e^{-tP})$ can be calculated by integrating the kernel on the diagonal,

$$\mathrm{Tr}\left(e^{-tP}\right) = \int K_t(x, x) \, dx,$$

the integration of the approximation of the kernel obtained by going through the procedure described above leads to an asymptotic expansion of the following form:

$$\mathrm{Tr}\left(e^{-tP}\right) \sim_{t\to 0^+} t^{-m/2} \sum_{n=0}^{\infty} a_{2n}(P)\, t^n, \tag{12}$$

where each coefficient a_{2n} is the integral of a density $a_{2n}(x, P)$ given by

$$a_{2n}(x, P) = \frac{1}{2\pi i} \int \int_{\gamma} e^{-\lambda} \mathrm{tr}(r_{2n}(x, \xi, \lambda))\, d\lambda\, d\xi.$$

In this integrand, the tr denotes the matrix trace which needs to be considered in the case of vector bundles.

It is a known fact that when P is a geometric operator such as the Laplacian of a metric, each $a_{2n}(x, P)$ can be written in terms of the Riemann curvature tensor, its contractions, and covariant derivatives, see, for example, [18]. However, in practice, as n grows, these terms become so complicated rapidly. One can refer to [36] for the formulas for the terms up a_6 derived using invariant theory.

3.3 Pseudodifferential calculus and heat kernel expansion for noncommutative tori

Now that we have illustrated the derivation of the heat kernel expansion (12), we explain briefly in this subsection that using the pseudodifferential calculus developed in [9] for C^*-dynamical systems, heat kernel expansions of Laplacians on noncommutative tori can be derived by taking a parallel approach. We note that, in [48], for *toric manifolds,* the Widom pseudodifferential calculus is adapted to their noncommutative deformations and it is used for the derivation of heat kernel expansions.

We first recall the pseudodifferential calculus on the algebra of noncommutative m-torus. A *pseudodifferential symbol* of order $d \in \mathbb{Z}$ on $\mathbb{T}^m_\Theta$ is a smooth mapping $\rho : \mathbb{R}^m \to C^\infty(\mathbb{T}^m_\Theta)$ such that for any multi-indices α and β of non-negative integers, there exists a constant $C_{\alpha,\beta}$ such that

$$||\partial_\xi^\beta \delta^\alpha \rho(\xi)|| \leq C_{\alpha,\beta}(1 + |\xi|)^{d-|\beta|}.$$

Here $||\cdot||$ denotes the C^*-algebra norm, which is the equivalent of the supremum norm in the commutative setting. Therefore this definition is the noncommutative analog of the definition given by (5) in the classical case. A symbol of order d is *elliptic* if $\rho(\xi)$ is invertible for large enough ξ and there exists a constant $C_\rho > 0$ such that

$$||\rho(\xi)^{-1}|| \leq C_\rho(1 + |\xi|)^{-d}.$$

Given a pseudodifferential symbol on $\mathbb{T}^m_\Theta$ the corresponding pseudodifferential operator $P_\rho : C^\infty(\mathbb{T}^m_\Theta) \to C^\infty(\mathbb{T}^m_\Theta)$ is defined in [9] by the oscillatory integral

$$P_\rho(a) = \iint e^{-is\cdot\xi}\rho(\xi)\,\alpha_s(a)\,ds\,d\xi, \qquad a \in C^\infty(\mathbb{T}^m_\Theta), \tag{13}$$

where α_s is the dynamics given by

$$\alpha_s(U^\alpha) = e^{is\cdot\alpha}U^\alpha.$$

For example, the symbol of a differential operator of the form $\sum_{|\alpha|\le d} a_\alpha\delta^\alpha$, $a_\alpha \in C^\infty(\mathbb{T}^m_\Theta)$ is $\sum_{|\alpha|\le d} a_\alpha\xi^\alpha$.

Given a positive elliptic operator P of order 2 acting on $C^\infty(\mathbb{T}^m_\Theta)$, such as the Laplacian of a metric, in order to derive an asymptotic expansion for $\mathrm{Tr}(e^{-tP})$ one can start by writing the Cauchy integral formula as we did in formula (8). However now one has to use the pseudodifferential calculus given by (13) to write $P-\lambda$ in terms of its symbol and thereby approximate its inverse. In this calculus, if ρ_1 and ρ_2 are, respectively, symbols of orders d_1 and d_2, then the composition $P_{\rho_1}P_{\rho_2}$ has a symbol of order d_1+d_2 with the following asymptotic expansion:

$$\sigma\left(P_{\rho_1}P_{\rho_2}\right) \sim \rho_1\circ\rho_2 := \sum_{\alpha\in\mathbb{Z}^m_{\ge 0}} \frac{1}{\alpha!}\partial_\xi^\alpha\rho_1\,\delta^\alpha\rho_2. \tag{14}$$

Having these tools available, one can then perform calculations as in the process illustrated in Section 3.2 to derive an asymptotic expansion for $\mathrm{Tr}(e^{-tP})$. That is, one writes $\sigma(P) = p_2+p_1+p_0$, where each p_j is homogeneous of order j, and finds recursively the terms r_j, $j=0,1,2,\ldots$, that are homogeneous of order $-2-j$ and

$$\sum_j r_j\circ((p_2-\lambda)+p_1+p_0) \sim 1.$$

This means that we are using the composition rule (14) to approximate the inverse of $P-\lambda$. The result of this process is a recursive formula similar to the one given by (10) and (11). That is, one finds that

$$r_0 = (p_2-\lambda)^{-1}. \tag{15}$$

and for $n=1,2,3,\ldots,$

$$r_n = -\left(\sum_{\substack{|\alpha|+j+2-k=n,\\ 0\le j<n,\,0\le k\le 2}} \frac{1}{\alpha!}\partial_\xi^\alpha r_j\,\delta^\alpha p_k\right) r_0. \tag{16}$$

Then one finds the small asymptotic expansion

$$\mathrm{Tr}(e^{-tP}) \sim_{t\to 0^+} t^{-m/2} \sum_{n=0}^{\infty} \varphi_0(a_{2n}) t^n,$$

where φ_0 is the canonical trace

$$\varphi_0 \left(\sum_{\alpha \in \mathbb{Z}^m} a_\alpha U^\alpha \right) = a_0$$

providing us with integration on the noncommutative torus $\mathbb{T}^m_\Theta$. The terms $a_{2n} \in C^\infty(\mathbb{T}^m_\Theta)$ can be calculated using (15) and (16) as follows:

$$a_{2n} = \frac{1}{2\pi i} \int_{\mathbb{R}^m} \int_\gamma e^{-\lambda} r_{2n}(\xi, \lambda)\, d\lambda\, d\xi. \tag{17}$$

We shall see in Section 4 that in order to perform this type of integrals in the noncommutative setting one encounters noncommutative features which will lead to the appearance of a functional calculus with a modular automorphism in the outcome of the integrals.

4 Gauss-Bonnet theorem and curvature for noncommutative 2-tori

The Gauss-Bonnet theorem for smooth oriented surfaces is a fundamental result that establishes a bridge between topology and differential geometry of surfaces. Given a surface, its Euler characteristic is a topological invariant which can be calculated by choosing an arbitrary triangulation on the surface and forming an alternating summation on the number of its vertices, edges, and faces. It is quite remarkable that the Euler characteristic is independent of the choice of triangulation and depends only on the genus of the surface. Clearly, under a diffeomorphism, or roughly speaking under changes on the surface that do not change the genus, the Euler characteristic remains unchanged. However the scalar curvature of the surface changes under such changes by diffeomorphisms, say when the surface is embedded in the three-dimensional Euclidean space and has inherited the metric of the ambient space. However, the striking fact, namely the statement of the Gauss-Bonnet theorem, is that the change of curvature on the surface occurs in a way that the increase and decrease of curvature over the surface compensate for each other to the effect that the curvature integrates to the Euler characteristic, up to multiplication by a universal constant that is independent of the surface. Hence, the total curvature, namely the integral of the scalar curvature over the surface, is a topological invariant.

4.1 Scalar curvature and Gauss-Bonnet theorem for $\mathbb{T}_\theta^2$

In noncommutative geometry, the analog of the Gauss-Bonnet theorem has been investigated for the noncommutative two torus. In this setting, the flat geometry of $\mathbb{T}_\theta^2$ was conformally perturbed by means of a conformal factor e^{-h}, where h is a selfadjoint element in $C^\infty(\mathbb{T}_\theta^2)$. In the late 1980s, a heavy calculation was performed by P. Tretkoff and A. Connes to find an expression for the analog of the total curvature of the perturbed metric on $\mathbb{T}_\theta^2$. The expression had a heavy dependence on the element h used for changing the metric, therefore it was not clear whether the analog of the Gauss-Bonnet theorem holds for $\mathbb{T}_\theta^2$, and they just recorded the result of their calculations in an MPI preprint [8]. However, following calculations for the spectral action in the presence of a dilaton [7] and developments in the theory of twisted spectral triples [20], there were indications that the complicated expression for the total curvature has to be independent of the element h. By further calculations, simplifications and using symmetries in the result, it was shown in [22] that the terms in the complicated expression for the total curvature indeed cancel each other out to 0, hence the analog of the Gauss-Bonnet theorem for $\mathbb{T}_\theta^2$. The conformal class of metrics that was used in [22] is associated with the simplest translation-invariant complex structure on $\mathbb{T}_\theta^2$, namely the complex structure associated with $i = \sqrt{-1}$. The Gauss-Bonnet theorem for $\mathbb{T}_\theta^2$ for the complex structure associated with an arbitrary complex number τ in the upper-half plane was established in [30].

After considering a general complex number τ in the upper half-plane to induce a complex structure and thereby a conformal structure on $\mathbb{T}_\theta^2$, and by conformally perturbing the flat metric in this class by a fixed conformal factor e^{-h}, $h = h^* \in C^\infty(\mathbb{T}_\theta^2)$, the Laplacian of the curved metric is shown [22, 30] to be anti-unitarily equivalent to the operator

$$\triangle_{\tau,h} = e^{h/2} \triangle_{\tau,0}\, e^{h/2},$$

where

$$\triangle_{\tau,0} = \delta_1^2 + 2\tau_1 \delta_1 \delta_2 + |\tau|^2 \delta_2^2$$

is the Laplacian of the flat metric in the conformal class determined by $\tau = \tau_1 + i\tau_2$ in the upper half-plane. The pseudodifferential symbol of $\triangle_{\tau,h}$ is the sum of the following homogeneous components of order 2, 1, and 0, in which we use $k = h/2$ for simplicity:

$$p_2(\xi) = \xi_1^2 k^2 + |\tau|^2 \xi_2^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2,$$

$$p_1(\xi) = 2\xi_1 k \delta_1(k) + 2|\tau|^2 \xi_2 k \delta_2(k) + 2\tau_1 \xi_1 k \delta_2(k) + 2\tau_1 \xi_2 k \delta_1(k),$$

$$p_0(\xi) = k\delta_1^2(k) + |\tau|^2 k \delta_2^2(k) + 2\tau_1 k \delta_1 \delta_2(k).$$

The analog of the scalar curvature is then the term $a_2(\triangle_{\tau,h}) \in C^\infty(\mathbb{T}_\theta^2)$ appearing in the small time ($t \to 0^+$) asymptotic expansion

$$\mathrm{Tr}(ae^{-t\triangle_{\tau,h}}) \sim t^{-1}\sum_{n=0}^{\infty} \varphi_0\left(a\, a_{2n}(\triangle_{\tau,h})\right) t^n, \qquad a \in C^\infty(\mathbb{T}_\theta^2). \tag{18}$$

By going through the process illustrated in Section 3.3 one can calculate a_2. However, there is a purely noncommutative obstruction for the calculation of the involved integrals in formula (17), namely one encounters integration of C^*-algebra valued functions defined on the Euclidean space, $\mathbb{R}^2$ in this case. By passing to a suitable variation of the polar coordinates, the angular integration can be performed easily, and the main obstruction remains in the radial integration which can be overcome by the following rearrangement lemma [22, 3, 21, 46]:

Lemma 4.1 *For any tuple* $m = (m_0, m_1, \ldots, m_\ell) \in \mathbb{Z}_{>0}^{\ell+1}$ *and elements* $\rho_1, \ldots, \rho_\ell \in C^\infty(\mathbb{T}_\theta^2)$*, one has*

$$\int_0^\infty \frac{u^{|m|-2}}{(e^h u+1)^{m_0}} \prod_1^\ell \rho_j (e^h u+1)^{-m_j}\, du = e^{-(|m|-1)h} F_m(\Delta_{(1)}, \ldots, \Delta_{(\ell)})\Big(\prod_1^\ell \rho_j\Big),$$

where

$$F_m(u_1, \ldots, u_\ell) = \int_0^\infty \frac{x^{|m|-2}}{(x+1)^{m_0}} \prod_1^\ell \left(x \prod_1^j u_h + 1\right)^{-m_j} dx,$$

and Δ *is the modular automorphism*

$$\Delta(a) = e^{-h} a e^h, \qquad a \in C(\mathbb{T}_\theta^2).$$

After applying this lemma to the numerous integrands with the help of computer programming, the result for the scalar curvature $a_2(\triangle_{\tau,h})$ was calculated in [21, 31] (Figure 1):

Theorem 4.1 *The scalar curvature* $a_2(\triangle_{\tau,h}) \in C^\infty(\mathbb{T}_\theta^2)$ *of a general metric in the conformal class associated with a complex number* $\tau = \tau_1 + i\tau_2$ *in the upper half-plane is given by*

$$\begin{aligned} a_2(\triangle_{\tau,h}) = K(\nabla)\left(\delta_1^2\left(\frac{h}{2}\right) + |\tau|^2\delta_2^2\left(\frac{h}{2}\right) + 2\tau_1\delta_1\delta_2\left(\frac{h}{2}\right)\right) \\ + H(\nabla,\nabla)\left(\delta_1\left(\frac{h}{2}\right)\delta_1\left(\frac{h}{2}\right) + |\tau|^2\delta_2\left(\frac{h}{2}\right)\delta_2\left(\frac{h}{2}\right)\right. \\ \left. + \tau_1\delta_1\left(\frac{h}{2}\right)\delta_2\left(\frac{h}{2}\right) + \tau_1\delta_2\left(\frac{h}{2}\right)\delta_1\left(\frac{h}{2}\right)\right), \end{aligned}$$

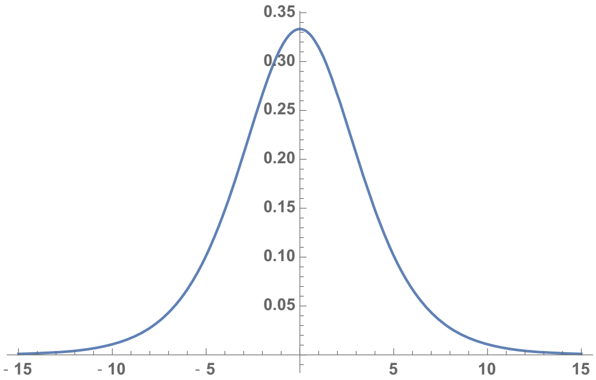


Fig. 1 Graph of K given by (19)

where

$$K(x) = \frac{2e^{x/2}(2 + e^x(-2 + x) + x)}{(-1 + e^x)^2 x}, \tag{19}$$

and

$$H(s,t) = -\frac{\frac{-t(s+t)\cosh s + s(s+t)\cosh t - (s-t)}{(s+t+\sinh s + \sinh t - \sinh(s+t))}}{\frac{st(s+t)\sinh(s/2)}{\sinh(t/2)\sinh^2((s+t)/2)}}. \tag{20}$$

Here the flat metric is conformally perturbed by e^{-h}, where $h = h^* \in C^\infty(\mathbb{T}_\theta^2)$, and ∇ is the logarithm of the modular automorphism $\Delta(a) = e^{-h}ae^h$, hence the derivation given by taking commutator with $-h$.

Using the symmetries of these functions describing the term $a_2(\triangle_{\tau,h})$ integrates to 0, hence the analog of the Gauss-Bonnet theorem. This result was proved in [22, 30] in a kind of simpler manner as by exploiting the trace property of φ_0 from the beginning of the symbolic calculations, only a one variable function was necessary to describe $\varphi_0(a_2(\triangle_{\tau,h}))$. However, for the description of a_2 one needs both one and two variable functions, which are given by (19) and (20). So we can state the Gauss-Bonnet theorem for $\mathbb{T}_\theta^2$ from [22, 30] as follows (Figure 2).

Theorem 4.2 *For any choice of the complex number τ in the upper half-plane and any conformal factor e^{-h}, where $h = h^* \in C^\infty(\mathbb{T}_\theta^2)$, one has*

$$\varphi_0(a_2(\triangle_{\tau,h})) = 0.$$

Hence the total curvature of $\mathbb{T}_\theta^2$ is independent of τ and h defining the metric.

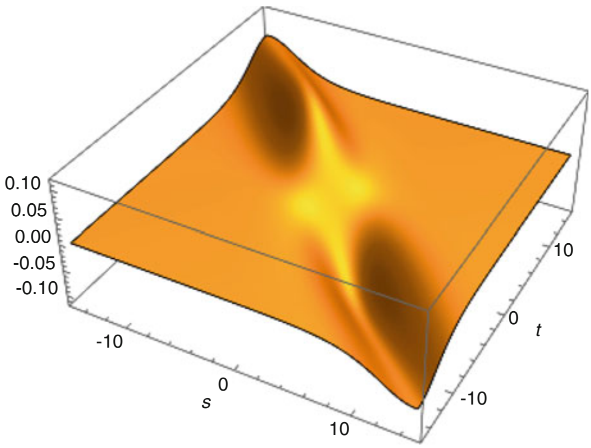


Fig. 2 Graph of H given by (20)

As we mentioned earlier, the validity of the Gauss-Bonnet theorem for $\mathbb{T}_\theta^2$ was suggested by developments on the spectral action in the presence of a dilaton [6] and also studies on twisted spectral triples [20]. In harmony with these developments, in fact a non-computational proof of the Gauss-Bonnet theorem can be given, as written in [21], in the spirit of conformal invariance of the value at the origin of the spectral zeta function of conformally covariant operators [4]. The argument is based on a variational technique: one can write a formula for the variation of the heat coefficients as one varies the metric conformally with e^{-sh}, where h is a dilaton, and the real parameter s goes from 0 to 1. However, the non-computational proof does not lead to an explicit formula for the curvature term $a_2(\triangle_{\tau,h})$. Hence the remarkable achievements in [22, 30, 21, 31] after heavy computer aided calculations include the explicit expression for the scalar curvature of $\mathbb{T}_\theta^2$ and the fact that the analog of the Gauss-Bonnet theorem holds for it.

4.2 The Laplacian on $(1,0)$-forms on $\mathbb{T}_\theta^2$ with curved metric

The analog of the Laplacian on $(1,0)$-forms is also considered in [21, 31] and the second term in its small time heat kernel expansion is calculated. The operator is anti-unitarily equivalent to the operator $\triangle_{\tau,h}^{(1,0)} = \bar{\partial} e^h \partial$, where $\partial = \delta_1 + \bar{\tau}\delta_2$ and $\bar{\partial} = \delta_1 + \tau\delta_2$. The symbol of this Laplacian is equal to $c_2(\xi) + c_1(\xi)$ where

$$c_2(\xi) = \xi_1^2 k^2 + 2\tau_1 \xi_1 \xi_2 k^2 + |\tau|^2 \xi_2^2 k^2,$$
$$c_1(\xi) = (\delta_1(k^2) + \tau\delta_2(k^2))\xi_1 + (\bar{\tau}\delta_1(k^2) + |\tau|^2\delta_2(k^2))\xi_2.$$

Therefore by using the same strategy of using computer aided symbol calculations one can calculate the terms appearing in the following heat kernel expansion:

$$\mathrm{Tr}\left(ae^{-t\triangle_{\tau,h}^{(1,0)}}\right) \sim t^{-1}\sum_{n=0}^{\infty}\varphi_0\left(a\, a_{2n}(\triangle_{\tau,h}^{(1,0)})\right)\, t^n, \qquad a\in C^{\infty}(\mathbb{T}_\theta^2).$$

The result for the second term in this expansion is that [21, 31]

$$\begin{aligned} a_2(\triangle_{\tau,h}^{(1,0)}) = {} & S(\nabla)\left(\delta_1^2\left(\frac{h}{2}\right)+|\tau|^2\delta_2^2\left(\frac{h}{2}\right)+2\tau_1\delta_1\delta_2\left(\frac{h}{2}\right)\right) \\ & +T(\nabla,\nabla)\left(\delta_1\left(\frac{h}{2}\right)\delta_1\left(\frac{h}{2}\right)+|\tau|^2\delta_2\left(\frac{h}{2}\right)\delta_2\left(\frac{h}{2}\right)\right. \\ & \left.+\,\tau_1\delta_1\left(\frac{h}{2}\right)\delta_2\left(\frac{h}{2}\right)+\tau_1\delta_2\left(\frac{h}{2}\right)\delta_1\left(\frac{h}{2}\right)\right) \\ & -i\tau_2 W(\nabla,\nabla)\left(\delta_1\left(\frac{h}{2}\right)\delta_2\left(\frac{h}{2}\right)-\delta_2\left(\frac{h}{2}\right)\delta_1\left(\frac{h}{2}\right)\right), \end{aligned}$$

where

$$S(x) = -\frac{4e^x(-x+\sinh x)}{(-1+e^{x/2})^2(1+e^{x/2})^2 x},$$

$$T(s,t) = -\cosh((s+t)/2)\times\frac{\frac{-t(s+t)\cosh s+s(s+t)\cosh t-(s-t)}{(s+t+\sinh s+\sinh t-\sinh(s+t))}}{\frac{st(s+t)\sinh(s/2)\sinh(t/2)}{\sinh^2((s+t)/2)}},$$

and

$$W(s,t) = \frac{-s-t+t\cosh s+s\cosh t+\sinh s+\sinh t-\sinh(s+t)}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$

Using a simple iso-spectrality argument for the operators $\triangle_{\tau,h}$ and $\triangle_{\tau,h}^{(1,0)}$ one can argue that $\varphi_0\left(a_2(\triangle_{\tau,h}^{(1,0)})\right)=0$, based on the Gauss-Bonnet theorem proved in [22, 30]. However, one can also use properties of the functions S, T, W to prove this directly (Figure 3).

5 Noncommutative residues for noncommutative tori and curvature of noncommutative 4-tori

In this section we discuss noncommutative residues and illustrate an application of a noncommutative residue defined for noncommutative tori in calculating the

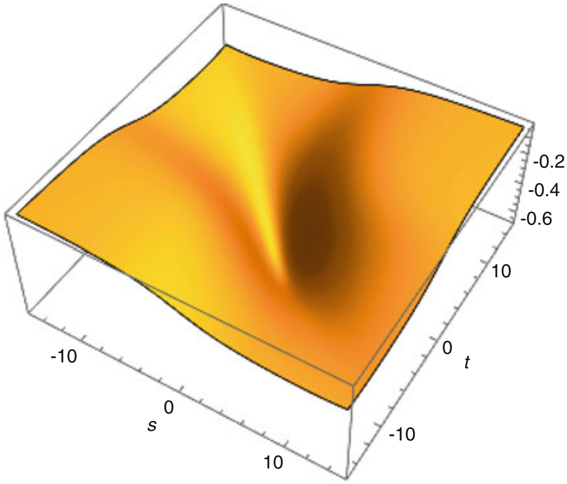


Fig. 3 Graph of W

scalar curvature of the noncommutative 4-torus in a convenient way with certain advantages.

5.1 Noncommutative residues

Noncommutative residues are trace functionals on algebras of pseudodifferential operators, which were first discovered by Adler and Manin in dimension 1 [1, 49]. In order to illustrate their construction in dimension 1 we consider the algebra $C^\infty(\mathbb{S}^1)$ of smooth functions on the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, and the differentiation $(-i)d/dx$, whose pseudodifferential symbol is $\sigma(\xi) = \xi$. We then consider the algebra of pseudodifferential symbols of the form

$$\sum_{n=-\infty}^{N} a_n(x)\xi^n, \qquad a_n(x) \in C^\infty(\mathbb{S}^1), \qquad N \in \mathbb{Z}.$$

The product rule of this algebra can be deduced from the following relations:

$$\xi a(x) = a(x)\xi + a'(x), \qquad a_n(x) \in C^\infty(\mathbb{S}^1),$$

which are dictated by the Leibniz property of differentiation. The Adler-Manin trace is the linear functional defined by

$$\sum_{n=-\infty}^{N} a_n(x)\xi^n \mapsto \int_{\mathbb{S}^1} a_{-1}(x)\,dx,$$

which is shown to be a trace functional on the algebra of pseudodifferential symbols on the circle [1, 49]. A twisted version of this trace was worked out in [27], motivated by the notion of twisted spectral triples [20].

Wodzicki generalized this functional, in a remarkable work, to higher dimensions [55]. Consider a closed manifold M of dimension m and the algebra of classical pseudodifferential operators M. A classical pseudodifferential symbol σ of order d has an expansion with homogeneous terms, of the form

$$\sigma(x,\xi) \sim \sum_{j=0}^{\infty} \sigma_{d-j}(x,\xi),$$

where $\sigma_{d-j}(x, t\xi) = t^{d-j}\sigma_{d-j}(x,\xi)$ for any $t > 0$. The composition rule of this algebra is induced by the composition rule for the symbol of pseudodifferential operators:

$$\sigma_{P_1 P_2}(x,\xi) \sim \sum_{\alpha\in\mathbb{Z}^m_{\geq 0}} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma_{P_1}(x,\xi)\, \partial_x^\alpha \sigma_{P_2}(x,\xi),$$

which we mentioned and used in Section 3 as well. Wodzicki's noncommutative residue WRes is the linear functional defined on the algebra of classical pseudodifferential symbols by

$$\mathrm{WRes}\left(\sum_{j=0}^{\infty} \sigma_{d-j}(x,\xi)\right) = \int_{S^*M} \mathrm{tr}(\sigma_{-m}(x,\xi))\, d^{m-1}\xi\, d^m x, \tag{21}$$

where S^*M is the cosphere bundle of the manifold with respect to a Riemannian metric. We stress that in this formula m is the dimension of the manifold M. It is proved that WRes is the unique trace functional on the algebra of classical pseudodifferential symbols on M [55].

The noncommutative residue has a spectral formulation as well. That is, one can fix a Laplacian $\triangle$ on M and define the noncommutative residue of a pseudodifferential operator P_σ to be the residue at $s = 0$ of the meromorphic extension of the zeta function defined, for complex numbers s with large enough real parts, by

$$s \mapsto \mathrm{Tr}(P_\sigma \triangle^{-s}).$$

This formulation is used in noncommutative geometry, when one works with the algebra of pseudodifferential operators associated with a spectral triple [19].

For noncommutative tori, the analog of formula (21) can be written and it was shown in [33] that it gives the unique *continuous* trace functional on the algebra of classical pseudodifferential operators on the noncommutative 2-torus. Although the argument written in [33] is for dimension 2, but it is general enough that works for any dimension, see, for example, [32] for the illustration in dimension 4. Given a classical pseudodifferential symbol $\rho : \mathbb{R}^m \to C^\infty(\mathbb{T}^m_\Theta)$ of order d on the noncommutative m-torus, by definition, there is an asymptotic expansion for $\xi \to \infty$ of the form

$$\rho(\xi) \sim \sum_{j=0}^{\infty} \rho_{d-j}(\xi),$$

where each ρ_{d-j} is positively homogeneous of order $d - j$. One can define the noncommutative residue Res of the corresponding pseudodifferential symbol as

$$\mathrm{Res}(P_\rho) = \int_{\mathbb{S}^{m-1}} \varphi_0 \left(\rho_{-m}\right) \, d\Omega, \tag{22}$$

where φ_0 is the canonical trace on $C(\mathbb{T}^m_\Theta)$ and $d\Omega$ is the volume form of the round metric on the $(m-1)$-dimensional sphere in $\mathbb{R}^m$. The same argument as the one given in [33] shows that Res is the unique continuous trace on the algebra of classical pseudodifferential symbols on $\mathbb{T}^m_\Theta$.

5.2 Scalar curvature of the noncommutative 4-torus

The Laplacian associated with the flat geometry of the noncommutative four torus $\mathbb{T}^4_\Theta$ is simply given by the sum of the squares of the canonical derivatives, namely:

$$\triangle_0 = \delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2.$$

After conformally perturbing the flat metric on $\mathbb{T}^4_\Theta$ by means of a conformal factor e^{-h}, for a fixed $h = h^* \in C^\infty(\mathbb{T}^4_\Theta)$, the perturbed Laplacian is shown in [32] to be anti-unitarily equivalent to the operator

$$\triangle_h = e^h \bar\partial_1 e^{-h} \partial_1 e^h + e^h \partial_1 e^{-h} \bar\partial_1 e^h + e^h \bar\partial_2 e^{-h} \partial_2 e^h + e^h \partial_2 e^{-h} \bar\partial_2 e^h,$$

where

$$\partial_1 = \delta_1 - i\delta_3, \qquad \partial_2 = \delta_2 - i\delta_4,$$

$$\bar\partial_1 = \delta_1 + i\delta_3, \qquad \bar\partial_2 = \delta_2 + i\delta_4.$$

The latter are the analogues of the Dolbeault operators.

The scalar curvature of the metric on $\mathbb{T}^4_\Theta$ encoded in $\triangle_h$ is the term $a_2 \in C^\infty(\mathbb{T}^4_\Theta)$ appearing in the following small time asymptotic expansion:

$$\mathrm{Tr}(ae^{-t\triangle_h}) \sim t^{-2}\sum_{n=0}^{\infty}\varphi_0(a\,a_{2n})t^n, \qquad a \in C^\infty(\mathbb{T}^4_\Theta).$$

The curvature term $a_2 \in C^\infty(\mathbb{T}^4_\Theta)$ was calculated in [32] by going through the procedure explained in Section 3.3. As we explained earlier, there is a purely noncommutative obstruction in this procedure that needs to be overcome by Lemma 4.1, the so-called rearrangement lemma. That is, one encounters integration over the Euclidean space of C^*-algebra valued functions. For this type of integrations, one can pass to polar coordinates and take care of the angular integrations with no problem. However, the redial integration brings forth the necessity of the rearrangement lemma.

Striking is the fact that after applying the rearrangement lemma to hundreds of terms, each of which involves a function from this lemma to appear in the calculations, the final formula for the curvature simplifies significantly with computer aid. In [28], by using properties of the noncommutative residue (22), it was shown that the curvature term $a_2 \in C^\infty(\mathbb{T}^4_\Theta)$ can be calculated as the integral over the 3-sphere of a homogeneous symbol. Therefore, with this method, the calculation of a_2 does not require radial integration, hence the calculation without using the rearrangement lemma and clarification of the reason for the significant simplifications. In fact, in [28], the term is shown to be a scalar multiple of $\int_{\mathbb{S}^3} b_2(\xi)\,d\Omega$, where b_2 is the homogeneous term of order -4 in the expansion of the symbol of the parametrix of $\triangle_h$. The result, in agreement with the calculation of [32], is that

$$a_2 = e^{-h}K(\nabla)\Big(\sum_{i=1}^{4}\delta_i^2(h)\Big) + e^{-h}H(\nabla,\nabla)\Big(\sum_{i=1}^{4}\delta_i(h)^2\Big) \in C^\infty(\mathbb{T}^4_\Theta), \tag{23}$$

where $\nabla = [-h,\cdot]$, and

$$K(x) = \frac{1-e^{-x}}{2x},$$
$$H(s,t) = -\frac{e^{-s-t}\left((-e^s-3)\,s\left(e^t-1\right)+(e^s-1)\left(3e^t+1\right)t\right)}{4st(s+t)}. \tag{24}$$

The simplicity of this calculation also revealed in [28] the following functional relation between the functions K and H (Figure 4).

Theorem 5.1 *Let $\tilde{K}(s) = e^sK(s)$ and $\tilde{H}(s,t) = e^{s+t}H(s,t)$, where the function K and H are given by* (24)*. Then*

$$\tilde{H}(s,t) = 2\frac{\tilde{K}(s+t)-\tilde{K}(s)}{t} + \frac{3}{2}\tilde{K}(s)\tilde{K}(t).$$

Another important result that we wish to recall from [32] is about the extrema of the analog of the Einstein-Hilbert action for $\mathbb{T}^4_\Theta$, namely $\varphi_0(a_2)$:

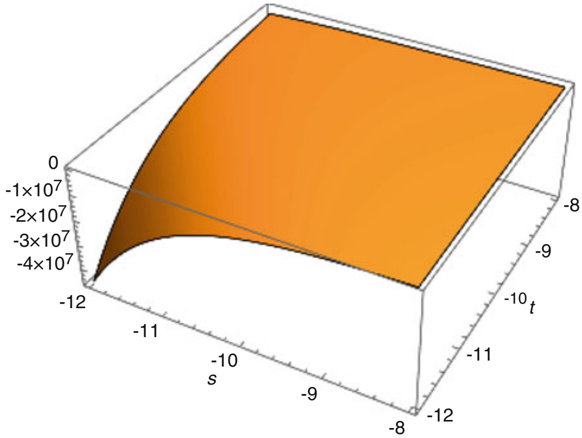


Fig. 4 Graph of H given by (24)

Theorem 5.2 *For any conformal factor e^{-h}, where $h = h^* \in C^\infty(\mathbb{T}^4_\Theta)$,*

$$\varphi_0(a_2) \leq 0,$$

where $a_2 \in C^\infty(\mathbb{T}^4_\Theta)$ is the scalar curvature given by (23). Moreover, we have $\varphi_0(a_2) = 0$ if and only if h is a scalar.

6 The Riemann curvature tensor and the term a_4 for noncommutative tori

The Riemann curvature tensor appears in the term a_4 in the heat kernel expansion for the Laplacian of any closed Riemannian manifold M. That is, if Δ_g is the Laplacian of a Riemannian metric g, which acts on $C^\infty(M)$, then

$$a_4(x, \Delta_g) = (4\pi)^{-1}(1/360)(-12\Delta_g R(x) + 5R(x)^2 - 2|Ric(x)|^2 + 2|Riem(x)|^2).$$

In this section we recall from [17] the formula obtained for the analog of the term a_4 in a noncommutative setting. Recall that in Section 4.1, we discussed the term a_2, namely the analog of the scalar curvature, for the noncommutative two torus when the flat metric is perturbed by a positive invertible element $e^{-h} \in C^\infty(\mathbb{T}^2_\theta)$, where $h = h^*$. These geometric terms appear in the expansion given by (18). Setting,

$$\ell = \frac{h}{2}$$

for the simplest conformal class (associated with $\tau = i$), the main calculation of [17] gives the term a_4 by a formula of the following form:

$$
\begin{aligned}
a_4(h) = -e^{2\ell}\Big(& K_1(\nabla)\left(\delta_1^2\delta_2^2(\ell)\right) + K_2(\nabla)\left(\delta_1^4(\ell)+\delta_2^4(\ell)\right) \\
& + K_3(\nabla,\nabla)\left((\delta_1\delta_2(\ell))\cdot(\delta_1\delta_2(\ell))\right) \\
& + K_4(\nabla,\nabla)\left(\delta_1^2(\ell)\cdot\delta_2^2(\ell)+\delta_2^2(\ell)\cdot\delta_1^2(\ell)\right) \\
& + K_5(\nabla,\nabla)\left(\delta_1^2(\ell)\cdot\delta_1^2(\ell)+\delta_2^2(\ell)\cdot\delta_2^2(\ell)\right) \\
& + K_6(\nabla,\nabla)\Big(\delta_1(\ell)\cdot\delta_1^3(\ell)+\delta_1(\ell)\cdot\left(\delta_1\delta_2^2(\ell)\right)+\delta_2(\ell)\cdot\delta_2^3(\ell) \\
& \qquad\qquad +\delta_2(\ell)\cdot\left(\delta_1^2\delta_2(\ell)\right)\Big) \\
& + K_7(\nabla,\nabla)\Big(\delta_1^3(\ell)\cdot\delta_1(\ell)+\left(\delta_1\delta_2^2(\ell)\right)\cdot\delta_1(\ell)+\delta_2^3(\ell)\cdot\delta_2(\ell) \\
& \qquad\qquad +\left(\delta_1^2\delta_2(\ell)\right)\cdot\delta_2(\ell)\Big) \\
& + K_8(\nabla,\nabla,\nabla)\left(\delta_1(\ell)\cdot\delta_1(\ell)\cdot\delta_2^2(\ell)+\delta_2(\ell)\cdot\delta_2(\ell)\cdot\delta_1^2(\ell)\right) \\
& + K_9(\nabla,\nabla,\nabla)\left(\delta_1(\ell)\cdot\delta_2(\ell)\cdot(\delta_1\delta_2(\ell))+\delta_2(\ell)\cdot\delta_1(\ell)\cdot(\delta_1\delta_2(\ell))\right) \\
& + K_{10}(\nabla,\nabla,\nabla)\left(\delta_1(\ell)\cdot(\delta_1\delta_2(\ell))\cdot\delta_2(\ell)+\delta_2(\ell)\cdot(\delta_1\delta_2(\ell))\cdot\delta_1(\ell)\right) \\
& + K_{11}(\nabla,\nabla,\nabla)\left(\delta_1(\ell)\cdot\delta_2^2(\ell)\cdot\delta_1(\ell)+\delta_2(\ell)\cdot\delta_1^2(\ell)\cdot\delta_2(\ell)\right) \\
& + K_{12}(\nabla,\nabla,\nabla)\left(\delta_1^2(\ell)\cdot\delta_2(\ell)\cdot\delta_2(\ell)+\delta_2^2(\ell)\cdot\delta_1(\ell)\cdot\delta_1(\ell)\right) \\
& + K_{13}(\nabla,\nabla,\nabla)\left((\delta_1\delta_2(\ell))\cdot\delta_1(\ell)\cdot\delta_2(\ell)+(\delta_1\delta_2(\ell))\cdot\delta_2(\ell)\cdot\delta_1(\ell)\right) \\
& + K_{14}(\nabla,\nabla,\nabla)\left(\delta_1^2(\ell)\cdot\delta_1(\ell)\cdot\delta_1(\ell)+\delta_2^2(\ell)\cdot\delta_2(\ell)\cdot\delta_2(\ell)\right) \\
& + K_{15}(\nabla,\nabla,\nabla)\left(\delta_1(\ell)\cdot\delta_1(\ell)\cdot\delta_1^2(\ell)+\delta_2(\ell)\cdot\delta_2(\ell)\cdot\delta_2^2(\ell)\right) \\
& + K_{16}(\nabla,\nabla,\nabla)\left(\delta_1(\ell)\cdot\delta_1^2(\ell)\cdot\delta_1(\ell)+\delta_2(\ell)\cdot\delta_2^2(\ell)\cdot\delta_2(\ell)\right) \\
& + K_{17}(\nabla,\nabla,\nabla,\nabla)\big(\delta_1(\ell)\cdot\delta_1(\ell)\cdot\delta_2(\ell)\cdot\delta_2(\ell) \\
& \qquad\qquad +\delta_2(\ell)\cdot\delta_2(\ell)\cdot\delta_1(\ell)\cdot\delta_1(\ell)\big) \\
& + K_{18}(\nabla,\nabla,\nabla,\nabla)\big(\delta_1(\ell)\cdot\delta_2(\ell)\cdot\delta_1(\ell)\cdot\delta_2(\ell) \\
& \qquad\qquad +\delta_2(\ell)\cdot\delta_1(\ell)\cdot\delta_2(\ell)\cdot\delta_1(\ell)\big) \\
& + K_{19}(\nabla,\nabla,\nabla,\nabla)\big(\delta_1(\ell)\cdot\delta_2(\ell)\cdot\delta_2(\ell)\cdot\delta_1(\ell) \\
& \qquad\qquad +\delta_2(\ell)\cdot\delta_1(\ell)\cdot\delta_1(\ell)\cdot\delta_2(\ell)\big) \\
& + K_{20}(\nabla,\nabla,\nabla,\nabla)\big(\delta_1(\ell)\cdot\delta_1(\ell)\cdot\delta_1(\ell)\cdot\delta_1(\ell) \\
& \qquad\qquad +\delta_2(\ell)\cdot\delta_2(\ell)\cdot\delta_2(\ell)\cdot\delta_2(\ell)\big)\Big).
\end{aligned} \tag{25}
$$

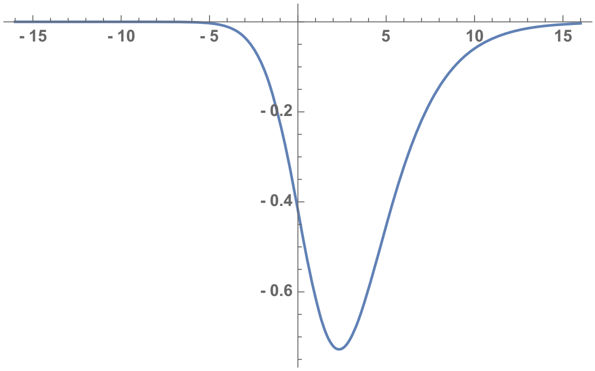


Fig. 5 Graph of K_1

We provide the explicit formulas for a few of the functions appearing in (25), and we refer the reader to [17] for the remaining functions, most of which have lengthy expressions. We have, for example, (Figure 5):

$$K_1(s_1) = -\frac{4\pi e^{\frac{3s_1}{2}} \left(\left(4e^{s_1} + e^{2s_1} + 1\right) s_1 - 3e^{2s_1} + 3\right)}{\left(e^{s_1} - 1\right)^4 s_1}, \tag{26}$$

and

$$K_3(s_1, s_2) = \frac{K_3^{\text{num}}(s_1, s_2)}{\left(e^{s_1} - 1\right)^2 \left(e^{s_2} - 1\right)^2 \left(e^{s_1+s_2} - 1\right)^4 s_1 s_2 \left(s_1 + s_2\right)}, \tag{27}$$

where the numerator is given by

$$\begin{aligned}
K_3^{\text{num}}(s_1, s_2) = 16\, e^{\frac{3s_1}{2} + \frac{3s_2}{2}} \pi \Big[& \left(e^{s_1} - 1\right)\left(e^{s_2} - 1\right)\left(e^{s_1+s_2} - 1\right) \\
& \times \Big\{ \Big(-5e^{s_1} - e^{s_2} + 6e^{s_1+s_2} - e^{2s_1+s_2} - 5e^{s_1+2s_2} \\
& \quad +3e^{2s_1+2s_2} + 3\Big) s_1 \\
& + \left(e^{s_1} + 5e^{s_2} - 6e^{s_1+s_2} + 5e^{2s_1+s_2} + e^{s_1+2s_2} - 3e^{2s_1+2s_2} - 3\right) s_2 \Big\} \\
& -2\left(e^{s_1} - e^{s_2}\right)\left(e^{s_1+s_2} - 1\right) \\
& \times \left(-e^{s_1} - e^{s_2} - e^{2s_1+s_2} - e^{s_1+2s_2} + 2e^{2s_1+2s_2} + 2\right) s_1 s_2 \\
& +2e^{s_1}\left(e^{s_2} - 1\right)^3 \left(e^{s_1} - e^{s_1+s_2} + 2e^{2s_1+s_2} - 2\right) s_1^2 \\
& -2e^{s_2}\left(e^{s_1} - 1\right)^3 \left(e^{s_2} - e^{s_1+s_2} + 2e^{s_1+2s_2} - 2\right) s_2^2 \Big].
\end{aligned}$$

6.1 Functional relations

One of the main results of [17] is the derivation of a family of conceptually predicted functional relations among the functions $K_1, \ldots, K_{20}$ appearing in (25). As we shall see shortly the functional relations are highly nontrivial. There are two main reasons for the derivation of the relations, both of which are extremely important. First, the calculation of the term a_4 involves a really heavy computer aided calculation, hence, the need for a way of confirming the validity of the outcome by checking that the expected functional relations are satisfied. Second, patterns and structural properties of the relations give significant information that help one to obtain conceptual understandings about the structure of the complicated functions appearing in the formula for a_4. In order to present the relations, we need to consider the modification of each function K_j in (25) to a new function $\widetilde{K}_j$ by the formula

$$\widetilde{K}_j(s_1, \ldots, s_n) = \frac{1}{2^n} \frac{\sinh((s_1 + \cdots + s_n)/2)}{(s_1 + \cdots + s_n)/2} K_j(s_1, \ldots, s_n),$$

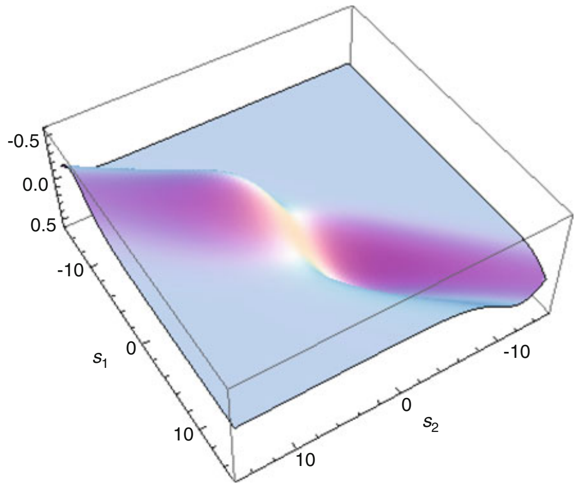
where $n \in \{1, 2, 3, 4\}$ is the number of variables, on which K_j depends. We also need to introduce the restriction of the functions K_j to certain hyperplanes by setting

$$k_j(s_1, \ldots, s_{n-1}) = K_j(s_1, \ldots, s_{n-1}, -s_1 - \cdots - s_{n-1}).$$

We shall explain shortly how these functional relations are predicted, using fundamental identities and lemmas [21, 17] (Figure 6).

Let us first list a few of the functional relations in which some auxiliary functions $G_n(s_1, \ldots, s_n)$ appear. These functions are mainly useful for relating the derivatives of e^h and those of h and we recall from [17] their recursive formula:

Fig. 6 Graph of K_3



Lemma 6.1 *The functions $G_n(s_1, \ldots, s_n)$ are given recursively by*

$$G_0 = 1,$$

and

$$G_n(s_1, \ldots, s_n) = \int_0^1 r^{n-1} e^{s_1 r}\, G_{n-1}(rs_2, rs_3, \ldots, rs_n)\, dr.$$

Explicitly, for $n = 1, 2, 3$, one has:

$$G_1(s_1) = \frac{e^{s_1} - 1}{s_1}, \tag{28}$$

$$G_2(s_1, s_2) = \frac{e^{s_1}\left(\left(e^{s_2} - 1\right) s_1 - s_2\right) + s_2}{s_1 s_2 \left(s_1 + s_2\right)},$$

$$G_3(s_1, s_2, s_3) = \frac{e^{s_1}\left(e^{s_2+s_3} s_1 s_2 (s_1+s_2) + (s_1+s_2+s_3)((s_1+s_2)s_3 - e^{s_2} s_1 (s_2+s_3))\right) - s_2 s_3 (s_2+s_3)}{s_1 s_2 (s_1+s_2) s_3 (s_2+s_3)(s_1+s_2+s_3)}.$$

We can now write the relations. The functional relation associated with the function K_1 is given by

$$\begin{aligned}\widetilde{K}_1(s_1) = &-\frac{1}{15}\pi G_1\left(s_1\right) + \frac{1}{4}e^{s_1} k_3\left(-s_1\right) + \frac{1}{4}k_3\left(s_1\right)\\ &+\frac{1}{2}e^{s_1} k_4\left(-s_1\right) + \frac{1}{2}k_4\left(s_1\right) - \frac{1}{2}e^{s_1} k_6\left(-s_1\right)\\ &-\frac{1}{2}k_6\left(s_1\right) - \frac{1}{2}e^{s_1} k_7\left(-s_1\right) - \frac{1}{2}k_7\left(s_1\right) - \frac{\pi\left(e^{s_1} - 1\right)}{15 s_1}.\end{aligned} \tag{29}$$

It is quite remarkable that such a nontrivial relation should exist among the functions, and it gets even more interesting when one looks at the case associated with a 2-variable function. For K_3 one finds the associated relation to be:

$$\begin{aligned}\widetilde{K}_3(s_1, s_2) = &\frac{1}{15}(-4)\pi G_2\left(s_1, s_2\right) + \frac{1}{2}k_8\left(s_1, s_2\right) + \frac{1}{4}k_9\left(s_1, s_2\right)\\ &-\frac{1}{4}e^{s_1+s_2} k_9\left(-s_1 - s_2, s_1\right)\\ &-\frac{1}{4}e^{s_1} k_9\left(s_2, -s_1 - s_2\right) - \frac{1}{4}k_{10}\left(s_1, s_2\right) - \frac{1}{4}e^{s_1+s_2} k_{10}\left(-s_1 - s_2, s_1\right)\\ &+\frac{1}{4}e^{s_1} k_{10}\left(s_2, -s_1 - s_2\right)\end{aligned} \tag{30}$$

$$
\begin{aligned}
&+\frac{1}{2}e^{s_1}k_{11}\left(s_2,-s_1-s_2\right)+\frac{1}{2}e^{s_1+s_2}k_{12}\left(-s_1-s_2,s_1\right)\\
&-\frac{1}{4}k_{13}\left(s_1,s_2\right)+\frac{1}{4}e^{s_1+s_2}k_{13}\left(-s_1-s_2,s_1\right)-\frac{1}{4}e^{s_1}k_{13}\left(s_2,-s_1-s_2\right)\\
&+\frac{1}{4}e^{s_2}G_1\left(s_1\right)k_3\left(-s_2\right)+\frac{1}{4}G_1\left(s_1\right)k_3\left(s_2\right)-G_1\left(s_1\right)k_6\left(s_2\right)\\
&-e^{s_2}G_1\left(s_1\right)k_7\left(-s_2\right)\\
&+\frac{\left(e^{s_1+s_2}-1\right)k_3\left(s_1\right)}{4\left(s_1+s_2\right)}+\frac{k_3\left(s_2\right)-k_3\left(s_1+s_2\right)}{4s_1}\\
&+\frac{k_3\left(s_1+s_2\right)-k_3\left(s_1\right)}{4s_2}\\
&+\frac{k_6\left(s_1\right)-k_6\left(s_1+s_2\right)}{s_2}+\frac{k_6\left(s_1+s_2\right)-k_6\left(s_2\right)}{s_1}\\
&+\frac{e^{s_1}\left(k_7\left(-s_1\right)-e^{s_2}k_7\left(-s_1-s_2\right)\right)}{s_2}\\
&+\frac{e^{s_2}\left(e^{s_1}k_7\left(-s_1-s_2\right)-k_7\left(-s_2\right)\right)}{s_1}\\
&-\frac{e^{s_2}\left(e^{s_1}k_3\left(-s_1-s_2\right)-k_3\left(-s_2\right)\right)}{4s_1}\\
&-\frac{e^{s_1}\left(k_3\left(-s_1\right)-e^{s_2}k_3\left(-s_1-s_2\right)\right)}{4s_2}\\
&-\frac{e^{s_1}\left(k_3\left(-s_1\right)+e^{s_2}k_3\left(s_1\right)-e^{s_2}k_3\left(-s_2\right)-k_3\left(s_2\right)\right)}{4\left(s_1+s_2\right)}.
\end{aligned}
$$

The rapid pace in growing complexity of the functional relations can be seen in the higher variable cases as, for example, the functional relation corresponding to the 3-variable function K_8 is the following expression:

$$
\begin{aligned}
\widetilde{K}_8(s_1,s_2,s_3)=&\ \frac{1}{15}(-2)\pi G_3\left(s_1,s_2,s_3\right)+\frac{1}{2}e^{s_3}G_2\left(s_1,s_2\right)k_4\left(-s_3\right) \qquad (31)\\
&-\frac{e^{s_3}\left(e^{s_2}s_1k_4(-s_2-s_3)+e^{s_2}s_2k_4(-s_2-s_3)-e^{s_1+s_2}s_2k_4(-s_1-s_2-s_3)-s_1k_4(-s_3)\right)}{2s_1s_2\left(s_1+s_2\right)}\\
&+\frac{1}{2}G_2\left(s_1,s_2\right)k_4\left(s_3\right)+\frac{G_1\left(s_1\right)\left(k_4\left(s_3\right)-k_4\left(s_2+s_3\right)\right)}{2s_2}\\
&+\frac{s_1k_4\left(s_3\right)-s_1k_4\left(s_2+s_3\right)-s_2k_4\left(s_2+s_3\right)+s_2k_4\left(s_1+s_2+s_3\right)}{2s_1s_2\left(s_1+s_2\right)}\\
&-\frac{1}{2}G_2\left(s_1,s_2\right)k_6\left(s_3\right)
\end{aligned}
$$

$$+\frac{G_1(s_1)(k_6(s_2)-k_6(s_2+s_3))}{4s_3}+\frac{k_6(s_2)-k_6(s_1+s_2)-k_6(s_2+s_3)+k_6(s_1+s_2+s_3)}{4s_1s_3}$$

$$+\frac{-s_3k_6(s_1)+s_2k_6(s_1+s_2)+s_3k_6(s_1+s_2)-s_2k_6(s_1+s_2+s_3)}{4s_2s_3(s_2+s_3)}$$

$$+\frac{-s_1k_6(s_3)+s_1k_6(s_2+s_3)+s_2k_6(s_2+s_3)-s_2k_6(s_1+s_2+s_3)}{2s_1s_2(s_1+s_2)}$$

$$+\frac{e^{s_2}G_1(s_1)(k_7(-s_2)-e^{s_3}k_7(-s_2-s_3))}{4s_3}$$

$$-\frac{e^{s_1}\left(s_3k_7(-s_1)-e^{s_2}s_2k_7(-s_1-s_2)-e^{s_2}s_3k_7(-s_1-s_2)+e^{s_2+s_3}s_2k_7(-s_1-s_2-s_3)\right)}{4s_2s_3(s_2+s_3)}$$

$$-\frac{e^{s_2}\left(e^{s_1}k_7(-s_1-s_2)-k_7(-s_2)+e^{s_3}k_7(-s_2-s_3)-e^{s_1+s_3}k_7(-s_1-s_2-s_3)\right)}{4s_1s_3}$$

$$+\frac{e^{s_3}G_1(s_1)(e^{s_2}k_7(-s_2-s_3)-k_7(-s_3))}{2s_2}$$

$$-\frac{1}{2}e^{s_3}G_2(s_1,s_2)k_7(-s_3)$$

$$+\frac{e^{s_3}\left(e^{s_2}s_1k_7(-s_2-s_3)+e^{s_2}s_2k_7(-s_2-s_3)-e^{s_1+s_2}s_2k_7(-s_1-s_2-s_3)-s_1k_7(-s_3)\right)}{2s_1s_2(s_1+s_2)}$$

$$+\frac{\left(-1+e^{s_1+s_2+s_3}\right)k_8(s_1,s_2)}{8(s_1+s_2+s_3)}+\frac{k_8(s_1,s_2+s_3)-k_8(s_1,s_2)}{8s_3}$$

$$-\frac{1}{8}e^{s_2+s_3}G_1(s_1)k_8(-s_2-s_3,s_2)$$

$$+\frac{e^{s_1+s_2+s_3}(k_8(-s_1-s_2-s_3,s_1)-k_8(-s_1-s_2-s_3,s_1+s_2))}{8s_2}$$

$$+\frac{1}{8}G_1(s_1)k_9(s_2,s_3)$$

$$+\frac{k_9(s_2,s_3)-k_9(s_1+s_2,s_3)}{8s_1}$$

$$+\frac{k_9(s_1+s_2,s_3)-k_9(s_1,s_2+s_3)}{8s_2}$$

$$
\begin{aligned}
&+\frac{1}{8}e^{s_2}G_1(s_1)\,k_{10}(s_3,-s_2-s_3)\\
&+\frac{e^{s_2}\left(k_{10}(s_3,-s_2-s_3)-e^{s_1}k_{10}(s_3,-s_1-s_2-s_3)\right)}{8s_1}\\
&+\frac{e^{s_1}\left(e^{s_2}k_{10}(s_3,-s_1-s_2-s_3)-k_{10}(s_2+s_3,-s_1-s_2-s_3)\right)}{8s_2}\\
&-\frac{1}{8}G_1(s_1)\,k_{11}(s_2,s_3)\\
&+\frac{k_{11}(s_1,s_2+s_3)-k_{11}(s_1+s_2,s_3)}{8s_2}\\
&+\frac{k_{11}(s_1+s_2,s_3)-k_{11}(s_2,s_3)}{8s_1}\\
&-\frac{1}{8}e^{s_2}G_1(s_1)\,k_{12}(s_3,-s_2-s_3)\\
&+\frac{1}{8}e^{s_2+s_3}G_1(s_1)\,k_{13}(-s_2-s_3,s_2)\\
&+\frac{e^{s_2+s_3}\left(k_{13}(-s_2-s_3,s_2)-e^{s_1}k_{13}(-s_1-s_2-s_3,s_1+s_2)\right)}{8s_1}\\
&-\frac{1}{16}k_{17}(s_1,s_2,s_3)-\frac{1}{16}e^{s_1+s_2}k_{17}(s_3,-s_1-s_2-s_3,s_1)\\
&-\frac{1}{16}e^{s_1}k_{19}(s_2,s_3,-s_1-s_2-s_3)\\
&-\frac{1}{16}e^{s_1+s_2+s_3}k_{19}(-s_1-s_2-s_3,s_1,s_2)\\
&-\frac{e^{s_2+s_3}\left(k_8(-s_2-s_3,s_2)-e^{s_1}k_8(-s_1-s_2-s_3,s_1+s_2)\right)}{8s_1}\\
&-\frac{e^{s_2}\left(k_{12}(s_3,-s_2-s_3)-e^{s_1}k_{12}(s_3,-s_1-s_2-s_3)\right)}{8s_1}\\
&-\frac{e^{s_3}G_1(s_1)\left(e^{s_2}k_4(-s_2-s_3)-k_4(-s_3)\right)}{2s_2}\\
&-\frac{G_1(s_1)\left(k_6(s_3)-k_6(s_2+s_3)\right)}{2s_2}\\
&-\frac{e^{s_1}\left(e^{s_2}k_{12}(s_3,-s_1-s_2-s_3)-k_{12}(s_2+s_3,-s_1-s_2-s_3)\right)}{8s_2}\\
&-\frac{\substack{e^{s_1+s_2+s_3}(k_{13}(-s_1-s_2-s_3,s_1)\\ -k_{13}(-s_1-s_2-s_3,s_1+s_2))}}{8s_2}
\end{aligned}
$$

$$-\frac{e^{s_1}\left(k_{11}\left(s_2,-s_1-s_2\right)-k_{11}\left(s_2+s_3,-s_1-s_2-s_3\right)\right)}{8s_3}$$

$$-\frac{e^{s_1+s_2}\left(k_{12}\left(-s_1-s_2,s_1\right)-e^{s_3}k_{12}\left(-s_1-s_2-s_3,s_1\right)\right)}{8s_3}$$

$$-\frac{e^{s_1+s_2+s_3}\left(k_8\left(s_1,s_2\right)-k_8\left(-s_2-s_3,s_2\right)\right)}{8\left(s_1+s_2+s_3\right)}$$

$$-\frac{e^{s_1}\left(k_{11}\left(s_2,-s_1-s_2\right)-k_{11}\left(s_2,s_3\right)\right)}{8\left(s_1+s_2+s_3\right)}$$

$$-\frac{e^{s_1+s_2}\left(k_{12}\left(-s_1-s_2,s_1\right)-k_{12}\left(s_3,-s_2-s_3\right)\right)}{8\left(s_1+s_2+s_3\right)}.$$

The interested reader can refer to [17] to see that the functional relations of the 4-variable functions get even more complicated. The main point, which will be elaborated further, is that all these functional relations are derived conceptually, and by checking that our calculated functions $K_1, \ldots, K_{20}$ satisfy these relations, the validity of the calculations and their outcome, such as the explicit formulas (26), (27), is confirmed.

6.2 A partial differential system associated with the functional relations

When one takes a close look at the functional relations, one notices that there are terms in the right-hand sides (in the finite difference expressions) with $s_1+\cdots+s_n$ in their denominators. For example, in (30) one can see that there is a term with s_1+s_2 in the denominator. The question answered in [17], which leads to a differential system with interesting properties, is what happens when one restricts the functional relations to the hyperplanes $s_1+\cdots+s_n=0$ by setting $s_1+\cdots+s_n=\varepsilon$ and letting $\varepsilon\to 0$. For example, the restriction of the functional relation (30) to the hyperplane $s_1+s_2=0$ yields:

$$\begin{aligned}\frac{1}{4}e^{s_1}k_3'\left(-s_1\right)-\frac{1}{4}k_3'\left(s_1\right)=\frac{1}{60s_1}\Big(&16\pi s_1G_2(s_1,-s_1)-30s_1k_8(s_1,-s_1)\\&+15s_1k_9(0,s_1)+15e^{s_1}s_1k_9(-s_1,0)\\&-15s_1k_9(s_1,-s_1)+15s_1k_{10}(0,s_1)\\&-15e^{s_1}s_1k_{10}(-s_1,0)+15s_1k_{10}(s_1,-s_1)\\&-30e^{s_1}s_1k_{11}(-s_1,0)-30s_1k_{12}(0,s_1)\end{aligned}\tag{32}$$

$$
\begin{aligned}
&-15 s_1 k_{13}(0, s_1) + 15 e^{s_1} s_1 k_{13}(-s_1, 0) \\
&+15 s_1 k_{13}(s_1, -s_1) - 15 s_1 G_1(s_1) k_3(-s_1) \\
&-15 e^{-s_1} s_1 G_1(s_1) k_3(s_1) + 60 s_1 G_1(s_1) k_6(-s_1) \\
&+60 e^{-s_1} s_1 G_1(s_1) k_7(s_1) - 15 e^{s_1} k_3(-s_1) \\
&-15 k_3(-s_1) - 15 e^{-s_1} k_3(s_1) - 15 k_3(s_1) \\
&+60 k_6(-s_1) + 60 k_6(s_1) + 60 e^{s_1} k_7(-s_1) \\
&+60 e^{-s_1} k_7(s_1) + 60 k_3(0) \\
&-120 k_6(0) - 120 k_7(0)\Big).
\end{aligned}
$$

The restriction of the functional relation (31) to the hyperplane $s_1 + s_2 + s_3 = 0$ yields

$$
\begin{aligned}
&\frac{1}{8} e^{s_1} \partial_2 k_{11}\,(s_2, -s_1 - s_2) - \frac{1}{8} e^{s_1+s_2} \partial_2 k_{12}\,(-s_1 - s_2, s_1) \\
&\quad -\frac{1}{8} \partial_1 k_8\,(s_1, s_2) + \frac{1}{8} e^{s_1+s_2} \partial_1 k_{12}\,(-s_1 - s_2, s_1) \\
&\quad = -\Big(\widetilde{K}_8(s_1, s_2, s_3) - \widetilde{K}_{8,\,\mathrm{s}}(s_1, s_2, s_3)\Big)\,\big|_{s_3=-s_1-s_2},
\end{aligned} \tag{33}
$$

where

$$
\begin{aligned}
\widetilde{K}_{8,\,\mathrm{s}}(s_1, s_2, s_3) = {} & \frac{1}{8\,(s_1+s_2+s_3)}\Big(- k_8\,(s_1, s_2) + e^{s_1+s_2+s_3} k_8\,(-s_2 - s_3, s_2) \\
& - e^{s_1} k_{11}\,(s_2, -s_1 - s_2) + e^{s_1} k_{11}\,(s_2, s_3) \\
& - e^{s_1+s_2} k_{12}\,(-s_1 - s_2, s_1) + e^{s_1+s_2} k_{12}\,(s_3, -s_2 - s_3)\Big).
\end{aligned}
$$

In order to see the general structure in a 4-variable case, we just mention that the restriction to the hyperplane $s_1 + s_2 + s_3 + s_4 = 0$ of the functional relation corresponding to the function $\widetilde{K}_{17}$ gives

$$
\begin{aligned}
&-\frac{1}{16} e^{s_1+s_2} \partial_3 k_{17}\,(s_3, -s_1 - s_2 - s_3, s_1) \\
&\quad +\frac{1}{16} e^{s_1} \partial_3 k_{19}\,(s_2, s_3, -s_1 - s_2 - s_3) \\
&\quad +\frac{1}{16} e^{s_1+s_2} \partial_2 k_{17}\,(s_3, -s_1 - s_2 - s_3, s_1) \\
&\quad -\frac{1}{16} e^{s_1+s_2+s_3} \partial_2 k_{19}\,(-s_1 - s_2 - s_3, s_1, s_2)
\end{aligned} \tag{34}
$$

$$-\frac{1}{16}\partial_1 k_{17}(s_1, s_2, s_3) + \frac{1}{16}e^{s_1+s_2+s_3}\partial_1 k_{19}(-s_1 - s_2 - s_3, s_1, s_2)$$
$$= -\left(\widetilde{K}_{17}(s_1, s_2, s_3, s_4) - \widetilde{K}_{17,\mathrm{s}}(s_1, s_2, s_3, s_4)\right)\Big|_{s_4=-s_1-s_2-s_3},$$

where

$$\widetilde{K}_{17,\mathrm{s}}(s_1, s_2, s_3, s_4) = \frac{1}{16(s_1 + s_2 + s_3 + s_4)}
\Big(- k_{17}(s_1, s_2, s_3)
- e^{s_1+s_2}k_{17}(s_3, -s_1 - s_2 - s_3, s_1)
+ e^{s_1+s_2}k_{17}(s_3, s_4, -s_2 - s_3 - s_4)
+ e^{s_1+s_2+s_3+s_4}k_{17}(-s_2 - s_3 - s_4, s_2, s_3)
- e^{s_1}k_{19}(s_2, s_3, -s_1 - s_2 - s_3) + e^{s_1}k_{19}(s_2, s_3, s_4)
- e^{s_1+s_2+s_3}k_{19}(-s_1 - s_2 - s_3, s_1, s_2)
+ e^{s_1+s_2+s_3}k_{19}(s_4, -s_2 - s_3 - s_4, s_2)\Big).$$

6.3 Action of cyclic groups in the differential system, invariant expressions and simple flow of the system

In the partial differential system of the form given by (32), (33), (34) the action of the cyclic groups $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ is involved. For example, in (32) one can see very easily that $\mathbb{Z}/2\mathbb{Z}$ is acting by

$$T_2(s_1) = -s_1, \qquad s_1 \in \mathbb{R}.$$

Using this fact, in [17], symmetries of some lengthy expressions are explored, which we recall in this subsection.

Theorem 6.1 *For any integers j_0, j_1 in $\{3, 4, 5, 6, 7\}$,*

$$e^{-\frac{s_1}{2}}\left(-(k'_{j_0}(s_1) + k'_{j_1}(s_1)) + e^{s_1}\left(k'_{j_0}(-s_1) + k'_{j_1}(-s_1)\right)\right),$$

is in the kernel of $1 + T_2$. Moreover, considering the finite difference expressions in the differential system corresponding to the following cases, one can find explicitly finite differences of the k_j that are in the kernel of $1 + T_2$:

(1) *When* $(j_0, j_1) = (3, 3)$.
(2) *When* $(j_0, j_1) = (4, 4)$.

(3) *When* $(j_0, j_1) = (5, 5)$.
(4) *When* $(j_0, j_1) = (6, 7)$.

In (33), the action of $\mathbb{Z}/3\mathbb{Z}$ is involved as we have the following transformation acting on the variables:

$$T_3(s_1, s_2) = (-s_1 - s_2, s_1). \tag{35}$$

Using the latter, symmetries of more complicated expressions are discovered in [17]:

Theorem 6.2 *For any integers* j_0, j_1, j_2 *in* $\{8, 9, \ldots, 16\}$,

$$\begin{aligned} e^{-\frac{2s_1}{3}-\frac{s_2}{3}}\Big(&- \partial_1(k_{j_0} + k_{j_1} + k_{j_2})\,(s_1, s_2) \\ &-e^{s_1+s_2}(\partial_2 - \partial_1)(k_{j_0} + k_{j_1} + k_{j_2})\,(-s_1 - s_2, s_1) \\ &+e^{s_1}\partial_2(k_{j_0} + k_{j_1} + k_{j_2})\,(s_2, -s_1 - s_2)\Big) \end{aligned}$$

is in the kernel of $1 + T_3 + T_3^2$. *Also there are finite differences of the functions* k_j *associated with the following cases that are in the kernel of* $1 + T_3 + T_3^2$*:*

(1) *When* $(j_0, j_1, j_2) = (8, 12, 11)$.
(2) *When* $(j_0, j_1, j_2) = (9, 13, 10)$.
(3) *When* $(j_0, j_1, j_2) = (14, 16, 15)$.

The action of $\mathbb{Z}/4\mathbb{Z}$ in the partial differential system can be seen in (34) since the following transformation is involved:

$$T_4(s_1, s_2, s_3) = (-s_1 - s_2 - s_3, s_1, s_2).$$

The symmetries of the functions with respect to this action are also analyzed in [17]:

Theorem 6.3 *For any pair of integers* j_0, j_1 *in* $\{17, 18, 19, 20\}$,

$$\begin{aligned} e^{-\frac{3s_1}{4}-\frac{s_2}{2}-\frac{s_3}{4}}\Big(&- \partial_1(k_{j_0} + k_{j_1})\,(s_1, s_2, s_3) \\ &-e^{s_1+s_2+s_3}(\partial_2 - \partial_1)(k_{j_0} + k_{j_1})(-s_1 - s_2 - s_3, s_1, s_2) \\ &-e^{s_1+s_2}(\partial_3 - \partial_2)(k_{j_0} + k_{j_1})(s_3, -s_1 - s_2 - s_3, s_1) \\ &+e^{s_1}\partial_3(k_{j_0} + k_{j_1})(s_2, s_3, -s_1 - s_2 - s_3)\Big) \end{aligned}$$

is in the kernel of $1 + T_4 + T_4^2 + T_4^3$. *Moreover, there are expressions given by finite differences of the* k_j *corresponding to the following cases that are in the kernel of* $1 + T_4 + T_4^2 + T_4^3$*:*

(1) *When* $(j_0, j_1) = (17, 19)$.
(2) *When* $(j_0, j_1) = (18, 18)$.
(3) *When* $(j_0, j_1) = (20, 20)$.

Moreover, in [17], it is shown that a very simple flow defined by

$$(s_1, s_2, \ldots, s_n) \mapsto (s_1 + t, s_2, \ldots, s_n),$$

combined with the action of the cyclic groups as described above, can be used to write the differential part of the partial differential system. In order to illustrate the idea, we just mention that, for example, in the case that the action of $\mathbb{Z}/3\mathbb{Z}$ is involved via the transformation (35), one defines the orbit $\mathcal{O}k$ of any 2-variable function k by

$$\mathcal{O}k(s_1, s_2) = (k(s_1, s_2), k(-s_1 - s_2, s_1), k(s_2, -s_1 - s_2)) \, .$$

Then one has to use the auxiliary function

$$\alpha_2(s_1, s_2) = e^{-\frac{2s_1}{3} - \frac{s_2}{3}},$$

to write

$$\left(\frac{d}{dt} \, \Big|_{t=0} \, \mathcal{O}k(s_1 + t, s_2) \right) \cdot (\mathcal{O}\alpha_2(s_1, s_2))$$

as a finite difference expression when $k = k_{j_0} + k_{j_1} + k_{j_2}$ and (j_0, j_1, j_2) is either (8, 12, 11), (9, 13, 10), or (14, 16, 15). One can refer to §4.3 of [17] for more details and to see the treatment of all cases in detail.

6.4 *Gradient calculations leading to functional relations*

Here we explain how the functional relations written in Section 6.1 were derived in [17]. In fact the idea comes from [21], where a fundamental identity was proved and by means of a functional relation, the 2-variable function of the scalar curvature term a_2 was written in terms of its 1-variable function. The main identity to use from [21] is that, if we consider the conformally perturbed Laplacian,

$$\triangle_h = e^{h/2} \triangle e^{h/2}.$$

then for the spectral zeta function defined by

$$\zeta_h(a, s) = \mathrm{Tr}(a \, \triangle_h^{-s}), \qquad s \in \mathbb{C}, \;\; \Re(s) \gg 0,$$

one has

$$\frac{d}{d\varepsilon} \, \Big|_{\varepsilon=0} \, \zeta_{h+\varepsilon a}(1, s) = -\frac{s}{2} \, \zeta_h \left(\widetilde{a}, s \right), \tag{36}$$

where

$$\widetilde{a} = \int_{-1}^{1} e^{uh/2} a e^{-uh/2}\, du.$$

One can then see that

$$\zeta_h(a,-1) = -\varphi_0(a\, a_4(h)), \qquad a \in C^\infty(\mathbb{T}_\theta^2),\ \ h = h^* \in C^\infty(\mathbb{T}_\theta^2).$$

Therefore, it follows from (36) that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \varphi_0(a_4(h+\varepsilon a)) = -\frac{1}{2}\zeta_h(\widetilde{a},-1) = \frac{1}{2}\varphi_0(\widetilde{a}\, a_4(h)) = -\varphi_0\left(a e^h\, \widetilde{a}_4(h)\right). \tag{37}$$

where $\widetilde{a}_4(h)$ is given by the same formula as (25) when the functions $K_j(s_1,\ldots,s_n)$ are replaced by

$$\widetilde{K}_j(s_1,\ldots,s_n) = \frac{1}{2^n}\frac{\sinh((s_1+\cdots+s_n)/2)}{(s_1+\cdots+s_n)/2} K_j(s_1,\ldots,s_n).$$

Hence, the gradient $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \varphi_0(a_4(h+\varepsilon a))$ can be calculated mainly by using the important identity (37).

There is a second way of calculating the gradient $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \varphi_0(a_4(h+\varepsilon a))$ which yields finite difference expressions. For this approach a series of lemmas were necessary as proved in [17], which are of the following type.

Lemma 6.2 *For any smooth function $L(s_1,s_2,s_3)$ and any elements x_1,x_2,x_3,x_4 in $C(\mathbb{T}_\theta^2)$, under the trace φ_0, one has:*

$$\begin{aligned} e^h &\left(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} L(\nabla_\varepsilon,\nabla_\varepsilon,\nabla_\varepsilon)(x_1\cdot x_2\cdot x_3)\right) x_4 \\ &= a e^h L^{\varepsilon}_{3,1}(\nabla,\nabla,\nabla,\nabla)(x_1\cdot x_2\cdot x_3\cdot x_4) \\ &\quad + a e^h L^{\varepsilon}_{3,2}(\nabla,\nabla,\nabla,\nabla)(x_2\cdot x_3\cdot x_4\cdot x_1) \\ &\quad + a e^h L^{\varepsilon}_{3,3}(\nabla,\nabla,\nabla,\nabla)(x_3\cdot x_4\cdot x_1\cdot x_2) \\ &\quad + a e^h L^{\varepsilon}_{3,4}(\nabla,\nabla,\nabla,\nabla)(x_4\cdot x_1\cdot x_2\cdot x_3), \end{aligned}$$

where

$$L^{\varepsilon}_{3,1}(s_1,s_2,s_3,s_4) := e^{s_1+s_2+s_3+s_4}\frac{L(-s_2-s_3-s_4,s_2,s_3)-L(s_1,s_2,s_3)}{s_1+s_2+s_3+s_4},$$

$$L^{\varepsilon}_{3,2}(s_1,s_2,s_3,s_4) := e^{s_1+s_2+s_3}\frac{L(s_4,-s_2-s_3-s_4,s_2)-L(-s_1-s_2-s_3,s_1,s_2)}{s_1+s_2+s_3+s_4},$$

$$L^{\varepsilon}_{3,3}(s_1, s_2, s_3, s_4) := e^{s_1+s_2} \frac{L(s_3, s_4, -s_2 - s_3 - s_4) - L(s_3, -s_1 - s_2 - s_3, s_1)}{s_1 + s_2 + s_3 + s_4},$$

$$L^{\varepsilon}_{3,4}(s_1, s_2, s_3, s_4) := e^{s_1} \frac{L(s_2, s_3, s_4) - L(s_2, s_3, -s_1 - s_2 - s_3)}{s_1 + s_2 + s_3 + s_4}.$$

Also, in order to perform necessary manipulations in the second calculation of the gradient $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \varphi_0(a_4(h + \varepsilon a))$, one needs lemmas of this type:

Lemma 6.3 *For any smooth function $L(s_1, s_2, s_3)$ and any elements x_1, x_2, x_3 in $C(\mathbb{T}^2_\theta)$, one has:*

$$\begin{aligned}
&\delta_j \left(L(\nabla, \nabla, \nabla)(x_1 \cdot x_2 \cdot x_3)\right) \\
&\quad = L(\nabla, \nabla, \nabla)(\delta_j(x_1) \cdot x_2 \cdot x_3) + L(\nabla, \nabla, \nabla)(x_1 \cdot \delta_j(x_2) \cdot x_3) \\
&\qquad + L(\nabla, \nabla, \nabla)(x_1 \cdot x_2 \cdot \delta_j(x_3)) + L^{\delta}_{3,1}(\nabla, \nabla, \nabla, \nabla)(\delta_j(h) \cdot x_1 \cdot x_2 \cdot x_3) \\
&\qquad + L^{\delta}_{3,2}(\nabla, \nabla, \nabla, \nabla)(x_1 \cdot \delta_j(h) \cdot x_2 \cdot x_3) \\
&\qquad + L^{\delta}_{3,3}(\nabla, \nabla, \nabla, \nabla)(x_1 \cdot x_2 \cdot \delta_j(h) \cdot x_3) \\
&\qquad + L^{\delta}_{3,4}(\nabla, \nabla, \nabla, \nabla)(x_1 \cdot x_2 \cdot x_3 \cdot \delta_j(h)),
\end{aligned}$$

where

$$L^{\delta}_{3,1}(s_1, s_2, s_3, s_4) := \frac{L(s_2, s_3, s_4) - L(s_1 + s_2, s_3, s_4)}{s_1},$$

$$L^{\delta}_{3,2}(s_1, s_2, s_3, s_4) := \frac{L(s_1 + s_2, s_3, s_4) - L(s_1, s_2 + s_3, s_4)}{s_2},$$

$$L^{\delta}_{3,3}(s_1, s_2, s_3, s_4) := \frac{L(s_1, s_2 + s_3, s_4) - L(s_1, s_2, s_3 + s_4)}{s_3},$$

$$L^{\delta}_{3,4}(s_1, s_2, s_3, s_4) := \frac{L(s_1, s_2, s_3 + s_4) - L(s_1, s_2, s_3)}{s_4}.$$

After performing the second gradient calculation in [17], and comparing it with the first calculation based on (37), the functional relations were derived conceptually.

6.5 *The term a_4 for non-conformally flat metrics on noncommutative four tori*

It was illustrated in [17] that, having the calculation of the term a_4 for the noncommutative two torus in place, one can conveniently write a formula for the term a_4 of a non-conformally flat metric on the noncommutative four torus that is the product of two noncommutative two tori. The metric is the noncommutative analog of the following metric. Let $(x_1, y_1, x_2, y_2) \in \mathbb{T}^4 = (\mathbb{R}/2\pi\mathbb{Z})^4$ be the coordinates of the ordinary four torus and consider the metric

$$g = e^{-h_1(x_1,y_1)} \left(dx_1^2 + dy_1^2\right) + e^{-h_2(x_2,y_2)} \left(dx_2^2 + dy_2^2\right),$$

where h_1 and h_2 are smooth real valued functions. The Weyl tensor is conformally invariant, and one can assure that the above metric is not conformally flat by calculating the components of its Weyl tensor and observing that they do not all vanish. The non-vanishing components are:

$$\begin{aligned} C_{1212} = &\frac{1}{6} e^{-h_1(x_1,y_1)} \partial_{y_1}^2 h_1(x_1, y_1) \\ &+ \frac{1}{6} e^{h_2(x_2,y_2)-2h_1(x_1,y_1)} \partial_{y_2}^2 h_2(x_2, y_2) \\ &+ \frac{1}{6} e^{-h_1(x_1,y_1)} \partial_{x_1}^2 h_1(x_1, y_1) \\ &+ \frac{1}{6} e^{h_2(x_2,y_2)-2h_1(x_1,y_1)} \partial_{x_2}^2 h_2(x_2, y_2), \end{aligned}$$

$$C_{1313} = -\frac{1}{2} e^{-h_2(x_2,y_2)+h_1(x_1,y_1)} C_{1212},$$

$$C_{2424} = C_{2323} = C_{1414} = C_{1313},$$

$$C_{3434} = e^{-2h_2(x_2,y_2)+2h_1(x_1,y_1)} C_{1212}.$$

Now, one can consider a noncommutative four torus of the form $\mathbb{T}^2_{\theta'} \times \mathbb{T}^2_{\theta''}$ that is the product of two noncommutative two tori. Its algebra has four unitary generators U_1, V_1, U_2, V_2 with the following relations: each element of the pair (U_1, V_1) commutes with each element of the pair (U_2, V_2), and there are fixed irrational real numbers θ' and θ'' such that:

$$V_1 U_1 = e^{2\pi i\theta'} U_1 V_1, \qquad V_2 U_2 = e^{2\pi i\theta''} U_2 V_2.$$

One can then choose conformal factors $e^{-h'}$ and $e^{-h''}$, where h' and h'' are selfadjoint elements in $C^\infty(\mathbb{T}^2_{\theta'})$ and $C^\infty(\mathbb{T}^2_{\theta''})$, respectively, and use them to perturb the flat metric of each component conformally. Then the Laplacian of the product geometry is given by

$$\triangle_{\varphi',\varphi''} = \triangle_{\varphi'} \otimes 1 + 1 \otimes \triangle_{\varphi''},$$

where $\triangle_{\varphi'}$ and $\triangle_{\varphi''}$ are, respectively, the Laplacians of the perturbed metrics on $\mathbb{T}^2_{\theta'}$ and $\mathbb{T}^2_{\theta''}$. Now one can use a simple Kunneth formula to find the term a_4 in the asymptotic expansion

$$\mathrm{Tr}(a\exp(-t\triangle_{\varphi',\varphi''})) \sim t^{-2}\Big((\varphi_0'\otimes\varphi_0'')(a\,a_0) + (\varphi_0'\otimes\varphi_0'')(a\,a_2)\,t \\ +(\varphi_0'\otimes\varphi_0'')(a\,a_4)\,t^2 + \cdots\Big) \tag{38}$$

in terms of the known terms appearing in the following expansions:

$$\mathrm{Tr}(a'\exp(-t\triangle_{\varphi'})) \sim t^{-1}\left(\varphi_0'(a'\,a_0') + \varphi_0'(a'\,a_2')\,t + \varphi_0'(a'\,a_4')\,t^2 + \cdots\right),$$
$$\mathrm{Tr}(a''\exp(-t\triangle_{\varphi''})) \sim t^{-1}\left(\varphi_0''(a''\,a_0'') + \varphi_0''(a''\,a_2'')\,t + \varphi_0''(a''\,a_4'')\,t^2 + \cdots\right).$$

The general formula is

$$a_{2n} = \sum_{i=0}^{n} a_{2i}'\otimes a_{2(n-i)}'' \in C^\infty(\mathbb{T}^2_{\theta'}\times\mathbb{T}^2_{\theta''}),$$

hence an explicit formula for a_4 of the noncommutative four torus with the product geometry explained above since there are explicit formulas for its ingredients.

In this case of the non-conformally flat metric on the product geometry, two modular automorphisms are involved in the formulas for the geometric invariants and this motivates further systematic research on *twistings* that involve two-dimensional modular structures, cf. [13].

7 Twisted spectral triples and Chern-Gauss-Bonnet theorem for ergodic C^*-dynamical systems

This section is devoted to the notion of twisted spectral triples and some details of their appearance in the context of noncommutative conformal geometry. In particular we explain construction of twisted spectral triples for ergodic C^*-dynamical systems and the validity of the Chern-Gauss-Bonnet theorem in this vast setting.

7.1 Twisted spectral triples

The notion of twisted spectral triples was introduced in [20] to incorporate the study of type III algebras using noncommutative differential geometric techniques. In the definition of this notion, in addition to a triple (A, H, D) of a $*$-algebra A, a

Hilbert space H, and an unbounded operator D on H which plays the role of the Dirac operator, one has to bring into the picture an automorphism σ of A which interacts with the data as follows. Instead of the ordinary commutators $[D, a]$ as in the definition of an ordinary spectral triple, in the twisted case one asks for the boundedness of the twisted commutators $[D, a]_\sigma = Da - \sigma(a)D$. More precisely, here also one assumes a representation of A by bounded operators on H such that the operator $Da - \sigma(a)D$ is defined on the domain of D for any $a \in A$, and that it extends by continuity to a bounded operator on H.

This twisted notion of a spectral triple is essential for type III examples as this type of algebras do not possess non-zero trace functionals, and ordinary spectral triples with suitable properties cannot be constructed over them for the following reason [20]. If (A, H, D) is an m^+-summable ordinary spectral triple then the following linear functional on A defined by

$$a \mapsto \mathrm{Tr}_\omega(a|D|^{-m})$$

gives a trace. The main reason for this is that the kernel of the Dixmier trace Tr_ω is a large kernel that contains all operators of the form $|D|^{-m}a - a|D|^{-m}$, $a \in A$, if the ordinary commutators are bounded. In fact we are using the *regularity* assumption on the spectral triple, which in particular requires boundedness of the commutators of elements of A with $|D|$ as well as with D (indeed this is a natural condition and the main point is that one is using ordinary commutators). Hence, trace-less algebras cannot fit into the paradigm of ordinary spectral triples.

It is quite amazing that in [20], examples are provided where one can obtain boundedness of twisted commutators $Da - \sigma(a)D$ and $|D|a - \sigma(a)|D|$ for all elements a of the algebra by means of an algebra automorphism σ, where the Dirac operator D has the m^+-summability property. Then they use the boundedness of the twisted commutators to show that operators of the form $|D|^{-m}a - \sigma^{-m}(a)|D|^{-m}$ are in the kernel of the Dixmier trace and the linear functional $a \mapsto \mathrm{Tr}_\omega(a|D|^{-m})$ yields a twisted trace on A.

7.2 Conformal perturbation of a spectral triple

One of the main examples in [20] that demonstrates the need for the notion of twisted spectral triples in noncommutative geometry is related to conformal perturbation of Riemannian metrics. That is, if D is the Dirac operator of a spin manifold equipped with a Riemannian metric g, then, after a conformal perturbation of the metric to $\tilde{g} = e^{-4h}g$ by means of a smooth real valued function h on the manifold, the Dirac operator of the perturbed metric $\tilde{g}$ is unitarily equivalent to the operator

$$\tilde{D} = e^h D e^h.$$

So this suggests that given an ordinary spectral triple (A, H, D) with a noncommutative algebra A, since the metric is encoded in the analog D of the Dirac operator, one can implement conformal perturbation of the metric by fixing a self-adjoint element $h \in A$ and by then replacing D with $\tilde{D} = e^h D e^h$. However, it turns out that the triple $(A, H, \tilde{D})$ is not necessarily a spectral triple any more, since, because of noncommutativity of A, the commutators $[\tilde{D}, a]$, $a \in A$, are not necessarily bounded operators. Despite this, interestingly, the remedy brought forth in [20] is to introduce the automorphism

$$\sigma(a) = e^{2h} a e^{-2h}, \qquad a \in A,$$

which yields the bounded twisted commutators

$$[\tilde{D}, a]_\sigma = \tilde{D} a - \sigma(a) \tilde{D}, \qquad a \in A.$$

7.3 *Conformal perturbation of the flat metric on* $\mathbb{T}_\theta^2$

Another important example, which is given in [22], shows that twisted spectral triples can arise in a more intrinsic manner, compared to the example we just illustrated, when a conformal perturbation is implemented. In [22], the flat geometry of $\mathbb{T}_\theta^2$ is perturbed by a fixed conformal factor e^{-h}, where $h = h^* \in C^\infty(\mathbb{T}_\theta^2)$. This is done by replacing the canonical trace φ_0 on $C(\mathbb{T}_\theta^2)$ (playing the role of the volume form) by the tracial state $\varphi(a) = \varphi_0(a e^{-h})$, $a \in C(\mathbb{T}_\theta^2)$. In order to represent the opposite algebra of $C(\mathbb{T}_\theta^2)$ on the Hilbert space $\mathcal{H}_\varphi$, obtained from $C(\mathbb{T}_\theta^2)$ by the GNS construction, one has to modify the ordinary action induced by right multiplication. That is, one has to consider the action defined by

$$a^{op} \cdot \xi = \xi e^{-h/2} a e^{h/2}.$$

It then turns out that with the new action, the ordinary commutators $[D, a]$, $a \in C^\infty(\mathbb{T}_\theta^2)^{op}$, are not bounded any more, where D is the Dirac operator

$$D = \begin{pmatrix} 0 & \partial_\varphi^* \\ \partial_\varphi & 0 \end{pmatrix} : \mathcal{H} \to \mathcal{H}.$$

Here,

$$\partial_\varphi = \delta_1 + i\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)},$$

where $\mathcal{H}^{(1,0)}$, the analogue of $(1, 0)$-forms, is the Hilbert space completion of finite sums $\sum a\partial(b)$, $a, b \in A_\theta^\infty$, with respect to the inner product

$$(a\partial b, c\partial d) = \varphi_0(c^* a(\partial b)(\partial d)^*),$$

and

$$\mathcal{H} = \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)}.$$

The remedy for obtaining bounded commutators is to use a twist given by the automorphism

$$\sigma(a^{op}) = e^{-h/2} a e^{h/2},$$

which leads to bounded twisted commutators [22]

$$[D, a^{op}]_\sigma = Da^{op} - \sigma(a^{op})D, \qquad a \in C^\infty(\mathbb{T}^2_\theta)^{op}.$$

7.4 Conformally twisted spectral triples for C*-dynamical systems

The example in Section 7.3 inspired the construction of twisted spectral triples for general ergodic C^*-dynamical systems in [29]. The Dirac operator used in this work, following more closely the geometric approach taken originally in [9], is the analog of the de Rham operator. An important reason for this choice is that an important goal in [29] was to confirm the validity of the analog of the Chern-Gauss-Bonnet theorem in the vast setting of ergodic C^*-dynamical systems.

In this subsection we consider a C^*-algebra $\mathcal{A}$ with a strongly continuous ergodic action α of a compact Lie group G of dimension n, and we let $\mathcal{A}^\infty$ denote the smooth subalgebra of $\mathcal{A}$, which is defined as:

$$\mathcal{A}^\infty = \{a \in \mathcal{A} : \text{the map } g \mapsto \alpha_g(a) \text{ is in } C^\infty(G, A)\}.$$

Following closely the construction in [9], we can define a space of *differential forms* on $\mathcal{A}$ by using the exterior powers of $\mathfrak{g}^*$, namely that for $k = 0, 1, 2, \ldots, n$, we set:

$$\Omega^k(\mathcal{A}, G) = \mathcal{A} \otimes \bigwedge^k \mathfrak{g}^*, \tag{39}$$

where $\mathfrak{g}^*$ is the linear dual of the Lie algebra $\mathfrak{g}$ of the Lie group G. We consider the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}^*$ induced by the Killing form, and extend it to an inner product on $\bigwedge^k \mathfrak{g}^*$ by setting

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det(\langle v_i, w_j \rangle).$$

After fixing an orthonormal basis $(\omega_j)_{j=1,\dots,n}$ for $\mathfrak{g}^*$, we equip the above differential forms with an exterior derivative $d : \Omega^k(\mathcal{A}, G) \to \Omega^{k+1}(\mathcal{A}, G)$ given by

$$\begin{aligned} d(a \otimes \omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_k}) = & \sum_{j=1}^{n} \partial_j(a) \otimes \omega_j \wedge \omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_k} \\ & - \frac{1}{2} \sum_{j=1}^{k} \sum_{\alpha,\beta} (-1)^{j+1} c^{i_j}_{\alpha\beta} a \otimes \omega_\alpha \wedge \omega_\beta \\ & \wedge \omega_{i_1} \wedge \cdots \wedge \omega_{i_{j-1}} \wedge \omega_{i_{j+1}} \wedge \cdots \wedge \omega_{i_k}, \end{aligned} \tag{40}$$

where the coefficients $c^i_{\alpha\beta}$ are the *structure constants* of the Lie algebra $\mathfrak{g}$ determined by the relations

$$[\partial_\alpha, \partial_\beta] = \sum_{i=1}^{n} c^i_{\alpha\beta} \partial_i$$

for the predual $(\partial_j)_{j=1,\dots,n}$ of $(\omega_j)_{j=1,\dots,n}$. This exterior derivative satisfies $d \circ d = 0$ on $\Omega^\bullet(\mathcal{A}, G)$, therefore we have a complex $(\Omega^\bullet(\mathcal{A}, G), d)$. This complex is called the Chevalley-Eilenberg cochain complex with coefficients in $\mathcal{A}$, one can refer to [44] for more details.

We now define an inner product on $\Omega^k(\mathcal{A}, G)$, for which we make use of the unique G-invariant tracial state φ_0 on $\mathcal{A}$, see [37]. The inner product is defined by

$$(a \otimes v_1 \wedge \cdots \wedge v_k, a' \otimes w_1 \wedge \cdots \wedge w_k)_0 = \varphi_0(a^* a') \det(\langle v_i, w_j \rangle). \tag{41}$$

We denote the Hilbert space completion of $\Omega^k(\mathcal{A}, G)$ with respect to this inner product by $\mathcal{H}_{k,0}$.

In order to implement a conformal perturbation, we fix a selfadjoint element $h \in \mathcal{A}^\infty$, define the following new inner product on $\Omega^k(\mathcal{A}, G)$:

$$(a \otimes v_1 \wedge \cdots \wedge v_k, a' \otimes w_1 \wedge \cdots \wedge w_k)_h = \varphi_0(a^* a' e^{(n/2-k)h}) \det(\langle v_i, w_j \rangle), \tag{42}$$

and denote the associated Hilbert space by $\mathcal{H}_{k,h}$.

One of the goals is to construct ordinary and twisted spectral triples by using the unbounded operator $d + d^*$, the analog of the de Rham operator, acting on the direct sum of all $\mathcal{H}_{k,h}$. Here the adjoint d^* of d is of course taken with respect to the conformally perturbed inner product $(\cdot, \cdot)_h$. The Hilbert spaces are simply related by the unitary maps $U_k : \mathcal{H}_{k,0} \to \mathcal{H}_{k,h}$ given on degree k forms by:

$$U_k(a \otimes v_1 \wedge \cdots \wedge v_k) = a e^{-(n/2-k)h/2} \otimes v_1 \wedge \cdots \wedge v_k.$$

Therefore, for simplicity, we use these unitary maps to transfer the operator $d + d^*$ to an unbounded operator D acting on the Hilbert space $\mathcal{H}$ that is the direct sum of all $\mathcal{H}_{k,0}$. We can now state the following result from [29].

Theorem 7.1 *Consider the above constructions associated with a C^*-algebra $\mathcal{A}$ with an ergodic action of an n-dimensional Lie group G. The operator D has a selfadjoint extension which is n^+-summable. With the representation of $\mathcal{A}^\infty$ on $\mathcal{H} = \bigoplus_k \mathcal{H}_{k,0}$ induced by left multiplication, the triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ is an ordinary spectral triple. However, when one represents the opposite algebra of $\mathcal{A}^\infty$ on $\mathcal{H}$ using multiplication from right, one obtains a twisted spectral triple with respect to the automorphism defined by $\beta(a^{op}) = e^h a e^{-h}$.*

The spectral triples described in the above theorem can in fact be equipped with the grading operator given by

$$\gamma(a \otimes v_1 \wedge \cdots \wedge v_k) = (-1)^k (a \otimes v_1 \wedge \cdots \wedge v_k).$$

Related to this grading, it is interesting to study the Fredholm index of the operator D, which is unitarily equivalent to $d + d^*$, when viewed as an operator from the direct sum of all even differential forms to the direct sum of all odd differential forms. We shall discuss this issue shortly.

7.5 The Chern-Gauss-Bonnet theorem for C^*-dynamical systems

In Section 4 we briefly discussed the Gauss-Bonnet theorem for surfaces, which states that for any closed oriented two-dimensional Riemannian manifold Σ with scalar curvature R, one has

$$\frac{1}{4\pi} \int_\Sigma R = \chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ. The Chern-Gauss-Bonnet theorem generalizes this result to higher even dimensional manifolds. That is, in higher dimensions as well, the Euler characteristic, which is a topological invariant, coincides with the integral of a certain geometric invariant, namely the *Pfaffian* of the curvature form. Given a closed oriented Riemannian manifold M of even dimension n, consider the Levi-Civita connection, which is the unique torsion-free metric-compatible connection on the tangent bundle TM. Let us denote the matrix of local 2-forms representing the curvature of this connection by Ω. The Chern-Gauss-Bonnet theorem states that the Pfaffian of Ω (the square root of the determinant defined on the space of anti-symmetric matrices) integrates over the manifold to the Euler characteristic of the manifold, up to multiplication by a universal constant:

$$\frac{1}{(2\pi)^n}\int_M \mathrm{Pf}(\Omega) = \chi(M).$$

Interestingly, there is a spectral way of interpreting such relations between local geometry and topology of manifolds. Relevant to our discussion is indeed the Fredholm index of the de Rham operator $d + d^*$, where d is the de Rham exterior derivative and d^* is its adjoint with respect to the metric on the differential forms induced by the Riemannian metric. The Fredholm index of $d + d^*$ should be calculated when the operator is viewed as a map from the direct sum of all even differential forms to the direct sum of all odd differential forms on M:

$$d + d^* : \Omega^{even} M = \bigoplus \Omega^{2k} M \to \Omega^{odd} M = \bigoplus \Omega^{2k+1} M$$

The index of this operator is certainly an important geometric quantity since the adjoint d^* of d heavily depends on the choice of metric on the manifold. Amazingly, using the *Hodge decomposition theorem*, one can find a canonical identification of the de Rham cohomology group $H^k(M)$ with the kernel of the Laplacian $\triangle_k = d^*d + dd^* : \Omega^k M \to \Omega^k M$. This can then be used to show that the index of $d + d^*$ is equal to the Euler characteristic of M. Moreover, one can appeal to the *McKean-Singer index theorem* to find curvature related invariants appearing in small time heat kernel expansions associated with $d + d^*$ to see that the index is given by the integral of curvature related invariants.

In [29], this spectral approach is taken to show that the analog of the Chern-Gauss-Bonnet theorem can be established for ergodic C^*-dynamical systems. Let us consider the setup and the constructions presented in Section 7.4 for a C^*-algebra $\mathcal{A}$ with an ergodic action of a compact Lie group G of dimension n. Then, one of the main results proved in [29] is the following statement. Here, d is given by (40), $h = h^* \in \mathcal{A}^\infty$ is the element that was used to implement with e^h a conformal perturbation of the metric, and the Hilbert space $\mathcal{H}_{k,h}$ is the completion of the k-differential forms $\Omega^k(\mathcal{A}, G)$ with respect to the perturbed metric.

Theorem 7.2 *The Fredholm index of the operator*

$$d + d^* : \bigoplus_k \mathcal{H}_{2k,h} \to \bigoplus_k \mathcal{H}_{2k+1,h}$$

is equal to the Euler characteristic $\chi(A, G)$ *of the complex* $(\Omega^\bullet(A, G), d)$*. Since* $\chi(A, G) = \sum_k (-1)^k dim\left(H^k(A, G)\right)$ *is the alternating sum of the dimensions of the cohomology groups, the index of* $d + d^*$ *is independent of the conformal factor* e^h *used for perturbing the metric.*

8 The Ricci curvature

Classically, scalar curvature is only a deem shadow of the full Riemann curvature tensor. In fact there is no evidence that Riemann considered anything else but the full curvature tensor, and, equivalently, the sectional curvature. Both were defined by him for a Riemannian manifold. The Ricci and scalar curvatures were later defined by contracting the Riemann curvature tensor with the metric tensor. Once the metric is given in a local coordinate chart, all three curvature tensors can be computed explicitly via algebraic formulas involving only partial derivatives of the metric tensor. This is a purely algebraic process, with deep geometric and analytic implications. It is also a top-down process, going from the full Riemann curvature tensor, to Ricci curvature, and then to scalar curvature.

The situation in the noncommutative case is reversed and we have to move up the ladder, starting from the scalar curvature first, which is the easiest to define spectrally, being given by the second term of the heat expansion for the scalar Laplacian, the square of the Dirac operator in general. Thus after treating the scalar curvature, which we recalled in previous sections together with examples, one should next try to define and possibly compute, in some cases, a Ricci curvature tensor. But how? In [34] and motivated by the local formulas for the asymptotic expansion of heat kernels in spectral geometry, the authors propose a definition of Ricci curvature in a noncommutative setting. One necessarily has to use the asymptotic expansion of Laplacians on functions and 1-forms and a version of the Weitzenböck formula.

As we shall see in this section, the Ricci operator of an oriented closed Riemannian manifold can be realized as a spectral functional, namely the functional defined by the zeta function of the full Laplacian of the de Rham complex, localized by smooth endomorphisms of the cotangent bundle and their trace. In the noncommutative case, this Ricci functional uniquely determines a density element, called the Ricci density, which plays the role of the Ricci operator. The main result of [34] provides a general definition and an explicit computation of the Ricci density when the conformally flat geometry of the curved noncommutative two torus is encoded in the de Rham spectral triple. In a follow-up paper [24], the Ricci curvature of a noncommutative curved three torus is computed. In this section we explain these recent developments in more detail.

8.1 A Weitzenböck formula

The Weitzenböck formula

$$\textbf{Hodge} - \textbf{Bochner} = \textbf{Ricci}$$

in conjunction with Gilkey's asymptotic expansion gives an opening to define the Ricci curvature in spectral terms. Let M be a closed oriented Riemannian manifold. Consider the de Rham spectral triple

$$(C^\infty(M), L^2(\Omega^{ev}(M)) \oplus L^2(\Omega^{odd}(M)), d+\delta, \gamma),$$

which is the even spectral triple constructed from the de Rham complex. Here d is the exterior derivative, δ is its adjoint acting on the exterior algebra, and γ is the $\mathbb{Z}_2$-grading on forms. The eigenspaces for eigenvalues 1 and -1 of γ are even and odd forms, respectively. The full Laplacian on forms $\Delta = d\delta + \delta d$ is the Laplacian of the Dirac operator $d+\delta$, and is the direct sum of Laplacians on p-forms, $\Delta = \oplus \Delta_p$. As a Laplace type operator, Δ can be written as $\nabla^*\nabla - E$ by Weitzenböck formula, where ∇ is the Levi-Civita connection extended to all forms and

$$E = -\frac{1}{2} c(dx^\mu) c(dx^\nu) \Omega(\partial_\mu, \partial_\nu).$$

Here c denotes the Clifford multiplication and Ω is the curvature operator of the Levi-Civita connection acting on exterior algebra. The restriction of E to one forms gives the Ricci operator.

8.2 *Ricci curvature as a spectral functional*

The Ricci curvature of a Riemannian manifold (M^m, g) is originally defined as follows. Let ∇ be the Levi-Civita connection of the metric g. The Riemannian operator and the curvature tensor are defined for vector fields X, Y, Z, W by

$$Riem(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$
$$\mathrm{Riem}(X,Y,Z,W) := g(Riem(X,Y)Z, W).$$

With respect to the coordinate frame $\partial_\mu = \frac{\partial}{\partial x^\mu}$, the components of the curvature tensor are denoted by

$$\mathrm{Riem}_{\mu\nu\rho\epsilon} := \mathrm{Riem}(\partial_\mu, \partial_\nu, \partial_\rho, \partial_\epsilon).$$

The components of the Ricci tensor Ric and scalar curvature R are given by

$$\mathrm{Ric}_{\mu\nu} := g^{\rho\epsilon} \mathrm{Riem}_{\mu\rho\epsilon\nu},$$
$$R := g^{\mu\nu} \mathrm{Ric}_{\mu\nu} = g^{\mu\nu} g^{\rho\epsilon} \mathrm{Riem}_{\mu\rho\epsilon\nu}.$$

Now these algebraic formulas have no chance to be extended to a noncommutative setting in general. One must thus look for a spectral alternative reformulation.

Let $P : C^\infty(V) \to C^\infty(V)$ be a positive elliptic differential operator of order two acting on the sections of a smooth Hermitian vector bundle V over M. The heat trace $\mathrm{Tr}(e^{-tP})$ has a short time asymptotic expansion of the form

$$\mathrm{Tr}(e^{-tP}) \sim \sum_{n=0}^{\infty} a_n(P) t^{\frac{n-m}{2}}, \qquad t \to 0^+,$$

where $a_n(P)$ are integrals of local densities

$$a_n(P) = \int \mathrm{tr}(a_n(x, P))dx.$$

Here $dx = dvol_x$ is the Riemannian volume form and tr is the fiberwise matrix trace. The endomorphism $a_n(x, P)$ can be uniquely determined by localizing the heat trace by an smooth endomorphism F of V. It is easy to see that the asymptotic expansion of the localized heat trace $\mathrm{Tr}(Fe^{-tP})$ is of the form

$$\mathrm{Tr}(Fe^{-tP}) \sim \sum_{n=0}^{\infty} a_n(F, P) t^{\frac{n-m}{2}}, \tag{43}$$

with

$$a_n(F, P) = \int \mathrm{tr}\big(F(x) a_n(x, P)\big)dx. \tag{44}$$

If P is a Laplace type operator i.e., its leading symbol is given by the metric tensor, then the densities $a_n(x, P)$ can be expressed in terms of the Riemannian curvature, an endomorphism E, and their derivatives. The endomorphism E measures how far the operator P is from being the Laplacian $\nabla^*\nabla$ of a connection ∇ on V, that is

$$E = \nabla^*\nabla - P. \tag{45}$$

The first two densities of the heat equation for such P are given by [36, Theorem 3.3.1]

$$a_0(x, P) = (4\pi)^{-m/2}\mathrm{I}, \tag{46}$$

$$a_2(x, P) = (4\pi)^{-m/2} \left(\frac{1}{6} R(x) + E\right). \tag{47}$$

For the scalar Laplacian $\triangle_0$, the connection is the de Rham differential $d : C^\infty(M) \to \Omega^1(M)$, and $E = 0$. Hence the first two first terms of the heat kernel of $\triangle_0$ are given by

$$a_0(x, \triangle_0) = (4\pi)^{-m/2}, \tag{48}$$

$$a_2(x, \triangle_0) = (4\pi)^{-m/2}\,\frac{1}{6}R(x). \tag{49}$$

For Laplacian on one forms $\triangle_1 : \Omega^1(M) \to \Omega^1(M)$, the Hodge-de Rham Laplacian, the connection in (45) is the Levi-Civita connection on the cotangent bundle. The endomorphism E is the negative of the Ricci operator, $E = -\mathrm{Ric}$, on the cotangent bundle, which is defined by raising the first index of the Ricci tensor (denoted by Ric as well),

$$\mathrm{Ric}_x(\alpha^\sharp, X) = \mathrm{Ric}_x(\alpha)(X), \quad \alpha \in T_x^*M,\ X \in T_xM.$$

Therefore, one has

$$a_0(x, \triangle_1) = (4\pi)^{-m/2}\mathrm{I}, \tag{50}$$

$$a_2(x, \triangle_1) = (4\pi)^{-m/2}\left(\frac{1}{6}R(x) - \mathrm{Ric}_x\right). \tag{51}$$

We can use the function $\mathrm{tr}(F)$ to localize the heat trace of the scalar Laplacian $\triangle_0$ and get the identity

$$a_2(\mathrm{tr}(F), \triangle_0) - a_2(F, \triangle_1) = (4\pi)^{-m/2}\int_M \mathrm{tr}\big(F(x)\mathrm{Ric}_x\big)dx. \tag{52}$$

This motivates the following definition.

Definition 8.1 ([34]) *The Ricci functional of the closed Riemannian manifold (M, g) is the functional on $C^\infty(\mathrm{End}(T^*M))$ defined as*

$$\mathcal{R}ic(F) = a_2(\mathrm{tr}(F), \triangle_0) - a_2(F, \triangle_1).$$

Proposition 8.1 *For a closed Riemannian manifold M of dimension m, we have the short time asymptotics*

$$Tr\left(\mathrm{tr}(F)e^{-t\triangle_0}\right) - Tr\left(Fe^{-t\triangle_1}\right) \sim \mathcal{R}ic(F)\,t^{1-\frac{m}{2}}.$$

Proof By (46) and (44), we have $\mathrm{tr}(F)a_0(x, \triangle_0) = \mathrm{tr}(F(x)a_0(x, \triangle_1))$. This implies that

$$a_0(\mathrm{tr}(F), \triangle_0) = a_0(F, \triangle_1), \qquad F \in C^\infty(\mathrm{End}(T^*M)). \tag{53}$$

The asymptotic expansion of the localized heat kernel (43) then shows that the first terms will cancel each other. The difference of the second terms, which are multiples of $t^{1-\frac{m}{2}}$, will become the first term in the asymptotic expansion of the differences of localized heat kernels. □

8.3 *Spectral zeta function and the Ricci functional*

The spectral zeta function of a positive elliptic operator P is defined as

$$\zeta(s, P) = \mathrm{Tr}(P^{-s}(\mathrm{I} - Q)), \quad \Re(s) \gg 0,$$

where Q is the projection on the kernel of P. Its localized version is $\zeta(s, F, P) = \mathrm{Tr}(FP^{-s}(\mathrm{I} - Q))$. These function have a meromorphic extension to the complex plane with isolated simple poles. Using the Mellin transform, one finds explicit relation between residue at the poles and coefficients of the heat kernel. This leads to the following expression for the Ricci functional in terms of zeta functions.

Proposition 8.2 *For a closed Riemannian manifold M of dimension $m > 2$, we have*

$$\mathcal{R}ic(F) = \Gamma\left(\frac{m}{2} - 1\right) \mathrm{res}_{\mathrm{s}=\frac{m}{2}-1}\Big(\zeta(\mathrm{s}, \mathrm{tr(F)}, \triangle_0) - \zeta(\mathrm{s}, \mathrm{F}, \triangle_1)\Big). \tag{54}$$

If M is two-dimensional, then

$$\mathcal{R}ic(F) = \zeta(0, \mathrm{tr}(F), \triangle_0) - \zeta(0, F, \triangle_1) + Tr(\mathrm{tr}(F)Q_0) - Tr(FQ_1), \tag{55}$$

where Q_j is the projection on the kernel of Laplacian $\triangle_j$, $j = 0, 1$.

It follows that the difference of zeta functions $\zeta(s, \mathrm{tr}(F), \triangle_0) - \zeta(s, F, \triangle_1)$ is regular at $m/2$, and its first pole is located at $s = m/2 - 1$.

To work with the Laplacian on one forms, we will use smooth endomorphisms F of the cotangent bundle. The smearing endomorphism $\tilde{F} = \mathrm{tr}(F)\mathrm{I}_0 \oplus F \in C^\infty(\mathrm{End}(\bigwedge^\bullet M))$, where I_0 denotes the identity map on functions, can be used to localize the heat kernel of the full Laplacian and

$$\mathcal{R}ic(F) = a_2(\gamma\tilde{F}, \triangle). \tag{56}$$

With the above notation, one can express the Ricci functional as special values of the (localized) spectral zeta functions

$$\mathcal{R}ic(F) = \begin{cases} \Gamma(\frac{m}{2} - 1)\mathrm{res}_{\mathrm{s}=\frac{m}{2}-1}\zeta(\mathrm{s}, \tilde{\mathrm{F}}\gamma, \triangle) & m > 2, \\ \\ \zeta(0, \gamma\tilde{F}, \triangle) + \mathrm{Tr}(\mathrm{tr}(F)Q_0) - \mathrm{Tr}(FQ_1) & m = 2. \end{cases} \tag{57}$$

The flat de Rham spectral triple of the noncommutative two torus can be perturbed by a Weyl factor e^{-h} with $h \in A_\theta^\infty$ a self adjoint element. This procedure gives rise to the de Rham spectral triple of a curved noncommutative torus. The Ricci functional is defined in a similar fashion as above, and it can be shown that there exists an element $\mathbf{Ric} \in A_\theta^\infty \otimes M_2(\mathbb{C})$, called the Ricci density, such that

$$\mathcal{R}ic(F) = \frac{1}{\Im(\tau)} \varphi(\mathrm{tr}(F\mathbf{Ric})e^{-h}), \quad F \in A_\theta^\infty \otimes M_2(\mathbb{C}).$$

8.4 The de Rham spectral triple for the noncommutative two torus

In this section, we describe the de Rham spectral triple of a noncommutative two torus A_θ equipped with a complex structure. This is a deformation of the Dolbeault complex that we used in Section. Consider the vector space $W = \mathbb{R}^2$, and let τ be a complex number in the upper half plane. Let g_τ be the positive definite symmetric bilinear form on W given by

$$g_\tau = \frac{1}{\Im(\tau)^2} \begin{pmatrix} |\tau|^2 & -\Re(\tau) \\ -\Re(\tau) & 1 \end{pmatrix}. \tag{58}$$

The inverse $g_\tau^{-1} = \begin{pmatrix} 1 & \Re(\tau) \\ \Re(\tau) & |\tau|^2 \end{pmatrix}$ of g_τ is a metric on the dual of W. The entries of g_τ^{-1} will be denoted by g^{jk}.

Let $\bigwedge^\bullet W_\mathbb{C}^*$ be the exterior algebra of $W_\mathbb{C}^* = (W \otimes \mathbb{C})^*$. The algebra $A_\theta^\infty \otimes \bigwedge^\bullet W_\mathbb{C}^*$ is the algebra of differential forms on the noncommutative two torus A_θ. In this framework, the Hilbert space of functions, denoted $\mathcal{H}^{(0)}$, is simply the Hilbert space given by the GNS construction of A_θ^∞ with respect to $\frac{1}{\Im(\tau)}\varphi$. Additionally, the Hilbert space of one forms, denoted $\mathcal{H}^{(1)}$, is the space $\mathcal{H}_0 \otimes (\mathbb{C}^2, g_\tau^{-1})$ with inner product given by

$$\langle a_1 \oplus a_2, b_1 \oplus b_2 \rangle = \frac{1}{\Im(\tau)} \sum_{j,k} g^{jk} \varphi(b_k^* a_j), \quad a_i,\ b_i \in A_\theta^\infty, \tag{59}$$

while the Hilbert space of two forms, denoted $\mathcal{H}^{(2)}$, is given by the GNS construction of A_θ^∞ with respect to $\Im(\tau)\,\varphi$.

The exterior derivative on elements of A_θ^∞ is given by

$$a \mapsto i\delta_1(a) \oplus i\delta_2(a), \quad a \in A_\theta^\infty. \tag{60}$$

It will be denoted by d_0, when considered as a densely defined operator from $\mathcal{H}^{(0)}$ to $\mathcal{H}^{(1)}$. The operator $d_1 : \mathcal{H}^{(1)} \to \mathcal{H}^{(2)}$ is defined on the elements of $A_\theta^\infty \oplus A_\theta^\infty$ as

$$a \oplus b \mapsto i\delta_1(b) - i\delta_2(a), \qquad a, b \in A_\theta^\infty. \tag{61}$$

The adjoints of the operators $d_0 : \mathcal{H}^{(0)} \to \mathcal{H}^{(1)}$ and $d_1 : \mathcal{H}^{(1)} \to \mathcal{H}^{(2)}$ are then given by

$$d_0^*(a \oplus b) = -i\delta_1(a) - i\Re(\tau)\delta_2(a) - i\Re(\tau)\delta_1(b) - i|\tau|^2\delta_2(b),$$
$$d_1^*(a) = (i|\tau|^2\delta_2(a) + i\Re(\tau)\delta_1(a)) \oplus (-i\Re(\tau)\delta_2(a) - i\delta_1(a)),$$

for all $a, b \in A_\theta^\infty$.

Definition 8.2 *The (flat) de Rham spectral triple of A_θ is the even spectral triple $(A_\theta, \mathcal{H}, D)$, where $\mathcal{H} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(2)} \oplus \mathcal{H}^{(1)}$, $D = \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix}$, and $d = d_0 + d_1^*$.*

Note that the operator d and its adjoint $d^* = d_1 + d_0^*$ act on $A_\theta^\infty \oplus A_\theta^\infty$ as

$$d = \begin{pmatrix} i\delta_1 & i|\tau|^2\delta_2 + i\Re(\tau)\delta_1 \\ i\delta_2 & -i\Re(\tau)\delta_2 - i\delta_1 \end{pmatrix}, \quad d^* = \begin{pmatrix} -i\delta_1 - i\Re(\tau)\delta_2 & -i\Re(\tau)\delta_1 - i|\tau|^2\delta_2 \\ -i\delta_2 & i\delta_1 \end{pmatrix}. \tag{62}$$

Note also that the de Rham spectral triple introduced in Definition 8.2 is isospectral to the de Rham complex spectral triple of the flat torus $\mathbb{T}^2$ with the metric given by (58).

8.5 The twisted de Rham spectral triple

The conformal perturbation of the metric on the noncommutative two torus is implemented by changing the tracial state φ by a noncommutative Weyl factor e^{-h}, where the dilaton h is a selfadjoint smooth element of the noncommutative two torus, $h = h^* \in A_\theta^\infty$. The conformal change of the metric by the Weyl factor e^{-h} will change the inner product on functions and on two forms as follows. On functions, the Hilbert space given by GNS construction of A_θ with respect to the positive linear functional $\varphi_0(a) = \frac{1}{\Im(\tau)}\varphi(ae^{-h})$ will be denoted by $\mathcal{H}_h^{(0)}$. Therefore the inner product of $\mathcal{H}_h^{(0)}$ is given by

$$\langle a, b\rangle_0 = \frac{1}{\Im(\tau)}\varphi(b^*ae^{-h}), \quad a, b \in A_\theta.$$

On one forms, the Hilbert space will stay the same as in (59), and will be denoted by $\mathcal{H}_h^{(1)}$. On the other hand, the Hilbert space of two forms, denoted by $\mathcal{H}_h^{(2)}$, is the Hilbert space given by the GNS construction of A_θ with respect to $\varphi_2(a) = \Im(\tau)\varphi(ae^h)$. Hence its inner product is given by

$$\langle a, b\rangle_2 = \Im(\tau)\varphi(b^* a e^h), \quad a, b \in A_\theta.$$

The positive functional $a \mapsto \varphi(ae^{-h})$, called the conformal weight, is a twisted trace of which modular operator is given by

$$\Delta(a) = e^{-h} a e^h, \quad a \in A_\theta.$$

The logarithm $\log \Delta$ of the modular operator will be denoted by ∇, and is given by $\nabla(a) = -[h, a]$. For more details the reader can check the previous sections.

The exterior derivatives are defined in the same way they were defined in the flat case (60) and (61). However, to emphasize that they are acting on different Hilbert spaces, we will denote them by $d_{h,0} : \mathcal{H}_h^{(0)} \to \mathcal{H}_h^{(1)}$ and $d_{h,1} : \mathcal{H}_h^{(1)} \to \mathcal{H}_h^{(2)}$.

Next, we consider the Hilbert spaces $\mathcal{H}_h^+ = \mathcal{H}_h^{(0)} \oplus \mathcal{H}_h^{(2)}$ and $\mathcal{H}_h^- = \mathcal{H}_h^{(1)}$, and the operator $d_h : \mathcal{H}_h^+ \to \mathcal{H}_h^-$, $d_h = d_{h,0} + d_{h,1}^*$. Therefore

$$d_h = \begin{pmatrix} i\delta_1 & \big(i|\tau|^2\delta_2 + i\Re(\tau)\delta_1\big) \circ R_{k^2} \\ i\delta_2 & \big(-i\Re(\tau)\delta_2 - i\delta_1\big) \circ R_{k^2} \end{pmatrix},$$

and its adjoint is given by

$$d_h^* = \begin{pmatrix} R_{k^2} \circ \big(i\delta_1 - i\Re(\tau)\delta_2\big) & R_{k^2} \circ \big(-i\Re(\tau)\delta_1 - i|\tau|^2\delta_2\big) \\ -i\delta_2 & i\delta_1 \end{pmatrix}.$$

We also consider the operator

$$D_h = \begin{pmatrix} 0 & d_h^* \\ d_h & 0 \end{pmatrix},$$

which acts on $\mathcal{H}_h = \mathcal{H}_h^+ \oplus \mathcal{H}_h^-$. Define the Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0 \oplus \mathcal{H}_0 \oplus \mathcal{H}_0$ and the unitary operator $W : \mathcal{H} \to \mathcal{H}_h$,

$$W = R_k \oplus R_{k^{-1}} \oplus \mathrm{I}_{\mathcal{H}_0 \oplus \mathcal{H}_0}.$$

The operator D_h can be transferred to an operator $\tilde{D}_h$ on $\mathcal{H}$ by the inner perturbation

$$\tilde{D}_h := W^* D_h W = \begin{pmatrix} 0 & R_k \circ d^* \\ d \circ R_k & 0 \end{pmatrix}.$$

In order to define the twisted, or modular, de Rham spectral triple for the noncommutative two torus, we employ the following constructions from [21]. Let

$(\mathcal{A}, \mathcal{H}^+ \oplus \mathcal{H}^-, D)$ be an even spectral triple with grading operator γ, where $D = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$ and $T : \mathcal{H}^+ \to \mathcal{H}^-$ is an unbounded operator with adjoint T^*. If $f \in \mathcal{A}$ is positive and invertible, then $(\mathcal{A}, \mathcal{H}, D_{(f,\gamma)})$ is a modular spectral triple with respect to the inner automorphism $\sigma(a) = faf^{-1}$, $a \in \mathcal{A}$ [21, Lemma 1.1], where the Dirac operator is given by

$$D_{(f,\gamma)} = \begin{pmatrix} 0 & fT^* \\ Tf & 0 \end{pmatrix}.$$

On the other hand, any modular spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with an automorphism σ admits a transposed modular spectral triple $(\mathcal{A}^{\mathrm{op}}, \bar{\mathcal{H}}, D^t)$ [21, Proposition 1.3], where $\mathcal{A}^{\mathrm{op}}$ is the opposite algebra of $\mathcal{A}$, $\bar{\mathcal{H}}$ is the dual Hilbert space, the action of $\mathcal{A}^{\mathrm{op}}$ on $\bar{\mathcal{H}}$ is the transpose of the action of $\mathcal{A}$ on $\mathcal{H}$, D^t is the transpose of D, and σ' is the automorphism of $\mathcal{A}^{\mathrm{op}}$ given by $\sigma'(a^{\mathrm{op}}) = (\sigma^{-1}(a))^{\mathrm{op}}$.

Proposition 8.3 *Let $k = e^{h/2}$, where $h = h^* \in A_\theta^\infty$. The triple $(A_\theta^{\mathrm{op}}, \mathcal{H}, \tilde{D}_h)$ is a modular spectral triple, where the automorphism of A_θ^{op} is given by*

$$a^{\mathrm{op}} \mapsto (k^{-1}ak)^{\mathrm{op}}, \quad a \in A_\theta^\infty,$$

and the representation of A_θ^{op} on $\mathcal{H}$ is given by the right multiplication of A_θ on $\mathcal{H}$. Moreover, the transposed of the modular spectral triple $(A_\theta^{\mathrm{op}}, \mathcal{H}, \tilde{D}_h)$ is isomorphic to the perturbed spectral triple

$$(A_\theta, \mathcal{H}, \bar{D}_h), \quad \bar{D}_h = \begin{pmatrix} 0 & kd \\ d^*k & 0 \end{pmatrix}, \tag{63}$$

where the operators d and d^* are as in (62).

Definition 8.3 *The modular spectral triple $(A_\theta, \mathcal{H}, \bar{D}_h)$ in (63) will be called the modular de Rham spectral triple of the noncommutative two torus with dilaton h.*

8.6 Ricci functional and Ricci curvature for the curved noncommutative torus

Using the pseudodifferential calculus with symbols in $A_\theta^\infty \otimes M_4(\mathbb{C})$, one shows that the localized heat trace of $\bar{D}_h^2$ has an asymptotic expansion with coefficients of the form

$$a_n(E, \bar{D}_h^2) = \varphi \circ \mathrm{tr}\left(E\, c_n(\bar{D}_h^2)\right), \qquad E \in A_\theta^\infty \otimes M_4(\mathbb{C}),$$

where $c_n(\bar{D}_h^2) \in A_\theta^\infty \otimes \big(M_2(\mathbb{C}) \oplus M_2(\mathbb{C})\big)$ and tr is the matrix trace. The Ricci functional can now be defined:

Definition 8.4 ([34]) *The Ricci functional of the modular de Rham spectral triple* $(A_\theta, \mathcal{H}, \bar{D}_h)$ *is the functional on* $A_\theta \otimes M_2(\mathbb{C})$ *defined as*

$$\mathcal{R}ic(F) = a_2(\gamma \tilde{F}, \bar{D}^2) = \zeta(0, \gamma \tilde{F}, \bar{D}_h^2) + Tr(\mathrm{tr}(F)Q_0) - Tr(FQ_1),$$

where $\tilde{F} = \mathrm{tr}(F) \oplus 0 \oplus F$*, and* Q_j *is the orthogonal projection on the kernel of* $\triangle_{h,j}$*, for* $j = 0, 1$.

Lemma 8.1 *There exists an element* **Ric** $\in A_\theta^\infty \otimes M_2(\mathbb{C})$ *such that for all* $F \in A_\theta^\infty \otimes M_2(\mathbb{C})$

$$\mathcal{R}ic(F) = \frac{1}{\Im(\tau)} \varphi(\mathrm{tr}(F\mathbf{Ric})e^{-h}).$$

Proof For any such F we have

$$a_2(\gamma \tilde{F}, \bar{D}^2) = a_2(\mathrm{tr}(F), \triangle_{h,0}) - a_2(F, \triangle_{h,1}).$$

Now $\mathrm{tr}(F)e^{-t\triangle_{h,0}} = \mathrm{tr}(Fe^{-t\triangle_{h,0}\otimes \mathrm{I}_2})$, and thus

$$a_2(\mathrm{tr}(F), \triangle_{h,0}) = a_2(F, \triangle_{h,0} \otimes I_2).$$

As a result, we have

$$\begin{aligned}
\mathbf{Ric}(F) &= a_2(\mathrm{tr}(F), \triangle_{h,0}) - a_2(F, \triangle_{h,1}) \\
&= \varphi\left(\mathrm{tr}\Big(F\big(c_2(\triangle_{h,0}) \otimes \mathrm{I}_2 - c_2(\triangle_{h,1})\big)\Big)\right) \\
&= \frac{1}{\Im(\tau)} \varphi\left(\Im(\tau)\mathrm{tr}\Big(F\big(c_2(\triangle_{h,0}) \otimes \mathrm{I}_2 - c_2(\triangle_{h,1})\big)\Big)e^h e^{-h}\right).
\end{aligned}$$

Hence,

$$\mathbf{Ric} = \Im(\tau)\Big(c_2(\triangle_{h,0}) \otimes \mathrm{I}_2 - c_2(\triangle_{h,1})\Big)e^h.$$

□

Definition 8.5 *The element* **Ric** *is called the Ricci density of the curved noncommutative torus with dilaton h.*

The terms $c_2(\triangle_{h,j})$ can be computed by integrating the symbol of the parametrix of $\triangle_{h,j}$. Since the operator $\triangle_{h,1}$ is a first order perturbation of $\triangle_\varphi^{(0,1)}$, we will only need to compute the difference $c_2(\triangle_{h,1}) - c_2(\triangle_\varphi^{(0,1)}) \otimes \mathrm{I}_2$. The terms

$c_2(\triangle_{h,0}) = c_2(k\triangle_0 k)$ and $c_2(\triangle_\varphi^{(0,1)})$ are computed previously in two places by Connes-Moscovici and Fathizadeh-Khalkhali, and their difference is given by

$$R^\gamma = \big(c_2(k\triangle k) \otimes \mathrm{I}_2 - c_2(\triangle_\varphi^{(0,1)})\big)e^h$$
$$= -\frac{\pi}{\Im(\tau)}\Big(K_\gamma(\nabla)(\triangle_0(\log k)) + H_\gamma\big(\nabla_1, \nabla_2\big)\,(\Box_{\Re}(\log k))$$
$$+S(\nabla_1, \nabla_2)(\Box_{\Im}(\log k))\Big)e^h.$$

Here,

$$\Box_{\Re}(\ell) = (\delta_1(\ell))^2 + \Re(\tau)\,(\delta_1(\ell)\delta_2(\ell) + \delta_2(\ell)\delta_1(\ell)) + |\tau|^2(\delta_2(\ell))^2,$$
$$\Box_{\Im}(\ell) = i\Im(\tau)(\delta_1(\ell)\delta_2(\ell) - \delta_2(\ell)\delta_1(\ell))$$

with $\ell = \log k$. Moreover,

$$K_\gamma(u) = \frac{\frac{1}{2} + \frac{\sinh(u/2)}{u}}{\cosh^2(u/4)},$$

$$H_\gamma(s,t) = \big(1 - \cosh((s+t)/2)\big) \times \frac{\begin{array}{c} t(s+t)\cosh(s) - s(s+t)\cosh(t) + (s-t) \\ (s+t+\sinh(s)+\sinh(t)-\sinh(s+t)) \end{array}}{st(s+t)\sinh\left(\frac{s}{2}\right)\sinh\left(\frac{t}{2}\right)\sinh\left(\frac{s+t}{2}\right)^2},$$

$$S(s,t) = \frac{(s+t-t\,\cosh(s) - s\,\cosh(t) - \sinh(s) - \sinh(t) + \sinh(s+t))}{s\,t\,\big(\sinh\left(\frac{s}{2}\right)\sinh\left(\frac{t}{2}\right)\sinh\left(\frac{s+t}{2}\right)\big)}.$$

The term S coincides with the function S found in [21, 31] for scalar curvature.

Now the main result of [34] can be stated as follows. It computes the Ricci curvature density of a curved noncommutative two torus with a conformally flat metric. The proof of this theorem is quite long and complicated and will not be reproduced here.

Theorem 8.1 ([34]) *Let $k = e^{h/2}$ with $h \in A_\theta^\infty$ a selfadjoint element. Then the Ricci density of the modular de Rham spectral triple with dilaton h is given by*

$$\mathbf{Ric} = \frac{\Im(\tau)}{4\pi^2}R^\gamma \otimes \mathrm{I}_2 - \frac{1}{4\pi}S(\nabla_1, \nabla_2)\big([\delta_1(\log k), \delta_2(\log k)]\big)e^h \otimes \begin{pmatrix} i\Im(\tau) & \Im(\tau)^2 \\ -1 & i\Im(\tau) \end{pmatrix}.$$

It is important to check the classical limit for consistency. In the commutative limit the Ricci density **Ric** is retrieved as $\lim_{(s,t)\to(0,0)} \mathbf{Ric}$. Since (cf. [21] for a proof)

$$\lim_{(s,t)\to(0,0)} R^{\gamma} = -\frac{\pi}{\Im(\tau)}\triangle_0(\log k),$$

and $[\delta_1(\log k), \delta_2(\log k)] = 0$, we have

$$\mathbf{Ric}|_{\theta=0} = \frac{-1}{4\pi}\triangle_0(\log k)e^h \otimes \mathrm{I}_2.$$

If we take into account the normalization of the classical case that comes from the heat kernel coefficients, this gives the formula for the Ricci operator in the classical case.

Unlike the commutative case, the Ricci density **Ric** in the noncommutative case is not a symmetric matrix. Indeed, it has non-zero off diagonal terms, which are multiples of $S(\nabla_1, \nabla_2)([\delta_1(\log k), \delta_2(\log k)])$. This phenomenon, observed in [34] for the first time, is obviously a consequence of the noncommutative nature of the space. It is an interesting feature of noncommutative geometry that, contrary to the commutative case, the Ricci curvature is not a multiple of the scalar curvature even in dimension two. This manifests itself in the existence of off diagonal terms in the Ricci operator **Ric** above.

It is clear that one can define in a similar fashion a Ricci curvature operator for higher dimensional noncommutative tori, as well as for noncommutative toric manifolds. Its computation in these cases poses an interesting problem. This problem now is completely solved for noncommutative three tori in [24]. It would also be interesting to find the analogue of the Ricci flow based on our definition of Ricci curvature functional. It should be noted that for noncommutative two tori a definition of Ricci flow, without a notion of Ricci curvature, is proposed in [3].

9 Beyond conformally flat metrics and beyond dimension four

In the study of spectral geometry of noncommutative tori one is naturally interested in going beyond conformally flat metrics and beyond dimension four. Even in the case of noncommutative two torus it is important to consider metrics which are not conformally flat. In fact while by uniformization theorem we know that any metric on the two torus is conformally flat, there is strong evidence that this is not so in the noncommutative case. This is closely related to the problem of classification of complex structures on the noncommutative two torus via positive Hochschild cocycles, which is still unsolved.

As far as higher dimensions go, our original methods do not allow us to treat the dimension as a variable in the calculations and obtain explicit formulas in all dimensions in a uniform manner. This is in sharp contrast with the classical case where formulas work in a uniform manner in all dimensions. In this section we report on a very recent development [35] where progress has been made on both fronts.

In the recent paper [35], using a new strategy based on Newton divided differences, it is shown how to consider non-conformal metrics and how to treat all higher dimensional noncommutative tori in a uniform way. In fact based on older methods it was not clear how to extend the computation of the scalar curvature to a general higher dimensional case. The class of non-conformal metrics introduced in [35] is quite large and leads to beautiful combinatorial identities for the curvature via divided differences. In this section we shall briefly sketch the results obtained in [35], following closely its organization of material.

9.1 Rearrangement lemma revisited

To compute and effectively work with integrals of the form

$$\int_0^\infty (uk^2+1)^{-m} b(uk^2+1)u^m du,$$

the rearrangement lemma was proved by Connes and Tretkoff in [22]. Here $k = e^{h/2}$, $h, b \in C^\infty(\mathbb{T}^2_\theta)$ and h is selfadjoint. The problem stems from the fact that h and b need not commute. Later on this lemma was generalized, for the sake of curvature calculations, for more than one b in [21, 31]. A detailed study of this lemma for more general integrands of the form

$$\int_0^\infty f_0(u,k)b_1 f_1(u,k)\rho_2 \cdots b_n f_n(u,k)du,$$

was given by M. Lesch in [46], with a new proof and a new point of view. This approach uses the multiplication map

$$\mu : a_1 \otimes a_2 \otimes \cdots \otimes a_n \mapsto a_1 a_2 \cdots a_n$$

from the projective tensor product $A^{\otimes_\gamma n}$ to A. The above integral is expressed as the contraction of the product of an element $F(k_{(0)}, \cdots, k_{(n)})$ of $A^{\otimes_\gamma (n+1)}$, with the element $b_1 \otimes b_2 \otimes \cdots b_n \otimes 1$ which is

$$\mu\Big(F(k_{(0)}, \cdots, k_{(n)})(b_1 \otimes b_2 \otimes \cdots b_n \otimes 1)\Big).$$

The above element is usually written in the so-called *contraction form*

$$F(k_{(0)}, \cdots, k_{(n)})(b_1 \cdot b_2 \cdot \cdots \cdot b_n). \tag{64}$$

The following version of the rearrangement lemma is stated in [35] with the domain of integration changed from $[0, \infty)$ to any domain in $\mathbb{R}^N$.

Lemma 9.1 (Rearrangement Lemma [35]) *Let A be a unital C^*-algebra, $h \in A$ be a selfadjoint element, and Λ be an open neighborhood of the spectrum of h in $\mathbb{R}$. For a domain U in $\mathbb{R}^N$, let $f_j : U \times \Lambda \to \mathbb{C}$, $0 \le j \le n$, be smooth functions such that $f(u, \lambda) = \prod_{j=0}^{n} f_j(u, \lambda_j)$ satisfies the following integrability condition: for any compact subset $K \subset \Lambda^{n+1}$ and every given multi-index α we have*

$$\int_U \sup_{\lambda \in K} |\partial_\lambda^\alpha f(u, \lambda)| du < \infty.$$

Then,

$$\int_U f_0(u, h) b_1 f_1(u, h) \cdots b_n f_n(u, h) du = F(h_{(0)}, h_{(1)}, \cdots, h_{(n)})(b_1 \cdot b_2 \cdots b_n), \tag{65}$$

where $F(\lambda) = \int_U f(u, \lambda) du$. □

In particular it follows that every expression in the contraction form with a Schwartz function $F \in \mathcal{S}(\mathbb{R}^{n+1})$ used in the operator part can be written as an integral. In fact if we set

$$f_n(\xi, \lambda) = \hat{f}(\xi) e^{i \xi_n \lambda}, \quad f_j(\xi, \lambda) = e^{i \xi_j \lambda}, \quad 0 \le j \le n - 1,$$

and $f(\xi, \lambda_0, \cdots, \lambda_n) = \prod_{j=0}^{n} f_j(\xi, \lambda_j)$, by the Fourier inversion formula, we have $F(\lambda) = \int f(\xi, \lambda) d\xi$. Then, Lemma 9.1 gives the equality

$$F(h_{(0)}, \cdots, h_{(n)})(b_1 \cdot b_2 \cdots b_n) = \int_{\mathbb{R}^n} e^{i \xi_0 h} b_1 e^{i \xi_1 h} b_2 \cdots b_n e^{i \xi_n h} \hat{f}(\xi) d\xi. \tag{66}$$

This is crucial for calculations in [35].

9.2 A new idea

As we saw in previous sections, to prove the Gauss-Bonnet theorem and to compute the scalar curvature of a curved noncommutative two torus in [22, 30] and [21, 31], the second density of the heat trace of the Laplacian D^2 of the Dirac operator had to be computed. First, the symbol of the parametrix of D^2 was computed, next a contour integral coming from Cauchy's formula for the heat operator had to be computed, and finally one had to integrate out the momentum variables. It was for this last step that the rearrangement lemma played an important role. Luckily, the contour integral could be avoided using a homogeneity argument.

A key observation in [35] is that one need not wait till the last step to have elements in the contraction form. It is just enough to start off with operators whose symbol is written in the contraction form

$$F(h_{(0)}, \cdots, h_{(n)})(b_1 \cdot b_2 \cdots b_n).$$

It is further noted that the symbol calculus can be effectively applied to differential operators whose symbols can be written in the contraction form. These operators are called *h-differential operators* in [35]. This is a new and larger class of differential operators that lends itself to precise spectral analysis. It is strictly larger than the class of Dirac Laplacians for conformally flat metrics on noncommutative tori which has been the subject of intensive studies lately.

Next, the Newton divided difference calculus was brought in to find the action of derivations on elements in contracted form (Theorem 9.1 below). For example, one has

$$\delta_j\big(f(h_{(0)}, h_{(1)})(b_1)\big) = f(h_{(0)}, h_{(1)})(\delta_j(b_1)) + \big[h_{(0)}, h_{(1)}; f(\cdot, h_{(2)})(\delta_j(h) \cdot b)\big] \\ +\big[h_{(1)}, h_{(2)}; f(h_{(0)}, \cdot)(b \cdot \delta_j(h))\big].$$

Using this fact, and applying the pseudodifferential calculus, one can compute the spectral densities of positive h-differential operators whose principal symbol is given by a functional metric. These operators are called *Laplace type h-differential operator* in [35].

This change in order of the computations, i.e. writing symbols in the contraction form first, led to a smoother computation symbolically, and played a fundamental in computing with more general functional metrics. It also paved the way for calculating the curvature in all higher dimensions for conformally flat and twisted product of flat metrics.

9.3 Newton divided differences

A nice application of the rearrangement lemma is to find a formula for the differentials of a smooth element written in contraction form. To this end, Newton divided differences were used in [35].

Let $x_0, x_1, \cdots, x_n$ be distinct points in an interval $I \subset \mathbb{R}$ and let f be a function on I. The nth-*order Newton divided difference* of f, denoted by $[x_0, x_1, \cdots, x_n; f]$, is the coefficient of x^n in the interpolating polynomial of f at the given points. In other words, if the interpolating polynomial is $p(x)$, then

$$p(x) = p_{n-1}(x) + [x_0, x_1, \cdots, x_n; f](x - x_0) \cdots (x - x_{n-1}),$$

where $p_{n-1}(x)$ is a polynomial of degree at most $n-1$. There is a recursive formula for the divided difference which is given by

$$[x_0; f] = f(x_0)$$
$$[x_0, x_1, \cdots, x_n; f] = \frac{[x_1, \cdots, x_n; f] - [x_0, x_1, \cdots, x_{n-1}; f]}{x_n - x_0}.$$

There is also an explicit formula for the divided difference:

$$[x_0, x_1, \cdots, x_n; f] = \sum_{j=0}^{n} \frac{f(x_j)}{\prod_{j \neq l}(x_j - x_l)}.$$

The *Hermite-Genocchi formula* gives an integral representation for the divided differences of an n times continuously differentiable function f as an integral over the standard simplex:

$$[x_0, \cdots, x_n; f] = \int_{\Sigma_n} f^{(n)}\Big(\sum_{j=0}^{n} s_j x_j\Big) ds. \tag{67}$$

Let δ be a densely defined, unbounded and closed derivation on a C^*-algebra A. If $a \in \mathrm{Dom}(\delta)$, then $e^{za} \in \mathrm{Dom}(\delta)$ for any $z \in \mathbb{C}$, and one has

$$\delta(e^{za}) = z \int_0^1 e^{zsa} \delta(a) e^{z(1-s)a} ds. \tag{68}$$

Using the rearrangement lemma, one can now express the differential of a smooth element given in contraction form. This result generalizes the expansional formula, also known as Feynman-Dyson formula, for e^{A+B}, and not only for elements of the form $f(h)$, but also for any element written in the contraction form.

Theorem 9.1 *Let δ be a closed derivation of a C^*-algebra A and $h \in \mathrm{Dom}(\delta)$ be a selfadjoint element. Let $b_j \in \mathrm{Dom}(\delta)$, $1 \leq j \leq n$, and let $f : \mathbb{R}^{n+1} \to \mathbb{C}$ be a smooth function. Then $f(h_{(0)}, \cdots, h_{(n)})(b_1 \cdot b_2 \cdot \cdots \cdot b_n)$ is in the domain of δ and*

$$\begin{aligned}
&\delta(f(h_{(0)}, \cdots, h_{(n)})(b_1 \cdot b_2 \cdots b_n)) \\
&\quad = \sum_{j=1}^{n} f(h_{(0)}, \cdots, h_{(n)})\big(b_1 \cdots b_{j-1} \cdot \delta(b_j) \cdot b_{j+1} \cdots b_n\big) \\
&\quad\quad + \sum_{j=0}^{n} f_j(h_{(0)}, \cdots, h_{(n+1)})\big(b_1 \cdots b_j \cdot \delta(h) \cdot b_{j+1} \cdots b_n\big),
\end{aligned}$$

where $f_j(t_0, \cdots, t_{n+1})$, which we call the partial divided difference, is defined as

$$f_j(t_0, \cdots, t_{n+1}) = \big[t_j, t_{j+1}; t \mapsto f(t_0, \cdots, t_{j-1}, t, t_{j+2}, \cdots, t_n)\big].$$

9.4 Laplace type h-differential operators and asymptotic expansions

Let us first recall this class of differential operators which is introduced in [35]. It extends the previous classes of differential operators on noncommutative tori, in particular Dirac Laplacians of conformally flat metrics.

Definition 9.1 ([35]) *Let $h \in C^\infty(\mathbb{T}_\theta^d)$ be a smooth selfadjoint element.*

(i) *An h-differential operator on $\mathbb{T}_\theta^d$ is a differential operator $P = \sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} \delta^{\boldsymbol{\alpha}}$, with $C^\infty(\mathbb{T}_\theta^d)$-valued coefficients $p_{\boldsymbol{\alpha}}$ which can be written in the contraction form*

$$p_{\boldsymbol{\alpha}} = P_{\boldsymbol{\alpha},\boldsymbol{\alpha}_1,\cdots,\boldsymbol{\alpha}_k}(h_{(0)}, \cdots, h_{(k)})(\delta^{\boldsymbol{\alpha}_1}(h) \cdots \delta^{\boldsymbol{\alpha}_k}(h)).$$

(ii) *A second order h-differential operator P is called a Laplace type h-differential operator if its symbol is a sum of homogeneous parts p_j of the form*

$$p_2 = P_2^{ij}(h)\xi_i\xi_j,$$
$$p_1 = P_1^{ij}(h_{(0)}, h_{(1)})(\delta_i(h))\xi_j,$$
$$p_0 = P_{0,1}^{ij}(h_{(0)}, h_{(1)})(\delta_i\delta_j(h)) + P_{0,2}^{ij}(h_{(0)}, h_{(1)}, h_{(2)})(\delta_i(h) \cdot \delta_j(h)),$$

where the principal symbol $p_2 : \mathbb{R}^n \to C^\infty(\mathbb{T}_\theta^d)$ is a $C^\infty(\mathbb{T}_\theta^d)$-valued quadratic form such that $p_2(\xi) > 0$ for all $\xi \in \mathbb{R}$.

We can allow the symbols to be matrix valued, that is $p_j : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{T}_\theta^d) \otimes M_n(\mathbb{C})$, provided that all $p_2(\xi) \in C^\infty(\mathbb{T}_\theta^d) \otimes \mathrm{I}_n$ for all non-zero $\xi \in \mathbb{R}$.

Many of the elliptic second order differential operators on noncommutative tori which were studied in the literature are Laplace type h-differential operators. For instance, the two differential operators on $\mathbb{T}_\theta^2$ whose spectral invariants are studied in [21, 31] are indeed Laplace type h-differential operator. In fact with $k = e^{h/2}$, these operators are given by

$$k\triangle k = k\delta\delta^* k, \qquad \triangle_\varphi^{(0,1)} = \delta^* k^2 \delta,$$

where $\delta = \delta_1 + \bar{\tau}\delta_2$ and $\delta^* = \delta_1 + \tau\delta_2$ for some complex number τ in the upper half plane.

Let P be a positive Laplace type h-differential operator. Using the Cauchy integral formula, one has

$$e^{-tP} = \frac{-1}{2\pi i} \int_\gamma e^{-t\lambda}(P - \lambda)^{-1} d\lambda, \quad t > 0,$$

for a suitable contour γ. Expanding the symbol of the parametrix $\sigma((P-\lambda)^{-1})$, one obtains a short time asymptotic expansion for localized heat trace for any $a \in C^\infty(\mathbb{T}_\theta^d)$:

$$\mathrm{Tr}(ae^{-tP}) \sim \sum_{n=0}^{\infty} c_n(a) t^{(n-d)/2}.$$

Here, $c_n(a) = \varphi(ab_n)$ with

$$b_n = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{-1}{2\pi i} \int_\gamma b_n(\xi,\lambda) d\lambda d\xi. \tag{69}$$

Using the rearrangement lemma (Lemma 9.1) and the fact that the contraction map and integration commute, one obtains

$$\begin{aligned} b_2 = &\left(\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{-1}{2\pi i} \int_\gamma B_{2,1}^{ij}(\xi,\lambda,h_{(0)},h_{(1)})\, e^{-\lambda} d\lambda d\xi \right) \big(\delta_i\delta_j(h)\big) \\ &+ \left(\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{-1}{2\pi i} \int_\gamma B_{2,2}^{ij}(\xi,\lambda,,h_{(0)},h_{(1)},,h_{(2)})\, e^{-\lambda} d\lambda d\xi \right) \\ &\times \big(\delta_i(h)\cdot\delta_j(h)\big). \end{aligned}$$

The dependence of $B_{2,k}^{ij}$ on λ comes only from different powers of B_0 in its terms, while its dependence on ξ_j's is the result of appearance of ξ_j as well as of B_0 in the terms. Therefore, the contour integral will only contain $e^{-\lambda}$ and product of powers of $B_0(t_j)$. Hence, we need to deal with a certain kind of contour integral for which we shall use the following notation and will call them *T-functions*:

$$T_{\boldsymbol{n};\boldsymbol{\alpha}}(t_0,\cdots,t_n) := \frac{-1}{\pi^{d/2}} \int_{\mathbb{R}^d} \xi_{n_1}\cdots\xi_{n_{2|\boldsymbol{\alpha}|-4}} \frac{1}{2\pi i} \int_\gamma e^{-\lambda} B_0^{\alpha_0}(t_0)\cdots B_0^{\alpha_n}(t_n) d\lambda d\xi, \tag{70}$$

where $\boldsymbol{n} = (n_1,\cdots,n_{2|\boldsymbol{\alpha}|-4})$ and $\boldsymbol{\alpha} = (\alpha_0,\cdots,\alpha_n)$. We recall the T-functions and their properties a bit later. There is an explicit formula for $b_2(P)$ which we now recall from [35]:

Proposition 9.1 *For a positive Laplace type h-differential operator P with the symbol given by* (9.1)*, the term $b_2(P)$ in the contraction form is given by*

$$\begin{aligned} b_2(P) = (4\pi)^{-d/2} \Big(&B_{2,1}^{ij}(h_{(0)},h_{(1)})\big(\delta_i\delta_j(h)\big) \\ &+ B_{2,2}^{ij}(h_{(0)},h_{(1)},h_{(2)})\big(\delta_i(h)\cdot\delta_j(h)\big)\Big), \end{aligned}$$

where the functions are defined by

$$
\begin{aligned}
B^{ij}_{2,1}(t_0,t_1) = &- T_{;1,1}(t_0,t_1) P^{ij}_{0,1}(t_0,t_1) + 2T_{k\ell;2,1}(t_0,t_1) P^{ik}_2(t_0) P^{j\ell}_1(t_0,t_1) \\
&+ T_{k\ell;2,1}(t_0,t_1)\, P^{ij}_2(t_0) \left[t_0, t_1;\, P^{k\ell}_2\right] \\
&- 4T_{k\ell mn;3,1}(t_0,t_1) P^{ik}_2(t_0) P^{j\ell}_2(t_0) \left[t_0, t_1;\, P^{mn}_2\right],
\end{aligned}
$$

and

$$
\begin{aligned}
B^{ij}_{2,2}(t_0,t_1,t_2) = &- T_{;1,1}(t_0,t_2) P^{ij}_{0,2}(t_0,t_1,t_2) \\
&+ T_{k\ell;1,1,1}(t_0,t_1,t_2)\, P^{ik}_1(t_0,t_1) P^{j\ell}_1(t_1,t_2) \\
&- 2T_{k\ell mn;2,1,1}(t_0,t_1,t_2)\, P^{im}_2(t_0) \left[t_0, t_1;\, P^{k\ell}_2\right] P^{jn}_1(t_1,t_2) \\
&+ 2T_{k\ell;2,1}(t_0,t_2)\, P^{ik}_2(t_0) \left[t_0, t_1;\, P^{j\ell}_1(\cdot, t_2)\right] \\
&+ 2T_{k\ell;2,1}(t_0,t_2)\, P^{jk}_2(t_0) \left[t_1, t_2;\, P^{i\ell}_1(t_0, \cdot)\right] \\
&+ T_{k\ell;1,1,1}(t_0,t_1,t_2)\, P^{ij}_1(t_0,t_1) \left[t_1, t_2;\, P^{k\ell}_2\right] \\
&- 2T_{k\ell mn;2,1,1}(t_0,t_1,t_2)\, P^{j\ell}_2(t_0) P^{ik}_1(t_0,t_1) \left[t_1, t_2;\, P^{mn}_2\right] \\
&- 2T_{k\ell mn;1,2,1}(t_0,t_1,t_2)\, P^{ik}_1(t_0,t_1) P^{j\ell}_2(t_1) \left[t_1, t_2;\, P^{mn}_2\right] \\
&- 2T_{k\ell mn;2,1,1}(t_0,t_1,t_2)\, P^{ij}_2(t_0) \left[t_0, t_1;\, P^{\ell m}_2\right] \left[t_1, t_2;\, P^{kn}_2\right] \\
&- 4T_{k\ell mn;2,1,1}(t_0,t_1,t_2)\, P^{ik}_2(t_0) \left[t_0, t_1;\, P^{j\ell}_2\right] \left[t_1, t_2;\, P^{mn}_2\right] \\
&+ 8T_{k\ell mnpq;3,1,1}(t_0,t_1,t_2)\, P^{ik}_2(t_0) P^{jn}_2(t_0) \left[t_0, t_1;\, P^{\ell m}_2\right] \\
&\quad \times \left[t_1, t_2;\, P^{pq}_2\right] \\
&+ 4T_{k\ell mnpq;2,2,1}(t_0,t_1,t_2) P^{ik}_2(t_0) \left[t_0, t_1;\, P^{\ell m}_2\right] P^{jn}_2(t_1) \\
&\quad \times \left[t_1, t_2;\, P^{pq}_2\right] \\
&+ 2T_{k\ell;2,1}(t_0,t_2)\, P^{ij}_2(t_0) \left[t_0, t_1, t_2;\, P^{k\ell}_2\right] \\
&- 8T_{k\ell mn;3,1}(t_0,t_2)\, P^{ik}_2(t_0) P^{j\ell}_2(t_0) [t_0, t_1, t_2;\, P^{mn}_2].
\end{aligned}
$$

□

The computation of the higher heat trace densities for a Laplace type h-operator can be similarly carried out, expecting many more terms in the results. This would give a way to generalize results obtained for the conformally flat noncommutative two torus in [17] where b_4 of the Laplacian D^2 of the Dirac operator D is computed. This problem won't be discussed further in this paper, but is certainly an interesting problem.

Evaluating T-functions (70), the only parts of formulas for $B_{2,1}^{i,j}$ and $B_{2,2}^{i,j}$ that need to be evaluated, is not always an easy task. In [35] a concise integral formula for T-functions is given and their properties are studied. For the contour integral in (70). it is clear that there are functions $f_{\alpha_1,\cdots,\alpha_n}$ such that

$$\frac{-1}{2\pi i}\int_\gamma e^{-\lambda} B_0^{\alpha_0}(t_0)\cdots B_0^{\alpha_n}(t_n)d\lambda = f_{\alpha_0,\cdots,\alpha_n}\left(\|\xi\|_{t_0}^2,\cdots,\|\xi\|_{t_n}^2\right).$$

Here, we denoted $P_2^{ij}(t_k)\xi_i\xi_j$ by $\|\xi\|_{t_k}^2$. Examples of such functions are

$$f_{1,1}(x_0,x_1) = -\frac{e^{-x_0}}{x_0-x_1} - \frac{e^{-x_1}}{x_1-x_0},$$
$$f_{2,1}(x_0,x_1) = -\frac{e^{-x_0}}{x_0-x_1} - \frac{e^{-x_0}}{(x_0-x_1)^2} + \frac{e^{-x_1}}{(x_1-x_0)^2}.$$

Lemma 9.2 ([35]) *Let $P_2(t)$ be a positive definite $d\times d$ matrix of smooth real functions. Then*

$$T_{\boldsymbol{n};\boldsymbol{\alpha}}(t_0,\cdots,t_n) = \frac{1}{2^{|\boldsymbol{\alpha}|-2}\boldsymbol{\beta}!}\int_{\Sigma_n}\prod_{j=0}^n s_j^{\alpha_j-1}\frac{\sum_{\boldsymbol{n}}\prod P^{-1}(s)_{n_i n_{\sigma(i)}}}{\sqrt{\det P(s)}}ds,$$

where $P(s)=\sum_{j=0}^n s_j P_2(t_j)$ and $\boldsymbol{\beta}=(\alpha_0-1,\cdots,\alpha_n-1)$. □

9.5 Functional metrics and scalar curvature

A natural question is if there exists a large class of noncommutative metrics whose Laplacians are h-differential operators and hence amenable to the spectral analysis developed in the last section. As we saw, conformally flat metrics on noncommutative tori is such a class. But there are more. One of the interesting concepts developed in [35] is the notion of a *functional metric* which is a much larger class than conformally flat metrics and whose Laplacian is still an h-differential operator. In this section we shall first recall this concept and reproduce the scalar curvature formula for these metrics developed in [35].

Definition 9.2 *Let h be a selfadjoint smooth element of a noncommutative d-torus and let $g_{ij}:\mathbb{R}\to\mathbb{R}$, $1\le i,j\le d$, be smooth functions such that the matrix $\big(g_{ij}(t)\big)$ is a positive definite matrix for every t in a neighborhood of the spectrum of h. We shall refer to $g_{ij}(h)$ as a functional metric on A_θ^d.*

The construction of the Laplacian on functions on $\mathbb{T}_\theta^d$ equipped with a functional metric $g=g_{ij}(h)$ follows the same pattern as in previous sections. Details can be found in [35], where the following crucial result is also proved. The Laplacian $\delta^*\delta : \mathcal{H}_{0,g} \to \mathcal{H}_{0,g}$ on elements of $C^\infty(\mathbb{T}^d_\theta)$ is given by

$$\delta_j(a) g^{jk}(h)\delta_k(|g|^{\frac{1}{2}}(h))|g|^{-\frac{1}{2}}(h) + \delta_j(a)\delta_k(g^{jk}(h)) + i\delta_k(\delta_j(a))g^{jk}(h).$$

To carry the spectral analysis of the Laplacian $\delta^*\delta : \mathcal{H}_{0,g} \to \mathcal{H}_{0,g}$, we switch to the antiunitary equivalent setting as follows. Let $\mathcal{H}_0$ be the Hilbert space obtained by the GNS construction from A^d_θ using the nonperturbed tracial state φ.

Proposition 9.2 *The operator $\delta^*\delta : \mathcal{H}_{0,g} \to \mathcal{H}_{0,g}$ is antiunitary equivalent to a Laplace type h-differential operator $\triangle_{0,g} : \mathcal{H}_0 \to \mathcal{H}_0$ whose symbol, when expressed in the contraction form, has the functional parts given by*

$$\begin{aligned}
P_2^{jk}(t_0) &= g^{jk}(t_0),\\
P_1^{jk}(t_0,t_1) &= |g|^{-\frac{1}{4}}(t_0)\big[t_0,t_1; |g|^{\frac{1}{4}}\big] g^{jk}(t_1) + \big[t_0,t_1; g^{jk}\big]\\
&\quad + |g|^{\frac{1}{4}}(t_0) g^{jk}(t_0)\big[t_0,t_1; |g|^{-\frac{1}{4}}\big],\\
P_{0,1}^{jk}(t_0,t_1) &= |g|^{\frac{1}{4}}(t_0) g^{jk}(t_0)\big[t_0,t_1; |g|^{-\frac{1}{4}}\big],\\
P_{0,2}^{jk}(t_0,t_1,t_2) &= |g|^{-\frac{1}{4}}(t_0)\big[t_0,t_1; g^{jk}|g|^{\frac{1}{2}}\big]\big[t_1,t_2; |g|^{-\frac{1}{4}}\big]\\
&\quad + 2|g|^{\frac{1}{4}}(t_0) g^{jk}(t_0)\big[t_0,t_1,t_2; |g|^{-\frac{1}{4}}\big].
\end{aligned}$$

An important case of the functional metric is the conformally flat metric

$$g_{ij}(t) = f(t)^{-1} g_{ij}, \tag{71}$$

where f is a positive smooth function and g_{ij}'s are the entries of a constant metric on $\mathbb{R}^d$. The functions given by Proposition 9.2, for the conformally flat metrics, give us the following:

$$\begin{aligned}
P_2^{jk}(t_0) &= g^{jk} f(t_0), \\
P_1^{jk}(t_0,t_1) &= g^{jk}\Big(f(t_0)^{\frac{d}{4}}\big[t_0,t_1; f^{1-\frac{d}{4}}\big] + f(t_0)^{1-\frac{d}{4}}\big[t_0,t_1; f^{\frac{d}{4}}\big]\Big),\\
P_{0,1}^{jk}(t_0,t_1) &= g^{jk} f(t_0)^{1-\frac{d}{4}}\big[t_0,t_1; f^{\frac{d}{4}}\big],\\
P_{0,2}^{jk}(t_0,t_1,t_2) &= g^{jk}\Big(f(t_0)^{\frac{d}{4}}\big[t_0,t_1; f^{1-\frac{d}{2}}\big]\big[t_1,t_2; f^{\frac{d}{4}}\big]\\
&\quad + 2 f(t_0)^{1-\frac{d}{4}}\big[t_0,t_1,t_2; f^{\frac{d}{4}}\big]\Big).
\end{aligned} \tag{72}$$

A careful examination of formula (72) shows that for any function $P_\bullet^{ij}$ there exists a function $P_\bullet$ such that $P_\bullet^{ij} = g^{ij} P_\bullet$. We have similar situation with the T-functions for conformally flat metrics.

Lemma 9.3 ([35]) *Let $\boldsymbol{\alpha}$ and $\boldsymbol{n} = (n_1, \cdots, n_{2|\boldsymbol{\alpha}|-4})$ be two multi-indices. Then the T-function $T_{\boldsymbol{n},\boldsymbol{\alpha}}$ for the conformally flat metric (71) is of the form*

$$T_{\boldsymbol{n},\boldsymbol{\alpha}}(t_0, \cdots, t_n) = \sqrt{|g|} \sum_{\boldsymbol{n}} \prod g_{n_i n_{\sigma(i)}} T_{\boldsymbol{\alpha}}(t_0, \cdots, t_n).$$

The function $T_{\boldsymbol{\alpha}}$ in dimension $d \neq 2$ is given by

$$T_{\boldsymbol{\alpha}}(t_0, \cdots, t_n) = \frac{(-1)^{|\boldsymbol{\alpha}|-1}\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}+|\boldsymbol{\alpha}|-2\right)} \partial_x^{\boldsymbol{\beta}}\left[x_0, \cdots, x_n; u^{1-\frac{d}{2}}\right]\Big|_{x_j = f(t_j)}, \tag{73}$$

where $\boldsymbol{\beta} = (\alpha_0 - 1, \cdots, \alpha_n - 1)$.

As an example, we have

$$\begin{aligned} T_{\alpha,1}(t_0, t_1) = & \frac{(-1)^{\alpha}\Gamma(\frac{d}{2}-1)}{2^{\alpha-1}\Gamma(\frac{d}{2}+\alpha-1)} \\ & \times \Big(\frac{f(t_1)^{1-\frac{d}{2}}}{(f(t_1)-f(t_0))^{\alpha}} \\ & \quad - \sum_{m=0}^{\alpha-1} \frac{(-1)^m \Gamma(\frac{d}{2}+m-1)}{\Gamma(\frac{d}{2}-1)m!} \frac{f(t_0)^{-\frac{d}{2}-m+1}}{(f(t_1)-f(t_0))^{\alpha-m}} \Big). \end{aligned}$$

Note that for dimension two, $T_{\alpha,1}(t_0, t_1)$ can be obtained by taking the limit of (73) as d approaches 2. When $f(t) = t$, we have

$$T_{\alpha,1}(t_0, t_1) = \frac{(-1)^{\alpha-1}}{2^{\alpha-1}\Gamma(\alpha)^2} \partial_{t_0}^{\alpha-1}\big[t_0, t_1; \log(u)\big].$$

Recall that the *scalar curvature density* of a given functional metric is defined by

$$R = (4\pi)^{\frac{d}{2}} b_2(\triangle_{0,g}).$$

This scalar curvature density is computed for two classes of examples in all dimensions: conformally flat metrics and twisted products of conformally flat metrics. Let us recall this result:

Theorem 9.2 ([35]) *The scalar curvature of the d-dimensional noncommutative tori $\mathbb{T}_\theta^d$ equipped with the metric $f(h)^{-1}g_{ij}$ is given by*

$$R = \sqrt{|g|}\Big(K_d(h_{(0)}, h_{(1)})(\triangle(h)) + H_d(h_{(0)}, h_{(1)}, h_{(2)})(\Box(h))\Big),$$

where $\triangle(h) = g^{ij}\delta_i\delta_j(h)$, $\square(h) = g^{ij}\delta_i(h)\cdot\delta_j(h)$. *The functions* K_d *and* H_d *are given by*

$$\begin{aligned} K_d(t_0,t_1) =& K_d^t(f(t_0), f(t_1))\big[t_0,t_1;f\big],\\ H_d(t_0,t_1,t_2) =& H_d^t(f(t_0), f(t_1), f(t_2))\big[t_0,t_1;f\big]\big[t_1,t_2;f\big]\\ &+2K_d^t(f(t_0), f(t_2))\big[t_0,t_1,t_2;f\big], \end{aligned} \tag{74}$$

where K_d^t *and* H_d^t *are the functions* K_d *and* H_d *when* $f(t)=t$. *For* $d\neq 2$, *they can be computed to be*

$$\begin{aligned} K_d^t(x,y) =& \frac{4\,x^{2-\frac{3d}{4}}y^{2-\frac{3d}{4}}}{d(d-2)(x-y)^3}\\ &\left((d-1)x^{\frac{d}{2}}y^{\frac{d}{2}-1}-(d-1)x^{\frac{d}{2}-1}y^{\frac{d}{2}}-x^{d-1}+y^{d-1}\right), \end{aligned}$$

and

$$\begin{aligned} H_d^t(x,y,z) =& \frac{2x^{-\frac{3d}{4}}y^{-d}z^{-\frac{3d}{4}}}{(d-2)d(x-y)^2(x-z)^3(y-z)^2}\\ &\times\Big(x^dy^dz^2(x-y)\left(3x^2y-2x^2z-4xy^2+4xyz-2xz^2+yz^2\right)\\ &+x^dy^{\frac{d}{2}+1}z^{\frac{d}{2}+1}(x-z)^2(z-y)(dx+(1-d)y)\\ &+x^dy^3z^d(z-x)^3+x^{\frac{d}{2}+1}y^{\frac{3d}{2}}z^2(x-y)(x-z)^2\\ &+2(d-1)x^{\frac{d}{2}+1}y^dz^{\frac{d}{2}+1}(x-y)(x-z)(z-y)(x-2y+z)\\ &-x^{\frac{d}{2}+1}y^{\frac{d}{2}+1}z^d(x-y)(x-z)^2((1-d)y+dz)\\ &-x^2y^{\frac{3d}{2}}z^{\frac{d}{2}+1}(x-z)^2(z-y)\\ &+x^2y^dz^d(y-z)\left(x^2y-2x^2z+4xyz-2xz^2-4y^2z+3yz^2\right)\Big). \end{aligned}$$

These functions for the dimension two are given by

$$\begin{aligned} K_2^t(x,y) =& -\frac{\sqrt{x}\sqrt{y}}{(x-y)^3}((x+y)\log(x/y)+2(y-x)),\\ H_2^t(x,y,z) =& \frac{2\sqrt{x}\sqrt{z}}{(x-y)^2(x-z)^3(y-z)^2}\\ &\times\Big(-(x-y)(x-z)(y-z)(x-2y+z)+y(x-z)^3\log(y) \end{aligned}$$

$$+ (y-z)^2(-2x^2+xy+yz)\log(x)$$
$$- (x-y)^2(xy+zy-2z^2)\log(z))\Big).$$

Note that the Function $K_d^t(x, y)$ is the symmetric part of the function

$$\frac{8x^{2-\frac{d}{4}}y^{2-\frac{3d}{4}}}{d(d-2)(x-y)^3}\left((d-1)y^{\frac{d}{2}-1} - x^{\frac{d}{2}-1}\right).$$

Similarly, $H_d^t(x, y, z)$ is equal to $(F_d(x, y, z) + F_d(z, y, x))/2$ where

$$\begin{aligned}F_d(x, y, z) =& \frac{4x^{-\frac{3d}{4}}y^{-d}z^{-\frac{3d}{4}}}{d(d-2)(x-y)^2(x-z)^3(y-z)^2}\\
&\times\Big(x^d y^d z^2(x-y)(3x^2y-2x^2z-4xy^2+4xyz-2xz^2+yz^2)\\
&+x^d y^{\frac{d}{2}+1}z^{\frac{d}{2}+1}(x-z)^2(z-y)(dx+(1-d)y)\\
&+\frac{1}{2}x^d y^3 z^d(z-x)^3\\
&+x^{\frac{d}{2}+1}y^{\frac{3d}{2}}z^2(x-y)(x-z)^2\\
&+2(d-1)x^{\frac{d}{2}+1}y^d z^{\frac{d}{2}+1}(x-y)^2(x-z)(z-y)\Big).\end{aligned}$$

As we recalled in earlier sections, in low dimensions two, three, and four, the curvature of the conformally flat metrics was computed in [21, 31, 28, 32, 24]. It is shown in [35] that the above general formula reproduces those results. We should first note that the functions found in all the aforementioned works are written in terms of the commutator $[h, \cdot]$, denoted by Δ. To produce those functions from our result, a linear substitution of the variables t_j in terms of new variables s_j is needed. On the other hand, it is important to note that the functions $K_d^t(x, y)$ and $H_d^t(x, y, z)$ are homogeneous rational functions of order $-\frac{d}{2}$ and $-\frac{d}{2} - 1$, respectively. Using formula (74), it is clear that the functions $K_d(t_0, t_1)$ and $H_d(t_0, t_1, t_2)$ are homogeneous of order $1 - \frac{d}{2}$ in $f(t_j)$'s. This is the reason that for function $f(t) = e^t$ and a linear substitution such as $t_j = \sum_{m=0}^{j} s_m$, a factor of some power of e^{s_0} comes out. This term can be replaced by a power of e^h multiplied from the left to the final outcome. This explains how the functions in the aforementioned papers have one less variable than our functions. In other words, we have

$$K_d(s_0, s_0{+}s_1) = e^{(1-\frac{d}{2})s_0}K_d(s_1),\;\; H_d(s_0, s_0{+}s_1, s_0{+}s_1{+}s_2) = e^{(1-\frac{d}{2})s_0}H_d(s_1, s_2).$$

For instance, function $K_d(s)$ is given by

$$K_d(s_1) = \frac{8e^{\frac{d+2}{4}s_1}\left((d-1)\sinh\left(\frac{s_1}{2}\right)+\sinh\left(\frac{(1-d)s_1}{2}\right)\right)}{d(d-2)d\,(e^{s_1}-1)^2\,s_1}.$$

Now, we can obtain functions in dimension two:

$$H_2(s_1) = -\frac{e^{\frac{s_1}{2}}\,(e^{s_1}\,(s_1-2)+s_1+2)}{(e^{s_1}-1)^2 s_1},$$

$$\begin{aligned}K_2(s_1,s_2) = \Big(&s_1(s_1+s_2)\cosh(s_2)-(s_1-s_2)\\ &\times(s_1+s_2+\sinh(s_1)+\sinh(s_2)-\sinh(s_1+s_2))\\ &-s_2(s_1+s_2)\cosh(s_1)\Big)\\ &\times\operatorname{csch}\left(\frac{s_1}{2}\right)\operatorname{csch}\left(\frac{s_2}{2}\right)\operatorname{csch}^2\left(\frac{s_1+s_2}{2}\right)/(4s_1s_2(s_1+s_2)).\end{aligned}$$

we have $-4H_2 = H$ and $-2K_2 = K$ where K and H are the functions found in [21, 31]. The difference is coming from the fact that the noncommutative parts of the results in [28, Section 5.1] are $\triangle(\log(e^{h/2})) = \frac{1}{2}\triangle(h)$ and $\square(\log(e^{h/2})) = \frac{1}{4}\square(h)$.

The functions for dimension four, with the same conformal factor $f(t) = e^t$ and substitution $t_j = \sum_{m=0}^{j} s_m$, gives the following which up to a negative sign are in complete agreement with the results from our papers [32, 28]:

$$H_4(s_1) = \frac{1-e^{s_1}}{2e^{s_1}s_1}, \quad K_4(s_1,s_2) = \frac{(e^{s_1}-1)\,(3e^{s_2}+1)\,s_2-(e^{s_1}+3)\,(e^{s_2}-1)\,s_1}{4e^{s_1+s_2}s_1s_2\,(s_1+s_2)}.$$

To recover the functions for curvature of a noncommutative three torus equipped with a conformally flat metric obtained in [42, 24], we need to set $f(t) = e^{2t}$ and $t_0 = s_0,\ t_1 = s_0 + s_1/3$ and $t_2 = s_0 + (s_1+s_2)/3$. Then up to a factor of e^{-s_0}, we have

$$H_3(s_1) = \frac{4-4e^{\frac{s_1}{3}}}{e^{\frac{s_1}{6}}\,(s_1e^{\frac{s_1}{3}}+1)}, \quad K_3(s_1,s_2) = \frac{6(e^{\frac{s_1}{3}}-1)(3e^{\frac{s_2}{3}}+1)s_2 - 6(e^{\frac{s_1}{3}}+3)(e^{\frac{s_2}{3}}-1)s_1}{e^{\frac{s_1+s_2}{6}}\,(e^{\frac{s_1+s_2}{3}}+1)s_1s_2(s_1+s_2)}.$$

Finally, one needs to check the classical limit of these formulas as $\theta \to 0$. In the commutative case, the scalar curvature of a conformally flat metric $\tilde{g} = e^{2h}g$ on a d-dimensional space reads

$$\tilde{R} = -2(d-1)e^{-2h}g^{jk}\partial_j\partial_k(h) - (d-2)(d-1)e^{-2h}g^{jk}\partial_j(h)\partial_k(h).$$

For $f(t) = e^{-2t}$, the limit is

$$\lim_{t_0,t_1\to t} K_d(t_0,t_1)=\frac{1}{3}(d-1)e^{(d-2)t}, \lim_{t_0,t_1,t_2\to t} H_d(t_0,t_1,t_2)=\frac{1}{6}(d-2)(d-1)e^{(d-2)t}.$$

We should also add that since $\delta_j \to -i\partial_j$ as $\theta \to 0$, we have $\triangle(h) \to -g^{jk}\partial_j\partial_k(h)$ and $\Box(h) \to -g^{jk}\partial_j(h)\partial_k(h)$. Therefore, these results recover the classical result up to a factor of $\sqrt{|g|}e^{dh}/6$. The factor $\sqrt{|g|}e^{dh}$ represents the volume form in the scalar curvature density and the factor $1/6$ is due to the choice of normalization in (9.5).

9.6 Twisted product, warped product, and scalar curvature

In this section, following [35], we shall recall the computation of the scalar curvature density of a noncommutative d-torus equipped with a class of functional metrics, which is called a twisted product metric.

Definition 9.3 ([35]) *Let g be an $r\times r$ and $\tilde{g}$ be a $(d-r)\times(d-r)$ positive definite real symmetric matrices and assume f is a positive function on the real line. The functional metric*

$$G = f(t)^{-1}g\oplus\tilde{g}, \tag{75}$$

is called a twisted product functional metric with the twisting element $f(h)^{-1}$.

Some examples of the twisted product metrics on noncommutative tori were already studied. The asymmetric two torus whose Dirac operator and spectral invariants are studied in [23] is a twisted product metric for $r=1$. The scalar and Ricci curvature of noncommutative three torus of twisted product metrics with $r=2$ are studied in [24]. It is worth mentioning that conformally flat metrics as well as warped metrics are two special cases of twisted product functional metrics. The following theorem is proved in [35].

Theorem 9.3 *The scalar curvature density of the d-dimensional noncommutative tori $\mathbb{T}^d_\theta$ equipped with the twisted product functional metric (75) with the twisting element $f(h)^{-1}$ is given by*

$$R=\sqrt{|g||\tilde{g}|}\Big(K_r(h_{(0)},h_{(1)})(\triangle(h))+H_r(h_{(0)},h_{(1)},h_{(2)})(\Box(h))$$
$$+\tilde{K}_r(h_{(0)},h_{(1)})(\tilde{\triangle}(h))+\tilde{H}_r(h_{(0)},h_{(1)},h_{(2)})(\tilde{\Box}(h))\Big),$$

where $\tilde{\triangle}(h)=\sum_{r<i,j}\tilde{g}^{ij}\delta_i\delta_j(h)$ and $\tilde{\Box}(h)=\sum_{r<i,j}\tilde{g}^{ij}\delta_i(h)\delta_j(h)$ and $\triangle$, $\Box$, K_r and H_r are given by Theorem 9.2. The functions $\tilde{K}_r$ and $\tilde{H}_r$ for $r\neq 2,4$ are given by

$$\tilde{K}_r(t_0,t_1)=\tilde{K}^t_r(f(t_0),f(t_1))\big[t_0,t_1;f\big],$$

$$\tilde{H}_r(t_0,t_1,t_2) = \tilde{H}_r^t(f(t_0), f(t_1), f(t_2))\big[t_0,t_1; f\big]\big[t_1,t_2; f\big] + 2\tilde{K}_r^t(f(t_0), f(t_2))\big[t_0,t_1,t_2; f\big].$$

The functions $\tilde{K}_r^t$ *and* $\tilde{H}_r^t$ *are*

$$\tilde{K}_r^t(x,y) = \frac{(2r-4)(x^2-y^2)x^{\frac{r}{2}}y^{\frac{r}{2}} + 4x^2y^r - 4x^ry^2}{(r-4)(r-2)x^{\frac{3}{4}r}y^{\frac{3}{4}r}(x-y)^3},$$

and

$$\begin{aligned}
\tilde{H}_r^t(x,y,z) = {} & \frac{2x^{-\frac{3r}{4}}y^{-r}z^{-\frac{3r}{4}}}{(r-4)(r-2)(x-y)^2(x-z)^3(y-z)^2} \\
& \times \Big(x^ry^rz^2(x-y)\Big(x^2+2x(y-2z)-4y^2+6yz-z^2\Big) \\
& + x^ry^{\frac{r}{2}}z^{\frac{r}{2}}(x-z)^2(y-z)\big((r-3)yz - x((r-3)z+y)\big) \\
& - x^ry^2z^r(x-z)^3 + x^{\frac{r}{2}}y^{\frac{3r}{2}}z^2(x-y)(x-z)^2 \\
& - x^{\frac{r}{2}}y^rz^{\frac{r}{2}}(x-y)(x-z)(y-z)\big((r-3)x^2 - 2x((r-2)y+(1-r)z) \\
& \quad + z((r-3)z-2(r-2)y)\big) \\
& + x^{\frac{r}{2}}y^{\frac{r}{2}}z^r(y-x)(x-z)^2(yz-(r-3)x(y-z)) \\
& + x^2y^{\frac{3r}{2}}z^{\frac{r}{2}}(x-z)^2(y-z) - x^2y^rz^r(y-z) \\
& \quad \times \Big(x^2-6xy+4xz+4y^2-2yz-z^2\Big)\Big).
\end{aligned}$$

When the selfadjoint element $h \in A_\theta^d$ has the property that $\delta_j(h) = 0$ for $1 \le j \le r$, we call the twisted product functional metric (75) a *warped functional metric* with the warping element $1/f(h)$.

Corollary 9.1 *The scalar curvature density of a warped product of* $\tilde{g}$ *and* g *with the warping element* $1/f(h)$ *is given by*

$$R = \sqrt{|g||\tilde{g}|}\Big(\tilde{K}_r(h_{(0)},h_{(1)})(\tilde{\triangle}(h)) + \tilde{H}_r(h_{(0)},h_{(1)},h_{(2)})(\tilde{\Box}(h))\Big).$$

Proof It is enough to see that $\triangle(h)$ and $\Box(h)$ vanish for the warped metric. □

For $r=2$ and $r=4$, functions $\tilde{H}_r$ and $\tilde{K}_r$ are the limit of the functions given in Theorem 9.3 as r approaches 2 or 4. This is because of the fact that for these values of r, some of $T_{\boldsymbol{\alpha}}^k$ functions are the limit case of formulas found earlier. For $r=2$ we have

$$\tilde{K}_2(x,y) = \frac{-x^2+y^2+2xy\log(\frac{x}{y})}{\sqrt{xy}(x-y)^3},$$

and

$$\begin{aligned}\tilde{H}_2(x,y,z) = &\frac{1}{2y\sqrt{xz}(x-y)^2(x-z)^3(y-z)^2}\\
&\times\Big(-y(x+y)(x-z)^3(y+z)\log(y)\\
&-z(x-y)^2\Big(-3x^2y+x^2z-8xy^2\\
&+10xyz-2xz^2+yz^2+z^3\Big)\log(z)\\
&+x(y-z)^2\Big(x^3+x^2y-2x^2z\\
&\quad+10xyz+xz^2-8y^2z-3yz^2\Big)\log(x)\\
&+2y(y-x)(x-z)(x+z)(z-y)(x-2y+z)\Big).\end{aligned}$$

For $r=4$, we have

$$\tilde{K}_4(x,y) = \frac{x^2-y^2-(x^2+y^2)\log(\frac{x}{y})}{xy(x-y)^3},$$

and

$$\begin{aligned}\tilde{H}_4(x,y,z) = &\frac{1}{2x(x-y)^2y^2(x-z)^3(y-z)^2z}\\
&\times\Big(\Big(x^2+y^2\Big)(x-z)^3\Big(y^2+z^2\Big)\log(y)\\
&+\log(x)(y-z)^2\Big(x^4y+x^4z-6x^3y^2-2x^3yz-2x^3z^2+3x^2y^3\\
&\quad+x^2y^2z+x^2yz^2+x^2z^3+2xy^3z-4xy^2z^2+3y^3z^2+y^2z^3\Big)\\
&-\log(z)(x-y)^2\Big(x^3y^2+x^3z^2+3x^2y^3-4x^2y^2z+x^2yz^2\\
&\quad-2x^2z^3+2xy^3z+xy^2z^2-2xyz^3+xz^4+3y^3z^2\\
&\quad-6y^2z^3+yz^4\Big)\\
&-2(x-y)(x-z)(y-z)\Big(x^3z+x^2y^2-2x^2z^2-2xy^3+2xy^2z\\
&\quad+xz^3-2y^3z+y^2z^2\Big)\Big).\end{aligned}$$

In [24, section 4.1], the scalar curvature density of twisted product functional metric on noncommutative three torus for $f(t) = e^{2t}$ and $r = 2$ is found. This result can be recovered from our formulas given in Theorem 9.3 by setting $t_0 = s_0$, $t_1 = s_0 + s_1/2$ and $t_2 = s_0 + s_1/2 + s_2/2$.

9.7 Dimension two and Gauss-Bonnet theorem

The following result which is proved in [35] shows that the total scalar curvature of a noncommutative two torus equipped with a functional metric g is independent of g. This result extends the Gauss-Bonnet theorem of [22, 30] earlier proved for conformally flat metrics. This is done by a careful study of the functions F_S^{ij} in dimension two, where it is shown that these functions vanish for the noncommutative two torus equipped with a functional metric g. This means that the total scalar curvature of $(\mathbb{T}_\theta^2, g)$ is independent of g. Similar to the case of conformally flat metrics, we call this result the Gauss-Bonnet theorem for functional metrics.

Theorem 9.4 (Gauss-Bonnet Theorem [35]) *The total scalar curvature $\varphi(R)$ of the noncommutative two tori equipped with a functional metric vanishes, hence it is independent of the metric.*

Let us summarize the results obtained in [35] where a new family of metrics, called functional metrics, on noncommutative tori is introduced and their spectral geometry is studied. A class of Laplace type operators for these metrics is introduced and their spectral invariants are obtained from the heat trace asymptotics. A formula for the second density of the heat trace is also obtained. In particular, the scalar curvature density and the total scalar curvature of functional metrics are explicitly computed in all dimensions for certain classes of metrics including conformally flat metrics and twisted product of flat metrics. Finally a Gauss-Bonnet type theorem for a noncommutative two torus equipped with a general functional metric is proved.

10 Matrix Gauss-Bonnet

As we emphasized in the previous section, it is quite important to go beyond conformally flat metrics, go beyond noncommutative tori, and beyond dimension four. For example, one naturally needs to consider noncommutative algebras that would represent higher genus noncommutative curves and other noncommutative manifolds. As far as noncommutative higher genus curves go, there is as yet no satisfactory theory, even at a topological level, and much less at a metric or spectral level. This is a largely uninvestigated area and we expect new methods and ideas will be needed to make further progress with these objects.

A reasonable class of noncommutative algebras are algebras of matrix valued functions on a smooth manifold. Now topologically they are Morita equivalent to commutative algebras and not so interesting, but their spectral geometry poses interesting questions. A first step was taken in [43] to address this question. In this paper a new class of noncommutative algebras that are amenable to spectral analysis, namely algebras of matrix valued functions on a Riemann surface of arbitrary genus, are studied. The Dirac operator is conformally rescaled by a diagonalizable matrix and a Gauss-Bonnet theorem is proved for them. This is the matrix Gauss-Bonnet in the title of this section. When the surface has genus one, scalar curvature is explicitly computed. We shall briefly sketch these results in this section.

Let M be a two-dimensional closed spin Riemannian manifold and consider the algebra of smooth matrix valued functions on M:

$$\mathcal{A} = C^{\infty}(M, M_n(\mathbb{C})).$$

The Dirac operator of M, $D : L^2(S) \to L^2(S)$ acts on the Hilbert space of spinors. The algebra $\mathcal{A}$ acts diagonally on the Hilbert space $\mathcal{H} = L^2(S) \otimes \mathbb{C}^n$ and we have a spectral triple.

Let $h \in \mathcal{A}$ be a positive element. We use h to perturb the spectral triple of $\mathcal{A}$ in the following way. Consider the operator $D_h = hDh$ as a conformally rescaled Dirac operator. Now D_h does not have bounded commutators with the elements of $\mathcal{A}$, but we still have a twisted spectral triple. This is similar to the situation with curved noncommutative tori. The question is if the Gauss-Bonnet theorem holds for D_h. One is also interested in knowing if the scalar curvature can be computed explicitly. The answer is positive as we sketch now.

To simplify the matters a bit, it is assumed that the Weyl conformal factor h is diagonalizable, that is $h = UHU^*$, where U is unitary and H is diagonal. Then we have

$$hDh = UHU^*DUHU^* = U\left(H\left(D + U^*[D, U]\right)H\right)U^*,$$

which shows that the spectrum of D_h and $D_{A,H} = H(D + A)H$ are equal. Here $A = U^*[D, U]$ is a matrix valued one-form on M and $D + A$ represents a fluctuation of the geometry represented by D. It is shown in [43] that the Gauss-Bonnet theorem holds for the family of conformally rescaled Dirac operators with possible fluctuations $D_{A,H} = H(D + A)H$ as above. Local expressions for the scalar curvature are computed as well. The results demonstrate that unlike the case of higher residues in [38], the expressions for the value of the ζ function at 0 are complicated also in the matrix case.

Let us consider first the canonical spectral triple for a flat torus $M = \mathbb{R}^2/\mathbb{Z}^2$. Its spin structure is defined by the Pauli spin matrices σ^1, σ^2 and its Dirac operator is

$$D = \sigma^1\delta_1 + \sigma^2\delta_2.$$

Here δ_1, δ_2 are the partial derivatives $\frac{1}{i}\frac{\partial}{\partial x}$ and $\frac{1}{i}\frac{\partial}{\partial y}$. To compute the resolvent kernel we work in the algebra of matrix valued pseudodifferential operators obtained by tensoring the algebra Ψ of pseudodifferential operators on a smooth manifold M by the algebra of n by n matrices.

The Resolvent The symbol of the Bochner Laplacian $D_{A,H}^2 = H(D+A)H^2(D+A)H$ is given by $\sigma_{D_{A,H}^2} = a_2 + a_1 + a_0$, where

$$
\begin{aligned}
a_2 &= H^4\xi^2,\\
a_1 &= i\epsilon_{ij}\sigma^3 2H^3\delta_i(H)\xi^j + 4H^3\delta_i(H)\xi^i - i\epsilon_{ij}\sigma^3 H^3 A_i H\xi^j\\
&\quad + H^3 A_i H\xi^i + i\epsilon_{ij}\sigma^3 HA_i H^3\xi^j + HA_i H^3\xi^i,\\
a_0 &= H^4(\Delta H) + H^3 A_j\delta_i(H) - H^3 i\sigma^3\epsilon_{ij}A_i\delta_j(H) + H^3\delta_i(A_i)H\\
&\quad + i\sigma^3 H^3\epsilon_{ij}\delta_j(A_i)H + 2H^2\delta_i(H)\delta_i(H) + 2H^2\delta_i(H)A_i H\\
&\quad + 2HA_i H^2\delta_i(H) + 2i\sigma^3 H^2\epsilon_{ij}\delta_i(H)A_j H\\
&\quad + i\sigma^3\epsilon_{ij}HA_i H^2\delta_i(H) + i\sigma^3\epsilon_{ij}HA_i H^2 A_j + HA_i H^2 A_i H.
\end{aligned}
$$

The first three terms of the symbols of $(D_{H,a})^{-2} = b_0 + b_1 + b_2 + \cdots$ are:

$$
\begin{aligned}
b_0 &= (a_2+1)^{-1},\\
b_1 &= -\left(b_0 a_1 + \partial_k(b_0)\delta_k(a_2)\right)b_0,\\
b_2 &= -\left(b_1 a_1 + b_0 a_0 + \partial_k(b_0)\delta_k(a_1) + \partial_k(b_1)\delta_k(a_2) + \frac{1}{2}\partial_k\partial_j(b_0)\delta_k\delta_j(a_2)\right)b_0.
\end{aligned}
$$

10.1 Matrix curvature

Let us call a matrix-valued function $R : \mathbb{T}^2 \to M_n(\mathbb{C})$ the scalar curvature if for any matrix valued function $f \in \mathcal{A}$ we have:

$$\zeta_{f,D}(0) = \int_{\mathbb{T}^2} \mathrm{Tr}\, fR,$$

where the localized spectral zeta function is defined by

$$\zeta_{f,D}(s) = \mathrm{Tr}\ f|D|^{-s}.$$

It is found that four terms contribute to the scalar curvature R [43]:

Terms not Depending on A They depend only on H and derivatives of H, and since they commute with each other, they can be computed as in the classical case.

$$\begin{aligned} b_2(H,\xi) =& 96\, b_0^5 \delta_i(H)\delta_i(H) H^{14}(\xi^2)^3 - 136\, b_0^4 \delta_i(H)\delta_i(H) H^{10}(\xi^2)^2 \\ &+ 46\, b_0^3 \delta_i(H)\delta_i(H) H^6(\xi^2) - 2\, b_0^2 \delta_i(H)\delta_i(H) H^2 \\ &- 8\, b_0^4 \Delta(H) H^{11}(\xi^2)^2 + 8\, b_0^3 \Delta(H) H^7(\xi^2) - b_0^2 \Delta(H) H^3, \end{aligned} \tag{76}$$

To integrate over the ξ space, we can use the formula

$$\int_0^\infty \frac{r^{2k+1} dr}{(1+a^2r^2)^{2k+3}} = \frac{1}{2(k+1)a^{2(k+1)}},$$

and obtain

$$R(H) = -\pi \left(\frac{1}{3} H^{-2} \delta_i(H)\delta_i(H) + \frac{1}{3} H^{-1} \Delta(H) \right). \tag{77}$$

To continue, we use the rearrangement lemma of [46]. Let

$$\Delta(x) = H^{-4} x H^4.$$

Terms Linear in A We have:

$$\begin{aligned} b_2^{(1)}(H,A) = & -b_0 h A_i b_0 \delta_i(H) H^2 + 5 b_0 h A_i b_0^2 \delta_i(H) H^6 \xi^2 \\ & - 4 b_0 h A_i b_0^3 \delta_i(H) H^{10} (\xi^2)^2 \\ & - b_0 H^3 A_i b_0 \delta_i(H) + 7 b_0 H^3 A_i b_0^2 \delta_i(H) H^4 \xi^2 \\ & - 4 b_0 H^3 A_i b_0^3 \delta_i(H) H^8 (\xi^2)^2 \\ & + 3 b_0^2 H^5 A_i b_0 \delta_i(H) H^2 \xi^2 - 4 b_0^2 H^5 A_i b_0^2 \delta_i(H) H^6 (\xi^2)^2 \\ & + b_0^2 H^7 A_i b_0 \delta_i(H) \xi^2 - 4 b_0^2 H^7 A_i b_0^2 \delta_i(H) H^4 (\xi^2)^2 \end{aligned}$$

and

$$\begin{aligned} b_2^{(1)}(H,A) = & -2 b_0 \delta_i(H) H^2 A_i b_0 H + 2 b_0^2 \delta_i(H) H^4 A_i b_0 H^3 \xi^2 \\ & + 6 b_0^2 \delta_i(H) H^6 A_i b_0 H \xi^2 - 4 b_0^3 \delta_i(H) H^8 A_i b_0 H^3 (\xi^2)^2 \\ & - 4 b_0^3 \delta_i(H) H^{10} A_i b_0 H (\xi^2)^2. \end{aligned}$$

Explicit computations give

$$R^{(1)}(H, A) = \sum_{i=1,2} 2\pi H G(\Delta)(A_i)\delta_i(H),$$

where G is the following function:

$$G(s) = \frac{(1+\sqrt{s})\sqrt{s}}{(s-1)^3}\big((s+1)\ln(s) - 2(s-1)\big),$$

and a second term,

$$R^{(2)}(H, A) = \sum_{i=1,2} -2\pi H^{-2}\delta_i(H)G(\Delta)(A_i)H,$$

with the same function $G(s)$. After taking the trace these terms cancel each other, and we get

$$\text{Tr}\,\Big(R^{(1)}(H, A) + R^{(2)}(H, A)\Big) = 0.$$

Terms Linear in $\delta_i(A_i)$ In this case we have:

$$b_2(H, \delta_i(A_j)) = -b_0 H^3\delta_i(A_i)b_0 H + b_0^2 H^5\delta_i(A_i)b_0 H^3\xi^2 \\ + b_0^2 H^7\delta_i(A_i)b_0 H\xi^2,$$

and integrating out the ξ variables, we get $\pi H^{-1}F(\Delta)(\delta_i(A_i))H$, where

$$F = -\frac{(1+\sqrt{s})\sqrt{s}}{(s-1)^2}\ln(s) + \frac{\sqrt{s}+1}{s-1}.$$

Again, it is not difficult to check that $F(1) = 0$ and the expression vanishes after taking the trace:

$$\text{Tr}\,\big(R(H, \delta_i(A_j))\big) = 0.$$

Quadratic Terms in A_i We have:

$$b_2(H, A^2) = -b_0 H A_i H^2 A_i b_0 H + b_0 H A_i b_0 H^6 A_i b_0 H\xi^2 \\ + b_0 H^3 A_i b_0 H^2 A_i b_0 H^3\xi^2.$$

Integrating over ξ we obtain:

$$R(H, A^2) = -\pi H^{-1}Q(\Delta^{(1)}, \Delta^{(2)})(A_i \cdot A_i)H$$

where

$$Q(s,t) = \frac{\sqrt{s}(\sqrt{t}+s)}{(s-1))(s-t)}\ \ln s - \frac{\sqrt{s}\sqrt{s}}{(s-t)\sqrt{t}}\ \ln t.$$

To compute the trace, let

$$F(s) = Q(s,1) = \frac{(s+1)\ln s + 2(1-s)}{(s-1)^2},$$

and observe that due to the trace property:

$$\begin{aligned}\operatorname{Tr}\left(H^{-1}F(\Delta)(A_i)A_iH\right) &= \operatorname{Tr}\left(A_iF(\Delta)(A_i)\right)\\ &= \operatorname{Tr}\left(F(\Delta^{-1})(A_i)A_i\right).\end{aligned}$$

Now since

$$\begin{aligned}F(\frac{1}{s}) &= \frac{-(\frac{1}{s}+1)\ln s + 2(1-\frac{1}{s})}{(\frac{1}{s}-1)^2}\\ &= \frac{-(s+1)\ln s - 2(1-s)}{(s-1)^2}\\ &= -F(s),\end{aligned}$$

one gets

$$\operatorname{Tr} R(H, A^2) = 0,$$

and so the quadratic term vanishes as well.

10.2 The Gauss-Bonnet theorem

The term which does not depend on A is a total derivative term:

$$\frac{1}{3}\delta_i\left(H^{-1}\delta_i(H)\right).$$

Using Stokes theorem, this term is seen to vanish as well after integration. Putting it all together, one thus obtains:

Proposition 10.1 *For the matrix conformally rescaled Dirac operator on the two-dimensional torus, $D_h = hDh$, where h is a globally diagonalizable positive matrix, the Gauss-Bonnet theorem holds:*

$$\zeta_{D_h}(0) = \zeta_D(0).$$

10.3 Higher genus matrix Gauss-Bonnet

Now we look at the general case where M is a closed Riemann surface with a spin structure and a Dirac operator D. Consider the operator

$$D_{H,A} = H(D + A)H,$$

for H a diagonal matrix valued function on M and A a matrix-valued one-form, identified here with its Clifford image.

We must now compute the value of $\zeta_{D^2_{H,A}}(0)$ using methods of pseudodifferential calculus. Let us denote the symbols of D^2_H as:

$$D^2_H = (HDH)^2 = a^H_2 + a^H_1 + a^H_0,$$

and the symbol of D^2 as

$$D^2 = a^o_2 + a^o_1 + a^o_0.$$

As in the case of the torus, the computation is divided into the cases of terms not depending on A, linear in A and quadratic in A.

Terms Independent of A Since H is globally diagonalizable, we can assume it is scalar. Thus we are reduced to a conformal rescaling of the classical Dirac operator. Since in this case the Gauss-Bonnet theorem holds, it remains only to see that the contribution to the Gauss-Bonnet term from the linear and quadratic terms in A vanishes.

Terms Linear in A Linear terms do arise in b_2 from the following terms:

$$\begin{aligned} &b_0a^H_1 b_0 a_1(A) b_0 + \partial^\xi_k(b_0)\partial^x_k(a^H_2) b_0 a_1(A) b_0 + b_0 a_1(A) b_0 a^H_1 b_0 \\ &\quad - b_0 a_0(A) b_0 - \partial^\xi_k(b_0)\partial^x_k(a_1(A)) b_0 - \partial^\xi_k(b_0 a_1(A) b_0)\partial^x_k(a^H_2) b_0. \end{aligned}$$

where $a_1(A), a_0(A)$ denote terms linear in A. Now, one can use normal coordinates at a given point x of M. The terms without derivatives reduce easily to the torus case. The only difficulty arises from terms with derivatives in x, that is, $\partial^x_k(a^H_2)$. and $\partial^x_k(a_1(A))$. Since $a^H_2 = H^4 g_{ij}\xi^i\xi^j$, and in normal coordinates the first derivatives of the metric vanish at the point x, we see that the only remaining term would be with the derivative of H^4, and again this term would be reduced to the term linear in A from the torus case.

Similar argument works also for the other term, $a_1(A)$, which is

$$\left(H^3 A_i H + H A_k H^3\right) \sigma^k \sigma^i \xi_i,$$

and since in $\partial_k^\xi(b_0)\partial_k^x(a_1(A))b_0$ there are no further σ matrices, one can compute first the trace over the Clifford algebra and write it as

$$\frac{1}{2}\left(H^3 A_i H + H A_k H^3\right) g^{ki} \xi_i.$$

Thus, in normal coordinates around x the expression is identical to the one for the flat torus. Therefore, the integration over ξ would yield the same result, and the density of the linear A contribution to the trace of b_2 vanishes at x. Consequently, the contribution to the Gauss-Bonnet term linear in A also vanishes.

Quadratic Terms The quadratic terms in A are

$$b_0 a_1(A) b_0 a_1(A) b_0 - b_0 a_0(A^2) b_0.$$

It is easy to see that in normal coordinates we have:

$$a_1(A) = \left(H^3 \sigma^j \xi_j (\sigma^i A_i) H + \sigma^i H A_i H^3 \sigma^j \xi_j\right),$$
$$a_0(A) = (\sigma^i H A_i H)(\sigma^k H A_k H).$$

Using normal coordinates, one reduces the ξ -integral to the situation already considered for the torus. Hence the density of Gauss-Bonnet term with quadratic contributions from A identically vanishes as well. This finishes the proof of Gauss-Bonnet for higher genus matrix valued functions with a general Dirac operator with fluctuation. This result was obtained in [43].

11 Curvature of the determinant line bundle

It would be interesting to know how far our hard analytic methods like pseudodifferential operators, spectral analysis and heat equation techniques, can be pushed in the noncommutative realm, at least for noncommutative tori and toric manifolds. So far we have seen that these analytic techniques, suitably modified and enhanced, has been quite successful in dealing with scalar and Ricci curvature. Along this idea, in [25] the curvature of the determinant line bundle on a family of Dirac operators for a noncommutative two torus is computed. Following Quillen's original construction for Riemann surfaces [53] and using zeta regularized determinant of Laplacians, the determinant line bundle is endowed with a natural Hermitian metric. By defining an analogue of Kontsevich-Vishik canonical trace, defined on Connes' algebra of classical pseudodifferential symbols for the noncommutative two torus, the curvature form of the determinant line bundle is computed through the second variation $\delta_w \delta_{\bar{w}} \log \det(\Delta)$. Calculus of symbols and the canonical trace were effectively used to bypass local calculations involving Green functions in [53] which is not applicable in the noncommutative case. In a sequel paper [26], the spectral eta function for certain families of Dirac operators on noncommutative 3-torus is studied and its regularity at zero is proved. By using variational techniques, it is shown that the eta function $\eta_D(0)$ is a conformal invariant. By studying the Laurent expansion at zero of $\mathrm{TR}(|D|^{-z})$, the conformal invariance of $\zeta'_{|D|}(0)$ for noncommutative 3-torus is proved. Finally, for the coupled Dirac operator, a local formula for the variation $\partial_A \eta_{D+A}(0)$ is derived which is the analogue of the so-called induced Chern-Simons term in quantum field theory literature.

In this section we shall recall and comment on results obtained in [25] on the curvature of the determinant line bundle on a noncommutative torus.

11.1 The determinant line bundle

Let $\mathcal{F} = \mathrm{Fred}(\mathcal{H}_0, \mathcal{H}_1)$ denote the set of Fredholm operators between Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$. It is an open subset, in norm topology, in the complex Banach space of all bounded linear operators between $\mathcal{H}_0$ and $\mathcal{H}_1$. The index map $index : \mathcal{F} \to \mathbb{Z}$ is a homotopy invariant and in fact defines a bijection between connected components of $\mathcal{F}$ and the set of integers $\mathbb{Z}$. It is well known that $\mathcal{F}$ is a classifying space for K-theory (Atiyah-Janich): for any compact space X we have a natural ring isomorphism $K^0(X) = [X, \mathcal{F}]$ between the K-theory of X and the set of homotopy classes of continuous maps from X to $\mathcal{F}$.

In [53] Quillen defines a holomorphic line bundle DET $\to \mathcal{F}$ over the space of Fredholm operators such that for any $T \in \mathcal{F}$

$$\mathrm{DET}_T = \Lambda^{max}(\ker(T))^* \otimes \Lambda^{max}(\mathrm{coker}(T)).$$

This is remarkable if we notice that $\ker(T)$ and $\mathrm{coker}(T)$ are not vector bundles due to discontinuities in their dimensions as T varies within $\mathcal{F}$.

It is tempting to think that since $c_1(\mathrm{DET})$ is the generator of $H^2(\mathcal{F}_0, \mathbb{Z}) \cong \mathbb{Z}$, $\mathcal{F}_0$ being the index zero operators, there might exist a natural Hermitian metric on DET whose curvature 2-form would be a representative of this generator. One problem is that the induced metric from $\ker(T)$ and $\ker(T^*)$ on DET is not even continuous. In [53] Quillen shows that for families of Cauchy-Riemann operators on a Riemann surface one can correct the Hilbert space metric by multiplying it by zeta regularized determinant and in this way one obtains a smooth Hermitian metric on the induced determinant line bundle. In [25] a similar construction for the noncommutative two torus is given as we explain later in this section.

11.2 The canonical trace and noncommutative residue

To carry the calculations, an analogue of the canonical trace of [45] for the noncommutative torus is constructed in [25]. First we need to extend our original algebra of pseudodifferential operators to *classical* pseudodifferential operators.

A smooth map $\sigma : \mathbb{R}^2 \to \mathcal{A}_\theta$ is called a classical symbol of order $\alpha \in \mathbb{C}$ if for any N and each $0 \leq j \leq N$ there exist $\sigma_{\alpha-j} : \mathbb{R}^2 \backslash \{0\} \to \mathcal{A}_\theta$ positive homogeneous of degree $\alpha - j$ and a symbol $\sigma^N \in \mathcal{S}^{\Re(\alpha)-N-1}(\mathcal{A}_\theta)$, such that

$$\sigma(\xi) = \sum_{j=0}^{N} \chi(\xi)\sigma_{\alpha-j}(\xi) + \sigma^N(\xi) \quad \xi \in \mathbb{R}^2. \tag{78}$$

Here χ is a smooth cut-off function on $\mathbb{R}^2$ which is equal to zero on a small ball around the origin, and is equal to one outside the unit ball. It can be shown that the homogeneous terms in the expansion are uniquely determined by σ. We denote the set of classical symbols of order α by $\mathcal{S}^\alpha_{cl}(\mathcal{A}_\theta)$ and the associated classical pseudodifferential operators by $\Psi^\alpha_{cl}(\mathcal{A}_\theta)$.

The space of classical symbols $\mathcal{S}_{cl}(\mathcal{A}_\theta)$ is equipped with a Fréchet topology induced by the semi-norms

$$p_{\alpha,\beta}(\sigma) = \sup_{\xi \in \mathbb{R}^2} (1 + |\xi|)^{-m+|\beta|} ||\delta^\alpha \partial^\beta \sigma(\xi)||. \tag{79}$$

The analogue of the Wodzicki residue for classical pseudodifferential operators on the noncommutative torus is defined in [33].

Definition 11.1 *The Wodzicki residue of a classical pseudodifferential operator P_σ is defined as*

$$\mathrm{Res}(\mathrm{P}_\sigma) = \varphi_0\left(\mathrm{res}(\mathrm{P}_\sigma)\right),$$

where $\mathrm{res}(\mathrm{P}_\sigma) := \int_{|\xi|=1} \sigma_{-2}(\xi) d\xi$.

It is evident from its definition that Wodzicki residue vanishes on differential operators and on non-integer order classical pseudodifferential operators.

To define the analogue of the canonical trace on non-integer order pseudodifferential operators on the noncommutative torus, one needs the existence of the so-called cut-off integral for classical symbols.

Proposition 11.1 *Let $\sigma \in \mathcal{S}^{\alpha}_{cl}(\mathcal{A}_\theta)$ and $B(R)$ be a disk of radius R around the origin. One has the following asymptotic expansion as $R \to \infty$*

$$\int_{B(R)} \sigma(\xi)d\xi \sim \sum_{j=0,\alpha-j+2\neq 0}^{\infty} \alpha_j(\sigma)R^{\alpha-j+2} + \beta(\sigma)\log R + c(\sigma),$$

where $\beta(\sigma) = \int_{|\xi|=1} \sigma_{-2}(\xi)d\xi$ and the constant term in the expansion, $c(\sigma)$, is given by

$$\int_{\mathbb{R}^n} \sigma^N + \sum_{j=0}^{N} \int_{B(1)} \chi(\xi)\sigma_{\alpha-j}(\xi)d\xi - \sum_{j=0,\alpha-j+2\neq 0}^{N} \frac{1}{\alpha-j+2} \int_{|\xi|=1} \sigma_{\alpha-j}(\omega)d\omega. \tag{80}$$

Definition 11.2 *The cut-off integral of a symbol $\sigma \in \mathcal{S}^{\alpha}_{cl}(\mathcal{A}_\theta)$ is defined to be the constant term in the above asymptotic expansion, and we denote it by $⨍ \sigma(\xi)d\xi$.*

The cut-off integral of a symbol is independent of the choice of N. It is also independent of the choice of the cut-off function χ.

Definition 11.3 *The canonical trace of a classical pseudodifferential operator $P \in \Psi^{\alpha}_{cl}(\mathcal{A}_\theta)$ of non-integral order α is defined as*

$$\mathrm{TR}(P) := \varphi_0\left(⨍ \sigma_P(\xi)d\xi\right).$$

Note that any pseudodifferential operator P of order less than -2 is a trace-class operator on $\mathcal{H}_0$ and its trace is given by

$$\mathrm{Tr}(P) = \varphi_0\left(\int_{\mathbb{R}^2} \sigma_P(\xi)d\xi\right).$$

On the other hand, for such operators the symbol is integrable and we have

$$⨍ \sigma_P(\xi)d\xi = \int_{\mathbb{R}^2} \sigma_P(\xi)d\xi. \tag{81}$$

Therefore, the TR-functional and operator trace coincide on classical pseudodifferential operators of order less than -2.

The canonical trace TR is an analytic continuation of the operator trace and using this fact one can prove that it is actually a trace.

Proposition 11.2 *Given a holomorphic family $\sigma(z) \in \mathcal{S}^{\alpha(z)}_{cl}(\mathcal{A}_\theta)$, $z \in W \subset \mathbb{C}$, the map*

$$z \mapsto ⨍ \sigma(z)(\xi)d\xi,$$

is meromorphic with at most simple poles. Its residues are given by

$$\mathrm{Res}_{\mathrm{z}=\mathrm{z}_0} \fint \sigma(\mathrm{z})(\xi)\mathrm{d}\xi = -\frac{1}{\alpha'(\mathrm{z}_0)} \int_{|\xi|=1} \sigma(\mathrm{z}_0)_{-2}\mathrm{d}\xi.$$

Using the above result one can show that if $A \in \Psi_{cl}^{\alpha}(\mathcal{A}_\theta)$ is of order $\alpha \in \mathbb{Z}$ and Q is a positive elliptic classical pseudodifferential operator of positive order q, then

$$\mathrm{Res}_{\mathrm{z}=0}\mathrm{TR}(\mathrm{AQ}^{-\mathrm{z}}) = \frac{1}{\mathrm{q}}\mathrm{Res}(\mathrm{A}).$$

Using this and the uniqueness of analytic continuation one can prove the trace property of TR. That is, $\mathrm{TR}(AB) = \mathrm{TR}(BA)$ for any $A, B \in \Psi_{cl}(\mathcal{A}_\theta)$, provided that $\mathrm{ord}(A) + \mathrm{ord}(B) \notin \mathbb{Z}$.

11.3 Log-polyhomogeneous symbols

In general, z-derivatives of a classical holomorphic family of symbols is not classical anymore and therefore one needs to introduce log-polyhomogeneous symbols which include the z-derivatives of the symbols of the holomorphic family $\sigma(AQ^{-z})$.

Definition 11.4 *A symbol σ is called a log-polyhomogeneous symbol if it has the following form*

$$\sigma(\xi) \sim \sum_{j\geq 0}\sum_{l=0}^{\infty} \sigma_{\alpha-j,l}(\xi)\log^l|\xi| \quad |\xi|>0, \tag{82}$$

with $\sigma_{\alpha-j,l}$ positively homogeneous in ξ of degree $\alpha - j$.

A prototypical example of an operator with such a symbol is $\log Q$ where $Q \in \Psi_{cl}^{q}(\mathcal{A}_\theta)$ is a positive elliptic pseudodifferential operator of order $q > 0$. The logarithm of Q can be defined by

$$\log Q = Q\left.\frac{d}{dz}\right|_{z=0} Q^{z-1} = Q\left.\frac{d}{dz}\right|_{z=0} \frac{i}{2\pi}\int_C \lambda^{z-1}(Q-\lambda)^{-1}d\lambda.$$

For an operator A with log-polyhomogeneous symbol as (82) we define

$$\mathrm{res}(\mathrm{A}) = \int_{|\xi|=1} \sigma_{-2,0}(\xi)\mathrm{d}\xi.$$

The following result can be proved along the lines of its classical counterpart in [52].

Proposition 11.3 *Let $A \in \Psi^{\alpha}_{cl}(\mathcal{A}_\theta)$ and Q be a positive, in general an admissible, elliptic pseudodifferential operator of positive order q. If $\alpha \in P$, then 0 is a possible simple pole for the function $z \mapsto \mathrm{TR}(AQ^{-z})$ with the following Laurent expansion around zero.*

$$\begin{aligned}\mathrm{TR}(AQ^{-z}) &= \frac{1}{q}\mathrm{Res(A)}\frac{1}{z} \\ &+ \varphi_0\left(⨍ \sigma(A)(\xi)\mathrm{d}\xi - \frac{1}{q}\mathrm{res}(A\log Q)\right) - Tr(A\Pi_Q) \\ &+ \sum_{k=1}^{K}(-1)^k \frac{(z)^k}{k!} \\ &\times \left(\varphi_0\left(⨍ \sigma(A(\log Q)^k)(\xi)d\xi - \frac{1}{q(k+1)}\mathrm{res}(A(\log Q)^{k+1})\right)\right. \\ &\left. - Tr(A(\log Q)^k\Pi_Q)\right) + o(z^K).\end{aligned}$$

where Π_Q is the projection on the kernel of Q.

For operators A and Q as in the previous Proposition, the *generalized zeta function* is defined by

$$\zeta(A, Q, z) = \mathrm{TR}(AQ^{-z}). \tag{83}$$

From Proposition 11.2, it follows that $\zeta(A, Q, z)$ is a meromorphic function with simple poles. Moreover, $\zeta(A, Q, z)$ is the analytic continuation of the spectral zeta function $\mathrm{Tr}(AQ^{-z})$. If A is a differential operator, the zeta function (83) is regular at $z = 0$ with a value

$$\varphi_0\left(⨍ \sigma(A)(\xi)d\xi - \frac{1}{q}\mathrm{res}(A\log Q)\right) - \mathrm{Tr}(A\Pi_Q).$$

11.4 Cauchy-Riemann operators on noncommutative tori

As we did before, we can fix a complex structure on A_θ by a complex number τ in the upper half plane. Consider the spectral triple

$$\left(A_\theta, \mathcal{H}_0 \oplus \mathcal{H}^{0,1}, D_0 = \begin{pmatrix} 0 & \bar\partial^* \\ \bar\partial & 0 \end{pmatrix}\right), \tag{84}$$

where $\bar{\partial} : A_\theta \to A_\theta$ is given by $\bar{\partial} = \delta_1 + \tau\delta_2$. The Hilbert space $\mathcal{H}_0$ is defined by the GNS construction from A_θ using the trace φ_0 and $\bar{\partial}^*$ is the adjoint of the operator $\bar{\partial}$.

As in the classical case, the Cauchy-Riemann operator on A_θ is the positive part of the twisted Dirac operator. All such operators define spectral triples of the form

$$\left(A_\theta, \mathcal{H}_0 \oplus \mathcal{H}^{0,1}, D_A = \begin{pmatrix} 0 & \bar{\partial}^* + \alpha^* \\ \bar{\partial} + \alpha & 0 \end{pmatrix}\right),$$

where $\alpha \in A_\theta$ is the positive part of a selfadjoint element

$$A = \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix} \in \Omega^1_{D_0}(A_\theta).$$

We recall that $\Omega^1_{D_0}(A_\theta)$ is the space of quantized one forms consisting of the elements $\sum a_i[D_0, b_i]$ where $a_i, b_i \in A_\theta$ [14]. Note that in this case the space $\mathcal{A}$ of Cauchy-Riemann operators is the space of (0, 1)-forms on A_θ.

11.5 *The curvature of the determinant line bundle for* A_θ

For any $\alpha \in \mathcal{A}$, the Cauchy-Riemann operator

$$\bar{\partial}_\alpha = \bar{\partial} + \alpha : \mathcal{H}_0 \to \mathcal{H}^{0,1}$$

is a Fredholm operator. We pull back the determinant line bundle DET on the space of Fredholm operators Fred$(\mathcal{H}_0, \mathcal{H}^{0,1})$, to get a line bundle $\mathcal{L}$ on $\mathcal{A}$. Following Quillen [53], one can define a Hermitian metric on $\mathcal{L}$ and the problem is to compute its curvature. Let us define a metric on the fiber

$$\mathcal{L}_\alpha = \Lambda^{max}(\ker \bar{\partial}_\alpha)^* \otimes \Lambda^{max}(\ker \bar{\partial}_\alpha^*)$$

as the product of the induced metrics on $\Lambda^{max}(\ker \bar{\partial}_\alpha)^*$ and $\Lambda^{max}(\ker \bar{\partial}_\alpha^*)$, with the zeta regularized determinant $e^{-\zeta'_{\Delta_\alpha}(0)}$. Here we define the Laplacian as $\Delta_\alpha = \bar{\partial}_\alpha^*\bar{\partial}_\alpha : \mathcal{H}_0 \to \mathcal{H}_0$, and its zeta function by

$$\zeta(z) = \mathrm{TR}(\Delta_\alpha^{-z}).$$

It is a meromorphic function and is regular at $z = 0$. Similar proof as in [53] shows that this defines a smooth Hermitian metric on the determinant line bundle $\mathcal{L}$.

On the open set of invertible operators each fiber of $\mathcal{L}$ is canonically isomorphic to $\mathbb{C}$ and the non-zero holomorphic section $\sigma = 1$ gives a trivialization. Also, according to the definition of the Hermitian metric, the norm of this section is given by

$$\|\sigma\|^2 = e^{-\zeta'_{\Delta_\alpha}(0)}. \tag{85}$$

11.6 Variations of LogDet and curvature form

A holomorphic line bundle equipped with a Hermitian inner product has a canonical connection compatible with the two structures. This is also known as the Chern connection. The curvature form of this connection is given by $\bar{\partial}\partial \log \|\sigma\|^2$, where σ is any non-zero local holomorphic section.

In the case at hand, the second variation $\bar{\partial}\partial \log \|\sigma\|^2$ on the open set of invertible Cauchy-Riemann operators must be computed. Let us consider a holomorphic family of invertible Cauchy-Riemann operators $D_w = \bar{\partial} + \alpha_w$, where α_w depends holomorphically on the complex variable w. The second variation of logdet, that is $\delta_{\bar{w}}\delta_w \zeta'_\Delta(0)$, is computed in [25] as we recall now.

Lemma 11.1 *For the holomorphic family of Cauchy-Riemann operators D_w, the second variation of $\zeta'(0)$ is given by*

$$\delta_{\bar{w}}\delta_w \zeta'(0) = \frac{1}{2}\varphi_0\left(\delta_w D \delta_{\bar{w}} \mathrm{res}(\log \Delta\, D^{-1})\right).$$

□

The final step is to compute $\delta_{\bar{w}}\mathrm{res}(\log \Delta\, D^{-1})$. This combined with the above lemma will show that the curvature form of the determinant line bundle equals the Kähler form on the space of connections. We refer the reader to [25] for the proof which is long and technical. We emphasize that the original Quillen proof, based on Green function calculations, cannot be extended to the noncommutative case.

Lemma 11.2 *With the above definitions and notations, we have*

$$\sigma_{-2,0}(\log \Delta\, D^{-1}) = \frac{(\alpha + \alpha^*)\xi_1 + (\bar{\tau}\alpha + \tau\alpha^*)\xi_2}{(\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2)(\xi_1 + \tau\xi_2)} - \log\left(\frac{\xi_1^2 + 2\Re(\tau)\xi_1\xi_2 + |\tau|^2\xi_2^2}{|\xi|^2}\right)\frac{\alpha}{\xi_1 + \tau\xi_2},$$

and

$$\delta_{\bar{w}}\mathrm{res}(\log(\Delta)D^{-1}) = \frac{1}{2\pi\Im(\tau)}(\delta_w D)^*.$$

Now we can state the main result of [25] which computes the curvature of the determinant line bundle in terms of the natural Kähler form on the space of connections.

Theorem 11.1 *The curvature of the determinant line bundle for the noncommutative two torus is given by*

$$\delta_{\bar{w}}\delta_w\zeta'(0) = \frac{1}{4\pi\Im(\tau)}\varphi_0\left(\delta_w D(\delta_w D)^*\right). \tag{86}$$

In order to recover the classical result of Quillen in the classical limit of $\theta = 0$, one has to notice that the volume form has changed due to a change of the metric. This means we just need to multiply the above result by $\Im(\tau)$.

12 Open problems

In this final section we formulate some of the open problems that we think are worthy of study for further understanding of local invariants of noncommutative manifolds.

1. Beyond dimension two and beyond conformally flat. The class of conformally flat metrics in dimensions bigger than two cover only a small part of all possible metrics. It would be very important to formulate large classes of metrics that are not conformally flat, but at the same time lend themselves to spectral analysis and to heat asymptotics techniques. It is also very important to have curvature formulas that work uniformly in all dimensions. The largest such class so far is the class of the so-called functional metrics introduced in [35] and surveyed in Section 9 of this paper. It is an interesting problem to further enlarge this class.
2. To extend the definition of curvature invariants to noncommutative spaces with non-integral dimension, including zero dimensional spaces. This would require rethinking the heat trace asymptotic expansion, and the nature of its leading and sub-leading terms. In particular since quantum spheres are zero dimensional, its spectrum is of exponential growth and does not satisfy the usual Weyl's asymptotic law. A first step would be to see how to formulate a Gauss-Bonnet type theorem for quantum spheres.
3. Weyl tensor and full curvature tensor. It is not clear that the classical differential geometry would, or should, give us a blueprint in the noncommutative case. One should be prepared for new phenomena. Having that in mind, one should still look for analogues of Weyl and full Riemann curvature tensors. The problem

is that the components of these tensors are quite entangled in the heat trace expansion, and separating and identifying their different components seem to be a hard task, if not impossible. One needs new ideas to make progress here.

4. Gauss-Bonnet terms in higher dimensions. The Gauss-Bonnet density in two dimension is particularly simple and is in fact equal to the scalar curvature multiplied by the volume form. In dimensions four and above this term is classically more complicated, being the Pfaffian of the curvature tensor. In dimension four it is a linear combination of norms of the Riemann tensor, the Ricci tensor, and the Ricci scalar. It is not clear how this can be expressed in terms of the heat kernel coefficients.
5. Higher genus noncommutative Riemann surfaces. It is highly desirable to define noncommutative Riemann surfaces of higher genus equipped with a spectral triple and check the Gauss-Bonnet theorem for them. This would greatly extend our understanding of local invariants of noncommutative spaces.
6. Noncommutative uniformization theorem. The study of curved noncommutative 2-tori suggests a natural problem in noncommutative geometry. At least for the class of noncommutative 2-tori it is desirable to know to what extent the uniformization theorem holds, or what form and shape it would take.
7. Analytic versus algebraic curvature. In classical differential geometry, as we saw in this paper, there are algebraic as well as analytic techniques (based on the heat equation) to define the scalar and Ricci curvature. The two approaches give the same results. This is not so in the noncommutative case. For noncommutative tori, when the deformation parameter satisfies some diophantine condition, Rosenberg in [54] proved a Levi-Civita type theorem and hence gets an algebraic definition of curvature. The resulting formula is very different from the formula of Connes-Moscovici-Fathizadeh-Khalkhali [21, 32] surveyed in this paper. It is important to see if there is any relation at all between these formulas and what this means for the study of curved noncommutative tori.

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