Confluence by Critical Pair Analysis Revisited

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Abstract. We present two methods for proving confluence of left-linear term rewrite systems. One is *hot-decreasingness*, combining the parallel/development closedness theorems with rule labelling based on a terminating subsystem. The other is *critical-pair-closing system*, allowing to boil down the confluence problem to confluence of a special subsystem whose duplicating rules are relatively terminating.

Keywords: Term rewriting · Confluence · Decreasing diagrams

1 Introduction

We present two results for proving confluence of first-order left-linear term rewrite systems, which extend and generalise three classical results: Knuth and Bendix' criterion [19] and strong and parallel closedness due to Huet [17]. Our idea is to reduce confluence of a term rewrite system $\mathcal{R}$ to that of a subsystem $\mathcal{C}$ comprising rewrite rules needed for *closing* the critical pairs of $\mathcal{R}$. In Sect. 3 we introduce *hot-decreasingness*, requiring that critical pairs can be closed using rules that are either below those in the peak or in a terminating subsystem $\mathcal{C}$. In Sect. 4 we introduce the notion of a *critical-pair-closing system* and present a confluence-preservation result based on relative termination $\mathcal{C}_d/\mathcal{R}$ of the duplicating part $\mathcal{C}_d$ of $\mathcal{C}$. For the left-linear systems we consider, our first criterion generalises both Huet's parallel closedness and Knuth and Bendix' criterion, and the second Huet's strong closedness. In Sect. 5, we assess viability of the new techniques, reporting on their implementation and empirical results.

Huet's parallel closedness result relies on the notion of overlap whose geometric intuition is subtle [1,24], and reasoning becomes intricate for development

closedness as covered by Theorem 2. We factor the classical theory of overlaps and critical pairs through the *encompassment* lattice in which overlapping redex-patterns *is* taking their join and the amount of overlap between redex-patterns is computed *via* their meet, thus allowing to reason algebraically about overlaps. Methodologically, our contribution here is the introduction of the lattice-theoretic language itself, relevant as it allows one to reason about occurrences of patterns[1] and their amount of (non-)overlap, omnipresent in deduction. Technically, whereas Huet's critical pair lemma [17] is well-suited for proving confluence of *terminating* TRSs, it is ill-suited to do so for *orthogonal* TRSs. Our lattice-theoretic results remedy this, allowing to decompose a reduction R both *horizontally* (as $R_1 \cdot R_2$) and *vertically* (as $R_1^{[x:=R_2]}$), enabling both termination and orthogonality reasoning in confluence proofs (Theorem 2).

In the last decade various classical confluence results for *term* rewrite systems have been factored through the decreasing diagrams method [28,30] for proving confluence of *abstract* rewrite systems, often leading to generalisations along the way: e.g. Felgenhauer's multistep labelling [13] generalises Okui's simultaneous closedness [27], the layer framework [12] generalises Toyama's modularity [33], critical pair systems [16] generalise both orthogonality [31] and Knuth and Bendix' criterion [19], and Jouannaud and Liu generalise, among others [20], parallel closedness, but in a way we do not know how to generalise to development closedness [29]. This paper fits into this line of research.[2]

We assume the reader is familiar with term rewriting [1,9,32] in general and confluence methods [17,19,30] in particular. Notions not explicitly defined in this paper can all be found in those works.

2   Preliminaries on Decreasingness and Encompassment

We recall the key ingredients of the decreasing diagrams method for proving confluence, see [20,26,30,32], and revisit the classical notion of critical pair, recasting its traditional account [1,17,19] based on redexes (substitution instances of left-hand sides) into one based on redex-patterns (left-hand sides).

Decreasingness. Consider an ARS comprising an I-indexed relation $\rightarrow = \bigcup_{\ell \in I} \rightarrow_\ell$ equipped with a well-founded strict order $\succ$. We refer to $\{\kappa \in I \mid \ell \succ \kappa\}$ by $\curlyvee\ell$, and to $\curlyvee\ell \cup \curlyvee\kappa$ by $\curlyvee\ell, \kappa$. For a subset J of I we define $\rightarrow_J$ as $\bigcup_{\ell \in J} \rightarrow_\ell$.

Definition 1. *A diagram for a peak $b \,_\ell\!\leftarrow a \rightarrow_\kappa c$ is* decreasing *if its closing conversion has shape $b \leftrightarrow^*_{\curlyvee\ell} \cdot \rightarrow^=_\kappa \cdot \leftrightarrow^*_{\curlyvee\ell,\kappa} \cdot \,^=_\ell\!\leftarrow \cdot \leftrightarrow^*_{\curlyvee\kappa} c$. An ARS in this setting is called* decreasing *if every peak can be completed into a decreasing diagram.*

One may think of decreasing diagrams as combining the diamond property [25, Theorem 1] (via the steps in the closing conversion with labels ℓ, κ) at the basis

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[1] Modelled in various ways, via e.g.: tree homomorphisms (tree automata [7]), term-operations (algebra), context-variables, labelling (rippling [5]), to name a few.

[2] For space reasons we have omitted the proof by decreasing diagrams of Theorem 3.

of confluence of *orthogonal* systems [6,31], with local confluence diagrams [25, Theorem 3] (via the conversions with labels $\curlyvee \ell, \kappa$) at the basis of confluence of *terminating* systems [19,21].

Theorem 1 ([28,30]). *An ARS is confluent if it is decreasing. Conversely, every countable ARS that is confluent, is decreasing for some set of indices I.*

For the converse part it suffices that the set of *labels I* is a doubleton, a result that can be reformulated without referring to decreasing diagrams, as follows.

Lemma 1 ([11]). *A countable confluent rewrite relation has a spanning forest.*

Here a spanning forest for $\to$ is a relation $\rightarrowtail \subseteq \to$ that is *spanning* ($\leftarrowtail\rightarrowtail^* = \leftrightarrow^*$) and a *forest*, i.e. deterministic ($b \leftarrowtail a \rightarrowtail c$ implies $b = c$) and acyclic.

Critical Peaks Revisited. We introduce *clusters* as the structures obtained *after* the matching of the left-hand side of a rule in a rewrite step, but *before* its replacement by the right-hand side. When proving the aforementioned results in Sects. 3 and 4, we use them as a tool to analyse overlaps and critical peaks. To illustrate our notions we use the following running example. We refer to [16,32] for the notion of multistep.

Example 1. In the TRS $\mathcal{R}$ with $\varrho(x): f(f(x)) \to g(x)$ the term $t = f(f(f(f(a))))$ allows the step $f(\varrho(f(a))): t \to f(g(f(a)))$ and multistep $\varrho(\varrho(a)): t \twoheadrightarrow g(g(a))$.

Here $f(\varrho(f(a)))$ and $\varrho(\varrho(a))$ are so-called proofterms, *terms* representing *proofs* of rewritability in rewriting logic [22,32]. The source of a proofterm can be computed by the 2nd-order substitution src of the left-hand side of the rule for the rule symbol[3] $f(\varrho(f(a)))^{\mathsf{src}} = f(\varrho(f(a)))^{[\![\varrho:=\lambda x. f(f(x))]\!]} = f(f(f(f(a))))$, and, *mutatis mutandis*, the same for the target via tgt. Proofclusters, introduced here, abstract from such proofterms by allowing to represent the matching and substitution phases of multisteps as well, by means of let-expressions.

Example 2. The multistep in Example 1 comprises three phases [28, Chapter 4]:

1. let $X, Y = \lambda x. f(f(x)), \lambda y. f(f(y))$ in $X(Y(a))$ denotes *matching* $f(f(x))$ twice;
2. let $X, Y = \lambda x. \varrho(x), \lambda x. \varrho(x)$ in $X(Y(a))$ denotes *replacing* by ϱ twice;
3. let $X, Y = \lambda x. g(x), \lambda x. g(x)$ in $X(Y(a))$ denotes *substituting* $g(x)$ twice.

To represent these we assume to have *proofterms $t, s, u, \ldots$* over a signature comprising *function* symbols $f, g, h, \ldots$, *rule* symbols $\varrho, \theta, \eta, \ldots$, 2nd-order variables $X, Y, Z, \ldots$, all having natural number arities, and 1st-order variables $x, y, z, \ldots$ (with arity 0). We call proofterms without 2nd-order variables or rule symbols, *1st-order* proofterms respectively *terms*, ranged over by $M, N, L, \ldots$.

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[3] src can be viewed as tree homomorphism [7], or as a term algebra $\varrho^{\mathcal{L}\hbar_\delta}(t) = \ell^{[x:=t]}$.

Definition 2. *A* proofcluster *is a let-expression* $\mathsf{let}\ \boldsymbol{X} = \boldsymbol{Q}\ \mathsf{in}\ t$, *where*

- $\boldsymbol{X}$ *is a vector* $X_1, \ldots, X_n$ *of (pairwise distinct) second-order variables;*
- $\boldsymbol{Q}$ *is a vector of length* n *of closed* λ-*terms* $Q_i = \lambda \boldsymbol{x_i}.s_i$, *where* s_i *is a proofterm and the length of the vector* $\boldsymbol{x_i}$ *of variables is the arity of* X_i; *and*
- t *is a proofterm, the* body, *with its 2nd-order variables among* $\boldsymbol{X}$.

Its denotation $[\![\mathsf{let}\ \boldsymbol{X} = \boldsymbol{Q}\ \mathsf{in}\ t]\!]$ *is* $t^{[\![\boldsymbol{X}:=\boldsymbol{Q}]\!]}$. *It is a* cluster *if* $s_1, \ldots, s_n, t$ *are terms.*

We let ς, ζ, ξ, ... range over (proof)clusters. They denote (proof)terms.

Example 3. Using ς, ζ, ξ for the three let-expressions in Example 2, each is a proofcluster and ς, ξ are clusters. Their denotations are the term $[\![\varsigma]\!] = f(f(f(f(a)))) = t$, proofterm $[\![\zeta]\!] = \varrho(\varrho(a))$, and term $[\![\xi]\!] = g(g(a))$.

We assume the usual variable renaming conventions, both for the 2nd-order ones in let-binders and the 1st-order ones in λ-abstractions. We say a proofcluster ς is *linear* if every (let or λ) binding binds exactly once, and *canonical* [23] if, when a binding variable occurs to the left of another such (of the same type), then the first bound occurrence of the former occurs before that of the latter in the pre-order walk of the relevant proofterm.

Example 4. Let ζ' and ξ' be the clusters $\mathsf{let}\ X = \lambda x.f(f(x))\ \mathsf{in}\ X(X(a))$ and $\mathsf{let}\ X, Y = \lambda yz.f(f(y)), \lambda x.f(f(x))\ \mathsf{in}\ Y(X(a, f(a)))$. Each of ς, ζ', ξ' denotes t in Example 1. The cluster ς is linear and canonical, ζ' is canonical but not linear (X occurs twice in the body), and ξ' is neither linear (z does not occur in $f(y)$) nor canonical (Y occurs outside of X in the body).

We adopt the convention that absent λ-binders are inserted linearly, canonically; $\mathsf{let}\ X = f(f(x))\ \mathsf{in}\ X(X(a))$ is ζ'. Clusters witness encompassment $\unrhd$ [9].

Proposition 1. $t \unrhd s$ *iff* $\exists u, X$ *s.t.* $[\![\mathsf{let}\ X = s\ \mathsf{in}\ u]\!] = t$ *and* X *occurs once in* u.

We define the *size* $\|t\|$ of a proofterm t in a way that is compatible with encompassment. Formally, $\|t\|$ is the pair comprising the number of non-1st-order-variable symbols in t, and the sum over the 1st-order variables x, of the *square* of the number of occurrences of x in t. Then $\|t\| > \|s\|$ if $t \rhd s$, where we (ab)use $>$ to denote the lexicographic product of the greater-than relation with itself, e.g. $\|g(a, a)\| = (3,0) > \|g(x, x)\| = (1,4) > \|g(x, y)\| = (1,2)$. For a proofcluster ς given by $\mathsf{let}\ \boldsymbol{x} = \boldsymbol{s}\ \mathsf{in}\ t$ its *pattern*-size $[\![\varsigma]\!]$ is $\sum_i \|s_i\|$ (adding component-wise, with empty sum (0,0)) and its *body*-size $\lfloor\varsigma\rfloor$ is $\|t\|$. Encompassment $\unrhd$ is at the basis of the theory of reducibility [7, Section 3.4.2]: t is reducible by a rule $\ell \to r$ iff $t \unrhd \ell$. For instance, $\mathsf{let}\ X = f(f(x))\ \mathsf{in}\ f(X(f(a)))$ is a witness to reducibility of t in Example 1. We call it, or simply $f(f(x))$, a *pattern* in t.

Definition 3. *Let* ς *be a canonical linear proofcluster* $\mathsf{let}\ \boldsymbol{X} = \boldsymbol{s}\ \mathsf{in}\ t$ *with term* t. *We say* ς *is a* multipattern *if each* s_i *is a non-variable 1st-order term, and* ς *is a* multistep *if each* s_i *has shape* $\varrho(\boldsymbol{x})$, *i.e. a rule symbol applied to a sequence of pairwise distinct variables. If* $\boldsymbol{X}$ *has length 1 we drop the prefix 'multi'.*

We use $\Phi, \Psi, \Omega, \ldots$ to range over multisteps, and $\phi, \psi, \omega, \ldots$ to range over steps. Taking their denotation yields the usual multistep [16,32] and step ARSs $\multimap\!\!\rightarrow$ and $\rightarrow$ underlying a TRS $\mathcal{R}$. These can be alternatively obtained by first applying src and tgt (of which only the former is guaranteed to yield a multipattern, by left-linearity) and then taking denotations: $[\![\Phi^{\mathsf{src}}]\!] = [\![\Phi]\!]^{\mathsf{src}}$ and $[\![\Phi^{\mathsf{tgt}}]\!] = [\![\Phi]\!]^{\mathsf{tgt}}$. Pattern- and body-sizes of multipatterns are compositional.

Proposition 2. *For multipatterns* $\varsigma, \boldsymbol{\varsigma}$ *if* $\varsigma = \varsigma_0^{[\boldsymbol{x} := \boldsymbol{\varsigma}]}$ *with each variable among* $\boldsymbol{x}$ *occurring once in the body of* ς_0, *then* $[\![\varsigma]\!] = \sum_i \varsigma_i$, *and* $\|\varsigma\| \geqslant \|\varsigma_i\|$ *for all* i, *with strict inequality holding in case the substitution is not a bijective renaming. Here multipattern-substitution substitutes in the body and combines let-bindings.*

Multipatterns are ordered by refinement $\sqsubseteq$.

Definition 4. *Let* ς *and* ζ *be multipatterns* let $\boldsymbol{X} = \boldsymbol{s}$ in t *and* let $\boldsymbol{Y} = \boldsymbol{u}$ in w. *We say* ς *refines* ζ *and write* $\varsigma \sqsubseteq \zeta$, *if there is a 2nd-order substitution* σ *on* $\boldsymbol{Y}$ *with* $w^\sigma = t$ *and* $[\![$let $\boldsymbol{X} = \boldsymbol{s}$ in $Y_i(\boldsymbol{y_i})^\sigma]\!] = u_i$ *for all* i, *with* $\boldsymbol{y_i}$ *the variables of* u_i.

Example 5. We have $\varsigma \sqsubseteq \varsigma'$ with ς' being let $Z = f(f(f(f(z))))$ in $Z(a)$, and ς as in Example 3, as witnessed by the 2nd-order substitution mapping Z to $\lambda x.X(Y(x))$.

Lemma 2. $\sqsubseteq$ *is a finite distributive lattice [8] on multipatterns denoting a 1st-order term* t, *with least element* $\bot$ *the empty let-expression* let $=$ in t, *and greatest element* $\top$ *of shape* let $X = t'$ in $X(\boldsymbol{x})$ *with* $\boldsymbol{x}$ *the vector of variables in* t.

Proof (Idea). Although showing that $\sqsubseteq$ is reflexive and transitive is easy, showing anti-symmetry or existence of/constructions for meets $\sqcap$ and joins $\sqcup$, directly is not. Instead, it *is* easy to see that each multipattern let $\boldsymbol{X} = \boldsymbol{s}$ in t is determined by the set of the (non-empty, convex,[4] pairwise disjoint) sets of node positions of its patterns s_i in t, and vice versa. For instance, the multipatterns ς and ς' in Example 5 are determined by $\{\{\varepsilon, 1\}, \{1 \cdot 1, 1 \cdot 1 \cdot 1\}\}$ and $\{\{\varepsilon, 1, 1 \cdot 1, 1 \cdot 1 \cdot 1\}\}$. Viewing multipatterns as sets in that way $\varsigma \sqsubseteq \zeta$ iff $\forall P \in \varsigma, \exists Q \in \zeta$ with $P \subseteq Q$. Saying $P, Q \in \varsigma \cup \zeta$ have *overlap* if $P \cap Q \neq \emptyset$, denoted by $P \between Q$, characterising meets and joins now also is easy: $\varsigma \sqcap \zeta = \{P \cap Q \mid P \in \varsigma, Q \in \zeta, \text{ and } P \between Q\}$, and $\varsigma \sqcup \zeta = \{\bigcup P_\between \mid P \in \varsigma \cup \zeta\}$, where $P_\between = \{Q \in \varsigma \cup \zeta \mid P \between^* Q\}$, i.e. the sets connected to P by successive overlaps. On this set-representation $\sqsubseteq$ can be shown to be a finite distributive lattice by set-theoretic reasoning, using that the intersection of two overlapping patterns is a pattern again[5]. For instance, $\bot$ *is* the empty set and $\top$ *is* the singleton containing the set of all non-variable positions in t. $\square$

The (proof of the) lemma allows to freely switch between viewing multisteps and multipatterns as let-expressions and as sets of sets of positions, and to reason about (non-)overlap of multipatterns and multisteps in lattice-theoretic terms.

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[4] Here convex means that for each pair of positions p, q in the set, all positions on the shortest path from p to q in the term tree are also in the set, cf. [32, Definition 8.6.21].

[5] This fails for, e.g., connected graphs; these may fall apart into non-connected ones.

We show any multistep Φ can be decomposed *horizontally* as ϕ followed by Φ/ϕ for any step $\phi \in \Phi$ [16,29], and *vertically* as some vector $\boldsymbol{\Phi}$ substituted in a prefix Φ_0 of Φ, and that peaks can be decomposed correspondingly.

Definition 5. *For a pair of multipatterns ς, ζ denoting the same term its amount of overlap*[6] *and non-overlap is $\varsigma \Cap \zeta = [\![\varsigma \sqcap \zeta]\!]$ respectively $\varsigma \Cup \zeta = [\![\varsigma \sqcup \zeta]\!]$, we say ς, ζ is* overlapping *if $\varsigma \sqcap \zeta \neq \bot$, and* critically *overlapping if moreover $\varsigma \sqcup \zeta = \top$ and $[\![\varsigma]\!] = [\![\zeta]\!]$ is linear. This extends to peaks $s \;_{\Phi}{\leftarrow}{\circ}{-}\; t \;{-}{\circ}{\rightarrow}_{\Psi}\; u$ via Φ^{src} and Ψ^{src}.*

Note ς, ζ is overlapping iff $\varsigma \Cap \zeta \neq (0,0)$. Critical peaks $s \;_{\phi}{\leftarrow}\; t \rightarrow_{\psi} u$ are classified by comparing the root-positions p_{ϕ}, p_{ψ} of their patterns with respect to the prefix order $\prec_o$, into being *outer–inner* ($p_{\phi} \prec_o p_{\psi}$), *inner–outer* ($p_{\psi} \prec_o p_{\phi}$), or *overlay* ($p_{\psi} = p_{\phi}$), and induce the usual [1,9,17,19,26,32] notion of critical *pair* (s,u).[7]

Definition 6. *A pair (ς', ζ') of overlapping patterns such that ς', ζ' are in the multipatterns ς, ζ with $\top = \varsigma \sqcup \zeta$, is called* inner, *if it is minimal among all such pairs, comparing them in the lexicographic product of $\prec_o$ with itself, via the root-positions of their patterns, ordering these themselves first by $\preceq_o$. This extends to pairs of steps in peaks of multisteps via* src.

Proposition 3. *If (ϕ, ψ) is an inner pair for a critical peak $\;_{\Phi}{\leftarrow}{\circ}{-}\; \cdot \;{-}{\circ}{\rightarrow}_{\Psi}$, and $\phi \in \Phi$, $\psi \in \Psi$ contract redexes at the same position, then $\phi = \Phi$ and $\psi = \Psi$.*

For patterns and peaks of ordinary steps, their join being top, entails they are overlapping, and the patterns in a join are joins of their constituent patterns.

Proposition 4. *Linear patterns ς, ζ are critically overlapping iff $\varsigma \sqcup \zeta = \top$.*

Lemma 3. *If $\xi = \varsigma \sqcup \zeta$ and $\varsigma, \zeta \sqsubseteq \xi$ are witnessed by the 2nd-order substitutions σ, τ, for multipatterns ς and ζ given by* let $X = \boldsymbol{t}$ in M *and* let $Y = \boldsymbol{s}$ in N, *then for all let-bindings $Z = \boldsymbol{u}$ of ξ, $\top_u = ($let $X = \boldsymbol{t}$ in $Z(\boldsymbol{z})^{\sigma}) \sqcup ($let $Y = \boldsymbol{s}$ in $Z(\boldsymbol{z})^{\tau})$.*

Lemma 4 (Vertical). *A peak $s \;_{\Phi}{\leftarrow}{\circ}{-}\; t \;{-}{\circ}{\rightarrow}_{\Psi}\; u$ of overlapping multisteps either is critical or it can be vertically decomposed as:*

$$s_0^{[\boldsymbol{x} := \boldsymbol{s}]} \;_{\Phi_0^{[\boldsymbol{x} := \boldsymbol{\Phi}]}}{\leftarrow}{\circ}{-}\; t_0^{[\boldsymbol{x} := \boldsymbol{t}]} \;{-}{\circ}{\rightarrow}_{\Psi_0^{[\boldsymbol{x} := \boldsymbol{\Psi}]}}\; u_0^{[\boldsymbol{x} := \boldsymbol{u}]}$$

for peaks $s_i \;_{\Phi_i}{\leftarrow}{\circ}{-}\; t_i \;{-}{\circ}{\rightarrow}_{\Psi_i}\; u_i$ with $\Phi \Cap \Psi \geqslant \Phi_i \Cap \Psi_i$ and $\Phi \Cup \Psi > \Phi_i \Cup \Psi_i$, for all i.

Let Φ, Ψ in $s \;_{\Phi}{\leftarrow}{\circ}{-}\; t \;{-}{\circ}{\rightarrow}_{\Psi}\; u$ be given by let $X = \boldsymbol{\varrho}(\boldsymbol{x})$ in M and let $Y = \boldsymbol{\theta}(\boldsymbol{y})$ in N, for rules $\varrho_i(\boldsymbol{x_i}) : \ell_i \rightarrow r_i$ and $\theta_j(\boldsymbol{y_j}) : g_j \rightarrow d_j$. Lemma 2 entails that if Φ, Ψ are non-overlapping their patterns are (pairwise) disjoint, so that the join $\Phi^{\mathsf{src}} \sqcup \Psi^{\mathsf{src}}$ is given by taking the (disjoint) union of the let-bindings: let $XY = \boldsymbol{\ell g}$ in L for some L such that $L^{[Y := g]} = M$ and $L^{[X := \ell]} = N$. We define the *join*[8] $\Phi \sqcup \Psi$ and

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[6] For the amount of overlap for *redexes* in parallel reduction $\twoheadrightarrow$, see e.g. [1,17,24].

[7] We exclude neither overlays of a rule with itself nor pairs obtained by symmetry.

[8] This does not create ambiguity with joins of multipatterns since if $\Phi \neq \Psi$, then $[\![\Phi]\!] \neq [\![\Psi]\!]$ unless the let-bindings of both are empty, so both are bottom.

residual Φ/Ψ by let $\boldsymbol{XY = \varrho(x)\theta(y)}$ in L respectively let $\boldsymbol{X = \varrho(x)}$ in $L^{[\![\boldsymbol{Y:=d}]\!]}$, where, as substituting the right-hand sides $\boldsymbol{d}$ may lose being linear and canonical, we implicitly canonise and linearise the latter by reordering and replicating let-bindings. Then $t \rightarrowtail_{\Phi \sqcup \Psi} \cdot {}_{\Phi/\Psi}\!\leftarrowtail u$, giving rise to the classical residual theory [2, 4,6,18], see [32, Section 8.7]. We let $\phi \in \Phi$ abbreviate $\exists \Psi. \Phi = \phi \sqcup \Psi$.

Example 6. The steps ϕ and ψ given by let $X = \lambda x.\varrho(x)$ in $X(f(f(a)))$ respectively let $X = \lambda x.\varrho(x)$ in $f(f(Y(a)))$, are non-overlapping, $\phi, \psi \in \zeta$, $\phi \sqcup \psi = \zeta$, and $f(f(g(a))) \rightarrowtail_{\phi/\psi} g(g(a))$, for ζ and ϱ as in Example 3.

Lemma 5 (Horizontal). *A peak $t\ {}_{\Phi}\!\leftarrowtail \cdot \rightarrowtail_{\Psi} s$ of multisteps either*

1. *is non-overlapping and then $t \rightarrowtail_{\Psi/\Phi} \cdot {}_{\Phi/\Psi}\!\leftarrowtail s$, with the rule symbols occurring in Ψ/Φ contained in Ψ (and those in Φ/Ψ contained in Φ); or*
2. *it can be horizontally decomposed: $t\ {}_{\Phi/\phi}\!\leftarrowtail \cdot {}_{\phi}\!\leftarrow \cdot \rightarrow_{\psi} \cdot \rightarrowtail_{\Psi/\psi} s$ for some peak ${}_{\phi}\!\leftarrow \cdot \rightarrow_{\psi}$ of overlapping steps $\phi \in \Phi$ and $\psi \in \Psi$.*

The above allows to refactor the proof of the critical pair lemma [17, Lemma 3.1] for left-linear TRSs, as an induction on the amount of *non-overlap* between the steps in the peak, such that the critical peaks form the *base* case:

Lemma 6. *A left-linear TRS is locally confluent if all critical pairs are joinable.*

Proof. We show every peak $\overline{\overline{}}_{\phi}\!\leftarrow \cdot \rightarrow_{\overline{\overline{}}\psi}$ of empty or single steps is joinable, by induction on the amount of non-overlap ($\phi \uplus \psi$) ordered by $>$. We distinguish cases on whether ϕ, ψ are overlapping ($\phi \Cap \psi \neq (0,0)$) or not. If ϕ, ψ do not have overlap, in particular when either ϕ or ψ is empty, then we conclude by Lemma 5(1). If ϕ, ψ do have overlap, then by Lemma 4 the peak either

– is critical and we conclude by assumption; or
– can be (vertically) *decomposed* into smaller such peaks $\overline{\overline{}}_{\phi_i}\!\leftarrow \cdot \rightarrow_{\overline{\overline{}}\psi_i}$. Since these are $>$-smaller, the induction hypothesis yields them joinable, from which we conclude by reductions and joins being closed under *composition*. □

Remark 1. Apart from enabling our proof of Theorem 2 below, we think this refactoring is methodologically interesting, as it extends to (parallel and) simultaneous critical pairs, then yielding, we claim, simple statements and proofs of confluence results [13,27] based on these and their higher-order generalisations.

3   Confluence by Hot-Decreasingness

Linear TRSs have a critical-pair criterion for so-called rule-labelling [16,30,35]: If all critical peaks are decreasing with respect to some rule-labelling, then the TRS is decreasing, hence confluent. We introduce the hot-labelling extending that result to left-linear TRSs. To deal with non-right-linear rules we make use of a rule-labelling for multisteps that is invariant under duplication, cf. [13,35].

Remark 2. Naïve extensions fail. Non-left-linear TRSs need not be confluent even without critical pairs [32, Exercise 2.7.20]. That non-right-linear TRSs need not be confluent even if all critical peaks are decreasing for rule-labelling, is witnessed by [16, Example 8].

Definition 7. *For a TRS $\mathcal{R}$, terminating subsystem $\mathcal{C} \subseteq \mathcal{R}$, and labelling of $\mathcal{R} - \mathcal{C}$-rules into a well-founded order $\succ$, hot-labelling $\mathring{\mathcal{L}}$ maps a multistep Φ:*
$$t \twoheadrightarrow_\mathcal{R} s$$

- *to the* term *t if Φ contains $\mathcal{C}$-rules only; and*
- *to the* set *of $\succ$-maximal $\mathcal{R} - \mathcal{C}$-rules in Φ otherwise.*

The hot-*order $\mathring{\succ}$ relates terms by $\to_\mathcal{C}^+$, sets by $\succ_{\mathsf{mul}}$, and all sets to all terms.*

Note $\mathring{\succ}$ is a well-founded order as series composition [3] of $\to_\mathcal{C}^+$ and $\succ_{\mathsf{mul}}$, which are well-founded orders by the assumptions on $\mathcal{C}$ and $\succ$. Taking the *set* of maximal rules in a multistep makes hot-labelling invariant under duplication. As with the notation $\curlyvee\ell$, we denote $\{\kappa \mid \ell \mathring{\succ} \kappa\}$ by $\mathring{\curlyvee}\ell$, and $\{\kappa \mid \ell \mathring{\succeq} \kappa\}$ by $\mathring{\widetilde{\curlyvee}}\ell$.

Definition 8. *A TRS $\mathcal{R}$ is* hot-decreasing *if its critical peaks are decreasing for the hot-labelling, for some $\mathcal{C}$ and $\succ$, such that each outer–inner critical peak $_\ell\leftarrow \cdot \to$ for label ℓ, is decreasing by a conversion of shape (oi): $\leftrightarrow_{\mathring{\curlyvee}\ell}^* \cdot \mathring{\widetilde{\curlyvee}}\ell\leftarrow\!\!\circ$.*

Theorem 2. *A left-linear TRS is confluent, if it is hot-decreasing.*

Before proving Theorem 2, we give (non-)examples and special cases.

Example 7. Consider the left-linear TRS $\mathcal{R}$:

ϱ_1: $nats \to 0 : inc(nats)$ ϱ_3: $inc(x : y) \to s(x) : inc(y)$ ϱ_5: $hd(x : y) \to x$
ϱ_2: $d(x) \to x : (x : d(x))$ ϱ_4: $inc(tl(nats)) \to tl(inc(nats))$ ϱ_6: $tl(x : y) \to y$

By taking $\mathcal{C} = \emptyset$, labelling rules by themselves, and ordering $\varrho_4 \succ \varrho_1, \varrho_3, \varrho_6$ the only critical peak $_{\{\varrho_4\}}\leftarrow \cdot \to_{\{\varrho_1\}}$ can be completed into the decreasing diagram:

$$tl(inc(nats)) \xleftarrow{\quad\{\varrho_4\}\quad} inc(tl(nats)) \xrightarrow{\quad\{\varrho_1\}\quad} inc(tl(0 : inc(nats)))$$
$$\{\varrho_1\}\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow\{\varrho_6\}$$
$$tl(inc(0 : inc(nats))) \xrightarrow[\{\varrho_3\}]{} tl(s(0) : inc(inc(nats))) \xrightarrow[\{\varrho_6\}]{} inc(inc(nats))$$

Since the peak is outer–inner, the closing conversion must be of (oi)-shape $\leftrightarrow_{\mathring{\curlyvee}\{\varrho_4\}}^* \cdot \mathring{\widetilde{\curlyvee}}\{\varrho_4\}\leftarrow\!\!\circ$. It is, so the system is confluent by Theorem 2.

Example 8. Consider the left-linear confluent TRS $\mathcal{R}$:

$$\varrho_1: f(a, a) \to b \qquad \varrho_3: f(c, x) \to f(x, x) \qquad \varrho_5: f(c, c) \to f(a, c)$$
$$\varrho_2: \quad a \to c \qquad \varrho_4: f(x, c) \to f(x, x)$$

Since b is an $\mathcal{R}$-normal form, the only way to join the outer–inner critical peak $b \,_{\varrho_1}\!\!\leftarrow f(a, a) \to_{\varrho_2} f(c, a)$ is by a conversion starting with a step $b \,_{\varrho_1}\!\!\leftarrow f(a, a)$. As its label must be identical to the same step in the peak, not smaller, whether

we choose ϱ_1 to be in $\mathcal{C}$ or not, the peak is not hot-decreasing, so Theorem 2 does not apply.

That hot-decreasingness in Theorem 2 cannot be weakened to (ordinary) decreasingness, can be seen by considering $\mathcal{R}'$ obtained by omitting ϱ_5 from $\mathcal{R}$. Although $\mathcal{R}'$ is not confluent [16, Example 8], by taking $\mathcal{C} = \varnothing$ and $\varrho_1 \succ \varrho_3, \varrho_4$, we can show that all critical peaks of $\mathcal{R}'$ are decreasing for the hot-labelling.

A special case of Theorem 2, is that a left-linear terminating TRS is confluent [19], if each critical pair is joinable, as can be seen by setting $\mathcal{C} = \mathcal{R}$.

Corollary 1. *A left-linear development closed TRS is confluent* [29, Corollary 24]*.*

Proof. A TRS is development closed if for every critical pair (t,s) such that t is obtained by an outer step, $t \leftarrowtail s$ holds. Taking $\mathcal{C} = \emptyset$ and labelling all rules the same, say by 0, yields that each outer–inner or overlay critical peak is labelled as $t \;{}_{\{0\}}{\leftarrow} \cdot {\rightarrow}_{\{0\}}\; s$, and can be completed as $t \;{}_{\{0\}}{\leftarrowtail}\; s$, yielding a hot-decreasing diagram of (oi)-shape. We conclude by Theorem 2. □

The proof of Theorem 2 uses the following structural properties of decreasing diagrams specific to the hot-labelling. The labelling was designed so they hold.

Lemma 7. *1. If the peak $s \;{}_{\ell}{\leftarrowtail}\; t \;{\twoheadrightarrow}_{\kappa}\; u$ is hot-decreasing, then it can be completed into a hot-decreasing diagram of shape $s \leftrightsquigarrow^*_{\mathring{\curlyvee}\ell} s' \;{\twoheadrightarrow}_{\kappa}\; s'' \leftrightsquigarrow^*_{\mathring{\curlyvee}\ell\kappa} u'' \;{}_{\ell}{\leftarrowtail}\; u' \leftrightsquigarrow^*_{\mathring{\curlyvee}\kappa} u$ such that the 1st-order variables in all terms in the diagram are contained in those of t.*

2. If the multisteps Φ, Ψ in the peak $s \;{}_{\Phi}{\leftarrowtail}\; t \;{\twoheadrightarrow}_{\Psi}\; u$ are non-overlapping, then the valley $s \;{\twoheadrightarrow}_{\Psi/\Phi} \cdot {}_{\Phi/\Psi}{\leftarrowtail}\; u$ completes it into a hot-decreasing diagram.

3. If the peak $s \leftarrowtail t \twoheadrightarrow u$ and vector of peaks $\boldsymbol{s} \leftarrowtail \boldsymbol{t} \twoheadrightarrow \boldsymbol{u}$ have hot-decreasing diagrams, so does the composition $s^{[\boldsymbol{x}:=\boldsymbol{s}]} \leftarrowtail t^{[\boldsymbol{x}:=\boldsymbol{t}]} \twoheadrightarrow u^{[\boldsymbol{x}:=\boldsymbol{u}]}$.

The proof of Theorem 2 refines our refactored proof (see Lemma 6) of Huet's critical pair lemma, by wrapping the induction on the amount of non-overlap ($\Cup$) between multisteps, into an outer induction on their amount of overlap ($\Cap$).

Proof (of Theorem 2). We show that every peak $s \;{}_{\Phi}{\leftarrowtail}\; t \;{\twoheadrightarrow}_{\Psi}\; u$ of multisteps Φ and Ψ can be closed into a hot-decreasing diagram, by induction on the pair $(\Phi\Cap\Psi, \Phi\Cup\Psi)$ ordered by the lexicographic product of $>$ with itself. We distinguish cases on whether or not Φ and Ψ have overlap.

If Φ and Ψ do not have overlap, Lemma 5(1) yields $s \;{\twoheadrightarrow}_{\Psi/\Phi} \cdot {}_{\Phi/\Psi}{\leftarrowtail}\; u$. This valley completes the peak into a hot-decreasing diagram by Lemma 7(2).

If Φ and Ψ do have overlap, then we further distinguish cases on whether or not the overlap is critical.

If the overlap is not critical, then by Lemma 4 the peak can be vertically decomposed into a number of peaks between multisteps Φ_i, Ψ_i that have an amount of overlap that is not greater, $\Phi \Cap \Psi \geqslant \Phi_i \Cap \Psi_i$, and a strictly smaller amount of non-overlap $\Phi \Cup \Psi > \Phi_i \Cup \Psi_i$. Hence the I.H. applies and yields that each such peak can be completed into a hot-decreasing diagram. We conclude by vertically recomposing them yielding a hot-decreasing diagram by Lemma 7(3).

If the overlap is critical, then by Lemma 5 the peak can be horizontally decomposed as $s \; {}_{\Phi/\phi}\!\leftarrow\!\!\circ\!\!-\; s' \; {}_{\phi}\!\leftarrow\; t \to_\psi u' \; -\!\!\circ\!\!\to_{\Psi/\psi} u$ for some peak $s' \; {}_{\phi}\!\leftarrow t \to_\psi u'$ of overlapping steps $\phi \in \Phi$ and $\psi \in \Psi$, i.e. such that $\Phi = \phi \sqcup \Phi' \; \Psi = \psi \sqcup \Psi'$ for some Φ', Ψ'. We choose (ϕ,ψ) to be *inner* among such overlapping pairs (see Definition 6), assuming w.l.o.g. that $p_\phi \preceq_o p_\psi$ for the root-positions p_ϕ, p_ψ of their patterns. We distinguish cases on whether or not p_ϕ is a strict prefix of p_ψ.

If $p_\phi = p_\psi$, then $\phi = \Phi$ and $\psi = \Psi$ by Proposition 3, so the peak *is* overlay, from which we conclude since such peaks are hot-decreasing by assumption.

Suppose $p_\phi \prec_o p_\psi$. We will construct a hot-decreasing diagram D for the peak $s \; {}_\Phi\!\leftarrow\!\!\circ\!\!-\; t \; -\!\!\circ\!\!\to_\Psi u$ out of several smaller such diagrams as illustrated in Fig. 1, using the multipattern $\varsigma = \Phi^{\mathsf{src}} \sqcup \psi^{\mathsf{src}}$ as a basic building block; it has as patterns those of Φ' and the join of the patterns of ϕ, ψ. To make ς explicit, unfold Φ and Ψ to let-expressions $\mathsf{let}\, \boldsymbol{X} = \boldsymbol{\varrho(x)}\, \mathsf{in}\, M$ respectively $\mathsf{let}\, \boldsymbol{Y} = \boldsymbol{\theta(y)}\, \mathsf{in}\, N$, for rules of shapes $\varrho_i(\boldsymbol{x_i}) : \ell_i \to r_i$ and $\theta_j(\boldsymbol{y_j}) : g_j \to d_j$. We let $\boldsymbol{X} = \boldsymbol{X'}X$ and $\boldsymbol{Y} = \boldsymbol{Y'}Y$ be such that X and Y are the 2nd-order variables corresponding to $\phi \in \Phi$ and $\psi \in \Psi$ for rules $\varrho(\boldsymbol{x}) : \ell \to r$ and $\theta(\boldsymbol{y}) : g \to d$. By the choice of (ϕ,ψ) as inner, ϕ^{src} is the *unique* pattern in Φ^{src} overlapping ψ^{src}. As a consequence we can write ς as $\mathsf{let}\, \boldsymbol{X'}Z = \boldsymbol{\ell'}\hat{t}\, \mathsf{in}\, L$, for some pattern $\hat{t}$, the join of the patterns of ϕ,ψ, such that σ maps Z to a term of shape $X(\boldsymbol{g_\psi})$ as ϕ is the outer step, and τ maps it to a term of shape $C[Y(\boldsymbol{\ell_\phi})]$,[9] where σ, τ witness $\Phi^{\mathsf{src}}, \psi^{\mathsf{src}} \sqsubseteq \varsigma$. That the other 2nd-order variables are $\boldsymbol{X'}$ follows by σ being the identity on them (their patterns do not overlap ψ), and that these are bound to the patterns $\boldsymbol{\ell'}$ by τ mapping them to 1st-order terms (only Z can be mapped to a non-1st-order term).
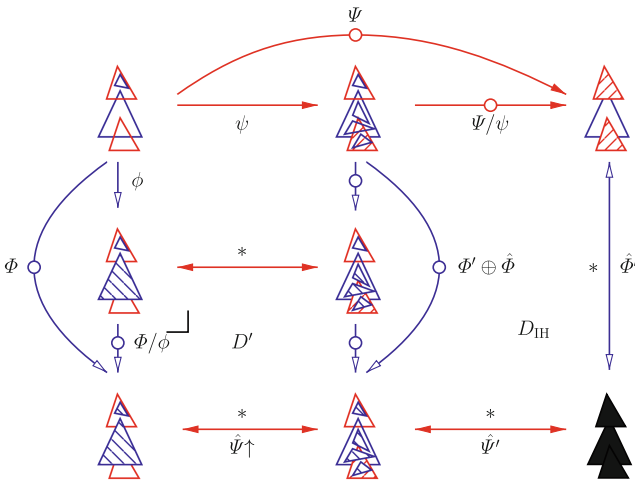

Fig. 1. Outer–inner critical peak construction

[9] C is a prefix of the left-hand side ℓ of ϱ. For instance, for a peak from $f(g(a))$ between $\varrho : f(g(a)) \to \ldots$ and $\theta : g(x) \to \ldots$, Z is mapped by σ to $X()$ and by τ to $f(Y(a))$.

We start with constructing a hot-decreasing diagram $\mathring{D}$ for the *critical* peak $\hat{s} \;_{\hat{\phi}}\!\!\leftarrow \hat{t} \rightarrow_{\hat{\psi}} \hat{u}$ encompassed by the peak between ϕ and ψ, as follows. We set $\hat{\phi}$ and $\hat{\psi}$ to $\mathsf{let}\, X = \varrho(\boldsymbol{x})\, \mathsf{in}\, Z(\boldsymbol{z})^\sigma$ respectively $\mathsf{let}\, Y = \theta(\boldsymbol{y})\, \mathsf{in}\, Z(\boldsymbol{z})^\tau$. This yields a peak as desired, which is outer–inner as $p_{\hat{\phi}} \prec_o p_{\hat{\psi}}$ by $p_\phi \prec_o p_\psi$, and critical by Lemma 3, hence by the hot-decreasingness assumption, it can be completed into a hot-decreasing diagram $\hat{D}$ by a conversion of (oi)-shape: $\hat{s} \leftrightarrow^*_{\;\Upsilon\mathring{\mathcal{L}}(\hat{\phi})} \hat{w} \;_{\check{\Upsilon}\mathring{\mathcal{L}}(\hat{\phi})}\!\!\leftarrow\!\!\circ\!\!-\; \hat{u}$.

Below we refer to its conversion and multistep as $\hat{\Psi}$ and $\hat{\Phi}$. Based on $\hat{D}$ we construct a hot-decreasing diagram D' (Fig. 1, left) for the peak $s \;_{\Phi}\!\!\leftarrow\!\!\circ\!\!-\; t \rightarrow_\psi u'$ by constructing a conversion $\hat{\Psi}\!\!\uparrow : s \leftrightarrow^* w''$ and a multistep $\Phi' \oplus \hat{\Phi} : u' \;-\!\!\circ\!\!\rightarrow w''$, with their composition (reversing the latter) of (oi)-shape.

The conversion $\hat{\Psi}\!\!\uparrow : s \leftrightarrow^* w''$ is constructed by lifting the closing conversion $\hat{\Psi}$ of the diagram $\hat{D}$ back into ς. Formally, for any multistep $\hat{\Omega}$ given by $\mathsf{let}\, \hat{\boldsymbol{Z}} = \boldsymbol{\eta}(\boldsymbol{w})\, \mathsf{in}\, \hat{L}$ for rules $\eta_k(\boldsymbol{w_k})$, occurring anywhere in $\hat{\Psi}$, we define its *lifting* $\hat{\Omega}\!\!\uparrow$ to be $\mathsf{let}\, \hat{\boldsymbol{Z}} = \boldsymbol{\eta}(\boldsymbol{w})\, \mathsf{in}\, L^{[\boldsymbol{X'}, Z := \boldsymbol{r'}, \hat{L}]}$. That is, we update ς by substituting[10] both $\hat{\Omega}$ (for Z, instead of binding that to $\hat{t}$) and the *right*-hand sides $\boldsymbol{r'}$ in its body. Because right-hand sides $\boldsymbol{r}$ need not be linear, the resulting proofclusters may have to be linearised (by replicating let-bindings) first to obtain multisteps. This extends to terms p by $p\!\!\uparrow = [\![(\mathsf{let}\, = \mathsf{in}\, p)\!\!\uparrow]\!]$. That this yields multisteps and terms that connect into a conversion $s = \hat{s}\!\!\uparrow \;\leftrightarrow^*_{\hat{\Psi}\uparrow}\; \hat{w}\!\!\uparrow = w''$ as desired follows by *computation*. E.g., $s = M^{[\boldsymbol{X'}, X := \boldsymbol{r'}, r]} = L^{[\boldsymbol{X'}, Z := \boldsymbol{r'}, \hat{s}]} = \hat{s}\!\!\uparrow$ using that σ witnesses $\Phi^{\mathsf{src}} \sqsubseteq \varsigma$ so that $M = L^\sigma$ and $\hat{s} = [\![\mathsf{let}\, X = r\, \mathsf{in}\, Z(\boldsymbol{z})^\sigma]\!]$. That the labels in $\hat{\Psi}\!\!\uparrow$ are strictly below $\mathring{\mathcal{L}}(\Phi)$ follows for *set*-labels from that lifting clearly does not introduce rule symbols and from that labels of rule symbols in $\hat{\Psi}$ are, by assumption, strictly below the label of the rule ϱ of ϕ. In case Φ is *term*-labelled, by t, it follows from closure of $\rightarrow_{\mathcal{C}}$-reduction under lifting (which also contracts Φ').

The multistep $\Phi' \oplus \hat{\Phi} : u' \;-\!\!\circ\!\!\rightarrow w''$ is the *combination* of the multisteps Φ' (the redex-patterns in Φ other than ϕ) and $\hat{\Phi}$, lifting the latter into ς. For $\hat{\Phi} : \hat{u} \;-\!\!\circ\!\!\rightarrow \hat{w}$ given by $\mathsf{let}\, \hat{\boldsymbol{X}} = \hat{\boldsymbol{\varrho}}(\hat{\boldsymbol{x}})\, \mathsf{in}\, \hat{M}$, it is defined as $\mathsf{let}\, \boldsymbol{X'}\hat{\boldsymbol{X}} = \boldsymbol{\varrho}(\boldsymbol{x})'\hat{\boldsymbol{\varrho}}(\hat{\boldsymbol{x}})\, \mathsf{in}\, L^{[Z := \hat{M}]}$. Per construction it only contracts rules in $\Phi', \hat{\Phi}$, so has a label in $\check{\Upsilon}\mathring{\mathcal{L}}(\Phi)$ by $\Phi = \phi \sqcup \Phi'$ and the label of $\hat{\Phi}$ is in $\check{\Upsilon}\mathring{\mathcal{L}}(\hat{\phi})$ by the (oi)-assumption. That $\Phi' \oplus \hat{\Phi} : u' \;-\!\!\circ\!\!\rightarrow w''$ follows again by *computation*, e.g. $[\![\mathsf{let}\, \boldsymbol{X'}\hat{\boldsymbol{X}} = \boldsymbol{r'}\hat{\boldsymbol{r}}\, \mathsf{in}\, L^{[Z := \hat{M}]}]\!] = L^{[\boldsymbol{X'}, Z := \boldsymbol{r'}, \hat{M}^{[\hat{\boldsymbol{X}} := \hat{\boldsymbol{r}}]}]} = L^{[\boldsymbol{X'}, Z := \boldsymbol{r'}, \hat{w}]} = \hat{w}\!\!\uparrow = w''$.

Finally, applying the I.H. to the peak $w'' \;_{\Phi' \oplus \hat{\Phi}}\!\!\leftarrow\!\!\circ\!\!-\; u' \;-\!\!\circ\!\!\rightarrow_{\Psi/\psi} u$ yields some hot-decreasing diagram D_{IH} (Fig. 1, right). Prefixing $\hat{\Psi}\!\!\uparrow$ to its closing conversion between w'' and u, then closes the original peak $s \;_{\Phi}\!\!\leftarrow\!\!\circ\!\!-\; t \;-\!\!\circ\!\!\rightarrow_\Psi u$ into a hot-decreasing diagram D, because labels of steps in $\hat{\Psi}\!\!\uparrow$ are in $\check{\Upsilon}\mathring{\mathcal{L}}(\Phi)$, $\mathring{\mathcal{L}}(\Phi) \mathrel{\succeq} \mathring{\mathcal{L}}(\Phi' \oplus \hat{\Phi})$ as seen above, and $\mathring{\mathcal{L}}(\Psi) \mathrel{\succeq} \mathring{\mathcal{L}}(\Psi/\psi)$. The I.H. applies since $\Phi \mathbin{\text{\rotatebox[origin=c]{90}{$\pitchfork$}}} \Psi > (\Phi' \oplus \hat{\Phi}) \mathbin{\text{\rotatebox[origin=c]{90}{$\pitchfork$}}} (\Psi/\psi)$: To see this, we define $L' = L^{[\boldsymbol{Z'} := \boldsymbol{\ell'}]}$ and $F' = \mathsf{let}\, \hat{\boldsymbol{X}} = \hat{\boldsymbol{\ell}}\, \mathsf{in}\, L'^{[Z := \hat{M}]}$ and collect needed ingredients (the joins are disjoint):

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[10] For this to be a valid 2nd-order substitution, the 1st-order variables of $\hat{\Omega}$ ($\hat{L}$) must be contained in those of $\hat{t}$, which we may assume by Lemma 7(1).

$$\begin{aligned}
D &= \varPhi^{\mathsf{src}} &&= (\mathsf{let}\ \boldsymbol{X'} = \boldsymbol{\ell'}\ \mathsf{in}\ L^{[Z:=\hat{t}]}) \sqcup \phi^{\mathsf{src}} &&= \varPhi'^{\mathsf{src}} \sqcup \phi^{\mathsf{src}} \\
E &= \varPsi^{\mathsf{src}} &&= (\mathsf{let}\ \boldsymbol{Y'} = \boldsymbol{g'}\ \mathsf{in}\ N^{[Y:=g]}) \sqcup \psi^{\mathsf{src}} &&= \varPsi'^{\mathsf{src}} \sqcup \psi^{\mathsf{src}} \\
D' &= (\varPhi' \oplus \hat{\varPhi})^{\mathsf{src}} &&= (\mathsf{let}\ \boldsymbol{X'} = \boldsymbol{\ell'}\ \mathsf{in}\ L^{[Z:=\hat{u}]}) \sqcup F' \\
E' &= (\varPsi/\psi)^{\mathsf{src}} &&= \mathsf{let}\ \boldsymbol{Y'} = \boldsymbol{g'}\ \mathsf{in}\ N^{[Y:=d]}
\end{aligned}$$

Using these one may reason with sets of patterns (not let-expressions as $t \neq s'$; the sets are positions in both t, s') as follows, relying on distributivity:

$$(D \sqcap E) \sqsupseteq (D_- \sqcap E) = (D_- \sqcap E') = (D'_+ \sqcap E') \sqsupseteq (D' \sqcap E') \tag{1}$$

where F is the singleton $\{\{p \in \phi^{\mathsf{src}} \mid p_\psi \not\preceq p\}\}$ having all positions in ϕ not below ψ's root, $D_- = \varPhi'^{\mathsf{src}} \sqcup F$, and $D'_+ = (D' - F') \sqcup F$. □

4 Confluence by Critical-Pair Closing Systems

We introduce a confluence criterion based on identifying for a term rewrite system $\mathcal{R}$ a subsystem $\mathcal{C}$ such that every $\mathcal{R}$-*critical peak* can be *closed* by means of $\mathcal{C}$-conversions, rendering the rules used in the peak redundant.

Definition 9. *A TRS $\mathcal{C}$ is* critical-pair closing *for a TRS $\mathcal{R}$, if $\mathcal{C}$ is a subsystem of $\mathcal{R}$ (namely $\mathcal{C} \subseteq \mathcal{R}$) and $s \leftrightarrow^*_{\mathcal{C}} t$ holds for all critical pairs (s,t) of $\mathcal{R}$.*

We phrase the main result of this section as a preservation-of-confluence result. We write $\to_{\mathcal{S}/\mathcal{R}}$ for $\twoheadrightarrow_{\mathcal{R}} \cdot \to_{\mathcal{S}} \cdot \twoheadrightarrow_{\mathcal{R}}$, and if it is terminating, $\mathcal{S}/\mathcal{R}$ is said to be *(relatively) terminating.* By $\mathcal{C}_d$ we denote the set of all duplicating rules in $\mathcal{C}$.

Theorem 3. *If $\mathcal{C}$ is a critical-pair-closing system for a left-linear TRS $\mathcal{R}$ such that $\mathcal{C}_d/\mathcal{R}$ is terminating, then $\mathcal{R}$ is confluent if $\mathcal{C}$ is confluent.*

Any left-linear TRS is critical-pair-closing for itself. However, the power of the method relies on choosing *small* $\mathcal{C}$. Before proving Theorem 3, we illustrate it by some (non-)examples and give a special case.

Example 9. Consider again the TRS $\mathcal{R}$ in Example 7. As we observed, the only critical pair originating from ϱ_4 and ϱ_1 is closed by $\to_{\varrho_1} \cdot \to_{\varrho_3} \cdot \to_{\varrho_6} \cdot {}_{\varrho_6}\leftarrow$. So the subsystem $\mathcal{C} = \{\varrho_1, \varrho_3, \varrho_6\}$ is a critical-pair-closing system for $\mathcal{R}$. As all $\mathcal{C}$-rules are linear, $\mathcal{C}_d/\mathcal{R}$ is vacuously terminating. Thus, by Theorem 3 it is sufficient to show confluence of $\mathcal{C}$. Because $\mathcal{C}$ has no critical pairs, the empty TRS $\emptyset$ is a critical-pair-closing TRS for $\mathcal{C}$. As $\emptyset/\mathcal{C}$ is terminating, confluence of $\mathcal{C}$ follows from that of $\emptyset$, which is trivial.

Observe how confluence was shown by successive applications of the theorem.

Remark 3. In our experiments (see Sect. 5), $\frac{3}{4}$ of the TRSs proven confluent by means of Theorem 3 used more than 1 iteration, with the maximum number of iterations being 6. For countable ARSs (see Corollary 2 below) 1 iteration suffices, which can be seen by setting $\mathcal{C}$ to the spanning forest obtained by Lemma 1. This provides the intuition underlying rule specialisation in Example 15 below.

Example 10. Although confluent, the TRS $\mathcal{R}$ in Example 8 does not have any confluent critical-pair-closing subsystem $\mathcal{C}$ such that $\mathcal{C}_d/\mathcal{R}$ is terminating, not even $\mathcal{R}$ itself: Because of b being in normal form in the critical pair induced by $b \ {}_{\varrho_1}\!\!\leftarrow f(a,a) \rightarrow_{\varrho_2} f(a,c)$, any such subsystem must contain ϱ_4, as one easily verifies, but ϱ_4 is both duplicating and non-terminating (looping).

Note that the termination condition of $\mathcal{C}_d/\mathcal{R}$ cannot be omitted from Theorem 3. Although the TRS $\mathcal{R}'$ in Example 8 is not confluent, it admits the confluent critical-pair-closing system $\{\varrho_1, \varrho_3, \varrho_4\}$.

Remark 4. The example is taken from [16] where it was used to show that decreasingness of *critical* peaks need not imply that of *all* peaks, for rule labelling. That example, in turn was adapted from Lévy's TRS in [17] showing that strong confluence need not imply confluence for left-linear TRSs.

Example 11. For *self-joinable* rules, i.e. rules that are self-overlapping and whose critical pairs need further applications of the rule itself to join, Theorem 3 is not helpful since the critical-pair-closing system $\mathcal{C}$ then contains the rule itself. Examples of self-joinable rules are associativity $(x \cdot y) \cdot z \rightarrow x \cdot (y \cdot z)$ and self-distributivity $(x \cdot y) \cdot z \rightarrow (x \cdot z) \cdot (y \cdot z)$, with confluence of the latter being known to be hard (currently no tool can handle it automatically).[11]

The special case we consider is that of TRSs that *are* ARSs, i.e. where all function symbols are nullary. The identification is justified by that any ARS in the standard sense [26,32] can be presented as $\rightarrow_{\mathcal{R}}$ for the TRS $\mathcal{R}$ having a nullary symbol for each object, and a rule for each step of the ARS. Since ARSs have no duplicating rules, Theorem 3 specialises to the following result.

Corollary 2. *If $\mathcal{C}$ is critical-pair-closing for ARS $\mathcal{R}$, $\mathcal{R}$ is confluent if $\mathcal{C}$ is.*

Example 12. Consider the TRS $\mathcal{R}$ given by $c \rightarrow a' \rightarrow a \rightarrow b$ and $a \rightarrow a' \rightarrow c$. It is an ARS having the critical-pair-closing system $\mathcal{C}$ given by the first part $c \rightarrow a' \rightarrow a \rightarrow b$. Since $\mathcal{C}$ is orthogonal it is confluent by Corollary 2, so $\mathcal{R}$ is confluent by the same corollary. In general, a confluent ARS may have many non-confluent critical-pair-closing systems. Requiring local confluence is no impediment to that: The subsystem $\mathcal{C}'$ of $\mathcal{R}$ obtained by removing $c \rightarrow a'$ allows to join all $\mathcal{R}$-critical peaks, but is not confluent; it simply is Kleene's example [32, Figure 1.2] showing that local confluence need not imply confluence.

For $\mathcal{C}_d/\mathcal{R}$ to be vacuously terminating it is sufficient that all rules are linear.

Example 13. Consider the linear TRS $\mathcal{R}$ consisting of $\rho_1\colon f(x) \rightarrow f(f(x))$, $\rho_2\colon f(x) \rightarrow g(x)$, and $\rho_3\colon g(x) \rightarrow f(x)$. The subsystem $\mathcal{C} = \{\rho_1, \rho_3\}$ is critical-pair-closing and has no critical pairs, so $\mathcal{R}$ is confluent.

From the above it is apparent that, whereas usual redundancy-criteria are based on *rules* being redundant, the theorem gives a sufficient criterion for *peaks* of steps being redundant. This allows one to leverage the power of extant confluence methods. Here we give a generalisation of Huet's strong closedness theorem [17] as a corollary of Theorem 3.

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[11] See problem 127 of http://cops.uibk.ac.at/results/?y=2019-full-run&c=TRS.

Definition 10. *A TRS $\mathcal{R}$ is* strongly closed *[17] if $s \twoheadrightarrow_{\mathcal{R}} \cdot \overset{=}{\underset{\mathcal{R}}{\twoheadleftarrow}} t$ and $s \twoheadrightarrow_{\overset{=}{\mathcal{R}}} \cdot {}_{\mathcal{R}}\twoheadleftarrow t$ hold for all critical pairs (s,t).*

Corollary 3. *A left-linear TRS $\mathcal{R}$ is confluent if there exists a critical-pair-closing system $\mathcal{C}$ for $\mathcal{R}$ such that $\mathcal{C}$ is linear and strongly closed.*

Example 14. Consider the linear TRS $\mathcal{R}$:

$\varrho_1 \colon h(f(x,y)) \to f(h(r(x)),y)$ $\varrho_2 \colon f(x,k(y,z)) \to g(p(y),q(z,x))$
$\varrho_3 \colon h(q(x,y)) \to q(x,h(r(y)))$ $\varrho_4 \colon q(x,h(r(y))) \to h(q(x,y))$
$\varrho_5 \colon h(g(x,y)) \to g(x,h(y))$
$\varrho_6 \colon \ a(x,y,z) \to h(f(x,k(y,z)))$ $\varrho_7 \colon \qquad a(x,y,z) \to g(p(y),q(z,h(r(x))))$

$\mathcal{C} = \{\varrho_1, \ldots, \varrho_5\}$ is critical-pair-closing for $\mathcal{R}$, since the $\mathcal{R}$-critical peak between ϱ_6 and ϱ_7 can be $\mathcal{C}$-closed: $h(f(x,k(y,z))) \to_{\varrho_1} f(h(r(x)),k(y,z)) \to_{\varrho_2} g(p(y),q(z,h(r(x))))$. Because $\mathcal{C}$ is strongly closed and also linear, confluence of $\mathcal{R}$ follows by Corollary 3.

Remark 5. Neither of the TRSs in Examples 13 and 14 is strongly closed. The former not, because $f(f(x)) \twoheadrightarrow_{\mathcal{R}} \cdot \overset{=}{\underset{\mathcal{R}}{\twoheadleftarrow}} g(x)$ does not hold, and the latter not because $g(p(y),q(z,h(r(x)))) \twoheadrightarrow_{\mathcal{R}} \cdot \overset{=}{\underset{\mathcal{R}}{\twoheadleftarrow}} h(f(x,k(y,z)))$ does not hold.

Having illustrated the usefulness of Theorem 3, we now present its proof. In TRSs there are *two* types of peaks: overlapping and non-overlapping ones. As Example 10 shows, confluence criteria only addressing the former need not generalise from ARSs to TRSs. Note that one of the peaks showing non-confluence of $\mathcal{R}'$, the one between ϱ_2 and ϱ_3 (ϱ_4), is non-overlapping. Therefore, restricting to a subsystem without ϱ_2 can only provide a partial analysis of confluence of $\mathcal{R}'$; the (non-overlapping) interaction between $\mathcal{C}$ and $\mathcal{R} - \mathcal{C}$ is not accounted for, and indeed that is fatal here. The intuition for our proof is that the problem is that the number of such interactions is unbounded due to the presence of the duplicating and non-terminating rule ϱ_3 (and ϱ_4) in $\mathcal{C}$, and that requiring termination of $\mathcal{C}_d/\mathcal{R}$ bounds that number and suffices to regain confluence. This is verified by showing that $\twoheadrightarrow_{\mathcal{C}} \cdot \multimap\!\!\twoheadrightarrow_{\mathcal{R}}$ has the diamond property.

Lemma 8. *Let $\to_{\mathcal{A}} = \bigcup_{a \in I} \to_a$ be a relation equipped with a well-founded order $\succ$ on a label set I, and let $\to_{\mathcal{B}}$ be a confluent relation with $\to_{\mathcal{B}} \subseteq \twoheadrightarrow_{\mathcal{A}}$. The relation $\to_{\mathcal{A}}$ is confluent if*

1. ${}_a\!\leftarrow \cdot \to_b \subseteq (\to_{\mathcal{A}} \cdot {}_{\mathcal{A}}\!\leftarrow) \cup \bigcup_{\{a,b\} \succ_{\mathsf{mul}} \{a',b'\}} ({}_{a'}\!\leftarrow \cdot \leftrightarrow^*_{\mathcal{B}} \cdot \to_{b'})$ for all $a,b \in I$; and
2. ${}_a\!\leftarrow \cdot \to_{\mathcal{B}} \subseteq (\twoheadrightarrow_{\mathcal{B}} \cdot {}_a\!\leftarrow) \cup \bigcup_{a \succ a'} (\twoheadrightarrow_{\mathcal{B}} \cdot {}_{a'}\!\leftarrow \cdot \leftrightarrow^*_{\mathcal{B}})$ for all $a \in I$.

Here $\succ_{\mathsf{mul}}$ stands for the multiset extension of $\succ$.

Proof (Sketch). Let $\rightarrowtail = \twoheadrightarrow_{\mathcal{B}} \cdot \to_{\mathcal{A}}$. We claim that ${}_a\!\leftarrow \cdot \to^m_{\mathcal{B}} \cdot {}^n_{\mathcal{B}}\!\leftarrow \cdot \to_b \subseteq \rightarrowtail \cdot \leftarrowtail$ holds for all labels a,b and numbers $m,n \geqslant 0$. The claim is shown by well-founded induction on $(\{a,b\}, m+n)$ with respect to the lexicographic product of $\succ_{\mathsf{mul}}$ and the greater-than order $>$ on $\mathbb{N}$. Thus, the diamond property of $\rightarrowtail$ follows from the claim and confluence of $\mathcal{B}$. As $\to_{\mathcal{A}} \subseteq \rightarrowtail \subseteq \twoheadrightarrow_{\mathcal{A}}$, we conclude confluence of $\mathcal{A}$ by e.g. [32, Proposition 1.1.11]. $\square$

Proof (of Theorem 3 by Lemma 8). Let I comprise pairs of a term and a natural number, and define $t \to_{(\hat{t},n)} s$ if $\hat{t} \twoheadrightarrow_{\mathcal{R}} t \multimap\hspace{-3pt}\twoheadrightarrow_{\mathcal{R}} s$ with n the maximal length of a development of the multistep,[12] and $\to_{\mathcal{B}} = \to_{\mathcal{C}}$, in Lemma 8. As well-founded order $\succ$ on indices we take the lexicographic product of $\mathcal{C}_d/\mathcal{R}$ and greater-than $>$. We only present the interesting case, leaving the others to the reader:

- Suppose $s \;_{(\hat{t},n)}\!\!\leftarrow t \to_{\mathcal{C}} u$ where the steps do not have overlap. Then by Lemma 5(1), $s \multimap\hspace{-3pt}\twoheadrightarrow_{\mathcal{C}} \cdot {}_{\mathcal{R}}\!\!\leftarrow\hspace{-3pt}\circ\; u$, so $s \twoheadrightarrow_{\mathcal{C}} \cdot {}_{\mathcal{R}}\!\!\leftarrow\hspace{-3pt}\circ\; u$. Distinguish cases on the type of the $\mathcal{C}$-rule employed in $t \to_{\mathcal{C}} u$.
  If the rule is duplicating, then $s \twoheadrightarrow_{\mathcal{C}} \cdot {}_{(u,m)}\!\!\leftarrow u$ for m the maximal length of a development of the $\multimap\hspace{-3pt}\twoheadrightarrow_{\mathcal{R}}$-step from u, and condition 2 is satisfied as $t \to_{\mathcal{C}_d} u$ implies $(\hat{t},n) \succeq (u,m)$.
  If the rule is non-duplicating, then $s \twoheadrightarrow_{\mathcal{C}} \cdot {}_{(\hat{t},n)}\!\!\leftarrow u$ as $\hat{t} \twoheadrightarrow_{\mathcal{R}} t \to_{\mathcal{R}} u$ by assumption and the length of the maximal development of the residual multistep does not increase when projecting over a linear rule. Again, condition 2 is satisfied. □

5 Implementation and Experiments

The presented confluence techniques have been implemented in the confluence tool Saigawa version 1.12 [14]. We used the tool to test the criteria on 432 left-linear TRSs in COPS [15] Nos. 1–1036, where we ruled out duplicated problems. Out of 432 systems, 224 are known to be confluent and 173 are non-confluent.

   We briefly explain how we automated the presented techniques. As illustrated in Example 9, Theorem 3 can be used as a stand-alone criterion. The condition $s \to_{\mathcal{R}}^* \cdot \overset{=}{_{\mathcal{R}}}\!\!\leftarrow t$ of strong closedness is tested by $s \to_{\mathcal{R}}^{\leqslant 5} \cdot \overset{=}{_{\mathcal{R}}}\!\!\leftarrow t$. For a critical peak $s \leftarrow \cdot \to t$ of $\mathcal{C}$, hot-decreasingness is tested by $s \twoheadrightarrow_{\mathcal{C}} \cdot {}_{\mathcal{C}}\!\!\leftarrow t$. For any other critical peak $s \;_{\ell}\!\!\leftarrow \cdot \to t$, we test the disjunction of $s \to_{\mathring{\curlyvee}\ell}^{\leqslant 5} \cdot {}_{\mathring{\curlyvee}\ell}\!\!\leftarrow\hspace{-3pt}\circ\; t$ and $s \twoheadrightarrow_{\mathcal{C}} \cdot {}_{\mathring{\curlyvee}\ell}\!\!\leftarrow\hspace{-3pt}\circ\; t$ if it is outer–inner one, and if it is overlay, the disjunction of $s \to_{\mathring{\curlyvee}\ell}^{\leqslant 5} \cdot {}_{\mathring{\curlyvee}\ell}\!\!\leftarrow\hspace{-3pt}\circ\; t$ and $s \twoheadrightarrow_{\mathcal{C}} \cdot {}_{\mathcal{C}}\!\!\leftarrow t$ is used. Order constraints for hot-labeling are solved by SMT solver Yices [10]. For proving (relative) termination we employ the termination tool NaTT version 1.8 [34]. Finally, suitable subsystems $\mathcal{C}$ used in our criteria are searched by enumeration.

   Table 1 gives a summary of the results.[13] The tests were run on a PC equipped with Intel Core i7-8500Y CPU (1.5 GHz) and 16 GB memory using a timeout of 60 seconds. For the sake of comparison we also tested Knuth and Bendix' theorem (kb), the strong closedness theorem (sc), and development closedness theorem (dc). As theoretically expected, they are subsumed by their generalizations.

6 Conclusion and Future Work

We presented two methods for proving confluence of TRSs, dubbed critical-pair-closing systems and hot-decreasingness. We gave a lattice-theoretic characterisation of overlap. Since many results in term rewriting, and beyond, are based on

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[12] By the Finite Developments Theorem lengths of such developments are finite [32].

[13] Detailed data are available from: http://www.jaist.ac.jp/project/saigawa/19cade/.

Table 1. Experimental results

	Theorem 2	Theorem 3	Corollary 3	kb	dc	sc
# proved (# timeouts)	101 (46)	81 (24)	94 (15)	45 (18)	34 (1)	62 (1)

reasoning about overlap, which is notoriously hard [24], we expect that formalising our characterisation could simplify or even enable formalising them. We expect that both methods generalise to commutation, extend to HRSs [21], and can be strengthened by considering rule *specialisations*.

Example 15. Analysing the TRS $\mathcal{R}$ of Example 10 one observes that for closing the critical pairs only (non-duplicating) *instances* of the duplicating rules ϱ_3 and ϱ_4 are used. Adjoining these specialisations allows the method to proceed: Adjoining $\varrho_3(a) : f(c, a) \to f(a, a)$ and $\varrho_4(a) : f(a, c) \to f(a, a)$ to $\mathcal{R}$ yields a (reduction-equivalent) TRS having critical-pair-closing system $\{\varrho_1, \varrho_3(a), \varrho_4(a), \varrho_5\}$. Since this is a linear system without critical pairs, it is confluent, so $\mathcal{R}$ is as well.

References

1. Baader, F., Nipkow, T.: Term Rewriting and All That. Cambridge University Press, Cambridge (1998)
2. Barendregt, H.: The Lambda Calculus: Its Syntax and Semantics, Studies in Logic and the Foundations of Mathematics, vol. 103. North-Holland (1985)
3. Bechet, D., de Groote, P., Retoré, C.: A complete axiomatisation for the inclusion of series-parallel partial orders. In: Comon, H. (ed.) RTA 1997. LNCS, vol. 1232, pp. 230–240. Springer, Heidelberg (1997). https://doi.org/10.1007/3-540-62950-5_74
4. Boudol, G.: Computational semantics of term rewriting systems. In: Nivat, M., Reynolds, J. (eds.) Algebraic Methods in Semantics, pp. 169–236. Cambridge University Press (1985)
5. Bundy, A., Basin, D., Hutter, D., Ireland, A.: Rippling: meta-level guidance for mathematical reasoning. In: Cambridge Tracts in Theoretical Computer Science, Cambridge University Press (2005). https://doi.org/10.1017/CBO9780511543326
6. Church, A., Rosser, J.: Some properties of conversion. Transact. Am. Math. Soc. **39**, 472–482 (1936)
7. Comon, H., et al.: Tree Automata Techniques and Applications (2007). http://www.grappa.univ-lille3.fr/tata
8. Davey, B., Priestley, H.: Introduction to Lattices and Order. Cambridge University Press, Cambridge (1990)
9. Dershowitz, N., Jouannaud, J.P.: Rewrite systems. In: van Leeuwen, J. (ed.) Handbook of Theoretical Computer Science, vol. B, Formal Models and Semantics, pp. 243–320. Elsevier (1990)
10. Dutertre, B.: Yices 2.2. In: Biere, A., Bloem, R. (eds.) CAV 2014. LNCS, vol. 8559, pp. 737–744. Springer, Cham (2014). https://doi.org/10.1007/978-3-319-08867-9_49

11. Endrullis, J., Klop, J., Overbeek, R.: Decreasing diagrams with two labels are complete for confluence of countable systems. In: Proceedings of 3rd FSCD. LIPIcs, vol. 108, pp. 14:1–14:15 (2018). https://doi.org/10.4230/LIPIcs.FSCD.2018.14

12. Felgenhauer, B., Middeldorp, A., Zankl, H., van Oostrom, V.: Layer systems for proving confluence. ACM Transact. Computat. Logic **16**, 1–32 (2015)

13. Felgenhauer, B.: Labeling multi-steps for confluence of left-linear term rewrite systems. In: Tiwari, A., Aoto, T. (eds.) Proceedings of 4th IWC, pp. 33–37 (2015)

14. Hirokawa, H., Klein, D.: Saigawa: A confluence tool. In: Proceedings of 1st IWC, p. 49 (2012). http://www.jaist.ac.jp/project/saigawa/

15. Hirokawa, N., Nagele, J., Middeldorp, A.: Cops and CoCoWeb: infrastructure for confluence tools. In: Galmiche, D., Schulz, S., Sebastiani, R. (eds.) IJCAR 2018. LNCS (LNAI), vol. 10900, pp. 346–353. Springer, Cham (2018). https://doi.org/10.1007/978-3-319-94205-6_23

16. Hirokawa, N., Middeldorp, A.: Decreasing diagrams and relative termination. J. Autom. Reasoning **47**(4), 481–501 (2011)

17. Huet, G.: Confluent reductions: abstract properties and applications to term rewriting systems. J. ACM **27**(4), 797–821 (1980)

18. Huet, G., Lévy, J.J.: Computations in orthogonal rewriting systems, I. In: Lassez, J.L., Plotkin, G. (eds.) Computational Logic: Essays in Honor of Alan Robinson, chap. 11. The MIT Press (1991)

19. Knuth, D.E., Bendix, P.B.: Simple word problems in universal algebras. In: Leech, J. (ed.) Computational Problems in Abstract Algebra, Proceedings of a Conference held at Oxford under the Auspices of the Science Research Council Atlas Computer Laboratory, 29 August–2 September 1967. pp. 263–297 (1970)

20. Liu, J.L.: Propriétés de Confluence des Règles de Réécriture par des Diagrammes Décroissants. Ph.D. thesis, Tsinghua University and l'Université Paris-Saclay préparée à l'École Polytechnique (2016)

21. Mayr, R., Nipkow, T.: Higher-order rewrite systems and their confluence. Theoret. Comput. Sci. **192**(1), 3–29 (1998). https://doi.org/10.1016/S0304-3975(97)00143-6

22. Meseguer, J.: Conditional rewriting logic as a unified model of concurrency. Theoret. Comput. Sci. **96**, 73–155 (1992)

23. Métivier, Y.: About the rewriting systems produced by the Knuth–Bendix completion algorithm. Inf. Process. Lett. **16**(1), 31–34 (1983)

24. Nagele, J., Middeldorp, A.: Certification of classical confluence results for left-linear term rewrite systems. In: Blanchette, J.C., Merz, S. (eds.) ITP 2016. LNCS, vol. 9807, pp. 290–306. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-43144-4_18

25. Newman, M.: On theories with a combinatorial definition of equivalence. Ann. Math. **43**(2), 223–243 (1942)

26. Ohlebusch, E.: Advanced Topics in Term Rewriting. Springer, Heidelberg (2002). https://doi.org/10.1007/978-1-4757-3661-8

27. Okui, S.: Simultaneous critical pairs and Church-Rosser property. In: Nipkow, T. (ed.) RTA 1998. LNCS, vol. 1379, pp. 2–16. Springer, Heidelberg (1998). https://doi.org/10.1007/BFb0052357

28. van Oostrom, V.: Confluence for Abstract and Higher-Order Rewriting. Ph.D. thesis, Vrije Universiteit, Amsterdam, March 1994

29. van Oostrom, V.: Developing developments. Theoret. Comput. Sci. **175**(1), 159–181 (1997)

30. van Oostrom, V.: Confluence by decreasing diagrams. In: Voronkov, A. (ed.) RTA 2008. LNCS, vol. 5117, pp. 306–320. Springer, Heidelberg (2008). https://doi.org/10.1007/978-3-540-70590-1_21
31. Rosen, B.: Tree-manipulating systems and Church-Rosser theorems. J. ACM **20**, 160–187 (1973)
32. Terese: Term Rewriting Systems. Cambridge University Press (2003)
33. Toyama, Y.: On the Church-Rosser property for the direct sum of term rewriting systems. J. ACM **34**(1), 128–143 (1987)
34. Yamada, A., Kusakari, K., Sakabe, T.: Nagoya termination tool. In: Dowek, G. (ed.) RTA 2014. LNCS, vol. 8560, pp. 466–475. Springer, Cham (2014). https://doi.org/10.1007/978-3-319-08918-8_32
35. Zankl, H., Felgenhauer, B., Middeldorp, A.: Labelings for decreasing diagrams. J. Autom. Reasoning **54**(2), 101–133 (2015)