Chapter 9
Solutions and Stability of Some Functional Equations on Semigroups

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Abstract In this paper we investigate the solutions and the Hyers-Ulam stability of the μ-Jensen functional equation

$$f(xy) + \mu(y) f(x\sigma(y)) = 2f(x), \ \ x, y \in S,$$

a variant of the μ-Jensen functional equation

$$f(xy) + \mu(y) f(\sigma(y)x) = 2f(x), \ \ x, y \in S,$$

and the μ-quadratic functional equation

$$f(xy) + \mu(y) f(x\sigma(y)) = 2f(x) + 2f(y), \ \ x, y \in S,$$

where S is a semigroup, σ is a morphism of S and μ: $S \longrightarrow \mathbb{C}$ is a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$.

Keywords Functional equation · Hyers-Ulam stability · μ-Jensen functional equation · μ-Quadratic functional equation

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9.1 Introduction

In 1940, Ulam [31] delivered a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he posed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms: Given a group G_1, a metric group (G_2, d), a number $\epsilon > 0$ and a mapping $f : G_1 \longrightarrow G_2$ which satisfies $d(f(xy), f(x)f(y)) < \epsilon$ for all $x, y \in G_1$, does there exist a homomorphism $g : G_1 \longrightarrow G_2$ and a constant $k > 0$, depending only on G_1 and G_2 such that $d(f(x), g(x)) < k\epsilon$ for all $x \in G_1$?

In the case of a positive answer to this problem, we say that the Cauchy functional equation $f(xy) = f(x)f(y)$ is stable for the pair (G_1, G_2).

The first affirmative partial answer was given in 1941 by Hyers [16] where G_1, G_2 are Banach spaces.

In 1950 Aoki [2] provided a generalization of Hyers' theorem for additive mappings and in 1978 Rassias [22] generalized Hyers' theorem for linear mappings by allowing the Cauchy difference to be unbounded.

Beginning around the year 1980, several results for Hyers-Ulam-Rassias stability of many functional equations have been proved by several mathematicians. For more details, we can refer for example to [3, 8–10, 12–14, 17, 19, 23–26].

Let S be a semigroup with identity element e. Let σ be an involutive morphism of S. That is σ is an involutive homomorphism:

$$\sigma(xy) = \sigma(x)\sigma(y) \text{ and } \sigma(\sigma(x)) = x \text{ for all } x, y \in S,$$

or σ is an involutive anti-homomorphism:

$$\sigma(xy) = \sigma(y)\sigma(x) \text{ and } \sigma(\sigma(x)) = x \text{ for all } x, y \in S.$$

We say that $f : S \longrightarrow \mathbb{C}$ satisfies the Jensen functional equation if

$$f(xy) + f(x\sigma(y)) = 2f(x), \tag{9.1}$$

for all $x, y \in S$.

A complex valued function f defined on a semigroup S is a solution of a variant of the Jensen functional equation if

$$f(xy) + f(\sigma(y)x) = 2f(x), \tag{9.2}$$

for all $x, y \in S$. Equations (9.1) and (9.2) coincide if f is central, and the central solutions are the maps of the form $f = a + c$, where $a : S \longrightarrow \mathbb{C}$ is an additive map such that $a(\sigma(x)) = -a(x)$ and where $c \in \mathbb{C}$ is a constant.

The Jensen functional equation (9.1) takes the form

$$f(xy) + f(xy^{-1}) = 2f(x) \tag{9.3}$$

for all $x, y \in S$ when $\sigma(x) = x^{-1}$ and S is a group. The new equation (9.2) is much simpler than (9.1). For a more general study we refer the reader to Ng's paper [21] and Stetkær's book [26].

The stability in the sense of Hyers-Ulam of the Jensen equations (9.1) and (9.3) has been studied by various authors for the case when S is an abelian group or a vector space. The interested reader is referred to the papers of Jung [18] and Kim [20].

In 2010, Faiziev and Sahoo [11] proved the Hyers-Ulam stability of Eq. (9.3) on some non-commutative groups such as metabelian groups and $T(2, K)$, where K is an arbitrary commutative field with characteristic different from two. They have shown as well that every semigroup can be embedded into a semigroup in which the Jensen equation is stable.

The quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \; x, y \in S \tag{9.4}$$

has been extensively studied (see for example [1, 17, 26]). It was generalized by Stetkær [25] to the more general equation

$$f(x + y) + f(x + \sigma(y)) = 2f(x) + 2f(y), \; x, y \in S. \tag{9.5}$$

A stability result for the quadratic functional equation (9.4) was derived by Cholewa [5] and by Czerwik [6]. Bouikhalene et al. [3] stated the stability theorem of Eq. (9.5). In [7] the stability of the quadratic functional equation

$$f(xy) + f(xy^{-1}) = 2f(x) + 2f(y), \; x, y \in S \tag{9.6}$$

was obtained on amenable groups.

Bouikhalene et al. [4] obtained the stability of the quadratic functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x) + 2f(y), \; x, y \in S \tag{9.7}$$

on amenable semigroups.

In this paper we consider the following functional equations:

The μ-Jensen functional equation

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x), \; x, y \in S, \tag{9.8}$$

a variant of the μ-Jensen functional equation

$$f(xy) + \mu(y)f(\sigma(y)x) = 2f(x), \; x, y \in S, \tag{9.9}$$

and the μ-quadratic functional equation

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x) + 2f(y), \; x, y \in S, \tag{9.10}$$

where $\mu: S \longrightarrow \mathbb{C}$ is a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$.

Our results are organized as follows. In Sects. 9.2 and 9.3 we give a proof of the Hyers-Ulam stability of the Jensen functional equation (9.1) and a variant of the Jensen functional equation (9.2) on an amenable semigroup. As an application (Sect. 9.4), we prove the Hyers-Ulam stability of the symmetric functional equation

$$f(xy) + f(yx) = 2f(x) + 2f(y),\ \ x, y \in G, \tag{9.11}$$

where G is an amenable group.

In Sects. 9.5 and 9.6 we prove that the μ-Jensen equation (9.8), respectively, the μ-quadratic functional equation (9.10) possesses the same solutions as Jensen's functional equation (9.1), respectively, the quadratic functional equation (9.7). Furthermore, we prove the equivalence of their stability theorems on semigroups.

Throughout this paper m denotes a linear functional on the space $B(S, \mathbb{C})$, namely the space of all bounded functions on S.

The linear functional m is called a left, respectively, right invariant mean if and only if

$$\inf_{x\in S} f(x) \le m(f) \le \sup_{x\in S} f(x);\ \ m({}_a f) = m(f);\ \text{respectively, } m(f_a) = m(f)$$

for all $f \in B(S, \mathbb{R})$ and $a \in S$, where ${}_a f$ and f_a are the left and right translates of f defined by ${}_a f(x) = f(ax)$; $f_a(x) = f(xa)$, $x \in S$.

A semigroup S which admits a left, respectively, right invariant mean on $B(S, \mathbb{C})$ will be called left, respectively, right amenable. If on the space $B(S, \mathbb{C})$ there exists a real linear functional which is simultaneously a left and right invariant mean, then we say that S is two-sided amenable or just amenable. We refer to [15] for the definition and properties of invariant means.

9.2 Stability of a Variant of the Jensen Functional Equation

In this section we investigate the Hyers-Ulam stability of the functional equation (9.2) on amenable semigroups.

Theorem 9.1 *Let S be an amenable semigroup with identity element e. Let σ be an involutive anti-homomorphism, and let $f : G \longrightarrow \mathbb{C}$ be a function. Assume that there exists $\delta \ge 0$ such that*

$$|f(xy) + f(\sigma(y)x) - 2f(x)| \le \delta \tag{9.12}$$

for all $x, y \in S$. Then, there exists a unique solution $J : S \longrightarrow \mathbb{C}$ of the functional equation (9.2) such that $J(\sigma(x)) = -J(x)$ and

$$|f(x) - J(x) - f(e)| \le \delta \tag{9.13}$$

for all $x \in S$*. Furthermore if* S *is a group and* $\sigma(x) = x^{-1}$ *then there exists a unique additive map* $a : S \longrightarrow \mathbb{C}$ *such that*

$$|f(x) - a(x) - f(e)| \leq \delta \tag{9.14}$$

for all $x \in S$*.*

Proof Let x, y be in S. Replacing x by $\sigma(x)$ in (9.12) we get

$$|f(\sigma(x)y) + f(\sigma(y)\sigma(x)) - 2f(\sigma(x))| \leq \delta \tag{9.15}$$

Adding (9.12) to (9.15), and using the triangle inequality we obtain that

$$|[f(xy) + f(\sigma(y)\sigma(x))] + [f(\sigma(y)x) + f(\sigma(x)y)] - 2[f(x) + f(\sigma(x))]| \leq 2\delta. \tag{9.16}$$

Hence

$$|f^e(xy) + f^e(\sigma(y)x) - 2f^e(x)| \leq \delta, \tag{9.17}$$

where

$$f^e(x) = \frac{f(x) + f(\sigma(x))}{2} \text{ for all } x \in S.$$

Subtracting (9.15) from (9.12), and using the triangle inequality we derive that

$$|f^o(xy) + f^o(\sigma(y)x) - 2f^o(x)| \leq \delta \tag{9.18}$$

for all $x, y \in S$, where

$$f^o(x) = \frac{f(x) - f(\sigma(x))}{2} \text{ for all } x \in S.$$

Setting $x = e$ in (9.17) we obtain

$$|f^e(y) - f^e(e)| \leq \frac{\delta}{2} \text{ for all } x, y \in S. \tag{9.19}$$

By replacing x by y in (9.18) and by the fact that f^o is odd we get

$$|f^o(yx) - f^o(\sigma(y)x) - 2f^o(y)| \leq \delta. \tag{9.20}$$

This implies that for each y fixed in S, the function $x \longrightarrow f^o(yx) - f^o(\sigma(y)x)$ is bounded. Since S is amenable, then there exists an invariant mean m on the space of complex bounded functions on S and we can define the new mapping on S by

$$\psi(y) = m\{_y f^o -_{\sigma(y)} f^o\}, \text{ for all } y \in S. \tag{9.21}$$

Using (9.21) and the fact that m is an invariant mean we get

$$\begin{aligned}\psi(yz)+\psi(\sigma(z)y) &= m\{_{yz}f^o -_{\sigma(z)\sigma(y)} f^o\} + m\{_{\sigma(z)y}f^o -_{\sigma(y)z} f^o\} \\ &= m\{_{yz}f^o -_{\sigma(y)z} f^o\} + m\{_{\sigma(z)y}f^o -_{\sigma(z)\sigma(y)} f^o\} \\ &= m\{_z[_y f^o -_{\sigma(y)} f^o]\} + m\{[_y f^o -_{\sigma(y)} f^o]_{\sigma(z)}\} \\ &= m\{_y f^o -_{\sigma(y)} f^o\} + m\{_y f^o -_{\sigma(y)} f^o\} \\ &= \psi(y)+\psi(y) = 2\psi(y)\end{aligned}$$

for all $x, y \in S$. The function

$$J(y) = \frac{\psi(y)}{2}$$

satisfies the variant of the Jensen functional equation (9.2), $J(\sigma(y)) = -J(y)$ for all $y \in S$, and we have the following inequality

$$\begin{aligned}|J(y) - f^o(y)| &= |\frac{1}{2}m\{yf^o -_{\sigma(y)} f^o - 2f(y)\}| \\ &\leq \frac{1}{2}\sup_{x\in S}|f^o(yx) - f^o(\sigma(y)x) - 2f^o(y)| \leq \frac{\delta}{2}.\end{aligned} \tag{9.22}$$

Finally, we obtain

$$\begin{aligned}|f(y) - J(y) - f(e)| &= |f^e(y) + f^o(y) - J(y) - f(e)| \\ &\leq |f^e(y) - f(e)| + |f^o(y) - J(y)| \leq \delta\end{aligned}$$

for all $y \in S$. This proves the first part of Theorem 9.1.

If S is a group and $\sigma(x) = x^{-1}$, then from [26, Proposition 12.29] we have $J = a$, where $a : S \longrightarrow \mathbb{C}$ is an additive map.

Now suppose that there exist two odd functions J_1 and J_2 satisfying the variant of the Jensen functional equation (9.2), and the following inequality

$$|f(y) - J_i(y) - f(e)| \leq \delta, \text{ with } i = 1, 2. \tag{9.23}$$

The function $J := J_1 - J_2$ is also a solution of the functional equation (9.2), that is

$$J(xy) + J(\sigma(y)x) = 2J(x) \text{ for all } x, y \in S. \tag{9.24}$$

By using the triangle inequality we get $|J(x)| \leq 2\delta$ for all $x \in S$.

Replacing y by x in (9.24) and using that $J(\sigma(x)) = -J(x)$ we get

$$J(x^2) = 2J(x) \tag{9.25}$$

and consequently, we get $J(x^{2^n}) = 2^n J(x)$ for all $n \in \mathbb{N}$. Since J is a bounded map then $J(x) = 0$ for all $x \in S$. This completes the proof of Theorem 2.1. □

The stability of Eq. (9.2) has been obtained in [4, Lemma 3.2], on amenable semigroups with identity element and under the condition that σ is an involutive homomorphism. In the following theorem we investigate the Hyers-Ulam stability of the functional equation (9.2) on amenable semigroups without identity element, and where σ is a homomorphism.

Theorem 9.2 *Let S be an amenable semigroup. Let σ be an involutive homomorphism of S and let $f : S \longrightarrow \mathbb{C}$ be a function. Assume that there exists $\delta \geq 0$ such that*

$$|f(xy) + f(\sigma(y)x) - 2f(x)| \leq \delta \tag{9.26}$$

for all $x, y \in S$. Then there exists a unique additive function $a : S \longrightarrow \mathbb{C}$ and $x_0 \in S$ such that

$$|f(x) - a(x) + f(x_0) - f(\sigma(x_0)) - f(x_0^2)| \leq 4\delta \tag{9.27}$$

for all $x \in S$.

Proof In the proof we use some ideas from Stetkær [28].

Let x, y, z be in S. If we replace x by xy and y by z in (9.26) we get

$$|f(xyz) + f(\sigma(z)xy) - 2f(xy)| \leq \delta. \tag{9.28}$$

By replacing x by $\sigma(z)x$ in (9.26) we get

$$|f(\sigma(z)xy) + f(\sigma(y)\sigma(z)x) - 2f(\sigma(z)x)| \leq \delta. \tag{9.29}$$

Replacing y by z in (9.26) and multiplying the result by 2 we get

$$|2f(xz) + 2f(\sigma(z)x) - 4f(x)| \leq 2\delta. \tag{9.30}$$

If we replace y by yz in (9.26) we get

$$|f(xyz) + f(\sigma(y)\sigma(z)x) - 2f(x)| \leq \delta. \tag{9.31}$$

Subtracting (9.31) from (9.29) and using the triangle inequality we get

$$|f(\sigma(z)xy) - 2f(\sigma(z)x) - f(xyz) + 2f(x)| \leq 2\delta. \tag{9.32}$$

Adding (9.30) and (9.32) and using the triangle inequality we obtain

$$|2f(xz) - 2f(x) + f(\sigma(z)xy) - f(xyz)| \leq 4\delta. \tag{9.33}$$

Subtracting (9.33) from (9.28) and applying the triangle inequality we get

$$|2f(xyz) - 2f(xy) - 2f(xz) + 2f(x)| \leq 5\delta, \tag{9.34}$$

which can be written as follows

$$|[2f(xyz) - 2f(x)] - [2f(xy) - 2f(x)] - [2f(xz) - 2f(x)]| \leq 5\delta. \tag{9.35}$$

Now, for each fixed x_0 in S we define on S the function $A_{x_0}(t) = 2f(x_0 t) - 2f(x_0)$. Therefore, the inequality (9.35) can be written as follows

$$|A_{x_0}(yz) - A_{x_0}(y) - A_{x_0}(z)| \leq 5\delta \text{ for all } y, z \in S. \tag{9.36}$$

Since S is an amenable semigroup then by Szekelyhidi [30] there exists a unique additive mapping $b : S \longrightarrow \mathbb{C}$ such that

$$|A_{x_0}(x) - b(x)| \leq 5\delta \text{ for all } x \in S. \tag{9.37}$$

Replacing y in (9.26) by yz we get

$$|f(xyz) + f(\sigma(yz)x) - 2f(x)| \leq \delta. \tag{9.38}$$

If we replace x by $\sigma(y)$ and y by $\sigma(z)x$ in (9.26) we derive

$$|f(\sigma(y)\sigma(z)x) + f(z\sigma(xy)) - 2f(\sigma(y))| \leq \delta. \tag{9.39}$$

Replacing x by z and y by $\sigma(xy)$ in (9.26) we get

$$|f(z\sigma(xy)) + f(xyz) - 2f(z)| \leq \delta. \tag{9.40}$$

Subtracting (9.39) from the sum of (9.38) and (9.40) and applying the triangle inequality we get

$$|2f(xyz) - 2f(x) - 2f(z) + 2f(\sigma(y))| \leq 3\delta. \tag{9.41}$$

By replacing x and y by x_0, and z by x in (9.41) we get

$$|2f(x_0^2 x) - 2f(x_0) - 2f(x) + 2f(\sigma(x_0))| \leq 3\delta, \tag{9.42}$$

which can be expressed as follows

$$|2f(x_0^2 x) - 2f(x_0^2) - 2f(x) - 2f(x_0) + 2f(\sigma(x_0)) + 2f(x_0^2)| \leq 3\delta. \tag{9.43}$$

Since $A_{x_0^2}(x) = 2f(x_0^2 x) - 2f(x_0^2)$, then we have

$$|A_{x_0^2}(x) - 2f(x) - 2f(x_0) + 2f(\sigma(x_0)) + 2f(x_0^2)| \leq 3\delta. \tag{9.44}$$

Subtracting (9.37) from (9.44) and using the triangle inequality we get

$$|f(x) - a(x) + f(x_0) - f(\sigma(x_0)) - f(x_0^2)| \leq 4\delta, \tag{9.45}$$

where $a = \frac{1}{2}b$. This completes the proof of Theorem 2.2. □

9.3 Hyers-Ulam Stability of Eq. (9.1) on Amenable Semigroups

In this section, we investigate the Hyers-Ulam stability of Eq. (9.1) on an amenable semigroup, where σ is an involutive anti-homomorphism.

Theorem 9.3 *Let S be an amenable semigroup with identity element e. Let σ be an involutive anti-homomorphism of S. Let $f : S \longrightarrow \mathbb{C}$ be a function which satisfies the following inequality*

$$|f(xy) + f(x\sigma(y)) - 2f(x)| \leq \delta \tag{9.46}$$

for all $x, y \in S$ and for some nonnegative δ. Then there exists a unique solution j of the Jensen equation (9.1) such that $j(\sigma(x)) = -j(x)$ and

$$|f(x) - j(x) - f(e)| \leq 3\delta \tag{9.47}$$

for all $x \in S$.

First, we prove the following useful lemma.

Lemma 9.1 *Let S be a semigroup. Let σ be an involutive anti-homomorphism of S. Let $f : S \longrightarrow \mathbb{C}$ be a function such that $f(\sigma(x)) = -f(x)$ for all $x \in S$ and for which there exists a solution g of the Drygas functional equation*

$$g(yx) + g(\sigma(y)x) = 2g(x) + g(y) + g(\sigma(y)),\ \ x, y \in S \tag{9.48}$$

such that $|f(x) - g(x)| \leq M$, for all $x \in S$ and for some non negative M. Then

$$g(x) = \lim_{n\to+\infty} 2^{-n} f(x^{2^n}) \text{ for all } x \in S. \tag{9.49}$$

Furthermore $g(\sigma(x)) = -g(x)$ for all $x \in S$ and g satisfies the Jensen functional equation

$$g(xy) + g(x\sigma(y)) = 2g(x) \text{ for all } x, y \in S.$$

Proof Replacing y by $x\sigma(x)$ in (9.48) we obtain

$$g((x\sigma(x))^2) + g((x\sigma(x))^2) = 2g(x\sigma(x)) + g(x\sigma(x)) + g(x\sigma(x)), \tag{9.50}$$

which implies that $g((x\sigma(x))^2) = 2g(x\sigma(x))$ for all $x \in S$.

By applying the induction assumption we get

$$2^n g(x\sigma(x)) = g((x\sigma(x))^{2^n}) \tag{9.51}$$

for all $n \in \mathbb{N}$ and for all $x \in S$.

Now, by the hypothesis, $f = g + b$ where b is a bounded function. Since f is odd we have $f = g^o + b^o$ and $g^e + b^e = 0$. Using (9.51) and the fact that

$$g((x\sigma(x))^{2^n}) = g^e((x\sigma(x))^{2^n})$$

we get

$$|g(x\sigma(x))| = 2^{-n}|g^e((x\sigma(x))^{2^n})| \le 2^{-n}|b^e(x\sigma(x))^{2^n}|. \tag{9.52}$$

Letting $n \to +\infty$ in the formula (9.52), we obtain that $g(x\sigma(x)) = 0$ and hence $g(\sigma(x)x) = 0$ for all $x \in S$.

Setting $y = x$ in (9.48) we get

$$g(x^2) = 2g(x) + g(x) + g(\sigma(x)). \tag{9.53}$$

If we replace x by $\sigma(x)$ in (9.53) we have

$$g(\sigma(x)^2) = 2g(\sigma(x)) + g(x) + g(\sigma(x)). \tag{9.54}$$

By adding (9.53) and (9.54) we get that $g^e(x^2) = 4g^e(x)$, and by induction it follows that

$$g^e(x^{2^n}) = 2^{2^n} g^e(x) \tag{9.55}$$

for all $x \in S$ and for all $n \in \mathbb{N}$.

Using (9.55) and the fact that $g^e + b^e = 0$ we have

$$g^e(x) = 2^{-2^n} g^e(x^{2^n}) = -2^{-2^n} b^e(x^{2^n}). \tag{9.56}$$

Therefore, we get

$$|g^e(x)| = |2^{-2^n} g^e(x^{2^n})| \le 2^{-2^n} |b^e(x^{2^n})|.$$

So by letting $n \to +\infty$ we obtain that $g^e(x) = 0$ for all $x \in S$, which proves that $g(\sigma(x)) = -g(x)$ for all $x \in S$.

Using (9.53) and that g is odd we get that $g(x^2) = 2g(x)$, and by induction we deduce that

$$g(x^{2^n}) = 2^n g(x) \tag{9.57}$$

for all $x \in S$, and for all $n \in \mathbb{N}$.

Using (9.57) we get

$$2^{-n} f(x^{2^n}) = 2^{-n}[g(x^{2^n}) + b^o(x^{2^n})] = g(x) + 2^{-n} b^o(x^{2^n}).$$

Thus

$$|g(x) - 2^{-n} f(x^{2^n})| \leq 2^{-n} |b^o(x^{2^n})|. \tag{9.58}$$

By letting $n \to +\infty$ we obtain

$$g(x) = \lim_{n \to +\infty} 2^{-n} f(x^{2^n}).$$

We will prove that g satisfies the Jensen functional equation (9.1).

Since $g(\sigma(x)) = -g(x)$ for all $x \in S$, the Drygas functional equation (9.48) can be written as follows

$$g(yx) + g(\sigma(y)x) = 2g(x), \;\; x, y \in S. \tag{9.59}$$

Replacing x by $\sigma(x)$ in (9.59) we get

$$g(y\sigma(x)) + g(\sigma(y)\sigma(x)) = 2g(\sigma(x)).$$

Using that $g(\sigma(x)) = -g(x)$ for all $x \in S$ we obtain

$$g(x\sigma(y)) + g(xy) = 2g(x), \;\; x, y \in S,$$

which means that g satisfies the Jensen functional equation (9.1). This completes the proof of Lemma 9.1. Now, we are ready to prove Theorem 9.3. Setting $x = e$ in (9.46) we get

$$|f^e(y) - f(e)| \leq \frac{\delta}{2} \tag{9.60}$$

for all $y \in S$.

The inequalities (9.46), (9.60) and the triangle inequality yield

$$
\begin{aligned}
|f(xy) + f(yx) - 2f(x) - 2f(y) + 2f(e)| &\le |f(xy) + f(x\sigma(y)) - 2f(x)| \\
&+ |f(yx) + f(y\sigma(x) - 2f(y)| + |2f(e) - f(x\sigma(y)) - f(y\sigma(x))| \le 3\delta.
\end{aligned} \tag{9.61}
$$

Hence, from (9.46), (9.60) and (9.61) we get

$$
\begin{aligned}
|f(yx) + f(\sigma(y)x) - 2f(x)| &\le |f(yx) + f(xy) - 2f(y) - 2f(x) + 2f(e)| \\
&+ |f(\sigma(y)x) + f(x\sigma(y)) - 2f(\sigma(y)) - 2f(x) + 2f(e)| \\
&+ |-f(xy) - f(x\sigma(y)) + 2f(x)| + |2f(y) + 2f(\sigma(y)) - 4f(e)| \le 9\delta.
\end{aligned} \tag{9.62}
$$

From (9.46) and (9.62) we obtain

$$
\begin{aligned}
2|f^o(yx) &+ f^o(y\sigma(x)) - 2f^o(y)| \\
&= |f(yx) - f(\sigma(x)\sigma(y)) + f(y\sigma(x)) - f(x\sigma(y)) - 2f(y) + 2f(\sigma(y))| \\
&\le |f(yx) + f(y\sigma(x)) - 2f(y)| + |f(x\sigma(y)) + f(\sigma(x)\sigma(y)) - 2f(\sigma(y))| \\
&\le 10\delta.
\end{aligned} \tag{9.63}
$$

Consequently we have

$$
|f^o(yx) + f^o(y\sigma(x)) - 2f^o(y)| \le 5\delta \tag{9.64}
$$

for all $x, y \in S$. Thus for fixed $y \in S$, the functions $x \longrightarrow f^o(yx) - f^o(x\sigma(y))$ and $x \longrightarrow f^o(xy) + f^o(x\sigma(y)) - 2f^o(x)$ are bounded on S.

Furthermore,

$$
\begin{aligned}
m\{f^o_{\sigma(y)\sigma(z)} + f^o_{\sigma(y)z} - 2f^o_{\sigma(y)}\} &= m\{(f^o_{\sigma(z)} + f^o_z - 2f^o)_{\sigma(y)}\} \\
&= m\{f^o_{\sigma(z)} + f^o_z - 2f^o\},
\end{aligned} \tag{9.65}
$$

where m is an invariant mean on S.

By using (9.62) we get that, for every fixed $y \in S$, the function

$$
x \longrightarrow f^o(yx) + f^o(\sigma(y)x) - 2f^o
$$

is bounded and

$$
\begin{aligned}
m\{_{zy}f^o +_{\sigma(z)y} f^o - 2_y f^o\} &= m\{_y(_z f^o +_{\sigma(z)} f^o - 2f^o)\} \\
&= m\{_z f^o +_{\sigma(z)} f^o - 2f^o\}.
\end{aligned} \tag{9.66}
$$

Now we define the new mapping

$$\phi(y) := m\{_y f^o - f^o_{\sigma(y)}\}, \ \ y \in S. \tag{9.67}$$

By using the definition of ϕ and m, the equalities (9.65) and (9.66), we obtain that

$$\begin{aligned} \phi(zy) \ + \phi(\sigma(z)y) &= m\{_{zy} f^o - f^o_{\sigma(y)\sigma(z)}\} + m\{_{\sigma(z)y} f^o - f^o_{\sigma(y)z}\} \\ &= m\{_{zy} f^o +_{\sigma(z)y} f^o - 2_y f^o\} - m\{f^o_{\sigma(y)\sigma(z)} + f^o_{\sigma(y)z} - 2f^o_{\sigma(y)}\} \\ &\quad + 2m\{_y f^o - f^o_{\sigma(y)}\} \\ &= m\{_z f^o +_{\sigma(z)} f^o - 2f^o\} - m\{f^o_{\sigma(z)} + f^o_z - 2f^o\} + 2m\{_y f^o - f^o_{\sigma(y)}\} \\ &= m\{_z f^o - f^o_{\sigma(z)}\} + m\{_{\sigma(z)} f^o - f^o_z\} + 2m\{_y f^o - f^o_{\sigma(y)}\} \\ &= 2\phi(y) + \phi(z) + \phi(\sigma(z)), \end{aligned} \tag{9.68}$$

which implies that ϕ is a solution of the Drygas functional equation (9.48). Furthermore, we have

$$\begin{aligned} |\frac{\phi}{2}(y) - f^o(y)| &= \frac{1}{2}|\phi(y) - 2f^o(y)| = \frac{1}{2}|m\{_y f^o - f^o_{\sigma(y)} - 2f^o(y)\}| \\ &\le \frac{1}{2}\sup_{x\in S}|f^o(yx) - f^o(x\sigma(y)) - 2f^o(y)| \\ &= \frac{1}{2}\sup_{x\in S}|f^o(yx) + f^o(y\sigma(x)) - 2f^o(y)| \\ &\le \frac{5}{2}\delta. \end{aligned} \tag{9.69}$$

By Lemma 9.1, it follows that the function $\frac{\phi}{2}$ is a solution of the Drygas functional equation (9.48) and $\frac{\phi}{2} - f^o$ is a bounded mapping, thus we have

$$\frac{\phi}{2} = \lim_{n\to+\infty} 2^{-n} f^o(x^{2^n}), \tag{9.70}$$

which implies that $\frac{\phi}{2}(\sigma(x)) = -\frac{\phi}{2}(x)$ for all $x \in S$, consequently $\frac{\phi}{2}$ is a solution of the Jensen functional equation (9.1). On the other hand, we have

$$\begin{aligned} |f(x) - \frac{\phi}{2} - f(e)| &= |f^e(x) + f^o(x) - \frac{\phi}{2} - f(e)| \\ &\le |f^e(x) - f(e)| + |f^o(x) - \frac{\phi}{2}| \\ &\le \frac{\delta}{2} + \frac{5\delta}{2} + 3\delta. \end{aligned} \tag{9.71}$$

We can use the same method as in Theorem 9.1 to prove the uniqueness of the derived solution. This completes the proof of Theorem 9.3. □

9.4 Application: Stability of the Symmetric Functional Equation (9.11)

In this section we use the result obtained in Sect. 9.3 to prove the stability of the symmetric functional equation (9.11).

Theorem 9.4 *Let G be an amenable group, and $f : G \longrightarrow \mathbb{C}$ a function. Assume that there exists a non-negative M such that*

$$|f(xy) + f(yx) - 2f(x) - 2f(y)| \leq M \tag{9.72}$$

for all $x, y \in G$. Then, there exists a unique solution $J : G \longrightarrow \mathbb{C}$ of the symmetric functional equation (9.11) such that

$$|f(x) - J(x) - f(e)| \leq 12M \text{ for all } x \in G. \tag{9.73}$$

Proof In the proof we use some ideas from Stetkær [26, Proposition 2.17].

Setting $x = y = e$ in (9.72) we get

$$|f(e)| \leq \frac{M}{2}. \tag{9.74}$$

If we replace y by x^{-1} in (9.72) we get

$$|f(e) - f(x) - f(x^{-1})| \leq \frac{M}{2}. \tag{9.75}$$

Subtracting (9.75) from (9.74) and using the triangle inequality we obtain

$$|f(x) + f(x^{-1})| \leq M. \tag{9.76}$$

Replacing x by xy and y by x^{-1} in (9.72) we derive

$$|f(xyx^{-1}) + f(y) - 2f(xy) - 2f(x^{-1})| \leq M. \tag{9.77}$$

Using (9.76), (9.77) and the triangle inequality we deduce that

$$|f(xyx^{-1}) + f(y) - 2f(xy) + 2f(x)| \leq 3M. \tag{9.78}$$

By replacing y by y^{-1} in (9.78) we get that

$$|f(xy^{-1}x^{-1}) + f(y^{-1}) - 2f(xy^{-1}) + 2f(x)| \leq 3M. \tag{9.79}$$

Adding (9.78) to (9.79) and using the triangle inequality we have that

$$|[f(xyx^{-1})+f((xyx^{-1})^{-1})]+[f(y)+f(y^{-1})]-2f(xy) \\ -2f(xy^{-1})+4f(x)| \leq 6M. \tag{9.80}$$

Using (9.76), (9.80) and the triangle inequality we obtain

$$|f(xy)+f(xy^{-1})-2f(x)| \leq 4M. \tag{9.81}$$

By applying Theorem 9.3 there exists $J\colon G \longrightarrow \mathbb{C}$, unique solution of the Jensen functional equation (9.3), that is

$$J(xy)+J(xy^{-1})=2J(x), \tag{9.82}$$

such that $J(x^{-1})=-J(x)$ and

$$|f(x)-J(x)-f(e)| \leq 12M \tag{9.83}$$

for all $x \in G$. Interchanging x and y in (9.82) we obtain

$$J(yx)+J(yx^{-1})=2J(y). \tag{9.84}$$

Adding (9.82) to (9.84) we get

$$J(xy)+J(yx)+J(xy^{-1})+J(yx^{-1})=2J(x)+2J(y). \tag{9.85}$$

Since $J(x^{-1})=-J(x)$ for all $x \in G$, then we deduce that

$$J(xy)+J(yx)=2J(x)+2J(y) \tag{9.86}$$

for all $x, y \in G$, which means that J satisfies the symmetric functional equation (9.11).

For the uniqueness of the solution J we use that if J is a solution of (9.86) then $J(x^{2^n})=2^nJ(x)$ for every integer n and for all $x \in G$, and by similar computations to those used above we deduce the rest of the proof. □

9.5 μ-Jensen Functional Equation

The trigonometric functional equations having a multiplicative function μ in front of terms like $f(x\sigma(y))$ or $f(\sigma(y)x)$ have been studied in many papers. The μ-d'Alembert's functional equation

$$f(xy)+\mu(y)f(xy^{-1})=2f(x)f(y),\ \ x, y \in S \tag{9.87}$$

which is an extension of d'Alembert's functional equation

$$f(xy) + f(xy^{-1}) = 2f(x)f(y),\ \ x, y \in S$$

has been treated systematically by Stetkær [27] on groups. The non-zero solutions of (9.87) on groups with involution are the normalized traces of certain representation of S on $\mathbb{C}^2$. On abelian groups the solutions of (9.87) are

$$f(x) = \frac{\gamma(x) + \mu(x)\gamma(x^{-1})}{2},\ \text{ where } \gamma : S \longrightarrow \mathbb{C}$$

is a multiplicative function (see [27]).

On abelian groups the solutions of μ-Wilson's functional equation

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x)g(y),\ \ x, y \in S$$

are studied in [9] and [29]. We refer also the interested reader to [8] and [10].

In the present section we prove that the μ-Jensen functional equations (9.8), (9.9) have a non-zero solution only if $\mu = 1$. We note that in this case σ is an arbitrary surjective homomorphism which is not necessary involutive.

Theorem 9.5 *Let S be a semigroup, $\sigma : S \longrightarrow S$ be a homomorphism, and μ be a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. If the functional equation*

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x),\ \ x, y \in S \tag{9.88}$$

has a non-zero solution then $\mu = 1$. That is, the μ-Jensen functional equation (9.88) possesses the same solutions to those of the Jensen functional equation (1.2).

Proof Making the substitutions (xy, z), $(x\sigma(y), z)$ in (9.88) we get respectively

$$f(xyz) + \mu(z)f(xy\sigma(z)) = 2f(xy), \tag{9.89}$$

$$f(x\sigma(y)z) + \mu(z)f(x\sigma(y)\sigma(z)) = 2f(x\sigma(y)). \tag{9.90}$$

Multiplying (9.90) by $\mu(y)$ we obtain

$$\mu(y)f(x\sigma(y)z) + \mu(yz)f(x\sigma(y)\sigma(z)) = 2\mu(y)f(x\sigma(y)). \tag{9.91}$$

Adding (9.89) and (9.91) and applying (9.88) we obtain

$$f(xyz) + \mu(z)f(xy\sigma(z)) + \mu(y)f(x\sigma(y)z) + \mu(yz)f(x\sigma(y)\sigma(z)) = 4f(x). \tag{9.92}$$

By using (9.88), Eq. (9.92) can be written as follows

$$2f(x) + \mu(z)[f(xy\sigma(z)) + \mu(y\sigma(z))f(x\sigma(y)z) = 4f(x). \tag{9.93}$$

Multiplying (9.93) by $\mu(\sigma(z))$ and using the fact that $\mu(z\sigma(z)) = 1$ we get after some simplification that

$$f(xy\sigma(z)) + \mu(y\sigma(z))f(x\sigma(y)z) = 2\mu(\sigma(z))f(x). \tag{9.94}$$

By replacing y in (9.88) by $y\sigma(z)$ we get

$$f(xy\sigma(z)) + \mu(y\sigma(z))f(x\sigma(y)\sigma^2(z)) = 2f(x). \tag{9.95}$$

Subtracting (9.95) from (9.94) we deduce that

$$\mu(y\sigma(z))[f(x\sigma(y)z) - f(x\sigma(y)\sigma^2(z))] = 2[\mu(\sigma(z)) - 1]f(x). \tag{9.96}$$

Multiplying the last identity by $\mu(\sigma(y)z)$ and using the fact that $\mu(z\sigma(z)) = 1$ we obtain that

$$f(x\sigma(y)z) - f(x\sigma(y)\sigma^2(z)) = 2\mu(\sigma(y))[1 - \mu(z)]f(x). \tag{9.97}$$

On the other hand, if we make the substitutions $(x\sigma(y), z)$ and $(x\sigma(y), \sigma(z))$ in (9.88) we deduce respectively

$$f(x\sigma(y)z) + \mu(z)f(x\sigma(y)\sigma(z)) = 2f(x\sigma(y)). \tag{9.98}$$

$$f(x\sigma(y)\sigma(z)) + \mu(\sigma(z))f(x\sigma(y)\sigma^2(z)) = 2f(x\sigma(y)). \tag{9.99}$$

Multiplying (9.99) by $\mu(z)$ and using the fact that $\mu(z\sigma(z)) = 1$ we derive that

$$\mu(z)f(x\sigma(y)\sigma(z)) + f(x\sigma(y)\sigma^2(z)) = 2\mu(z)f(x\sigma(y)). \tag{9.100}$$

Subtracting (9.100) from (9.98) we obtain

$$f(x\sigma(y)z) - f(x\sigma(y)\sigma^2(z)) = 2[1 - \mu(z)]f(x\sigma(y)). \tag{9.101}$$

By comparing (9.101) and (9.97) we deduce that

$$2\mu(\sigma(y))[1 - \mu(z)]f(x) = 2[1 - \mu(z)]f(x\sigma(y)), \tag{9.102}$$

from which we get

$$[1 - \mu(z)][\mu(\sigma(y))f(x) - f(x\sigma(y))] = 0. \tag{9.103}$$

If we suppose that $\mu \neq 1$, then from (9.103) we deduce that

$$f(x\sigma(y)) = \mu(\sigma(y))f(x) \tag{9.104}$$

for all $x, y \in S$. If we combine Eqs. (9.104) and (9.88) we get

$$f(xy) + \mu(y)\mu(\sigma(y))f(x) = 2f(x). \tag{9.105}$$

Since $\mu(y\sigma(y)) = 1$ we deduce that $f(xy) = f(x)$ for all $y \in S$. Therefore (9.88) becomes

$$(\mu(y) - 1)f(x) = 0$$

which means that either $f = 0$ or $\mu = 1$. Since $\mu \neq 1$, then we get $f = 0$, which contradicts the assumption that $f \neq 0$. □

Theorem 9.6 *Let S be a semigroup, let $\sigma : S \longrightarrow S$ be a homomorphism, and μ be a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. If the variant of the μ-Jensen functional equation*

$$f(xy) + \mu(y)f(\sigma(y)x) = 2f(x), \ \ x, y \in S \tag{9.106}$$

has a non-zero solution, then $\mu = 1$.

Proof The computations used in [10] for $g = 1$ show that for all fixed a in S, the mapping $x \longrightarrow f(ax) - f(a)$ is additive.

On the other hand, by replacing y by yz in (9.106) we get

$$f(xyz) + \mu(yz)f(\sigma(yz)x) = 2f(x). \tag{9.107}$$

If we replace x by $\sigma(y)$ and y by $\sigma(z)x$ in (9.106) and multiply the result obtained by $\mu(yz)$ we deduce that

$$\mu(yz)f(\sigma(yz)x) + \mu(xy)f(z\sigma(xy)) = 2\mu(yz)f(\sigma(y)). \tag{9.108}$$

By replacing x by z and y by $\sigma(xy)$ in (9.106) and multiplying the result obtained by $\mu(xy)$ we get

$$\mu(xy)f(z\sigma(xy)) + f(xyz) = 2\mu(xy)f(z). \tag{9.109}$$

By subtracting the sum of (9.107) and (9.109) from (9.108) we obtain

$$f(xyz) = f(x) + \mu(xy)f(z) - \mu(yz)f(\sigma(y)). \tag{9.110}$$

Since for each fixed a in S the function $x \longrightarrow f(a^2x) - f(a^2)$ is additive then the new function

$$\begin{aligned} x \longrightarrow & \mu(a^2)f(x) - \mu(a)\mu(x)f(\sigma(a)) + 2f(a) - 2f(a^2) \\ & = \mu(a)[\mu(a)f(x) - \mu(x)f(\sigma(a))] + 2f(a) - 2f(a^2) \end{aligned}$$

is additive. Since $\mu \neq 0$, then we deduce that f is central. That is $f(xy) = f(yx)$ for all $x, y \in S$. For the rest of the proof we use Theorem 9.5. □

Theorem 9.7 *Let S be a semigroup, $\sigma : S \longrightarrow S$ be an anti-homomorphism which is surjective and $\mu : S \longrightarrow \mathbb{C}$ be a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. If the μ-Jensen functional equation*

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x), \ \ x, y \in S \tag{9.111}$$

has a non-zero solution, then $\mu = 1$.

Proof Making the substitutions (xy, z), $(x\sigma(y), z)$ in (9.111) and multiplying the second result by $\mu(y)$ we get respectively

$$f(xyz) + \mu(z)f(xy\sigma(z)) = 2f(xy), \tag{9.112}$$

$$\mu(y)f(x\sigma(y)z) + \mu(yz)f(x\sigma(y)\sigma(z)) = 2\mu(y)f(x\sigma(y)). \tag{9.113}$$

Adding (9.112) to (9.113) and using (9.111) we obtain

$$f(xyz) + \mu(z)f(xy\sigma(z)) + \mu(y)f(x\sigma(y)z) + \mu(yz)f(x\sigma(y)\sigma(z)) = 4f(x). \tag{9.114}$$

If we replace y in (9.111) by yz we get

$$f(xyz) + \mu(yz)f(x\sigma(z)\sigma(y)) = 2f(x). \tag{9.115}$$

Subtracting (9.115) from (9.114) we obtain

$$\mu(yz)[f(x\sigma(y)\sigma(z)) - f(x\sigma(z)\sigma(y))] + \mu(z)f(xy\sigma(z)) + \mu(y)f(x\sigma(y)z) = 2f(x). \tag{9.116}$$

Taking $y = z$ in the last identity we find

$$\mu(y)[f(xy\sigma(y)) + f(x\sigma(y)y)] = 2f(x). \tag{9.117}$$

On the other hand, if we subtract (9.112) from (9.115) and multiply the result by $\mu(\sigma(z))$ and use the fact that $\mu(z\sigma(z)) = 1$ we get

$$\mu(y)f(x\sigma(z)\sigma(y)) - f(xy\sigma(z)) = 2\mu(\sigma(z))f(x) - 2\mu(\sigma(z))f(xy). \tag{9.118}$$

Replacing x in (9.111) by $x\sigma(z)$ implies

$$f(x\sigma(z)y) + \mu(y)f(x\sigma(z)\sigma(y)) = 2f(x\sigma(z)). \tag{9.119}$$

The subtraction of (9.118) from (9.119) yields

$$f(x\sigma(z)y) + f(xy\sigma(z)) = 2f(x\sigma(z)) - 2\mu(\sigma(z))f(x) + 2\mu(\sigma(z))f(xy). \tag{9.120}$$

Since σ is surjective, then by taking $t = \sigma(z)$ in (9.120) we obtain

$$f(xty) + f(xyt) = 2f(xt) + 2\mu(t)f(xy) - 2\mu(t)f(x) \tag{9.121}$$

for all $x, t, y \in S$. Replacing t in (9.121) by y, and y by $\sigma(y)$ and multiplying the resulting formulas obtained by $\mu(y)$ and using the fact that $\mu(y\sigma(y)) = 1$ we get

$$\begin{aligned}\mu(y)[f(xy\sigma(y)) &+ f(x\sigma(y)y)] \\ &= 2\mu(y)f(xy) + 2\mu^2(y)f(x\sigma(y)) - 2\mu^2(y)f(x).\end{aligned} \tag{9.122}$$

If we subtract (9.122) from (9.117) we deduce

$$2\mu(y)[f(xy) + \mu(y)f(x\sigma(y))] - 2\mu^2(y)f(x) = 2f(x). \tag{9.123}$$

Using (9.111) we get

$$[\mu(y) - 1]^2 f(x) = 0 \tag{9.124}$$

for all x and y in S. This means that if f is a non-zero solution of (9.121) then $\mu = 1$. □

9.6 Solutions of μ-Quadratic Functional Equation

In this section we consider the μ-quadratic functional equation (1.10), and we prove a similar result as in the precedent section for the μ-quadratic functional equation (9.10).

Theorem 9.8 *Let S be a semigroup, $\sigma : S \longrightarrow S$ be a homomorphism, and μ be a multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. If the μ-quadratic functional equation*

$$f(xy) + \mu(y)f(x\sigma(y)) = 2f(x) + 2f(y),\ \ x, y \in S \tag{9.125}$$

has a non-zero solution, then $\mu = 1$. That is, the μ-quadratic functional equation (9.125) possesses the same solutions to those of the quadratic functional equation (1.4)

Proof Making the substitutions (xy, z), $(x\sigma(y), z)$ in (9.125) we get respectively

$$f(xyz) + \mu(z)f(xy\sigma(z)) = 2f(xy) + 2f(z). \tag{9.126}$$

$$f(x\sigma(y)z) + \mu(z)f(x\sigma(y)\sigma(z)) = 2f(x\sigma(y)) + 2f(z). \tag{9.127}$$

Multiplying (9.127) by $\mu(y)$ we get

$$\mu(y)f(x\sigma(y)z) + \mu(yz)f(x\sigma(y)\sigma(z)) = 2\mu(y)f(x\sigma(y)) + 2\mu(y)f(z). \tag{9.128}$$

Adding (9.126) to (9.128) we obtain

$$\begin{aligned}[f(xyz) + \mu(yz)f(x\sigma(y)\sigma(z))] + [\mu(z)f(xy\sigma(z)) + \mu(y)f(x\sigma(y)z)]\\ = 2[f(xy) + \mu(y)f(x\sigma(y))] + 2[1 + \mu(y)]f(z).\end{aligned} \tag{9.129}$$

Replacing y by yz in (9.125) we get

$$f(xyz) + \mu(yz)f(x\sigma(y)\sigma(z)) = 2f(x) + 2f(yz). \tag{9.130}$$

Multiplying (9.125) by 2 we derive

$$2[f(xy) + \mu(y)f(x\sigma(y))] = 4f(x) + 4f(y). \tag{9.131}$$

If we subtract (9.130) from the sum of (9.129) and (9.131) we obtain

$$\begin{aligned}\mu(z)f(xy\sigma(z)) + \mu(y)f(x\sigma(y)z) + 2f(yz)\\ = 2f(x) + 4f(y) + 2[1 + \mu(y)]f(z).\end{aligned} \tag{9.132}$$

On the other hand, if we replace y by $y\sigma(z)$ in (9.125) we get

$$f(xy\sigma(z)) + \mu(y\sigma(z))f(x\sigma(y)\sigma^2(z)) = 2f(x) + 2f(y\sigma(z)). \tag{9.133}$$

Multiplying the last equality by $\mu(z)$ and using the fact that $\mu(z\sigma(z)) = 1$ we get

$$\mu(z)f(xy\sigma(z)) + \mu(y)f(x\sigma(y)\sigma^2(z)) = 2\mu(z)f(x) + 2\mu(z)f(y\sigma(z)). \tag{9.134}$$

Subtracting (9.134) from (9.132) we deduce that

$$\begin{aligned}\mu(y)[f(x\sigma(y)z) - f(x\sigma(y)\sigma^2(z))] + 2[f(yz) + \mu(z)f(y\sigma(z))] \\ = 2[1-\mu(z)]f(x) + 4f(y) + 2(1+\mu(y))f(z).\end{aligned} \tag{9.135}$$

If we make the substitution (y, z) in (9.125) and multiply the result obtained by 2 we derive

$$2[f(yz) + \mu(z)f(y\sigma(z))] = 4[f(y) + f(z)]. \tag{9.136}$$

The subtraction of (9.136) from (9.135) implies after some simplification

$$\mu(y)[f(x\sigma(y)z) - f(x\sigma(y)\sigma^2(z))] = 2[1-\mu(z)]f(x) + 2(\mu(y)-1)f(z). \tag{9.137}$$

On the other hand, if we make the substitutions $(x\sigma(y), z)$ and $(x\sigma(y), \sigma(z))$ in (9.125) we get respectively

$$f(x\sigma(y)z) + \mu(z)f(x\sigma(y)\sigma(z)) = 2f(x\sigma(y)) + 2f(z). \tag{9.138}$$

$$f(x\sigma(y)\sigma(z)) + \mu(\sigma(z))f(x\sigma(y)\sigma^2(z)) = 2f(x\sigma(y)) + 2f(\sigma(z)). \tag{9.139}$$

Multiplying (9.139) by $\mu(z)$ and using the fact that $\mu(z\sigma(z)) = 1$ we get

$$\mu(z)f(x\sigma(y)\sigma(z)) + f(x\sigma(y)\sigma^2(z)) = 2\mu(z)f(x\sigma(y)) + 2\mu(z)f(\sigma(z)). \tag{9.140}$$

Subtracting (9.140) from (9.138) we obtain

$$\begin{aligned}f(x\sigma(y)z) - f(x\sigma(y)\sigma^2(z)) \\ = 2f(x\sigma(y))[1-\mu(z)] + 2f(z) - 2\mu(z)f(\sigma(z)).\end{aligned} \tag{9.141}$$

Multiplying the last equation by $\mu(y)$ we obtain

$$\begin{aligned}\mu(y)[f(x\sigma(y)z) - f(x\sigma(y)\sigma^2(z))] = 2\mu(y)[1-\mu(z)]f(x\sigma(y)) \\ + 2\mu(y)f(z) - 2\mu(yz)f(\sigma(z)).\end{aligned} \tag{9.142}$$

Now, if we subtract (9.142) from (9.137) we deduce that

$$2[1-\mu(z)]f(x)-2f(z)=2\mu(y)[1-\mu(z)]f(x\sigma(y))-2\mu(yz)f(\sigma(z)), \tag{9.143}$$

from which we get

$$[1-\mu(z)][f(x)-\mu(y)f(x\sigma(y))]=f(z)-\mu(yz)f(\sigma(z)). \tag{9.144}$$

Taking $y=z$ in (9.144) we obtain

$$[1-\mu(y)][f(x)-\mu(y)f(x\sigma(y))]=f(y)-\mu(y^2)f(\sigma(y)) \tag{9.145}$$

for all $x,y\in S$.

Setting $\beta(y)=1-\mu(y)$ and multiplying (9.125) by $\beta(y)$ and adding the result obtained to (9.145) we derive that

$$\beta(y)[f(xy)-f(x)-2f(y)]=f(y)-\mu(y^2)f(\sigma(y)). \tag{9.146}$$

The last equation can be written as follows

$$\beta(y)f(xy)=\beta(y)f(x)+[2\beta(y)+1]f(y)-\mu(y^2)f(\sigma(y)). \tag{9.147}$$

Replacing y in (9.147) by $\sigma(y)$, and multiplying the result obtained by $\mu(y^2)$ and using the fact that $\mu(z\sigma(z))=1$ we find

$$\begin{aligned}\mu(y^2)\beta(\sigma(y))f(x\sigma(y))) =& \mu(y^2)\beta(\sigma(y))f(x)\\ &+\mu(y^2)[2\beta(\sigma(y))+1]f(\sigma(y))-f(\sigma^2(y)).\end{aligned} \tag{9.148}$$

Since $\mu(y\sigma(y))=1$ we get that

$$\mu(y)\beta(\sigma(y))=\mu(y)[1-\mu(\sigma(y))]=\mu(y)-1=-\beta(y),$$

and thus Eq. (9.148) can be written in the form

$$\begin{aligned}\mu(y)\beta(y)f(x\sigma(y)) =& \mu(y)\beta(y)f(x)\\ &+[2\mu(y)\beta(y)-\mu(y^2)]f(\sigma(y))+f(\sigma^2(y)).\end{aligned} \tag{9.149}$$

Adding (9.149) and (9.147) and using (9.125) we get

$$\begin{aligned}\beta(y)[2f(x)+2f(y)] =& [\beta(y)+\mu(y)\beta(y)]f(x)+[2\beta(y)+1]f(y)\\ &+[2\mu(y)\beta(y)-2\mu(y^2)]f(\sigma(y))+f(\sigma^2(y)).\end{aligned} \tag{9.150}$$

Thus

$$\beta(y)f(x) = f(y) + 2\mu(y)[\beta(y) - \mu(y)]f(\sigma(y)) + f(\sigma^2(y)) \tag{9.151}$$

for all x, y in S.

If $\mu \neq 1$ then there exists $y_0 \in S$ such that $\beta(y_0) \neq 0$ and from (9.151) we deduce that $f(x) = c$, for all $x \in S$, where

$$c = \frac{1}{\beta(y_0)}[f(y_0) + 2\mu(y_0)[\beta(y_0) - \mu(y_0)]f(\sigma(y_0)) + f(\sigma^2(y_0))],$$

which means that f is a constant. From (9.125) we deduce that $f = 0$, which contradicts the assumption that $f \neq 0$. This completes the proof of Theorem 9.8.

□

9.7 Stability of the μ-Jensen Functional Equation

In this section we study the stability of μ-Jensen functional equation (9.8), where σ is a surjective homomorphism, and μ is a bounded multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$.

Theorem 9.9 *Let S be a semigroup, $\sigma : S \longrightarrow S$ be a homomorphism, and μ be a bounded multiplicative function such that $\mu(x\sigma(x)) = 1$ for all $x \in S$. If there exists a non-negative scalar δ such that*

$$|f(xy) + \mu(y)f(x\sigma(y)) - 2f(x)| \leq \delta \tag{9.152}$$

for all $x, y \in S$, then either f is unbounded or $\mu = 1$.

Furthermore, the μ-Jensen functional equation (9.8) is stable if and only if the Jensen functional equation (1.1) is stable.

Proof Making the substitutions (xy, z), $(x\sigma(y), z)$ in (9.152) we get respectively

$$|f(xyz) + \mu(z)f(xy\sigma(z)) - 2f(xy)| \leq \delta, \tag{9.153}$$

$$|f(x\sigma(y)z) + \mu(z)f(x\sigma(y)\sigma(z)) - 2f(x\sigma(y))| \leq \delta. \tag{9.154}$$

The multiplicative mapping μ is bounded, thus there exists a nonnegative real M such that $|\mu(x)| \leq M$ for all $x \in S$. Multiplying (9.154) by $\mu(y)$ we get

$$|\mu(y)f(x\sigma(y)z) + \mu(yz)f(x\sigma(y)\sigma(z)) - 2\mu(y)f(x\sigma(y))| \leq M\delta. \tag{9.155}$$

Adding (9.153) and (9.155) and using the triangle inequality we obtain

$$|[f(xyz)+\mu(yz)f(x\sigma(y)\sigma(z))]+[\mu(z)f(xy\sigma(z))+\mu(y)f(x\sigma(y)z)] \\ -2[f(xy)+\mu(y)f(x\sigma(y))]|\leq(1+M)\delta. \tag{9.156}$$

Replacing y by yz in (9.152) we obtain

$$|f(xyz)+\mu(yz)f(x\sigma(y)\sigma(z))-2f(x)|\leq\delta. \tag{9.157}$$

Multiplying (9.152) by 2 we get

$$|2[f(xy)+\mu(y)f(x\sigma(y))]-4f(x)|\leq 2\delta. \tag{9.158}$$

If we subtract (9.157) from the sum of (9.156) and (9.158) and use the triangle inequality we obtain

$$|\mu(z)[f(xy\sigma(z))+\mu(y\sigma(z))f(x\sigma(y)z)-2f(x)|\leq(4+M)\delta. \tag{9.159}$$

Multiplying the last inequality by $\mu(\sigma(z))$ and using the fact that $\mu(z\sigma(z))=1$ we get after some simplification

$$|f(xy\sigma(z))+\mu(y\sigma(z))f(x\sigma(y)z)-2\mu(\sigma(z))f(x)|\leq(4M+M^2)\delta. \tag{9.160}$$

On the other hand, if we replace y in (9.152) by $y\sigma(z)$ we get

$$|f(xy\sigma(z))+\mu(y\sigma(z))f(x\sigma(y)\sigma^2(z))-2f(x)|\leq\delta. \tag{9.161}$$

Subtracting (9.161) from (9.160) we deduce that

$$|\mu(y\sigma(z))[f(x\sigma(y)z)-f(x\sigma(y)\sigma^2(z))-2[\mu(\sigma(z))-1]f(x)| \\ \leq(1+4M+M^2)\delta. \tag{9.162}$$

Multiplying the last identity by $\mu(\sigma(y)z)$ and using the fact that $\mu(z\sigma(z))=1$ we obtain

$$|f(x\sigma(y)z)-f(x\sigma(y)\sigma^2(z))-2\mu(\sigma(y))[1-\mu(z)]f(x)| \\ \leq(M+4M^2+M^3)\delta. \tag{9.163}$$

On the other hand, if we make the substitutions $(x\sigma(y),z)$ and $(x\sigma(y),\sigma(z))$ in (9.152) we get respectively

$$|f(x\sigma(y)z)+\mu(z)f(x\sigma(y)\sigma(z))-2f(x\sigma(y))|\leq\delta, \tag{9.164}$$

$$|f(x\sigma(y)\sigma(z)) + \mu(\sigma(z))f(x\sigma(y)\sigma^2(z)) - 2f(x\sigma(y))| \leq \delta. \tag{9.165}$$

Multiplying (9.165) by $\mu(z)$ and using $\mu(z\sigma(z)) = 1$ we derive that

$$|\mu(z)f(x\sigma(y)\sigma(z)) + f(x\sigma(y)\sigma^2(z)) - 2\mu(z)f(x\sigma(y))| \leq M\delta. \tag{9.166}$$

Subtracting (9.166) from (9.164) and using the triangle inequality we obtain

$$|f(x\sigma(y)z) - f(x\sigma(y)\sigma^2(z)) - 2f(x\sigma(y))[1 - \mu(z)]| \leq (1 + M)\delta. \tag{9.167}$$

If we subtract (9.167) from (9.163) we deduce that

$$\begin{aligned} |2[\mu(\sigma(y))[1 - \mu(z)]f(x) - 2f(x\sigma(y)))[1 - \mu(z)]| \\ \leq (1 + 2M + 4M^2 + M^3)\delta, \end{aligned} \tag{9.168}$$

from which we get

$$|[1 - \mu(z)][\mu(\sigma(y))f(x) - f(x\sigma(y))]| \leq (1 + 2M + 4M^2 + M^3)\frac{\delta}{2}. \tag{9.169}$$

If we suppose that $\mu \neq 1$, then there exists $z_0 \in S$ such that $\mu(z_0) \neq 1$. From (9.169) we deduce that

$$|f(x\sigma(y)) - \mu(\sigma(y))f(x)| \leq \phi\delta \tag{9.170}$$

for all $x, y \in S$, where

$$\phi = \frac{1}{2(1 - \mu(z_0))}(1 + 2M + 4M^2 + M^3).$$

If we multiply (9.170) by $\mu(y)$ and use the fact that $\mu(x\sigma(x)) = 1$, we obtain

$$|\mu(y)f(x\sigma(y)) - f(x)| \leq M\phi\delta. \tag{9.171}$$

Subtracting (9.152) from (9.171) and using the triangle inequality we get

$$|f(xy) - f(x)| \leq M(\phi + 1)\delta \tag{9.172}$$

for all $y \in S$. Replacing y by $\sigma(y)$ in (9.172) and multiplying the result by $\sigma(y)$ we obtain

$$|\mu(y)f(x\sigma(y)) - \mu(y)f(x)| \leq M^2(\phi + 1)\delta. \tag{9.173}$$

Subtracting (9.152) from the sum of (9.172) and (9.173) and using the triangle inequality we deduce

$$|[1-\mu(y)]f(x)| \le (M^2+M)(\phi+1)\delta+\delta. \tag{9.174}$$

Since $\mu \neq 1$ we deduce that f is a bounded function. This completes the proof of Theorem 9.9. □

9.8 Stability of the μ-Quadratic Functional Equation

In this section we investigate the stability of the μ-quadratic functional equation (1.10).

Theorem 9.10 *Let S be a semigroup, let $\sigma : S \longrightarrow S$ be a homomorphism, and μ be a bounded multiplicative function such that $\mu(x\sigma(x)) = 1$. If there exists a non-negative scalar δ such that*

$$|f(xy)+\mu(y)f(x\sigma(y))-2f(x)-2f(y)| \le \delta,\ \ x,y \in S, \tag{9.175}$$

then either f is unbounded or $\mu = 1$.

Furthermore, the μ-quadratic functional equation (1.10) *is stable if and only if the quadratic functional equation (9.7) is stable.*

Proof Making the substitutions (xy, z), $(x\sigma(y), z)$ in (9.175) we get respectively

$$|f(xyz)+\mu(z)f(xy\sigma(z))-2f(xy)-2f(z)| \le \delta. \tag{9.176}$$

$$|f(x\sigma(y)z)+\mu(z)f(x\sigma(y)\sigma(z))-2f(x\sigma(y))-2f(z)| \le \delta. \tag{9.177}$$

Multiplying (9.177) by $\mu(y)$ we get

$$\begin{aligned}|\mu(y)f(x\sigma(y)z)+\mu(yz)f(x\sigma(y)\sigma(z))-2\mu(y)f(x\sigma(y))-2\mu(y)f(z)|\\ \le M\delta.\end{aligned} \tag{9.178}$$

Adding (9.176) and (9.178) and using the triangle inequality we obtain

$$\begin{aligned}|[f(xyz)+\mu(yz)f(x\sigma(y)\sigma(z))]+[\mu(z)f(xy\sigma(z))+\mu(y)f(x\sigma(y)z)]\\ -2[f(xy)+\mu(y)f(x\sigma(y))]-2[1+\mu(y)]f(z)| \le (1+M)\delta.\end{aligned} \tag{9.179}$$

Replacing y by yz in (9.175) we get

$$|f(xyz) + \mu(yz)f(x\sigma(y)\sigma(z)) - 2f(x) - 2f(yz)| \leq \delta. \tag{9.180}$$

Multiplying (9.175) by 2 we get

$$|2[f(xy) + \mu(y)f(x\sigma(y))] - 4f(x) - 4f(y)| \leq 2\delta. \tag{9.181}$$

If we subtract (9.180) from the sum of (9.179) and (9.181) and use the triangle inequality we obtain

$$\begin{aligned}|\mu(z)f(xy\sigma(z)) + \mu(y)f(x\sigma(y)z) + 2f(yz) - 2f(x) - 4f(y)\\ -2[1 + \mu(y)]f(z)| \leq (4 + M)\delta.\end{aligned} \tag{9.182}$$

On the other hand, if we replace y in (9.175) by $y\sigma(z)$ we deduce that

$$|f(xy\sigma(z)) + \mu(y\sigma(z))f(x\sigma(y)\sigma^2(z)) - 2f(x) - 2f(y\sigma(z))| \leq \delta. \tag{9.183}$$

Multiplying the last inequality by $\mu(z)$ and using the fact that $\mu(z\sigma(z)) = 1$ we get

$$\begin{aligned}|\mu(z)f(xy\sigma(z)) + \mu(y)f(x\sigma(y)\sigma^2(z)) - 2\mu(z)f(x) - 2\mu(z)f(y\sigma(z))|\\ \leq M\delta.\end{aligned} \tag{9.184}$$

Subtracting (9.184) from (9.182) and using the triangle inequality we obtain that

$$\begin{aligned}|\mu(y)[f(x\sigma(y)z) - f(x\sigma(y)\sigma^2(z))] + 2[f(yz) + \mu(z)f(y\sigma(z))]\\ - 2[1 - \mu(z)]f(x) - 4f(y) - 2(1 + \mu(y))f(z)| \leq (4 + 2M)\delta.\end{aligned} \tag{9.185}$$

If we make the substitution (y, z) in (9.175) and multiply the result by 2 we obtain

$$|2[f(yz) + \mu(z)f(y\sigma(z))] - 4[f(y) + f(z)] \leq 2\delta. \tag{9.186}$$

The subtraction of (9.186) from (9.185) and the triangle inequality provide after some simplification that

$$\begin{aligned}|\mu(y)[f(x\sigma(y)z) - f(x\sigma(y)\sigma^2(z))] + 2[\mu(z) - 1]f(x)\\ +2(1 - \mu(y))f(z)| \leq (6 + 2M)\delta.\end{aligned} \tag{9.187}$$

On the other hand, if we make the substitutions $(x\sigma(y), z)$ and $(x\sigma(y), \sigma(z))$ in (9.175) we get respectively

$$|f(x\sigma(y)z) + \mu(z)f(x\sigma(y)\sigma(z)) - 2f(x\sigma(y)) - 2f(z)| \leq \delta. \tag{9.188}$$

$$|f(x\sigma(y)\sigma(z)) + \mu(\sigma(z))f(x\sigma(y)\sigma^2(z)) - 2f(x\sigma(y)) - 2f(\sigma(z))| \leq \delta. \tag{9.189}$$

Multiplying (9.189) by $\mu(z)$ and using the fact that $\mu(z\sigma(z)) = 1$ we get that

$$|\mu(z)f(x\sigma(y)\sigma(z)) + f(x\sigma(y)\sigma^2(z)) - 2\mu(z)f(x\sigma(y)) - 2\mu(z)f(\sigma(z))| \leq M\delta. \tag{9.190}$$

Subtracting (9.190) from (9.188) and using the triangle inequality we obtain

$$\begin{aligned}|f(x\sigma(y)z) - f(x\sigma(y)\sigma^2(z)) - 2f(x\sigma(y))[1-\mu(z)] - 2f(z) \\ +2\mu(z)f(\sigma(z))| \leq (1+M)\delta.\end{aligned} \tag{9.191}$$

Multiplying the last identity by $\mu(y)$ we obtain

$$\begin{aligned}|\mu(y)[f(x\sigma(y)z) - f(x\sigma(y)\sigma^2(z))] - 2\mu(y)[1-\mu(z)]f(x\sigma(y)) \\ -2\mu(y)f(z) + 2\mu(yz)f(\sigma(z))| \leq (M+M^2)\delta.\end{aligned} \tag{9.192}$$

If we subtract (9.192) from (9.187) and use the triangle inequality we obtain that

$$\begin{aligned}|2[\mu(z)-1]f(x) + 2(1-\mu(y))f(z) + 2\mu(y)[1-\mu(z)]f(x\sigma(y)) \\ +2\mu(y)f(z) - 2\mu(yz)f(\sigma(z))| \leq (6+3M+M^2)\delta,\end{aligned} \tag{9.193}$$

from which we get

$$\begin{aligned}|[\mu(z)-1][f(x) - \mu(y)f(x\sigma(y))] + f(z) - \mu(yz)f(\sigma(z))| \\ \leq (6+3M+M^2)\frac{\delta}{2}.\end{aligned} \tag{9.194}$$

Setting $y = z$ in (9.194) we obtain

$$|\beta(y)[f(x) - \mu(y)f(x\sigma(y))] + f(y) - \mu(y^2)f(\sigma(y))| \leq \alpha \tag{9.195}$$

where $\beta(y) = \mu(y) - 1$ for all $y \in S$, and

$$\alpha = (6+3M+M^2)\frac{\delta}{2}.$$

Adding (9.195) to (9.175) multiplied by $\beta(y)$ and using the triangle inequality we obtain

$$|\beta(y)[f(xy) - f(x) - 2f(y)] + f(y) - \mu(y^2)f(\sigma(y))| \leq \alpha + (M+1)\delta. \tag{9.196}$$

The last inequality can be written in the form

$$|\beta(y)f(xy) - \beta(y)f(x) - [2\beta(y) - 1]f(y) - \mu(y^2)f(\sigma(y))| \le \alpha + (M + 1)\delta. \tag{9.197}$$

Replacing y in (9.197) by $\sigma(y)$, and multiplying the result by $\mu(y^2)$ and using the fact that $\mu(z\sigma(z)) = 1$ we derive

$$|\mu(y^2)\,\beta(\sigma(y))f(x\sigma(y)) - \mu(y^2)\beta(\sigma(y))f(x) - \mu(y^2)[2\beta(\sigma(y)) - 1]f(\sigma(y)) - f(\sigma^2(y))| \le M^2\alpha + (M^2 + M^3)\delta. \tag{9.198}$$

Since $\mu(y\sigma(y)) = 1$ we get that

$$\mu(y)\beta(\sigma(y)) = \mu(y)[\mu(\sigma(y)) - 1] = 1 - \mu(y) = -\beta(y),$$

and thus inequality (9.197) can be expressed as follows

$$|\mu(y)\beta(y)f(x\sigma(y)) - \mu(y)\beta(y)f(x) - [2\mu(y)\beta(y) + \mu(y^2)]f(\sigma(y)) + f(\sigma^2(y))| \le M^2\alpha + (M^2 + M^3)\delta. \tag{9.199}$$

Subtracting (9.175) multiplied by $\beta(y)$ from the sum of (9.199) and (9.197) and using the triangle inequality we get

$$|\beta(y)[2f(x) + 2f(y)] - [\beta(y) + \mu(y)\beta(y)]f(x) - [2\beta(y) - 1]f(y) - [2\mu(y)\beta(y) + 2\mu(y^2)]f(\sigma(y)) + f(\sigma^2(y))| \le (1 + M^2)\alpha + (M^3 + M^2 + M + 2)\delta. \tag{9.200}$$

Simplifying the last inequality we obtain

$$|\beta^2(y)f(x) - f(y) - 2\mu(y)f(\sigma(y)) - f(\sigma^2(y))| \le (1 + M^2)\alpha + (M^3 + M^2 + M + 2)\delta. \tag{9.201}$$

Using the triangle inequality we deduce that

$$|\beta^2(y)f(x)| \le |f(y) + 2\mu(y)f(\sigma(y)) + f(\sigma^2(y))| + (M^2 + 1)\alpha + (M^3 + M^2 + M + 2)\delta \tag{9.202}$$

for all x, y in S.

If $\mu \neq 1$ then there exists $y_0 \in S$ such that $\beta(y_0) \neq 0$. From (9.202) we deduce that f is bounded. This completes the proof of Theorem 9.10. □

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