Chapter 21
On Exact and Approximate Orthogonalities Based on Norm Derivatives

Ali Zamani and Mahdi Dehghani

Abstract We survey mainly recent results on the orthogonality relations in normed linear spaces related to norm derivatives. We will focus on fundamental properties of norm derivatives orthogonality, differences and connections between these orthogonality types, and geometric results and problems closely related to them.

Keywords Norm derivative · Orthogonality · Approximate orthogonality · Orthogonality preserving mappings · Approximate orthogonality preserving property · Stability

Mathematics Subject Classification (2010) Primary 46B20; Secondary 46C50, 47B49

21.1 Introduction

One of the most well-known concept in study of the geometry of normed linear spaces is the notion of orthogonality. This concept and its connection with several geometric properties of normed linear spaces, like strict convexity (rotundity) and smoothness has been studied extensively. It is known that in an inner product space $(H, \langle\cdot,\cdot\rangle)$ there is one orthogonality relation derived from inner product. In fact, the vectors $x, y \in H$ are orthogonal (written as $x \perp y$) if and only if $\langle x, y\rangle = 0$.

The situation is completely different in general normed linear spaces. However, there is not a unique way to define the notion of orthogonality in general normed

A. Zamani (✉)
Department of Mathematics, Farhangian University, Tehran, Iran

M. Dehghani
Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan, Iran
e-mail: m.dehghani@kashanu.ac.ir

J. Brzdęk et al. (eds.), *Ulam Type Stability*,
https://doi.org/10.1007/978-3-030-28972-0_21

linear spaces. Since 1934 many mathematicians have introduced different generalized orthogonality in normed linear spaces for which, all of them are generalizations of orthogonality in an inner product space.

In 1934, Roberts [67] introduced the first orthogonality in real normed linear spaces. Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, whose dimension is at least 2. A vector $x \in X$ is said to be orthogonal in the sense of Roberts to a vector $y \in X$, denoted by $x \perp_R y$ if

$$\|x - ty\| = \|x + ty\| \qquad (t \in \mathbb{K}).$$

Later, in 1935 Birkhoff [8] introduced one of the most important orthogonality type. This notion of orthogonality was developed by James in [38, 39]. (Actually, this notion was much earlier considered by Carathéodory, see [2].) A vector $x \in X$ is said to be orthogonal to a vector $y \in X$ in the sense of Birkhoff–James, written as $x \perp_B y$, if

$$\|x + ty\| \geq \|x\| \qquad (t \in \mathbb{K}).$$

The geometrical interpretation is that the line passing through x in the direction of y supports (at the point x) the ball centred at 0 and with radius $\|x\|$. Note that Roberts orthogonality implies Birkhoff–James orthogonality. In [39] James elaborated how the notions like smoothness, rotundity, etc., of a normed linear space can be studied using Birkhoff–James orthogonality.

Also, James showed an example of a normed plane in which at least one of any two vectors, which are Roberts orthogonal to each other, must be the origin cf. [38]. Due to this situation, James introduced in 1945 isosceles orthogonality and Pythagorean orthogonality [38]. A vector $x \in X$ is said to be isosceles orthogonal to a vector $y \in X$ denoted by $x \perp_I y$ if

$$\|x + y\| = \|x - y\|.$$

Furthermore, a vector $x \in X$ is said to be Pythagorean orthogonal to a vector $y \in X$ denoted by $x \perp_P y$ if

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

For normed linear spaces, isosceles and Pythagorean orthogonality are not equivalent. They are also not equivalent to Roberts orthogonality. Of course, in an inner product space we have

$$\perp_B = \perp_I = \perp_P = \perp_R = \perp .$$

However, properties like symmetry, homogeneity, additivity, etc., of the orthogonality in inner product spaces do not always carry over to generalized orthogonalities. For example, it is known that Birkhoff–James orthogonality is homogeneous and not symmetric, while isosceles orthogonality and Pythagorean orthogonality are symmetric but not homogeneous, which shows (besides further properties) that these types of orthogonalities are different. We refer the reader to [2, 4, 38–40] and the references therein for basic properties of these type of orthogonalities. Also, a classification of different types of orthogonality in normed linear spaces, their main properties, and the relations between them can be found in e.g., survey paper [68] (see also [24, 25, 37]).

Recall that a normed linear space X is called smooth if each point of the unit sphere $\mathbb{S}_X$ has a unique supporting hyperplane to the closed unit ball $\mathbb{B}_X$, or equivalently, if to each nonzero $x \in X$ there exits a unique $x^* \in X^*$ satisfying $\|x^*\| = 1$ and $x^*(x) = \|x\|$ (see e.g., [3, 31]). Here, X^* denotes as usual the (topological) dual of X. In the case of real normed linear space $(X, \|\cdot\|)$, it has been proved that X is smooth if the $\|\cdot\|$ has the Gateaux derivative in X, i.e.,

$$G_{\pm}(x, y) := \lim_{t \to 0^{\pm}} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in X$; see e.g., [48].

One of the prominent reasons for importance of Birkhoff–James orthogonality is its application to characterize smooth normed linear spaces. Considering existence properties of Birkhoff–James orthogonality, we recall here the following result from [39].

Lemma 21.1 ([39, Corollary 2.2 and Theorem 4.1]) *Let X be a normed linear space and let $x, y \in X$ with $x \neq 0$. Then there exists $t \in \mathbb{K}$ such that $x \perp_B (tx + y)$. In particular, t is unique if and only if X is smooth.*

The concept of semi-inner product space was introduced by Lumer [53] and then the main properties of it were discovered in [34, 55, 64]. It has been proved in [53] that in any normed linear space $(X, \|\cdot\|)$ there exists a mapping $[\cdot|\cdot] : X \times X \to \mathbb{K}$ satisfying the properties:

(i) $[\alpha x + y|z] = \alpha[x|z] + [y|z]$ for all $x, y, z \in X$ and all $\alpha \in \mathbb{K}$;
(ii) $[x|\beta y] = \bar{\beta}[x|y]$ for all $x, y \in X$ and all $\beta \in \mathbb{K}$;
(iii) $[x|x] = \|x\|^2$ for all $x \in X$;
(iv) $|[x|y]| \leq \|x\|\|y\|$ for all $x, y \in X$.

Such a mapping is called a semi-inner product in X. It is known, however, that in a normed linear space there exists exactly one semi-inner product if and only if the space is smooth. More characterizations of smooth normed linear spaces by the notion of semi-inner products could be found in [31].

For vectors $x, y \in X$, the semi-inner product orthogonality is defined as follows:

$$x \perp_s y \quad \text{if and only if} \quad [y|x] = 0.$$

We remark that for any semi-inner product that generates the norm, we have $\perp_s \subset \perp_B$. Nevertheless, the reverse implication is generally not true; see e.g, [31].

For more information about semi-inner product spaces and its relation with Birkhoff–James orthogonality the reader is refereed to [31] and the references therein.

In 1986 norm derivatives were defined by Amir [4] in a real normed linear space $(X, \|\cdot\|)$ as follows:

$$\rho_{\pm}(x, y) := \lim_{t\to 0^{\pm}} \frac{\|x+ty\|^2 - \|x\|^2}{2t} = \|x\| \lim_{t\to 0^{\pm}} \frac{\|x+ty\| - \|x\|}{t}.$$

These functionals extend inner products and many geometrical properties of inner product spaces could be formulated in normed linear spaces by means of norm derivatives. The problem of finding necessary and sufficient conditions for a normed linear space to be an inner product one has been investigated by many mathematicians. There are many different ways to characterize inner product spaces among normed linear spaces. In 1935, Jordan and von Neumann [42] proved that the norm on a linear space X is induced by an inner product if and only if it satisfies the parallelogram law. Another way to obtain characterizations of inner product spaces is to force the orthogonality relation on a normed linear space to fulfill some properties of the natural orthogonality of inner product spaces. Day [26] and James [40] obtained some new characterizations of inner product spaces by means of isosceles and Birkhoff–James orthogonality. For instance, they proved that a normed linear space X, whose dimension is at least three, is an inner product space if and only if Birkhoff–James orthogonality is symmetric in X. Also, it has been proved in [38] that isosceles orthogonality is homogeneous in a normed linear space if and only if this space is an inner product space. In particular, Tapia [70, 71] characterized inner product spaces in terms of norm derivatives. More precisely, he proved that a normed linear space X is an inner product space if and only if $G_{+}(\cdot, \cdot)$ is linear in the first variable if and only if $G_{+}(\cdot, \cdot)$ is symmetric.

Norm derivatives play an important role in describing the geometric properties of normed linear spaces. The basic geometric properties such as strict convexity and smoothness of normed linear spaces have been characterized by many mathematicians using the notion of norm derivatives. As the most famous descriptions for smooth real normed linear spaces based on norm derivatives, we point out here the following result from [3].

Lemma 21.2 ([3, Remark 2.1.1]) *Let X be a real normed linear space. Then X is smooth if and only if $\rho_{-}(x, y) = \rho_{+}(x, y)$ for all $x, y \in X$.*

Orthogonality relations which are taken from norm derivatives provide a good framework for developing studies of the geometric structure of normed linear spaces. During the last years many papers concerning various aspects of orthogonalities related to norm derivatives have appeared. In this paper we want to give some overview on these results as well as to collect a number of items from the literature dealing with the subject. Our paper can also be taken as an update of existing surveys and monographs; see [16, 22, 68].

21.2 Exact and Approximate Norm Derivatives Orthogonalities

In this section we assume that the considered normed linear spaces are real and their dimensions are not less than 2.

21.2.1 ρ-Orthogonality

Let $(X, \|\cdot\|)$ be a normed linear space and let $x, y \in X$. The orthogonality relations associated to the functionals ρ_- and ρ_+ are defined by

$$x \perp_{\rho_-} y \quad \text{if and only if} \quad \rho_-(x, y) = 0;$$
$$x \perp_{\rho_+} y \quad \text{if and only if} \quad \rho_+(x, y) = 0.$$

In 1987, Miličić [56] introduced a new orthogonality relation as follows

$$x \perp_\rho y \quad \text{if and only if} \quad \rho(x, y) = 0,$$

where the functional $\rho(\cdot,\cdot) := \langle \cdot,\cdot\rangle_g : X \times X \to \mathbb{R}$ was defined by

$$\rho(x, y) = \langle y, x\rangle_g = \frac{\rho_-(x, y) + \rho_+(x, y)}{2}.$$

Among the just defined three orthogonality relations only $\perp_\rho$ is homogeneous (i.e., for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{R}$, if $x \perp_\rho y$, then $\alpha x \perp_\rho \beta y$) and none of them is symmetric. It has been proved in [3] that the relations $\perp_{\rho_\pm}$ and $\perp_\rho$ in a normed linear space X are symmetric if and only if X is an inner product space. First, we remind several properties of these functions, which are used to obtain different characterizations of inner product spaces and smooth normed linear spaces.

Theorem 21.1 *Let $(X, \|\cdot\|)$ be a normed linear space, and let $x, y \in X$. Then*

(i) $\rho_\pm(x, x) = \|x\|^2$ *and* $\rho_-(x, y) \le \rho_+(x, y)$.
(ii) *For all* $t \in \mathbb{R}$, $\rho_\pm(tx, y) = \rho_\pm(x, ty) = \begin{cases} t\rho_\pm(x, y) & t \ge 0 \\ t\rho_\mp(x, y) & t \le 0. \end{cases}$
(iii) $|\rho_\pm(x, y)| \le \|x\|\,\|y\|$.
(iv) *For all* $t \in \mathbb{R}$, $\rho_\pm(x, tx + y) = t\|x\|^2 + \rho_\pm(x, y)$.

In [3], Alsina et al. provided a complete description of these orthogonality relations. The relation of Birkhoff–James orthogonality, $\rho_\pm$-orthogonality and ρ-orthogonality has been obtained in [3] as follows:

Theorem 21.2 ([3, Propositions 2.2.2-3]) *Let X be a normed linear space. Then* $\perp_{\rho_\pm} \subset \perp_B$ *and* $\perp_\rho \subset \perp_B$.

In particular, the equalities $\perp_B = \perp_{\rho_-}$, $\perp_B = \perp_{\rho_+}$ *and* $\perp_B = \perp_\rho$ *in* X *are equivalent to the smoothness of* X.

Now, we recall that norm derivatives characterize Birkhoff–James orthogonality in the following sense.

Theorem 21.3 ([4, 39]) *Let* $(X, \|\cdot\|)$ *be a normed linear space,* $x, y \in X$ *and let* $\alpha \in \mathbb{R}$. *Then the following conditions are equivalent:*

(i) $x \perp_B (y - \alpha x)$.
(ii) $\rho_-(x, y) \le \alpha \|x\|^2 \le \rho_+(x, y)$.

In particular, $x \perp_B y$ *if and only if* $\rho_-(x, y) \le 0 \le \rho_+(x, y)$.

There is a deep connection of smooth normed linear spaces and the orthogonality relations related norm derivatives. Chmieliński and Wójcik in [21] clarified that the relations $\perp_{\rho_\pm}$ and $\perp_\rho$ are generally incomparable. More precisely, they proved that these orthogonality relations are comparable in a normed linear space X if and only if X is smooth.

Theorem 21.4 ([21, Theorem 1]) *Let* X *be a normed linear space. Then the following conditions are equivalent:*

$(i)\ \perp_{\rho_+} \subset \perp_{\rho_-}$. $\quad (ii)\ \perp_{\rho_-} \subset \perp_{\rho_+}$. $\quad (iii)\ \perp_{\rho_+} = \perp_{\rho_-}$.

$(iv)\ \perp_{\rho_+} \subset \perp_\rho$. $\quad (v)\ \perp_\rho \subset \perp_{\rho_+}$. $\quad (vi)\ \perp_{\rho_+} = \perp_\rho$.

$(vii)\ \perp_{\rho_-} \subset \perp_\rho$. $\quad (viii)\ \perp_\rho \subset \perp_{\rho_-}$. $\quad (ix)\ \perp_{\rho_-} = \perp_\rho$. $\quad (x)\ X\ is\ smooth.$

Finally, we remark that the connection between the relations $\perp_\rho$ and $\perp_s$ were given in [21].

Theorem 21.5 ([21, Theorem 2]) *Let* X *be a normed linear space and let* $[\cdot|\cdot]$ *be a given semi-inner product in* X. *Then the following conditions are equivalent:*

$$(i)\ \perp_\rho \subset \perp_s. \quad (ii)\ \perp_s \subset \perp_\rho. \quad (iii)\ \perp_\rho = \perp_s. \quad (iv)\ \rho(\cdot,\cdot) = [\cdot|\cdot].$$

21.2.2 ρ_*-Orthogonality

Another type of an orthogonality relation connected to norm derivatives that was introduced in [11] is ρ_*-orthogonality. In this section we will review elementary properties of ρ_*-orthogonality. Also, some characterizations of smooth normed linear spaces in terms of ρ_*-orthogonality which has been obtained in [60] are reviewed.

Definition 21.1 ([11]) Let X be a normed linear space. Then a vector $x \in X$ is called ρ_*-orthogonal to a vector $y \in X$, denoted by $x \perp_{\rho_*} y$ if

$$\rho_*(x, y) := \rho_-(x, y)\rho_+(x, y) = 0.$$

First, we represent some elementary properties of the functional ρ_*.

Proposition 21.1 ([60, Proposition 2.1]) *Let $(X, \|\cdot\|)$ be a normed linear space. Then*

(i) $\rho_*(tx, y) = \rho_*(x, ty) = t^2\rho_*(x, y)$ *for all* $x, y \in X$ *and all* $t \in \mathbb{R}$.
(ii) $|\rho_*(x, y)| \leq \|x\|^2\|y\|^2$ *for all* $x, y \in X$.
(iii) For all nonzero vectors $x, y \in X$, if $x \perp_{\rho_*} y$, then x and y are linearly independent.
(iv) $\rho_*(x, tx + y) = t^2\|x\|^4 + 2t\|x\|^2\rho(x, y) + \rho_*(x, y)$ *for all* $x, y \in X$ *and all* $t \in \mathbb{R}$.

It is clear that $\perp_{\rho_-} \cup \perp_{\rho_+} = \perp_{\rho_*} \subset \perp_B$ and so the equality $\perp_B = \perp_{\rho_*}$ implies the smoothness of the norm. Also, it is noticed in [60] that the relations $\perp_\rho$ and $\perp_{\rho_*}$ are incomparable. In fact, according to the following theorem, these orthogonality relations in a normed linear space X are comparable if and only if X is smooth.

Theorem 21.6 ([60, Theorem 3.1]) *Let X be a normed linear space. Then the following conditions are equivalent:*

$(i)\ \perp_B \subset \perp_{\rho_*}.\quad (ii)\ \perp_B = \perp_{\rho_*}.\quad (iii)\ \perp_\rho \subset \perp_{\rho_*}.$

$(iv)\ \perp_{\rho_*} \subset \perp_\rho.\quad (v)\ \perp_{\rho_*} = \perp_\rho.\quad (vi)\ \perp_{\rho_*} \subset \perp_{\rho_+}.$

$(vii)\ \perp_{\rho_*} \subset \perp_{\rho_-}.\quad (viii)\ \perp_{\rho_*} = \perp_{\rho_-}.\quad (ix)$ *X is smooth.*

Moreover, the connection between semi-inner product orthogonality and ρ_*-orthogonality has been established in the following theorem.

Theorem 21.7 ([60, Proposition 2.2]) *Let X be a normed linear space and let $[\cdot|\cdot]$ be a given semi-inner product in X. Then the following conditions are equivalent:*

(i) $\perp_{\rho_*} = \perp_s$.
(ii) $\perp_{\rho_*} \subset \perp_s$.
(iii) $\rho_*(x, y) = [y|x]^2$ *for all* $x, y \in X$.

Let us now suppose that $\perp$ is a binary relation on a real vector space X satisfying

(O1) Totality of $\perp$ for zero: $x \perp 0$ and $0 \perp x$ for all $x \in X$;
(O2) Independence: if $x, y \in X \setminus \{0\}$ and $x \perp y$, then x and y are linearly independent;
(O3) Homogeneity: if $x, y \in X$ and $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
(O4) The Thalesian property: let P be a two-dimensional subspace of X. If $x \in P$ and $\mu \geq 0$, then there exists $y \in P$ such that $x \perp y$ and $x + y \perp \mu x - y$.

The pair $(X, \perp)$ is called an orthogonality space in the sense of Rätz [66]. Inner product spaces and normed linear spaces with Birkhoff–James orthogonality are typical examples of orthogonality spaces. Also, it has been proved in [3] that ρ-orthogonality is an orthogonality space. Using Proposition 21.1, it easy to check that the conditions (O1)–(O3) are true for ρ_*-orthogonality and the following theorem ensures that ρ_*-orthogonality has the Talesian property. Therefore a normed linear space with ρ_*-orthogonality is an orthogonality space in the sense of Rätz.

Theorem 21.8 ([60, Theorem 4.2]) *For any two-dimensional subspace P of a normed linear space X and for every $x \in P$, $\mu \geq 0$, there exists a vector $y \in P$ such that*

$$x \perp_{\rho_*} y \qquad \text{and} \qquad x + y \perp_{\rho_*} \mu x - y.$$

Let X be a normed linear space and let $(G, +)$ be an Abelian group. Let us recall that a mapping $A : X \longrightarrow G$ is called additive if $A(x + y) = A(x) + A(y)$ for all $x, y \in X$, a mapping $B : X \times X \longrightarrow G$ is called biadditive if it is additive in both variables and a mapping $Q : X \longrightarrow G$ is called quadratic if $Q(x+y)+Q(x-y) = 2Q(x) + 2Q(y)$ for all $x, y \in X$. As an immediate consequence of Theorem 21.8 and [3, Theorem 2.8.1], we deduce the following assertion.

Corollary 21.1 *Let X be a normed linear space and let $(G, +)$ be an Abelian group. A mapping $f : X \longrightarrow G$ satisfies the condition*

$$x \perp_{\rho_*} y \Longrightarrow f(x + y) = f(x) + f(y) \qquad (x, y \in X)$$

if and only if there exist an additive mapping $A : X \longrightarrow G$ and a biadditive and symmetric mapping $B : X \times X \longrightarrow G$ such that

$$f(x) = A(x) + B(x, x) \qquad (x \in X)$$

and

$$x \perp_{\rho_*} y \Longrightarrow B(x, y) = 0 \qquad (x, y \in X).$$

Finally, as a consequence of Theorem 21.8 and [58, Theorem 3], we have the following result.

Corollary 21.2 *Let X be a normed linear space and let $(G, +)$ be an Abelian group. Suppose that Y is a real Banach space. If $f : X \longrightarrow G$ is a mapping fulfilling*

$$x \perp_{\rho_*} y \Longrightarrow \|f(x + y) - f(x) - f(y)\| \leq \varepsilon \qquad (x, y \in X)$$

for some $\varepsilon > 0$*, then there exist exactly an additive mapping* $A : X \longrightarrow Y$ *and exactly a quadratic mapping* $Q : X \longrightarrow Y$ *such that*

$$\|f(x) - f(0) - A(x) - Q(x)\| \le \frac{68}{3}\varepsilon \qquad (x \in X).$$

21.2.3 Some Generalized Norm Derivatives Orthogonality

In [88] an orthogonality relation as an extension of $\rho_\pm$ and ρ-orthogonality that is called ρ_λ-orthogonality has been introduced. We start this section by reviewing some main results which obtained about this orthogonality relation in [88].

Let X be a normed linear space and let $\lambda \in [0, 1]$. Then a vector $x \in X$ is said to be ρ_λ-orthogonal to a vector $y \in X$ denoted by $x \perp_{\rho_\lambda} y$ if

$$\rho_\lambda(x, y) := \lambda\rho_-(x, y) + (1 - \lambda)\rho_+(x, y) = 0.$$

It is evident that ρ_0 and ρ_1-orthogonality coincide with ρ_+ and ρ_--orthogonality, respectively. Also, $\rho_{\frac{1}{2}}$-orthogonality is equivalent to ρ-orthogonality. As an extension of Theorem 21.2, it has been proved that ρ_λ-orthogonality always implies Birkhoff–James orthogonality.

Proposition 21.2 ([88, Theorem 2.5]) *Let X be a normed linear space and let* $\lambda \in [0, 1]$. *Then* $\perp_{\rho_\lambda} \subset \perp_B$.

It is noticed in [88, Example 2.8] that for nonsmooth normed linear spaces, the orthogonalities $\perp_{\rho_\lambda}$ and $\perp_B$ may not coincide. However, analogously to Theorem 21.2, the equality $\perp_{\rho_\lambda} = \perp_B$ in a normed linear space X implies the smoothness of X.

Theorem 21.9 ([88, Theorem 2.7]) *Let X be a normed linear space and let* $\lambda \in [0, 1]$. *Then the following statements are equivalent:*

$$(i)\ \perp_B \subset \perp_{\rho_\lambda}. \quad (ii)\ \perp_B = \perp_{\rho_\lambda}. \quad (iii)\ X \text{ is smooth}.$$

Moreover, the relations $\perp_{\rho_\pm}$, $\perp_\rho$ and $\perp_{\rho_\lambda}$ are generally incomparable; cf. [88, Example 2.10]. The following theorems give some characterizations of smooth normed linear spaces in terms of ρ_λ-orthogonality.

Theorem 21.10 ([88, Theorem 2.12]) *Let X be a normed linear space and let* $\lambda \in (0, 1]$. *Then the following conditions are equivalent.*

$$(i)\ \perp_{\rho_\lambda} \subset \perp_{\rho_+}. \quad (ii)\ \perp_{\rho_+} \subset \perp_{\rho_\lambda}. \quad (iii)\ \perp_{\rho_\lambda} = \perp_{\rho_+}. \quad (iv)\ X \text{ is smooth}.$$

Theorem 21.11 ([88, Theorem 2.13]) *Let X be a normed linear space and let $\lambda \in [0, 1)$. Then the following conditions are equivalent:*

$$(i)\ \perp_{\rho_\lambda} \subset \perp_{\rho_-}.\quad (ii)\ \perp_{\rho_-} \subset \perp_{\rho_\lambda}.\quad (iii)\ \perp_{\rho_\lambda} = \perp_{\rho_-}.\quad (iv)\ X \text{ is smooth}.$$

Theorem 21.12 ([88, Theorem 2.11]) *Let X be a normed linear space and let $\lambda \in [0, 1]$ such that $\lambda \neq \frac{1}{2}$. Then the following conditions are equivalent:*

$$(i)\ \perp_{\rho} \subset \perp_{\rho_\lambda}.\quad (ii)\ \perp_{\rho_\lambda} \subset \perp_{\rho}.\quad (iii)\ \perp_{\rho_\lambda} = \perp_{\rho}.\quad (iv)\ X \text{ is smooth}.$$

More generally, a new orthogonality relation based on norm derivatives which is a generalization of the above orthogonalities has been introduced and studied in [28]. We will continue this section to review this orthogonality and its relation with other types of orthogonality relations which have already introduced.

Definition 21.2 ([28]) Let X be a normed linear space, and let $\lambda \in [0, 1]$, $\upsilon = \frac{1}{2k-1}$ with $k \in \mathbb{N}$. For $x, y \in X$, consider the functional $\rho_\lambda^\upsilon : X \times X \to \mathbb{R}$ which is defined by

$$\rho_\lambda^\upsilon(x, y) := \lambda \rho_-^\upsilon(x, y)\rho_+^{1-\upsilon}(x, y) + (1-\lambda)\rho_+^\upsilon(x, y)\rho_-^{1-\upsilon}(x, y).$$

A vector $x \in X$ is called ρ_λ^υ-orthogonal to a vector $y \in X$, denoted by $x \perp_{\rho_\lambda^\upsilon} y$, if $\rho_\lambda^\upsilon(x, y) = 0$.

It is obvious that for a real inner product space, ρ_λ^υ-orthogonality coincides with the standard orthogonality given by the inner product. Therefore ρ_λ^υ-orthogonality can be considered as a generalization of orthogonality of inner product spaces in real normed linear spaces. We have

$$\rho_0^\upsilon(x, y) = \rho_+^\upsilon(x, y)\rho_-^{1-\upsilon}(x, y) \text{ and } \rho_1^\upsilon(x, y) = \rho_-^\upsilon(x, y)\rho_+^{1-\upsilon}(x, y)$$

for all $\upsilon = \frac{1}{2k-1}$ $(k \in \mathbb{N})$. Hence it is easy to see that $\perp_{\rho_0^\upsilon} = \perp_{\rho_*}$ and $\perp_{\rho_1^\upsilon} = \perp_{\rho_*}$ for all $\upsilon = \frac{1}{2k+1}$ $(k \in \mathbb{N})$. On the other hand, we have $\rho_\lambda^1(x, y) = \rho_\lambda(x, y)$ and therefore ρ_λ^1-orthogonality coincides with ρ_λ-orthogonality for all $\lambda \in [0, 1]$. We point out here the elementary properties of the functional ρ_λ^υ.

Proposition 21.3 ([28, Theorem 2.1]) *Let $(X, \|\cdot\|)$ be a normed linear space and let $x, y \in X$. Then*

(i) $\rho_\lambda^\upsilon(x, x) = \|x\|^2$.

(ii) $\rho_\lambda^\upsilon(tx, y) = \rho_\lambda^\upsilon(x, ty) = \begin{cases} t\rho_\lambda^\upsilon(x, y) & t \geq 0 \\ t\rho_{1-\lambda}^\upsilon(x, y) & t \leq 0. \end{cases}$

(iii) $|\rho_\lambda^\upsilon(x, y)| \leq \|x\|\|y\|$.

(iv) Let t be a real number such that $\rho_*(x, tx + y) \neq 0$. If

$$K := K(x, y, t) = \frac{\rho_-(x, tx + y)}{\rho_+(x, tx + y)},$$

then

$$\rho_\lambda^v(x, tx + y) = t\|x\|^2(\lambda K^v + (1 - \lambda)K^{-v}) + \lambda K^v \rho_+(x, y) \\ +(1 - \lambda)K^{-v}\rho_-(x, y).$$

It is clear that $\perp_{\rho_\pm} \subset \perp_{\rho_\lambda^v}$ and so $\perp_{\rho_*} \subset \perp_{\rho_\lambda^v}$. However, some illustrative example have been prepared in [28] which show that the relations $\perp_\rho$, $\perp_{\rho_\lambda}$ and $\perp_{\rho_\lambda^v}$ are generally incomparable. This fact lead us to the following descriptions of smooth normed linear spaces.

Theorem 21.13 ([28, Theorem 2.14]) *Let X be a normed linear space and let $\lambda \in [0, 1] \setminus \{\frac{1}{2}\}$ and $v = \frac{1}{2k-1}$ $(k \in \mathbb{N})$. Then the following conditions are equivalent:*

(i) $\perp_\rho \subset \perp_{\rho_\lambda^v}$. (ii) $\perp_{\rho_\lambda^v} \subset \perp_\rho$. (iii) $\perp_{\rho_\lambda^v} = \perp_\rho$. (iv) *X is smooth.*

It is worth noting that the situation is different for the case $\lambda = \frac{1}{2}$ and in this case, we have $\perp_\rho \subset \perp_{\rho_{\frac{1}{2}}^v}$. Indeed, for each $x, y \in X$, if $x \perp_\rho y$, then $\rho_-(x, y) = -\rho_+(x, y)$. Hence

$$\rho_{\frac{1}{2}}^v(x, y) = \frac{1}{2}\Big[(-1)^v \rho_+^v(x, y)\rho_+^{1-v}(x, y) + (-1)^{1-v}\rho_+^v(x, y)\rho_+^{1-v}(x, y)\Big] \\ = \frac{1}{2}[-\rho_+(x, y) + \rho_+(x, y)] = 0.$$

Theorem 21.14 ([28, Theorem 2.16]) *Let X be a normed linear space and let $\lambda \in (0, 1)$ and $v = \frac{1}{2k+1}$ $(k \in \mathbb{N})$. Then the following conditions are equivalent:*

(i) $\perp_{\rho_\lambda^v} \subset \perp_{\rho_\lambda}$. (ii) $\perp_{\rho_\lambda} \subset \perp_{\rho_\lambda^v}$ $(\lambda \neq \frac{1}{2})$. (iii) $\perp_{\rho_\lambda} = \perp_{\rho_\lambda^v}$. (iv) *X is smooth.*

Theorem 21.15 ([28, Theorem 2.17]) *Let X be a normed linear space and let $\lambda \in [0, 1]$ and $v = \frac{1}{2k-1}$ $(k \in \mathbb{N})$. Then the following conditions are equivalent:*

(i) $\perp_{\rho_\lambda^v} \subset \perp_{\rho_-}$ *(except for $\perp_{\rho_1^1} = \perp_{\rho_-}$).*
(ii) $\perp_{\rho_\lambda^v} \subset \perp_{\rho_+}$ *(except for $\perp_{\rho_0^1} = \perp_{\rho_+}$).*
(iii) X is smooth.

The following result is an analogue of Theorems 21.2 and Proposition 21.2 which describes the relation between Birkhoff–James orthogonality and ρ_λ^v-orthogonality.

Proposition 21.4 ([28, Proposition 2.9]) *Let X be a normed linear space and let $\lambda \in [0, 1]$ and $v = \frac{1}{2k-1}$ $(k \in \mathbb{N})$. Then $\perp_{\rho_\lambda^v} \subset \perp_B$.*

Also, as stated in [28], for non-smooth normed linear spaces, Birkhoff–James orthogonality and ρ_λ^v-orthogonality may not coincide. Now, as an analogue of Theorems 21.2 and 21.9, we prove that the equality $\perp_{\rho_\lambda^v} = \perp_B$ in normed linear spaces yields the smoothness of the norm. In fact, all the results which mentioned in Theorem 21.2 and Proposition 21.2 are given from the next theorem for the particular modes of λ and v.

Theorem 21.16 *Let X be a normed linear space, $\lambda \in [0, 1]$ and let $v = \frac{1}{2k-1}$ ($k \in \mathbb{N}$). Then the following conditions are equivalent:*

(i) X is smooth.
(ii) $\perp_B \subset \perp_{\rho_\lambda^v}$.

Proof The implication (i)$\Rightarrow$(ii) is clear. Now, we prove the implication (ii)$\Rightarrow$(i). Suppose that $\lambda \in [0, 1]$ such that $\lambda \neq \frac{1}{2}$ and (ii) holds. It follows from (ii) and Theorem 21.2 that $\perp_\rho \subset \perp_B \subset \perp_{\rho_\lambda^v}$ and so Theorem 21.13 concludes that X is smooth.

Now, assume that $\lambda = \frac{1}{2}$. If $x, y \in X$ and $x \neq 0$, then we obtain from Lemma 21.1 that there is $t \in \mathbb{R}$ such that $x \perp_B (tx + y)$ and so (ii) implies that there is $t \in \mathbb{R}$ such that $\rho_\lambda^v(x, tx + y) = 0$. If $\rho_*(x, tx + y) \neq 0$, then it follows from Proposition 21.3 (iv) that

$$K^v \rho_+(x, tx + y) + K^{-v} \rho_-(x, tx + y) = 0.$$

So, we have $K^{2v-1} = -1$. Accordingly, $K = \frac{\rho_-(x,tx+y)}{\rho_+(x,tx+y)} = -1$ and so $t = \frac{-\rho(x,y)}{\|x\|^2}$. Consequently, Birkhoff–James orthogonality is right-unique. Therefore X is smooth, by Lemma 21.1.

Also, if $\rho_*(x, tx + y) = 0$, then $\rho_-(x, tx + y)\rho_+(x, tx + y) = 0$. Therefore, we obtain $t = \frac{-\rho_\pm(x,y)}{\|x\|^2}$. Hence Birkhoff–James orthogonality is right-unique, and so X is smooth.

21.2.4 The λ-Angularly Property of Norms

The concept of angle and the question how to measure angles are interesting from the geometrical view points; see e.g. [7, 61, 62] and the references therein. In this section, we study an angle function based on ρ_λ. Let us begin with some observations. In a real inner product space $\big(H, \langle \cdot, \cdot \rangle\big)$, the angle $\theta(x, y)$ between two non-zero elements x, y is defined by

$$\theta(x, y) = \arccos\left(\frac{\langle x, y\rangle}{\|x\|\|y\|}\right).$$

Now, let $(X, \|\cdot\|)$ be a real normed linear space, and let $\lambda \in [0, 1]$. For all non-zero elements $x, y \in X$ we have $-1 \leq \frac{\rho_\lambda(x,y)}{\|x\|\|y\|} \leq 1$. Hence we can define the notion of λ-angle between the non-zero elements x and y.

Definition 21.3 The number

$$\theta_\lambda(x, y) := \arccos\left(\frac{\rho_\lambda(x, y)}{\|x\|\|y\|}\right).$$

is called the λ-angle between the element x and the element y in a normed linear space.

We will refrain from referring to the λ-angle between x and y, since the λ-angle from x to y may not coincide with the λ-angle from y to x. Notice that $\theta_\lambda(x, y)$ does not depend on the lengths of x and y. Also, if the norm in X arises from an inner product, it is easy to see that λ-angles agree with angles defined by the inner product.

Definition 21.4 Two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, on X have the λ-angularly property if there exists a constant C such that for all non-zero elements $x, y \in X$,

$$\tan\left(\frac{\theta_{\lambda,2}(x, y)}{2}\right) \leq C \tan\left(\frac{\theta_{\lambda,1}(x, y)}{2}\right).$$

Here $\theta_{\lambda,1}(x, y)$ and $\theta_{\lambda,2}(x, y)$ are the λ-angles from x to y relative to $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Also, $\tan(\frac{\pi}{2})$ is taken to be $+\infty$.

Our definition is motivated by the Wielandt and generalized Wielandt inequalities, which can be applied in matrix analysis and multivariate analysis, where angles between elements correspond to statistical correlation; see e.g. [74].

Remark 21.1 Suppose the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ have the λ-angularly property on X. Then the norms $\|\cdot\|_2$ and $\|\cdot\|_1$ have the $(1-\lambda)$-angularly property on X. Indeed, for every non-zero $x, y \in X$ we have

$$\begin{aligned}\tan\left(\frac{\theta_{1-\lambda,1}(x, y)}{2}\right) &= -\tan\left(\frac{\theta_{\lambda,1}(x, -y)}{2}\right)\\ &\leq -\frac{1}{C}\tan\left(\frac{\theta_{\lambda,2}(x, -y)}{2}\right) = \frac{1}{C}\tan\left(\frac{\theta_{1-\lambda,2}(x, y)}{2}\right).\end{aligned}$$

In the following theorem we show that λ-angularly property of norms share a geometric property.

Recall that a normed linear space $(X, \|\cdot\|)$ is strictly convex (rotund) if and only if $x \neq y$ and $\|x\| = \|y\| = 1$ together imply that $\|tx + (1-t)y\| < 1$ for all $0 < t < 1$. To get the next result we use some ideas of [44].

Theorem 21.17 *Suppose the norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *have the* λ*-angularly property on* X*. Then the following statements are equivalent:*

(i) $(X, \|\cdot\|_1)$ *is strictly convex.*
(ii) $(X, \|\cdot\|_2)$ *is strictly convex.*

Proof (i)⇒(ii) Since a normed linear space is strictly convex if every boundary point of the unit ball is an extreme point (see [31]), hence it is enough to show that if $\frac{x}{\|x\|_1}$ is an extreme point of the $\|\cdot\|_1$-unit ball, then $\frac{x}{\|x\|_2}$ is an extreme point of the $\|\cdot\|_2$-unit ball. Suppose $\frac{x}{\|x\|_2}$ is not an extreme point of the $\|\cdot\|_2$-unit ball. Then there are points y and z in X such that $\frac{x}{\|x\|_2} = \frac{y+z}{2}$ and the closed line segment from y to z is contained in the $\|\cdot\|_2$-unit ball. If $s \in [0, 1]$ then the points $(1-s)y + sz$ and $sy + (1-s)z$ are on the line segment and hence in the $\|\cdot\|_2$-unit ball. Thus,

$$\begin{aligned}2 = \|y+z\|_2 &= \|(1-s)y + sz + sy + (1-s)z\|_2\\ &\le \|(1-s)y + sz\|_2 + \|sy + (1-s)z\|_2 \le 1 + 1 = 2.\end{aligned}$$

It follows that $\|(1-s)y + sz\|_2 = \|sy + (1-s)z\|_2 = 1$. In particular, we observe that $\|y\|_2 = \|z\|_2 = 1$. Hence

$$\begin{aligned}\rho_{\lambda,2}(y,z) &= \lambda\rho_{-,2}(y,z) + (1-\lambda)\rho_{+,2}(y,z)\\ &= \lambda\|y\|_2 \lim_{t\to 0^-} \frac{\|y+tz\|_2 - \|y\|_2}{t} + (1-\lambda)\|y\|_2 \lim_{t\to 0^+} \frac{\|y+tz\|_2 - \|y\|_2}{t}\\ &= \lambda \lim_{s\to 0^-} \frac{\left\|y + \frac{s}{1-s}z\right\|_2 - 1}{\frac{s}{1-s}} + (1-\lambda) \lim_{s\to 0^+} \frac{\left\|y + \frac{s}{1-s}z\right\|_2 - 1}{\frac{s}{1-s}}\\ &= \lambda \lim_{s\to 0^-} \frac{\|(1-s)y + sz\|_2 - (1-s)}{s}\\ &\quad + (1-\lambda) \lim_{s\to 0^+} \frac{\|(1-s)y + sz\|_2 - (1-s)}{s}\\ &= \lambda + (1-\lambda) = 1.\end{aligned}$$

It follows that $\rho_{\lambda,2}(y,z) = 1$, $\cos\big(\theta_{\lambda,2}(y,z)\big) = 1$, and $\tan\Big(\frac{\theta_{\lambda,2}(x,y)}{2}\Big) = 0$. By the λ-angularly property, $\tan\Big(\frac{\theta_{\lambda,1}(x,y)}{2}\Big) = 0$ as well. This implies $\cos\big(\theta_{\lambda,1}(y,z)\big) = 1$ and hence $\rho_{\lambda,1}(y,z) = \|y\|_1\|z\|_1$. From [88, Theorem 2.2] we obtain

$$\|y\|_1\|z\|_1 = \rho_{\lambda,1}(y,z) \le (\|y+z\|_1 - \|y\|_1)\|y\|_1 \le \|z\|_1\|y\|_1,$$

and hence $(\|y+z\|_1 - \|y\|_1)\|y\|_1 = \|z\|_1\|y\|_1$, i.e., $\|y+z\|_1 = \|y\|_1 + \|z\|_1$. On the other hands, we have

$$\frac{x}{\|x\|_1} = \frac{\frac{y+z}{2}\|x\|_2}{\left\|\frac{y+z}{2}\|x\|_2\right\|_1} = \frac{y+z}{\|y+z\|_1} = \frac{\|y\|_1}{\|y\|_1 + \|z\|_1}\frac{y}{\|y\|_1} + \frac{\|z\|_1}{\|y\|_1 + \|z\|_1}\frac{z}{\|z\|_1},$$

which is a convex combination of the points $\frac{y}{\|y\|_1}$ and $\frac{z}{\|z\|_1}$. Thus, $\frac{x}{\|x\|_1}$ is an interior point of the line segment from $\frac{y}{\|y\|_1}$ to $\frac{z}{\|z\|_1}$. Since the endpoints of this segment lie in the $\|\cdot\|_1$-unit ball, so the convexity shows that the entire line segment lies in the $\|\cdot\|_1$-unit ball. Thus $\frac{x}{\|x\|_1}$ is not an extreme point of the $\|\cdot\|_1$-unit ball, which is a contradiction.

By using a similar argument we get (ii)⇒(i).

The next theorem may be viewed as a stability result for the λ-angularly property of norms.

Theorem 21.18 *Suppose the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ have the λ-angularly property on X and let $\|\cdot\|_3 = \|\cdot\|_1 + \|\cdot\|_2$. Then the following statements hold.*

(i) The norms $\|\cdot\|_3$ and $\|\cdot\|_1$ have the λ-angularly property.
(ii) The norms $\|\cdot\|_3$ and $\|\cdot\|_2$ have the $(1-\lambda)$-angularly property.

Proof

(i) Let $x, y \in X \setminus \{0\}$. Let $\rho_{\lambda,i}(x,y)$ and $\theta_{\lambda,i}(x,y)$ be the functional ρ_λ and the λ-angle from x to y with respect to the norm $\|\cdot\|_i$, for $i = 1, 2, 3$. We have

$$\begin{aligned}
\rho_{\lambda,3}(x,y) &= \lambda\rho_{-,3}(x,y) + (1-\lambda)\rho_{+,3}(x,y) \\
&= \lambda\|x\|_3 \lim_{t\to 0^-}\frac{\|x+ty\|_3 - \|x\|_3}{t} + (1-\lambda)\|x\|_3 \lim_{t\to 0^+}\frac{\|x+ty\|_3 - \|x\|_3}{t} \\
&= \lambda\|x\|_3 \lim_{t\to 0^-}\frac{\|x+ty\|_1 + \|x+ty\|_2 - \|x\|_1 - \|x\|_2}{t} \\
&\qquad + (1-\lambda)\|x\|_3 \lim_{t\to 0^+}\frac{\|x+ty\|_1 + \|x+ty\|_2 - \|x\|_1 - \|x\|_2}{t} \\
&= \lambda\|x\|_3\frac{\rho_{-,1}(x,y)}{\|x\|_1} + \lambda\|x\|_3\frac{\rho_{-,2}(x,y)}{\|x\|_2} \\
&\qquad + (1-\lambda)\|x\|_3\frac{\rho_{+,1}(x,y)}{\|x\|_1} + (1-\lambda)\|x\|_3\frac{\rho_{+,2}(x,y)}{\|x\|_2} \\
&= \frac{\|x\|_3}{\|x\|_1}\Big(\lambda\rho_{-,1}(x,y) + (1-\lambda)\rho_{+,1}(x,y)\Big) \\
&\qquad + \frac{\|x\|_3}{\|x\|_2}\Big(\lambda\rho_{-,2}(x,y) + (1-\lambda)\rho_{+,2}(x,y)\Big) \\
&= \frac{\|x\|_3}{\|x\|_1}\rho_{\lambda,1}(x,y) + \frac{\|x\|_3}{\|x\|_2}\rho_{\lambda,2}(x,y).
\end{aligned}$$

Therefore

$$\rho_{\lambda,3}(x, y) = \frac{\|x\|_3}{\|x\|_1}\rho_{\lambda,1}(x, y) + \frac{\|x\|_3}{\|x\|_2}\rho_{\lambda,2}(x, y),$$

whence

$$\begin{aligned}\cos\theta_{\lambda,3}(x, y) &= \frac{\rho_{\lambda,3}(x, y)}{\|x\|_3\,\|y\|_3}\\ &= \frac{\rho_{\lambda,1}(x, y)}{\|x\|_1\,\|y\|_3} + \frac{\rho_{\lambda,2}(x, y)}{\|x\|_2\,\|y\|_3}\\ &= \frac{\|y\|_1}{\|y\|_3}\cos\theta_{\lambda,1}(x, y) + \frac{\|y\|_2}{\|y\|_3}\cos\theta_{\lambda,2}(x, y).\end{aligned}$$

Thus

$$\cos\theta_{\lambda,3}(x, y) = \frac{\|y\|_1}{\|y\|_3}\cos\theta_{\lambda,1}(x, y) + \frac{\|y\|_2}{\|y\|_3}\cos\theta_{\lambda,2}(x, y). \tag{21.1}$$

Now, by (21.1) and the fact that $\frac{1+r}{1+t} \leq 1 + \frac{r}{t}$ for all $r, t > 0$, we have

$$\begin{aligned}\tan\left(\frac{\theta_{\lambda,3}(x, y)}{2}\right) &= \sqrt{\frac{1-\cos\theta_{\lambda,3}(x, y)}{1+\cos\theta_{\lambda,3}(x, y)}}\\ &\leq \sqrt{1 + \frac{\tan\left(\frac{\theta^2_{\lambda,2}(x,y)}{2}\right)}{\tan\left(\frac{\theta^2_{\lambda,1}(x,y)}{2}\right)}}\tan\left(\frac{\theta_{\lambda,1}(x, y)}{2}\right)\\ &\leq \sqrt{1+C^2}\tan\left(\frac{\theta_{\lambda,1}(x, y)}{2}\right).\end{aligned}$$

Hence

$$\tan\left(\frac{\theta_{\lambda,3}(x, y)}{2}\right) \leq \sqrt{1+C^2}\tan\left(\frac{\theta_{\lambda,1}(x, y)}{2}\right).$$

So, the norms $\|\cdot\|_3$ and $\|\cdot\|_1$ have the λ-angularly property.

(ii) Since the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ have the λ-angularly property, Remark 21.1 shows that the norms $\|\cdot\|_2$ and $\|\cdot\|_1$ have the $(1-\lambda)$-angularly property. Thus from (i) we conclude that the norms $\|\cdot\|_3$ and $\|\cdot\|_2$ have the $(1-\lambda)$-angularly property.

21.2.5 Approximate Norm Derivatives Orthogonalities

In an inner product space $(H, \langle\cdot,\cdot\rangle)$ an approximate orthogonality (ε-orthogonality) of vectors $x, y \in H$ was naturally defined in [13, 30] by

$$x \perp^{\varepsilon} y \quad \text{if and only if} \quad |\langle x, y\rangle| \leq \varepsilon\|x\|\|y\|.$$

For $\varepsilon \geq 1$, it is clear that every pair of vectors are ε-orthogonal, so the interesting case is when $\varepsilon \in [0, 1)$.

Now, let $(X, \|\cdot\|)$ be a normed linear space and let $x, y \in X$. Analogously, for a given semi-inner product $[\cdot|\cdot]$ on X the approximate semi-orthogonality relation was defined in [15, 30] by

$$x \perp_s^{\varepsilon} y \quad \text{if and only if} \quad |[y|x]| \leq \varepsilon\|x\|\|y\|.$$

The first notion of approximate Birkhoff–James orthogonality has been proposed by Dragomir [29] as follows:

$$x \perp_D^{\varepsilon} y \quad \text{if and only if} \quad \|x + ty\| \geq (1-\varepsilon)\|x\| \qquad (t \in \mathbb{K}).$$

Chmieliński [12] also introduced another notion of approximate Birkhoff–James orthogonality, defined in the following way:

$$x \perp_B^{\varepsilon} y \quad \text{if and only if} \quad \|x + ty\|^2 \geq \|x\|^2 - 2\varepsilon\|x\|\|ty\| \qquad (t \in \mathbb{K}).$$

We would like to remark that in a normed linear space, both types of approximate Birkhoff–James orthogonality are homogeneous. For more information about these types of approximate orthogonality and their properties the reader is referred to [12, 29].

Inspired by approximate Birkhoff–James orthogonality, for a normed linear space, others notions of approximate orthogonality were considered. One of them is the approximate Roberts orthogonality. In fact, the authors in [86] introduced two versions of approximate Roberts orthogonality as follows:

$$x \perp_R^{\varepsilon} y \Leftrightarrow \left|\|x + ty\|^2 - \|x - ty\|^2\right| \leq 4\varepsilon\|x\|\|ty\| \qquad (t \in \mathbb{R})$$

and

$$x\,{}^{\varepsilon}\!\perp_R y \Leftrightarrow \left|\|x + ty\| - \|x - ty\|\right| \leq \varepsilon(\|x + ty\| + \|x - ty\|) \qquad (t \in \mathbb{R}).$$

It can be remarked that these two orthogonality relations are related to analogous definitions for isosceles orthogonality introduced in [20] (see also [85]). Another one is the approximate Pythagorean orthogonality which has been investigated in [77]:

$$x \perp_P^{\varepsilon} y \Leftrightarrow \Big|\|x+y\|^2 - \|x\|^2 - \|y\|^2\Big| \leq 2\varepsilon\|x\|\|y\|.$$

Also, we remember two generalized types of approximate isosceles orthogonality, namely approximate cI-orthogonality, in normed linear spaces were considered in [87]. For a fixed $c \neq 0$, the first one is

$$x\, {}^{\varepsilon}\!\perp_{cI}\, y \Leftrightarrow \Big|\|x+cy\|^2 - \|x-cy\|^2\Big| \leq 4\varepsilon\|x\|\|cy\|,$$

and the second one is

$$x \perp_{cI}^{\varepsilon} y \Leftrightarrow \Big|\|x+cy\| - \|x-cy\|\Big| \leq \varepsilon(\|x+cy\| + \|x-cy\|).$$

In a similar way Chmieliński and Wójcik [22] introduced the notions of an approximate $\rho_{\pm}$ and ρ-orthogonality as follows:

$$x \perp_{\rho_{\pm}}^{\varepsilon} y \quad \text{if and only if} \quad |\rho_{\pm}(x, y)| \leq \varepsilon\|x\|\|y\|,$$
$$x \perp_{\rho}^{\varepsilon} y \quad \text{if and only if} \quad |\rho(x, y)| \leq \varepsilon\|x\|\|y\|.$$

Similarly, the approximate ρ_*-orthogonality has been defined and studied in [27]:

$$x \perp_{\rho_*}^{\varepsilon} y \quad \text{if and only if} \quad |\rho_*(x, y)| \leq \varepsilon^2\|x\|^2\|y\|^2.$$

Obviously, if the norm in X comes from an inner product, then

$$\perp^{\varepsilon} = \perp_s^{\varepsilon} = \perp_B^{\varepsilon} = \perp_R^{\varepsilon} = \perp_P^{\varepsilon} = {}^{\varepsilon}\!\perp_{cI} = \perp_{\rho_{\pm}}^{\varepsilon} = \perp_{\rho}^{\varepsilon} = \perp_{\rho_*}^{\varepsilon}.$$

Also, it is clear that for $\varepsilon = 0$ all the above approximate orthogonalities coincide with the related exact orthogonalities.

Chmieliński and Wójcik generalized Theorem 21.3 for approximate Birkhoff–James orthogonality in [22] as follows:

Theorem 21.19 ([22, Thorem 3.1]) *Let* $(X, \|\cdot\|)$ *be a normed linear space and let* $\varepsilon \in [0, 1)$. *Then, for arbitrary* $x, y \in X$ *and* $\alpha \in \mathbb{R}$ *the following condition are equivalent:*

(i) $x \perp_B^{\varepsilon} (y - \alpha x)$.
(ii) $\rho_-(x, y) - \varepsilon\|x\|\|y - \alpha x\| \leq \alpha\|x\|^2 \leq \rho_+(x, y) + \varepsilon\|x\|\|y - \alpha x\|$.

In particular, $x \perp_B^{\varepsilon} y$ *if and only if*

$$\rho_-(x, y) - \varepsilon\|x\|\|y\| \leq 0 \leq \rho_+(x, y) + \varepsilon\|x\|\|y\|.$$

They also identified the relationship between $\perp_s^{\varepsilon}$, $\perp_{\rho_{\pm}}^{\varepsilon}$, $\perp_{\rho}^{\varepsilon}$ and $\perp_B^{\varepsilon}$ in the following theorem.

Theorem 21.20 ([21, 22]) *Let* $\varepsilon \in [0, 1)$. *For an arbitrary normed linear space* X *and* $\diamondsuit \in \{s, \rho_-, \rho_+, \rho\}$ *we have* $\perp_{\diamondsuit}^{\varepsilon} \subset \perp_{B}^{\varepsilon}$.

Of course, for non-smooth normed linear spaces, the approximate orthogonalities $\perp_{\rho_\pm}^{\varepsilon}$ and $\perp_{\rho}^{\varepsilon}$ are incomparable. The following generalization of Theorem 21.4 has been proved in [22].

Theorem 21.21 ([22, Theorem 3.3]) *Let* X *be a normed linear space and let* $\varepsilon \in [0, 1)$. *Then the following conditions are equivalent:*

$$(i)\ \perp_{\rho_+}^{\varepsilon} \subset \perp_{\rho_-}^{\varepsilon}. \quad (ii)\ \perp_{\rho_-}^{\varepsilon} \subset \perp_{\rho_+}^{\varepsilon}. \quad (iii)\ \perp_{\rho_+}^{\varepsilon} = \perp_{\rho_-}^{\varepsilon}.$$

$$(iv)\ \perp_{\rho_+}^{\varepsilon} \subset \perp_{\rho}^{\varepsilon}. \quad (v)\ \perp_{\rho}^{\varepsilon} \subset \perp_{\rho_+}^{\varepsilon}. \quad (vi)\ \perp_{\rho_+}^{\varepsilon} = \perp_{\rho}^{\varepsilon}.$$

$$(vii)\ \perp_{\rho_-}^{\varepsilon} \subset \perp_{\rho}^{\varepsilon}. \quad (viii)\ \perp_{\rho}^{\varepsilon} \subset \perp_{\rho_-}^{\varepsilon}. \quad (ix)\ \perp_{\rho_-}^{\varepsilon} = \perp_{\rho}^{\varepsilon}. \quad (x)\ X \text{ is smooth}.$$

Some illustrated examples were provided in [21, 22] which show that equalities in Theorem 21.20 need not to hold in non-smooth normed linear spaces. Actually, using this fact and Theorem 21.21 it has been proved in [22] that the smoothness of a normed linear space X resulted also from $\perp_{\rho_\pm}^{\varepsilon} = \perp_{B}^{\varepsilon}$ and $\perp_{\rho}^{\varepsilon} = \perp_{B}^{\varepsilon}$ for some $\varepsilon \in [0, 1)$. In fact, the following theorem is a generalization of Theorem 21.2.

Theorem 21.22 ([22, Theorem 3.4]) *Let* X *be a normed linear space and let* $\varepsilon \in [0, 1)$. *If* $\perp_{\rho_\pm}^{\varepsilon} = \perp_{B}^{\varepsilon}$ *or* $\perp_{\rho}^{\varepsilon} = \perp_{B}^{\varepsilon}$, *then* X *is smooth.*

Moreover, an approximate version of Theorem 21.5 has been prepared as follows:

Theorem 21.23 ([22, Theorem 3.5]) *Let* X *be a normed linear space and let* $[\cdot|\cdot]$ *be a fixed semi-inner product in* X. *For* $\varepsilon \in [0, 1)$ *the following conditions are equivalent:*

$$(i)\ \perp_{\rho}^{\varepsilon} \subset \perp_{s}^{\varepsilon}. \quad (ii)\ \perp_{s}^{\varepsilon} \subset \perp_{\rho}^{\varepsilon}. \quad (iii)\ \perp_{\rho}^{\varepsilon} = \perp_{s}^{\varepsilon}. \quad (iv)\ \langle \cdot, \cdot \rangle_g = [\cdot|\cdot].$$

Another characterization of smooth normed linear spaces using comparison of approximate $\rho_\pm$-orthogonality and approximate semi-inner product has been presented in [78].

Theorem 21.24 *Let* X *be a normed linear space and let* $[\cdot|\cdot]$ *be a fixed semi-inner product in* X. *For* $\varepsilon \in [0, 1)$ *the following conditions are equivalent.*

$$(i)\ \perp_{\rho_+}^{\varepsilon} \subset \perp_{s}^{\varepsilon}. \quad (ii)\ \perp_{s}^{\varepsilon} \subset \perp_{\rho_+}^{\varepsilon}. \quad (iii)\ \perp_{\rho_+}^{\varepsilon} = \perp_{s}^{\varepsilon}.$$

$$(iv)\ \perp_{\rho_-}^{\varepsilon} \subset \perp_{s}^{\varepsilon}. \quad (v)\ \perp_{s}^{\varepsilon} \subset \perp_{\rho_-}^{\varepsilon}. \quad (vi)\ \perp_{\rho_-}^{\varepsilon} = \perp_{s}^{\varepsilon}.$$

$$(vii)\ X \text{ is smooth}.$$

In [69] Stypuła and Wójcik by introducing the constants

$$\mathscr{E}^{\rho}(X) := \inf\left\{\varepsilon \in [0,1] : \quad \perp_{\rho_+} \subset \perp^{\varepsilon}_{\rho_-}\right\}$$

and

$$\mathscr{R}(X) := \sup\left\{\|x-y\| : \quad \mathrm{conv}\{x,y\} \subset \mathbb{S}_X\right\}$$

provided some different characterizations of rotundity and smoothness of dual spaces. We have, of course, $0 \le \mathscr{E}^{\rho}(X) \le 1$ and $0 \le \mathscr{R}(X) \le 2$. Observe that,

$$\mathscr{E}^{\rho}(X) = 0 \quad \text{if and only if} \quad X \text{ is smooth}$$

and

$$\mathscr{R}(X) = 0 \quad \text{if and only if} \quad X \text{ is rotund.}$$

A well-known theorem states that if X^* is rotund, then X is smooth. The following theorem states this well-known result in terms of constants $\mathscr{E}^{\rho}(X)$ and $\mathscr{R}(X)$.

Theorem 21.25 ([69, Corollary 2.6]) *Let X be a real normed linear space. Then*

$$\mathscr{E}^{\rho}(X) \le \mathscr{R}(X^*).$$

Moreover, if X is a reflexive Banach space, then

$$\mathscr{E}^{\rho}(X) \le \mathscr{R}(X^*) \le 2\mathscr{E}^{\rho}(X).$$

Hence,

(i) if X is reflexive, X^* is rotund if and only if X is smooth;
(ii) if X is reflexive, X^* is smooth if and only if X is rotund.

Now, let us review the results obtained related to approximate ρ_*-orthogonality from [27]. It is easy to check that the approximate ρ_*-orthogonality is homogenous. Also, if $x \perp^{\varepsilon}_{\rho_+} y$ and $x \perp^{\varepsilon}_{\rho_-} y$, then $x \perp^{\varepsilon}_{\rho_*} y$. Indeed, by the arithmetic-geometric means inequality, we get

$$|\rho_*(x,y)| = |\rho_-(x,y)\rho_+(x,y)| \le \left(\frac{|\rho_-(x,y)| + |\rho_+(x,y)|}{2}\right)^2 \le \varepsilon^2\|x\|^2\|y\|^2.$$

We notice that the relations $\perp^{\varepsilon}_{\rho_\pm}$, $\perp^{\varepsilon}_{\rho}$ and $\perp^{\varepsilon}_{\rho_*}$ are generally incomparable, see [27, Example 2.1]. Also, the relation between $\perp^{\varepsilon}_{\rho_*}$ and $\perp^{\varepsilon}_{B}$ has been identified as follows:

Theorem 21.26 ([27, Theorem 2.3]) *Let X be a normed linear space and let* $\varepsilon \in [0, 1)$. *Then* $\perp^{\varepsilon}_{\rho_*} \subset \perp^{\varepsilon}_{B}$.

It is noticed in [27, Example 2.4] that for nonsmooth normed linear spaces, the orthogonalities $\perp^{\varepsilon}_{\rho_*}$ and $\perp^{\varepsilon}_{B}$ may not coincide.

Theorem 21.27 ([27, Remark 2.5]) *Let X be a normed linear space and let* $\varepsilon \in [0, 1)$. *If* $\perp^{\varepsilon}_{B} \subset \perp^{\varepsilon}_{\rho_*}$*, then X is smooth.*

To finish this section we consider analogously, the notion of approximate ρ^{v}_{λ}-orthogonality which is studied in [1]. In fact, naturally, for $\varepsilon \in [0, 1)$, $\lambda \in [0, 1]$ and $v = \frac{1}{2k-1}$ ($k \in \mathbb{N}$), we say that a vector $x \in X$ is approximate ρ^{v}_{λ}-orthogonal to a vector $y \in X$, in short $x \perp^{\varepsilon}_{\rho^{v}_{\lambda}} y$, if

$$|\rho^{v}_{\lambda}(x, y)| \leq \varepsilon \|x\| \|y\|.$$

In particular, for $v = 1$, we have $x \perp^{\varepsilon}_{\rho_\lambda} y$ if and only if $|\rho_\lambda(x, y)| \leq \varepsilon \|x\| \|y\|$. Note that the relations $x \perp^{\varepsilon}_{\rho_0} y$, $x \perp^{\varepsilon}_{\rho_1} y$ and $x \perp^{\varepsilon}_{\rho_{\frac{1}{2}}} y$ coincide with the relations $x \perp^{\varepsilon}_{\rho_+} y$, $x \perp^{\varepsilon}_{\rho_-} y$ and $x \perp^{\varepsilon}_{\rho} y$, respectively.

In [1] some illustrated examples have been presented to show that the relations $\perp^{\varepsilon}_{\rho_\pm}$, $\perp^{\varepsilon}_{\rho}$, $\perp^{\varepsilon}_{\rho_*}$, $\perp^{\varepsilon}_{\rho_\lambda}$ and $\perp^{\varepsilon}_{\rho^{v}_{\lambda}}$ are incomparable in general normed linear spaces.

The following result is a generalization of Theorem 21.20.

Theorem 21.28 ([1, Theorem 2.4]) *Let X be a normed linear space and let* $\varepsilon \in [0, 1)$, $\lambda \in [0, 1]$ *and* $v = \frac{1}{2k-1}$ ($k \in \mathbb{N}$). *Then* $\perp^{\varepsilon}_{\rho^{v}_{\lambda}} \subset \perp^{\varepsilon}_{B}$.

According to [1], there are non-smooth normed linear spaces such that $\perp^{\varepsilon}_{B} \not\subset \perp^{\varepsilon}_{\rho_\lambda}$. Analogously to Theorem 21.22, it has been proved in the following theorem that approximate Birkhoff–James orthogonality and approximate ρ^{v}_{λ}-orthogonality in a normed linear space X are equivalent if and only if X is smooth.

Theorem 21.29 ([1, Theorem 2.7]) *Let X be a normed linear space and let* $\varepsilon \in [0, 1)$, $\lambda \in [0, 1]$ *and* $v = \frac{1}{2k-1}$ ($k \in \mathbb{N}$). *If* $\perp^{\varepsilon}_{B} \subset \perp^{\varepsilon}_{\rho^{v}_{\lambda}}$*, then X is smooth.*

21.3 Orthogonality Preserving Property and Applications in the Geometry of Normed Linear Spaces

21.3.1 Linear Mappings Preserving Orthogonality

The problem of determining the structure of linear mappings between normed linear spaces, which leave certain properties invariant, has been considered in several papers. These are the so-called linear preserver problems, see [10, 52] and the references therein. The study on linear orthogonality preserving mappings can be considered as a part of the theory of linear preservers. The orthogonality preserving property have been intensively studied recently in connection with functional analysis and operator theory; cf. [13, 23, 47, 72, 81, 89, 91].

Let H and K be inner product spaces. A mapping $T : H \to K$ is called orthogonality preserving if

$$x \perp y \Rightarrow Tx \perp Ty \qquad (x, y \in X).$$

Such mappings can be very irregular, far from being continuous or linear (see [13, Example 2]). For that reason we restrict ourselves to linear mappings only. On the other hand, for linear orthogonality preserving mappings we have a simple characterization.

Theorem 21.30 ([13, Theorem 1]) *Let H and K be (real or complex) inner product spaces. For a nonzero linear mapping $T : H \to K$ the following conditions are equivalent (with some $\gamma > 0$):*

(i) T is a similarity (scalar multiple of a linear isometry), i.e., $\|Tx\| = \gamma\|x\|$ for all $x \in H$.
(ii) $\langle Tx, Ty\rangle = \gamma^2\langle x, y\rangle$ for all $x, y \in H$.
(iii) T is orthogonality preserving.

Orthogonality preserving mappings have been widely studied in the setting of inner product C^*-modules, see [5, 6, 32, 35, 43, 49–51, 59, 89]. In particular, further generalizations of Theorem 21.30 can be found in [18, 33]. Similar investigations have been carried out in normed linear spaces for sesquilinear form (instead of inner products) in paper [79].

Let X and Y be normed linear spaces and let $T : X \to Y$ be a linear and continuous operator. The norm of T is defined as usual:

$$\|T\| = \sup\big\{\|Tx\| : \|x\| = 1\big\} = \inf\big\{M > 0 : \|Tx\| \le M\|x\|,\ x \in X\big\}.$$

Similarly, we define

$$[T] := \inf\big\{\|Tx\| : \|x\| = 1\big\} = \sup\big\{m \ge 0 : \|Tx\| \ge m\|x\|,\ x \in X\big\}.$$

Now, let $\diamondsuit, \heartsuit \in \{B, I, s, \rho_\pm, \rho, \rho_*, \rho_\lambda, \rho_\lambda^\upsilon\}$. We say that a mapping $T : X \to Y$ (exactly) preserves $(\diamondsuit, \heartsuit)$-orthogonality if

$$x \perp_\diamondsuit y \Rightarrow Tx \perp_\heartsuit Ty \qquad (x, y \in X).$$

In particular, we say that T is $\diamondsuit$-orthogonality preserving if

$$x \perp_\diamondsuit y \Rightarrow Tx \perp_\diamondsuit Ty \qquad (x, y \in X).$$

Koehler and Rosenthal [45, Theorem 1] showed that a linear operator from a normed linear space into itself is an isometry if and only if it preserves some semi-inner product. Blanco and Turnšek [9, Remark 3.2] and Chmieliński [15, Theorem 2.5] extended it to different normed linear spaces. Namely, we have the following result.

Theorem 21.31 *Let X and Y be normed linear spaces. For a linear mapping $T : X \longrightarrow Y$ and some $\gamma > 0$ the following conditions are equivalent:*

(i) *T is a similarity.*
(ii) *$[Tx, Ty]_Y = \gamma^2 [x, y]_X$ for all $x, y \in X$.*
(iii) *T is s-orthogonality preserving.*

The conditions (ii) *and* (iii) *should be understood that they are satisfied with respect to some semi-inner products $[\cdot,\cdot]_X$ and $[\cdot,\cdot]_Y$ in X and Y, respectively.*

It has been proved by Koldobsky [46] that a linear mapping $T : X \to X$ preserving B-orthogonality has to be a similarity. In [36, Theorem 1], Ionică using the connections between the Birkhoff–James orthogonality and norm derivatives gave an alternative proof of the above results in the case of different real normed linear spaces (see also [65]). The respective result for both real and complex cases was given by Blanco and Turnšek in [9, Theorem 3.1]. Very recently, Wójcik in [83] presented a somewhat simpler proof of this theorem.

Theorem 21.32 *Let X and Y be (real or complex) normed linear spaces. A linear mapping $T : X \to Y$ is B-orthogonality preserving if and only if it is a scalar multiple of a linear isometry.*

The following result gives a characterization of inner product spaces.

Theorem 21.33 ([15, Theorem 2.9]) *Let X be a normed linear space. Suppose that there exists an inner product space K and a linear mapping T from X into K or from K onto X such that T preserves B-orthogonality. Then X is an inner product space.*

Martini and Wu [54, Lemma 4] proved the following result.

Theorem 21.34 *Let X and Y be two normed linear spaces. If a linear mapping $T : X \longrightarrow Y$ preserves I-orthogonality, then it also preserves B-orthogonality.*

Combining Theorems 21.32 and 21.34 actually lead us to the following result.

Corollary 21.3 *Let X and Y be normed linear spaces, and let $T : X \longrightarrow Y$ be a nonzero linear mapping. Then the following conditions are equivalent:*

(i) *T is I-orthogonality preserving.*
(ii) *T is a scalar multiple of a linear isometry.*

Remark 21.2 Notice that Corollary 21.3 also has been proved in [16, Theorem 4.5].

The next theorem gives characterizations of inner product spaces by properties of linear operators related to B-orthogonality and I-orthogonality.

Theorem 21.35 ([84, Theorems 9, 10]) *Let X and Y be two normed linear spaces. Each one of the following conditions implies that X and T(X) are inner product spaces.*

(i) There exists a nonzero linear mapping $T : X \longrightarrow Y$ which preserve (I, B)-orthogonality.
(ii) There exists a nonzero linear mapping $T : X \longrightarrow Y$ which preserve (B, I)-orthogonality.

The orthogonality preserving mappings have been considered also in [63]. The paper [63] shows another way to consider the orthogonality preserving mappings. Some other results on B-orthogonality preserving mapping can be found in [19, 82, 84].

21.3.2 Mappings Which Exactly Preserve Norm Derivatives Orthogonality

The aim of this subsection is to present results concerning the linear mappings which preserve norm derivatives orthogonality. We survey on the results presented in [11, 21, 22, 28, 60, 75, 88], as well as give some new and more general ones. In 2010, Chmieliński and Wójcik [21] studied norm derivatives orthogonality preserving mappings. They proved that for arbitrary normed linear spaces X and Y, if a linear mapping $T : X \longrightarrow Y$ preserves ρ_--orthogonality or preserves ρ_+-orthogonality then it is a similarity. Later, Wójcik [75] showed that a linear mapping preserving ρ-orthogonality has to be a similarity. These results give

Theorem 21.36 *Let X and Y be normed linear spaces, and let $T : X \longrightarrow Y$ be a nonzero linear mapping. Then the following conditions are equivalent:*

(i) T preserves ρ_+-orthogonality.
(ii) T preserves ρ_--orthogonality.
(iii) T preserves ρ-orthogonality.
(iv) $\|Tx\| = \|T\|\,\|x\|$ for all $x \in X$.
(v) $\rho_+(Tx, Ty) = \|T\|^2\rho_+(x, y)$ for all $x, y \in X$.
(vi) $\rho_-(Tx, Ty) = \|T\|^2\rho_-(x, y)$ for all $x, y \in X$.
(vii) $\rho(Tx, Ty) = \|T\|^2\rho(x, y)$ for all $x, y \in X$.

As for the ρ_*-orthogonality preserving mapping the following characterization has been given in [60] (see also [11]).

Theorem 21.37 *Let X, Y be normed linear spaces and let $T : X \longrightarrow Y$ be a nonzero linear mapping. Then the following conditions are equivalent:*

(i) T preserves ρ_*-orthogonality.
(ii) T preserves (ρ_*, B)-orthogonality.
(iii) T preserves (B, ρ_*)-orthogonality.

(iv) $\|Tx\| = \|T\| \, \|x\|$ *for all* $x \in X$.
(v) $\rho_*(Tx, Ty) = \|T\|^4 \, \rho_*(x, y)$ *for all* $x, y \in X$.
If $X = Y$*, then each one of these assertions is also equivalent to*
(vi) *there exists a semi-inner product* $[\cdot|\cdot] : X \times X \longrightarrow \mathbb{R}$ *satisfying*

$$[Tx, Ty]_X = \|T\|^2 [x, y]_X \qquad (x, y \in X).$$

Recall that a normed linear space $(X, \|\cdot\|)$ satisfies the δ-parallelogram law for some $\delta \in [0, 1)$, if the double inequality

$$2(1-\delta)\|z\|^2 \leq \|z+w\|^2 + \|z-w\|^2 - 2\|w\|^2 \leq 2(1+\delta)\|z\|^2$$

holds for all $z, w \in X$; cf. [17]. Also a normed linear space $(X, \|\cdot\|)$ is equivalent to an inner product space if there exist an inner product in X and a norm $|||\cdot|||$ generated by this inner product such that

$$\frac{1}{k}\|x\| \leq |||x||| \leq k\|x\| \qquad (x \in X)$$

holds for some $k \geq 1$; see [41].

Corollary 21.4 ([60, Corollary 2.7]) *Any one of the following assertions implies that* X *is equivalent to an inner product space.*

(i) *There exist a normed linear space* Y *satisfying the* δ*-parallelogram law for some* $\delta \in [0, 1)$ *and a nonzero linear mapping* $T : X \longrightarrow Y$ *such that* T *preserves* ρ_**-orthogonality.*
(ii) *There exist a normed linear space* Y *satisfying the* δ*-parallelogram law for some* $\delta \in [0, 1)$ *and a nonzero surjective linear mapping* $S : Y \longrightarrow X$ *such that* S *preserves* ρ_**-orthogonality.*

We remark that the converse of Corollary 21.4 holds also true. Indeed, if X is equivalent to an inner product space, then we can choose $\delta = 0$, $Y = X$ and $T = id$, the identity operator on X. Recall that a normed linear space $(X, \|\cdot\|)$ is called uniformly smooth if X satisfies the property that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in X$ with $\|x\| = 1$ and $\|y\| \leq \delta$, then $\|x+y\|+\|x-y\| \leq 2+\varepsilon\|y\|$; cf. [3].

The modulus of smoothness of X is the function ϱ_X defined for every $t > 0$ by the formula

$$\varrho_X(t) := \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1 \,:\, \|x\| = 1,\ \|y\| = t\right\}.$$

Furthermore, X is called uniformly convex if for every $0 < \varepsilon \leq 2$ there is some $\delta > 0$ such that for any two vectors with $\|x\| = \|y\| = 1$, the condition $\|x-y\| \geq \varepsilon$ implies that $\left\|\frac{x+y}{2}\right\| \leq 1-\delta$.

The modulus of convexity of X is the function σ_X defined by

$$\sigma_X(\varepsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| \,:\, \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\right\}.$$

Let X, Y be normed linear spaces. If a linear mapping $T : X \longrightarrow Y$ preserves ρ_*-orthogonality, then from Theorem 21.37 we conclude that T must be a similarity. Thus, the spaces X and Y have to share some geometrical properties. In particular, the modulus of convexity σ_X and modulus of smoothness ϱ_X must be preserved, i.e., $\sigma_X = \sigma_{T(X)}$ and $\varrho_X = \varrho_{T(X)}$. As a consequence, we have the following result.

Corollary 21.5 *Let X be a normed linear space. Suppose that there exists a normed linear space Y which is a uniformly convex (uniformly smooth) space, a strictly convex space, or an inner product space and a nontrivial linear mapping T from X into Y (or from Y onto X) such that T preserves ρ_*-orthogonality. Then X is, respectively, a uniformly convex (uniformly smooth) space, a strictly convex space, an inner product space.*

Recently, the authors of the paper [88] considered the class of linear mappings preserving ρ_λ-orthogonality. They showed that each such a mapping must be a similarity. Namely, they proved the following result.

Theorem 21.38 ([88, Theorem 3.4]) *Let X and Y be normed linear spaces and $\lambda \in [0, 1]$. Let $T : X \longrightarrow Y$ be a nonzero linear mapping. Then the following conditions are equivalent:*

(i) *T preserves ρ_λ-orthogonality.*
(ii) $\|Tx\| = \|T\|\,\|x\|$ *for all* $x \in X$.
(iii) $\rho_\lambda(Tx, Ty) = \|T\|^2\,\rho_\lambda(x, y)$ *for all* $x, y \in X$.

Let X be a normed linear space endowed with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$, which generate respective functionals $\rho_{\diamond,1}$ and $\rho_{\diamond,2}$, where $\diamond \in \{\lambda, *\}$. Following [3, Definition 2.4.1], we say that functionals $\rho_{\diamond,1}$ and $\rho_{\diamond,2}$ are equivalent if there exist constants $0 < m \leq M$ such that

$$m|\rho_{\diamond,1}(x, y)| \leq |\rho_{\diamond,2}(x, y)| \leq M|\rho_{\diamond,1}(x, y)| \qquad (x, y \in X).$$

Taking $X = Y$ and $T = id$, one obtains, from Theorems 21.37 and 21.38, the following result.

Corollary 21.6 *Let X be a normed linear space endowed with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$, which generate respective functionals $\rho_{\diamond,1}$ and $\rho_{\diamond,2}$ with $\diamond \in \{\lambda, *\}$ and $\lambda \in [0, 1]$. Then the following conditions are equivalent:*

(i) *The functionals $\rho_{\diamond,1}$ and $\rho_{\diamond,2}$ are equivalent.*
(ii) *The spaces $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are isometrically isomorphic.*

Next, we formulate one of our main results.

Theorem 21.39 *Let X, Y be normed linear spaces, $\lambda \in [0, 1]$ and let $\upsilon = \frac{1}{2k-1}$ ($k \in \mathbb{N}$). If $T : X \longrightarrow Y$ be a nonzero linear mapping, then the following conditions are equivalent:*

(i) T preserves ρ_λ^υ-orthogonality.
(ii) T preserves $(\rho_\lambda^\upsilon, B)$-orthogonality.
(iii) $\|Tx\| = \|T\|\,\|x\|$ for all $x \in X$.
(iv) $\rho_\lambda^\upsilon(Tx, Ty) = \|T\|^2\,\rho_\lambda^\upsilon(x, y)$ for all $x, y \in X$.

Proof (i)⇒(ii) Suppose that $x, y \in X$ and $x \perp_{\rho_\lambda^\upsilon} y$. Then $Tx \perp_{\rho_\lambda^\upsilon} Ty$, by (i). It follows from Proposition 21.4 that $Tx \perp_B Ty$. Thus T preserves $(\rho_\lambda^\upsilon, B)$-orthogonality.

(ii)⇒(iii) Suppose that (ii) holds and fix $x, y \in X \setminus \{0\}$. If x and y are linearly dependent, then $\frac{\|Tx\|}{\|x\|} = \frac{\|Ty\|}{\|y\|}$. Now, assume that x and y are linearly independent. For any $t \in \mathbb{R}$, it is easy to see that $\rho_\pm\Big(x + ty, \frac{-\rho_\pm(x+ty, y)}{\|x+ty\|^2}(x + ty) + y\Big) = 0$ and hence

$$\rho_\lambda^\upsilon\Big(x + ty, \frac{-\rho_\pm(x + ty, y)}{\|x + ty\|^2}(x + ty) + y\Big) = 0.$$

It follows from Proposition 21.4 that $Tx + tTy \perp_B \frac{-\rho_\pm(x+ty,y)}{\|x+ty\|^2}(Tx + tTy) + Ty$. By Theorem 21.3, we get

$$\begin{aligned}\rho_-\Big(&Tx + tTy, \frac{-\rho_\pm(x + ty, y)}{\|x + ty\|^2}(Tx + tTy) + Ty\Big)\\ &\le 0\\ &\le \rho_+\Big(Tx + tTy, \frac{-\rho_\pm(x + ty, y)}{\|x + ty\|^2}(Tx + tTy) + Ty\Big).\end{aligned}$$

This implies

$$\frac{-\rho_-(x + ty, y)}{\|x + ty\|^2}\|Tx + tTy\|^2 + \rho_-(Tx + tTy, Ty) \le 0 \qquad (t \in \mathbb{R}) \tag{21.2}$$

and

$$0 \le \frac{-\rho_+(x + ty, y)}{\|x + ty\|^2}\|Tx + tTy\|^2 + \rho_+(Tx + tTy, Ty) \qquad (t \in \mathbb{R}). \tag{21.3}$$

Let us define

$$\varphi_{x,y}(t) := \frac{\|Tx + tTy\|}{\|x + ty\|} \qquad (t \in \mathbb{R}).$$

Then simple computations show that

$$(\varphi_{x,y})'_{\pm}(t) = \frac{\rho_{\pm}(Tx + tTy, Ty)\|x + ty\| - \rho_{\pm}(x + ty, y)\|Tx + tTy\|}{\|x + ty\|^2}.$$

From (21.2) and (21.3) it follows that

$$0 \leq (\varphi_{x,y})'_{-}(t) \quad \text{and} \quad (\varphi_{x,y})'_{+}(t) \leq 0 \qquad (t \in \mathbb{R}).$$

Hence $\varphi_{x,y}$ is constant on $\mathbb{R}$. Therefore,

$$\frac{\|Tx\|}{\|x\|} = \varphi_{x,y}(0) = \lim_{t\to\infty} \varphi_{x,y}(t) = \frac{\|Ty\|}{\|y\|}.$$

Now, we fix a unit vector y_0 in X. For every nonzero vector $x \in X$, we conclude that $\frac{\|Tx\|}{\|x\|} = \|Ty_0\|$. Hence $\|Tx\| = \|Ty_0\|\|x\|$ for all $x \in X$. Therefore (iii) is valid.

The other implications are trivial.

Let us adopt the notion of Birkhoff orthogonal set of x from [3]:

$$[x]^B_{\|\cdot\|} = \{y \in X \;:\; x \perp_B y\}.$$

We now define the $\diamondsuit$-orthogonal set of x as follows:

$$[x]^{\diamondsuit}_{\|\cdot\|} = \{y \in X \;:\; x \perp_{\diamondsuit} y\},$$

where $\diamondsuit \in \{I, s, \rho_*, \rho^{\upsilon}_{\lambda}\}$.

Theorem 21.40 *Let X be a normed linear space endowed with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$, and let $\lambda \in [0, 1]$ and $\upsilon = \frac{1}{2k-1}$ $(k \in \mathbb{N})$. For every $x \in X$, the following conditions are equivalent:*

$(i)\ [x]^B_{\|\cdot\|_1} = [x]^B_{\|\cdot\|_2}$. $\qquad (ii)\ [x]^s_{\|\cdot\|_1} = [x]^s_{\|\cdot\|_2}$.

$(iii)\ [x]^I_{\|\cdot\|_1} = [x]^I_{\|\cdot\|_2}$. $\qquad (iv)\ [x]^{\rho_*}_{\|\cdot\|_1} = [x]^{\rho_*}_{\|\cdot\|_2}$.

$(v)\ [x]^{\rho^{\upsilon}_{\lambda}}_{\|\cdot\|_1} = [x]^{\rho^{\upsilon}_{\lambda}}_{\|\cdot\|_2}$.

Proof (i)⇒(v) Suppose that (i) holds and define $T = id : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ to be the identity map. Then T is B-orthogonal preserving. It follows from Theorem 21.32 that there exists $M > 0$ such that $\|Tx\|_2 = \|x\|_2 = M\|x\|_1$, which implies that $[x]^{\rho^{\upsilon}_{\lambda}}_{\|\cdot\|_1} = [x]^{\rho^{\upsilon}_{\lambda}}_{\|\cdot\|_2}$ $(x \in X)$.

(v)⇒(i) If (v) holds, then $T = id : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is $\rho^{\upsilon}_{\lambda}$-orthogonal preserving. It follows from Theorem 21.39 that there exists $M > 0$ such that $\|x\|_2 = \|Tx\|_2 = M\|x\|_1$, which ensures that $[x]^B_{\|\cdot\|_1} = [x]^B_{\|\cdot\|_2}$ $(x \in X)$.

The other implications can be proved similarly.

In [75, Theorem 5.1] Wójcik proved that a real normed linear space X is smooth if and only if there exist a real normed linear space Y and a nonvanishing linear mapping $T : X \longrightarrow Y$, such that T preserves $(\diamondsuit, \heartsuit)$-orthogonality for some $\diamondsuit, \heartsuit \in \{\rho_-, \rho_+, \rho\}$ with $\diamondsuit \neq \heartsuit$.

In the sequel, from [60, Theorems 3.2-3] and [28, Theorems 2.20-22] we are going to provide some characterizations of smooth real normed linear spaces in terms of linear mappings that preserve ρ_* and ρ_λ^υ-orthogonality to other types of orthogonality relations.

Theorem 21.41 *Let X be a real normed linear space and let $\lambda \in [0, 1]$ and $\upsilon = \frac{1}{2k-1}$ $(k \in \mathbb{N})$. Then the following conditions are equivalent:*

(i) *X is smooth.*
(ii) *There exist a normed linear space Y and a nonvanishing linear mapping $T : X \longrightarrow Y$ such that T preserves (ρ_*, ρ_+)-orthogonality.*
(iii) *There exist a normed linear space Y and a nonvanishing linear mapping $T : X \longrightarrow Y$ such that T preserves (ρ_*, ρ_-)-orthogonality.*
(iv) *There exist a normed linear space Y and a nonvanishing linear mapping $T : X \longrightarrow Y$ such that T preserves (ρ_*, ρ)-orthogonality.*
(v) *There exist a normed linear space Y and a nonvanishing linear mapping $T : X \longrightarrow Y$ such that T preserves (ρ, ρ_*)-orthogonality.*
(vi) *There exist a normed linear space Y and a nonvanishing linear mapping $T : X \longrightarrow Y$ such that T preserves $(\rho_\lambda^\upsilon, \rho_-)$-orthogonality.*
(vii) *There exist a normed linear space Y and a nonvanishing linear mapping $T : X \longrightarrow Y$ such that T preserves $(\rho_\lambda^\upsilon, \rho_+)$-orthogonality.*
(viii) *There exist a normed linear space Y and a nonvanishing linear mapping $T : X \longrightarrow Y$ such that T preserves $(\rho_\lambda^\upsilon, \rho_\lambda)$-orthogonality.*
(ix) *There exist a normed linear space Y and a nonvanishing linear mapping $T : X \longrightarrow Y$ such that T preserves $(\rho_\lambda^\upsilon, \rho)$-orthogonality.*

21.3.3 Approximate Orthogonality Preserving Mapping

Ulam [73] raised the general problem of when a mathematical object which satisfies a certain property approximately must be close, in some sense, to one that satisfies this property accurately. Approximately orthogonality preserving mappings in the framework of inner product spaces have been studied in this setting, see [13, 14, 23, 47, 72, 81, 90, 91].

Let H and K be two inner product spaces and let $\delta, \varepsilon \in [0, 1)$. A mapping $T : H \to K$ is called a (δ, ε)-orthogonality preserving if

$$x \perp^\delta y \Rightarrow Tx \perp^\varepsilon Ty \qquad (x, y \in H).$$

Often $\delta = 0$ has been considered. Therefore, we say that T is ε-orthogonality preserving if

$$x \perp y \Rightarrow Tx \perp^{\varepsilon} Ty \qquad (x, y \in H).$$

Obviously, if $\delta = \varepsilon = 0$, then T is orthogonality preserving. Hence, the natural question is whether a (δ, ε)-orthogonality preserving linear mapping T must be close to a linear orthogonality preserving mapping. The following result was proved in [13, Theorem 2] (see also [72, Remark 2.1]).

Theorem 21.42 ([13, Theorem 2]) *Let H and K be two Hilbert spaces, and let $T : H \to K$ be a nonzero linear ε-orthogonality preserving mapping with $\varepsilon \in [0, 1)$. Then T is injective, continuous and, with some $\gamma > 0$, T satisfies the functional inequality*

$$\Big|\langle Tx, Ty\rangle - \gamma\langle x, y\rangle\Big| \leq \frac{4\varepsilon}{1+\varepsilon}\min\Big\{\gamma\|x\|\|y\|, \|Tx\|\,\|Ty\|\Big\} \qquad (x, y \in H).$$

Conversely, if $T : H \to K$ satisfies

$$\Big|\langle Tx, Ty\rangle - \gamma\langle x, y\rangle\Big| \leq \varepsilon\min\Big\{\gamma\|x\|\|y\|, \|Tx\|\,\|Ty\|\Big\} \qquad (x, y \in H)$$

with $\varepsilon \geq 0$ and with $\gamma > 0$, then T is a quasi-linear mapping and ε-orthogonality preserving.

Recently, Moslehian et al. [61] have been obtained the following result.

Theorem 21.43 ([61, Theorem 3.10]) *Let H and K be two real Hilbert spaces and* $\dim H < \infty$. *Let $T : H \to K$ be a linear mapping with $0 < [T]$. Then there exists γ such that T satisfies*

$$\Big|\langle Tx, Ty\rangle - \gamma\langle x, y\rangle\Big| \leq \Big(1 - \frac{[T]^2}{\|T\|^2}\Big)\|T\|^2\,\|x\|\,\|y\| \qquad (x, y \in H).$$

Moreover, $[T]^2 \leq |\gamma| \leq 2\|T\|^2 - [T]^2$.

In 2007, Turnšek proved the following

Theorem 21.44 ([72, Theorem 2.3]) *Let H and K be two Hilbert spaces, $T : H \to K$ be a nonzero ε-orthogonality preserving linear mapping, $\varepsilon \in [0, 1)$, and $T = U|T|$ be its polar decomposition. Then U is an isometry and*

$$\Big\|T - \|T\|U\Big\| \leq \Big(1 - \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\Big)\|T\|.$$

Wójcik extended Theorem 21.44 as follows. (The same result is later obtained in [91] by using a different approach.)

Theorem 21.45 ([81, Theorem 5.4]) *Let H be Hilbert space, $T : H \to H$ be a nonzero linear ε-orthogonality preserving mapping and let $\varepsilon \in [0, 1)$. Then there exists linear mapping $S : H \to H$ preserving orthogonality such that*

$$\left\| T - S \right\| \le \frac{1}{2}\Big(1 - \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\Big)\|T\|.$$

Moreover, $\|S\| = \frac{1}{2}(\|T\| + [T])$ *and* $\|T - S\| = \frac{1}{2}(\|T\| - [T])$.

Kong and Cao [47] considered the class of (δ, ε)-orthogonality preserving linear mappings. They proved the following result.

Theorem 21.46 *Let $\delta, \varepsilon \in [0, 1)$. Let H, K be Hilbert spaces and let $T : H \to K$ be a nonzero (δ, ε)-orthogonality preserving linear mapping. Then there exists $\lambda_0 \in \{z \in \mathbb{C} : \frac{\delta+1}{2} \le |z| \le \delta + 2\}$ such that*

$$\sqrt{\frac{|\lambda_0|^2 - \varepsilon|\lambda_0|^2}{(\delta+1)^2 + \varepsilon(\delta+1)^2}}\|T\|\|x\| \le \|Tx\| \le \|Tx\|\|x\| \qquad (x \in H).$$

An stronger version of the previous theorem proved by Wójcik in [81].

Theorem 21.47 ([81, Theorem 3.4]) *Let $\delta, \varepsilon \in [0, 1)$. Let H, K be Hilbert spaces and let $T : H \to K$ be a nonzero (δ, ε)-orthogonality preserving linear mapping. Then T is injective, continuous and $\delta \le \varepsilon$. Moreover the following inequality is true:*

$$\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\sqrt{\frac{1+\delta}{1-\delta}}\|T\|\|x\| \le \|Tx\| \le \|Tx\|\|x\| \qquad (x \in H).$$

As for a stability problem, we would like to know, whether each (δ, ε)-orthogonality preserving linear mapping T can be approximated by a linear orthogonality preserving map S. Kong and Cao [47] proved the following result.

Theorem 21.48 *Let $\delta, \varepsilon \in [0, 1)$. Let H, K be Hilbert spaces and let $T : H \to K$ be linear mapping (δ, ε)-orthogonality preserving. Let $T = U|T|$ be its polar decomposition. Then U is an isometry and there exists $\lambda_0 \in \mathbb{C}$ such that*

$$\left\| T - \|T\|U \right\| \le \Big(1 - \sqrt{\frac{|\lambda_0|^2 - \varepsilon|\lambda_0|^2}{(\delta+1)^2 + \varepsilon(\delta+1)^2}}\Big)\|T\|.$$

Wójcik extended Theorem 21.48 as follows.

Theorem 21.49 *Let $\delta, \varepsilon \in [0, 1)$. Let H be Hilbert space, and let $T : H \to K$ be linear mapping (δ, ε)-orthogonality preserving. Then there exists linear mapping $S : H \to H$ preserving orthogonality such that*

$$\left\| T - S \right\| \le \frac{1}{2}\left(1 - \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\sqrt{\frac{1+\delta}{1-\delta}}\right)\|T\|.$$

Moreover, $\|S\| = \frac{1}{2}(\|T\| + [T])$ *and* $\|T - S\| = \frac{1}{2}(\|T\| - [T])$.

For Hilbert C^*-modules some analogous results can be found in [35, 59]. Recently, Chmieliński et al. [23] have been verified the approximate orthogonality preserving property for two linear mappings. Similar investigations have been carried out for pairs of mappings on inner product C^*-modules in [33].

Approximate orthogonality preserving property has been considered also in the setting of normed linear spaces with respect to various definitions of orthogonality in general normed linear spaces.

Let X, Y be two normed linear spaces, $\delta, \varepsilon \in [0, 1)$ and

$$\diamondsuit, \heartsuit \in \{B, cI, R, P, \rho_{\pm}, \rho, \rho_*, \rho_\lambda, \rho_\lambda^v\}.$$

A mapping $T : X \to Y$ is called a (δ, ε)-$(\diamondsuit, \heartsuit)$-orthogonality preserving if

$$x \perp_\diamondsuit^\delta y \Rightarrow Tx \perp_\heartsuit^\varepsilon Ty, \qquad (x, y \in X).$$

In particular, we say that T is ε-$\diamondsuit$-orthogonality preserving if

$$x \perp_\diamondsuit y \Rightarrow Tx \perp_\diamondsuit^\varepsilon Ty, \qquad (x, y \in X).$$

Mojškerc and Turnšek [57] considered the class of linear mappings approximately preserving the Birkhoff–James orthogonality. They proved the following result.

Theorem 21.50 ([57, Theorem 3.5 and Remark 3.1]) *Let X, Y be two normed linear spaces, $\varepsilon \in [0, \frac{1}{16})$ and let $T : X \to Y$ be a nonzero ε-B-orthogonality preserving linear mapping. Then T is injective, continuous and*

$$(1 - 16\varepsilon)\|T\|\|x\| \le \|Tx\| \le \|T\|\|x\| \qquad (x \in X).$$

Also, if X and Y are real normed linear spaces, then the constant $(1 - 16\varepsilon)$ *can be replaced by* $(1 - 8\varepsilon)$ *with* $\varepsilon \in [0, \frac{1}{8})$.

The following result was proved in [87]:

Theorem 21.51 ([87, Theorem 3.2]) *Let X, Y be two real normed linear spaces, and let $0 < b \le a$ and $\delta, \varepsilon \in [0, \frac{b}{a})$. Let $T : X \longrightarrow Y$ be a nonzero linear (δ, ε)-(aI, bI)-orthogonality preserving mapping. Then $\delta \le \frac{a-b+\varepsilon(a+b)}{a+b-\varepsilon(a-b)}$ and T is injective, continuous and satisfies*

$$\frac{(1+\delta)(b-\varepsilon a)}{(1-\delta)(a+\varepsilon b)}\gamma\|x\| \le \|Tx\| \le \frac{(1-\delta)(a+\varepsilon b)}{(1+\delta)(b-\varepsilon a)}\gamma\|x\|$$

for all $x \in X$ and for all $\gamma \in [\,[T], \|T\|\,]$.

Taking $a = b = 1$ and $\delta = 0$ we get from Theorem 21.51 the following result.

Theorem 21.52 ([20, Theorem 3.2]) *Let X and Y be two real normed linear spaces, and let $\varepsilon \in [0, 1)$. Let $T : X \longrightarrow Y$ be a nonzero linear ε-I-orthogonality preserving mapping. Then T is injective, continuous and satisfies*

$$\frac{1-\varepsilon}{1+\varepsilon}\|T\|\|x\| \leq \|Tx\| \leq \frac{1+\varepsilon}{1-\varepsilon}[T]\|x\| \qquad (x \in X). \tag{21.4}$$

In the next Theorem we formulate a result from Theorem 21.52.

Theorem 21.53 ([20, Theorem 3.6]) *Let X and Y be two real normed linear spaces, and let $\varepsilon \in [0, 1)$. For a nontrivial linear mapping $T : X \longrightarrow Y$ the following conditions are equivalent:*

(i) *T preserves ε-I-orthogonality.*
(ii) $\frac{1-\varepsilon}{1+\varepsilon}\|T\|\|x\| \leq \|Tx\| \leq \frac{1+\varepsilon}{1-\varepsilon}[T]\|x\|$ *for all $x \in X$.*
(iii) $\frac{1-\varepsilon}{1+\varepsilon}\gamma\|x\| \leq \|Tx\| \leq \frac{1+\varepsilon}{1-\varepsilon}\gamma\|x\|$ *for all $x \in X$ and for all $\gamma \in [\,[T], \|T\|\,]$.*
(iv) $\|Tx\|\,\|y\| \leq \frac{1+\varepsilon}{1-\varepsilon}\|Ty\|\,\|x\|$ *for all $x, y \in X$.*
(v) $\|T\| \leq \frac{1+\varepsilon}{1-\varepsilon}[T]$.

As consequences of Theorem 21.51, we have the following results.

Corollary 21.7 ([87, Corollary 3.3]) *Let X, Y be two real normed linear spaces, and let $0 < b \leq a$ and $\varepsilon, \delta \in [0, \frac{b}{a})$. Let $T : X \longrightarrow Y$ be a linear (δ, ε)-(aI, bI)-orthogonality preserving mapping with $0 \leq \frac{a-b+\varepsilon(a+b)}{a+b-\varepsilon(a-b)} < \delta$. Then $T = 0$.*

Corollary 21.8 ([87, Corollary 3.4]) *Let X, Y be two real normed linear spaces, and let $0 < b \leq a$ and $\varepsilon, \delta \in [0, \frac{b}{a})$. Let $T : X \longrightarrow Y$ be a nonzero linear (δ, ε)-(aI, bI)-orthogonality preserving mapping. If a linear mapping $S : X \to Y$ satisfies $\|S - T\| \leq \theta\|T\|$, then $\|S\| \leq \eta[S]$, where $\eta = \frac{(1-\delta)^2(a+\varepsilon b)^2+\theta(1-\delta^2)(a+\varepsilon b)(b-\varepsilon a)}{(1+\delta)^2(b-\varepsilon a)^2-\theta(1-\delta^2)(a+\varepsilon b)(b-\varepsilon a)}$.*

Wójcik in [77] was obtain the following result for the stability of the orthogonality preserving mappings for the finite-dimensional spaces.

Theorem 21.54 ([77]) *Let X and Y be finite-dimensional real normed linear spaces, and let $\diamondsuit \in \{D, I, R, P\}$. Then, for an arbitrary $\theta > 0$, there exists $\varepsilon > 0$ such that for any linear ε-$\diamondsuit$-orthogonality preserving mapping $T : X \longrightarrow Y$ there exists a linear $\diamondsuit$-orthogonality preserving mapping $S : X \longrightarrow Y$ such that*

$$\|T - S\| \leq \theta \min\{\|T\|, \|S\|\}.$$

Some other results for the stability of the orthogonality preserving property in normed linear spaces can be found in [16, 20, 57, 68, 76].

Approximate orthogonality preserving mappings have been also considered for norm derivatives orthogonality relations.

Remark 21.3 Let X, Y be normed linear spaces and let $T : X \to Y$ be linear mapping. Then, T approximately preserves ρ_+-orthogonality if and only if T approximately preserves ρ_--orthogonality. Indeed, suppose that T approximately preserves ρ_+-orthogonality and let $x \perp_{\rho_-} y$. Thus $-x \perp_{\rho_+} y$, hence $-Tx \perp^{\varepsilon}_{\rho_+} Ty$ and finally $Tx \perp^{\varepsilon}_{\rho_-} Ty$, i.e., T approximately preserves ρ_--orthogonality. The proof of the reverse is the same.

Chmieliński and Wójcik in [22, Theorem 5.1] proved that an approximate $\rho_\pm$-orthogonality preserving mapping is an approximate B-orthogonality preserving mapping. Next, Wójcik [80, Theorem 5.5] obtained a same result for approximate ρ-orthogonality preserving mappings. More precisely, he proved that the property that a linear mapping approximately preserves the Birkhoff–James orthogonality is equivalent to that it approximately preserves the ρ and $\rho_\pm$-orthogonality (the proof of which is by no means elementary). The same result was proved in [27, Theorem 2.7] for approximate ρ_*-orthogonality preserving mappings. Thus, from Theorem 21.50, we obtain the following characterization of linear mappings approximately preserving the orthogonality relations.

Theorem 21.55 *Let X, Y be two real normed linear space and let $\varepsilon \in [0, \frac{1}{8})$. If $T : X \to Y$ is a nonzero linear mapping, then the following conditions are equivalent:*

(i) T is ε-ρ_--orthogonality preserving.
(ii) T is ε-ρ_+-orthogonality preserving.
(iii) T is ε-ρ-orthogonality preserving.
(iv) T is ε-ρ_*-orthogonality preserving.
(v) T is ε-B-orthogonality preserving.

Moreover, each of the above conditions implies that T is injective, continuous and

$$(1 - 8\varepsilon)\|T\|\|x\| \le \|Tx\| \le \|T\|\|x\| \qquad (x \in X).$$

Note that, in particular for $\diamondsuit, \heartsuit \in \{B, \rho_\pm, \rho, \rho_*\}$ with $\diamondsuit \neq \heartsuit$, the property that a linear mapping approximately preserves the $\diamondsuit$-orthogonality is equivalent to that it approximately preserves the $\heartsuit$-orthogonality. Although $\perp^{\varepsilon}_{\diamondsuit}$ and $\perp^{\varepsilon}_{\heartsuit}$ need not be equivalent unless we assume the smoothness of the norm.

Taking $X = Y$ and $T = id$, one obtains, from Theorem 21.55, the following result.

Corollary 21.9 *Let $\varepsilon \in [0, \frac{1}{8})$ and $\diamondsuit \in \{B, \rho_\pm, \rho, \rho_*\}$. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms in a linear space X. By $\perp_1$ and $\perp_2$ we denote the $\diamondsuit$-orthogonality with respect to one of the two norms. If $\perp_1 \subset \perp^{\varepsilon}_2$, then both norms are equivalent and, with some $\gamma > 0$, we have*

$$(1 - 8\varepsilon)\gamma\|x\|_1 \le \|x\|_2 \le \gamma\|x\|_1 \qquad (x \in X).$$

Recently, the class of linear mappings approximately preserving ρ_λ^υ-orthogonality has been studied in [1]. Although $\perp^\varepsilon_{\rho_\lambda^\upsilon}$ need not be equivalent to $\perp^\varepsilon_B$, unless we assume the smoothness of the norm, it has been proved in [1] the following result.

Theorem 21.56 ([1, Theorem 3.4]) *Let X and Y be normed linear spaces and let $\varepsilon \in [0, 1)$, $\lambda \in [0, 1]$ and $\upsilon = \frac{1}{2k+1}$ $(k \in \mathbb{N})$. If $T : X \to Y$ is a nonzero linear mapping, then the following conditions are equivalent:*

(i) T is ε-ρ_λ^υ-orthogonality preserving.
(ii) T is ε-B-orthogonality preserving.

Moreover, each of the above conditions implies that T is injective, continuous and

$$(1 - 8\varepsilon)\|T\|\|x\| \le \|Tx\| \le \|T\|\|x\| \qquad (x \in X).$$

As an special case, the authors in [1] proved a similar result for approximately ρ_λ-orthogonality preserving linear mappings.

Corollary 21.10 ([1, Corollary 3.6]) *Let X and Y be normed linear spaces and let $\varepsilon \in [0, 1)$ and $\lambda \in [0, 1]$. If $T : X \to Y$ is a nonzero linear mapping, then the following conditions are equivalent:*

(i) T is ε-ρ_λ-orthogonality preserving.
(ii) T is ε-B-orthogonality preserving.

Moreover, each of the above conditions implies that T is injective, continuous and

$$(1 - 8\varepsilon)\|T\|\|x\| \le \|Tx\| \le \|T\|\|x\| \qquad (x \in X).$$

In particular, if we take $\varepsilon = 0$ in the foregoing corollary, then we obtain that every ρ_λ-orthogonality preserving mapping is a similarity. We should notify that this result was already shown in [88, Theorem 3.4] with a different approach.

Acknowledgement The authors would like to thank the referee for her/his valuable suggestions and comments.

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