Stabilities of MIQD and MIQA Functional Equations via Fixed Point Technique

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Abstract In this chapter, we investigate the stabilities of multiplicative inverse quadratic difference and multiplicative inverse quadratic adjoint functional equations in the setting of non-Archimedean fields via fixed point method.

1 Introduction and Preliminaries

The question posed by Ulam [15] in 1940 is the basis for the theory of stability of functional equations. The question raised by Ulam was answered by Hyers [5] which made a cornerstone in the study of stability of functional equations. The result obtained by Hyers is termed as Hyers–Ulam stability or ϵ-stability of functional equation. Then, Hyers' result was generalized by Aoki [1]. Also, Hyers' result was modified by Rassias [10] considering the upper bound as sum of powers of norms (Hyers–Ulam–T. Rassias stability). Rassias [11] established Hyers' result by taking the upper bound as product of powers of norms (Ulam–Gavruta–J. Rassias stability). In 1994, the stability result was further generalized into simple form by Gavruta [4] by replacing the upper bound by a general control function (generalized Hyers–Ulam stability).

In recent times, Ravi and Suresh [12] have investigated the generalized Hyers–Ulam stability of multiplicative inverse quadratic functional equation in two variables of the form

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$$R(u+v) = \frac{R(u)R(v)}{R(u)+R(v)+2\sqrt{R(u)R(v)}} \tag{1}$$

in the setting of real numbers. It is easy to verify that the multiplicative inverse quadratic function $R(u) = \frac{1}{u^2}$ is a solution of Eq. (1). For further stability results using fixed point method concerning different types of functional equations and rational functional equations, one may refer [2, 6, 7, 9, 13, 14].

Here, we evoke a few fundamental notions of non-Archimedean field and fixed point alternative theorem in non-Archimedean spaces. Throughout this chapter, let us assume that N and R are the sets of natural numbers and real numbers, respectively.

Definition 1 Let F be a field with a mapping (valuation) $|\cdot|$ from F into $[0,\infty)$. Then F is said to be a non-Archimedean field if the upcoming requirements persist:

(i) $|k| = 0$ if and only if $k = 0$;
(ii) $|k_1k_2| = |k_1||k_2|$;
(iii) $|k_1 + k_2| \leq max\{|k_1|, |k_2|\}$

for all $k, k_1, k_2 \in F$.

It is evident that $|1| = |-1| = 1$ and $|k| \leq 1$ for all $k \in N$. Furthermore, we presume that $|\cdot|$ is non-trivial, that is, there exists an $\alpha_0 \in F$ such that $|\alpha_0| \neq 0, 1$.

Suppose E is a vector space over a scalar field F with a non-Archimedean non-trivial valuation $|\cdot|$. A function $||\cdot|| : E \longrightarrow R$ is a *non-Archimedean norm* (*valuation*) if it satisfies the ensuing requirements:

(i) $||u|| = 0$ if and only if $u = 0$;
(ii) $||\rho u|| = |\rho|||u|| \quad (\rho \in F, u \in E)$;
(iii) the strong triangle inequality (ultrametric); namely,

$$||u+v|| \leq \max\{||u||, ||v||\} \quad (u, v \in E).$$

Then $(E, ||\cdot||)$ is known as a non-Archimedean space. By virtue of the inequality

$$\|u_\ell - u_k\| \leq \max\left\{\left\|u_{j+1} - u_j\right\| : k \leq j \leq \ell - 1\right\} \quad (\ell > k),$$

a sequence $\{u_k\}$ is Cauchy if and only if $\{u_{k+1} - u_k\}$ converges to 0 in a non-Archimedean space. If every Cauchy sequence is convergent in the space, then it is called as complete non-Archimedean space.

Definition 2 Assume H is a nonempty set. Suppose $d : H \times H \to [0,\infty]$ satisfies the following properties:

(i) $d(\alpha, \beta) = 0$ if and only if $\alpha = \beta$;
(ii) $d(\alpha, \beta) = d(\beta, \alpha)$ (symmetry);
(iii) $d(\alpha, \gamma) \leq \max\{d(\alpha, \beta), d(\beta, \gamma)\}$ (strong triangle inequality)

for all $\alpha, \beta, \gamma \in H$. Then (H, d) is called a generalized non-Archimedean metric space. Suppose every Cauchy sequence in A is convergent, then (A, d) is called complete.

Example 1 Let F be a non-Archimedean field. Assume A and B are two non-Archimedean spaces over F. If B is complete and $\phi : A \longrightarrow [0, \infty)$, for every $s, t : A \longrightarrow B$, define

$$d(s,t) = \inf\{\delta > 0 : |s(u) - t(u)| \leq \delta\phi(u), \text{ for all } u \in A\}.$$

Using Theorem 2.5 [3], Mirmostafaee [8] proposed new version of the alternative fixed point principle in the setting of non-Archimedean space as follows:

Theorem 1 ([8] (Alternative Fixed Point Principle in Non-Archimedean Scheme)) *Suppose (H, d) is a non-Archimedean generalized metric space. Let a mapping $\Lambda : H \longrightarrow H$ be a strictly contractive, (that is, $d(\Lambda(u), \Lambda(v)) \leq \rho d(v, u)$, for all $u, v \in H$ and a Lipschitz constant $\rho < 1$), then either*

(i) $d\left(\Lambda^p(u), \Lambda^{p+1}u\right) = \infty$ *for all* $p \geq 0$*, or;*
(ii) there exists some $p_0 \geq 0$ *such that* $d\left(\Lambda^p(u), \Lambda^{p+1}(u)\right) < \infty$ *for all* $p \geq p_0$*;*

the sequence $\{\Lambda^p(u)\}$ is convergent to a fixed point $u^\star$ of Λ; $u^\star$ is the distinct invariant point of Λ in the set $Y = \{y \in X : d(\Lambda^{p_0}(u), v) < \infty\}$ and $d(v, u^\star) \leq d(v, \Lambda(v))$ for all v in this set.

In this chapter, we consider the following functional equations

$$R_q\left(\frac{u+v}{2}\right) - R_q(u+v) = \frac{3R_q(u)R_q(v)}{R_q(u) + R_q(v) + 2\sqrt{R_q(u)R_q(v)}} \tag{2}$$

and

$$R_q\left(\frac{u+v}{2}\right) + R_q(u+v) = \frac{5R_q(u)R_q(v)}{R_q(u) +_q (v) + 2\sqrt{R_q(u)R_q(v)}}. \tag{3}$$

Clearly, the multiplicative inverse quadratic function $R_q(u) = \frac{1}{u^2}$ satisfies Eqs. (2) and (3). Hence, Eqs. (2) and (3) are called as Multiplicative Inverse Quadratic Difference (MIQD) functional equation and Multiplicative Inverse Quadratic Adjoint (MIQA) functional equation, respectively. We prove the stabilities of the above Eqs. (2) and (3) in non-Archimedean fields by fixed point approach.

Let us presume that E and F are a non-Archimedean field and a complete non-Archimedean field, respectively, in this chapter. In the sequel, we represent $E^* = E\backslash\{0\}$, where E is a non-Archimedean field. For the sake of easy computation, we describe the difference operators $\Delta_1 R_q, \Delta_2 R_q : E^* \times E^* \longrightarrow F$ by

$$\Delta_1 R_q(u,v) = R_q\left(\frac{u+v}{2}\right) - R_q(u+v) - \frac{3R_q(u)R_q(v)}{R_q(u) + R_q(v) + 2\sqrt{R_q(u)R_q(v)}}$$

and

$$\Delta_2 R_q(u, v) = R_q\left(\frac{u+v}{2}\right) + R_q(u+v) - \frac{5R_q(u)R_q(v)}{R_q(u) + R_q(v) + 2\sqrt{R_q(u)R_q(v)}}$$

for all $u, v \in E^*$.

2 Solution of Eqs. (2) and (3)

In this section, we attain the solution of functional equations (2) and (3). In the following, we denote $R\backslash\{0\}$ by R^*.

Theorem 2 *A mapping $R_q : R^* \longrightarrow R$ satisfies Eq.* (1) *if and only if $R_q : R^* \longrightarrow R$ satisfies Eq.* (2) *if and only if $R_q : R^* \longrightarrow R$ satisfies Eq.* (3)*. Therefore, every solution of Eqs.* (2) *and* (3) *is also a multiplicative inverse quadratic mapping.*

Proof Let $R_q : R^* \longrightarrow R$ satisfy Eq. (1). Switching v into u in (1), we obtain

$$R_q(2u) = \frac{1}{4}R_q(u) \tag{4}$$

for all $u \in R^*$. Now, letting u to $\frac{u}{2}$ in (4), one finds

$$R_q\left(\frac{u}{2}\right) = 4R_q(u) \tag{5}$$

for all $u \in R^*$. Again, substituting (u, v) by $(\frac{u}{2}, \frac{v}{2})$ in (1) and applying (5), we obtain

$$R_q\left(\frac{u+v}{2}\right) = \frac{4R_q(u)R_q(v)}{R_q(u) + R_q(v) + 2\sqrt{R_q(u)R_q(v)}} \tag{6}$$

for all $u, v \in R^*$. Subtracting (1) from (6), we arrive at (3).

Now, suppose $R_q : R^* \longrightarrow R$ satisfies Eq. (3). Plugging v by u in (3), we obtain

$$R_q(2u) = \frac{1}{4}R_Q(u) \tag{7}$$

for all $u \in R^*$. Now, replacing u by $\frac{u}{2}$ in (7), we get

$$R_q\left(\frac{u}{2}\right) = 4R_q(u) \tag{8}$$

for all $u \in R^*$. Using (8) in (3), we obtain

$$R_q(u+v) = \frac{R_q(u)R_q(v)}{R_q(u) + R_q(v) + \sqrt{R_q(u)R_q(v)}} \tag{9}$$

for all $u, v \in R^*$. Now, summing (9) with (3), we lead to (3).

Lastly, suppose $R_q : R^* \longrightarrow R$ satisfies Eq. (3). Letting $v = u$ in (3), we obtain

$$R_q(2u) = \frac{1}{4}R_q(u) \tag{10}$$

for all $u \in R^*$. Replacing u by $\frac{u}{2}$ in (10), we obtain

$$R_q\left(\frac{u}{2}\right) = 4R_q(u) \tag{11}$$

for all $u \in R^*$. In lieu of (11) and (3), we arrive at (1), which completes the proof.

3 Stabilities of Eqs. (2) and (3)

In this section, we investigate stabilities of Eqs. (2) and (3) via fixed point method in non-Archimedean fields.

Theorem 3 *Assume a mapping $R_q : E^* \longrightarrow F$ satisfies the inequality*

$$\left|\Delta_1 R_q(u, v)\right| \leq \varphi(u, v) \tag{12}$$

for all $u, v \in E^*$, where $\varphi : E^* \times E^* \longrightarrow F$ is a given function. If $0 < L < 1$,

$$|2|^{-2}\varphi\left(2^{-1}u, 2^{-1}v\right) \leq L\varphi(u, v) \tag{13}$$

for all $u, v \in E^*$, then there exists a unique multiplicative inverse quadratic mapping $r_d : E^* \longrightarrow F$ satisfying Eq. (2) and

$$|R_q(u) - r_d(u)| \leq L|2|^2\varphi(u, u) \tag{14}$$

for all $u \in E^*$.

Proof Plugging (u, v) by $\left(\frac{u}{2}, \frac{u}{2}\right)$ in (12), we obtain

$$\left|R_q(u) - 2^{-2}R_q\left(2^{-1}u\right)\right| \leq \varphi\left(2^{-1}u, 2^{-1}u\right) \tag{15}$$

for all $u \in E^*$. Let $\mathcal{A} = \{p | p : E^* \longrightarrow F\}$, and define

$$d(p, q) = \inf\{\gamma > 0 : |p(u) - q(u)| \leq \gamma\varphi(u, u), \ \text{ for all } u \in E^*\}.$$

In lieu of Example 1, we find that d turns into a complete generalized non-Archimedean complete metric on $\mathcal{A}$. Let $\Gamma : \mathcal{A} \longrightarrow \mathcal{A}$ be a mapping defined by

$$\Gamma(p)(u) = 2^{-2} p\left(2^{-1}u\right)$$

for all $u \in E^*$ and $p \in \mathcal{A}$. Then Γ is strictly contractive on $\mathcal{A}$, in fact if $|p(u) - q(u)| \leq \gamma\varphi(u,u), \quad (u \in E^*)$, then by (13), we obtain

$$\begin{aligned}|\Gamma(p)(u) - \Gamma(q)(u)| &= |2|^{-2}\left|p\left(2^{-1}u\right) - q\left(2^{-1}u\right)\right| \\ &\leq \gamma|2|^{-2}\varphi\left(2^{-1}u, 2^{-1}u\right) \\ &\leq \gamma L\varphi(u,u) \quad (u \in E^*).\end{aligned}$$

From the above, we conclude that

$$(\Gamma(p), \Gamma(q)) \leq Ld(p,q) \;\; (p, q \in \mathcal{A}).$$

Consequently, the mapping d is strictly contractive with Lipschitz constant L. Using (15), we have

$$\begin{aligned}|\rho(u)(s) - u(s)| &= \left|3^{-11}u\left(3^{-1}s\right) - u(s)\right| \\ &\leq \zeta\left(\frac{s}{3}, \frac{s}{3}\right) \leq |3|^{11}L\zeta(s,s) \quad (s \in G^*).\end{aligned}$$

This indicates that $d(\Gamma(R_q), R_q) \leq L|2|^2$. Due to Theorem 1 (ii), Γ has a distinct invariant point $r_d : E^* \longrightarrow F$ in the set $G = \{g \in F : d(u, g) < \infty\}$ and for each $u \in E^*$, $r_d(u) = \lim_{s\to\infty} \Gamma^s R_q(u) = \lim_{s\to\infty} 2^{-2s} R_q\left(2^{-s}u\right) \quad (u \in E^*)$. Therefore, for all $u, v \in E^*$,

$$\begin{aligned}|\Delta_1 r_d(u,v)| &= \lim_{s\to\infty} |2|^{-2s}\left|\Delta_1 R_q\left(2^{-s}u, 2^{-s}v\right)\right| \\ &\leq \lim_{s\to\infty} |2|^{-2s}\varphi\left(2^{-s}u, 2^{-s}v\right) \\ &\leq \lim_{s\to\infty} L^s\varphi(u,v) = 0\end{aligned}$$

which shows that r_d is multiplicative inverse quadratic. Theorem 1 (ii) implies $d(R_q, r_d(u)) \leq d(\Gamma(R_q), R_q)$, that is, $|R_q(u) - r_d(u)| \leq |2|^2 L\varphi(u,u) \quad (u \in E^*)$. Let $r_d' : E^* \longrightarrow F$ be a multiplicative inverse quadratic mapping which satisfies (14), then r_d' is a fixed point of Γ in $\mathcal{A}$. However, by Theorem 1, Γ has only one invariant in G. This completes the distinctiveness allegation of the theorem.

The following theorem is dual of Theorem 3. We skip the proof as it is analogous to Theorem 3.

Theorem 4 *Suppose the mapping* $R_q : E^* \longrightarrow F$ *satisfies the inequality* (13). *If* $0 < L < 1$,

$$|2|^2\varphi(2u, 2v) \leq L\varphi(u, v),$$

for all $u, v \in E^*$, *then there exists a unique multiplicative inverse quadratic mapping* $r_d : E^* \longrightarrow F$ *satisfying Eq.* (2) *and*

$$|R_q(u) - r_d(u)| \leq L\varphi\left(\frac{u}{2}, \frac{u}{2}\right),$$

for all $u \in E^*$.

The following corollaries follow directly from Theorems 3 and 4. In the following corollaries, we assume that $|2| < 1$ for a non-Archimedean field E.

Corollary 1 *Let* ϵ*(independent of* u, v*)*≥ 0 *be a constant exists for a mapping* $R_q : E^* \longrightarrow F$ *such that the functional inequality satisfies*

$$\left|\Delta_1 R_q(u, v)\right| \leq \epsilon,$$

for all $u, v \in E^*$. *Then there exists a unique multiplicative inverse quadratic mapping* $r_d : E^* \longrightarrow F$ *satisfying Eq.* (2) *and*

$$|R_q(u) - r_d(u)| \leq \epsilon,$$

for all $u \in E^*$.

Proof Assuming $\varphi(u, v) = \epsilon$ and selecting $L = |2|^{-2}$ in Theorem 3, we get the desired result.

Corollary 2 *Let* $\lambda \neq -2$ *and* $c_1 \geq 0$ *be real numbers exists for a mapping* $R_q : E^* \longrightarrow F$ *such that the following inequality holds*

$$\left|\Delta_1 R_q(u, v)\right| \leq c_1\left(|u|^\lambda + |v|^\lambda\right),$$

for all $u, v \in E^*$. *Then there exists a unique multiplicative inverse quadratic mapping* $r_d : E^* \longrightarrow F$ *satisfying Eq.* (2) *and*

$$\left|R_q(u) - r_d(u)\right| \leq \begin{cases} \frac{|2|c_1}{|2|^\lambda}|u|^\lambda, & \lambda > -2 \\ |2|^3c_1|u|^\lambda, & \lambda < -2 \end{cases}$$

for all $u \in E^*$.

Proof Consider $\varphi(u, v) = c_1 \left(|u|^{\lambda} + |v|^{\lambda}\right)$ in Theorems 3 and 4 and then assume $L = |2|^{-\lambda-2}$, $\lambda > -2$ and $L = |2|^{\lambda+2}$, $\lambda < -2$, respectively, the proof follows directly.

Corollary 3 *Let $c_2 \geq 0$ and $\lambda \neq -2$ be real numbers, and $R_q : E^* \longrightarrow F$ be a mapping satisfying the functional inequality*

$$\left|\Delta_1 R_q(u, v)\right| \leq c_2 |u|^{\lambda/2} |v|^{\lambda/2},$$

for all $u, v \in E^*$. Then there exists a unique multiplicative inverse quadratic mapping $r_d : E^* \longrightarrow F$ satisfying Eq. (2) and

$$|R_q(u) - r_d(u)| \leq \begin{cases} \frac{c_2}{|2|^{\lambda}} |u|^{\lambda}, & \lambda > -2 \\ |2|^2 c_2 |u|^{\lambda}, & \lambda < -2 \end{cases}$$

for all $u \in E^*$.

Proof It is easy to prove this corollary, by taking $\varphi(u, v) = c_2 |u|^{\lambda/2} |v|^{\lambda/2}$ and then choosing $L = |2|^{-\lambda-2}$, $\lambda > -2$ and $L = |2|^{\lambda+2}$, $\lambda < -2$, respectively in Theorems 3 and 4.

In the sequel, using fixed point technique, we investigate the stabilities of Eq. (3) in the framework of non-Archimedean fields. Since the proof of the subsequent results is akin to the results of Eq. (2), for the sake of completeness, we state only theorems and skip their proofs.

Theorem 5 *Let $R_q : E^* \longrightarrow F$ be a mapping satisfying the inequality*

$$\left|\Delta_2 R_q(u, v)\right| \leq \xi(u, v) \tag{16}$$

for all $u, v \in E^*$, where $\xi : E^* \times E^* \longrightarrow [0, \infty)$ is an arbitrary function. If $0 < L < 1$,

$$|2|^{-2} \xi\left(2^{-1} u, 2^{-1} v\right) \leq L \xi(u, v),$$

for every $u, v \in E^*$, then there exists a unique multiplicative inverse quadratic mapping $r_a : E^* \longrightarrow F$ gratifying Eq. (3) and

$$|R_q(u) - r_a(u)| \leq L |2|^2 \xi(u, u),$$

for each $u \in E^*$.

Theorem 6 *Let $R_q : E^* \longrightarrow F$ be a mapping satisfying the inequality (16). If $0 < L < 1$,*

$$|2|^2 \xi(2u, 2v) \leq L \xi(u, v),$$

for every $u, v \in E^*$, *then there exists a unique multiplicative inverse quadratic mapping* $r_a : E^* \longrightarrow F$ *satisfying Eq.* (3) *and*

$$|R_q(u) - r_a(u)| \leq L\xi\left(\frac{u}{2}, \frac{u}{2}\right)$$

for each $u \in E^*$.

Corollary 4 *Let* θ (*independent of* u, v)≥ 0 *be a constant. Suppose a mapping* $R_q : E^* \longrightarrow F$ *satisfies the inequality*

$$\left|\Delta_2 R_q(u, v)\right| \leq \theta$$

for every $u, v \in E^*$. *Then there exists a unique multiplicative inverse quadratic mapping* $r_a : E^* \longrightarrow F$ *satisfying Eq.* (3) *and*

$$|R_q(u) - r_a(u)| \leq \theta,$$

for each $u \in E^*$.

Corollary 5 *Let* $\alpha \neq -2$ *and* $\delta_1 \geq 0$ *be real numbers. If* $R_q : E^* \longrightarrow F$ *is a mapping satisfying the inequality*

$$\left|\Delta_2 R_q(u, v)\right| \leq \delta_1\left(|u|^\alpha + |v|^\alpha\right),$$

for every $u, v \in E^*$, *then there exists a unique multiplicative inverse quadratic mapping* $r_a : E^* \longrightarrow F$ *satisfying Eq.* (3) *and*

$$|R_q(u) - r_a(u)| \leq \begin{cases} \frac{|2|\delta_1}{|2|^\alpha}|u|^\alpha, & \alpha > -2 \\ |2|^3\delta_1|u|^\alpha, & \alpha < -2 \end{cases}$$

for each $u \in E^*$.

Corollary 6 *Let* $R_q : E^* \longrightarrow F$ *be a mapping and* $\delta_2 \geq 0$ *and* $\alpha \neq -2$ *be real numbers. If the mapping* R_q *satisfies the functional inequality*

$$\left|\Delta_2 R_q(u, v)\right| \leq \delta_2|u|^{\alpha/2}|v|^{\alpha/2}$$

for every $u, v \in E^*$, *then there exists a unique multiplicative inverse quadratic mapping* $r_a : E^* \longrightarrow F$ *satisfying Eq.* (3) *and*

$$|R_q(u) - r_a(u)| \leq \begin{cases} \frac{\delta_2}{|2|^\alpha}|u|^\alpha, & \alpha > -2 \\ |2|^2\delta_2|u|^\alpha, & \alpha < -2 \end{cases}$$

for each $u \in E^*$.

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